Study of heat engines in short time regime by using Maxwell-Cattaneo equation

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Certificate of Examination

This is to certify that the dissertation titled **" Study of heat engines in short time regime by using Maxwell-Cattaneo equation"** submitted by **Keshav Nagpal** (Reg. No. MS13060) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Sanjeev Kumar	Dr. Ananth Venkatesan	Dr. Ramandeep Singh Johal (Supervisor)

Dated: April 19, 2018

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Ramandeep Singh Johal at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Keshav Nagpal
(Candidate)

Dated: April 19, 2018

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Ramandeep Singh Johal
(Supervisor)

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Abstract

Carnot in 1824, set up the cornerstone for the classical heat engines by giving the maximum possible efficiency, which is well known as the Carnot efficiency

$$\eta_c = 1 - \frac{T_C}{T_H}$$

But one can not obtain the Carnot efficiency in reality because the power production in the Carnot engine is zero due to infeasibly large cycle times. The bound for the heat engine at non-zero power was set up by Curzon and Ahlborn in 1975, given by

$$\eta_{CA} = 1 - \sqrt{\frac{T_C}{T_H}}$$

For it, Curzon and Ahlborn consider Carnot-like engine cycling for finite time , in which the heat flux transfer from the reservoirs was assumed to follow linear law (Newton law) of the heat flux transfer. But one can take any valid form of the heat flux transfer law for same model which can change the results. De Vos and N. V. Orlov have set up upper and lower bounds to the efficiency by considering the inverse law of heat flux transfer, given by

$$\frac{\eta_C}{2} \leq \eta \leq \frac{\eta_C}{2 - \eta_C}$$

In this work, we included Maxwell-Cattaneo equation to generalize the model taken by Curzon and Ahlborn, by using two heat flux transfer laws, Newton law and inverse law. We have done analytical and numerical calculations for two heat transfer laws. For analytical calculations, we have included an assumption that the relaxation time of the working substance is larger than the operational time of the heat engine. The assumption helped us to study the cases in short time regime(with respect to relaxation time). And for numerical calculations, we have used "Mathematica".

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Chapter 1

Introduction

Background

Heat engines are the devices which convert heat energy into mechanical work. The first theory which explains the working of these heat engines is classical equilibrium thermodynamics. Classical equilibrium thermodynamics has been developed in 19th century. It basically gives the macroscopic description of systems in equilibrium. For a system to remain in equilibrium the processes it undergoes should be quasi-static and reversible in nature. Which means processes under classical equilibrium thermodynamics take infinitely long time to occur. In thermodynamics theory, there are mainly four types of processes which are isothermal process, isochoric process, adiabatic process and isobaric process. Depending upon types of processes a heat engine undergoes, they have different names.

The most common among all the heat engines is the Carnot heat engine [13] whose stages are combination of isothermal and adiabatic processes. The working substance in Carnot heat engine undergoes a cyclic process from higher temperature to lower temperature. It takes heat from an infinite hot bath at temperature T_H, then convert the heat energy into useful work and rejects extra heat to an infinite cold sink at temperature T_C. The working of the Carnot heat engine is based on main assumption that all the processes are reversible and quasi-static. Also there is no dissipation in the system. So all the processes in Carnot heat engine take infinitely long time to occur and the engine stay in equilibrium through out the cycle. By using mentioned assumption Carnot in 1824, set up cornerstone for classical heat engines by giving maximum possible efficiency, which is well known as Carnot efficiency

$$\eta_c = 1 - \frac{T_C}{T_H} \tag{1.1}$$

But the Carnot heat engine is unrealistic. Because practically one can not make an engine which takes infinite long time for its cycle and has no dissipation in the system. Moreover power (defined as work output per cycle time) production in case of the Carnot heat engine is zero due to infinite long cycle time but in reality heat engines operate at finite rates and produce finite power. Secondly, operation of the real heat engines always entails some dissipation in the system. Also the real processes are irreversible in nature. The working of the real heat engines can be explained by the theory of finite-time thermodynamics, which mainly get developed after 1975.

The study of flow of energy and entropy in the systems operating irreversibly and in the finite time has been termed " finite-time thermodynamics". Study of the heat engines operating under this regime is one of the leading research interests in the thermodynamics. A major objective of finite-time thermodynamics is to understand irreversible, finite-time processes and to establish the general, natural bounds on the efficiency, basically efficiency at maximum power for such processes. Over the years there have been many attempts to develop models for the explanation of the working of the heat engines under finite-time thermodynamics. Low dissipation model, Onsager theory, linear irreversible thermodynamics and endoreversible thermodynamics are some of the examples. My work deals with the " Endoreversible thermodynamics".

Endoreversible thermodynamics is one of the fields under finite-time thermodynamics. Its name was given by Rubin [10], [11] in 1979 . He defined an " endoreversible engine to be an engine such that during its operation its working fluid goes reversible transformations." Endoreversible thermodynamics is a non-equilibrium approach for the study of the heat engines which consider system(engine) as network of internally reversible(endo-reversible) subsystems. So endoreversible heat engine is an internally reversible and externally irreversible cyclic devices which exchange energy with its surroundings. The most important example of such heat engines is the Curzon-Ahlborn heat engine.

Curzon-Ahlborn heat engine [1] is the Carnot-like heat engine , which is irreversibly coupled to the infinite hot bath at temperature T_H and the infinite cold bath at temperature T_C. In this, Curzon and Ahlborn have taken the linear heat flux law known as the Newton law as the only source of the irreversibility. Their model gives efficiency at maximum power which is given by

$$\eta_{CA} = 1 - \sqrt{\frac{T_C}{T_H}} \tag{1.2}$$

This is known as the Curzon-Ahlborn efficiency and like the Carnot efficiency, it depends only the temperature of the heat baths, but does not depend upon the heat transfer coefficient. This result is a very good approximation for efficiencies of the real power plants.

The problem with the Curzon-Ahlborn efficiency is that it is not general like the Carnot efficiency. The main reason for its non generality is that the model considers only the Newton law for heat flux transfer. But what happens when we change the form of the heat flux transfer law? Does the result remain same? Many authors ([14],[16],[23]) have tried to study the Curzon-Ahlborn heat engine by taking different heat flux transfer laws which includes inverse law, radiation law, general law etc. Among all these, the work of De Vos [6] and Orlov [7] is very attractive. They have taken the inverse law for heat flux transfer in the Curzon-Ahlborn heat engine and have derived upper and lower bounds, for the efficiency

$$\frac{\eta_C}{2} \leq \eta \leq \frac{\eta_C}{2-\eta_C} \tag{1.3}$$

where η_C is Carnot efficiency.

The form of all heat flux transfer laws mentioned above, is actually based on an assumption that the rate of transfer of heat from hotter body to colder body is time independent. Which means the hotter particles from the hotter body get transfer to the colder body at the infinite speed when they come in contact with each other. But speed of any particle can not be more than the speed of light. So to overcome this problem Maxwell and Cattaneo have derived an equation known as the Maxwell-Cattaneo equation. The main result of this equation is that it gives the time of contact and the relaxation time dependent form of the heat fluxes and thus remove the problem of the infinite speed of particles.

In this work, we include the Maxwell-Cattaneo equation [4] to generalize the endoreversible model under finite time regime, by using two heat flux transfer laws, Newton law and inverse law. In order to obtain the analytical solutions, we have assumed that the relaxation time of the working substance is larger than the operational time of the heat engine. So we have studied the heat engines in the short time regime(with respect to relaxation time) using two types heat flux transfer laws and by using the Maxwell-Cattaneo equation.

