Index Theory

Himanshu Yadav

A dissertation submitted for the partial fulfilment of BS-MS dual degree in Science

Indian Institute of Science Education and Research Mohali

April 2018

Certificate of Examination

This is to certify that the dissertation titled **Index Theory** submitted by **Himanshu Yadav** (Reg. No. MS12060) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Alok Maharana   Dr. Lingaraj Sahu   Dr. Kapil Hari Paranjape
(Supervisor)

Dated: April 20, 2018

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Kapil Hari Paranjape at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

Himanshu Yadav
(Candidate)

Dated: April 20, 2018

In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Kapil Hari Paranjape
(Supervisor)

Acknowledgment

I would like to express my sincere gratitude to my adviser Dr. Kapil H. Paranjape for his support and motivation to carry out my MS thesis. His guidance has helped me throughout this project and I am extremely thankful to him for this opportunity and steering me in the right direction.

I would also like to thank my batch mates who improved my understanding by helpful discussions.

Finally, I would like to thank my friends and family and most importantly my parents who continuously encouraged me and have guided me through life.

Notation

I	Identity Operator
$\mathrm{B}(X, Y)$	Set of compact operators from X into Y.
$\mathrm{B}(X)$	Compact operators from X to X.
X, Y	Banach spaces
$\mathrm{L}(X, Y)$	Banach space of all continuous bounded operators from X to Y.
$\mathcal{F}(X, Y)$	Set of all Fredholm operators between X and Y.
$\mathrm{w}(\phi)$	Winding number of the symbol of a Toeplitz operator
$\mathbb{Z}$	Set of integers
S^1	Circle in a complex plane

Abstract

In this thesis, I have discussed different types of operators and their indices. Beginning with basics of compact operators and their properties, I gave an exposition on Fredholm operators between Banach spaces. I have also shown that compact operators are the furthest from Fredholm operators in infinite dimensional space. And finally, I studied Toeplitz operators which are Fredholm operators with an invertible symbol.

.

Contents

Chapter 1

Compact Operators

1.1 Introduction

Operator theory deals with operators on Banach spaces and its connection with other mathematical subjects. The class of compact operators is resulted directly from the study of the integral equations. The integral operators are the most classical examples of compact operators. The main characteristic of these operators, is that they show similar behaviour with the operators in finite dimensional spaces and thus they can be easily analyzed. Let us now begin with some basic definitions and results about compact operators.

1.2 Preliminaries

Definition 1.1 (Relatively Compact)
A relatively compact subspace Y of a metric space X is a subset whose closure is compact.

Definition 1.2 (Precompact)
Let X be a metric space, $Y \subseteq X$ is said to be precompact or totally bounded if for every $\epsilon > 0$ there exists finitely many points $(x_1, x_2, x_3, ...x_N)$ such that $\bigcup_{i=1}^{N} B(x_i, \epsilon)$ contains Y.
For a complete metric space the notion of relatively compact and precompact are

the same since if Y is a precompact subset of a complete metric space X, then Y is also totally bounded and it is complete because it is closed in X which makes it compact/relatively compact.

1.2.1 Compact Operator

X and Y are Banach spaces. If a linear operator T is such that image of any bounded subset of X under T is relatively compact subset of Y, then T is said to be compact.

Equivalently, a linear operator $\mathsf{T} : X \to Y$ is compact if and only if any of the following is true:

- Image of the closed unit ball in X under T is relatively compact in Y.
- For any sequence $(x_n)_{n\in\mathbb{N}}$ in the unit ball in X, the sequence $(Tx_n)_{n\in\mathbb{N}}$ contains a Cauchy subsequence.

1.2.2 Examples

- The identity operator is a compact operator if and only if the space is finite dimensional.
- Every $m \times n$ matrix corresponds to a compact operator.
- Every bounded finite rank operator is compact.
  Proof Let $\mathsf{T} : X \to Y$ be a bounded finite rank operator that means $im(\mathsf{T})$ is finite-dimensional. And for any bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ is bounded in the image, so this sequence must contain a convergent subsequence by Bolzano-Weierstrass theorem[Sim15]. Hence T is compact.

