On a Conjecture on Linear Systems

A thesis
submitted by

SONICA

in partial fulfilment of
the requirements for the degree of

Doctor of Philosophy

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November 2015

dedicated

to

my parents

Declaration

The work presented in this thesis has been carried out by me under the guidance of Professor Kapil Hari Paranjape at Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Date:
Place:

Sonica

In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Kapil H. Paranjape
(Supervisor)

Acknowledgements

First of all I would like to thank my supervisor Prof. Kapil H. Paranjape, without whose patience, guidance and support, none of this work would have been possible. He has been a source of inspiration. His advice and encouragement has been an outstanding influence in my graduate student life. My sincere gratitude to visionary Director Prof. N. Sathyamurthy for inspiration. I thank Prof. Peter Newstead for his valuable comments on work done in this thesis. I am also grateful to Prof. I. B. S. Passi, Dr. Chanchal Kumar, Dr. Dinesh Khurana, Prof. Sudesh K. Khanduja and Dr. Lingaraj Sahu who have given me courses during first year of my PhD.

I acknowledge NBHM for supporting various ATM schools that I attended and benefited a lot. I thank DST for providing me travel grant to attend "Student Tropical Algebraic Geometry Symposium 2015" held at Brown University, Providence, USA. I thank "Ramanujan Mathematical Society" to give me an opportunity to present my research work in 30th annual conference of "RMS" held at IISER Mohali.

I am grateful to IISER Mohali for providing advanced infrastructure and excellent research environment. My gratitude to the library facility of IISER Mohali and Dr. P. Visakhi for providing subscription to various journals, which plays an essential role in research. I acknowledge financial support provided by UGC during my PhD.

Thanks to my family for their affection and support, and especially to my parents for believing in me and helping me achieve all that I have done so far. My father helped me take my first mathematical steps and helped to shape the way I think. I am thankful to my husband Mr. Nitin Phull for his emotional support. My deepest gratitude to my parents-in-law for their love and support.

I would like to thank all my friends Shalini Mam, Preety, Shelly, Dilpreet, Preetinder, Ajay, Shiv, Kanika and Khushwant for being there when I needed them, and for making sure that I had a life outside of math. The entire staff of the math department was very warm and supportive. Special thanks to Dr. Chanchal Kumar, Dr. Yashonidhi Pandey and Dr. Alok Maharana who fielded my innumerable queries.

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Contents

Glossary of notations

k	:	a field.
V	:	finite dimensional vector space.
$\mathbb{C}$	:	the field of complex numbers.
V˘	:	dual of a vector space.
$\dim V$	:	dimension of a vector space.
$\operatorname{codim} V$	:	codimension of a vector space.
S^*V	:	symmetric algebra over a vector space.
$\mathcal{K}_{p,q}(B,V)$	:	(p,q)th Koszul cohomology group of a graded module.
γ_C	:	Clifford index of a curve.
$\lvert D\rvert$	:	linear system of a divisor.
K	:	canonical line bundle on a curve.
g	:	genus of a curve.
$H^i(C,L)$	:	cohomology group.
$h^i(C,L)$	:	dimension of the cohomology group.
$\mathcal{O}_C(D)$	:	sheaf of holomorphic functions with divisor bounded by D.
$\deg D$	:	degree of a divisor.
$[r]$	:	greatest integer part of a real number.
Φ_K	:	canonical map of a curve.
f^*G	:	pullback of a vector bundle.
f_*G	:	push forward of a vector bundle.
$\det G$	:	determinant of a vector bundle.
E_K	:	pullback of universal quotient bundle by the canonical map.
$\Gamma(C,L)$	:	global sections of a line bundle.
$\wedge^i G$	:	exterior power of a vector bundle.
$\sum_i$	:	cone of locally decomposable sections.
$\mathbb{P}^n$	:	projective space of dimension n.
$\chi(L)$	:	Euler characteristic of a line bundle.
τ	:	hyperplane section.
g^r_d	:	a linear system of dimension r and degree d.
β_{ij}	:	graded betti numbers.
M_{ij}	:	k- vector space of $\dim \beta_{ij}$.
ζ^b	:	Eagon-Northcott complex.
$\binom{n}{r}$	:	n choose r.
Γ	:	graph morphism.
Δ	:	diagonal morphism.
$Sl_2(\mathbb{C})$	:	special linear group of order 2 over $\mathbb{C}$.
(L,V)	:	generated linear system.
$\square$	:	end (or omission) of proof.

Chapter 1

Green's conjecture

Introduction

In 1984, M.Green introduced Koszul cohomology in his paper [Gre84]. In this foundational paper, he introduced Koszul cohomology group $\mathcal{K}_{p,q}(C, L)$ associated to a line bundle L over a smooth projective variety C. He studied various properties of these groups and established certain vanishing theorems. In the appendix to this paper [Gre84], the condition of vanishing of Koszul cohomology group is related to a numerical invariant named as Clifford index of the projective curve C.

The term Clifford index was defined by H.H. Martens for a smooth projective curve of genus $g \geq 2$ with $H^0(C, L) \geq 2$ and $H^1(C, L) \geq 2$.

Green [Gre84] stated a conjecture that relate the Koszul cohomology of a smooth projective curve over $\mathbb{C}$ to its Clifford index. This conjecture became an important guideline for future research. A lot of work has been done on this conjecture. In 1988, K. Paranjape and S. Ramanan [PR88] gave an equivalent formulation to Green conjecture. In 1992, K. Hulek, K. Paranjape and S. Ramanan [HPR92] proved stronger version of Green conjecture for curves with Clifford index 1 and in 1998, A. Hirshowitz and S. Ramanan [HR98] gave new evidence for Green conjecture on syzygies of canonical curve. Then in 2002, C. Voisin [Voi02] achieved a major breakthrough by proving Green conjecture for curves of even genus lying on a K3 surface and in 2005, she [Voi05] proved it for curves of odd genus lying on K3 surface. Then finally in 2011, M. Aprodu and G. Farkas [AF11] couped with results of Voisin and Hirshowitz - Ramanan provided a complete solution to Green conjecture for smooth curves on arbitrary K3 surfaces. In 2012, Eusen and Schreyer [ES12] gave a remark on conjecture of K. Paranjape and S. Ramanan.

This chapter is devoted to a review of a number of basic definitions and results on Koszul cohomology and Clifford index, which are mainly included to fix the notations and to obtain a self contained presentation. In §1.1, 1.2 and 1.3, Koszul complex, Koszul cohomology group $\mathcal{K}_{p,q}(C, L)$ in algebraic and geometric context respectively [AN10], [Gre84]are defined. In §1.4, Clifford index and its various properties are given and in §1.5, Green's conjecture is stated.

1.1 The Koszul Complex

Let V be a vector space of dimension $r+1$ over a field k. Given a nonzero element $x \in$ V$\check{}$, the corresponding map

$$\langle x, \rangle : V \to k$$

extends uniquely to an antiderivation

$$i_x : \wedge^* V \to \wedge^* V$$

of the exterior algebra. This derivation is defined inductively by putting $i_x|_V = \langle x, \rangle : V \to k$ and

$$i_x(v \wedge v_1 \wedge \cdots \wedge v_{p-1}) = \langle x, v \rangle . v_1 \wedge \cdots \wedge v_{p-1} - v \wedge i_x(v_1 \wedge \cdots \wedge v_{p-1}).$$

The resulting maps

$$i_x : \wedge^p V \to \wedge^{p-1} V$$

are called contraction (or inner product) maps ; they are dual to the exterior product maps

$$\lambda_x : \wedge^{p-1} \mathrm{V}\check{} \to \wedge^p \mathrm{V}\check{}$$

and satisfy $i_x \circ i_x = 0$. Hence we obtain a complex

$$K_\bullet(x) : (0 \to \wedge^{r+1} V \to \ldots \to \wedge^p V \xrightarrow{i_x} \wedge^{p-1} V \to \wedge^{p-2} V \to \ldots \to k \to 0)$$

called the Koszul complex .

Remark 1.1. For any $\alpha \in k^*$, the complexes $K_\bullet(x)$ and $K_\bullet(\alpha x)$ are isomorphic. Hence the Koszul complex $K_\bullet(x)$ depends only on the point $[x] \in \mathbb{P}(\mathrm{V}\check{})$

Lemma 1.1.1. *Given nonzero elements $\xi \in V$, $x \in V\check{}$, let $\lambda_\xi : \wedge^{p-1}V \xrightarrow{\wedge\xi} \wedge^p V$ be the map given by the wedge product with ξ. We have*

$$i_x \circ \lambda_\xi + \lambda_\xi \circ i_x = \langle x, \xi \rangle . id.$$

Proof: It is sufficient to verify the statement on decomposable elements.

$$\begin{aligned}
(i_x \circ \lambda_\xi)(v_1 \wedge v_2 \wedge \cdots \wedge v_p) &= i_x(\xi \wedge v_1 \wedge v_2 \wedge \cdots \wedge v_p) \\
&= \langle x, \xi \rangle . v_1 \wedge v_2 \wedge \cdots \wedge v_p + \\
&\quad \sum_i (-1)^{i-1} \langle x, v_i \rangle . \xi \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_p \\
(\xi \circ i_x)(v_1 \wedge v_2 \wedge \cdots \wedge v_p) &= \sum_i (-1)^i \langle x, v_i \rangle . \xi \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_p
\end{aligned}$$

and the statement follows. □

Corollary 1.1.2. *For any non zero element $x \in V\check{}$, the Koszul complex $K.(x)$ is an exact complex of k- vector spaces.*

Proof: Choose $\xi \in V$ such that $\langle x, \xi \rangle = 1$ and apply lemma 1.1.1. □

1.2 Koszul Cohomology in algebraic context

Let B be a graded module over the symmetric algebra S^*V. Let $i : \wedge^p V \to \wedge^{p-1}V \otimes V$ be the dual of the wedge product map $\lambda : \wedge^{p-1}V\check{} \otimes V\check{} \to \wedge^p V\check{}$. Since

$$\wedge^{p-1}V \otimes V \cong Hom(V\check{}, \wedge^{p-1}V)$$

Thus, under this identification, we have

$$\begin{array}{ccc}
\wedge^p V & \xrightarrow{\lrcorner i} & \wedge^{p-1}V \otimes V \\
v_1 \wedge \cdots \wedge v_p & \mapsto & (x \mapsto i_x(v_1 \wedge \cdots \wedge v_p)).
\end{array}$$

The graded S^*V module structure of B induces maps $m_q : V \otimes B_q \to B_{q+1}$ for all q. Define a map

$$d_{p,q} : \wedge^p V \otimes B_q \to \wedge^{p-1}V \otimes B_{q+1}$$

by the composition

$$\begin{tikzcd}
\wedge^p V \otimes B_q \arrow[r, "\lrcorner i \otimes id"] \arrow[dr, "d_{p,q}"'] & \wedge^{p-1} V \otimes V \otimes B_q \arrow[d, "id \otimes m_q"] \\
& \wedge^{p-1} V \otimes B_{q+1}
\end{tikzcd}$$

Definition 1.2.1. *The Koszul cohomology group $\mathcal{K}_{p,q}(B,V)$ of B is the cohomology at the middle term of the complex*

$$\wedge^{p+1} V \otimes B_{q-1} \xrightarrow{d_{p+1,q-1}} \wedge^p V \otimes B_q \xrightarrow{d_{p,q}} \wedge^{p-1} V \otimes B_{q+1}$$

i.e.