Document structure

In chapter 2, all of the basics required for our work is given. In chapter 3, detail analytical calculations in short time regime for Newton law case and its numerical calculation for general case is given. In chapter 4, detail analytical calculations in short time regime for inverse law case and its numerical calculation for short time regime only is given. Chapter 5, includes short conclusion of our work, together with an outlook about the further improvements of work.

Chapter 2

Basics

Description

The aim of this chapter is to describe all the basics which I have used for my thesis work. It includes the description of the, Carnot cycle, endoreversible thermodynamics, Curzon-Ahlborn model of heat engine, inverse law calculation and Maxwell-Cattaneo equation.

2.1 Carnot cycle

Carnot cycle is the most common used cycle for the study of the heat engines. It is an four stage cycle. Each stage belongs to a process as shown in [Fig(2.1)]. All the processes in Carnot cycle are reversible and quasi-static.

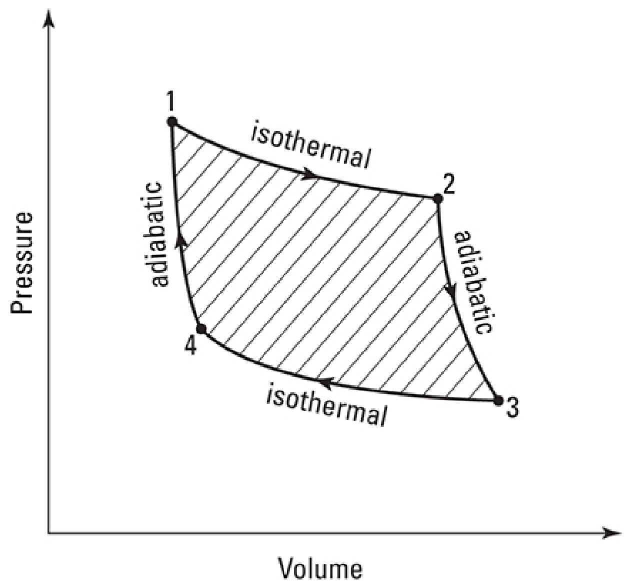

Figure 2.1: Carnot cycle

2.2 Endoreversible Thermodynamics

Endoreversible thermodynamics is a field of finite time thermodynamics which helps to study the real heat engines by making some realistic assumptions. The real heat engines are always irreversible in nature and this field consider only external irreversibility in a engine but do not consider any internal irreversibility. It mainly aims at making an more realistic assumptions about the heat flux transfer between the working substance and the reservoirs in finite time. If a Carnot-like engine is considered then this field makes following assumptions:

1. Isothermal stages are of finite time and the only source of irreversibility during these stages is the heat flux transfer from the reservoirs.

2. Adiabatic stages are of very short time and are reversible. So there is no irreversibility during these stages.

The model which follows "Endoreversible thermodynamics" assumptions can be named as the endoreversible model. The results of this model depend upon the type of heat flux transfer law used. Following is the description for some of the heat flux transfer laws.

1. Newton Law - This law states that the form of heat flux transfer between two connected contacts depends linearly on temperature difference between them.
$$q_N = k(T_1 - T_2) \tag{2.1}$$
Here k is thermal conductivity between two contacts, T_1 is temperature of hotter body and T_2 is temperature of colder body. This law is mainly applied to the solids for the conduction but it can be also applied for the convection for small temperature differences. Newton law is the most interesting and widely used law for the study of the endoreversible model.

2. Radiation law - If very high temperature source is used, then the form of the heat flux transfer between two bodies follows the radiation law. Typically it is described by the Steafen-Boltzman law for black body radiation
$$q_R = k_1(T_1^4 - T_2^4) \tag{2.2}$$
Where k_1 is Steafen-Boltzman constant, T_1 and T_2 have same meaning as above.

3. <u>Inverse law</u>- According to this law, the heat flux transfer between two bodies is inversely proportional to the temperature difference between them

$$q_I = k(\frac{1}{T_2} - \frac{1}{T_1}) \tag{2.3}$$

   Like Newton law, this law also can be used for the conduction or convection phenomenon.

4. <u>General Law</u>- A general heat transfer law can be written as

$$q_G = K_1(T_1^n - T_2^n) \tag{2.4}$$

   this includes the Newton law for ($K_1 = k$, $n = 1$) , the radiation law for ($K_1 = k_1$, $n = 4$) and the inverse law for ($K_1 = k$, $n = -1$).

2.3 Curzon-Ahlborn heat engine

Curzon-Ahlborn heat engine [Fig.(2.2)] is an best example of the endoreversible model. Its an regular heat engine, which converts heat energy into mechanical work. The working substance in this heat engine follows Carnot cycle [Fig(2.1)] which is irreversibly coupled to the infinite hot bath at temperature T_H and the infinite cold sink at temperature T_C. Two isothermal stages operate for finite time and two adiabatic stages operate for very short time. During isothermal expansion stage the temperature of working substance remains T_{1w} which is higher than the temperature of working substance in the isothermal compression stage T_{2w} as shown in [Fig.(2.2)]. This engine is assumed to avoid any heat leaks, heat friction etc. The other main assumption is that the heat flux transfer between the infinite heat bath or sink, and working substance follows the Newton law of heat flux transfer (q_N).

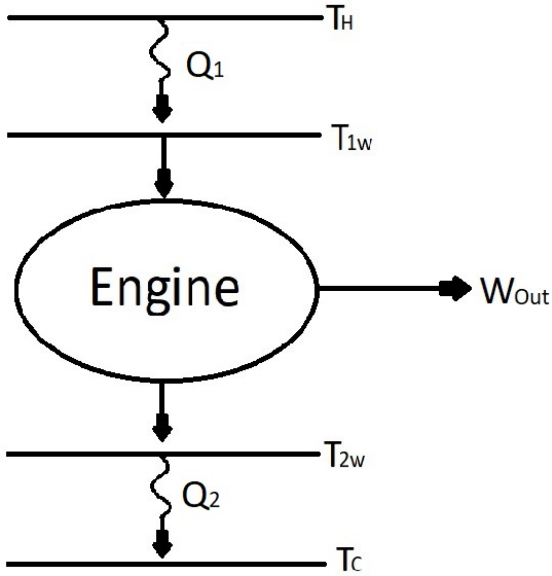

Figure 2.2: Curzon-Ahlborn heat engine.

So, the heat flux transfer forms during the isothermal expansion and the isothermal compression stages are

$$q_1 = \alpha(T_H - T_{1w}) \qquad\qquad q_2 = \beta(T_{2w} - T_C) \tag{2.5}$$

By using $Q_1 = \int_0^{t_1} q_1 dt$ and $Q_2 = \int_0^{t_2} q_2 dt$, total heat transfer for the isothermal expansion stage (Q_1) and total heat transfer for the isothermal compression stage (Q_2) are given by

$$Q_1 = \alpha(T_H - T_{1w})t_1 \qquad\qquad Q_2 = \beta(T_{2w} - T_C)t_2 \tag{2.6}$$

Here t_1 is the time of contact of the heat bath with the working substance during the isothermal expansion stage and t_2 is the time of contact of the cold sink with the working substance during the isothermal compression stage. The adiabatic expansion stage and the adiabatic compression stage are of negligible time. One can get the entropy balance equation, which is also known as the endoreversiblity condition from two adiabatic stages, given by

$$\frac{Q_1}{T_{1w}} = \frac{Q_2}{T_{2w}} \tag{2.7}$$

Using (2.6) and (2.7),the relation between t_1 and t_2 is

$$t_1 = t_2 \frac{\beta y T_{1w}}{\alpha x T_{2w}} \tag{2.8}$$

where $x = T_H - T_{1w}$ and $y = T_{2w} - T_C$. Now by using (2.8) and by using $P_{tot} = W_{tot}/t_{tot}$, where $W_{tot} = Q_1 - Q_2$ is total work per cycle and $t_{tot} = t_1 + t_2$ is total time per cycle, the expression for total power (P_{tot}) is obtained

$$P_{tot} = \frac{\alpha\beta xy\big(T_H - T_C - x - y\big)}{\beta y\big(T_H - x\big) + \alpha x\big(T_C + y\big)} \tag{2.9}$$