1.3 Important remarks about compact operators

- Let T_1, $\mathsf{T}_2 : X \to Y$ be compact operators then $\mathsf{T}_1 + \mathsf{T}_2$ is compact.
  Proof: Let $\{x_n\} \subseteq X$ be a bounded sequence. Since T_1 is compact, $\{\mathsf{T}_1 x_n\}$ has a subsequence $\{\mathsf{T}_1 x_{n_k}\}$ which is convergent and it converges to y. Also $\{x_{n_k}\}$

is a bounded sequence. Since T_2 is compact, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $\{\mathsf{T}_2 x_{n_{k_l}}\}$ is convergent. So we can say $\{\mathsf{T}_1 x_{n_k} + \mathsf{T}_2 x_{n_{k_l}}\}$ is convergent. Hence $\mathsf{T}_1 + \mathsf{T}_2$ is compact.

- The scalar multiple of a compact operator is compact.
  Proof: Let T is a compact operator on X and $\mathtt{k}$ be a scalar. Assume $\{x_n\} \subseteq X$ is a bounded sequence. Since T is compact, let $\{x_n\} \subseteq X$ be a bounded sequence means that $\{\mathsf{T}x_n\}$ has a subsequence $\{\mathsf{T}x_{n_k}\}$ which converges to y. As sequence $\{\mathsf{kT}x_n\}$ for the operator kT also converges, kT is compact.

- If T is compact, then T is bounded.
  Proof: Assume that T is not bounded such that $||T(v_n)|| \mapsto \infty$, where v_n is a sequence of bounded vectors in some vector space V with $||v_n|| < 1$ and hence $\{Tv_n\}$ may not have a convergent subsequence which implies T is not compact.

- The converse to the above statement need not be true. For example consider the identity operator:

$$\mathsf{I} : X \to X$$

  If X is not finite dimensional then basis of unit vector with unit norm is linear independent and do not converge. So we establish that I is compact if and only if dimension of X is finite.

- Let X be a Banach space and T and S be in $\mathrm{B}(X)$. If T is compact then so are ST and TS.
  Proof: Consider the mapping ST. Let $\{x_n\}$ be a bounded sequence in X, then there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ that converges in X.

$$Tx_{n_k} \to y^* \in X.$$

  Since S is continuous, it follows that $\{STx_{n_k}\} \to S(y^*)$. Hence the sequence $\{STx_{n_k}\}$ converges in X and so ST is compact.
  To show that TS is compact, take a bounded sequence $\{x_n\}$ in X and note that $\{Sx_n\}$ is bounded also (since S is continuous). Thus there exists a subsequence $\{TSx_{n_k}\}$ which converges in X. Hence TS is compact as well.

- An isometry is compact if and only if it is a finite rank operator.
  Proof: An isometry is a linear transformation which preserves length. Having

said that this statement is equivalent to the one in example 3, page 2. Proof follows the same.

- Restriction of a compact operator to a closed subspace is again a compact operator. (Proof on page 111,[Sim15])

- Suppose X is a Banach space. If $\{T_n\}$ is a sequence of compact operators in $\mathsf{B}(X)$ and $|T_n - \mathsf{T}|$ converges to 0 for some $\mathsf{T} \in \mathsf{B}(X)$ then T is a compact operator as well.
  Proof: Let B be the unit ball in X. Given any $\epsilon > 0$, choose n large enough that $|T_n - T| < \epsilon$. Since T_n is a sequence of compact operators, $T_n(B)$ can be covered by a finite number of balls of radii ϵ of the form $B(T_{x_i}, \epsilon)$. Then $\mathsf{T}(B)$ can be covered by balls of radius 2ϵ of the form $B(T_{x_i}, 2\epsilon)$ proving T a compact operator.

- Let X be a Banach space, K be a bounded linear map from X onto itself. Then all nearby maps of the form K-A, where $|A| <$ e for e small enough also maps X onto itself. (Refer to section 20.1, theorem 1 [Lax02] for the proof).

Compact operators, as a generalization of operators in finite dimensional spaces, show a relatively simple structure. There is another class of operators known as Fredholm Operators, which we will discuss in detail in next chapter, can be regarded as a kind of anti-compact operators.

1.4 Compact Operators: Fredholm Alternative

Before moving on to Fredholm operators, there is yet another important property of compact operators, the Fredholm Alternative. It provides necessary information on the solvability of a certain category of linear equations.
For finite rank cases it is stated as:
Let $\mathsf{K} : R^n \to R^n$, then $(\mathsf{I} - \mathsf{K})x = 0$ or $\mathsf{I} - \mathsf{K}$ is one-one if and only if $(\mathsf{I} - \mathsf{K})x - y = 0$ or $\mathsf{I} - \mathsf{K}$ is onto.
For infinite dimensional spaces; $(\mathsf{I} - \mathsf{K})x$ is invertible if and only if $\mathsf{I} - \mathsf{K}$ is onto.
This extends to the existence of solutions of the equations of the type $(\lambda\mathsf{K} + \mathsf{I})x = y$ where λ is a scalar. This known as general Fredholm alternative theorem is similar

to previous cases. It says that $\mathsf{I} - \lambda \mathsf{K}$ is one-one if and only if it is onto. Fredholm alternative theorem helps in understanding that uniqueness yields existence.

