# A study of homological properties of certain classes of monomial ideals

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**Dedicated**

**To**

**My Beloved Parents**

**Declaration**

The work presented in this thesis has been carried out by me under the supervision of Dr. Chanchal Kumar at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Date:
Place: Ajay Kumar

In my capacity as a supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

**Dr. Chanchal Kumar**
(Supervisor)

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# Contents

# Chapter 1

# Introduction

This dissertation is an interplay between commutative algebra and combinatorics. The symbiosis of combinatorics and commutative algebra have proved beneficial for many branches of mathematics. Many problems in combinatorics can be proved using tools of commutative algebra. Richard Stanley was the first mathematician who introduced commutative algebra techniques to solve combinatorial problems. To each simplicial complex $\Delta$ on $n$ vertices, Stanley associated a quotient ring $k[\Delta]$ of the standard polynomial ring $k[x_1, x_2, \ldots, x_n]$ over a field $k$, called the *Stanley-Reisner ring* of $\Delta$ in such a manner that the combinatorial properties of the simplicial complex $\Delta$ are intimately related with the algebraic properties of the Stanley-Reisner ring $k[\Delta]$. More generally, one can associate finitely generated $k$-algebras to certain combinatorial problems and it was observed by Stanley that these $k$-algebras are Cohen-Macaulay or Gorenstein could be crucial in solving these combinatorial problems. Also the formal power series associated with some counting problems can be thought of as a Hilbert series of a finitely generated commutative $k$-algebra. Stanley [27, 28] proved the upper bound conjecture(UBC) and Anand-Dumir-Gupta(ADG) conjecture, which are fine examples of the symbiosis of combinatorics and commutative algebra. This symbiosis, which is commonly referred as combinatorial commutative algebra is a very active area of research, which is indicated by a vast amount of literature published in the form of research papers and monographs. Combinatorics and Commutative Algebra [28], Monomial Algebra [32], Combinatorial Commutative Algebra [20], Monomial Ideals [13] are a few of standard books in this area.

Stanley-Reisner rings have been extensively studied in the last few decades. Fröberg [11] studied the Stanley-Reisner rings having linear resolutions. He characterized the Stanley-Reisner rings having 2-linear resolutions in terms of triangulated graphs. Bruns and Hibi [4, 6] extended the work of Fröberg and characterized Stanley-Reisner ring $k[\Delta]$ with pure resolutions for $\dim(\Delta) = 1$ or 2. Eagon and Reiner [8] established a relationship between a Stanley-Reisner ideal $I_\Delta$ having linear resolution and Cohen-Macaulayness of the Alexander dual $\Delta^*$. This result has been generalized by Herzog-Hibi [12] replacing the ideal having linear resolution by the notion of *componentwise linear ideal* and Cohen-Macaulay by *sequentially Cohen-Macaulay*. Hochster [14] gave a formula for graded Betti numbers of Stanley-Reisner rings. The Betti numbers of an ideal $I$ in a polynomial ring $R = k[x_1, x_2, \ldots, x_n]$ gives us information about the homological structure of the quotient ring $R/I$. The $i^{th}$ Betti number $\beta_i(I)$ is a homological invariant of the ideal $I$ in $R$ and is given by the formula

$$\beta_i(I) = \dim_k Tor_i{}^R(I, k) \ \forall\ i \geq 0.$$

Equivalently, $\beta_i(I)$ is the rank of the $i^{th}$ free module in a minimal free resolution of $I$. It is not always easy to find a minimal free resolution of a monomial ideal. However, formulae for the Betti numbers can be determined in certain type of monomial ideals.

In a beautiful paper [9], Eliahou and Kervaire constructed the minimal resolutions of a class of monomial ideals called *stable ideals*. By definition, a monomial ideal $I$ in $k[x_1, x_2, \ldots, x_n]$ is called a *stable* if for every monomial $\mathbf{w} \in I$ and index $i < m = max(\mathbf{w})$, the monomial $x_i\mathbf{w}/x_m$ again belongs to $I$, where $max(\mathbf{w})$ denotes the largest index of the variables dividing $\mathbf{w}$. The minimal resolution of a stable ideal is now termed as its *Eliahou-Kervaire resolution*. A combinatorial formula for the Betti numbers of stable ideals has also been obtained by them. In fact,

$$\beta_i(I) = \sum_{\mathbf{u} \in G(I)} \binom{max(\mathbf{u}) - 1}{i},$$

where $G(I)$ is the canonical generating system of $I$. In the same paper [9], Eliahou and Kervaire introduced a concept of *splittable monomial ideals.* Since a power $\mathfrak{m}^d$ of the maximal ideal $\mathfrak{m} = \langle x_1, x_2, \ldots, x_n \rangle$ in $R$ is stable and having a natural splitting, Betti numbers of power $\mathfrak{m}^d$ can easily be calculated. In fact,

$$\beta_i(\mathfrak{m}^d) = \binom{d+n-1}{d+i}\binom{d+i-1}{i}.$$

The Betti numbers of power $\mathfrak{m}^d$ of maximal ideal can also be calculated using the Eagon-Northcott complex [7].

Bayer and Sturmfell [2] constructed a cellular free complex $\mathbb{F}_*(X)$ (1.1.2) associated to a labeled cell complex $X$. By a *labeled cell complex* $X$ we mean a polyhedral cell complex whose vertices are labeled with the monomial generators of some monomial ideal $I$. They described a necessary and sufficient condition for the free complex $\mathbb{F}_*(X)$ to be a free resolution of $I$. If $\mathbb{F}_*(X)$ is a free resolution of $I$, then it is called a *cellular resolution* of $I$ supported on $X$. Further with the help of cellular resolutions, they constructed the canonical free resolutions, so called *hull resolutions* of monomial ideals. For $\mathbf{a} \in \mathbb{N}^n$ and $t \in \mathbb{R}$, the convex hull of points $t^{\mathbf{a}} = (t^{a_1}, t^{a_2}, \ldots, t^{a_n})$ such that $\mathbf{x}^{\mathbf{a}} \in I$ is a polyhedron $\mathcal{P}_t \in \mathbb{R}^n$. The polyhedral cell complex consisting of all bounded faces of $\mathcal{P}_t$ is independent of $t$ for sufficiently large $t$. This polyhedral cell complex is called the *hull complex* $\mathcal{H}(I)$ of $I$ and the cellular free complex $\mathbb{F}_*(\mathcal{H}(I))$ is called the *hull resolution* of $I$. The hull resolution of $I$ is a free resolution of $I$ but it need not be minimal. However if $I$ is a *generic monomial ideal*([17], Definition 6.5) then the hull complex of $I$ coincides with the *Scarf complex* $\Delta_I$ of $I$ and the hull resolution is the minimal resolution $\mathbb{F}_*(\Delta_I)$. Let $I$ be a monomial ideal with minimal generating set $\{n_1, n_2, \ldots, n_s\}$. Then by a Scarf complex $\Delta_I$, we mean a simplicial complex consisting of the subsets $F \subseteq \{1, 2, \ldots, s\}$ such that $lcm\{n_i | i \in F\}$ is unique ([2, 20]).

In the present thesis work, we have studied relationships between combinatorial and algebraic properties of certain classes of monomial ideals that are obtained from combinatorial objects such as *Permutohedron* and *Multipermutohedron.* We have computed the multigraded Betti numbers of an Alexander dual of a multipermutohedron ideal. We use the standard formula for multigraded Betti numbers of a monomial ideal $I$,

$$\beta_{i-1,\mathbf{b}}(I) = \dim_k \widetilde{H}_{|Supp(\mathbf{b})|-i-1}(K_{\mathbf{b}}(I); k); \quad i \geq 1,$$

where $K_{\mathbf{b}}(I)$ is the *lower Koszul simplicial complex* of $I$ (Definition 1.1.18) and support $Supp(\mathbf{b}) = \{i : b_i > 0\}$. In fact, $\beta_{i-1}(I) = \sum_{\mathbf{b}} \beta_{i-1,\mathbf{b}}$. We observed that the lower Koszul simplicial complex of an Alexander dual of a multipermutohedron ideal in a given degree is a join of skeletons of simplices. This observation turned out to be crucial for computing multigraded Betti numbers of the dual. We also characterize minimality of the cellular resolution of dual of multipermutohedron ideal supported on a "subcomplex" associated to the first barycentric subdivision of an $(n-1)$-simplex. A similar characterization of minimality of the cellular resolution of multipermutohedron ideal is given in [17]. Further, for an Artinian quotient of dual of multipermutohedron ideal, the number of standard monomials is given by number of *generalized parking functions*.

We intended to compute the Betti numbers of all higher powers of the multipermutohedron ideals. However, this problem turned out to be too difficult and at the end, we could only compute the Betti numbers of sum of two multipermutohedron ideals and their Alexander duals. There are large classes of ideals for which explicit formulae for Betti numbers are known. Eliahou and Kervaire [9] have described a class of *splittable monomial ideals*. In fact if $I$ is a monomial ideal with splitting $U$ and $V$, then $\beta_i(I) = \beta_i(U) + \beta_i(V) + \beta_{i-1}(U \cap V)$; $i \geq 0$. Actually the sum of two multipermutohedron ideals is not a splittable monomial ideal. Thus it seems interesting to calculate their multigraded Betti numbers.

We have introduced and studied certain classes of monomial ideals called, *split-multipermutohedron ideals* and *hypercubic ideals*, which are certain variants of multipermutohedron ideals. We have calculated the Betti numbers of these ideals and their Alexander duals. The standard monomials of an Artinian quotient of the Alexander dual of a hypercubic ideal corresponds to certain combinatorial objects called, *restricted $\lambda$-parking functions*. Using free resolution of this quotient, we have obtained an explicit combinatorial formula for counting the restricted $\lambda$-parking functions.

## 1.1 Basic Notions of Commutative Algebra

This section consists of basic concepts of commutative algebra. We have compiled important concepts and results needed to make the dissertation self-contained. These results have been taken from the books by Bruns and Herzog [5], Miller and Sturmfels[20], Herzog and Hibi [13], Villareal [32].

**Definition 1.1.1.** Let $R = k[x_1, x_2, \dots, x_n]$ be a $\mathbb{N}^n$-graded polynomial ring and $M$ be a $\mathbb{N}^n$-graded $R$-module. If the vector space dimension $\dim_k(M_{\mathbf{a}})$ is finite for all $\mathbf{a} \in \mathbb{N}^n$, then the formal power series

$$H(M; \mathbf{x}) = \sum_{\mathbf{a} \in \mathbb{N}^n} \dim_k(M_{\mathbf{a}}).\mathbf{x}^{\mathbf{a}}$$

is the $\mathbb{N}^n$-*graded Hilbert series* of M. If $x_i = t$ for all $i$, then we get the $\mathbb{Z}$-graded Hilbert series.

**Remark 1.1.2.**

1. The $\mathbb{N}^n$-graded Hilbert series of $R = k[x_1, x_2, \dots, x_n]$ is given by $H(R; \mathbf{x}) = \prod_{i=1}^{n} \frac{1}{1-x_i}$. Also $H(R(-\mathbf{a}); \mathbf{x}) = \frac{\mathbf{x}^{\mathbf{a}}}{\prod_{i=1}^{n}(1-x_i)} = \mathbf{x}^{\mathbf{a}}.H(R; \mathbf{x})$.
2. Hilbert series is additive i.e. $H(M; \mathbf{x}) = H(M'; \mathbf{x}) + H(M'', \mathbf{x})$, for a graded short exact sequence of the form $0 \to M' \to M \to M'' \to 0$.

### 1.1.1 Minimal Free Resolutions

**Definition 1.1.3.** A complex $\mathcal{F}$ of $R$-modules is a sequence of modules $F_i$ and homomorphisms $\partial_i : F_i \to F_{i-1}$ such that the composition $\partial_i \circ \partial_{i+1}$ are all zero. Then the $R$-module

$$H_i(\mathcal{F}) = \frac{ker(\partial_i : F_i \to F_{i-1})}{Im(\partial_{i+1} : F_{i+1} \to F_i)}$$

is called the $i^{th}$ homology module. The homology modules of a complex is a measure of the extent of deviation of the complex from being exact.

A *free resolution* of $R$-module $M$ is a complex

$$\mathcal{F} : \ldots . \to F_n \stackrel{\partial_n}{\to} F_{n-1} \to \ldots \to F_1 \stackrel{\partial_1}{\to} F_0 \to 0$$

of free $R$-modules such that $coker\partial_1 = M$ and $\mathcal{F}$ is exact (except at $0^{th}$ position) Further image of $\partial_i, im\ \partial_i = ker\ \partial_{i-1}, i \geq 1$ is called the $i^{th}$ *syzygy module* of $M$. A resolution $\mathcal{F}$ is a graded free resolution if $R$ is a graded ring, the $F_i$ are graded free modules and the maps are homogeneous of degree 0. Of course, only graded modules can have graded free resolutions. If for some $n < \infty$ we have $F_{n+1} = 0$, but $F_i \neq 0$ for $0 \leq i \leq n$, then we shall say that $\mathcal{F}$ is a *finite free resolution* of length $n$.

**Definition 1.1.4.** A free(graded) complex

$$\mathcal{F} : \ldots . \to F_n \stackrel{\partial_n}{\to} F_{n-1} \to \ldots \to F_1 \stackrel{\partial_1}{\to} F_0$$

over a polynomial ring $R = k[x_1, \ldots x_n]$ is minimal if the differentials in the complex $\mathcal{F} \bigotimes R/\mathfrak{m}$ are all zero; that is , for each $n$, the image $\partial_n : F_n \to F_{n-1}$ is contained in $\mathfrak{m}F_{n-1}$, where $\mathfrak{m} = \langle x_1, x_2, \ldots, x_n \rangle$.

**Theorem 1.1.5.** *If $R = k[x_1, x_2, \ldots, x_n]$, then every finitely generated graded $R$-module has a finite graded free resolution of length $\leq n$.*

**Definition 1.1.6.** Let $M$ and $N$ be two $R$-modules. Consider the chain complex

$$\mathcal{C} \otimes N : \ldots \to C_{n+1} \otimes_R N \to C_n \otimes_R N \to \ldots \to C_1 \otimes_R N \to C_0 \otimes_R N \to 0,$$

where

$$\mathcal{C} \ldots \to C_{n+1} \to C_n \to \ldots \to C_1 \to C_0 \to 0$$

is a projective resolution of $M$. Then the $i^{th}$ homology of the complex $\mathcal{C} \otimes N$ is called the $i^{th}$ *torsion module* of $M$ and $N$ denoted by $Tor_i^R(M, N)$ and we have $Tor_0^R(M, N) = M \otimes_R N$.

**Definition 1.1.7.** Let $R = k[x_1, x_2, \ldots, x_n]$ and a free complex

$$\mathcal{F} : 0 \to F_l \stackrel{\partial_l}{\to} F_{l-1} \to \ldots \to F_1 \stackrel{\partial_1}{\to} F_0 \to 0$$

be a minimal free resolution of a finitely generated $\mathbb{N}^n$-graded module $M$ with

$$F_i = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} R(-\mathbf{a})^{\beta_{i,\mathbf{a}}},$$

then the $i^{th}$ *Betti number* of $M$ in degree $\mathbf{a}$ is the invariant $\beta_{i,\mathbf{a}} = \beta_{i,\mathbf{a}}(M)$. In other words, the Betti numbers measures the minimum number of generators required in degree $\mathbf{a}$ for any $i^{th}$ syzygy module of $M$.

**Lemma 1.1.8.** *The $i^{th}$ Betti number of an $\mathbb{N}^n$-graded module $M$ in degree* $\mathbf{a}$ *equals the vector space dimension* $\dim_k Tor_i^R(k, M)_{\mathbf{a}}$.