Next step is to maximize (2.9) with respect to x and y by using

$$\frac{dP_{tot}}{dx} = 0 \qquad , \qquad \frac{dP_{tot}}{dy} = 0 \tag{2.10}$$

By solving two equations obtained from (2.10), relation between x and y is

$$y = \sqrt{\frac{\alpha T_C}{\beta T_H}}x \tag{2.11}$$

By inserting (2.11) into one of the equation obtained from (2.10), optimal value of x, x_{max} and then by inserting x_{max} into (2.10), y_{max} is calculated. The expressions for x_{max} and y_{max} are

$$x_{max} = \frac{1 - \sqrt{T_H T_C}}{1 + \sqrt{\alpha/\beta}} \tag{2.12}$$

$$y_{max} = \frac{\sqrt{T_H T_C} - 1}{1 + \sqrt{\beta/\alpha}} \tag{2.13}$$

Next step is to calculate the efficiency at the maximum power. In general efficiency is defined as

$$\eta = \frac{Q_1 - Q_2}{Q_1} \tag{2.14}$$

On solving (2.14) and by using (2.6), the expression for maximum efficiency can be written as

$$\eta_{max} = 1 - \frac{T_C + y_{max}}{T_H - x_{max}} \tag{2.15}$$

By inserting x_{max} and y_{max} expressions from (2.12) and (2.13) respectively into (2.15), efficiency at maximum power is,

$$\eta_{CA} = 1 - \sqrt{\frac{T_C}{T_H}} \tag{2.16}$$

This is known as the Curzon-Ahlborn efficiency, because of which it is denoted as η_{CA}. It can also be written in terms of the Carnot efficiency as

$$\eta_{CA} = 1 - \sqrt{1 - \eta_C} \tag{2.17}$$

The plot of η_C v/s η_{CA} is shown in fig(2.3)

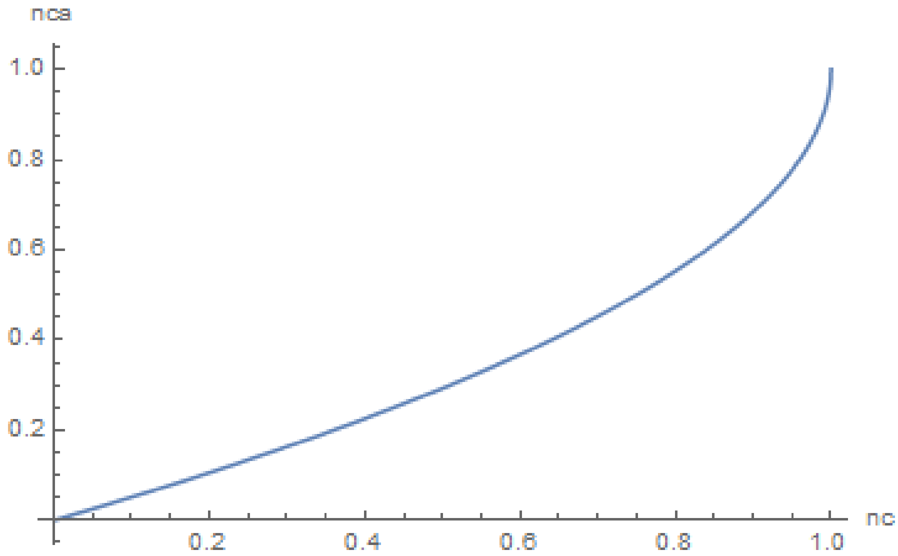

Figure 2.3: X axis represents the Carnot efficiency and Y axis represent the Curzon-Ahlborn efficiency.

In final step, by inserting (2.12) and (2.13) into (2.9), maximum power is calculated

$$P_{max} = \alpha\beta \left[\frac{\sqrt{T_H} - \sqrt{T_C}}{\sqrt{\alpha} + \sqrt{\beta}} \right]^2 \tag{2.18}$$

which can be written in terms of η_C as

$$P_{max} = \alpha\beta T_H \left[\frac{1 - \sqrt{1 - \eta_C}}{\sqrt{\alpha} + \sqrt{\beta}} \right]^2 \tag{2.19}$$

whose plot with respect to η_{CA} is shown in fig(2.4)

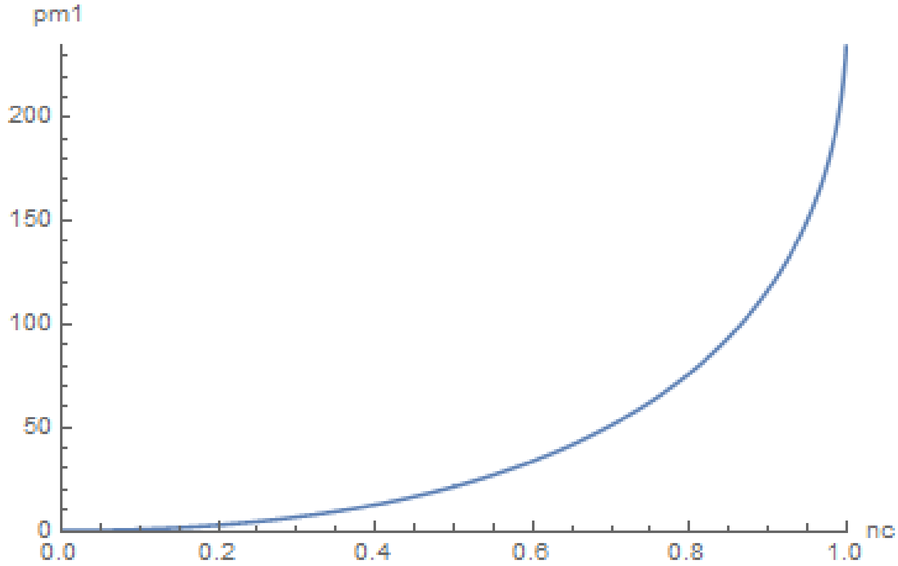

Figure 2.4: X axis represents the Carnot efficiency and Y axis represents the maximum power. The graph is plotted for $\alpha = 1$, $\beta = 1$ and $T_H = 1000$

2.4 Inverse law

The efficiency obtained by the Curzon and Ahlobrn is independent of the thermal conductivity constants and like the Carnot efficiency, it only depends on the heat baths temperatures. On one hand, this result is very attractive but on the other hand this result is not very general. More general result can be obtained by using the inverse law of heat flux transfer (q_I) in the endoreversible model rather than the Newton law of heat flux transfer.
The model taken in this case is similar to the Curzon-Ahlborn model, with including the assumption that there is no heat leaks , heat friction etc. in the system. By taking notations for temperatures, the time of contacts and the conductivity constants similar to the section (2.3), the heat flux transfer for the isothermal expansion stage and for the isothermal compression stage is given by,

$$q_1' = \alpha\Big(\frac{1}{T_{1w}} - \frac{1}{T_1}\Big) \qquad , \qquad q_2' = \beta\Big(\frac{1}{T_2} - \frac{1}{T_{2w}}\Big) \tag{2.20}$$