Chapter 2

Index of Fredholm Operators

2.1 Introduction

One of the most fundamental problems in mathematics is to solve linear equations of the form

$$\mathsf{T}f = g$$

say solving equations like $\mathsf{T}v = w$ where v and w belong to two vector spaces, or like solving $\mathsf{D}f = g$ for some suitable differential operator between Banach spaces of functions. The main goal of index theory is to assign an index, a number to operators, like to T as above mentioned which encapsulates information about both existence and uniqueness of solutions simultaneously.

2.2 Index Theory In Finite Dimensions

Let V and W be finite dimensional vector spaces and let $\mathsf{T} : V \to W$ be a linear transformation. In order to understand the solutions to equations $\mathsf{T}v = w$, we look at these two points:

- Uniqueness of solutions corresponds to the injectivity of T. Since T is injective exactly when $\dim\ker(\mathsf{T}) = 0$, the larger the dimension of $\ker(\mathsf{T})$, the less injective T is.

- Similarly, existence of solutions of $\mathsf{T}v = w$ can be understood by the surjectivity of T.

$$\text{coker}(\mathsf{T}) = W/im(\mathsf{T})$$

  Hence we can measure surjectivity by considering the dimension of coker(T). If $\dim \text{coker}(\mathsf{T}) = 0$, then $im(\mathsf{T})$ is all of W and so T is surjective. The larger $\dim \text{coker}(\mathsf{T})$ is, the less surjective T is.

Definition:
Let V and W be finite dimensional vector spaces and let $\mathsf{T} : V \to W$ be linear, then index of T is

$$\text{ind}(\mathsf{T}) = \dim \ker(\mathsf{T}) - \dim \text{coker}(\mathsf{T})$$

Proposition 2.1:
Let V and W be finite dimensional vector spaces and let $\mathsf{T} : V \to W$ be linear, then

$$\text{ind}(\mathsf{T}) = \dim(V) - \dim(W)$$

Proof: Since

$$\text{coker}(\mathsf{T}) = W/im(\mathsf{T})$$

We have $\dim \text{coker}(\mathsf{T}) = \dim W - \dim(im\mathsf{T})$
Hence,

$$\text{ind}(\mathsf{T}) = \dim \ker(\mathsf{T}) - \dim \text{coker}(\mathsf{T})$$

$$\text{ind}(\mathsf{T}) = \dim \ker(\mathsf{T}) + \dim(im(\mathsf{T})) - \dim W$$

By Rank-Nullity theorem, which states that rank and nullity of a matrix add upto number of columns of the matrix,

$$\dim \ker(\mathsf{T}) + \dim(im(\mathsf{T})) = \dim V$$

Therefore,

$$\text{ind}(\mathsf{T}) = \dim(V) - \dim(W)$$

We saw from the above proposition that in finite dimensional space the concept of index depends on dimensions of domain and codomain. Therefore, we now try to go into infinite dimensional spaces.

2.3 Fredholm Operator In Infinite Dimensional Spaces

In the infinite dimensional space, dimension of kernel or cokernel of an operator may be infinite and index might not be well defined for all operators.
For instance, take $0 : X \to Y$; both kernel and cokernel are infinite dimensional if $X$and Y are infinite dimensional. So we want kernel and cokernel to be finite dimensional and then comes the concept of Fredholm Operators. In the next section, we define Fredholm operators acting on Banach spaces, its properties and index of Fredholm Operators.

2.3.1 Fredholm operator

A Fredholm operator is an operator that arises in the Fredholm theory of integral equations.
Let T be a bounded linear operator from X to Y, where X and Y are both Banach spaces.
If $\dim\ker(\mathsf{T}))$ and $\dim\operatorname{coker}(\mathsf{T})$ are finite and range is closed, then T is said to be a Fredholm operator. This operator is almost injective as it has only finite dimensional kernal.
Dimension of $\ker(\mathsf{T})$ provides us the extent to which $\mathsf{T}x = y$ fails to have a unique solution and dimension of $\operatorname{coker}(\mathsf{T})$ provides the extent to which $\mathsf{T}x = y$ fails to have a solution, same as discussed in the beginning of this chapter.
Remarks
Here $\mathsf{T} : X \to Y$, other notations are as above.