### 1.1.2 Simplicial homology

Let $\Delta$ be a simplicial complex on $\{1, \ldots, n\}$. We may consider the reduced chain complex of $k$-vector spaces.

$$\widetilde{\mathcal{C}}.(\Delta; k) : 0 \to C_{n-1} \stackrel{\partial_{n-1}}{\to} C_{n-2} \to \ldots \to C_1 \stackrel{\partial_1}{\to} C_0 \stackrel{\partial_0}{\to} C_{-1} \to 0,$$

where $C_i = \bigoplus_{\substack{\sigma \in \Delta \\ \dim \sigma = i}} k e_\sigma$. The boundary maps $\partial_i$ are defined by setting $\text{sign}(j, \sigma) = (-1)^{r-1}$ if $j$ is the $r^{th}$ element of the set $\sigma \subseteq \{1, \ldots, n\}$, written in increasing order, and

$$\partial_i(e_\sigma) = \sum_{j \in \sigma} sign(j, \sigma) e_{\sigma \smallsetminus j}$$

If $i < -1$ or $i > n-1$, then $C_i = 0$ and $\partial_i = 0$ by definition. It can be checked that $\partial_i \circ \partial_{i+1} = 0$.

**Definition 1.1.9.** For each integer $i$, the $k$-vector space

$$\widetilde{H}_i(\Delta; k) = ker(\partial_i)/im(\partial_{i+1})$$

in homological degree $i$ is the $i^{th}$ reduced homology of $\Delta$ over $k$.

**Definition 1.1.10.** For a monomial ideal $I$ and a degree $\mathbf{b} \in \mathbb{N}^n$, define

$$K^{\mathbf{b}}(I) = \{\text{squarefree vectors } \tau | \mathbf{x}^{\mathbf{b}-\tau} \in \text{I}\}$$

to be *upper Koszul simplicial complex* of $I$ in degree $\mathbf{b}$.

**Theorem 1.1.11.** *Given a vector* $\mathbf{b} \in \mathbb{N}^n$*, the* $i^{th}$ *Betti numbers of* $I$ *and* $S/I$ *in degree* $\mathbf{b}$ *can be expressed as*

$$\beta_{i,\mathbf{b}}(I) = \beta_{i+1,\mathbf{b}}(S/I) = \dim_k \widetilde{H}_{i-1}(K^{\mathbf{b}}(I); k).$$

**Proof.** For a complete proof, we refer to [20](Theorem 1.34). □

### 1.1.3 Alexander duals of monomial ideals

Let $n$ be a positive integer and $[n] = \{1, 2, \ldots, n\}$. Let $\Delta$ be a simplicial complex on the vertex set $[n]$. The *Alexander dual* $\Delta^*$ of $\Delta$ is a simplicial complex on $[n]$ given by

$$\Delta^* = \{A \subseteq [n] : [n] - A \notin \Delta\}.$$

For any subset $A \subseteq [n]$, $\mathbf{x}_A = \prod_{i \in A} x_i$ is a squarefree monomial in the polynomial ring $R = k[x_1, x_2, \ldots, x_n]$ over a field $k$. The *Stanley-Reisner ideal* $I_\Delta$ of the simplicial complex $\Delta$ is defined to be the squarefree monomial ideal

$$I_\Delta = \langle \mathbf{x}_A : A \text{ is a minimal nonface of } \Delta \rangle$$

in $R$. Now the *Alexander dual of the squarefree monomial ideal* $I_\Delta$ is defined to be the Stanley-Reisner ideal $I_{\Delta^*}$ of the Alexander dual $\Delta^*$.

**Example 1.1.12.** Consider a simplicial complex $\Delta = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}\}$. Then $\Delta^* = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$.

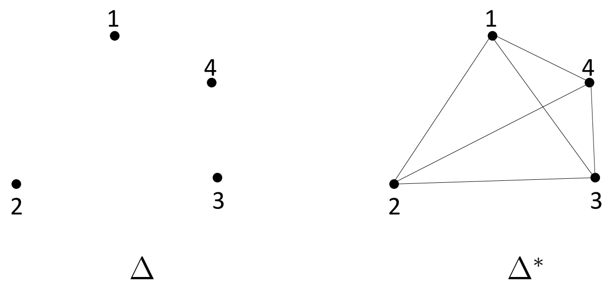



Figure 1.1: Simplicial complex $\Delta$ and its dual.

Thus $I_\Delta = \langle x_i x_j : 1 \le i < j \le 4 \rangle$ and $I_{\Delta^*} = \langle x_1x_2x_3, x_1x_2x_4, x_2x_3x_4, x_1x_3x_4 \rangle$

In fact, there is a close homological connection between $\Delta$ and its $\Delta^*$ known as "combinatorial Alexander duality".

**Theorem 1.1.13.** (Combinatorial Alexander duality) *Let $\Delta$ be a simplicial complex on the vertex set $[n]$ such that $\Delta \neq 2^{[n]}$. Then*

$$\widetilde{H}_i(\Delta^*; k) \cong \widetilde{H}^{n-i-3}(\Delta; k).$$

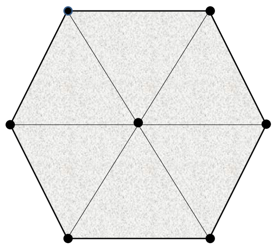
Combinatorial Alexander duality is a special case of topological Alexander duality, which states that there is an isomorphism between the reduced homology of a proper closed topological subspace of a sphere and the reduced cohomology of complement of the subspace.

**Example 1.1.14.** Let $\Delta$ be defined as in Example 1.1.12, then $\Delta^*$ is 1-skeleton of a 3-simplex. Now $S^2 - \Delta$ retracts to $\Delta^*$. Thus homology of dual $\Delta^*$ and the complement $S^2 - \Delta$ are same.

**Definition 1.1.15.** Let $\sigma \in \Delta$ be a face. The *link* of $\sigma$ inside $\Delta$ is

$$link_\Delta(\sigma) = \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta \text{ and } \tau \cap \sigma = \emptyset\}.$$

**Example 1.1.16.** Let $v$ be the vertex at the center of a hexagon as shown in the figure. Then *link* $(v)$ is the subcomplex constituting the boundary of the hexagon.



**Proposition 1.1.17.** (**Hochster's formula, dual version**) All nonzero Betti numbers of $I_\Delta$ and $S/I_\Delta$ lie in squarefree degree $\sigma$

$$\beta_{i,\sigma}(I_\Delta) = \beta_{i+1,\sigma}(S/I_\Delta) = \dim_k \widetilde{H}_{i-1}(link_{\Delta^*}(\bar{\sigma}); k),$$

where $\bar{\sigma} = \{1, 2, \ldots, n\} \smallsetminus \sigma$.

**Proof.** See [20], Corollary 1.40. □

**Definition 1.1.18.** For each vector $\mathbf{b} \in \mathbb{N}^n$, define $\mathbf{b}'$ by subtracting 1 from each nonzero coordinate of $\mathbf{b}$. Given a monomial ideal $I$ and a degree $\mathbf{b} \in \mathbb{N}^n$, the *(lower) Koszul simplicial complex* of $S/I$ in degree $\mathbf{b}$ is

$$K_{\mathbf{b}}(I) = \{\text{squarefree vectors } \tau \preceq \mathbf{b} | \ \mathbf{x}^{\mathbf{b}'+\tau} \notin \mathbf{I}\}.$$

The following result can be thought of as a dual version of Theorem 1.1.11.

**Theorem 1.1.19.** *Given a vector* $\mathbf{b} \in \mathbb{N}^n$ *with support supp*$(\mathbf{b}) = \{i \mid b_i \neq 0\}$, *the Betti numbers of* $I$ *and* $S/I$ *in degree* $\mathbf{b}$ *can be expressed as*

$$\beta_{i-1,\mathbf{b}}(I) = \beta_{i,\mathbf{b}}(S/I) = \dim_k \widetilde{H}^{|supp(\mathbf{b})|-i-1}(K_{\mathbf{b}}(I); k).$$

**Proof.** See [20], Theorem 5.11. □

**Remark 1.1.20.** Since we are working over a field $k$, one may substitute reduced homology for reduced cohomology when calculating Betti numbers, since they have the same dimension.

The notion of Alexander duals of a squarefree monomial ideals has been extended to monomial ideals by Miller [21]. Let $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{N}^n$. Then $\mathbf{x^b}$ denotes the monomial $\prod_{i=1}^n x_i^{b_i}$ and $\mathbf{m^b}$ denotes the monomial ideal $\langle x_i^{b_i} : b_i > 0\rangle$ in the standard polynomial ring $R = k[x_1, x_2, \ldots, x_n]$ over a field $k$. Consider a monomial ideal $I$ in the polynomial ring $R$. Then $I$ has a unique minimal set of monomial generators and all the (monomial) primary components of $I$ are unique. Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $\mathbb{N}^n$ such that $\{\mathbf{x^b} : \mathbf{b} \in \mathcal{A}\}$ be the set of minimal generators of $I$ and $\{\mathbf{m^b} : \mathbf{b} \in \mathcal{B}\}$ be the set of (monomial) primary components of $I$. Thus, we have

$$I = \langle \mathbf{x^b} : \mathbf{b} \in \mathcal{A}\rangle = \bigcap \{\mathbf{m^b} : \mathbf{b} \in \mathcal{B}\}.$$

Choose $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ such that $\mathbf{a} - \mathbf{b} \in \mathbb{N}^n$ for all $\mathbf{b} \in \mathcal{A}$. In other words, all the minimal generator $\mathbf{x^b}$ of $I$ divide $\mathbf{x^a}$. Whenever $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ with $\mathbf{a} - \mathbf{b} \in \mathbb{N}^n$, we set $\mathbf{a} \ominus \mathbf{b} \in \mathbb{N}^n$ by defining its $i^{th}$ coordinate

$$(\mathbf{a} \ominus \mathbf{b})_i = \begin{cases} a_i + 1 - b_i & \text{if } b_i > 0, \\ 0 & \text{if } b_i = 0. \end{cases}$$

**Definition 1.1.21.** The *Alexander dual* $I^{[\mathbf{a}]}$ of the monomial ideal $I$ with respect to $\mathbf{a}$ is defined to be the monomial ideal

$$I^{[\mathbf{a}]} = \bigcap \{\mathbf{m}^{\mathbf{a} \ominus \mathbf{b}} : \mathbf{b} \in \mathcal{A}\}.$$

Equivalently, $I^{[\mathbf{a}]} = \langle \mathbf{x}^{\mathbf{a} \ominus \mathbf{b}} : \mathbf{b} \in \mathcal{B}\rangle$.

**Example 1.1.22.** Let $\mathbf{a} = (3, 3)$. Then

$$\begin{aligned} I &= \langle x^3, x^2y\rangle \\ &= \langle x^2\rangle \cap \langle y, x^3\rangle \end{aligned} \quad \Rightarrow \quad \begin{aligned} I^{[\mathbf{a}]} &= \langle x^2, xy^3\rangle \\ &= \langle x\rangle \cap \langle x^2, y^3\rangle. \end{aligned}$$

**Example 1.1.23.** Consider an ideal $I = \langle xy^2z^3, xy^3z^2, x^2yz^3, x^2y^3z, x^3yz^2, x^3y^2z\rangle$. Then we have

$$I = \langle x^3, y^3, z^3\rangle \cap \langle x^2, y^2\rangle \cap \langle x^2, z^2\rangle \cap \langle y^2, z^2\rangle \cap \langle x\rangle \cap \langle y\rangle \cap \langle z\rangle.$$

The Alexander dual $I^{[(3,3,3)]}$ is given by

$$I^{[(3,3,3)]} = \langle xyz, x^2y^2, x^2z^2, y^2z^2, x^3, y^3, z^3\rangle.$$

**Remarks 1.1.24.**

1. The Alexander dual is indeed a duality in the sense that $(I^{[\mathbf{a}]})^{[\mathbf{a}]} = I$. Also, the Alexander dual $(I_\Delta)^{[\mathbf{1}]}$ of a Stanley-Reisner ideal $I_\Delta$ with respect to $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{N}^n$ is precisely $I_{\Delta^*}$. Therefore, the notion of Alexander duality of monomial ideals introduced by Miller turns out to be an appropriate generalization.

2. Let $\mathbf{a}_I$ be the exponent on the LCM of all minimal generators of the monomial ideal $I$. Then we define the (*tight*) *Alexander dual* $I^* = I^{[\mathbf{a}_I]}$. The only inadequacy of this notion is that $(I^*)^*$ need not equal $I$, unlike $(I^{[\mathbf{a}]})^{[\mathbf{a}]} = I$ .

**Example 1.1.25.** Let $I = \left\langle x^3, x^2y\right\rangle$ be a monomial ideal. Then $\mathbf{a}_I = (3, 1) \Rightarrow I^* = \left\langle x^2, xy\right\rangle$. Now $\mathbf{a}_{I^*} = (2, 1)$, therefore $(I^*)^* = \left\langle x^2, xy\right\rangle \neq I$.

The minimal generating set of an Alexander dual of a monomial ideal with respect to a vector $\mathbf{a}$ can be described using the following proposition.

**Proposition 1.1.26.** *Suppose that all minimal generators of ideal $I$ divide $\mathbf{x}^{\mathbf{a}}$. If $\mathbf{b} \leq \mathbf{a}$ then $\mathbf{x}^{\mathbf{b}}$ lies outside $I$ if and only if $\mathbf{x}^{\mathbf{a}-\mathbf{b}} \in I^{[\mathbf{a}]}$.*

**Proof.** See [20], Prop 5.23. □

### 1.1.4 Convex polytopes

**Definition 1.1.27.** A subset $K$ of $\mathbb{R}^n$ is *convex* if for any two points $x_0, x_1 \in K$ the line segment with end points $x_0$ and $x_1$, that is, the set of points $x = (1 - \lambda)x_0 + \lambda x_1, \lambda \in \mathbb{R}, 0 \leq \lambda \leq 1$ belongs to $K$. The intersection of any non-empty family of convex sets is again convex. Now, *the convex hull*, $Conv(X)$, of a subset $X \subset \mathbb{R}^n$ to be the intersection of all convex sets $K \subset \mathbb{R}^d$ which contains $X$. The convex hull of $X$ can also be described as the set of all *convex combinations* of finite subsets of $X$, that is, as the set of linear combinations $\lambda_1 x_1 + \lambda_1 x_2 + \ldots + \lambda_r x_r$, with $x_i \in X, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1$. A *polytope* is the convex hull of a finite set of points in $\mathbb{R}^n$.

There is another alternative description of a polytope as the intersection of a finite number of closed half-spaces. Let $\mathbf{a} \in \mathbb{R}^d, \mathbf{a} \neq 0$ and $\beta \in \mathbb{R}$, the set

$$H = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{a}, \mathbf{x} \rangle = \beta\},$$

where $\langle \mathbf{a}, \mathbf{x} \rangle = \sum_{i=1}^{d} a_i x_i$, is a *hyperplane* with normal vector $\mathbf{a}$. Thus, we define the *closed half space* as

$$H^{+} = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{a}, \mathbf{x} \rangle \geq \beta\}$$

and

$$H_{-} = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{a}, \mathbf{x} \rangle \leq \beta\}.$$

For a detailed study of convex polytopes we refer to [34].

### 1.1.5 Cellular Resolution

**Definition 1.1.28.** A *polyhedral cell complex* $X$ is a finite collection of convex polytopes in $\mathbb{R}^n$, called faces of X, satisfying two properties:

1. If $\mathcal{P}$ is a polytope in $X$ and F is a face of $\mathcal{P}$, then F is in $X$.

2. If $\mathcal{P}$ and $\mathcal{Q}$ are in $X$, then $\mathcal{P} \cap \mathcal{Q}$ is a face of both $\mathcal{P}$ and $\mathcal{Q}$.