Using $Q_1' = \int_0^{t_1} q_1' dt$ and $Q_2' = \int_0^{t_2} q_2' dt$, the total heat transfer for the isothermal expansion stage (Q_1') and the total heat transfer for the isothermal compression stage (Q_2') has been calculated

$$Q_1' = \alpha\Big(\frac{1}{T_{1w}} - \frac{1}{T_1}\Big)t_1 \qquad , \qquad Q_2' = \beta\Big(\frac{1}{T_2} - \frac{1}{T_{2w}}\Big)t_2 \tag{2.21}$$

Now by using the endoreversibility condition,

$$\frac{Q_1'}{T_{1w}} = \frac{Q_2'}{T_{2w}}, \tag{2.22}$$

relation between t_1 and t_2 is obtained

$$t_1 = \frac{\beta\left(\frac{1}{T_2} - \frac{1}{T_{2w}}\right)}{\alpha\left(\frac{1}{T_{1w}} - \frac{1}{T_1}\right)} \frac{T_{1w}}{T_{2w}} t_2 \tag{2.23}$$

Now by using the definition of the power and (2.21), (2.22), the expression for the total power is

$$P'_{tot} = \frac{\alpha\beta\left(\frac{1}{T_{1w}} - \frac{1}{T_1}\right)\left(\frac{1}{T_2} - \frac{1}{T_{2w}}\right)\left(T_{1w} - T_{2w}\right)}{\alpha\left(\frac{1}{T_{1w}} - \frac{1}{T_1}\right)T_{2w} + \beta\left(\frac{1}{T_2} - \frac{1}{T_{2w}}\right)T_{1w}} \tag{2.24}$$

The equation (2.24) can be written in a little nicer way by using the definition of the efficiency $\eta = 1 - \frac{Q_2'}{Q_1'}$, which on simplifying become

$$\eta = 1 - \frac{T_{2w}}{T_{1w}}, \tag{2.25}$$

$$\Rightarrow T_{2w} = T_{1w}(1-\eta) \tag{2.26}$$

By inserting (2.26) into (2.24) and simplifying it, the expression of the power can be written as

$$P'_{tot} = \frac{\alpha\eta}{\left(\frac{1}{T_{1w}} - \frac{1}{T_1}\right)^{-1} + \frac{\alpha}{\beta}\left(\frac{1}{T_2} - \frac{1}{T_{2w}}\right)^{-1}\left(1-\eta\right)} \tag{2.27}$$

Equation (2.27) has two variables now, η and T_{1w}. Next step is to optimize (2.27) with respect to T_{1w} and to obtain its expression in terms of η. So by using $dP'_{tot}/dT_{1w} = 0$, T_{1w} is,

$$T_{1w} = \frac{\left[\sqrt{\frac{\alpha}{\beta}} + (1-\eta)^{-1}\right]T_H T_C}{T_H + T_C\sqrt{\frac{\alpha}{\beta}}} \tag{2.28}$$

By inserting (2.28) into (2.27) and simplifying it, expression of power can be written in terms of η, only

$$P'_{tot} = \frac{\alpha\eta\left[T_H(1-\eta) - T_C\right]}{T_H T_C\left[\sqrt{\frac{\alpha}{\beta}} + 1\right]^2} \tag{2.29}$$

The plot of P'_{tot} v/s η is given in [Fig.(2.5)]. The plot shows two extreme ends where the power and the efficiency both are zero and one maximum point, which is the efficiency at maximum power.

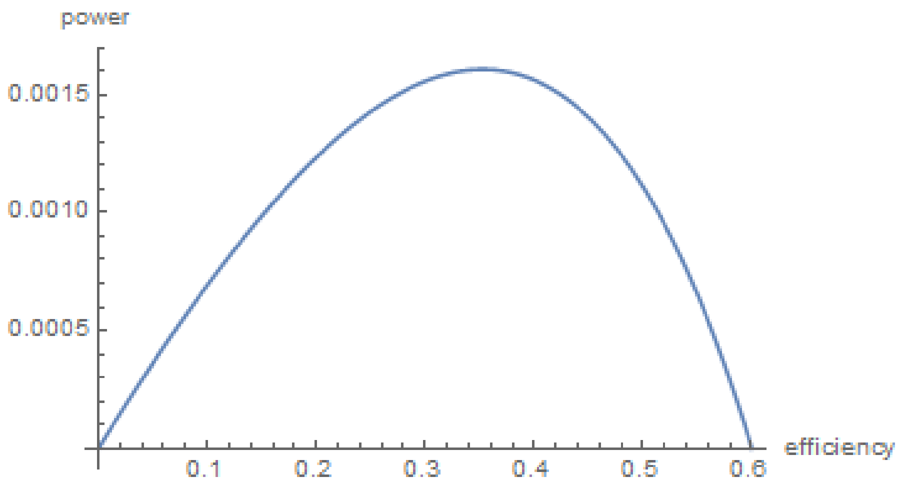

Figure 2.5: X axis represents the general efficiency and Y axis represents the power. The graph is plotted for $\alpha = 2$, $\beta = 2$, $T_H = 100$ and $T_C = 40$

The efficiency at the maximum power is calculated, by using, $dP'_{tot}/d\eta' = 0$,

$$\eta'_{max} = \frac{\left[1 + \sqrt{\frac{\alpha}{\beta}}\right]\left[1 - \frac{T_C}{T_H}\right]}{2 + \sqrt{\frac{\alpha}{\beta}}\left[1 + \frac{T_C}{T_H}\right]} \tag{2.30}$$

which further can be written in terms of the Carnot efficiency ($\eta_C = 1 - T_C/T_H$),

$$\eta'_{max} = \frac{\left[1 + \sqrt{\frac{\alpha}{\beta}}\right]\eta_C}{2 + \sqrt{\frac{\alpha}{\beta}}\left[2 - \eta_C\right]} \tag{2.31}$$

Equation (2.30) gives upper and lower bounds to the efficiency at the maximum power. These bounds are obtained by using the limits $\alpha/\beta \to 0$ and $\beta/\alpha \to 0$. The limit $\alpha/\beta \to 0$ gives $\eta_C/2$ and the limit $\beta/\alpha \to 0$ gives $\eta_C/(2 - \eta_C)$. This can be expressed as

$$\frac{\eta_C}{2} \leq \eta'_{max} \leq \frac{\eta_C}{2 - \eta_C} \tag{2.32}$$

The other limit is the symmetric limit which is calculated by inserting $\alpha = \beta$ into (2.30)

$$\eta'_{max} = \frac{2\eta_C}{4 - \eta_C} \tag{2.33}$$

The plot of all the efficiencies at the maximum power is shown in [Fig.(2.6)]

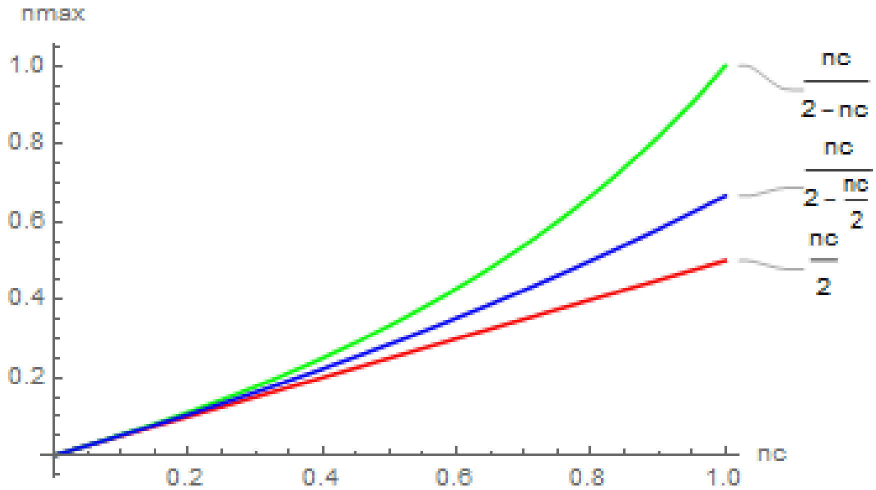

Figure 2.6: X axis represents Carnot efficiency and Y axis represents efficiency at maximum power for 1) $\alpha/\beta \to 0$, 2) $\beta/\alpha \to 0$ and 3) $\alpha = \beta$