- If $\mathsf{T}x = y$ always has a solution for arbitrary y then

$$im(\mathsf{T}) = Y$$

$$\dim\operatorname{coker}(\mathsf{T}) = 0$$

- If $\mathsf{T}x = y$ always has an unique solution once it has a solution then

$$\dim \ker(\mathsf{T}) = 0$$

.

- The adjoint of an operator is obtained by taking the complex conjugate followed by transposing the operator. In particular, an operator T is Fredholm if and only if T^* is Fredholm.
  Suppose that T is Fredholm, we know that $\ker \mathsf{T}^* = (im\mathsf{T})^\perp$. Since $im\mathsf{T}$ is closed and $X = im\mathsf{T} \bigoplus (im\mathsf{T})^\perp$, it follows that $\ker \mathsf{T}^* = (im\mathsf{T})^\perp$ which is equivalent to $\operatorname{coker} \mathsf{T}$. Therefore, T is Fredholm if and only if T^* is Fredholm

- The composition of two Fredholm operators is a Fredholm operator as well.

Example
The shift operator S takes a function $f(x)$ to its translation $f(x+a)$, a being some scalar. Let

$$\mathsf{S} : L^2(\mathbb{Z}_+) \to L^2(\mathbb{Z}_+)$$

be the right shift map acting on the elements of $L^2(\mathbb{Z}_+)$ with $a = 1$. Elements of this space are the sequences like $(c_{00}, c_{01}, ...)$ of complex numbers with square summable complex values i.e. $\sum |c_n|^2 < \infty$.
If every sequence in $L^2(\mathbb{Z}_+)$ is treated as an element then $Sc_n = c_{n+a}$ with $f(x) = c_n$ and $f(x+a) = c_{n+a}$

$$\dim \ker(\mathsf{S}) = 0$$

$$\dim \operatorname{coker}(\mathsf{S}) = 1$$

Since both kernel and cokernel have finite dimension and the image is closed, it is a Fredholm operator.
Another case where shift operator is Fredholm is for the Hilbert spaces. Let $\{e_1, e_2, ...\}$ be an orthonormal basis for H, and let $\mathsf{M} : H \to H$ be the operator defined by $\mathsf{M}e_j = e_{j+1}$. The image of M is $im\mathsf{M} = \operatorname{Span}\{e_2, e_3, ...\}$.
Hence, $\operatorname{coker} \mathsf{M} = H/im\mathsf{M} = \operatorname{Span}\{e_1\}$, and so $\dim \operatorname{coker} \mathsf{M} = 1$. Since M is injective, $\dim \ker \mathsf{M} = 0$.
Definition (Semi- Fredholm Operator)
Let X and Y be two Banach spaces and let $\mathsf{L}(X, Y)$ denote the Banach space of all bounded operators from X into Y. An operator T is said to be semi-Fredholm

operator if $\mathsf{T}X$ is closed and at least one of the spaces $\ker \mathsf{T}$ and $\operatorname{coker} \mathsf{T}$ is of finite dimension.

2.3.2 Properties of Fredholm Operators

Lemma 2.1: Let T: $X \to Y$ be a operator such that range admits a finite complementary subspace. Then the range of T is closed.
Proof: Let C be a complement of the range. Since it is finite dimensional, it is closed. $\ker(\mathsf{T})$ is a closed subspace, so $X/\ker(\mathsf{T})$ is a Banach space. Replacing X by $X/\ker(\mathsf{T})$ we assume that T is injective.
If we now consider the map $\mathsf{S} : X \bigoplus C \to Y$ by

$$\mathsf{S}(x, c) = \mathsf{T}(x) + c$$

which is a bounded linear isomorphism because

$$||\mathsf{T}x + c|| \leq ||\mathsf{T}x|| + ||c||$$

$$||\mathsf{T}x|| + ||c|| \leq ||\mathsf{T}||||x|| + ||c||$$

. Let $\mathsf{K} = max\{||\mathsf{T}||, 1\}$, then we get

$$||\mathsf{T}||||x|| + ||c|| \leq \mathsf{K}(||x|| + ||c||)$$

and so S is bounded and hence continuous. Moreover $\mathsf{T}x + c = 0$ implies $(x, c) = 0$. Since the intersection of $\mathsf{T}x$ and c is zero, it is one to one. It is also onto and S is an invertible isomorphism, so it is a homeomorphism by the Open Mapping theorem (if T is a bounded operator and is surjective, then T is open).
Therefore $im(\mathsf{T}) = \mathrm{S}(\mathrm{X}\bigoplus\{0\})$ is closed.