We now define labeling of a polyhedral cell complex

**Definition 1.1.29.** Let $X$ be a polyhedral cell complex with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$. A *labeling* on $X$ consists of the following data:

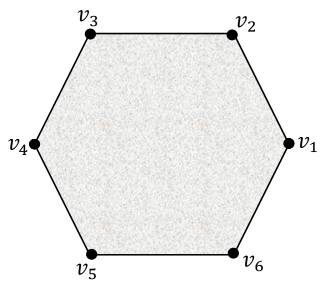
*i*) Each vertex $v$ of $X$ corresponds to a monomial $\mathbf{x}^{\mathbf{a}_v}$ called the *monomial label* of $v$.

*ii*) Monomial label on each face $F$ of $X$ is $\mathbf{x}^{\mathbf{a}_F} = lcm(\mathbf{x}^{\mathbf{a}_v} | v \in F)$.

*iii*) for the empty face $\emptyset$, $\mathbf{x}^{\mathbf{a}_\emptyset} = 1$.

A polyhedral cell complex together with a vertex labeling is called a *labeled cell complex* and we call $\mathbf{a}_F$ the *degree* of the face $F$.

**Example 1.1.30.** Consider the polyhedral cell complex $X$ consisting of faces of a hexagon




Suppose the labels on the vertices are given by

$$\mathbf{a}_{v_1} = (3,1,2),\ \mathbf{a}_{v_4} = (1,3,2),$$

$$\mathbf{a}_{v_2} = (3,2,1),\ \mathbf{a}_{v_5} = (1,2,3),$$

$$\mathbf{a}_{v_3} = (2,3,1),\ \mathbf{a}_{v_6} = (2,1,3).$$

The labeling on edges is given by

$$\mathbf{a}_{\langle v_1,v_2\rangle} = (3,2,2),\ \mathbf{a}_{\langle v_4,v_5\rangle} = (1,3,3),$$

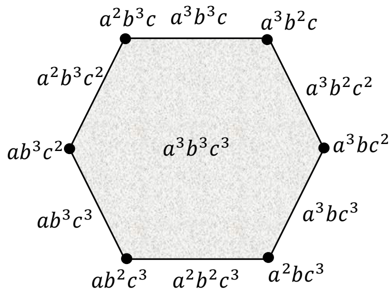
$$\mathbf{a}_{\langle v_2,v_3\rangle} = (3,3,1),\ \mathbf{a}_{\langle v_5,v_6\rangle} = (2,2,3),$$

$$\mathbf{a}_{\langle v_3,v_4\rangle} = (2,3,2),\ \mathbf{a}_{\langle v_6,v_1\rangle} = (3,1,3).$$

and the labeling on the hexagon is given by

$$\mathbf{a}_{\langle v_1,v_2,v_3,v_4,v_5,v_6\rangle} = (3,3,3).$$

The labeled hexagon appears as in the following figure.




Now we proceed to define the reduced chain complex of a polyhedral cell complex. In order to define a reduced chain complex of a polyhedral cell complex $X$, we need to define orientation on its faces. Orientation on the faces of the cell complex $X$ is chosen so that for every oriented face $G$ and its facet $H$, we could define the signature $sign(H,G)$, which is $+1$ if orientation on $G$ induces orientation on $H$, otherwise it is $-1$. Further, the boundary of a face $G$ of $X$ is given by

$$\partial(G) = \sum_{facets\ H \subseteq G} sign(H,G)H, \tag{1.1.1}$$

where the boundary map $\partial$ satisfies the obvious property that $\partial \circ \partial = 0$.

Now for an oriented labeled cell complex $X$, we proceed to define a free complex $\mathbb{F}_*(\mathbf{X})$ and determine condition under which $\mathbb{F}_*(\mathbf{X})$ becomes a free resolution of $R/I(X)$, where $I(X)$ is a monomial ideal $\langle \mathbf{x}^{\mathbf{a}_v} : v \in V\rangle$ generated by vertex labeling on $X$.

Let $\mathcal{F}_i(X)$ denotes the set of $i$-dimensional faces of $X$. Then we consider the chain complex of free $R$-modules

$$\mathbb{F}_*(\mathbf{X}) : \ldots \to F_i \xrightarrow{\partial} F_{i-1} \xrightarrow{\partial} \ldots \to F_0, \tag{1.1.2}$$

where

$$F_i = \bigoplus_{F \in \mathcal{F}_{i-1}(X)} R(-\mathbf{a}_F)F, \ \partial F = \sum_{facets\ G \subseteq F} sign(G, F)\mathbf{x}^{\mathbf{a}_F - \mathbf{a}_G} G.$$

Since $X$ is an oriented cell complex the signature $sign(G, F)$ is consistently defined for every pair $(G, F)$, where $G$ is a facet of $F$. We see that $\partial : F_i \to F_{i-1}$ is in fact an $\mathbb{N}^n$-graded $R$-module homomorphism. Thus $\mathbb{F}_*(\mathbf{X})$ is an $\mathbb{N}^n$-graded chain complex of free $R$-modules. It is clear that if $\mathbb{F}_*(\mathbf{X})$ is an exact $R$-complex, then it gives us a free resolution of $R/I(X)$.

**Definition 1.1.31.** *The free complex $\mathbb{F}_*(\mathbf{X})$ supported on the polyhedral cell complex $X$ is called the cellular free complex supported on $X$ and $\mathbb{F}_*(\mathbf{X})$ is called a cellular resolution if it is acyclic*(or exact).

**Definition 1.1.32.** Given two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$, we write $\mathbf{a} \preceq \mathbf{b}$ and say that $\mathbf{a}$ *precedes* $\mathbf{b}$ if $\mathbf{b} - \mathbf{a} \in \mathbb{N}^n$. For each $\mathbf{b} \in \mathbb{N}^n$, let $X_{\preceq \mathbf{b}}$ be the subcomplex of $X$ consisting of the faces with degree $\mathbf{a}_F \preceq \mathbf{b}$ and let $X_{\prec \mathbf{b}}$ be the subcomplex of $X_{\preceq \mathbf{b}}$ obtained by deleting the faces of degree $\mathbf{b}$.

**Theorem 1.1.33.** *The cellular free complex $\mathbb{F}_*(\mathbf{X})$ supported on $X$ is a cellular resolution if and only if $X_{\preceq \mathbf{b}}$ is acyclic over $k$ for all $\mathbf{b} \in \mathbb{N}^n$. When $\mathbb{F}_*(\mathbf{X})$ is acyclic, it is a free resolution of $R/I(X)$, where $I(X) = \langle \mathbf{x}^{\mathbf{a}_v} \mid v \in X \text{ is a vertex } \rangle$ is generated by the monomial labels on vertices. Moreover the cellular resolution $\mathbb{F}_*(\mathbf{X})$ is a minimal resolution if and only if any two comparable faces $F' \subset F$ of the complex $X$ have distinct degrees $\mathbf{a}_F \neq \mathbf{a}_{F'}$.*

**Proof.** See [2](Proposition 1.2). □

## 1.2 Betti numbers of multipermutohedron ideals

Let $\mathbb{N}$ be a set of non-negative integers. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ with $0 \leq u_1 < u_2 < \ldots < u_n$. For any permutation $\pi$ of $\mathbf{u}$, $\pi\mathbf{u} = (\pi u_1, \ldots, \pi u_n) \in \mathbb{N}^n$. These $n!$ vertices $\pi\mathbf{u}$ span an $(n-1)$-dimensional polytope in $\mathbb{R}^n$ called a *Permutohedron* $P(\mathbf{u})$. For $n = 4$ and $\mathbf{u} = (1, 2, 3, 4)$, permutohedron is a 3-dimensional polytope as shown in the following figure.

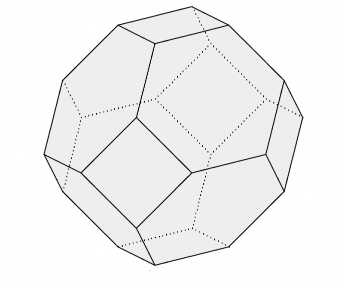


Figure 1.2: Permutohedron having 24 vertices, 36 edges, 14 faces.

The associated ideal generated by the monomial vertex labels $\mathbf{x}^{\pi\mathbf{u}}$ of $P(\mathbf{u})$ is called a *permutohedron ideal.*

Now consider $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ such that the first $m_1$ coordinates be equal to $u_1$ and next $m_2$ coordinates be equal to $u_{m_1+1}$ and so on. In other words,

$$u_1 = \ldots = u_{s_1} < u_{s_1+1} = \ldots = u_{s_2} < u_{s_2+1} = \ldots < u_{s_{l-1}+1} = \ldots = u_n,$$

where $s_i = \sum_{\alpha=1}^{i} m_\alpha$ for $0 \leq i \leq l$ and $s_0 = 0$. Note that $m_i \geq 1$ and $s_l = \sum_{i=1}^{l} m_i = n$. Now set $\mathbf{m}(\text{or } \mathbf{m_u}) = (m_1, m_2, \ldots, m_l) \in \mathbb{N}^l$. For a permutation $\pi$ of $\mathbf{u}$, let $\pi\mathbf{u} = (\pi u_1, \ldots, \pi u_n) \in \mathbb{R}^n$. The convex hull of all the points $\pi\mathbf{u}$; $\pi$ a permutation of $\mathbf{u}$, is an $(n-1)$-dimensional (except when $u_1 = u_2 = \ldots = u_n$) polytope $P_{\mathbf{m}}(\mathbf{u})$ called *a multipermutohedron.* Note that for $\mathbf{m} = \mathbf{1}$, $P_{\mathbf{1}}(\mathbf{u})$ is a permutohedron. Each vertex $\pi\mathbf{u}$ of the multipermutohedron $P(\mathbf{u})$ is naturally labeled with the monomial $\mathbf{x}^{\pi\mathbf{u}}$ making it a labeled polyhedral cell complex. The associated monomial ideal $I(\mathbf{u}) = \langle \mathbf{x}^{\pi\mathbf{u}} : \pi$ a permutation of $\mathbf{u}\rangle$ is called a *multipermutohedron ideal.*

### 1.2.1 Facial description of multipermutohedron ideal

The facial description of a multipermutohedron is given by Billera-Sarangarajan [3]. A similar facial description has been given in [17]. Firstly we give the facial description of permutohedron ideal. Every $i$-face of $P_{\mathbf{1}}(\mathbf{u})$ corresponds to a chain of subsets of $[n]$ of length $n-i$.

$$\emptyset = \sigma_0 \subset \sigma_1 \subset \ldots \subset \widehat{\sigma}_{j_1} \subset \sigma_{j_1+1} \subset \ldots \subset \widehat{\sigma}_{j_i} \subset \sigma_{j_i+1} \subset \ldots \subset \sigma_n = [n], \qquad (1.2.1)$$

The inequality description of $P_{\mathbf{1}}(\mathbf{u})$ is as follows(see [20])

$$P_{\mathbf{1}}(\mathbf{u}) = \left\{ v \in \mathbb{R}^n : \sum_{j=1}^{n} v_j = \sum_{j=1}^{n} u_j \text{ and } \sum_{j\in\sigma} v_j \geq u_1 + \ldots + u_{|\sigma|},\ \forall\ \sigma \subset [n] \right\}.$$

The $i$-face given by chain 1.2.1 is actually the $i$-face

$$\left\{ v \in P_{\mathbf{1}}(\mathbf{u}) : \sum_{j\in\sigma_k} v_j = u_1 + \ldots + u_k\ \forall\ k \in [n] - \{j_1, \ldots, j_i\} \right\}$$

in the inequality description of $P_{\mathbf{1}}(\mathbf{u})$.

Now we give the facial description of multipermutohedron ideal. Consider such a chain of subsets of $[n]$

$$\emptyset = \sigma_0 \subset \sigma_1 \subset \ldots \subset \sigma_{s_1} \subset \sigma_{s_1+1} \subset \ldots \subset \sigma_{s_i} \subset \sigma_{s_i+1} \subset \ldots \subset \sigma_n = [n]. \qquad (1.2.2)$$

Using the inequality description of permutohedron it has been shown in [17] that a vertex of $P_{\mathbf{m}}(\mathbf{u})$ depends on the sets $\emptyset = \sigma_0, \sigma_{s_1}, \sigma_{s_2}, \ldots, \sigma_{s_j}, \ldots, \sigma_{s_l} = [n]$ (with $\sigma_0 = \emptyset$; $\sigma_n = [n]$ fixed) in the given chain (1.2.2) and not on the intermediate members in between $\sigma_{s_{j-1}} \subset \sigma_{s_j}$ for any $j$. Thus we mark the subsets $\sigma_{s_j}$ $(0 \leq j \leq n)$ in the chain (1.2.2) by underlining them. This shows that a 0-face (or a vertex) of $P_{\mathbf{m}}(\mathbf{u})$ is represented by a *marked* chain of subsets of $[n]$ of length $n$

$$\emptyset = \underline{\sigma_0} \subset \sigma_1 \subset \ldots \subset \underline{\sigma_{s_1}} \subset \sigma_{s_1+1} \subset \ldots \subset \underline{\sigma_{s_i}} \subset \sigma_{s_i+1} \subset \ldots \subset \underline{\sigma_n} = [n], \quad (1.2.3)$$

with the understanding that any two such chains are *equivalent* if their corresponding marked subsets are equal. Thus the vertices of $P_{\mathbf{m}}(\mathbf{u})$ are in one-one correspondence with the equivalence classes of the marked chains of subsets of $[n]$ of length $n$ of the form (1.2.3). Therefore, the number of vertices of $P_{\mathbf{m}}(\mathbf{u})$ is given by

$$f_0(P_{\mathbf{m}}(\mathbf{u})) = \prod_{\alpha=1}^{l} \binom{s_\alpha}{s_{\alpha-1}} = \frac{n!}{\prod_{\alpha=1}^{l} m_\alpha!}. \quad (1.2.4)$$

Now we describe 1-faces (or edges) of $P_{\mathbf{m}}(\mathbf{u})$. It has been shown in [17] that a 1-face (or an edge) of $P_{\mathbf{m}}(\mathbf{u})$ corresponds to a marked chains of subsets of $[n]$ of length $n-1$ obtained by deleting any one of marked subsets $\sigma_{s_j}$ (except $\sigma_0 = \emptyset$ and $\sigma_n = [n]$) from an equivalence class of marked chain of subsets of $[n]$ of length $n$ representing a vertex and marking the subsets $\sigma_{s_j-1}$ and $\sigma_{s_j+1}$ if they are not already marked. Again two such marked chain of subsets of $[n]$ of length $n-1$ are equivalent if their marked subsets are the same. Thus a 1-face of $P_{\mathbf{m}}(\mathbf{u})$ is represented by an equivalence class of marked chain of subsets of $[n]$ of length $n-1$ of the form

$$\emptyset = \underline{\sigma_0} \subset \sigma_1 \subset \ldots \subset \underline{\sigma_{s_1}} \subset \ldots \subset \underline{\sigma_{s_j-1}} \subset \widehat{\sigma}_{s_j} \subset \underline{\sigma_{s_j+1}} \subset \ldots \subset \underline{\sigma_n} = [n]. \quad (1.2.5)$$

This gives an inductive procedure to describe faces of a multipermutohedron in terms of equivalence classes of marked chains of subsets of $[n]$. Now, in order to describe $i$-faces of $P_{\mathbf{m}}(\mathbf{u})$, we introduce a notion of $\mathbf{m}$-*admissible subsets*. A subset $[p,q] = \{x \in \mathbb{N} : p \leq x \leq q\}$ is called an *integral interval*, provided $p, q \in \mathbb{N}$ with $p \leq q$. If $p < q$, we write $(p, q]$ for $[p+1, q]$.