$$\mathcal{K}_{p,q}(B,V) = \frac{ker d_{p,q}}{im d_{p+1,q-1}}$$

Note 1.2. *In defining $d_{p,q}$, we are using the convention $\wedge^p V = 0$ if $p < 0$ or $p > \dim V$*

By above convention, we have automatically, $\mathcal{K}_{p,q}(B,V) = 0$ if $p < 0$ or $p > \dim V$

An element $x \in$ V˘induces a derivation

$$\partial_x : S^* V \to S^* V$$

of degree -1 on the symmetric algebra, which is defined inductively by the rule

$$\partial_x(v.v_1 \ldots v_{p-1}) = \partial_x(v).v_1 \ldots v_{p-1} + v.\partial_x(v_1 \ldots v_{p-1})$$

If we choose coordinates $X_0, X_1, \ldots, X_r \in V$, with duals $x_i \in$ V˘, the resulting map

$$\partial_{x_k} : S^p V \to S^{p-1} V$$

sends a homogeneous polynomial f of degree p to the partial derivative $\frac{\partial f}{\partial X_k}$.
Using the natural map

$$S^{q+1} V \xrightarrow{\partial} S^q V \otimes V \cong Hom(V˘, S^q V)$$

which is given by

$$f \mapsto (x \mapsto \partial_x(f))$$

and the wedge product map $\lambda : \wedge^{p-1}V \otimes V \to \wedge^p V$, we define the map

$$D : \wedge^{p-1}V \otimes S^{q+1}V \to \wedge^p V \otimes S^q V$$

as the composition

$$\begin{tikzcd} \wedge^{p-1}V \otimes S^{q+1}V \arrow[r, "id\otimes\partial"] \arrow[dr, "D"'] & \wedge^{p-1}V \otimes S^q V \otimes V \cong \wedge^{p-1}V \otimes V \otimes S^q V \arrow[d, "\lambda\otimes id"] \\ & \wedge^p V \otimes S^q V \end{tikzcd}$$

Proposition 1.2.2. *We have* $\mathcal{K}_{0,0}(S^*V, V) \cong k$ *and* $\mathcal{K}_{p,q}(S^*V, V) = 0$ *for all* $(p,q) \neq (0,0)$.

Proof: $\mathcal{K}_{0,0}(S^*V, V) \cong k$ follows from the definition.
To prove the second part, choose coordinates $X_0, X_1, \ldots, X_r$ on V and note that

$$D : \wedge^p V \otimes S^{q+1}V \to \wedge^{p+1}V \otimes S^q V$$

is given by

$$X_{i_1} \wedge \ldots \wedge X_{i_p} \otimes f \mapsto \sum_{k=0}^{r} X_k \wedge X_{i_1} \wedge \ldots \wedge X_{i_p} \otimes \frac{\partial f}{\partial X_k}$$

and

$$d_{p,q} : \wedge^{p+1}V \otimes S^q V \to \wedge^p V \otimes S^{q+1}V$$

is given by

$$X_{i_1} \wedge \ldots \wedge X_{i_{p+1}} \otimes f \mapsto \sum_{k=1}^{p+1} X_{i_1} \wedge \ldots \wedge \hat{X_{i_k}} \wedge \ldots \wedge X_{i_{p+1}} \otimes X_{i_k} f$$

The Euler formula

$$\sum_{k=0}^{r} X_k \frac{\partial f}{\partial X_k} = q.f$$

implies that

$$D \circ d_{p,q} + d_{p,q} \circ D = (p+q).id,$$

hence the Koszul complex is exact. $\square$

1.3 Definitions in the geometric context

The above construction will be of interest to us primarily in the case

$$\begin{cases} C & \text{a projective curve over } \mathbb{C}, \\ L \to C & \text{a holomorphic line bundle,} \\ V = H^0(C, L) & \text{a finite dimensional vector space} \end{cases}$$

Definition 1.3.1. *The Koszul cohomology group $\mathcal{K}_{p,q}(C, L)$ is the Koszul cohomology of the graded S^*V- module*

$$B = \bigoplus_{q \in \mathbb{Z}} H^0(C, L^q)$$

i.e. $\mathcal{K}_{p,q}(C, L)$ is the cohomology at the middle term of the complex

$$\wedge^{p+1} V \otimes H^0(C, L^{q-1}) \xrightarrow{d_{p+1,q-1}} \wedge^p V \otimes H^0(C, L^q) \xrightarrow{d_{p,q}} \wedge^{p-1} V \otimes H^0(C, L^{q+1})$$

where the differential

$$\wedge^p V \otimes H^0(C, L^q) \xrightarrow{d_{p,q}} \wedge^{p-1} V \otimes H^0(C, L^{q+1})$$

is given by

$$d_{p,q}(v_1 \wedge \ldots \wedge v_p \otimes s) = \sum_i (-1)^i v_1 \wedge \ldots \wedge \hat{v_i} \wedge \ldots \wedge v_p \otimes (v_i.s).$$

Denote $\mathcal{K}_{p,q}(C, L) = \mathcal{K}_{p,q}(B, V)$

1.4 Clifford Index

The Clifford index is a numerical invariant associated with a curve of genus greater than or equal to 2. In this section we will give the definition and some elementary properties of this invariant. Let C be a smooth projective curve over a field k. Let K be the canonical line bundle on C. The origin of this notion of Clifford index is in the proof of Clifford's Theorem.

Theorem 1.3. *Clifford Theorem [Nar92]: Let D be an effective special divisor on C (so that $h^0(K - D) > 0$). Let d be the degree of D. Then*

$$\dim |D| \leq \tfrac{1}{2} d = \tfrac{1}{2} \deg D.$$

Moreover, if equality holds, then D must be 0, or $D \sim K$, or C must be hyperelliptic.

Definition 1.4.1. *Let C be a smooth projective curve of genus $g \geq 2$ over a field k. H.H.Martens has defined the Clifford index γ_C of C as*

$$\gamma_C = min\{\deg(D) - 2\dim |D|\}$$

where D runs over all divisors of C such that $h^0(C, \mathcal{O}_C(D)) \geq 2$ and $h^1(C, \mathcal{O}_C(D)) \geq 2$.

Proposition 1.4.2. *Let $0 \leq i \leq g-2$ ($i \in \mathbb{Z}$). Consider the following generalized Clifford Inequality,*

$$dim H^0(C, L) \leq \tfrac{\deg(L)-i}{2} + 1 \qquad (C_i)$$

Then the following are equivalent:

1. *If $h^0(C, L) \geq 2$ and $h^1(C, L) \geq 2$ then (C_i) holds.*
2. *If $i \leq \deg(L) \leq 2g-2-i$ then (C_i) holds.*
3. *If $i < \deg(L) < 2g-2-i$ then (C_i) holds.*

The largest integer i such that every line bundle L satisfies the condition $(1)(or(2)or(3))$ of the proposition is called the Clifford Index of C.

Proof: By Riemann-Roch theorem, we have

$$h^0(C, L) - h^1(C, L) = 1 - g + \deg L \tag{1.1}$$

and by Serre' duality theorem, we have

$$h^0(C, K \otimes L^{-1}) = h^1(C, L)$$

Applying Riemann-Roch theorem on line bundle $(K \otimes L^{-1})$ and subtracting it from (1.1), we get

$$h^0(C, L) - \frac{\deg L}{2} = h^0(C, K \otimes L^{-1}) - \frac{\deg(K \otimes L^{-1})}{2}$$

Thus the inequality (C_i) for L is equivalent to the same inequality for $K \otimes L^{-1}$.
The same is true for the hypothesis $(1) - (3)$. Thus we may confine ourselves to the study of L

such that $0 \le \deg L \le g-1$.
Assuming (1), we see that we need to check (2) only when $\dim H^0(C,L) \ge 2$. But then, using $0 \le \deg(L) \le (g-1)$, we have $\dim H^1(C,L) \ge 2$. Hence, in this case (1) implies (2). Since (2) implies (3) is obvious we only need to check (3) implies (1). By (3) any line bundle of degree $(i+1)$ has at most one section up to scalar multiples. Since any line bundle L with $\deg(L) \le i$ is a subsheaf of some line bundle M with $\deg(M) = (i+1)$, such an L cannot have 2 or more linearly independent sections, hence does not come under the preview of (1). Thus, the assumption that $\dim H^0(C,L) \ge 2$ implies that $\deg(L) > i$. Since $\deg(L) \le (g-1) < g \le (2g-2-i)$ by assumption we have the required inequality (C_i) for any such L.

□

Lemma 1.4.3. *: Let d be the least integer with $(\gamma+1) \le d \le (g-1)$ such that there is a divisor D of degree d with $\dim |D| = \frac{d-\gamma}{2}$. Then $d \le (3\gamma+2)$, or what is the same $\dim |D| \le (\gamma+1)$.*

Proof: Suppose the assertion is false. Then we have $\dim |D| \ge \gamma+2$ and we have $\dim |K - D| \ge \gamma+2$ as well. Choose any effective divisor D' of degree $\gamma+1$ in D.
Then we can find effective divisors D_1 and D_2 containing D' which are linearly equivalent to D and $K-D$ respectively with $D_1 \ne D_2$.
Let

$$D'' = gcd(D_1, D_2)$$

$$D''(a) = min(D_1(a), D_2(a))$$

Now consider the exact sequence

$$0 \to \mathcal{O}(D'') \to \mathcal{O}(D_1) \bigoplus \mathcal{O}(D_2) \to \mathcal{O}(D_1 + D_2 - D'') \to 0$$

Exact cohomology sequence gives

$$0 \to H^0(C, \mathcal{O}_{D''}) \to H^0(C, \mathcal{O}_{D_1}) \bigoplus H^0(C, \mathcal{O}_{D_2}) \to H^0(C, \mathcal{O}_{D_1+D_2-D''}) \to \dots$$

We have

$$\dim |D''| + \dim |D_1 + D_2 - D''| \ge \dim |D_1| + \dim |D_2|$$

Since $\mathcal{O}_C(D_1 + D_2) \cong K$, we have

$$\deg D'' - 2\dim |D''| \le \deg D_1 - 2\dim |D_1| = \gamma$$

Hence $\deg D'' - 2\dim|D''| = \gamma$ and $\deg D''$ is strictly less than d. This contradicts the assumption that $\dim|D| \geq \gamma + 2$. □

As an easy consequence of the lemma, we have

Corollary 1.4. *1. $\gamma_C = 0$ iff C is a hyperelliptic curve.*

2. $\gamma_C = 1$ iff C is either a trigonal curve (C is a degree 3 cover of $\mathbb{P}^1$) or a plane curve of degree 5 (and g=6).

From the solution of the Brill-Noether problem on the existence of special divisors [ACGH85], we have

Lemma 1.4.4. *: $\gamma_C \leq [\frac{g-1}{2}]$ for all curves C and equality holds for the general curve.*

Proof: The existence theorems [ACGH85] say that

1. Every curve of genus g has a divisor D with $\deg(D) = d$ and $\dim|D| = r$, whenever $g \geq (r+1)(g+r-d)$. If $d = [\frac{g+3}{2}]$ and $r = 1$, then this inequality is satisfied. Thus $d - 2r = [\frac{g-1}{2}]$, which shows the first part.

2. If the general curve of genus g has a divisor D with $\deg(D) = d$ and $\dim|D| = r$, then $g \geq (r+1)(g+r-d)$, so that $d - 2r \geq \frac{rg}{r+1} - r$. Since $\gamma_C =$ minimum of $d - 2r$ over all divisors of degree $d \leq g-1$ with $r \geq 1$, we may assume that $2r \leq g-1$; but then it is easily checked that $\frac{rg}{r+1} - r \geq [\frac{g-1}{2}]$, thus proving the second part of the lemma.