Riesz Lemma: If a unit ball in X is precompact then X is finite dimensional.
Result: If T has closed range then $\operatorname{coker}(\mathsf{T})^* = \ker(\mathsf{T}^*)$.(For proof [ocw15])

Lemma 2.2: Let $\mathsf{K} : X \to X$ be a compact operator then $\mathsf{I} + \mathsf{K}$ is a Fredholm operator.
Proof: Let B be the unit ball in $ker(\mathsf{I} + \mathsf{K})$. Then $B = \mathrm{K}(B)$, B is image of a bounded set under a compact operator, it is precompact. But B is closed, so it is

compact. By Riesz lemma, a subspace of $\ker(\mathsf{I} + \mathsf{K})$ is finite dimensional.
To show that $im(\mathsf{I} + \mathsf{K})$ is closed we consider that if $\{x_n\}$ is a bounded sequence such that $\{x_n + \mathsf{K}\ x_n\}$ converges to $y \in Y$ then there is $x \in X$ so that $x + \mathsf{K}x = y$. Since $\{x_n\}$ is bounded there is a subsequence $\{x_{n_m}\}$ so that $\{\mathsf{K}x_{n_m}\}$ converges but then $\{x_{n_m}\}$ converges. Thus the operator $\mathsf{I} + \mathsf{K}$ is a semi Fredholm operator. Applying the same arguement to the adjoint $\mathsf{I} + \mathsf{K}^*$ and using the result mentioned above, we get $\mathsf{I} + \mathsf{K}$ is Fredholm.

2.4 Index Of Fredholm Operator

Definition Let $\mathsf{T} : X \to Y$ be a Fredholm operator in a Banach space, then the index of this operator is defined as the finite integer.

$$\text{ind}\,\mathsf{T} = \dim\ker\mathsf{T} - \dim\text{coker}\,\mathsf{T}$$

- If $\text{ind}\,\mathsf{T} = 0$ this means that existence and uniqueness are equivalent.
- If T is invertible, then $\dim\ker\mathsf{T}$ and $\dim\text{coker}\,\mathsf{T}$ are 0 which in turn means that $\text{ind}\ \mathsf{T} = 0$

So a necessary condition for a Fredholm operator to be invertible is that its index be 0.[Dep01]

Remarks

- The set of all Fredholm operators from X to Y is denoted as $\mathcal{F}(X, Y)$. $\mathsf{B}(X, Y)$ is space of compact operators acting between X and Y as mentioned earlier.

Proposition 2.2

- Compact perturbations do not change Fredholmness and the index. If $\mathsf{K} \in \mathsf{B}(X, Y)$ and $\mathsf{A} \in \mathcal{F}(X, Y)$, then $\mathsf{A} + \mathsf{K} \in \mathcal{F}(X, Y)$ and $\text{ind}(\mathsf{A} + \mathsf{K}) = \text{ind}(\mathsf{A})$.
- Zero index is achieved only by compact perturbations of invertible operators. More precisely, if $\mathsf{A} \in \mathcal{F}(X, Y)$ then $\text{ind}(\mathsf{A}) = 0$ if and only if $\mathsf{A} = \mathsf{A}_0 + \mathsf{K}$ for some invertible operator A_0 and some compact operator K.

- If $\mathsf{K} : X \to X$ is a compact operator, then $\mathsf{I} - \lambda\mathsf{K}$ is Fredholm for every $\lambda \in \mathbb{R}$. (Proof is similar to lemma 2.2). This property indicates the fact that Fredholm operators in infinite space are the furthest from compact operators which will be proved by Atkinson's theorem.

Atkinson's Theorem

Fredholmness is equivalent to invertible modulo compact operators means given a bounded operator $\mathsf{A} : X \to Y$ the following are equivalent:

- A is Fredholm.
- A is an invertible modulo compact operator, that means there exists operator $\mathsf{B} \in \mathsf{L}(Y, X)$ and compact operators K and J such that $\mathsf{BA} = 1 + \mathsf{K}$, $\mathsf{AB} = 1 + \mathsf{J}$.