**Definition 1.2.1.** Let $\mathbf{m} = (m_1, m_2, \ldots, m_l)$ with $1 \leq m_i$ and $\sum\limits_{\alpha=1}^{l} m_\alpha = n$. A subset $J \subseteq [n-1]$ is said to be $\mathbf{m}$-*admissible* if $([0, n] - J) \cap [s_{j-1}, s_j]$ is either an empty set or an integral interval for $1 \leq j \leq l$. The set of all $\mathbf{m}$-admissible subsets is denoted by $\mathcal{A}_{\mathbf{m}}$. If $J \subseteq [n-1]$ is $\mathbf{m}$-admissible with size $\mid J \mid= i$, we write

$J \in \mathcal{A}_{\mathbf{m}}(\mid i \mid)$.

Let $J \in \mathcal{A}_{\mathbf{m}}(\mid i \mid)$ and $[0,n] - J = \{\lambda_0, \lambda_1, \ldots, \lambda_{n-i-1}, \lambda_{n-i} = n\}$; $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_{n-i} = n$. Since $([0,n] - J) \cap [s_{j-1}, s_j]$ is either empty or an integral interval, let $C_J = \{j \in [1,l] : ([0,n] - J) \cap [s_{j-1}, s_j] \neq \emptyset\}$ and for each $j \in C_J$, $([0,n] - J) \cap [s_{j-1}, s_j] = [\nu_j, \mu_j]$ with $\nu_j \leq \mu_j$. For every such $J \in \mathcal{A}_{\mathbf{m}}(\mid i \mid)$, one associates an equivalence class of marked chain of subsets of $[n]$ of length $n - i$ given by

$$\emptyset = \underline{\sigma_0} \subset \sigma_{\lambda_1} \subset \sigma_{\lambda_2} \subset \ldots \subset \sigma_{\lambda_{n-i-1}} \subset \underline{\sigma_{\lambda_{n-i}}} = [n] \tag{1.2.6}$$

and a subset $\sigma_\lambda$ in this chain is marked if $\lambda = \nu_j$ or $\lambda = \mu_j$ for some $j \in C_J$. We now have a description for the faces of $P_{\mathbf{m}}(\mathbf{u})$.

**Lemma 1.2.2.** *The $i$-faces of the multipermutohedron $P_{\mathbf{m}}(\mathbf{u})$ are in one-one correspondence with the equivalence classes of marked chains of subsets of $[n]$ of length $n - i$ associated with some $J \in \mathcal{A}_{\mathbf{m}}(\mid i \mid)$.*

**Proof.** See [17](Lemma 2.2). □

### 1.2.2 Multigraded Betti numbers

Multigraded Betti numbers of multipermutohedron ideals have been calculated in [17]. We need the following definition to describe the formula for the multigraded Betti numbers of multipermutoedron ideals.

**Definition 1.2.3.** A subset $J = \{j_1, j_2, \ldots, j_t\} \subseteq [n]$ is said to be $\mathbf{m}$(or $\mathbf{m_u}$)-*isolated* if $j_t = n$ and $J \cap (s_{j-1}, s_j]$ is either empty or singleton for $1 \leq j \leq l$. Thus for each $\alpha$, there is a unique $i_\alpha$ with $s_{i_\alpha - 1} + 1 \leq j_\alpha \leq s_{i_\alpha}$. In other words, $J$ contains at most one point from each of the integral intervals $(s_{j-1}, s_j]$ $(1 \leq j \leq l)$, which is the reason for the name $\mathbf{m}$(or $\mathbf{m_u}$)-*isolated* . For $\mathbf{u} = (u_1, \ldots, u_n)$, set $\mathbf{b}(J) = \sum_{\alpha=1}^{t} u_{j_\alpha} E(j_{\alpha-1}, j_\alpha)$ and set $\mathbf{m_u}$-weight $wt_{\mathbf{m_u}}(J) = wt_{\mathbf{m_u}}(\mathbf{b}(J)) = \sum_{\alpha=1}^{t} (s_{i_\alpha - 1} - j_{\alpha-1})$, where $j_0 = 0$. The set of all $\mathbf{m_u}$-isolated subsets of $[n]$ is denoted by $\mathcal{I}_{\mathbf{m_u}}$. If $J \subseteq [n]$ is an $\mathbf{m_u}$-isolated subset with $wt_{\mathbf{m_u}}(J) = i$, we write $J \in \mathcal{I}_{\mathbf{m_u}}(\langle i \rangle)$.

**Theorem 1.2.4.** ([17], Theorem 3.5) For $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{N}^n$ and $i \geq 1$, let $\beta_{i,\mathbf{b}}(I(\mathbf{u}))$ be an $i$-th multigraded Betti number of multipermutohedran ideal $I(\mathbf{u})$ in the degree $\mathbf{b}$. Then $\beta_{i,\mathbf{b}}(I(\mathbf{u}))$ are as follows:

i) For $J = \{j_1, j_2, \ldots, j_t\} \in \mathcal{I}_{\mathbf{m_u}}$,

$$\beta_{i,\mathbf{b}(J)}(I(\mathbf{u})) = \Big[\prod_{\alpha=1}^{t} \binom{j_\alpha - j_{\alpha-1} - 1}{s_{i_\alpha - 1} - j_{\alpha-1}}\Big]\delta_{i,wt_{\mathbf{m_u}}(J)},$$

where $J \cap (s_{i_\alpha-1}, s_{i_\alpha}] = \{j_\alpha\}$. If $\pi$ is a permutation of $\mathbf{b}(J)$, then

$$\beta_{i,\pi\mathbf{b}(J)}(I(\mathbf{u})) = \beta_{i,\mathbf{b}(J)}(I(\mathbf{u})).$$

ii) If $\mathbf{b} \neq \pi\mathbf{b}(J)$ for any $J \in \mathcal{I}_{\mathbf{m_u}}(\langle i \rangle)$ and any permutation $\pi$ of $\mathbf{b}(J)$, then

$$\beta_{i,\mathbf{b}}(I(\mathbf{u})) = 0.$$

**Corollary 1.2.5.** ([17], Corollary 3.7) Let $\beta_i(I(\mathbf{u})$ be the $i$-th Betti number of a multipermutohedron ideal $I(\mathbf{u})$. For $J = \{\lambda_1, \lambda_2, \ldots, \lambda_t\} \in \mathcal{I}_{\mathbf{m}}(\langle i \rangle)$, we set

$$\beta_i^J = \prod_{\alpha=1}^{t}\Big[\binom{\lambda_\alpha - \lambda_{\alpha-1} - 1}{s_{i_\alpha-1} - \lambda_{\alpha-1}}\binom{\lambda_\alpha}{\lambda_{\alpha-1}}\Big], \text{ where } J \cap (s_{i_\alpha-1}, s_{i_\alpha}] = \{\lambda_\alpha\}.$$

Then $\beta_i(I(\mathbf{u}) = \sum_{J \in \mathcal{I}_{\mathbf{m}}(\langle i \rangle)} \beta_i^J$.

Now as a consequence of corollary 1.2.5 and lemma 1.2.2 the minimality of the cellular resolution supported on multipermutohedron has been characterized in [17].

**Theorem 1.2.6.** ([17], Theorem 3.9) The cellular free complex associated to a multipermutohedron $P_{\mathbf{m}}(\mathbf{u})$ is the minimal resolution of $R/I(\mathbf{u})$ if and only if $m_\alpha = 1$ for $2 \leq \alpha \leq l$.

Now we give some examples to illustrate Theorem 1.2.6.

**Example 1.2.7.** Let $\mathbf{u} = (a, a, b, b), 0 < a < b$. Then the multipermutohedron $P(a, a, b, b)$ is a (solid) octahedron consisting of six vertices, twelve edges, eight triangular 2-dimensional faces and one 3-dimensional face as shown in figure 1.3. We will show that the cellular resolution supported by $P(a, a, b, b)$ is the nonminimal free resolution of multipermutohedron ideal $I(a, a, b, b)$. We check this by determining all $\mathbf{m_u}$-isolated subsets. We see that subsets $J_0 = \{2, 4\}, J_1 = \{1, 4\}$ and $J_2 = \{4\}$ are unique $\mathbf{m_u}$-isolated subsets of weights $0, 1$ and 2, respectively. Thus the nonzero Betti numbers of $I(a, a, b, b)$ are given by

$$\beta_0 = \beta_0^{J_0} = 6, \quad \beta_1 = \beta_1^{J_1} = 8 \text{ and } \beta_2 = \beta_2^{J_2} = 3.$$

Since $f_1(P(a, a, b, b)) = 12 > 8 = \beta_1(I(a, a, b, b))$, the cellular resolution supported by $P(a, a, b, b)$ is nonminimal.

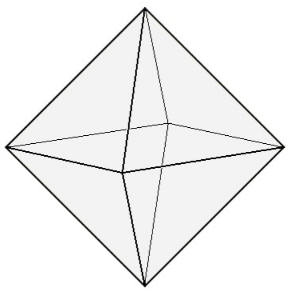


Figure 1.3: Octahedron

**Example 1.2.8.** Let $\mathbf{u} = (a, a, b, c), 0 < a < b < c$. Then the multipermutohedron $P(a, a, b, c)$ is a 3-dimensional polytope(truncated octahedron) consisting of twelve vertices, eighteen edges, eight 2-dimensional faces (four hexagonal and four triangular faces) and one 3-dimensional face as shown in figure 1.4. In this case the cellular resolution supported by $P(a, a, b, c)$ is the minimal resolution of the multipermutohedron ideal $I(a, a, b, c)$ in $k[x, y, z, w]$. We check this by determining all $\mathbf{m_u}$-isolated subsets. The subset $J_0 = \{2, 3, 4\}$ is the unique $\mathbf{m_u}$-isolated subset of weight 0 and $\beta_0^{J_0} = 12$. The subsets $J_1 = \{1, 3, 4\}$ and $J_1' = \{2, 4\}$ are $\mathbf{m_u}$-isolated subsets of weight 1 with $\beta_1^{J_1} = 12$ and $\beta_1^{J_1'} = 6$. The subsets $J_2 = \{1, 4\}$ and $J_2' = \{3, 4\}$ are $\mathbf{m_u}$-isolated subsets of weight 2 with $\beta_2^{J_2} = 4$ and $\beta_2^{J_2'} = 4$.

Finally, $J_3 = \{4\}$ is the unique $\mathbf{m_u}$-isolated subset of weight 3 with $\beta_3^{J_3} = 1$. Thus the nonzero Betti numbers of $I(a, a, b, c)$ are given by

$$\beta_0 = 12, \quad \beta_1 = 12 + 6 = 18, \quad \beta_2 = 4 + 4 = 8 \text{ and } \beta_3 = 1.$$

Since $f_i(P(a, a, b, c)) = \beta_i(I(a, a, b, c))$ for all $i$, the cellular resolution supported by $P(a, a, b, c)$ is minimal.

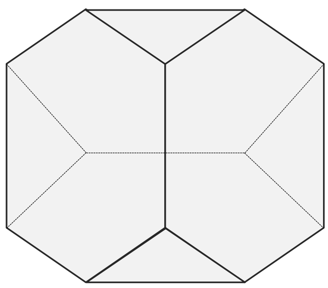


Figure 1.4: Truncated octahedron

## 1.3 Overview of the thesis

In this section, we give a brief overview of the thesis. This thesis contains five chapters of which, the first is an Introduction. Before describing other chapters we shall fix some notations and give important definitions for sake of clarity. Let $R = k[x_1, x_2, \ldots, x_n]$ be the standard polynomial ring over a field $k$. For $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{N}^n$, let $\mathbf{x^b}$ be the monomial $\prod_{i=1}^{n} x_i^{b_i} \in R$. Consider a monomial ideal $I$ in the polynomial ring $R$. Choose $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ so that all the minimal generators of $I$ divide the monomial $\mathbf{x^a}$. The Alexander dual $I^{[\mathbf{a}]}$ of $I$ with respect to $\mathbf{a}$ is a monomial ideal in the polynomial ring $R$. The Alexander dual of a (general) monomial ideal has been introduced by E. Miller (Definition 1.1.21) extending the definition of Alexander dual for squarefree monomial ideal.

Let $I(\mathbf{u})$ be a multipermutohedron ideal. For any integer $c \geq 1$, we consider

$\mathbf{u_n} + \mathbf{c} - \mathbf{1} = (u_n + c - 1, u_n + c - 1, \ldots, u_n + c - 1)$ and the Alexander dual $I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ is given by

$$I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]} = \Big\langle \left(\prod_{i\in A} x_i\right)^{u_n - u_{|A|}+c} : \emptyset \neq A \subset [n] \Big\rangle,$$

except the generating set need not be minimal. If $u_1 \geq 1$, then the quotient $R' = R/I(\mathbf{u})^{[\mathbf{u_n}-\mathbf{c}+\mathbf{1}]}$ is an Artinian $k$-algebra. So one would like to know, what is the dimension $\dim_k(R')$ or equivalently, the number of standard monomials in $R' = R/I(\mathbf{u})^{[\mathbf{u_n}-\mathbf{c}+\mathbf{1}]}$? A solution of this problem is known and it lies in counting *generalized parking functions*. The dimension $\dim_k(R')$ equals the number of $\lambda$-parking functions (Definition 2.2.3), where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$; $\lambda_i = u_n - u_i + c$.

Using a free resolution of $R'$ and its multigraded Hilbert series, we give a simple proof of the Steck determinant formula for counting $\lambda$-parking functions. More precisely, we prove the following result to count the number of $\lambda$-parking functions (Theorem 3.1.1).

**Theorem 1.3.1.** *Let $\mu_{ij} = \frac{(\lambda_{n-i+1})^{j-i+1}}{(j-i+1)!}$ if $1 \leq i \leq j+1$ and $\mu_{ij} = 0$ if $j+1 < i \leq n$. Then the number of $\lambda$-parking functions of length $n$ is given by $(n!)\det[\mu_{ij}]_{n\times n}$.*

To describe a combinatorial formula for all the multigraded Betti numbers of the Alexander dual $I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ of the multipermutohedron ideal we need the following notion. Let $p, q \in \mathbb{N}$ and $p \leq q$. Then $[p, q]$ denotes an integral interval $\{r \in \mathbb{N} : p \leq r \leq q\}$. We also write $(p, q]$ for $[p + 1, q]$. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. For $0 \leq j \leq i$, set $E(j, i) = \sum_{\alpha=j+1}^{i} e_\alpha$.

**Definition 1.3.2.** Let $J = \{j_1, j_2, \ldots, j_t\} \subseteq [n]$ with $0 = j_0 < j_1 < j_2 < \ldots < j_t \leq n$. Then $J$ is said to be a *dual* $\mathbf{m_u}$*-isolated* if $J \cap (s_{j-1}, s_j]$ is either empty or singleton for $1 \leq j \leq l$. Thus for each $\alpha$, there is a unique $i_\alpha$ with $s_{i_\alpha-1} + 1 \leq j_\alpha \leq s_{i_\alpha}$. For $\mathbf{u} = (u_1, \ldots, u_n)$, set $\widetilde{\mathbf{b}}(J) = \sum_{\alpha=1}^{t} \lambda_{j_\alpha} E(j_{\alpha-1}, j_\alpha)$, $\lambda_i = u_n - u_i + c$ and set *dual* $\mathbf{m_u}$*-weight* $dwt_{\mathbf{m_u}}(J) = dwt_{\mathbf{m_u}}(\widetilde{\mathbf{b}}(J)) = \left[\sum_{\alpha=1}^{t}(j_\alpha - s_{i_\alpha-1})\right] - 1$. Also, the size of the support $|Supp(\mathbf{b}(J))| = j_t$. The set of all dual $\mathbf{m_u}$-isolated subsets of $[n]$ is denoted by $\mathcal{I}^*_{\mathbf{m_u}}$.