□

The following result of E. Ballico [Bal86] which was conjectured by M. Green and R. Lazarsfeld [GL85] follows from the theory of Limit Linear Series of D. Eisenbud and J. Harris [EH86]

Lemma 1.4.5. *: Every integer γ in the range $[0, \frac{g-1}{2}]$ is the Clifford Index of some curve of genus g. In fact, if X is a general curve with a morphism to $\mathbb{P}^1$ of degree $\gamma + 2$, then its Clifford Index is γ.*

1.5 Green's Conjecture

Let C be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$ and K be the canonical line bundle on C.

Proposition 1.5.1. *[Gre84] Let L be a line bundle such that $h^0(C, L) \geq 2$, $h^1(C, L) \geq 2$. Then*

$$\mathcal{K}_{p,1}(C, K) \neq 0 \textit{ for all } p \leq g - \gamma_L - 2.$$

Proof: Put $L_1 = L$, $L_2 = K \otimes L^*$, $r_i = r(L_i)(i = 1, 2), d = \deg(L)$. By Riemann-Roch we have $r_2 - r_1 = g - d - 1$, hence $r_1 + r_2 - 1 = g - d + 2r(L) - 2 = g - \gamma_L - 2$. Using Theorem and Corollary given in appendix [Gre84], we obtain

$$\mathcal{K}_{p,1}(C, K) \neq 0 \text{ for all } p \leq g - \gamma_L - 2.$$

□

The strongest non vanishing result of this type is obtained by taking the minimal value of γ_L. This gives the implication

$$p \leq g - \gamma_C - 2 \implies \mathcal{K}_{p,1}(C, K) \neq 0. \tag{1.2}$$

Green [Gre84] conjectures that the converse of (1.2) holds.

Conjecture 1.5.1. *[Gre84] Let C be a smooth projective curve. Then*

$$\mathcal{K}_{p,1}(C, K) = 0 \Leftrightarrow p \geq g - \gamma_C - 1.$$

Chapter 2

Results for complete linear systems

Introduction

Let C be a smooth projective curve of genus $g \geq 2$ over a field k. Let K be the canonical line bundle on C. As explained in §(1.5), chapter 1, In [Gre84], Green made a conjecture which relates two aspects Koszul cohomology (an algebraic aspect) and Clifford Index γ_C (a geometric aspect) of a curve. Paranjape and Ramanan [PR88] made an effort to understand Green's conjecture. They studied the vector bundle E_K, where E_K is given by the pullback of the universal quotient bundle on $\mathbb{P}^{g-1}$ by the canonical map $\Phi_K : C \to \mathbb{P}^{g-1}$. They gave an equivalent formulation to Green's conjecture: To prove Green's conjecture is equivalent to prove that the map $\wedge^i\Gamma(C, E_K) \to \Gamma(C, \wedge^i E_K)$ is surjective $\forall\ i \leq \gamma_C$.
Paranjape and Ramanan [PR88] also studied the stability properties of vector bundle E_K. E_K is semi stable (even stable if C is not hyperelliptic). The main result of [PR88] is that all sections of $\wedge^i E_K$ which are locally decomposable are in the image of $\wedge^i\Gamma(E_K)\ \forall\ i \leq \gamma_C$. Thus, if locally decomposable sections of $\wedge^i E_K$ spans the space $\Gamma(\wedge^i E_K)$ then the map $\wedge^i\Gamma(C, E_K) \to \Gamma(C, \wedge^i E_K)$ is clearly surjective.
Let $\sum_{i,K}$ be the cone of locally decomposable sections of $\wedge^i E_K$. Then, in view of all above observations and results, Hulek, Paranjape and Ramanan [HPR92] stated a conjecture.

Conjecture 2.0.1. *$\sum_{i,K}$ spans $\Gamma(\wedge^i E_K)\ \forall\ i$ and for all curves.*

This is stronger than Green's conjecture. They proved it for curves with Clifford index 1 (trigonal curves and plane quintics). Conjecture 2.0.1 is trivial in case of hyperelliptic curves, since E_K is the $(g-1)$- fold direct sum of the hyperelliptic line bundle.

In a remark to conjecture 2.0.1 Eusen and Schreyer [ES12] asked a more general question, that if N is a stable globally generated vector bundle on C and $\sum_{i,N}$ be the cone of locally decomposable sections of $\wedge^i N$ then

Conjecture 2.0.2. *$\sum_{i,N}$ spans $\Gamma(\wedge^i N)\ \forall\ i$ and for all curves.*

They gave counter examples to this more general conjecture [ES12].
In this chapter, we study conjecture 2.0.2 in the context of general linear systems on a hyperelliptic curve C.

Theorem 2.1. *[Ana] Let C be a smooth hyperelliptic curve of genus $g \geq 2$ and let L be a globally generated line bundle on C of degree $d \geq 2g+1$ such that $H^1(L \otimes T^{-2}) = 0$, where T is the hyperelliptic line bundle on C. The evaluation map gives rise to an exact sequence*

$$0 \to E^* \to \Gamma(L)_C \to L \to 0 \tag{2.1}$$

where E^* is locally free of $\operatorname{rank} h^0(L) - 1$. Let $\sum_i$ be the cone of locally decomposable sections of $\wedge^i E$. Then $\sum_i$ spans $\Gamma(\wedge^i E)\ \forall\ i$.

From now onwards, throughout this chapter, C is a smooth hyperelliptic curve of genus $g \geq 2$ and L is a globally generated line bundle on C of degree $d \geq 2g+1$ satisfying $H^1(L \otimes T^{-2}) = 0$ and K is the canonical line bundle on C.
Hyperelliptic curves have 2:1 map to $\mathbb{P}^1$. The bundle E and its exterior powers are the main object of above stated theorem. So, taking advantage of this special feature of hyperelliptic curves, sections of $\wedge^i E$ on C will be related to sections of some suitable vector bundle on $\mathbb{P}^1$.
In §(2.1), we study the geometry of hyperelliptic curves and obtain a suitable vector bundle on $\mathbb{P}^1$ to which the sections of $\wedge^i E$ are related.
In §(2.2), syzygies of the curve C are computed and are used to compute the dimension of the space of sections of $\wedge^i E$ on C. Also the dimension of the corresponding space of sections on $\mathbb{P}^1$ are computed.
Since $\sum_i$, the set of locally decomposable sections of $\wedge^i E$ is a scheme defined by requiring that at each point the sections satisfy the equations of the Grassmannian cone in its Plucker embedding. They are obtained in the following way: Let F be a subbundle of $\operatorname{rank} i$ of E and $s \in \Gamma(detF)$. Then s can be treated as a section of $\wedge^i E$ as well, where it is locally decomposable. In §(2.3), we construct such a subbundle F.
In §(2.4), we provide the proof of the main theorem.

2.1 Geometry of the hyperelliptic curve

Consider the following proposition

Proposition 2.2. *[Har77] Let C be a hyperelliptic curve of genus $g \geq 2$. Then C has a unique g_2^1. If $f_0 : C \to \mathbb{P}^1$ is the corresponding morphism of degree 2 then the canonical morphism $f : C \to \mathbb{P}^{g-1}$ consists of f_0 followed by the $(g-1)$-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^{g-1}$. In particular, the image $C' = f(C)$ is a rational normal curve of degree $g-1$, and f is a morphism of degree 2 onto C'. Finally, every effective canonical divisor on C is a sum of $g-1$ divisors in the unique g_2^1, so we write $|K| = \sum_1^{g-1} g_2^1$*

□

In our case, since C is hyperelliptic of genus $g \geq 2$. Thus by proposition 2.2 g_2^1 on C is unique. Let $\pi : C \to \mathbb{P}^1$ be the corresponding morphism of degree 2, $T := \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ is the unique g_2^1 and canonical line bundle on C is given by T^{g-1}
Consider the $\mathrm{rank}\, 2$ vector bundle W on $\mathbb{P}^1$, where $W := \pi_* L$.
Since,

$$\chi(L) = \chi(\pi_* L)$$

So,

$$d + 1 - g = rk(W)(\frac{\deg W}{rkW} + 1 - g_{\mathbb{P}^1}) = 2(\frac{\deg W}{2} + 1)$$

which gives

$$\deg W = d - g - 1$$

Thus,

$$\begin{aligned} detW &\cong \mathcal{O}_{\mathbb{P}^1}(d-g-1) \\ W &\cong W^*(d-g-1) \end{aligned}$$

Since $\deg W = d - g - 1$, thus there is a unique integer $x \leq \frac{d-g-1}{2}$ such that

$$W \cong \mathcal{O}_{\mathbb{P}^1}(x) \bigoplus \mathcal{O}_{\mathbb{P}^1}(d-g-1-x) \tag{2.2}$$

Remark 2.3. 1. x is the least integer such that

$$H^1(W(-2-x)) = H^1(\pi_* L(-2-x)) = H^1(L \otimes T^{-2-x}) \neq 0$$

In particular, this implies that $\deg(L \otimes T^{-2-x}) \leq 2g-2$ and thus we have

$$\frac{d-2g-2}{2} \leq x \leq \frac{d-g-1}{2}$$

2. Since $H^1(L \otimes T^{-2}) = 0$, thus $x > 0$, so we have

$$max.\{1, \frac{d-2g-2}{2}\} \leq x \leq \frac{d-g-1}{2}$$

which implies both W and $W(-1)$ is globally generated.

3. Also, $H^1(L \otimes T^{-2}) = 0$ implies $H^1(L) = 0$, thus by Riemann- Roch theorem, we have

$$h^0(L) = d - g + 1 \tag{2.3}$$

and $\mathrm{rank}(E) = h^0(L) - 1 = d - g \geq 3$ (since $d \geq 2g+1$ and $g \geq 2$)

4. We have

$$\begin{aligned} \Gamma(W(-1)) &\cong \Gamma(\pi_*(L \otimes T^{-1})) \\ &\cong \Gamma(L \otimes T^{-1}) \end{aligned}$$

$H^1(L \otimes T^{-2}) = 0$ gives $H^1(L \otimes T^{-1}) = 0$
Thus, by Riemann-Roch theorem, we have

$$h^0(L \otimes T^{-1}) = d - g - 1$$

i.e., we have

$$h^0(W(-1)) = d - g - 1 \tag{2.4}$$

Since $W(-1)$ is globally generated and $\Gamma(W) \cong \Gamma(L)$, so we have a surjection

$$\Gamma(L)_{\mathbb{P}^1} \to W \to 0,$$

which is an isomorphism for sections. Since $W(-1)$ is globally generated bundle on $\mathbb{P}^1$, W is very ample, i.e., we get an inclusion

$$\mathbb{P}(W^*) \hookrightarrow \mathbb{P}(\Gamma(L)^*)) =: \mathbb{P}. \tag{2.5}$$

Also we have a surjection

$$\pi^* W \to L \to 0$$

In other words, we have a subbundle of $\pi^*(W^*)$ that is isomorphic to L^{-1}. This gives a morphism from C to $\mathbb{P}(W^*)$ with the property that the pullback of $\mathcal{O}_W(1)$ to C is L. Also the composite of this morphism with the projection $p : \mathbb{P}(W^*) \to \mathbb{P}^1$ is π. Since the induced map

$$\Gamma(\mathbb{P}(W^*), \mathcal{O}_W(1)) \cong \Gamma(\mathbb{P}^1, W) \to \Gamma(C, L)$$

is an isomorphism. Thus C is actually embedded in $\mathbb{P}(W^*)$. Let us denote the image of $\mathbb{P}(W^*)$ in $\mathbb{P}$ by S. We return to the ruled surface $p : \mathbb{P}(W^*) \to \mathbb{P}^1$. By (2.5), there is an embedding $\mathbb{P}(W^*) \subset \mathbb{P} = \mathbb{P}(\Gamma(L)^*)$ with hyperplane section $\tau = \mathcal{O}_W(1)$. Note that $\tau^2 = \deg W = d-g-1$. Since C is a secant (2-section) of $\mathbb{P}(W^*)$, its class is of the form $\mathcal{O}_W(2) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(m)$. To compute m, we note that $d = C.\tau = 2\tau^2 + m$. Thus $m = 2g - d + 2$.
Altogether, we have the following proposition