2.5 Maxwell-Cattaneo equation

Fourier law of heat conduction linearly relates temperature gradient ∇T to the heat flux according to the following equation

$$q = -\lambda \nabla T \tag{2.34}$$

where λ is the heat conductivity, depending generally upon the temperature. If we put above equation into the energy balance equation, written in absence of the source term, as

$$\rho \frac{du}{dt} = -\nabla . q, \tag{2.35}$$

and relating the specific internal energy u to the temperature by means of $du = c_v dT$, with c_v being the heat capacity per unit mass at the constant volume, one obtains,

$$\rho c_v \frac{dT}{dt} = \nabla . (\lambda \nabla T) \tag{2.36}$$

where ρ is the density and t is the contact time between two different temperature bodies. Above equation is well known equation for heat conduction, whose name is "Fourier equation". From mathematical point of view, this equation is a parabolic equation. Although this equation is well tested for most of the practical problems, it fails to describe the transient temperature field in situations involving short times, high frequencies and small wavelengths. The main reason for such failure lies in the statement of Fourier's law, according to which when two bodies at different temperatures comes in contact with each other, there is instantaneously rise in the heat flux everywhere in the entire system. In other terms, any temperature disturbance will propagates at the infinite velocity. But physically, it is expected, and it is experimentally observed, that a change in temperature gradient should be felt after some build-up or relaxation time, and that disturbances travel at the finite velocity. To overcome this problem, Cattaneo in 1948 proposed a damped version of Fourier's law by introducing a heat flux relaxation term, given by

$$q + \tau \frac{dq}{dt} = -\lambda \nabla T$$

where τ is the heat flux relaxation time term. We can see from above equation that if τ is very small and is negligible then this equation will be reduced to Fourier's law. Because Maxwell have already studied relaxational effects on heat conductors in 1867, this equation is also known as Maxwell-Cattaneo equation.

Our main interests lies in the general solution of the Maxwell-Cattaneo equation which gives time dependent heat flux form,

$$q_t = \lambda \nabla T + u e^{-t/\tau}, \tag{2.37}$$

where u is the integration constant.

Chapter 3

Extension of Curzon-Ahlborn calculations in short time regime

3.1 Introduction

This chapter includes our work of extending the Curzon-Ahlborn calculations in short time regime. For this we have used general solution of the Maxwell-Cattaneo equation and applied it to the Curzon-Ahlborn model of the heat engine. To get analytical calculations in short time regime we have used an assumption whose details are given in section(3.2). We have also done numerical calculation for the general case, without taking any assumption, whose details is given in section(3.3).

3.2 Newton law in short time regime

We started by taking the general solution of the Maxwell-Cattaneo equation

$$q_t = \lambda \nabla T + u e^{-t/\tau} \tag{3.1}$$

where u is the integration constant and applied it to the Curzon-Ahlborn heat engine. As mentioned in the section (2.3) the Curzon-Ahlborn heat engine is a four stage heat engine, which follow the Carnot cycle. Its isothermal expansion stage lasts for time t_1 and its isothermal compression stage lasts for time t_2. During the isothermal expansion stage there is the heat flux transfer from the infinite hot bath at temperature T_H to the working substance at temperature T_{1w} and during the isothermal compression stage there is release of the heat flux from the working substance at temperature T_{2w} to the infinite cold sink at temperature T_C. The form of these heat flux transfers have taken to be linear, according to the Newton law. So when we apply Maxwell-Cattaneo equation for the heat flux on the isothermal expansion and the isothermal compression stages we get general form of the heat fluxes,

$$q_{ie} = \alpha x(1 - e^{-t_1'/\tau}) \qquad , \qquad q_{ic} = \beta y(1 - e^{-t_2'/\tau}) \tag{3.2}$$

where $t_1' \epsilon [0, t_1]$ and $t_2' \epsilon [0, t_2]$and x, y, α, β, τ have same definitions as in the Curzon-Ahlborn calculations, mentioned in the section (2.3). If we look at the plot of q_{ie} or q_{ic} with respect to the contact time t_1 or t_2, [Fig(3.1)], we can divide the plot in two segment.

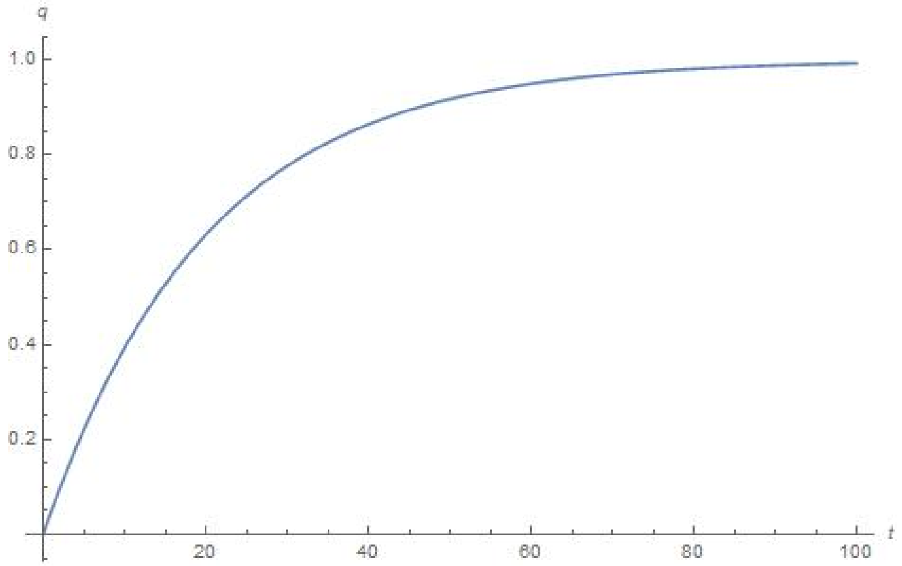

Figure 3.1: This graph is plotted for the heat flux transfer in the isothermal expansion stage. The plot for the isothermal compression stage will be same. Here X axis represents the time of contact of the working substance with the infinite heat bath during the isothermal expansion stage and Y axis represents the heat flux transfer during it. In this the value of $\alpha = 1$, $T_H = 10$ and $T_{1w} = 9$

In the first segment the heat flux actually starts building up from time zero and reach at the transition point. In the second segment, after the transition point, the heat flux becomes constant with respect to time. The form of the heat flux transfer which Curzon and Ahlborn have taken lies in the second segment which means, it is constant with respect to time. But we are interested in the form of the heat flux lies in the first segment. So for it, we have taken an important assumption for our further calculations.