Using the above definition we prove the following result to calculate the multigraded Betti numbers of the Alexander dual of a multipermutohedron ideal (Theorem 2.3.4 & Corollary 2.3.6).

**Theorem 1.3.3.** *For $\mathbf{b} \in \mathbb{N}^n$ and $i \geq 1$, let $\beta_{i-1,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$ be an $(i-1)^{th}$ multigraded Betti number of $I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ in the degree $\mathbf{b}$. If $u_1 \geq 1$, then the multigraded Betti numbers $\beta_{i-1,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$ are given as follows:*

1. *For $J = \{j_1, j_2, \ldots, j_t\} \in \mathcal{I}^*_{\mathbf{m_u}}$,*

$$\beta_{i-1,\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = \Big[\prod_{\alpha=1}^{t} \binom{j_\alpha - j_{\alpha-1} - 1}{s_{i_\alpha - 1} - j_{\alpha-1}}\Big] \delta_{i-1, dwt_{\mathbf{m_u}}(J)},$$

   *where $J \cap (s_{i_\alpha - 1}, s_{i_\alpha}] = \{j_\alpha\}$. If $\pi$ is a permutation of $\widetilde{\mathbf{b}}(J)$, then*

$$\beta_{i-1,\pi\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = \beta_{i-1,\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}).$$

2. *If $\mathbf{b} \neq \pi\widetilde{\mathbf{b}}(J)$ for any $J \in \mathcal{I}^*_{\mathbf{m_u}}(\langle i-1\rangle)$ and any permutation $\pi$ of $\widetilde{\mathbf{b}}(J)$, then*

$$\beta_{i-1,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = 0.$$

**Corollary 1.3.4.** *Let $\beta_{i-1}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$ be the $i-1$-th Betti number of the Alexander dual $I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$. Suppose $u_1 \geq 1$ and for $J = \{j_1, j_2, \ldots, j_t\} \in \mathcal{I}^*_{\mathbf{m_u}}(\langle i-1\rangle)$, we set*

$$\beta^J_{i-1} = \prod_{\alpha=1}^{t} \Big[\binom{j_\alpha - j_{\alpha-1} - 1}{s_{i_\alpha - 1} - j_{\alpha-1}} \binom{j_{\alpha+1}}{j_\alpha}\Big], \text{ where } J \cap (s_{i_\alpha - 1}, s_{i_\alpha}] = \{j_\alpha\} \text{ and } j_{t+1} = n.$$

*Then* $\beta_{i-1}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = \sum_{J \in \mathcal{I}^*_{\mathbf{m_u}}(\langle i-1\rangle)} \beta^J_{i-1}.$

We also have defined a polyhedral cell complex $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ obtained by modifying the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ and investigated the minimality of the cellular resolution supported by $\mathbf{Bd^m}(\Delta_{n-1})$. More precisely, we prove the following result (Theorem 2.3.9).

**Theorem 1.3.5.** *The cellular resolution supported by* $\mathbf{Bd^m}(\Delta_{n-1})$ *is the minimal resolution of* $R/I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ *if and only if* $m_\alpha = 1$ *for* $2 \leq \alpha \leq l$.

In the third chapter, we have computed multigraded Betti numbers of a sum of two multipermutohedron ideals and their Alexander duals. Since the multigraded Betti numbers of a monomial ideal are given in terms of the reduced homology groups of the lower Koszul simplicial complex the Mayer-Vietoris sequence can be used to compute $\widetilde{H}_i(K_{\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]})$. Also from the above formula of multigraded Betti numbers of an Alexander dual of a multipermutohedron ideal we observe that for any degree $\mathbf{b} \in \mathbf{N}^n$, there is at most one non-zero multigraded Betti number. Therefore many terms in the Mayer-Vietoris sequence are zero. Thus we can calculate the multigraded Betti numbers of $(I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}$ (Theorem 3.2.1). On the similar lines multigraded Betti numbers of sum of two multipermutohedron ideals have also been calculated (Theorem 3.1.1).

In the fourth chapter we defined a notion of *split-multipermutohedron ideal.* Let $n = r + s$ with $r, s \geq 1$ be a positive integer and $\mathfrak{S}_n$ denotes the set of all permutations of $\{1, 2, \ldots, n\}$. Consider the set H of all permutations of $\{1, 2, \ldots, n\}$ of type $(\sigma_1, \sigma_2)$, where $\sigma_1$ is a permutation of $\{1, 2, \ldots, r\}$ and $\sigma_2$ is a permutation of $\{r+1, \ldots, r+s = n\}$. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$. The convex hull of points $\sigma\mathbf{u}$ for $\sigma \in$H is also a polytope, which is the product of multipermutohedrons $P(\mathbf{v}) \times P(\mathbf{w})$, where $\mathbf{v} = (u_1, \ldots, u_r)$ and $\mathbf{w} = (u_{r+1}, \ldots, u_n)$. The associated monomial ideal $\mathcal{I} = \langle \mathbf{x}^{\sigma_1\mathbf{v}}.\mathbf{y}^{\sigma_2\mathbf{w}} \ : (\sigma_1, \sigma_2) \in \text{H} \ \rangle \subseteq k[x_1, x_2, \ldots, x_r, y_1, \ldots, y_s]$, where $y_j = x_{r+j}$ is a called *split-multipermutohedron ideal.* We have computed the multigraded Betti numbers of split-multipermutohedron ideals and their Alexander duals and we have the following results (Theorems 4.1.3 & 4.2.3).

**Theorem 1.3.6.** *The multigraded Betti numbers of a split-multipermutohedron ideal* $\mathcal{I}$ *exist only in the multidegree* $\mathbf{b}$ *of the form* $\mathbf{b} = (\mathbf{b_v}, \mathbf{b_w})$ *and*

$$\beta_{i,\mathbf{b}}(\mathcal{I}) = \beta_{p,\mathbf{b_v}}(I(\mathbf{v}))\beta_{q,\mathbf{b_w}}(I(\mathbf{w})),$$

*where* $\mathbf{b_v} = \mathbf{b}(J)$, $\mathbf{b_w} = \mathbf{b}(J')$ *for* $J \in \mathcal{I}_{\mathbf{m_v}}$, $J' \in \mathcal{I}_{\mathbf{m_w}}$ *and* $wt_{\mathbf{m_v}}(\mathbf{b_v}) = p$, $wt_{\mathbf{m_w}}(\mathbf{b_w}) = q$ *with* $p + q = i$.

**Theorem 1.3.7.** *Let* $\mathbf{a} = u_r E(0,r) + u_n E(r,n) = (\mathbf{v_r}, \mathbf{w_s})$ *and* $\mathcal{I}^{[\mathbf{a}]}$ *be the Alexander dual of split-multipermutohedron ideal* $\mathcal{I}$ *with respect to* $\mathbf{a}$*. The multigraded Betti numbers of* $R/\mathcal{I}^{[\mathbf{a}]}$ *exists only in the multidegree* $\mathbf{b}$ *of the form* $\mathbf{b} = (\mathbf{b_v}, \mathbf{b_w})$ *and*

$$\beta_{i,\mathbf{b}}(R/\mathcal{I}^{[\mathbf{a}]}) = \beta_{p,\mathbf{b_v}}(R_1/I(\mathbf{v})^{[\mathbf{v_r}]})\beta_{q,\mathbf{b_w}}(R_2/I(\mathbf{w})^{[\mathbf{w_s}]}),$$

*where* $\mathbf{b_v} = \widetilde{\mathbf{b}}(J)$, $\mathbf{b_w} = \widetilde{\mathbf{b}}(J')$ *for* $J \in \mathcal{I}^*_{\mathbf{m_v}}$, $J' \in \mathcal{I}^*_{\mathbf{m_w}}$ *and* $p = dwt_{\mathbf{m_v}}(\mathbf{b_v})$, $q = dwt_{\mathbf{m_w}}(\mathbf{b_w})$ *with* $p + q = i$.

In the fifth and final chapter, we have defined so called, *hypercubic ideals*. Let $W$ be the set of permutations of $\{1, 2, \ldots, n\}$ such that apart from the leading element, the number $k$ can be placed only if either $k+1$ or $k-1$ already appears. In other words, $W$ is inductively defined as the set of all permutations $\sigma \in \mathfrak{S}_n$ such that $\sigma(1)$ is arbitrary and $\sigma(j) = k$ for $j > 1$ if either $\sigma(i) = k+1$ or $\sigma(i) = k-1$ for some $i < j$. It is easy to see that $|W| = 2^{n-1}$. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ with $1 \leq u_1 < u_2 < \ldots < u_n$, then the convex hull of the $2^{n-1}$ points $\sigma\mathbf{u}, \sigma \in W$ is an $(n-1)$-dimensional hypercube in $\mathbb{R}^n$. Thus the monomial ideal $J(\mathbf{u}) = \langle \mathbf{x}^{\sigma\mathbf{u}} : \sigma \in W \rangle$ is called a *hypercubic ideal.* The cellular free complex supported on the hypercube is a minimal free resolution of $J(\mathbf{u})$. Thus the $i^{th}$ Betti number $\beta_i(J(\mathbf{u}))$ is the number of $i$-faces of the hypercube $= 2^{n-i-1}\binom{n-1}{i}$. We will prove the following theorem about the Alexander dual of hypercubic ideal (Theorem 5.1.4).

**Theorem 1.3.8.** *Let* $\mathbf{u_n} = u_n E(0,n) = (u_n, u_n, \ldots, u_n)$*. Then the Alexander dual* $J(\mathbf{u})^{[\mathbf{u_n}]}$ *of hypercubic ideal is given by*

$$J(\mathbf{u})^{[\mathbf{u_n}]} = \langle \prod_{j \in T} x_j^{\mu_{j,T}} \mid \emptyset \neq T = \{j_1, j_2, \ldots, j_t\} \subseteq [n]; j_1 < j_2 < \ldots < j_t \rangle,$$

*where* $\mu_{j_1,T} = u_n - u_t + 1$ *and* $\mu_{j_i,T} = u_n - u_{t+j_i-i} + 1$ *for* $i \in \{2, 3, \ldots, t\}$.

The vertices of the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $(n-1)$-simplex can be naturally labeled with the minimal generators of the Alexander dual $J(\mathbf{u})^{[\mathbf{u_n}]}$

of hypercubic ideals and the cellular free complex $\mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1}))$ supported on $\mathbf{Bd}(\Delta_{n-1})$ is a minimal free resolution of $R/J(\mathbf{u})^{[\mathbf{u_n}]}$. Thus $i^{th}$ Betti number of $J(\mathbf{u})^{[\mathbf{u_n}]}$ is the number of $i$-faces of the first barycentric subdivision of an $(n-1)$-simplex $= (i+1)!S_{n+1,i+2}$, where $S_{n+1,i+2}$ is the *Stirling number of the second type.* We recall that the Stirling number $S_{n,k}$ of second type is the number of partitions of the set $\{1,\ldots,n\}$ into $k$ nonempty blocks.

A hypercubic ideal $J(\mathbf{u})$ has the following minimal property. For each non empty set $B \subseteq \mathfrak{S}_n$, we associate a monomial ideal $I_B = \langle \mathbf{x}^{\sigma\mathbf{u}} | \sigma \in B \rangle$. Clearly $I_{\mathfrak{S}_n} = I(\mathbf{u})$ is a permutohedron ideal and the cellular free complex $\mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1}))$ supported on $\mathbf{Bd}(\Delta_{n-1})$ is a minimal free resolution of the quotient $R/I(\mathbf{u})^{[\mathbf{u_n}]}$. Now consider the set

$$\begin{aligned} \mathcal{A} \;=\; & \{B \subseteq \mathfrak{S}_n | \mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1})) \text{ supported on } \mathbf{Bd}(\Delta_{n-1}) \text{ is a minimal free} \\ & \text{resolution of the quotient } R/I_B^{[\mathbf{u_n}]}\}. \end{aligned}$$

Then $W \in \mathcal{A}$ is a minimal element of $\mathcal{A}$. Thus hypercubic ideal $J(\mathbf{u}) = I_W$ has a minimal property (Theorem 5.1.6).

**Theorem 1.3.9.** *For a non empty subset $B$ of $\mathfrak{S}_n$, the ideal $I_W = J(\mathbf{u})$ has a property that if $B \supseteq W$, then the cellular free complex $\mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1}))$ supported on $\mathbf{Bd}(\Delta_{n-1})$ is a minimal free resolution of the quotient $R/I_B^{[\mathbf{u_n}]}$ and if $B \subsetneq W$, then the cellular resolution of $R/I_B^{[\mathbf{u_n}]}$ is not minimally supported on $\mathbf{Bd}(\Delta_{n-1})$.*

For $u_1 \geq 1$, the quotient $R/J(\mathbf{u})^{[\mathbf{u_n}]}$ is an Artinian $k$-algebra and its standard monomials corresponds to so called, *restricted $\lambda$-parking functions.*

**Definition 1.3.10.** A sequence $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ of positive integers is called a *restricted $\lambda$-parking function* of length $n$, if there exists a permutation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathfrak{S}_\mathfrak{n}$ such that $p_{\alpha_i} - 1 < \mu_{\alpha_i, T_i}$ with $T_i = [n] - \{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}\}$ for $i = 1, 2, \ldots, n$. The quantity $\mu_{\alpha_i, T_i}$ is as in the Theorem 1.3.8.

Again using free resolution and Hilbert series of $R/J(\mathbf{u})^{[\mathbf{u_n}]}$, we have obtained a combinatorial formula for number of restricted $\lambda$-parking functions. We have the following theorem (Proposition 5.2.5).

**Theorem 1.3.11.** *Let $J(\mathbf{u})$ be a hypercubic ideal and $J(\mathbf{u})^{[\mathbf{u_n}]}$ be its Alexander dual with respect to $\mathbf{u_n}$ and $R' = R/J(\mathbf{u})^{[\mathbf{u_n}]}$, Then the number of restricted $\lambda$-parking functions of length $n$ is given as follows.*

$$\sum_{i=1}^{n}(-1)^{n-i} \sum_{\emptyset \subsetneq A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_i = [n]} \prod_{q=1}^{i} \prod_{j \in A_q - A_{q-1}} \mu_{j,\, A_q},$$

*where $\mu_{j,\, A_q}$ is defined above.*

# Chapter 2

# Multipermutohedron Ideals and their Alexander Duals

In this chapter we study the Alexander duals of multipermutohedron ideals. The standard monomials of an Artinian quotient of such a dual correspond bijectively to some $\lambda$-parking functions, and many interesting properties of these Artinian quotients are obtained by Postnikov and Shapiro [23]. Using the multigraded Hilbert series of an Artinian quotient of an Alexander dual of multipermutohedron ideals, we obtained a simple proof of Steck determinant formula for enumeration of $\lambda$-parking functions. A combinatorial formula for all the multigraded Betti numbers of an Alexander dual of multipermutohedron ideals are also obtained.