Proposition 2.4. *There are inclusions*

$$C \subset S \subset \mathbb{P}$$

with the following properties:

1. *the restriction of $\mathcal{O}_{\mathbb{P}}(1)$ to S is $\mathcal{O}_W(1)$;*
2. *the restriction of $\mathcal{O}_{\mathbb{P}}(1)$ to C is L;*
3. *both restrictions induce isomorphisms of the corresponding linear systems;*
4. *the divisor class on S defined by C is $\mathcal{O}_W(2) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(-d + 2g + 2)$.*

Notation: We will use the notation U for the vector space $\Gamma(L \otimes T^{-1})$, i.e. we have

$$\Gamma(W(-1)) \cong U \tag{2.6}$$

and by (2.4), we have

$$\dim U = d - g - 1 \tag{2.7}$$

2.2 Computation of dimensions

In order to prove the conjecture, we want to relate the sections of $\wedge^i E$ to the sections of a suitable vector bundle on $\mathbb{P}^1$

Lemma 2.5. *Let F be a vector bundle on $\mathbb{P}^1$ that is globally generated. Then the evaluation sequence is*

$$0 \to \Gamma(F(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \to \Gamma(F)_{\mathbb{P}^1} \to F \to 0$$

Proof: F is a sum of line bundles of degree ≥ 0. Thus remains to check for line bundles, which is easy.

□

We want to apply this lemma to W .

$$0 \to \Gamma(W(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \to \Gamma(W)_{\mathbb{P}^1} \to W \to 0 \tag{2.8}$$

Pulling back the evaluation sequence for W on $\mathbb{P}^1$ to C and using (2.6) and the fact that $\Gamma(\pi_* L) \cong \Gamma(L)$, we get

$$0 \to U \otimes T^{-1} \to \Gamma(L)_C \to \pi^* W \to 0 \tag{2.9}$$

Also, we have a surjective map $\pi^* W \to L \to 0$, Let Y be the kernel of $\pi^* W \to L \to 0$
i.e. we have

$$0 \to Y \to \pi^* W \to L \to 0$$

$$\begin{aligned}
\wedge^2(\pi^* W) &\cong Y \otimes L \\
\pi^*(\wedge^2 W) &\cong Y \otimes L \\
\pi^*(\mathcal{O}_{\mathbb{P}^1}(d-g-1)) &\cong Y \otimes L \\
T^{d-g-1} &\cong Y \otimes L \\
Y &\cong L^{-1} \otimes T^{d-g-1}
\end{aligned}$$

Thus, we have

$$0 \to L^{-1} \otimes T^{d-g-1} \to \pi^* W \to L \to 0 \tag{2.10}$$

we get a following commutative diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & U \otimes T^{-1} \arrow[r] \arrow[d, equal] & E^* \arrow[r] \arrow[d] & L^{-1} \otimes T^{d-g-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U \otimes T^{-1} \arrow[r] & \Gamma(L)_C \arrow[r] \arrow[d] & \pi^* W \arrow[r] \arrow[d] & 0 \\
 & & L \arrow[r, equal] \arrow[d] & L \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
\tag{2.11}
$$

where the left vertical map is the evaluation map.

Dualise the diagram (2.11), we get

$$
\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & L \otimes T^{-(d-g-1)} \arrow[r] \arrow[u] & E \arrow[r] \arrow[u] & U^* \otimes T \arrow[r] & 0 \\
0 \arrow[r] & \pi^* W^* \arrow[r] \arrow[u] & \Gamma(L)^*_C \arrow[r] \arrow[u] & U^* \otimes T \arrow[r] \arrow[u, equal] & 0 \\
 & L^{-1} \arrow[r, equal] \arrow[u] & L^{-1} \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}
\tag{2.12}
$$

The first line of (2.12) gives rise to an exact sequence

$$0 \to L \otimes T^{-(d-g-1)} \otimes \wedge^{i-1} U^* \otimes T^{i-1} \to \wedge^i E \to \wedge^i U^* \otimes T^i \to 0 \tag{2.13}$$

Since, $\pi^* W^*$ is a $\mathrm{rank}\, 2$ bundle we only get a filtration consisting of the following two exact sequences:

$$0 \to \wedge^2 \pi^* W^* \otimes \wedge^{i-2} U^* \otimes T^{i-2} \to \wedge^i \Gamma(L)^*_C \to L_i \to 0 \tag{2.14}$$

$$0 \to \pi^* W^* \otimes \wedge^{i-1} U^* \otimes T^{i-1} \to L_i \to \wedge^i U^* \otimes T^i \to 0 \tag{2.15}$$

Since the second horizontal sequence of (2.12) is the pullback via π of the dual of the sequence (2.8). Thus both the above sequences come from $\mathbb{P}^1$, i.e. there exists a vector bundle L_i' on $\mathbb{P}^1$, such that $L_i = \pi^* L_i'$ and the sequences

$$0 \to \wedge^2 W^* \otimes \wedge^{i-2} U^* \otimes \mathcal{O}(i-2) \to \wedge^i \Gamma(L)^*_{\mathbb{P}^1} \to L_i' \to 0 \tag{2.16}$$

$$0 \to W^* \otimes \wedge^{i-1} U^* \otimes \mathcal{O}(i-1) \to L_i' \to \wedge^i U^* \otimes \mathcal{O}(i) \to 0 \tag{2.17}$$

are such that (2.14) and(2.15) are pullback of (2.16) and (2.17) respectively.
Dualising (2.1), we have

$$0 \to L^{-1} \to \Gamma(L)^*_C \to E \to 0$$

Thus the map $\wedge^i \Gamma(L)^*_C \to \wedge^i E$ is surjective.
Also we have $\wedge^i \Gamma(L)^*_C \to L_i \to 0$. The maps $\wedge^i \Gamma(L)^*_C \to \wedge^i E \to 0$ factors through $L_i = \pi^* L_i'$.
Thus we get the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & L \otimes T^{-(d-g-1)} \otimes \wedge^{i-1} U^* \otimes T^{i-1} \arrow[r] \arrow[u] & \wedge^i E \arrow[r] \arrow[u] & \wedge^i U^* \otimes T^i \arrow[r] & 0 \\
0 \arrow[r] & \pi^* W^* \otimes \wedge^{i-1} U^* \otimes T^{i-1} \arrow[r] \arrow[u] & \pi^* L_i' \arrow[r] \arrow[u] & \wedge^i U^* \otimes T^i \arrow[r] \arrow[u, equal] & 0 \\
 & L^{-1} \otimes \wedge^{i-1} U^* \otimes T^{i-1} \arrow[r, equal] \arrow[u] & L^{-1} \otimes \wedge^{i-1} U^* \otimes T^{i-1} \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd} \tag{2.18}$$

where the top horizontal sequence is (2.13), middle horizontal sequence is (2.15), left vertical sequence is obtained by dualising (2.10) and tensoring it with $\wedge^{i-1} U^* \otimes T^{i-1}$.

Let us compute the dimensions of the spaces $\Gamma(L_i')$ and $\Gamma(\wedge^i E)$ for $i \leq d-g$ $(= \operatorname{rank} E)$

Lemma 2.6. *When $d \geq 2g+1$, we have*

$$\dim \Gamma(L_i') = \binom{d-g+1}{i} + \binom{d-g-1}{i-2}(d-i-g)$$

for $i \leq d-g$

Proof: Consider (2.16)

$$0 \to \wedge^2 W^* \otimes \wedge^{i-2} U^* \otimes \mathcal{O}(i-2) \to \wedge^i \Gamma(L)^*_{\mathbb{P}^1} \to L'_i \to 0$$

Since

$$detW \cong \mathcal{O}_{\mathbb{P}^1}(d-g-1).$$

Thus

$$detW^* \cong \mathcal{O}_{\mathbb{P}^1}(-(d-g-1))$$

Thus, we get

$$0 \to \wedge^{i-2} U^* \otimes \mathcal{O}(i-d+g-1) \to \wedge^i \Gamma(L)^* \to L'_i \to 0$$

Since,

$$i \leq d-g, h^0(\mathcal{O}(i-d+g-1)) = 0$$

Therefore,

$$h^0(L'_i) = h^0(\wedge^i \Gamma(L)^*) + h^1(\wedge^{i-2} U^* \otimes \mathcal{O}(i-d+g-1))$$

By (2.7), we have

$$\dim U = d-g-1$$

and By (2.3)

$$h^0(L) = d-g+1$$

$$h^1(\mathcal{O}(i-d+g-1)) = d-i-g$$

Thus,

$$\dim \Gamma(L'_i) = \tbinom{d-g+1}{i} + \tbinom{d-g-1}{i-2}(d-i-g).$$

□

2.2.1 Syzygies of the curve

The syzygies of canonically embedded curves were computed by Schreyer [Sch86]. Based on the parallel idea, we compute the syzygies of the curve C. For this, let

$$R = \bigoplus_{i=1}^{\infty} \Gamma(C, L^i)$$

be the homogeneous coordinate ring of C w.r.t L and

$$S = Sym\Gamma(C, L) = \bigoplus_{n \geq 0} \Gamma(\mathcal{O}_{\mathbb{P}^{d-g}}(n)).$$

Let

$$0 \to F_t \to \cdots \to F_0 \to R \to 0, \tag{2.19}$$

be a minimal free resolution of the graded S- module R. Then $F_i = \bigoplus_j S(-j)^{\beta_{ij}} = \bigoplus_j M_{ij} \otimes S(-j)$, where M_{ij} is a k- vector space of $\dim \beta_{ij}$ and $S(-j)$ is the free S- module with one generator in degree j.
The resolution (2.19) is equivalent to the free resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbb{P}^{d-g}}$- module:

$$0 \to \bigoplus_j \mathcal{O}(-j)^{\beta_{d-g-1,j}} \to \cdots \to \bigoplus_j \mathcal{O}(-j)^{\beta_{0,j}} \to \mathcal{O}_C \to 0$$

To find this resolution, one starts with the exact sequence

$$0 \to \mathcal{O}_S(-2\tau + (d - 2g - 2)f) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$$

(see Proposition (2.4)). The idea is to first resolve the sheaves $\mathcal{O}_S$ and $\mathcal{O}_S(-2\tau + (d - 2g - 2)f)$ resp. as $\mathcal{O}_{\mathbb{P}^{d-g}}$ modules and then form a mapping cone . The result turns out to be a minimal resolution of $\mathcal{O}_C$.
Firstly, we will recall from [Eis05] the description of the syzygies of these sheaves.
Let $\xi = \mathcal{O}(e_1) \bigoplus \mathcal{O}(e_2) \bigoplus \cdots \bigoplus \mathcal{O}(e_s)$ be a locally free sheaf of rank s on $\mathbb{P}^1$, and let p_ξ : $\mathbb{P}(\xi) \to \mathbb{P}^1$ denote the corresponding $\mathbb{P}^{s-1}$ bundle. A rational normal scroll X of type $S(e_1, e_2, \cdots, e_s)$ with $e_1 \geq e_2 \geq \cdots \geq e_s \geq 0$ and

$$f = e_1 + e_2 + \cdots + e_s \geq 2$$

is the image of $\mathbb{P}(\xi)$ in $\mathbb{P}^r = \mathbb{P}(H^0(\mathbb{P}(\xi), \mathcal{O}_{\mathbb{P}(\xi)}(1)))$:

$$j : \mathbb{P}(\xi) \to X \subset \mathbb{P}^r, r = f + s - 1$$

The Picard group of $\mathbb{P}(\xi)$ is generated by the hyperplane class $H = [j^*\mathcal{O}_{\mathbb{P}^r}(1)]$ and the ruling $R = [p_\xi^*\mathcal{O}_{\mathbb{P}^1}(1)]$:

$$Pic\mathbb{P}(\xi) = \mathbb{Z}H \bigoplus \mathbb{Z}R,$$

the intersection product is given by

$$H^s = f, H^{s-1}.R = 1, R^2 = 0.$$

We recall from [Eis05], the description of the syzygies of the sheaves

$$\mathcal{O}_X(aH+bR) := j_*\mathcal{O}_{\mathbb{P}(\xi)}(aH+bR), \quad a,b \in \mathbb{Z}$$

regarded as $\mathcal{O}_{\mathbb{P}^r}$- modules, at least in case $b \geq -1$.