Proof: Assume first the existence of B, K and J. Since identity plus compact operator is a Fredholm operator(Lemma 2.2), we deduce that the kernel of A is finite dimensional (since it is included in the kernel of $1 + \mathsf{K}$; $1+$ K being Fredholm with finite dimensional kernel) similarly, the cokernel of A is also finite dimensional. Since cokernel is finite it means range is closed. Hence A is Fredholm.
Assume now that A is Fredholm. Choose a complement E_1 of $\ker \mathsf{A}$ in E and a complement F_1 of $im\mathsf{A}$ in F. Then $A_1 = A\restriction_{E_1}$ is an isomorphism from E_1 into $im\mathsf{A}$ and we define B such that $\mathsf{B} = (A_1^{-1})$ on $im\mathsf{A}$ and $\mathsf{B} = 0$ on F_1. Then the resulting K will be a projection onto $\ker \mathsf{A}$ and $1 + \mathsf{J}$ will be a projection onto $im\mathsf{A}$. Hence K and J have the required properties.

Proposition 2.3

If $\mathsf{T} : X \to Y$; X & Y are finite dimensional spaces, then T is a Fredholm operator with $\operatorname{ind} \mathsf{T} = \dim X - \dim Y$.
Operators in a finite dimensional space are Fredholm. For similar proof, refer proposition 2.1.

Important results

If T, S are Fredholm operators and K is a compact operator then the following results holds:

- $\operatorname{ind}(\mathsf{TS}) = \operatorname{ind} \mathsf{T} + \operatorname{ind} \mathsf{S}$. This is known as the multiplicative property of the index.

- If T is Fredholm, then so is it's adjoint and $\text{ind}(\mathsf{T}^*) = -\text{ind}(\mathsf{T})$. Since $\text{ind}(\mathsf{T}) = \dim\ker\mathsf{T} - \dim\ker\mathsf{T}^*$ and $\text{ind}(\mathsf{T}^*) = \dim\ker(\mathsf{T}^*) - \dim\ker(\mathsf{T}^{**})$, we get $\text{ind}(\mathsf{T}^*) = -\text{ind}(\mathsf{T})$

- $\text{ind}\,(\mathsf{T}+\mathsf{K}) = \text{ind}\,\mathsf{T}$

- If T is an isomorphism then $\text{ind}(\mathsf{T}+\mathsf{K}) = 0$. Since I is a special case of isomorphism, $\text{ind}(\mathsf{I}+\mathsf{K})$ is zero.

Invariance of index under small perturbations

Let T be a Fredholm operator, if S is such that $||\mathsf{S}|| < \mathsf{c}$ for a positive scalar c, then $\mathsf{T}+\mathsf{S}$ is a Fredholm operator which satisfies

$$\text{ind}\,(\mathsf{T}+\mathsf{S}) = \text{ind}(\mathsf{T})$$

Theorem 2.1: The Fredholm index map $\text{ind} : \mathcal{F}(X) \to \mathbb{Z}$ is continuous, and is locally constant. Also given any Fredholm operator T, there is an open neighborhood U of Fredholm operators containing T such that $\text{ind}(\mathsf{S}) = \text{ind}(\mathsf{T})$ for all $\mathsf{S} \in U$.

Proof

Refer to Lemma 1.4.4 for the proof [Gil74].

One implication of this theorem is that the index is constant on connected components of $\mathcal{F}(X)$. Suppose that T and S are two Fredholm operators which are connected by a path in $\mathcal{F}(X)$. Since the Fredholm index is locally constant, at every point along the path we can find open neighborhoods of constant index. Since path is a compact space, we can cover the entire path with a finite number of such open sets. The index is constant on such open sets and also on the union of any two intersecting neighborhoods. And each neighbourhood intersects with atleast one other neighbourhood and by connectedness it results into the constant index throughout the path. So to say the index map is continuous and locally constant, by connectedness is constant on connected components.

The converse also holds and so in other words, the index partitions the space of Fredholm operators into connected components.

Explicitly, in the case of Hilbert space H there arises a bijection between connected components of $\mathcal{F}(H)$ and $\mathbb{Z}$.