## 2.1 Alexander duals of multipermutohedron ideals

Let $I(1, 2, \ldots, n)$ be a permutohedron ideal. The Alexander dual of the permutohedron ideal $I(1, 2, \ldots, n)$ with respect to $\mathbf{n} = (n, n, \ldots, n)$ is given by $I(1, 2, \ldots, n)^{[\mathbf{n}]} = \left\langle \left(\prod_{i \in A} x_i\right)^{n-|A|+1} : \emptyset \neq A \subset [n] \right\rangle$. In fact, the quotient $R/I(1, 2, \ldots, n)^{[\mathbf{n}]}$ of an Alexander dual of the permutohedron ideal is an Artinian $k$-algebra and the dimension of this quotient is $\dim_k(R/I(1, 2, \ldots, n)^{[\mathbf{n}]}) = (n + 1)^{n-1}$. In other words, the number of standard monomials in the Artinian quotient $R/I(1, 2, \ldots, n)^{[\mathbf{n}]}$ is precisely $(n+1)^{n-1}$, which is the number of labeled trees on $(n+1)$ vertices. Thus the monomial ideal $I(1,2,\ldots,n)^{[\mathbf{n}]}$ is called a *tree ideal.* The vertices of the first barycentric subdivision of an $(n-1)$-simplex can be naturally labeled with the minimal generators of the tree ideal $I(1,2,\ldots,n)^{[\mathbf{n}]}$ and the free resolution of tree ideal supported by the *first barycentric subdivision* $\mathbf{Bd}(\Delta_{n-1})$ of an $(n-1)$-simplex $\Delta_{n-1}$ is minimal.

The (tight) Alexander dual $I(\mathbf{u})^* = I(\mathbf{u})^{[\mathbf{u_n}]}$, where $\mathbf{u_n} = (u_n, u_n, \ldots, u_n)$. We have considered the Alexander duals $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ of the multipermutohedron ideal $I(\mathbf{u})$ with respect to $\mathbf{u_n} + \mathbf{c} - \mathbf{1} = (u_n + c - 1, \ldots, u_n + c - 1)$ for $c \geq 1$. We now describe the Alexander dual of multipermutohedron ideal in terms of its minimal generating set by the following lemma.

**Lemma 2.1.1.** *The minimal generators of the Alexander dual $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ of the multipermutohedron ideal is given by*

$$I(\mathbf{u})^{[\mathbf{u_n+c-1}]} = \Big\langle \Big(\prod_{j\in A} x_j\Big)^{u_n - u_{|A|} + c} : A \subseteq [n],\ |A| = s_i + 1 \text{ for } 0 \leq i < l \text{ and } u_{|A|} \geq 1 \Big\rangle.$$

*Therefore, the quotient $R/(I(\mathbf{u})^{[\mathbf{u_n+c-1}]})$ is an Artinian $k$-algebra if and only if $u_1 \geq 1$.*

**Proof.** Suppose all the minimal generators of a monomial ideal $I$ in $R$ divides $\mathbf{x^a}$. Then, for any $\mathbf{b} = (b_1, \ldots, b_n)$ with $\mathbf{a} - \mathbf{b} \in \mathbb{N}^n$, the monomial $\mathbf{x^b} \notin I$ if and only if $\mathbf{x^{a-b}} \in I^{[\mathbf{a}]}$ (Proposition 1.1.26). Clearly, $\mathbf{x^{a-b}}$ is a minimal generator of $I^{[\mathbf{a}]}$ precisely when $\mathbf{b}$ is maximal. If $u_{s_{i+1}} \geq 1$ then $\mathbf{b} = (u_{s_i+1} - 1)E(0, s_i + 1) + (u_n + c - 1)E(s_i + 1, n)$ or its permutation is a maximal with $\mathbf{a} - \mathbf{b} \in \mathbb{N}^n$ and $\mathbf{x^b} \notin I(\mathbf{u})$. Suppose $\mathbf{x^b} \in I(\mathbf{u})$, which means $\mathbf{b} \succeq \mathbf{u}$. Which implies that $u_{s_{i+1}} - 1 \geq u_{s_{i+1}}$, which is a contradiction. This proves that $\mathbf{x^b} \notin I(\mathbf{u})$. Now we show that $\mathbf{b}$ is maximal with $\mathbf{b} \preceq \mathbf{a}$ and $\mathbf{x^b} \notin I(\mathbf{u})$. Consider the vector $\mathbf{b} + \mathbf{e}_j$ for $1 \leq j \leq s_{i+1}$. Then our aim is to produce some $\mathbf{c} = \sigma\mathbf{u}$ for some $\sigma \in \mathfrak{S}_n$ such that $\mathbf{b} + \mathbf{e}_j \geq \mathbf{c}$. Note that it is enough to take $j = s_{i+1}$. Then $\mathbf{b} + \mathbf{e}_{s_{i+1}} = (u_{s_{i+1}} - 1)E(0, s_i) + u_{s_{i+1}}E(s_i, s_{i+1}) + (u_n + c - 1)E(s_{i+1}, n) \geq \mathbf{u}$. Thus we see that $\mathbf{x}^{\mathbf{u_n+c-1-b}}$ is a minimal generator of $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$. $\square$

**Example 2.1.2.** Let $\mathbf{u} = (2, 2, 4, 4, 5)$ and $c = 2$. Then we have $\mathbf{a} = \mathbf{u_n} + \mathbf{c} - \mathbf{1} = (6, 6, 6, 6, 6)$ and

$$\begin{aligned} I^{[\mathbf{a}]} &= \langle x^5, y^5, z^5, t^5, w^5, (xyz)^3, (xyt)^3, (xyw)^3, \\ &\quad (xzt)^3, (xzw)^3, (xtw)^3, (yzt)^3, (yzw)^3, (ytw)^3, (ztw)^3, (xyztw)^2\rangle. \end{aligned}$$

**Remark 2.1.3.** It follows from Lemma 2.1.1 that Alexander dual of a multipermutohedron ideal is a sum of special multipermutohedron ideals. In fact, the tree ideal $I(1, 2, \ldots, n)^{[\mathbf{n}]}$ is a sum of multipermutohedron ideals

$$I(0, \ldots, 0, n) + I(0, \ldots, 0, n-1, n-1) + \ldots + I(0, 2, 2, \ldots, 2) + I(1, 1, \ldots, 1).$$

This gives another motivation for studying multipermutohedron ideals.

## 2.2 $\lambda$-parking functions

The quotient $R' = R/I(\mathbf{u})^{[\mathbf{u_n-c+1}]}$ is an Artinian $k$-algebra, if $u_1 \geq 1$. We have seen that the $\dim_k(R/I(1, 2, \ldots, n)^{[\mathbf{n}]}) = (n+1)^{n-1}$. In case of multipermutohedron ideal $I(\mathbf{u})$ one would like to know, what is the dimension $\dim_k(R')$ or equivalently, the number of standard monomials in $R' = R/I(\mathbf{u})^{[\mathbf{u_n-c+1}]}$? Answer to this problem is known and it lies in counting $\lambda$-parking functions. The dimension $\dim_k(R')$ equals the number of $\lambda$-parking functions, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$; $\lambda_i = u_n - u_i + c$. Using a free resolution of $R'$ and its multigraded Hilbert series, we give a simple proof of the Steck determinant formula for counting $\lambda$-parking functions (Theorem 3.1.1). Actually $\lambda$-parking functions are generalization of parking function, so we firstly recall parking functions.

**Definition 2.2.1.** Suppose there are $n$ cars $C_1, C_2, \ldots, C_n$ and there are linearly ordered $n$ parking spaces that can be numbered as $1, 2, \ldots, n$. Suppose each car $C_i$ has a preferred parking place, say $a_i \in [n]$ and it goes directly to the place $a_i$ for parking and if $a_i$ is already occupied than the car $C_i$ can be parked to the next available space in the linear ordering. A sequence $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is called a *parking function* of length $n$ if all cars can be parked. Equivalently, it can be seen that a sequence $(a_1, a_2, \ldots, a_n)$ of positive integers is a *parking function of length* $n$ if its non-decreasing rearrangement $b_1 \leq b_2 \leq \ldots \leq b_n$ satisfies $b_i \leq i \ \forall \ i$ [30].

**Theorem 2.2.2.** *The number of parking functions of length $n$ is $(n+1)^{n-1}$.*

An easy proof of the above theorem is due to Polak [10].

**Definition 2.2.3.** For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, a sequence $(p_1, p_2, \ldots, p_n)$ of positive integers is said to be a $\lambda$-*parking function of length $n$* if its nondecreasing rearrangement $q_1 \leq q_2 \leq \ldots \leq q_n$ satisfy $q_i \leq \lambda_{n-i+1}$, $\forall i$ . A $\lambda$-parking function for $\lambda = (n, n-1, n-2, \ldots, 1)$ is clearly a parking function.

**Lemma 2.2.4.** *A monomial $\mathbf{x}^{\mathbf{p}} = x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$ is a standard monomial in the Artinian $k$-algebra $R' = R/(I(\mathbf{u})^{[\mathbf{u_n+c-1}]})$ if and only if $\mathbf{p}+\mathbf{1} = (p_1+1, p_2+1, \ldots, p_n+1)$ is a $\lambda$-parking function for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) = (u_n - u_1 + c, u_n - u_2 + c, \ldots, u_n - u_n + c)$. Thus, the multigraded Hilbert series of $R'$ is given by $H(R', \mathbf{x}) = \sum_{\mathbf{p} \in \Lambda_n} \mathbf{x}^{\mathbf{p-1}}$, where $\Lambda_n$ is the set of all $\lambda$-parking functions of length $n$. Also,* $\dim_k(R') = H(R', \mathbf{1}) = |\Lambda_n|$.

**Proof.** Suppose $\mathbf{p}+\mathbf{1} = (p_1+1, p_2+1, \ldots, p_n+1)$ is not a $\lambda$-parking function. Thus there is a nondecreasing rearrangement $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ of $\mathbf{p}+\mathbf{1}$ such that $q_j > \lambda_{n-j+1} = u_n - u_{n-j+1} + c$ for some $j$. Equivalently, there are at least $n-j+1$ indices $i_1, i_2, \ldots, i_{n-j+1}$ such that $p_{i_r} \geq u_n - u_{n-j+1} + c$ for $1 \leq r \leq n-j+1$. This condition holds if and only if $\mathbf{x}^{\mathbf{p}}$ is divisible by $(\prod_{i \in A} x_i)^{u_n - u_{n-j+1} + c}$ with $A = \{i_1, i_2, \ldots, i_{n-j+1}\}$. Hence, $\mathbf{x}^{\mathbf{p}} \in I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$. This shows that $\mathbf{p}+\mathbf{1} = (p_1+1, p_2+1, \ldots, p_n+1)$ is not a $\lambda$-parking function $\Leftrightarrow$ $\mathbf{x}^{\mathbf{p}}$ is not a standard monomial in $R'$. Since the multigraded Hilbert series is the sum of all standard monomials, the second and third parts of the lemma follows. $\square$

We now proceed to give a proof of Steck determinant formula for counting $\lambda$-parking functions. Consider the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $n-1$-simplex $\Delta_{n-1}$. A vertex of $\mathbf{Bd}(\Delta_{n-1})$ corresponds to a nonempty subset $A \subseteq [n]$

and hence it is naturally labeled with the monomial $(\prod_{\alpha\in A} x_\alpha)^{u_n-u_{|A|}+c}$ in $R$. Also, an $(i-1)$-dimensional face of $\mathbf{Bd}(\Delta_{n-1})$ corresponds to a tuple $(A_1, A_2, \ldots, A_i)$ of nonempty subsets of $[n]$ with $\emptyset = A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_i$ and the monomial label on this $i-1$-face is

$$\prod_{j=1}^{i}\left(\prod_{\alpha\in A_j-A_{j-1}} x_\alpha\right)^{u_n-u_{|A_j|}+c}.$$

Thus, $\mathbf{Bd}(\Delta_{n-1})$ is a labeled simplicial complex. Let $X$ be the labeled polyhedral cell complex and $\mathbb{F}_*(X)$ be the associated cellular chain complex(see, Equation 1.1.2), then the multigraded Hilbert series of $R/I(\mathbf{X})$ is given by

$$\begin{aligned} H(R/I(\mathbf{X}),\mathbf{x}) &= \sum_{i=0}^{\dim(\mathbf{X})+1}(-1)^i H(F_i,\mathbf{x}) \\ &= \sum_{i=0}^{\dim(\mathbf{X})+1}(-1)^i \sum_{\sigma\in\mathcal{F}_{i-1}} \frac{\mathbf{x}^{\nu(\sigma)}}{(1-x_1)(1-x_2)\ldots(1-x_n)}. \end{aligned}$$

The cellular free complex $\mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1}))$ associated to the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ is in fact a free resolution of the quotient $R' = R/(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$ ([23]). This resolution is usually nonminimal, but it can be used to calculate the multigraded Hilbert series $H(R',\mathbf{x})$ of the quotient $R'$. We have,

$$H(R',\mathbf{x}) = \frac{1}{\prod_{j=1}^{n}(1-x_j)}\sum_{i=0}^{n}(-1)^i \sum_{(A_1,\ldots,A_i)\in\mathcal{F}_{i-1}}\left[\prod_{j=1}^{i}\left(\prod_{\alpha\in A_j-A_{j-1}} x_\alpha\right)^{u_n-u_{|A_j|}+c}\right]. \tag{2.2.1}$$

**Proposition 2.2.5.** *The number of $\lambda$-parking functions of length $n$ is given by*

$$|\Lambda_n| = (n!)\sum_{i=0}^{n}(-1)^{n-i}\left\{\sum_{0=t_0<t_1<\ldots<t_{i-1}<t_i=n}\left(\prod_{j=1}^{i}\frac{(\lambda_{t_j})^{t_j-t_{j-1}}}{(t_j-t_{j-1})!}\right)\right\}.$$

**Proof.** From Lemma 2.2.4, we have

$$|\Lambda_n| = H(R', \mathbf{1}) = \lim_{\substack{x_1\to 1,\\ \dots,\\ x_n\to 1}} H(R', \mathbf{x}) = \lim_{\substack{x_1\to 1,\\ \dots,\\ x_n\to 1}} \frac{Q(\mathbf{x})}{\prod_{j=1}^{n}(1-x_j)},$$

where the polynomial $Q(\mathbf{x})$, in view of Equation 2.2.1, is

$$Q(\mathbf{x}) = \sum_{i=0}^{n}(-1)^i \sum_{(A_1,\dots,A_i)\in\mathcal{F}_{i-1}(\mathbf{Bd}(\Delta_{n-1}))} \left[\prod_{j=1}^{i}\left(\prod_{\alpha\in A_j - A_{j-1}} x_\alpha\right)^{u_n - u_{|A_j|}+c}\right].$$

Now applying L'Hospital's rule, we see that

$$|\Lambda_n| = \frac{1}{(-1)^n}\left.\frac{\partial^n Q(\mathbf{x})}{\partial x_1 \partial x_2 \dots \partial x_n}\right|_{\mathbf{x}=\mathbf{1}}.$$

In the partial derivative $\frac{\partial^n Q(\mathbf{x})}{\partial x_1\partial x_2\dots\partial x_n}$, term corresponding to the tuple $(A_1,\dots,A_i)$ survives only if $|A_i| = n$. Putting $|A_j| = t_j, \lambda_j = u_n - u_j + c$, and observing that the number of $i-1$-faces $(A_1,\dots,A_i)$ with $|A_j| = t_j$ is precisely $\frac{n!}{\prod_{j=1}^{i}(t_j - t_{j-1})!}$, we get the desired result. □