Let

$$\Phi : F \to G$$

be a map of locally free sheaves of rank f' and g', $f' \geq g'$, respectively on a smooth variety V. We recall from [BE75] the family of complexes $\zeta^b, b \geq -1$ of locally free sheaves on V, which resolve the b^{th}- symmetric power of coker Φ under suitable hypothesis on Φ.
Define the jth term in the complex ζ^b by

$$\zeta_j^b = \begin{cases} \wedge^j F \otimes S_{b-j}G, & for \quad 0 \leq j \leq b \\ \wedge^{j+g'-1}F \otimes D_{j-b-1}G^* \otimes \wedge^{g'}G^*, & for \quad j \geq b+1 \end{cases}$$

and differential

$$\zeta_j^b \to \zeta_{j-1}^b$$

by the multiplication with $\Phi \in H^0(V, F^* \otimes G)$ for $j \neq b+1$ and $\wedge^{g'}\Phi \in H^0(V, \wedge^{g'}F^* \otimes \wedge^{g'}G)$ for $j = b+1$ in the appropriate term of the exterior $(\wedge F)$, symmetric (S.G) or divided power (D.G) algebra.

Proposition 2.7. *[Eis05] $\zeta^b(a)$ for $b \geq -1$ is the minimal resolution of $\mathcal{O}_X(aH+bR)$ as an $\mathcal{O}_{\mathbb{P}^r}$- module, where $\zeta^b(a) = \zeta^b \otimes \mathcal{O}_{\mathbb{P}^r}(a)$*

2.2.2 Minimal Resolution of $\mathcal{O}_C$

We have

$$C \subset S \subset \mathbb{P} = \mathbb{P}(\Gamma(C,L)^*)$$

C is contained in a 2- dimensional rational normal scroll S of type $S(e_1,e_2)$ and degree $f = e_1+e_2 = d-g-1 \geq 2$.
C is a divisor of class

$$C \sim 2H - (f-(g+1))R \quad on \ S$$

The mapping cone [Sch86]

$$\zeta^{f-(g+1)}(-2) \to \zeta^0$$

is the minimal resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{\mathbb{P}^{d-g}}$- module.
We consider

$$\Phi : F \otimes \mathcal{O}_{\mathbb{P}^{d-g}}(-1) \to G \otimes \mathcal{O}_{\mathbb{P}^{d-g}}$$

be the map of locally free sheaves, where F is a vector space of dimension $f = d - g - 1$ and G is a vector space of dimension 2

Firstly, we will compute

$$\zeta^{f-(g+1)}(-2) = \zeta^{d-2g-2} \otimes \mathcal{O}(-2)$$

Now,

$$\zeta_j^{d-2g-2} = \begin{cases} \wedge^j(F \otimes \mathcal{O}(-1)) \otimes S_{d-2g-2-j}(G \otimes \mathcal{O}), & 0 \leq j \leq d-2g-2 \\ \wedge^{j+1}(F \otimes \mathcal{O}(-1)) \otimes D_{j-d+2g+2-1}(G \otimes \mathcal{O})^* \otimes \wedge^2(G \otimes \mathcal{O})^*, & j \geq d-2g-1 \end{cases}$$

Since $j+1$ can be atmost $d-g-1$. Thus, we have

$$\zeta_j^{d-2g-2} = \begin{cases} \wedge^j(F \otimes \mathcal{O}(-1)) \otimes S_{d-2g-2-j}(G \otimes \mathcal{O}), & 0 \leq j \leq d-2g-2 \\ \wedge^{j+1}(F \otimes \mathcal{O}(-1)) \otimes D_{j-d+2g+2-1}(G \otimes \mathcal{O})^* \otimes \wedge^2(G \otimes \mathcal{O})^*, & d-2g-1 \leq j \leq d-g-2 \end{cases}$$

$$\zeta_j^{d-2g-2} = \begin{cases} \wedge^j(F \otimes \mathcal{O}(-1)) \otimes S_{d-2g-2-j}(G \otimes \mathcal{O}), & 0 \leq j \leq d-2g-2 \\ \wedge^j(F \otimes \mathcal{O}(-1)) \otimes D_{j-d+2g}(G \otimes \mathcal{O})^* \otimes \wedge^2(G \otimes \mathcal{O})^*, & d-2g-1 \leq j \leq d-g-1 \end{cases}$$

Similarly we can compute ζ_j^0,

$$\zeta_j^0 = \begin{cases} \wedge^j(F \otimes \mathcal{O}(-1)) \otimes S_{0-j}(G \otimes \mathcal{O}) & j = 0 \\ \wedge^{j+1}(F \otimes \mathcal{O}(-1)) \otimes D_{j-1}(G \otimes \mathcal{O})^* \otimes \wedge^2(G \otimes \mathcal{O})^* & 1 \leq j \leq d-g-2 \end{cases}$$

The minimal free resolution of $\mathcal{O}_C$ is

$$\begin{aligned} 0 \to \mathcal{O}(-(d-g-1))^g \to \cdots \to \wedge^{j+1}(F \otimes \mathcal{O}(-1)) \otimes D_{j-d+2g+2-1}(G \otimes \mathcal{O})^* \otimes \wedge^2(G \otimes \mathcal{O})^* \to \cdots \to \\ \wedge^j(F \otimes \mathcal{O}(-1)) \otimes S_{d-2g-2-j}(G \otimes \mathcal{O}) \to \cdots \to \wedge^{j+1}(F \otimes \mathcal{O}(-1)) \otimes D_{j-1}(G \otimes \mathcal{O})^* \otimes \wedge^2(G \otimes \mathcal{O})^* \\ \to \cdots \to \wedge^j(F \otimes \mathcal{O}(-1)) \otimes S_{0-j}(G \otimes \mathcal{O}) \to \mathcal{O}_C \to 0 \end{aligned} \tag{2.20}$$

We will use this resolution to compute the $\dim \Gamma(\wedge^i E)$. For this, consider (2.19), the minimal free resolution of R and recall the results from Chapter 1 section 2 of the Ph.D. thesis (1992) of Prof. Kapil H. Paranjape (University of Bombay, Bombay, India), we have

$$M_{p,p+q} = coker(\wedge^{p+1} V \otimes \Gamma(C, K \otimes L^{q-1}) \to \Gamma(C, \wedge^p E^* \otimes L^q))$$

where $M_{p,p+q} = (Tor_p^S(\mathbb{C}, R))_{p+q}$
and $\dim(Tor_p^S(\mathbb{C}, R))_{p+q} = \beta_{p,p+q}$.
Since $H^1(L) = 0$, so we have

$$M_{p,p+2} \approx H^1(\wedge^{p+1}E^* \otimes L)$$

$$M^*_{p,p+2} \approx H^0(\wedge^{p+1}E \otimes L^{-1} \otimes K)$$

Lemma 2.8. *When $d \geq 2g+1$, we have*

$$\dim \Gamma(\wedge^i E) = \tbinom{d-g+1}{i} + \tbinom{d-g-1}{i-2}(d-i-g)$$

for $i \leq d-g$

Proof: Consider (2.13)

$$0 \to L \otimes T^{-(d-g-1)} \otimes \wedge^{i-1}U^* \otimes T^{i-1} \to \wedge^i E \to \wedge^i U^* \otimes T^i \to 0$$

i.e.

$$0 \to L \otimes T^{-(d-g-i)} \otimes \wedge^{i-1}U^* \to \wedge^i E \to \wedge^i U^* \otimes T^i \to 0$$

Thus,

$$\begin{aligned} h^0(\wedge^i E) &= \left[h^0(L \otimes T^{-(d-g-i)}) - h^1(L \otimes T^{-(d-g-i)})\right]\tbinom{d-g-1}{i-1} + \tbinom{d-g-1}{i}h^0(T^i) \\ &\quad -\tbinom{d-g-1}{i}h^1(T^i) + h^1(\wedge^i E) \\ &= \tbinom{d-g-1}{i-1}(g-d+2i+1) + \tbinom{d-g-1}{i}(i+1) - \tbinom{d-g-1}{i}(g-i) + h^1(\wedge^i E) \end{aligned}$$

Now,

$$\begin{aligned} H^1(\wedge^i E) &= H^1(\wedge^{d-g-i}E^* \otimes L) \qquad (\operatorname{rank} E = d-g) \\ &= H^0(\wedge^{d-g-i}E \otimes L^{-1} \otimes K)^* \end{aligned}$$

Thus,

$$h^1(\wedge^i E) = h^0(\wedge^{d-g-i}E \otimes L^{-1} \otimes K)$$

Since

$$M^*_{p,p+2} \approx H^0(\wedge^{p+1}E \otimes L^{-1} \otimes K)$$

Thus, we have

$$M^*_{d-g-i-1,d-g-i+1} = H^0(\wedge^{d-g-i}E \otimes L^{-1} \otimes K)$$

We have $d \geq 2g+1$ and $i \leq d-g$ which implies $i \leq g$. Thus, for $j = (d-g-1)-i \geq d-2g-1$, we have from (2.20),

$$\dim M^*_{d-g-i-1,d-g-i+1} = \dim \wedge^{d-g-1-i}(F \otimes \mathcal{O}(-1)) \otimes D_{g-1-i}(G \otimes \mathcal{O})^* \otimes \wedge^2 (G \otimes \mathcal{O})^* = \binom{d-g-1}{i}(g-i)$$

i.e.

$$h^1(\wedge^i E) = \binom{d-g-1}{i}(g-i)$$

Thus

$$\begin{aligned}
\dim \Gamma(\wedge^i E) &= \binom{d-g-1}{i-1}(g-d+2i+1) + \binom{d-g-1}{i}(i+1) \\
&= \binom{d-g-1}{i-1}(g-d+2i+1) + \binom{d-g-1}{i}(i+1) + \binom{d-g+1}{i} - \binom{d-g+1}{i} \\
&= \binom{d-g+1}{i} + \binom{d-g-1}{i-2}(d-i-g)
\end{aligned}$$

$\square$

Proposition 2.9. *For $i \leq d-g$, the map $\Gamma(L'_i) \to \Gamma(\wedge^i E)$ induced by diagram (2.18) is an isomorphism.*

Proof: We get an exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \Gamma(L \otimes T^{-(d-g-1)+(i-1)}) \otimes \wedge^{i-1}U^* & \to & \Gamma(\wedge^i E) & \longrightarrow & \wedge^i U^* \otimes \Gamma(T^i) & \longrightarrow & \dots \\
 & & \alpha_1 \uparrow & & \alpha_2 \uparrow & & \alpha_3 \uparrow & & \\
0 & \longrightarrow & \Gamma(W^*(i-1)) \otimes \wedge^{i-1}U^* & \longrightarrow & \Gamma(L'_i) & \longrightarrow & \wedge^i U^* \otimes \Gamma(\mathcal{O}(i)) & \to & \dots
\end{array}$$

where $\alpha_i's$ are induced by diagram (2.18).