Proposition 2.4
Let A: $X \to Y$ and $X = M \oplus N$, $Y = M^{'} \oplus N^{'}$ and suppose,

$$\text{A} = \begin{bmatrix} A_1 & X \\ 0 & A_2 \end{bmatrix}$$

relative to the two decompositions of X and Y. If A_1 is invertible and N, $N^{'}$ are finite dimensional then A is a Fredholm operator and

$$\operatorname{ind} \mathsf{A} = \dim N - \dim N^{'}.$$

2.5 Relations In General Index Theory

Definitions:

- **Unilateral shifts** The shift operators acting on one-sided infinite sequences are called unilateral shifts.
- **Finite index** If the index is a well defined integer, it is known as finite index.
- **Degenerate map** A degenerate map is the one which has finite range.
- **Pseudoinverse map** Two bounded linear maps $\mathsf{T} : U \to V$ and $\mathsf{S} : V \to U$ are called pseudo inverse of each other if $\mathsf{ST} = \mathsf{I}+\mathsf{K}$ and $\mathsf{TS} = \mathsf{I}+\mathsf{J}$ where K and J are compact maps of U and V respectively into themselves.

Example 2: Let S be a unilateral shift of multiplicity α, put $\mathsf{A} = S \oplus S^*$. A is Fredholm if α is finite with trivial index.
Theorem 2.2
A bounded linear map $\mathsf{T} : U \to V$ has finite index if and only if T has a pseudo inverse.
Proof Refer to [Lax02] Section 27.1, theorem 1.
Remark
Let $\mathsf{T} : U \to V$ and $\mathsf{R} : V \to W$ be bounded maps with finite indices then RT has finite index equal to the sum of indices of R and T.

$$\mathrm{ind}(\mathsf{RT}) = \mathrm{ind}(\mathsf{R}) + \mathrm{ind}(\mathsf{T})$$

Theorem 2.3
Suppose $\mathsf{T} : U \to V$ has finite index, L is a compact linear map from U to V. Then $\mathsf{T}+\mathsf{L}$ has a finite index and $\mathrm{ind}\,(\mathsf{L}+\mathsf{T}) = \mathrm{ind}\,(\mathsf{T})$.
Proof: Since T has finite index it has pseudo inverse S.S is also pseudo inverse to $(\mathsf{T}+\mathsf{L})$.

$$\mathsf{S}(\mathsf{T}+\mathsf{L}) = \mathsf{ST} + \mathsf{SL} = \mathsf{I} + \mathsf{K} + \mathsf{SL}$$

Since L is a compact map, so is SL. We have already seen that sum of indices of two operators which are pseudoinverse to each other is trivial. (A and B are pseudoinverses to each other, then $\mathrm{ind}(\mathsf{A}) + \mathrm{ind}(\mathsf{B}) = 0$ [Lax02]) we have

$$\mathrm{ind}\,(\mathsf{T}+\mathsf{L}) = -\,\mathrm{ind}\,\mathsf{S} = \mathrm{ind}\,\mathsf{T}$$

Theorem 2.4 Let $\mathsf{T} : U \to V$ be a linear map of finite index and $\mathsf{G} : U \to V$ is a degenerate linear map, then $\mathsf{T} + \mathsf{G}$ has finite index and

$$\mathrm{ind}\,(\mathsf{T}+\mathsf{G}) = \mathrm{ind}(\mathsf{T})$$

So far, in last two chapters, we presented two important classes of operators, the compact operators and the Fredholm operators. These operators in infinite dimensional spaces, assist in easily solving linear equations of the form $\mathsf{T}f = g$. The central role in this solution plays the concept of the index of a Fredholm operator. Whenever T is a Fredholm operator with zero index, it can be decomposed directly as a sum of an invertible and a compact operator, so that the equation could be subsequently solved via the Fredholm Alternative. Even if the index of T is nonzero, the equations could be solved and have the same solutions with the initial ones. In the next chapter, we will look at another type of operator called Toeplitz operator and establish a connection between Fredholm operators and Toeplitz operators.

Chapter 3

Toeplitz Operators

3.1 Introduction

The theory of index is not only limited to Fredholm operators. There is another class of operators called Toeplitz operators whose index can be calculated explicitly. In this chapter we will briefly discuss about Toeplitz operators, its properties and also when can a Toeplitz operator be a Fredholm operator. We will also establish that index theory is an important tool connecting analytic(index) and topological properties(winding number).