**Theorem 2.2.6** (Steck)**.** *Let $A = (\mu_{ij})$ be a $n\times n$ matrix where,*

$$\mu_{ij} = \begin{cases} \frac{(\lambda_{n-i+1})^{j-i+1}}{(j-i+1)!} & \text{if } 1 \le i \le j+1, \\ 0 & \text{if } j+1 < i \le n. \end{cases}$$

$$\begin{aligned} \textit{Then } |\Lambda_n| &= (n!)\det A \\ &= n!\det\begin{bmatrix} \frac{\lambda_n}{1!} & \frac{\lambda_n^2}{2!} & \dots & \frac{\lambda_n^{n-1}}{(n-1)!} & \frac{\lambda_n^n}{n!} \\ 1 & \frac{\lambda_{n-1}^1}{1!} & \dots & \frac{\lambda_{n-1}^{n-2}}{(n-2)!} & \frac{\lambda_{n-1}^{n-1}}{(n-1)!} \\ 0 & 1 & \dots & \frac{\lambda_{n-2}^{n-3}}{(n-3)!} & \frac{\lambda_{n-2}^{n-2}}{(n-2)!} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \frac{\lambda_1}{1!} \end{bmatrix}. \end{aligned}$$

**Proof.** Let $\mathbf{v}_r = \sum_{j=0}^{r} \frac{(\lambda_{n-j})^{r-j}}{(r-j)!} e_{j+1}$ for $1 \leq r \leq n$ and $e_{n+1} = 0$, where $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$. The column vector $\mathbf{v}_r$ is the $r^{th}$ column of the $n \times n$ matrix $[\mu_{ij}]$. Thus

$$\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \dots \wedge \mathbf{v}_n = (\det[\mu_{ij}])\, e_1 \wedge e_2 \wedge \dots \wedge e_n.$$

It is a straightforward verification that the exterior product $\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \dots \wedge \mathbf{v}_n$ equals

$$\sum_{i=0}^{n} (-1)^{n-i} \left\{ \sum_{0=t_0<t_1<\dots<t_{i-1}<t_i=n} \left( \prod_{j=1}^{i} \frac{(\lambda_{t_j})^{t_j-t_{j-1}}}{(t_j - t_{j-1})!} \right) \right\} e_1 \wedge e_2 \wedge \dots \wedge e_n.$$

Since exterior product is distributive and $e_i \wedge e_i = 0$, terms in the product $\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \dots \wedge \mathbf{v}_n$ are obtained by choosing one term from each vector $\mathbf{v}_r$ so that their product give rise to a multiple of $e_1 \wedge e_2 \wedge \dots \wedge e_n$. For $0 \leq i \leq n$ and a tuple $(t_1, t_2, \dots, t_i)$ with $0 = t_0 < t_1 \dots < t_{i-1} < t_i = n$, we choose a term $\mathbf{f}_r$ from the vector $\mathbf{v}_r$ $(1 \leq r \leq n)$ as follows:

$$\mathbf{f}_r = \begin{cases} \frac{(\lambda_{t_j})^{t_j-t_{j-1}}}{(t_j-t_{j-1})!} e_{n-t_j+1} & \text{if } r = n - t_{j-1}, \\ e_{r+1} & \text{if } r \neq n - t_{j-1}. \end{cases}$$

Then $\mathbf{f}_1 \wedge \mathbf{f}_2 \wedge \dots \wedge \mathbf{f}_n$ is clearly equal to

$$\left( \prod_{j=1}^{i} \frac{(\lambda_{t_j})^{t_j-t_{j-1}}}{(t_j - t_{j-1})!} \right) \left( \prod_{j=1}^{i} (-1)^{t_j-t_{j-1}-1} \right) e_1 \wedge e_2 \wedge \dots \wedge e_n. \tag{2.2.2}$$

As $\prod_{j=1}^{i} (-1)^{t_j-t_{j-1}-1} = (-1)^{n-i}$ and the product $\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \dots \wedge \mathbf{v}_n$ is obtained by summing quantity (2.2.2) over all the possible values of $i$ and $(t_1, t_2, \dots, t_i)$, using Proposition 2.2.5, we get

$$(n!)\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \dots \wedge \mathbf{v}_n = |\Lambda_n| e_1 \wedge e_2 \wedge \dots \wedge e_n.$$

This completes the proof. □

**Remark 2.2.7.** The formula for counting $\lambda$-parking functions was known earliar (see [23]). We have obtained an alternative proof of this formula.

## 2.3 Multigraded Betti Numbers

In this section we will calculate the Betti numbers of an Alexander dual of a multipermutohedron ideal. Betti numbers of multipermutohedron ideals has been calculated in [17]. Postnikov and Shapiro studied the monomial ideal $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ in [23], without referring it as an Alexander dual. They explicitly constructed a finite free resolution of so called *monotone monomial ideals* and this free resolution is minimal if the monomial ideal is *strictly monotone*. In particular, the ideal $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ is strictly monotone if $\mathbf{m} = (1, 1, \ldots, 1)$, or equivalently $u_1 < u_2 < \ldots < u_n$, and in this case, the minimal resolution of $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ is the cellular resolution supported by the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $(n-1)$-simplex $\Delta_{n-1}$ with the vertex label $(\prod_{i\in A} x_i)^{u_n - u_{|A|}+c}$ on the vertex corresponding to $\emptyset \neq A \subseteq [n]$ (see [23], Corollary 6.4 and Corollary 12.2). The minimal resolution of the monomial ideal $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ and their Betti numbers for the case $\mathbf{m_u} \neq (1, 1, \ldots, 1)$ are not discussed in [23].

Let $I$ be a monomial ideal in the polynomial ring $k[x_1, x_2, \ldots, x_n]$ and $\mathbf{b} \in \mathbb{N}^n$, then the multigraded Betti numbers of $I$ in degree $\mathbf{b}$ are given by

$$\beta_{i,\mathbf{b}}(I) = \dim_k \widetilde{H}_{i-1}(K^{\mathbf{b}}(I); k) \text{ and } \beta_{i-1,\mathbf{b}}(I) = \dim_k \widetilde{H}^{|Supp(\mathbf{b})|-i-1}(K_{\mathbf{b}}(I); k);\;\; i \geq 1,$$

where the support $Supp(\mathbf{b}) = \{i : b_i > 0\}$ (See Theorem 1.1.11 and Theorem 1.1.19) . We will be primarily using lower Koszul simplicial complexes in computing multigraded Betti numbers of the Alexander dual of multipermutohedron ideal $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$. The minimal generators of an Alexander dual $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ are of the form $(\prod_{j\in A} x_j)^{u_n - u_{|A|}+c}$, where $A \subseteq [n], |A| = s_i + 1$ for $0 \leq i < l$ and $u_{|A|} \geq 1$. Thus

$$\beta_{0,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{u_n+c-1}]}) = 1$$

for $\mathbf{b} = (u_n - u_{s_i+1} + c)E(0, s_i + 1)$ or its permutation, where $0 \leq i < l$ if $u_1 \geq 1$ or $0 < i < l$ if $u_1 = 0$. Therefore,

$$\beta_0(I(\mathbf{u})^{[\mathbf{u_n+c-1}]}) = \begin{cases} \sum_{i=0}^{l-1} \binom{n}{s_i+1} & \text{if } u_1 \geq 1, \\ \sum_{i=1}^{l-1} \binom{n}{s_i+1} & \text{if } u_1 = 0. \end{cases}$$

The dimensions of reduced homology $k$-vector spaces of $i$-skeleton of $n$-simplex and of the join of such skeletons are used in the computation of multigraded Betti numbers. The following simple lemma is well-known to combinatorial topologists (see [16], Theorem 12.3). We have reproduced its proof as in [17] for the sake of completeness

**Lemma 2.3.1.** *Let $\Delta_n = \langle v_0, v_1, \ldots, v_n\rangle$ be an $n$-simplex and $\Delta_n^{(i)} = \{F \in \Delta_n : dim(F) \leq i\}$ be its $i$-skeleton. Suppose $\Delta_{n_j} = \langle v_0^j, v_1^j, \ldots, v_{n_j}^j\rangle; 1 \leq j \leq t$, is a disjoint family of simplices and $\Gamma = \Delta_{n_1}^{(i_1)} * \Delta_{n_2}^{(i_2)} * \ldots * \Delta_{n_t}^{(i_t)}$ is the join of $i_j$-skeleton of $n_j$-simplex $\Delta_{n_j}$ for $1 \leq j \leq t$ with $\dim(\Gamma) = \sum_{j=1}^{t} i_j + (t-1)$. Let $\delta_{i,j}$ be the Kronecker delta. Then*

i) $\dim_k \widetilde{H}_j(\Delta_n^{(i)}; k) = \binom{n}{i+1}\delta_{i,j}$,

ii) $\dim_k \widetilde{H}_j(\Gamma; k) = \left[\prod_{\alpha=1}^{t} \binom{n_\alpha}{i_\alpha+1}\right]\delta_{\dim(\Gamma),j}$.

**Proof.** Since every $j$-cycle of an $n$-simplex $\Delta_n$, is a $j$-boundary and there are no $j$-dimensional faces of $\Delta_n^{(i)}$ for $j > i$, we have $\widetilde{H}_j(\Delta_n^{(i)}; k) = 0$ for $j \neq i$. For $\{j_0, j_1, \ldots, j_{i+1}\} \subseteq [0, n]$ with $j_0 < j_1 < \ldots < j_{i+1}$;

$$\partial\langle v_{j_0}, v_{j_1}, \ldots, v_{j_{i+1}}\rangle = \sum_{\alpha=0}^{i+1}(-1)^\alpha \langle v_{j_0}, \ldots, \widehat{v}_{j_\alpha}, \ldots, v_{j_{i+1}}\rangle$$

is an $i$-cycle of $\Delta_n^{(i)}$ as $\partial \circ \partial = 0$ for boundary operators. If $0 < j_0$, then

$$\partial\langle v_0, v_{j_0}, \ldots, v_{j_{i+1}}\rangle = \langle v_{j_0}, \ldots, v_{j_{i+1}}\rangle - \sum_{\alpha=0}^{i+1}(-1)^\alpha \langle v_0, v_{j_0}, \ldots, \widehat{v}_{j_\alpha}, \ldots, v_{j_{i+1}}\rangle.$$

Using $\partial \circ \partial = 0$, we get

$$\partial\langle v_{j_0}, v_{j_1}, \ldots, v_{j_{i+1}}\rangle = \sum_{\alpha=0}^{i+1}(-1)^\alpha \partial\langle v_0, v_{j_0}, \ldots, \widehat{v}_{j_\alpha}, \ldots, v_{j_{i+1}}\rangle.$$

These are all independent relations among generators. Thus $i$-cycles of $\Delta_n^{(i)}$ of the form $\partial\langle v_0, v_{j_1}, \ldots, v_{j_{i+1}}\rangle$ constitute a basis for $\widetilde{H}_i(\Delta_n^{(i)}; k)$, which shows that $dim_k(\widetilde{H}_i(\Delta_n^{(i)}; k)) = \binom{n}{i+1}$. This proves (i).

For $\Gamma = \Delta_{n_1}^{(i_1)} * \Delta_{n_2}^{(i_2)} * \ldots * \Delta_{n_t}^{(i_t)}$, $dim(\Gamma) = \sum_{\alpha=1}^{t} i_\alpha + (t-1)$. Thus, if $j > dim(\Gamma)$, then $\widetilde{H}_j(\Gamma; k) = 0$. Assuming $j \leq dim(\Gamma)$, every $j$-cycle of $\Gamma$ is a $k$-linear combination of the $j$-cycles of the form $c_1 * c_2 * \cdots * c_t$, where $c_\alpha$ is a $j_\alpha$-cycle of $\Delta_{n_\alpha}^{(i_\alpha)}$ with $j = \sum_{\alpha=1}^{t} j_\alpha + (t-1)$. If $j < dim(\Gamma)$, then there exists an $\alpha$ with $j_\alpha < i_\alpha$. Thus $c_\alpha$ is a boundary, say $\partial \bar{c}_\alpha = c_\alpha$. Now, we have

$$\partial(c_1 * \cdots * \bar{c}_\alpha * \cdots * c_t) = \pm c_1 * \cdots * c_\alpha * \cdots * c_t.$$

Thus $\widetilde{H}_j(\Gamma; k) = 0$ for $j < dim(\Gamma)$. If $j = \sum_{\alpha=1}^{t} i_\alpha + (t-1)$, then $j$-cycles of the form $c_1 * c_2 * \cdots * c_t$ constitute a basis of $\widetilde{H}_j(\Gamma; k)$, where $c_\alpha$ runs over a basis of $\widetilde{H}_{i_\alpha}(\Delta_{n_\alpha}^{(i_\alpha)}; k)$ for all $\alpha$. Now in view of (i), part (ii) follows. □

Let $p, q \in \mathbb{N}$ and $p \leq q$. Then $[p, q]$ denotes an integral interval $\{r \in \mathbb{N} : p \leq r \leq q\}$. We also write $(p, q]$ for $[p+1, q]$. In order to describe multigraded Betti numbers of the Alexander dual $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$, we need the definition of dual $\mathbf{m_u}$-*isolated* set.

**Definition 2.3.2.** Let $J = \{j_1, j_2, \ldots, j_t\} \subseteq [n]$ with $0 = j_0 < j_1 < j_2 < \ldots < j_t \leq n$. Then $J$ is said to be a *dual* $\mathbf{m_u}$-*isolated* if $J \cap (s_{j-1}, s_j]$ is either empty or singleton for $1 \leq j \leq l$. Thus for each $\alpha$, there is a unique $i_\alpha$ with $s_{i_\alpha - 1} + 1 \leq j_\alpha \leq s_{i_\alpha}$. In other words, $J$ contains at most one point from each of the integral intervals $(s_{j-1}, s_j]$ $(1 \leq j \leq l)$, which is the reason for the name $\mathbf{m_u}$-*isolated*. For $\mathbf{u} = (u_1, \ldots, u_n)$, set $\widetilde{\mathbf{b}}(J) = \sum_{\alpha=1}^{t} \lambda_{j_\alpha} E(j_{\alpha-1}, j_\alpha)$, $\quad \lambda_i = u_n - u_i + c$ and set *dual* $\mathbf{m_u}$-*weight* $dwt_{\mathbf{m_u}}(J) = dwt_{\mathbf{m_u}}(\widetilde{\mathbf{b}}(J)) = \left[\sum_{\alpha=1}^{t} (j_\alpha - s_{i_\alpha - 1})\right] - 1$. Also, the size of the support $|Supp(\mathbf{b}(J))| = j_t$. The set of all dual $\mathbf{m_u}$-isolated subsets of $[n]$ is denoted by $\mathcal{I}^*_{\mathbf{m_u}}$. If $J \subseteq [n]$ is a dual $\mathbf{m_u}$-isolated subset with $dwt_{\mathbf{m_u}}(J) = i$, we write $J \in \mathcal{I}^*_{\mathbf{m_u}}(\langle i \rangle)$.

**Example 2.3.3.** Let $\mathbf{u} = (u_1, u_2, u_3, u_4) = (a, a, b, b)$, then $s_0 = 0, s_1 = 2, s_2 = 4$. The dual $\mathbf{m_u}$-isolated subsets in this case are, $J_0 = \{1\}, J_1 = \{2\}, J_0^{'} = \{3\}, J_1^{'} = \{4\}, J_1^{''} = \{1, 3\}, J_2 = \{1, 4\}, J_2^{'} = \{2, 3\}, J_3 = \{2, 4\}$, where $J_i$(or $J_i^{'}, J_i^{''}$) has dual weight $i$.

In the following theorem, multigraded Betti numbers of the Alexander dual $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ are computed using the notion of dual $\mathbf{m_u}$-isolated subsets.