- α_1 is injective.
  Consider the left vertical sequence of (2.18)

$$0 \to L^{-1} \otimes \wedge^{i-1}U^* \otimes T^{i-1} \to \pi^* W^* \otimes \wedge^{i-1}U^* \otimes T^{i-1}$$
$$\to L \otimes T^{-(d-g-1)} \otimes \wedge^{i-1}U^* \otimes T^{i-1} \to 0$$

This gives rise to

$$
\begin{aligned}
0 \to \Gamma(L^{-1} \otimes T^{i-1}) \otimes \wedge^{i-1}U^* &\to \Gamma(W^*(i-1)) \otimes \wedge^{i-1}U^* \\
&\xrightarrow{\alpha_1} \Gamma(L \otimes T^{-(d-g-1)+(i-1)}) \otimes \wedge^{i-1}U^* \to \dots
\end{aligned}
$$

Since

$$
\begin{aligned}
\Gamma(W^* \otimes \mathcal{O}_{\mathbb{P}^1}(i-1)) &\cong \Gamma(W \otimes \mathcal{O}_{\mathbb{P}^1}(-d+g+i)) \\
&\cong \Gamma(\pi_* L \otimes \mathcal{O}_{\mathbb{P}^1}(-d+g+i)) \\
&\cong \Gamma(\pi_*(L \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(-d+g+i))) \\
&\cong \Gamma(L \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(-d+g+i)) \\
&\cong \Gamma(L \otimes T^{-d+g+i})
\end{aligned}
$$

Thus α_1 is injective.

- α_3 is injective by definition.

Hence α_2 is injective and since both the spaces have the same dimension, the map $\Gamma(L'_i) \to \Gamma(\wedge^i E)$ is an isomorphism.

□

2.3 Construction of a subbundle of E

We want to prove theorem(2.1). We shall first do this for $i = 2$. The main point is to construct sufficiently many locally decomposable sections that are not globally decomposable.
Consider $p : \mathbb{P}(W^*) \to \mathbb{P}^1$, the natural projection. For every $a \in \mathbb{P}^1$, the fibre $l_a = p^{-1}(a)$ is a secant of the curve C. Let W_a^* be the fibre of W^* at a. Since W is globally generated thus we have $\Gamma(W)_{\mathbb{P}^1} \to W \to 0$. $W = \pi_* L$ and $\Gamma(\pi_* L) \cong \Gamma(L)$ thus we have $\Gamma(L)_{\mathbb{P}^1} \to W \to 0$, which gives $W^* \to \Gamma(L)^*_{\mathbb{P}^1}$ and we can identify $\Gamma(L)^*$ with $\Gamma(E)$. Also, we have

$$0 \to E^* \to \Gamma(L)_C \to L \to 0,$$

which gives $\Gamma(L)^*_C \to E$ i.e. a map $\Gamma(E)_C \to E$. Thus we get a map

$$(W_a)^*_C \to \Gamma(E)_C \to E$$

which is composite of the inclusion of W_a^* in $\Gamma(E)$ and the evaluation map.

Let $F(a)$ be the subbundle of E generated by the image of W_a^*. A section of $\Gamma(E)$ is non-zero at every point of C if it corresponds to a point of $\mathbb{P}(\Gamma(L)^*)$ not on the curve C, while a section corresponding to a point say $x \in C$ vanishes exactly at x.
Hence the map $W_a^* \to F(a)$ is an isomorphism outside $C \bigcap l_a$ but has $\operatorname{rank} 1$ over $C \bigcap l_a$.The induced map $\wedge^2 W_a^* \to \wedge^2 F(a)$ has simple zeros exactly over $C \bigcap l_a$. Hence $F(a)$ has $\operatorname{rank} 2$ and $\wedge^2 F(a) = T$.
The vector bundle $F(a)$ has W_a^* as its space of sections i.e. $\dim \Gamma(F(a)) = 2$. On the other hand $\dim \Gamma(\wedge^2 F(a)) = \dim \Gamma(T) = 2$. Thus we get a 2- dimensional subspace of $\Gamma(\wedge^2 E)$ consisting of locally decomposable sections of which only the 1- dimensional subspace $\wedge^2 \Gamma(F(a)) \subset \Gamma(\wedge^2 F(a))$ consists of globally decomposable sections.
The next step is to globalise this construction, i.e. to vary the point a. We consider the graph inclusion $\Gamma \subset C \times \mathbb{P}^1$ given by the map π. This divisor belongs to the line bundle $p_1^* T \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(1)$, where p_1 and p_2 are the natural projections to C, resp. $\mathbb{P}^1$. the direct image by p_2 of the bundle morphism $p_2^* W^* \to \Gamma(E)_{C \times \mathbb{P}^1}$ yields the map $W^* \to \Gamma(E)_{\mathbb{P}^1}$, and hence a map $\wedge^2 W^* \to \wedge^2 \Gamma(E)_{\mathbb{P}^1}$.
On the other hand the bundle homomorphism $p_2^* W^* \to \Gamma(E)_{C \times \mathbb{P}^1} \to p_1^* E$ fails to be injective precisely over Γ. Thus, we get a morphism
$p_2^*(\wedge^2 W^*) \otimes \mathcal{O}(\Gamma) \to p_1^*(\wedge^2 E)$. Taking direct image by p_2 gives a morphism
$\wedge^2 W^* \otimes \Gamma(T) \otimes \mathcal{O}(1) \to \Gamma(\wedge^2 E)_{\mathbb{P}^1}$. For every $a \in \mathbb{P}^1$ this induces a map $\Gamma(T) \to \Gamma(\wedge^2 E)_{\mathbb{P}^1}$ and this gives exactly the space of locally decomposable sections described above.

Altogether, we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \wedge^2 W^* & \longrightarrow & \wedge^2 W^* \otimes \Gamma(T) \otimes \mathcal{O}_{\mathbb{P}^1}(1) & \to & \wedge^2 W^* \otimes \mathcal{O}_{\mathbb{P}^1}(2) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \wedge^2 \Gamma(E)_{\mathbb{P}^1} & \longrightarrow & \Gamma(\wedge^2 E)_{\mathbb{P}^1} & \longrightarrow & D^2_{\mathbb{P}^1} & \longrightarrow & 0 \end{array} \tag{2.21}$$

where $D^2_{\mathbb{P}^1} := \frac{\Gamma(\wedge^2 E)}{\wedge^2 \Gamma(E)}$

where the top horizontal row is the evaluation sequence for $\mathcal{O}_{\mathbb{P}^1}(1)$, which is

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \Gamma(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0$$

Tensoring with $\wedge^2 W^* \otimes \mathcal{O}_{\mathbb{P}^1}(1)$, we get

$$0 \to \wedge^2 W^* \to \wedge^2 W^* \otimes \Gamma(T) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \to \wedge^2 W^* \otimes \mathcal{O}_{\mathbb{P}^1}(2) \to 0$$

and use $\Gamma(T) \cong \Gamma(\mathcal{O}_{\mathbb{P}^1})$.
We have to show that the locally decomposable sections constructed above together with $\wedge^2\Gamma(E)$ generate $\Gamma(\wedge^2 E)$. For this, we consider the map $\wedge^2 W^* \otimes \mathcal{O}_{\mathbb{P}^1}(2) \to D^2_{\mathbb{P}^1}$. We want to show that this map is injective (as a bundle map) and that the resulting rational curve in $\mathbb{P}(D^2)$ is the rational normal curve of degree $d-g-3$ (recall that $\dim D^2 = d-g-2$). This is sufficient since the rational normal curve of degree n in $\mathbb{P}^n$ spans $\mathbb{P}^n$.

Our aim is to do this by entirely reducing the problem to computations on $\mathbb{P}^1$, resp. $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 2.10. *[HPR92] Let $\mathcal{O}_{\mathbb{P}^1}(-n) \to \Gamma(\mathcal{O}_{\mathbb{P}^1}(n))^*_{\mathbb{P}^1}$ be a non-zero $Sl_2(\mathbb{C})$- equivariant morphism. Then this morphism defines an embedding of $\mathbb{P}^1$ into $\mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^1}(n)^*))$ as a rational normal curve of degree n.*

We return to the bundle W. Sequence (2.16) gives for $i = 2$ the following sequence:

$$0 \to \wedge^2 W^* \to \wedge^2\Gamma(W)^*_{\mathbb{P}^1} \to L'_2 \to 0 \tag{2.22}$$

Consider $\mathbb{P}^1 \times \mathbb{P}^1$ together with projections q_1 and q_2 resp.
Taking pullback of (2.22) via q_1 and q_2 resp., we get a map $q_2^* \wedge^2 W^* \to q_1^* L'_2$ that vanishes along the diagonal $\triangle \subset \mathbb{P}^1 \times \mathbb{P}^1$.
Hence, we get a morphism

$$q_2^* \wedge^2 W^* \otimes \mathcal{O}(\triangle) \to q_1^* L'_2$$

Applying q_{2*}, we get a map

$$\wedge^2 W^* \otimes \Gamma(\mathcal{O}(1)) \otimes \mathcal{O}(1) \to \Gamma(L'_2)_{\mathbb{P}^1}$$

This gives rise to a commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\wedge^2 W^* & \longrightarrow & \wedge^2\Gamma(W)^*_{\mathbb{P}^1} \\
\downarrow & & \downarrow \\
\wedge^2 W^* \otimes \Gamma(\mathcal{O}(1)) \otimes \mathcal{O}(1) & \longrightarrow & \Gamma(L_2')_{\mathbb{P}^1} \\
\downarrow & & \downarrow \\
\wedge^2 W^* \otimes \mathcal{O}(2) & \longrightarrow & H^1(\wedge^2 W^*) = \Gamma(\wedge^2 W \otimes \mathcal{O}(-2))^* \\
\downarrow & & \downarrow \\
0 & & 0
\end{array} \tag{2.23}$$

where the left hand column is the Euler sequence on $\mathbb{P}^1$ twisted by $\wedge^2 W^* \otimes \mathcal{O}(1)$, the right hand column comes from (2.22) and the map $\wedge^2 W^* \to \wedge^2\Gamma(W)^*$ is the natural one. This diagram is $Sl_2(\mathbb{C})$ equivariant, where $Sl_2(\mathbb{C})$ acts on $\mathbb{P}^1$ in the usual way and on $\mathbb{P}^1 \times \mathbb{P}^1$ by the diagonal action. In particular the morphism $\wedge^2 W^* \otimes \mathcal{O}(2) \to \Gamma(\wedge^2 W \otimes \mathcal{O}(-2))^*$ is $Sl_2(\mathbb{C})$ equivariant, by Lemma(2.11), it defines an embedding of $\mathbb{P}^1$ into $\mathbb{P}(\Gamma(\wedge^2 W \otimes \mathcal{O}(-2))^*)$ as a rational normal curve of degree $d-g-3$.