We assume that the contact time for the isothermal expansion stage t_1 and the contact time for the isothermal compression stage t_2 is very small with respect to the relaxation time for the heat flux. Mathematically it is,

$$t_1 << \tau \qquad , \qquad t_2 << \tau \tag{3.3}$$

This assumption lead us to study the model in short time regime. Now we can write the expression for the heat fluxes q_{ie} and q_{ic} in short time regime. By using the assumption we can expand the exponential terms in (3.2) upto first order, which gives

$$\overline{q}_{ie} = \frac{\alpha x t_1}{\tau} \qquad , \qquad \overline{q}_{ic} = \frac{\beta y t_2}{\tau} \tag{3.4}$$

Now we write the total heat transfer form for the isothermal expansion stage and for the isothermal compression stage by using $\overline{Q}_{ie} = \int_0^{t_1} q_{ie} dt'_1$ and $\overline{Q}_{ic} = \int_0^{t_2} q_{ic} dt'_2$ and we get,

$$\overline{Q_{ie}} = \frac{\alpha x t_1^2}{2\tau} \qquad , \qquad \overline{Q_{ic}} = \frac{\beta y t_2^2}{2\tau} \tag{3.5}$$

In next step we want relation between the contact times t_1 and t_2 , for it we used the endoreversible condition, as mentioned in section(2.3)

$$\frac{\overline{Q_{ie}}}{T_{1w}} = \frac{\overline{Q_{ic}}}{T_{2w}}, \tag{3.6}$$

By inserting (3.5) into (3.6) we get

$$\frac{\alpha x t_1^2}{2\tau T_{1w}} = \frac{\beta y t_2^2}{2\tau T_{2w}} \tag{3.7}$$

On solving (3.7) for t_2 we get relation between t_1 and t_2,

$$t_2 = t_1 \sqrt{\frac{\alpha x T_{2w}}{\beta y T_{1w}}} \tag{3.8}$$

Next we calculated the power form by using $\overline{P}_{tot} = W_{tot}/t_{tot}$ where $W_{tot} = \overline{Q}_{ie} - \overline{Q}_{ic}$ and $t_{tot} = t_1 + t_2$ and by using (3.5), (3.8) we get

$$\overline{P_{tot}} = \frac{t_1 \alpha x \sqrt{\beta y}\Big(T_H - x - y - T_C\Big)}{2\tau\sqrt{T_H - x}\Big[\sqrt{\beta y (T_H - x)} + \sqrt{\alpha x (y + T_C)}\Big]} \tag{3.9}$$

Now task is to maximize (3.9) with respect to x and y by using $d\overline{P}_{tot}/dx = 0$ and $d\overline{P}_{tot}/dy = 0$ We used "Mathematica" to get this calculations and obtained relation between x and y on solving $d\overline{P}_{tot}/dx = 0$ and $d\overline{P}_{tot}/dy = 0$,

$$x = T_H - \sqrt{\frac{T_H}{T_C}}(T_C + y) \tag{3.10}$$

Further we solved for y by inserting (3.10) into equation obtained from $d\overline{P}_{tot}/dy = 0$. Finally we obtained an cubic equation with respect to y, given by

$$4\Big(\frac{\beta}{\alpha}+1\Big)y^3 = 8y^2r - 5yr^2 + r^3 \tag{3.11}$$

where r is $\sqrt{T_C T_H} - T_C$. On solving (3.11) for y, we get only one valid root. We then simplified this root in three ways

1. By using limit $\alpha/\beta \to 0$, that is when the thermal conductivity in the isothermal compression stage is much higher than the thermal conductivity in the isothermal expansion stage. The expressions for y_{max} and x_{max} for it are

$$y_{max} = 0 \qquad , \qquad x_{max} = T_H - \sqrt{T_H T_C} \tag{3.12}$$

2. By using limit $\beta/\alpha \to 0$, that is when the thermal conductivity in the isothermal expansion stage is much higher than the thermal conductivity in the isothermal compression stage. The expressions for y_{max} and x_{max} for it are

$$y_{max} = \sqrt{T_C T_H} - T_C \qquad , \qquad x_{max} = 0 \tag{3.13}$$

3. And by using $\alpha = \beta$, that is the symmetric case. The expressions for y_{max} and x_{max} for it are

$$y_{max} = 0.302\Big(\sqrt{T_C T_H} - T_C\Big)\ ,\ x_{max} = T_H - \sqrt{\frac{T_H}{T_C}}\Big(0.698 T_C - 0.302\sqrt{T_H T_C}\Big) \tag{3.14}$$

Now task is to obtain efficiency at maximum power for all cases, by using respective x_{max} and y_{max} values. The basic formula for efficiency is given by

$$\eta = \frac{W_{output}}{Q_{input}} \tag{3.15}$$

where $W_{output} = Q_{ie} - Q_{ic}$ and $Q_{input} = Q_{ie}$, the expressions for Q_{ie} and Q_{ic} is given in (3.5). If we simplify (3.15) and write in its maximum form then it becomes

$$\eta_{max} = 1 - \frac{T_C + y_{max}}{T_H - x_{max}} \tag{3.16}$$

Now by inserting respective y_{max} and x_{max} values for all the cases in (3.16), we get same result for all cases, given by

$$\overline{\eta}_{max} = 1 - \sqrt{\frac{T_C}{T_H}} \tag{3.17}$$

Which is similar to the Curzon-Ahlborn efficiency (η_{CA}). The result is quite exciting because two approaches for calculations are very different from each other but at last both give same result for the efficiency at the maximum power.

3.3 Numerical calculations

We have used "Mathematica" for numerical calculations. It has done for two cases, short time regime case and for the general case. For short time regime case, it is possible to obtain the analytical solution as shown in section (3.2) but for the general case, analytical solution is difficult to obtain. So the main aim of these calculations is to solve the general case.

Here general case is the case which includes no assumption and takes the general form of heat flux transfer for isothermal expansion stage and for isothermal compression stages, given in (3.2) . Now the general form of the total heat transfer for two stages can be obtained by using $Q_{ie} = \int_0^{t_1} q_{ie} dt'_1$ and $Q_{ic} = \int_0^{t_2} q_{ic} dt'_2$. So we get,

$$Q_{ie} = \alpha x\Big(t_1 + \tau(e^{-t_1/\tau} - 1)\Big) \qquad , \qquad Q_{ic} = \beta y\Big(t_2 + \tau(e^{-t_2/\tau} - 1)\Big) \tag{3.18}$$

Next we obtained the general relation between t_1 and t_2 by using endoreversible condition

$$\frac{Q_{ie}}{T_{1w}} = \frac{Q_{ic}}{T_{2w}} \tag{3.19}$$

which is further inserted in the power formula $P = (Q_{ie} - Q_{ic})/(t_1 + t_2)$ from which we obtained the general expression for the power. The expression obtained depends upon α, β, t_1 , τ, x and y. Then we used numerical code to maximize this power form with respect to x and y .

3.3.1 Result

The first result of this calculation is shown in [Fig.(3.2)]. In this we have plotted obtained values for the efficiency at the maximum power with respect to the Carnot efficiency.

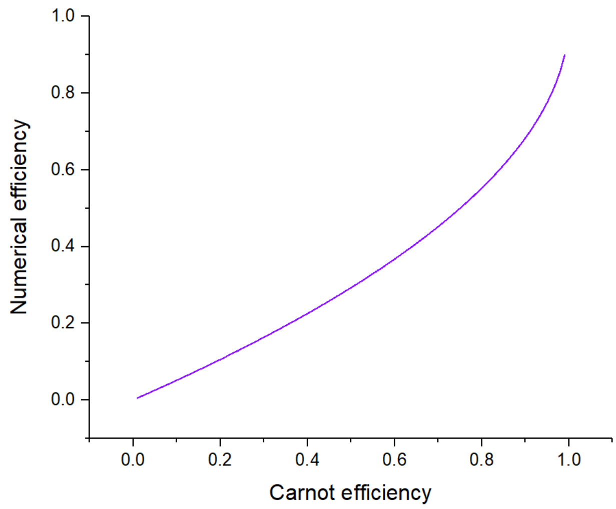

Figure 3.2: Here $\alpha = 1$, $\beta = 1$, $\tau = 10$, $T_H = 100$ and $t_1 = 1$

The curve obtained from this is similar to the curve η_C v/s η_{CA} as shown in [Fig(2.3)]. So in first result, the conclusion is that the solution for the efficiency at maximum power for general case is the Curzon-Ahlborn efficiency (η_{CA}).