3.2 Toeplitz Operator

3.2.1 Basics

A Toeplitz matrix is a matrix that is constant on each line parallel to the main diagonal. Entries of the matrix are determined as $a_{ij} = (a_{i-j})$.
For example,

$$A = \begin{bmatrix} 2 & 9 & 0 & 1 & 0 \\ -1 & 2 & 9 & 0 & 1 \\ 5 & -1 & 2 & 9 & 0 \\ 2 & 5 & -1 & 2 & 9 \\ 1 & 2 & 5 & -1 & 2 \end{bmatrix}$$

is a Toeplitz matrix.
A Toeplitz operator is essentially an infinite dimensional analogue of Toeplitz matrices. Toeplitz operator looks like

$$T = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & & \\ a_1 & a_0 & a_{-1} & a_{-2} & . & \\ a_2 & a_1 & a_0 & a_{-1} & .. & \\ .. & a_2 & a_1 & a_0 & .. & \\ .. & .. & .. & .. & .. & \end{bmatrix}$$

Associated to a Toeplitz operator T is an infinite sequence of complex numbers $\{a_n\}_{-\infty}^{+\infty}$. We define ϕ, the symbol of T as a continuous complex function from S^1 to $\mathbb{C}$ as

$$\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

The image of ϕ is a loop in the complex plane.
A Toeplitz operator acts on the space called Hardy space defined as $H^2 = H^2(S^1) =$ Hilbert subspace of $L^2(S^1)$ spanned by $\{e^{i\theta n}; n \geq 0\}$
In other words, the Hardy space H^2 is defined to be the closed linear span in $L^2(S^1, \sigma)$ of $\{z^n : 0 \leq n\}$. For $\phi \in L^\infty(S^1, \sigma)$, the Toeplitz operator with symbol ϕ, denoted T_ϕ , is the operator on H^2 defined by $\mathsf{T}_\phi h = \mathsf{P}(\phi h)$, P is the orthogonal projection of $L^2(D, \sigma)$ onto H^2. Also T_ϕ is a bounded operator on H^2.
The operator T is bounded if the entries of the respective Toeplitz matrix are the fourier coefficients of ϕ.

3.2.2 Fredholm Toeplitz Operator

In this section, we are going to look how and when is a Toeplitz operator a Fredholm operator.

Proposition 3.1:
A Toeplitz operator T is Fredholm if and only if its symbol ϕ, defined above, is non-zero everywhere.
Example Consider T with 1's on the lower diagonal and zero everywhere.

$$\mathrm{T} = \begin{bmatrix} 0 & 0 & 0 & 0 & .. \\ 1 & 0 & 0 & 0 & ... \\ 0 & 1 & 0 & 0 & .. \\ 0 & 0 & 1 & 0 & .. \\ .. & . & .. & 1 & .. \end{bmatrix}$$

Here $a_1 = 1$ and $a_n = 0$ everywhere else. $\phi(z) = z$ is the symbol of this operator. Since z is never zero on S^1 as because image of S^1 under z is just S^1; the function z does not do anything to S^1. Hence T is a Fredholm Operator.

3.2.3 Index Of Toeplitz Operator

Winding number denoted as $w(\phi)$, is the number of times the curves $\phi(S^1)$ loops around the origin going counter-clockwise. For instance in the example above, $w(z)$=1 since image of S^1 is S^1.
Winding number of the symbol of the Toeplitz operator is a topological invariant, that is, it is preserved in hoemomorphism. Winding number of ϕ is related with the Fredholm index of the Toeplitz operator with symbol ϕ by the theorem mentioned below.

Toeplitz Index Theorem
The Toeplitz index theorem gives an explicit index computation of an important class of number of continuous functions. Stated as, let T be a Toeplitz operator with non-zero symbol ϕ. Then

$$ind(\mathsf{T}) = -w(\phi)$$

As we can see, the above theorem exemplifies the connections between analytic and topological ideas. There is yet another portion in index theory connecting analysis and topology, namely Atiyah Singer index theorem. It deals with a type of differential operator
The Atiyah Singer index theorem tells us that how many solutions there are to a

system of differential equations, has a concrete answer in topology. Furthermore, it is safe to say that index theory has many appliactions such as in string theory. I would like to conclude with the statement of the theorem.

Atiyah Singer Index Theorem Let $\mathsf{P}(f) = 0$ be a system of elliptic differential operator on an n-dimensional compact smooth C^∞ boundaryless manifold. Then analytical index(P) = topological index(P)

Bibliography

[Dep01] UCSD Math Department, *Compact and fredholm operators*, 2001.

[Gil74] Peter B. Gilkey, *Invariace theory, heat equation and atiyah-singer theorem*, 1974.

[Lax02] Peter D. Lax, *Functional analysis*, October 2002.

[ocw15] ocw.mit, *Lectures 16, 17*, 2015.

[Sim15] G.F. Simmons, *Topology and modern analysis*, 2015.