Lemma 2.11. *[HPR92] Diagram (2.23) gives rise to a commutative and exact diagram*

$$\begin{array}{ccccccccc}
0 & \to & \Gamma(\wedge^2 W \otimes \mathcal{O}(-2)) & \longrightarrow & \Gamma(L_2')^* & \longrightarrow & \wedge^2\Gamma(W) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \Gamma(\wedge^2 W \otimes \mathcal{O}(-2)) & \to & \Gamma(\wedge^2 W \otimes \mathcal{O}(-1)) \otimes \Gamma(\mathcal{O}(1)) & \to & \Gamma(\wedge^2 W) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{2.24}$$

Proposition 2.12. $\Gamma(\wedge^2 E)$ *is generated by locally decomposable sections.*

Proof: We have constructed maps $p_2^*(\wedge^2 W^*) \otimes \mathcal{O}(\Gamma) \to p_1^*(\wedge^2 E)$ on $C \times \mathbb{P}^1$ and $q_2^* \wedge^2 W^* \otimes \mathcal{O}(\triangle) \to q_1^* L_2'$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Consider the diagram

$$\begin{array}{ccccc}
C & \xleftarrow{p_1} & C \times \mathbb{P}^1 & \xrightarrow{p_2} & \mathbb{P}^1 \\
\downarrow\pi & & \downarrow\pi\times id & & \| \\
\mathbb{P}^1 & \xleftarrow{q_1} & \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{q_2} & \mathbb{P}^1
\end{array} \tag{2.25}$$

Pulling the morphism $q_2^* \wedge^2 W^* \otimes \mathcal{O}(\triangle) \to q_1^* L_2'$ on $\mathbb{P}^1 \times \mathbb{P}^1$ back to $C \times \mathbb{P}^1$, we get a morphism $p_2^*(\wedge^2 W^*) \otimes \mathcal{O}(\Gamma) \to p_1^*(\pi^* L_2')$. By construction the diagram

$$\begin{array}{ccc} p_2^*(\wedge^2 W^*) \otimes \mathcal{O}(\Gamma) & \to & p_1^*(\pi^* L_2') \\ \| & & \downarrow \\ p_2^*(\wedge^2 W^*) \otimes \mathcal{O}(\Gamma) & \to & p_1^*(\wedge^2 E) \end{array} \tag{2.26}$$

commutes where the map $p_1^*(\pi^* L_2') \to p_1^* \wedge^2 E$ is the pullback via p_1 of the corresponding map in diagram (2.18).Pushing this down via $\pi \times id$ to $\mathbb{P}^1 \times \mathbb{P}^1$ leads to the commutative diagram

$$\begin{array}{ccccc} q_2^* \wedge^2 W^* \otimes \mathcal{O}(\triangle) & \longrightarrow & & \longrightarrow & q_1^* L_2' \\ \| & & & & \downarrow \\ q_2^* \wedge^2 W^* \otimes \mathcal{O}(\triangle) & \to & q_2^* \wedge^2 W^* \otimes (\pi \times id)_* \mathcal{O}(\Gamma) & \to & q_1^*(\pi_* \pi^* L_2') \\ \| & & \| & & \downarrow \\ q_2^* \wedge^2 W^* \otimes \mathcal{O}(\triangle) & \to & q_2^* \wedge^2 W^* \otimes (\pi \times id)_* \mathcal{O}(\Gamma) & \to & q_1^*(\pi_* \wedge^2 E) \end{array} \tag{2.27}$$

Now taking q_{2*} of the outermost square we get

$$\begin{array}{ccc} \wedge^2 W^* \otimes \Gamma(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(1) & \longrightarrow & \Gamma(\mathbb{P}^1, L_2') \\ \| & & \downarrow \\ \wedge^2 W^* \otimes \Gamma(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(1) & \to & \Gamma(C, \wedge^2 E) \end{array} \tag{2.28}$$

where the right hand vertical map is an isomorphism from Proposition 2.12.
Thus in order to compute the diagram

$$\begin{array}{ccc} \wedge^2 W^* & \longrightarrow & \wedge^2 \Gamma(W)^*_{\mathbb{P}^1} \\ \downarrow & & \downarrow \\ \wedge^2 W^* \otimes \Gamma(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(1) & \longrightarrow & \Gamma(\wedge^2 E) \end{array} \tag{2.29}$$

we can compute

$$\begin{array}{ccc} \wedge^2 W^* & \longrightarrow & \wedge^2 \Gamma(W)^*_{\mathbb{P}^1} \\ \downarrow & & \downarrow \\ \wedge^2 W^* \otimes \Gamma(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(1) & \longrightarrow & \Gamma(L_2')_{\mathbb{P}^1} \end{array} \tag{2.30}$$

and the result follows from Lemma 2.11. $\square$

2.4 Proof of main result

Main Result (Theorem 2.1) *Let C be a smooth hyperelliptic curve of genus $g \geq 2$ and let L be a globally generated line bundle on C of degree $d \geq 2g+1$ such that $H^1(L \otimes T^{-2}) = 0$, where T is the hyperelliptic line bundle on C. The evaluation map gives rise to an exact sequence*

$$0 \to E^* \to \Gamma(L)_C \to L \to 0$$

where E^* is locally free of $\operatorname{rank} h^0(L) - 1$. Let $\sum_i$ be the cone of locally decomposable sections of $\wedge^i E$. Then $\sum_i$ spans $\Gamma(\wedge^i E)$ $\forall$ i.

Proof: We shall first show that for $2 \leq i \leq d-g$, there is a natural epimorphism

$$\wedge^{i-2}\Gamma(W)^* \otimes \Gamma(L_2')_{\mathbb{P}^1} \to \Gamma(L_i')_{\mathbb{P}^1} \to 0$$

Setting $i = 2$ in (2.16), we get

$$0 \to \wedge^2 W^* \to \wedge^2 \Gamma(L)^*_{\mathbb{P}^1} \to L_2' \to 0$$

Twisting with $\wedge^{i-2}\Gamma(W)^*$, we get an exact sequence

$$0 \to \wedge^{i-2}\Gamma(W)^* \otimes \wedge^2 W^* \to \wedge^{i-2}\Gamma(W)^* \otimes \wedge^2\Gamma(W)^* \to \wedge^{i-2}\Gamma(W)^* \otimes L_2' \to 0$$

(since $\Gamma(L) \cong \Gamma(W)$) Combining this with (2.16), we get a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \wedge^{i-2}\Gamma(W)^* \otimes \wedge^2 W^* & \longrightarrow & \wedge^{i-2}\Gamma(W)^* \otimes \wedge^2\Gamma(W)^* & \to & \wedge^{i-2}\Gamma(W)^* \otimes L_2' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \wedge^{i-2}\Gamma(W(-1))^* \otimes \mathcal{O}(i-2) \otimes \wedge^2 W^* & \longrightarrow & \wedge^i\Gamma(W)^* & \longrightarrow & L_i' & \longrightarrow & 0 \end{array} \tag{2.31}$$

Here the middle vertical map is the canonical one and the left hand vertical map is given by taking $\wedge^{i-2}$ of the dual evaluation sequence

$$0 \to W^* \to \Gamma(W)^* \otimes \mathcal{O}_{\mathbb{P}^1} \to \Gamma(W(-1))^* \otimes \mathcal{O}_{\mathbb{P}^1}(1) \to 0$$

Taking $\wedge^{i-2}$ of the above sequence, we get

$$0 \to \wedge^2 W^* \otimes \wedge^{i-2}\Gamma(W(-1))^* \otimes \mathcal{O}_{\mathbb{P}^1}(i-2) \to \wedge^{i-2}\Gamma(W)^* \to F_{i-2} \to 0$$

$$0 \to F_{i-2} \to \wedge^{i-2}\Gamma(W)^* \otimes \mathcal{O}_{\mathbb{P}^1} \to \wedge^{i-2}\Gamma(W(-1))^* \otimes \mathcal{O}_{\mathbb{P}^1}(i-2) \to 0$$

Tensoring above sequence with $\wedge^2 W^*$, we get

$$0 \to F_{i-2} \otimes \wedge^2 W^* \to \wedge^{i-2}\Gamma(W)^* \otimes \mathcal{O}_{\mathbb{P}^1} \otimes \wedge^2 W^* \to \wedge^{i-2}\Gamma(W(-1))^* \otimes \mathcal{O}_{\mathbb{P}^1}(i-2) \otimes \wedge^2 W^* \to 0$$

Taking the associated cohomology sequence of (2.31), we get the following commutative diagram:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \wedge^{i-2}\Gamma(W)^* \otimes \wedge^2\Gamma(W)^* & \longrightarrow & \wedge^i\Gamma(W)^* \\ \downarrow & & \downarrow \\ \wedge^{i-2}\Gamma(W)^* \otimes \Gamma(L_2') & \longrightarrow & \Gamma(L_i') \\ \downarrow & & \downarrow \\ H^1(\wedge^{i-2}\Gamma(W)^* \otimes \wedge^2 W^*) & \to & H^1(\wedge^{i-2}\Gamma(W(-1))^* \otimes \mathcal{O}(i-2) \otimes \wedge^2 W^*) \\ \downarrow & & \downarrow \\ 0 & & \vdots \end{array} \tag{2.32}$$

Here W is a $\mathrm{rank}\,2$ vector bundle on $\mathbb{P}^1$ of degree $d-g-1$.

$$\begin{aligned} detW &\cong \mathcal{O}(d-g-1) \\ detW^* &\cong \mathcal{O}(-d+g+1) \\ i.e. \wedge^2 W^* &\cong \mathcal{O}(-d+g+1) \end{aligned}$$

Since $2 \leq d-g-1$, thus $\Gamma(\wedge^2 W) = 0$.

The top horizontal map is clearly surjective. The bottom horizontal map is surjective since H^2 vanishes on $\mathbb{P}^1$. By standard diagram chasing the middle horizontal map must be surjective thus giving our first claim.
By construction the natural diagram

$$\begin{array}{ccc} \wedge^{i-2}\Gamma(W)^* \otimes \Gamma(L_2') & \to & \wedge^{i-2}\Gamma(E) \otimes \Gamma(\wedge^2 E) \\ \downarrow & & \downarrow \\ \Gamma(L_i') & \longrightarrow & \Gamma(\wedge^i E) \end{array} \tag{2.33}$$

commutes.

Since $\Gamma(W)^* \cong \Gamma(L)^*$ and $\Gamma(L)^*$ can be identified with $\Gamma(E)$. By proposition (2.12), the horizontal maps are isomorphisms. Hence the natural map $\wedge^{i-2}\Gamma(E) \otimes \Gamma(\wedge^2 E) \to \Gamma(\wedge^i E)$ is surjective and our claim follows from Proposition (2.12).

Chapter 3

Results for sub-linear systems

Introduction

In chapter 2, the conjecture 2.0.2 is studied in context of complete linear systems on a hyperelliptic curve. In this chapter, we study conjecture 2.0.2 in case of sub-linear systems on a hyperelliptic curve and some results are obtained in case of sub-linear systems with codimension 1.

I would like to acknowledge Prof. Peter Newstead for suggesting this question.