In the second result, we have plotted obtained maximum power values with respect to the Carnot efficiency. This is shown in [Fig(3.3)].

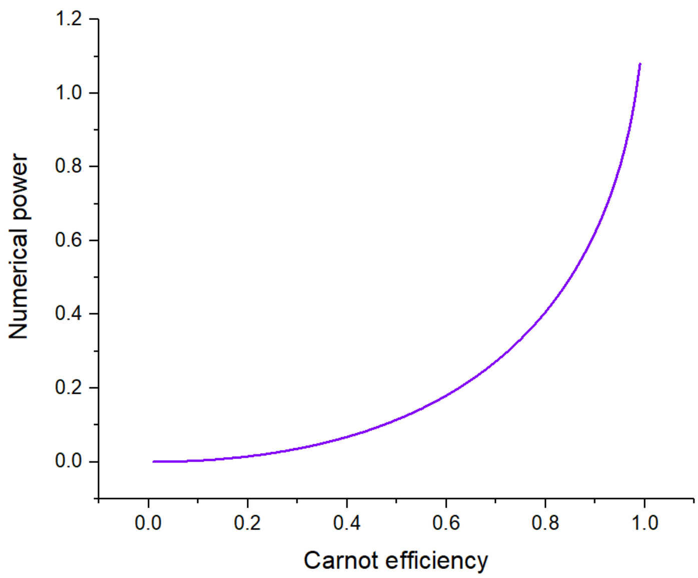

Figure 3.3: Here $\alpha = 1$, $\beta = 1$, $\tau = 10$, $T_H = 100$ and $t_1 = 1$

The curve is plotted for specific values of t_1, τ, α and β and it follows the same trend as the curve η_{CA} v/s P_{max}, shown in [Fig.(2.4)] for the Curzon-Ahlborn case.

In the third and last result, we have plotted obtained maximum power values with respect to the relaxation time. The plot is shown in [Fig.(3.4)]

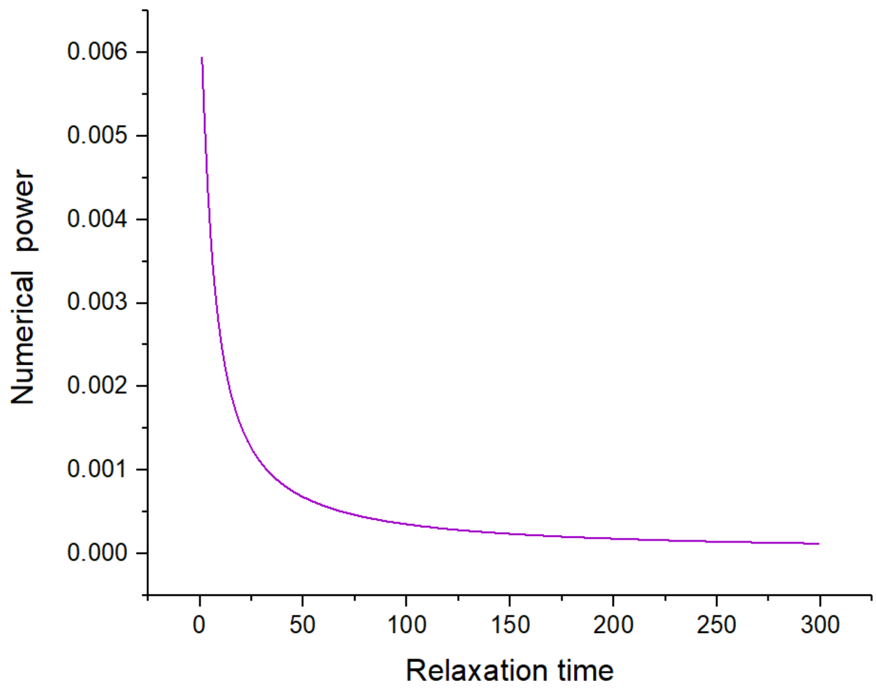

Figure 3.4: Here $\alpha = 1$, $\beta = 1$, $T_H = 10$, $T_C = 9$ and $t_1 = 10$

This plot tells that by increasing the relaxation time, the maximum power decreases and reaches to zero after reasonable increase in the relaxation time value.

In the conclusion of this chapter, the results in short time regime and for the general case are reproduced for the endoreversible model using the Newton law in finite-time regime and the study of the dependence of the maximum power on the relaxation time is done.

Chapter 4

Extension of inverse law calculations in short time regime

4.1 Introduction

This chapter includes our work of extending the inverse law calculations for the endoreversible model in short time regime(as mentioned in section (3.2)). The starting point and the way of the calculations is same as in section (3.2). The only change is the law of the heat flux transfer taken and the power optimization parameters. The details of calculation is given in section (4.2). We have also done numerical calculations of this case whose results are given in section (4.3).

4.2 Inverse law in short time regime

According to the inverse law, the heat flux transfer between two bodies is inversely proportional to the temperature difference between them. By applying this definition and eq.(2.37) to the endoreversible model (fig(2.2)), the general expressions for the heat flux transfer for the isothermal expansion stage and for the isothermal compression stage are written as

$$q'_{ie} = \alpha\Big(\frac{1}{T_{1w}} - \frac{1}{T_H}\Big)\Big(1 - e^{-t'_1/\tau}\Big) \qquad , \qquad q'_{ic} = \beta\Big(\frac{1}{T_C} - \frac{1}{T_{2w}}\Big)\Big(1 - e^{-t'_2/\tau}\Big) \quad (4.1)$$

where $t'_1 \,\epsilon\, [0, t_1]$ and $t'_2 \,\epsilon\, [0, t_2]$and α, β, τ have same definitions as in the Curzon-Ahlborn calculations, mentioned in the section (2.3). To write the expressions of (4.1) into short time regime we have used the assumption (3.3) and have expanded the exponentials in (4.1) to first order, which gave,

$$\tilde{q_{ie}} = \frac{\alpha\Big(\frac{1}{T_{1w}} - \frac{1}{T_H}\Big)t_1}{\tau} \qquad , \qquad \tilde{q_{ic}} = \frac{\beta\Big(\frac{1}{T_C} - \frac{1}{T_{2w}}\Big)t_2}{\tau} \quad (4.2)$$

Next the total heat transfer during two isothermal stages is obtained by using $\tilde{Q_{ie}} = \int_0^{t_1} q'_{ie} dt'_1$ and $\tilde{Q_{ic}} = \int_0^{t_2} q'_{ic} dt'_2$.

$$\tilde{Q_{ie}} = \frac{\alpha\Big(\frac{1}{T_{1w}} - \frac{1}{T_H}\Big)t_1^2}{2\tau} \qquad , \qquad \tilde{Q_{ic}} = \frac{\beta\Big(\frac{1}{T_C} - \frac{1}{T_{2w}}\Big)t_2^2}{2\tau} \tag{4.3}$$