The main point of this chapter is the following question:

Question 3.1. *Let C be a hyperelliptic curve of genus $g \geq 2$ and let (L, V) be a linear system with L a generated line bundle of degree $d \geq 2g + 1$ on C such that $H^1(L \otimes T^{-2}) = 0$, where T is the hyperelliptic line bundle on C and $V \subset H^0(L)$ a linear subspace of dimension $n + 1$ which generates L. The evaluation map gives rise to an exact sequence*

$$0 \to E_V^* \to V \otimes \mathcal{O}_{\mathcal{C}} \to L \to 0 \tag{3.1}$$

where E_V^* is locally free of $\operatorname{rank} n$. Let $\sum_i$ be the cone of locally decomposable sections of $\wedge^i E_V$, then $\sum_i$ spans $\Gamma(\wedge^i E_V)$ for all i.

In this chapter, we discuss question 3.1in case of sub-linear system with $\operatorname{codim} 1$. From now onwards, throughout this chapter, C is a smooth hyperelliptic curve of genus $g \geq 2$ and (L, V) a linear system with L a generated line bundle of degree $d \geq 2g+1$ on C such that $H^1(L \otimes T^{-2}) = 0$ and $V \subset H^0(L)$ a linear subspace of dimension $n + 1$ which generates L and $\operatorname{codim} V = 1$. K is the canonical line bundle on C.

The idea of the proof is similar to the case of complete linear systems with minor changes in computation of the dimensions. In §(3.1), we recall the arguments and results obtained in chapter 2 that holds for sub-linear system too.
In §(3.2), we compute the dimension of the space of sections of the vector bundle on $\mathbb{P}^1$ which is related to the space of sections of the vector bundle $\wedge^i E_V$ on C. In §(3.3), we will see that under some assumptions, result holds for sub-linear system with $\operatorname{codim} 1$.

3.1 Notations and Results obtained for complete linear systems

We fix with the following notations of chapter 2
C is a hyperelliptic curve of genus $g \geq 2$, $\pi : C \to \mathbb{P}^1$ is the associated 2- sheeted covering and $T := \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$
Here (L, V) is a linear system with L a generated line bundle of degree $d \geq 2g + 1$ on C and $W := \pi_* L$.
By repeating the arguments given in §(2.1), we obtain

$$\deg W = d - g - 1$$

and

$$h^0(L) = d - g + 1 \tag{3.2}$$

Since $\operatorname{codim} V = 1$ which gives $n = d - g - 1$ Thus,

$$\operatorname{rank}(E_V) = n = d - g - 1 \geq 2 \tag{3.3}$$

The following results which are obtained in chapter 2 also holds in case of sub-linear systems.

1. $W(-1)$ is globally generated.

2. There are inclusions

$$C \subset S \subset \mathbb{P}$$

where $\mathbb{P} := \mathbb{P}(V^*)$ and $S :=$ image of $\mathbb{P}(W^*)$ in $\mathbb{P}$ with the following properties:

(a) the restriction of $\mathcal{O}_{\mathbb{P}}(1)$ to S is $\mathcal{O}_W(1)$;

(b) the restriction of $\mathcal{O}_{\mathbb{P}}(1)$ to C is L;

(c) both restrictions induce isomorphisms of the corresponding linear systems.

3.2 Computation of dimensions

Similar to §2.2, we want to relate the sections of $\wedge^i E_V$ to the sections of a suitable vector bundle on $\mathbb{P}^1$.
Since L is generated by $V \subset H^0(L)$. Thus we have the surjection map

$$V \otimes \mathcal{O}_{\mathcal{C}} \to L \to 0$$

Though W is globally generated, but if W is generated by $V \subset \Gamma(L) \cong \Gamma(W)$ then we have,

$$V \otimes \mathcal{O}_{\mathbb{P}^1} \to W \to 0 \tag{3.4}$$

Let K_V be the kernel of (3.4). Thus we have the evaluation sequence for W

$$0 \to K_V \to V \otimes \mathcal{O}_{\mathbb{P}^1} \to W \to 0 \tag{3.5}$$

Pulling back the evaluation sequence for W on $\mathbb{P}^1$ to C, we get

$$0 \to \pi^* K_V \to V \otimes \mathcal{O}_C \to \pi^* W \to 0 \tag{3.6}$$

Working on the parallel lines as for the complete linear system in §(2.2), we obtain the following sequences of vector bundles involving $\wedge^i E_V$ and M_i' (the corresponding vector bundle on $\mathbb{P}^1$) respectively.

$$0 \to L \otimes T^{-(d-g-1)} \otimes \wedge^{i-1} \pi^* K_V^* \to \wedge^i E_V \to \wedge^i \pi^* K_V^* \to 0 \tag{3.7}$$

$$0 \to \wedge^2 W^* \otimes \wedge^{i-2} K_V^* \to \wedge^i V_{\mathbb{P}^1}^* \to M_i' \to 0 \tag{3.8}$$

Sequences 3.7 and 3.8 refer to the corresponding sequences 2.13 and 2.16 respectively of the complete linear system. Also, we get the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \to & L\otimes T^{-(d-g-1)}\otimes\wedge^{i-1}\pi^*K_V^* & \longrightarrow & \wedge^i E_V & \longrightarrow & \wedge^i\pi^*K_V^* & \to & 0\\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \pi^*W^*\otimes\wedge^{i-1}\pi^*K_V^* & \longrightarrow & \pi^*M_i' & \longrightarrow & \wedge^i\pi^*K_V^* & \to & 0\\
 & & \uparrow & & \uparrow & & & & \\
 & & L^{-1}\otimes\wedge^{i-1}\pi^*K_V^* & = & L^{-1}\otimes\wedge^{i-1}\pi^*K_V^* & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.9}$$

which is similar to the diagram (2.18), the vector bundle on $\mathbb{P}^1$ is denoted by M_i' corresponding to the vector bundle $\wedge^i E_V$ on C, which in case of $\wedge^i E$ is denoted by L_i'.

Consider (3.5)

$$0\to K_V\to V\otimes\mathcal{O}_{\mathbb{P}^1}\to W\to 0$$

$$\deg K_V=-\deg W=-(d-g-1)$$

and

$$\operatorname{rank} K_V=\dim V-\operatorname{rank} W=(n+1)-2=d-g-2$$

Thus, K_V is a rank $d-g-2$ vector bundle on $\mathbb{P}^1$ of degree $-(d-g-1)$

$$K_V\cong\mathcal{O}_{\mathbb{P}^1}(a_1)\bigoplus\mathcal{O}_{\mathbb{P}^1}(a_2)\bigoplus\cdots\bigoplus\mathcal{O}_{\mathbb{P}^1}(a_{d-g-2})$$

where

$$\sum_{i=1}^{d-g-2}a_i=-(d-g-1)$$

and $a_i \leq 0 \; \forall \, i$ which implies

$$\begin{aligned}
K_V &\cong \mathcal{O}_{\mathbb{P}^1}(-1) \bigoplus \mathcal{O}_{\mathbb{P}^1}(-1) \bigoplus \cdots \bigoplus \mathcal{O}_{\mathbb{P}^1}(-1) \bigoplus \mathcal{O}_{\mathbb{P}^1}(-2) \\
&\cong \bigoplus_{d-g-3} \mathcal{O}_{\mathbb{P}^1}(-1) \bigoplus \mathcal{O}_{\mathbb{P}^1}(-2) \\
K_V^* &\cong \bigoplus_{d-g-3} \mathcal{O}_{\mathbb{P}^1}(1) \bigoplus \mathcal{O}_{\mathbb{P}^1}(2) \\
\wedge^{i-2} K_V^* &\cong \bigoplus_{l} \mathcal{O}_{\mathbb{P}^1}(i-2) \bigoplus_{m} \mathcal{O}_{\mathbb{P}^1}(i-1)
\end{aligned}$$

where $l = \binom{d-g-3}{i-2}$ and $m = \binom{d-g-3}{i-3}$

Let us compute the dimensions of the space $\Gamma(M_i')$ for $i \leq d-g-1 = \operatorname{rank} E_V$.

Lemma 3.1. *When $d \geq 2g+1$, we have*

$$\dim \Gamma(M_i') = \binom{d-g}{i} + \binom{d-g-3}{i-2}(d-1-g) - \binom{d-g-2}{i-2}$$

for $i \leq d-g-1$

Proof: Consider (3.8)

$$0 \to \wedge^2 W^* \otimes \wedge^{i-2} K_V^* \to \wedge^i V_{\mathbb{P}^1}^* \to M_i' \to 0$$

Since

$$detW \cong \mathcal{O}_{\mathbb{P}^1}(d-g-1).$$

Thus

$$detW^* \cong \mathcal{O}_{\mathbb{P}^1}(-(d-g-1))$$

Thus, we get

$$0 \to \mathcal{O}(-d+g+1) \otimes \wedge^{i-2} K_V^* \to \wedge^i V^* \to M_i' \to 0$$

Denote $\mathcal{O}(-d+g+1)\otimes\wedge^{i-2}K_V^*$ by B

$$\begin{aligned}
B &\cong \mathcal{O}(-d+g+1)\otimes\wedge^{i-2}K_V^* \\
&\cong \mathcal{O}(-d+g+1)\otimes(\bigoplus_l \mathcal{O}_{\mathbb{P}^1}(i-2)\bigoplus_m \mathcal{O}_{\mathbb{P}^1}(i-1)) \\
&\cong \bigoplus_l \mathcal{O}_{\mathbb{P}^1}(-d+g+i-1)\bigoplus_m \mathcal{O}_{\mathbb{P}^1}(-d+g+i) \\
&\cong \bigoplus_l \mathcal{O}_{\mathbb{P}^1}(i-(d-g+1))\bigoplus_m \mathcal{O}_{\mathbb{P}^1}(i-(d-g))
\end{aligned}$$

Since,

$$i<d-g, h^0(B)=0$$

$\operatorname{rank} B=\binom{n-1}{i-2}=\binom{d-g-2}{i-2}$
By Riemann- Roch Theorem, we have

$$\begin{aligned}
h^0(B)-h^1(B) &= \operatorname{rank} B(\frac{\deg B}{\operatorname{rank} B}+1-g_{\mathbb{P}^1}) \\
0-h^1(B) &= \deg B+\operatorname{rank} B \\
h^1(B) &= -\deg(B)-\operatorname{rank}(B)
\end{aligned}$$

So,

$$h^1(B)=-[\binom{d-g-3}{i-2}(i-(d-g+1)+\binom{d-g-3}{i-3}(i-(d-g))]-\binom{d-g-2}{i-2}$$

Therefore,

$$h^0(M_i')=h^0(\wedge^i V^*)+h^1(B)$$

$$h^0(M_i')=\binom{n+1}{i}-\binom{d-g-3}{i-2}(i-(d-g+1)-\binom{d-g-3}{i-3}(i-(d-g))-\binom{d-g-2}{i-2}$$

Thus,

$$\dim\Gamma(M_i')=\binom{d-g}{i}+\binom{d-g-3}{i-2}(d-1-g)-\binom{d-g-2}{i-2}$$

□

3.3 Proof of main result

If for $d \geq 2g+1$,

$$\dim \Gamma(\wedge^i E_V) = \dim \Gamma(M_i') = \binom{d-g}{i} + \binom{d-g-3}{i-2}(d-1-g) - \binom{d-g-2}{i-2}$$

for all $i \leq d-g-1$, then the map $\Gamma(M_i') \to \Gamma(\wedge^i E_V)$ induced by diagram (3.9) is an isomorphism. (refer to proposition 2.9)

and we construct a subbundle F of E_V. The construction is same as for the case of complete linear system, all the arguments and results given in §2.3 and 2.4 works for the sub-linear system. We only make the replacements of vector bundle E by E_V, L_i' by M_i' and $\Gamma(L)$ by V. Then the answer to question 3.1 is affirmative for sub-linear system with $\operatorname{codim} 1$ (refer to §2.4 with above mentioned replacements).

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