Now by using the endoreversibility condition,

$$\frac{\tilde{Q_{ie}}}{T_{1w}} = \frac{\tilde{Q_{ic}}}{T_{2w}} \tag{4.4}$$

and by using (4.3), relation between t_1 and t_2 can be written as

$$t_2 = t_1 \sqrt{\frac{\alpha\Big(\frac{1}{T_{1w}} - \frac{1}{T_H}\Big)T_{2w}}{\beta\Big(\frac{1}{T_C} - \frac{1}{T_{2w}}\Big)T_{1w}}} \tag{4.5}$$

Next by using (4.5) and the definition of the power($\tilde{P_{tot}} = W_{output}/t_{tot}$) where $W_{output} = \tilde{Q_{ie}} - \tilde{Q_{ic}}$ and $t_{tot} = t_1 + t_2$, we get the expression for the power in short time regime.

$$\tilde{P_{tot}} = \frac{t_1\alpha\Big(\frac{1}{T_{1w}} - \frac{1}{T_H}\Big)\sqrt{\beta\big(\frac{1}{T_C} - \frac{1}{T_{2w}}\big)}\Big(T_{1w} - T_{2w}\Big)}{2\tau\sqrt{T_{1w}}\Big[\sqrt{\beta\big(\frac{1}{T_C} - \frac{1}{T_{2w}}\big)T_{1w}} + \sqrt{\alpha\big(\frac{1}{T_{1w}} - \frac{1}{T_H}\big)T_{2w}}\Big]} \tag{4.6}$$

Next we optimized (4.6) with respect to T_{1w} and T_{2w} by using $d\tilde{P_{tot}}/dT_{1w} = 0$ and $d\tilde{P_{tot}}/dT_{2w} = 0$. For $d\tilde{P_{tot}}/dT_{1w} = 0$ we get,

$$\Big(2T_HT_{1w} - 4T_HT_{2w} + 2T_{1w}T_{2w}\Big)\sqrt{\beta\Big(\frac{1}{T_C} - \frac{1}{T_{2w}}\Big)T_1w} + \Big(T_{1w}^2 - 2T_HT_{2w} + T_{1w}T_{2w}\Big)\sqrt{\alpha\Big(\frac{1}{T_{1w}} - \frac{1}{T_H}\Big)T_{2w}} = 0 \tag{4.7}$$

and for $d\tilde{P_{tot}}/dT_{2w} = 0$ we get,

$$\Big(2T_{2w}^2 - 2T_CT_{2w}\Big)\sqrt{\beta\Big(\frac{1}{T_C} - \frac{1}{T_{2w}}\Big)T_{1w}} + \Big(T_{2w}^2 + T_{1w}T_{2w} - 2T_{1w}T_C\Big)\sqrt{\alpha\Big(\frac{1}{T_{1w}} - \frac{1}{T_H}\Big)T_{2w}} = 0 \tag{4.8}$$

Now by dividing (4.7) with (4.8) and solving, we get relation between T_1w and T_2w to be,

$$T_{1w} = \frac{2}{\frac{2}{T_{2w}} + \frac{1}{T_H} - \frac{1}{T_C}} \tag{4.9}$$

Next by inserting (4.9) into (4.8) and simplifying it, we get

$$\sqrt{\alpha\Big(-\frac{1}{2}\Big(\frac{1}{T_H} + \frac{1}{T_C}\Big)T_{2w} + 1\Big)}\Big(-4T_C + 4T_{2w} + T_{2w}^2\Big(\frac{1}{T_H} - \frac{1}{T_C}\Big)\Big) + 2\sqrt{2\beta\Big(\frac{1}{T_C} - \frac{1}{T_{2w}}\Big)\Big(\frac{1}{T_H} - \frac{1}{T_C} + \frac{2}{T_{2w}}\Big)}\Big(-T_CT_{2w} + T_{2w}^2\Big) = 0 \tag{4.10}$$

Next task is to solve (4.10) for T_{2w} and thus to get optimum value for T_{2w}. We have solved it by using the limits $\alpha/\beta \to 0$ and $\beta/\alpha \to 0$. The limit $\alpha/\beta \to 0$ give one valid solution which is,

$$T_{2w} = T_C \tag{4.11}$$

and the limit $\beta/\alpha \to 0$ gives two valid solutions which are

$$T_{2w} = \frac{2T_H T_C}{T_H + T_C} \tag{4.12}$$

and

$$T_{2w} = \frac{2\left(T_H - \sqrt{T_H T_C}\right)}{\frac{T_H}{T_C} - 1} \tag{4.13}$$

Next by inserting (4.11) in (4.9) the optimum value of $T_1 w$ for $\alpha/\beta \to 0$ is,

$$T_{1w} = \frac{2T_H T_C}{T_H + T_C} \tag{4.14}$$

by inserting (4.12) into (4.9), the first optimum value of T_{1w} for $\beta/\alpha \to 0$ is

$$T_{1w} = T_H \tag{4.15}$$

and by inserting (4.13) into (4.9), the second optimum value of $T_1 w$ for limit $\beta/\alpha \to 0$ is,

$$T_{1w} = \frac{2T_H T_C\left(1 - \sqrt{\frac{T_C}{T_H}}\right)}{\left(T_H - T_C\right)\sqrt{\frac{T_C}{T_H}}} \tag{4.16}$$

Now by using the definition of efficiency($\eta = W_{Output}/Q_{input}$), where $W_{Output} = \tilde{Q_{ie}} - \tilde{Q_{ic}}$ and $Q_{input} = \tilde{Q_{ie}}$ and using (4.3) and (4.5), the expression for the general efficiency can be written as

$$\eta = 1 - \frac{T_{2w}}{T_{1w}} \tag{4.17}$$

By inserting (4.11) and (4.14) into (4.16), the efficiency at the maximum power(η_{max}) for the limit $\alpha/\beta \to 0$ is,

$$\eta_{max} = \frac{\eta_C}{2} \tag{4.18}$$

by inserting (4.12)(i.e for first solution) and (4.15) into (4.17), the efficiency at the maximum power(η_{max}) for the limit $\beta/\alpha \to 0$ is,

$$\eta_{max} = \frac{\eta_C}{2 - \eta_C} \tag{4.19}$$

and by inserting (4.13)(i.e for second solution) and (4.16) into (4.17), the efficiency at the maximum power(η_{max}) for the limit $\beta/\alpha \to 0$ is,

$$\eta_{max} = 1 - \sqrt{1 - \eta_C} \tag{4.20}$$

Now as the limit $\beta/\alpha \to 0$ is giving two optimum values for the efficiency at the maximum power, so the question is which one among (4.19) or (4.20) is maximum or minimum. We have tried to answer it analytically and numerically but the question is still remaining to be get answered properly.

4.3 Further work which can be done

1. One can try to obtain general form for the efficiency at the maximum power in short time regime.

2. One can numerically maximize the power expression for the short time case and the general case(i.e without using any assumption) and can study the behaviour of the maximum power or efficiency at the maximum power with respect to Carnot efficiency. The other interesting study which can be done is on the behaviour of the maximum power with respect to the relaxation time.

Chapter 5

Conclusion

We have introduced Maxwell-Cattaneo equation into endoreversible model for two heat flux transfer laws, Newton law and inverse law. For the analytical calculation of two cases, we have introduced an assumption, which gave results in short time regime. The analytical results obtained for two cases are similar to the results in finite time regime(i.e without using Maxwell-Cattaneo equation). We have also done numerical calculations for both short time cases and we have successfully obtained numerical solution for general case(i.e without using any assumption) for Newton law.

In further work, one can apply the Maxwell-Cattaneo equation to the other models in finite-time thermodynamics like low dissipation model ([2], [15]) etc. or one can apply Maxwell -Cattaneo equation by taking finite heat sources ([17], [18]) instead of infinite heat sources.

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