**Theorem 2.3.4.** *For* $\mathbf{b} \in \mathbb{N}^n$ *and* $i \geq 1$, *let* $\beta_{i-1,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{u_n+c-1}]})$ *be an* $i-1$*-th multigraded Betti number of* $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ *in the degree* $\mathbf{b}$. *If* $u_1 \geq 1$, *then the multigraded Betti numbers* $\beta_{i-1,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{u_n+c-1}]})$ *are given as follows:*

1. *For* $J = \{j_1, j_2, \ldots, j_t\} \in \mathcal{I}^*_{\mathbf{m_u}}$,

$$\beta_{i-1,\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n+c-1}]}) = \Big[\prod_{\alpha=1}^{t} \binom{j_\alpha - j_{\alpha-1} - 1}{s_{i_\alpha-1} - j_{\alpha-1}}\Big]\delta_{i-1,dwt_{\mathbf{m_u}}(J)},$$

*where* $J \cap (s_{i_\alpha-1}, s_{i_\alpha}] = \{j_\alpha\}$. *If* $\pi$ *is a permutation of* $\widetilde{\mathbf{b}}(J)$, *then*

$$\beta_{i-1,\pi\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n+c-1}]}) = \beta_{i-1,\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n+c-1}]}).$$

2. *If* $\mathbf{b} \neq \pi\widetilde{\mathbf{b}}(J)$ *for any* $J \in \mathcal{I}^*_{\mathbf{m_u}}(\langle i-1\rangle)$ *and any permutation* $\pi$ *of* $\widetilde{\mathbf{b}}(J)$, *then*

$$\beta_{i-1,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{u_n+c-1}]}) = 0.$$

**Proof.** The lower Koszul simplicial complex $K_{\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n+c-1}]})$ of the Alexander dual $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ of the multipermutohedron ideal $I(\mathbf{u})$ is claimed to be the join of skeletons of simplices

$$\Delta_{j_1-1}^{(s_{i_1-1}-1)} * \Delta_{j_2-j_1-1}^{(s_{i_2-1}-j_1-1)} * \ldots * \Delta_{j_\alpha-j_{\alpha-1}-1}^{(s_{i_\alpha-1}-j_{\alpha-1}-1)} * \ldots * \Delta_{j_t-j_{t-1}-1}^{(s_{i_t-1}-j_{t-1}-1)},$$

where simplex $\Delta_{j_\alpha-j_{\alpha-1}-1}$ is spanned by vertices $\{e_\nu : j_{\alpha-1}+1 \leq \nu \leq j_\alpha\}$. This claim is proved by a straightforward verification. Consider the vector

$$\mathbf{v} = E(0, s_{i_1-1}) + E(j_1, s_{i_2-1}) + E(j_2, s_{i_3-1}) + \ldots + E(j_{t-1}, s_{i_t-1}).$$

Then $\mathbf{v}$ is the vector

$$\mathbf{v} = \Big(1,\ldots,1,0,\ldots,0,\vdots 1,\ldots,1,0,\ldots,0,\vdots\ldots\ldots\vdots 1,\ldots,1,0,\ldots,0,\vdots 0,\ldots,0\Big)$$

consisting of exactly $t$ strands of 1's followed by 0's together with $n - j_t$ zeros at the end. The length of the $\alpha$th strand is $j_\alpha - j_{\alpha-1}$ and precisely first

$s_{i_\alpha-1} - j_{\alpha-1}$ entries of the $\alpha$th strand are 1's followed by 0's. Now, set $\widetilde{\mathbf{b}}'(J) = \sum_{\alpha=1}^{t}(\lambda_{j_\alpha} - 1)E(j_{\alpha-1}, j_\alpha)$. Clearly, $\widetilde{\mathbf{b}}'(J) + \mathbf{v}$ cannot be bigger than or equal to an exponent of any minimal generator of $I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$. Thus $\mathbf{v} \in K_{\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$. Let $\pi_j$ be a permutation of the $j$th strand of the vector $\mathbf{v}$ and let $\pi$ be the product of the (disjoint) permutations $\pi_j$ for $1 \leq j \leq t$. Then we also have $\pi\mathbf{v} \in K_{\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$. If $\mathbf{v}'$ is another vector obtained from $\mathbf{v}$ by replacing at least one of the 0 by 1, then $\widetilde{\mathbf{b}}'(J) + \pi\mathbf{v}'$ becomes bigger than or equal to an exponent of some minimal generator of $K_{\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$. This proves the claim.

The dimension $\dim(K_{\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})) = \sum_{\alpha=1}^{t}(s_{i_{\alpha-1}} - j_{\alpha-1} - 1) + (t-1) = j_t - \sum_{\alpha=1}^{t}(j_\alpha - s_{i_{\alpha-1}}) - 1$. The multigraded Betti number

$$\beta_{i-1,\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = \dim_k(\widetilde{H}^{j_t-i-1}(K_{\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}); k)) \text{ for } i \geq 1.$$

Clearly, $i - 1 = dwt(\widetilde{\mathbf{b}}(J)) \Leftrightarrow j_t - i - 1 = \dim(K_{\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}))$. Thus from the above result on homology groups of the join of skeletons of simplexes, the first part of (1) follows. Since minimal generators of $I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ are invariant under a permutations, we have $\beta_{i,\pi\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = \beta_{i,\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$, for a permutation $\pi$ of $\widetilde{\mathbf{b}}(J)$.

Let $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{N}^n$ such that $\mathbf{b} \neq \pi\widetilde{\mathbf{b}}(J)$ for any permutation $\pi$. Changing $\mathbf{b}$ by a permutation, we may assume that $b_1 \geq b_2 \geq \ldots \geq b_n$. The nonzero Betti numbers of a monomial ideal exist in a multidegree $\mathbf{b}$ only if the monomial $\mathbf{x}^{\mathbf{b}}$ is a LCM of some set of minimal generators of the monomial ideal. Therefore, $\mathbf{b} = \sum_{\alpha=1}^{t} \lambda_{j_\alpha} E(j_{\alpha-1}, j_\alpha)$ for some $J' = \{j_1, j_2, \ldots, j_t\} \in \mathcal{I}_{\mathbf{m}}^*$. But by the given condition, $J' \notin \mathcal{I}_{\mathbf{m}}^*(\langle i-1 \rangle)$. Thus $\beta_{i-1,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = 0$. This proves (2). $\square$

**Remark 2.3.5.** The Theorem 2.3.4 looks quite similar to the Theorem 1.2.4. This is not surprising as there is a general duality for Betti numbers of a monomial ideal $I$ and its Alexander dual $I^{[\mathbf{a}]}$. In fact,

$$\beta_{n-i,\mathbf{b}}(R/I) = \beta_{i,\mathbf{a}+\mathbf{1}-\mathbf{b}}(I^{[\mathbf{a}]}) \tag{2.3.1}$$

for $b = (b_1, \ldots, b_n)$ with $1 \leq b_i \leq a_i$, $\forall i$ ([20],Theorem 5.48). However, this duality does not give all the multigraded Betti numbers of the dual $I^{[\mathbf{a}]}$. Using Theorem 1.1.2. and Theorem 1.1.5. we verify this duality for multipermutohedron ideals. Take $j_t = n$ in Theorem 1.1.5., we get

$$\begin{aligned}\widetilde{\mathbf{b}}(J) &= \sum_{\alpha=1}^{t}(u_n - u_{j_\alpha} + c)E(j_{\alpha-1}, j_\alpha) \\ &= (u_n + c - 1)E(0,n) + 1E(0,n) - u_{j_\alpha}E(0,n) \\ &= \mathbf{a} - \mathbf{b}(J) + \mathbf{1}\end{aligned}$$

and $wt(\mathbf{b}(J)) = \sum_{\alpha=1}^{t}(s_{i_\alpha - 1} - j_{\alpha-1}) = j_t - \left[\sum_{\alpha=1}^{t}(j_\alpha - s_{i_\alpha - 1})\right] - 1$
$= n - dwt(\widetilde{\mathbf{b}}(J)) - 1$

Thus we have,

$$\beta_{n-i,\mathbf{b(J)}}(R/I(\mathbf{u})) = \beta_{i,\mathbf{a}-\mathbf{b}(J)+\mathbf{1}} I(\mathbf{u})^{[\mathbf{u_n+c-1}]}, \text{ where } i = dwt(\widetilde{\mathbf{b}}(J))).$$

and for $\mathbf{b} \neq \mathbf{b}(J)$ both sides of equation 2.3.1 are 0.

**Corollary 2.3.6.** *Let $\beta_{i-1}(I(\mathbf{u})^{[\mathbf{u_n+c-1}]})$ be the $i-1$-th Betti number of the Alexander dual $I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$. Suppose $u_1 \geq 1$ and for $J = \{j_1, j_2, \dots, j_t\} \in \mathcal{I}^*_{\mathbf{m_u}}(\langle i-1\rangle)$, we set*

$$\beta^J_{i-1} = \prod_{\alpha=1}^{t}\left[\binom{j_\alpha - j_{\alpha-1} - 1}{s_{i_\alpha - 1} - j_{\alpha-1}}\binom{j_{\alpha+1}}{j_\alpha}\right], \text{ where } J \cap (s_{i_\alpha-1}, s_{i_\alpha}] = \{j_\alpha\} \text{ and } j_{t+1} = n.$$

*Then* $\beta_{i-1}(I(\mathbf{u})^{[\mathbf{u_n+c-1}]}) = \sum_{J \in \mathcal{I}^*_{\mathbf{m_u}}(\langle i-1\rangle)} \beta^J_{i-1}.$

**Proof.** Let $Per(\mathbf{b}(J))$ be the set of all permutations of $\mathbf{b}(J)$. Then, in view of Theorem 2.3.4, we have

$$\begin{aligned}\beta_{i-1}(I(\mathbf{u})^{[\mathbf{u_n+c-1}]}) &= \sum_{\mathbf{b}\in\mathbb{N}^n} \beta_{i-1,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{u_n+c-1}]}) \\ &= \sum_{J \in \mathcal{I}^*_{\mathbf{m_u}}(\langle i-1\rangle)}\left[\sum_{\pi \in Per(\mathbf{b}(J))} \beta_{i-1,\pi\mathbf{b}(J)}(I(\mathbf{u})^{[\mathbf{u_n+c-1}]})\right] \\ &= \sum_{J \in \mathcal{I}^*_{\mathbf{m_u}}(\langle i-1\rangle)} \beta^J_{i-1},\end{aligned}$$

where $\beta^J_{i-1} = \sum\limits_{\pi \in Per(\mathbf{b}(J))} \beta_{i-1,\pi\mathbf{b}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$. For the set $J = \{j_1, j_2, \ldots, j_t\} \in \mathcal{I}^*_{\mathbf{m_u}}(\langle i-1\rangle)$ with $J \cap (s_{i_\alpha - 1}, s_{i_\alpha}] = \{j_\alpha\}$, we have

$$\beta_{i-1,\pi\mathbf{b}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = \prod_{\alpha=1}^{t} \binom{j_\alpha - j_{\alpha-1} - 1}{s_{i_\alpha - 1} - j_{\alpha-1}},$$

for all $\pi \in Per(\mathbf{b}(J))$. The number of permutations $\pi$ of $\mathbf{b}(J)$ is

$$\mid Per(\mathbf{b}(J)) \mid = \frac{n!}{\prod_{\alpha=1}^{t}(j_{\alpha+1} - j_\alpha)!} = \prod_{\alpha=1}^{t} \binom{j_{\alpha+1}}{j_\alpha}.$$

Therefore,

$$\beta^J_{i-1} = \Big[\prod_{\alpha=1}^{t} \binom{j_\alpha - j_{\alpha-1} - 1}{s_{i_\alpha - 1} - j_{\alpha-1}}\Big]\Big[\prod_{\alpha=1}^{t} \binom{j_{\alpha+1}}{j_\alpha}\Big].$$

This completes the proof. □

**Remark 2.3.7.** Let $\widetilde{\mathcal{I}}^*_{\mathbf{m_u}}(\langle i-1\rangle) = \{J \in \mathcal{I}^*_{\mathbf{m_u}}(\langle i-1\rangle) : J \cap (s_0, s_1] = \emptyset\}$. Then for $u_1 = 0$, the formula for Betti numbers of the Alexander dual $I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ as in Theorem 2.3.4 or Corollary 2.3.6 remain valid by just replacing $\mathcal{I}^*_{\mathbf{m_u}}(\langle i-1\rangle)$ with $\widetilde{\mathcal{I}}^*_{\mathbf{m_u}}(\langle i-1\rangle)$.

We have already obtained formula for the zeroth Betti number of the Alexander dual $I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$. For the first Betti number, we need to determine all dual $\mathbf{m_u}$-isolated subsets $J \in \mathcal{I}^*_{\mathbf{m_u}}(\langle 1\rangle)$ of dual weight 1. Let $J_\alpha = \{s_\alpha + 2\}$ for $0 \leq \alpha < l$ with $s_{\alpha+1} - s_\alpha \geq 2$ and $J_{\nu,\omega} = \{s_\nu + 1, s_\omega + 1\}$ with $0 \leq \nu < \omega < l$. Then $J_\alpha, J_{\nu,\omega} \in \mathcal{I}^*_{\mathbf{m_u}}(\langle 1\rangle)$ and

$$\beta_1^{J_\alpha} = (s_\alpha + 1)\binom{n}{s_\alpha + 2} \text{ while } \beta_1^{J_{\nu,\omega}} = \binom{n}{s_\omega + 1}\binom{s_\omega + 1}{s_\nu + 1}.$$

Thus

$$\begin{aligned} \beta_1(I(\mathbf{u})^{[\mathbf{u_n+c-1}]}) &= \sum_{\substack{0\le\alpha<l,\\ s_{\alpha+1}-s_\alpha\ge 2}} \beta_1^{J_\alpha} + \sum_{0\le\nu<\omega<l} \beta_1^{J_{\nu,\omega}} \\ &= \sum_{\substack{0\le\alpha<l,\\ s_{\alpha+1}-s_\alpha\ge 2}} (s_\alpha+1)\binom{n}{s_\alpha+2} + \qquad (2.3.2) \\ & \quad \sum_{0\le\nu<\omega<l} \binom{n}{s_\omega+1}\binom{s_\omega+1}{s_\nu+1}, \end{aligned}$$

provided $u_1 \geq 1$. On the other hand, if $u_1 = 0$, then

$$\beta_1(I(\mathbf{u})^{[\mathbf{u_n+c-1}]}) = \sum_{\substack{0<\alpha<l,\\ s_{\alpha+1}-s_\alpha\ge 2}} (s_\alpha+1)\binom{n}{s_\alpha+2} + \sum_{0<\nu<\omega<l} \binom{n}{s_\omega+1}\binom{s_\omega+1}{s_\nu+1}.$$

We have seen that the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $n-1$-simplex $\Delta_{n-1}$ supports a free resolution of the quotient $R/I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ of the Alexander dual of the multipermutohedron ideal. Now consider a polyhedral cell complex $\mathbf{Bd^m}(\Delta_{n-1})$ obtained by modifying the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ as follows: First assume that $u_1 \geq 1$. In this case, the vertices of the polyhedral cell complex $\mathbf{Bd^m}(\Delta_{n-1})$ are precisely the barycenters corresponding to the subsets $A \subseteq [n]$ with $|A| = s_i + 1$ for $0 \leq i < l$ and the edges corresponds to the chain of subsets $A \subset B$ of $[n]$ with $|A| = s_\nu + 1$ and $|B| = s_\omega + 1$ for $0 \leq \nu < \omega < l$, or the subsets $C$ of $[n]$ of the form $C = A \cup B$ with $|C| = s_\alpha + 2, |A| = |B| = s_\alpha + 1$, and $s_{\alpha+1} - s_\alpha \geq 2$. Higher dimensional faces of $\mathbf{Bd^m}(\Delta_{n-1})$ are spanned by the vertices and edges so that the polyhedral cell complex gives a subdivision of the $n-1$-simplex $\Delta_{n-1}$. Thus the dimension of $\mathbf{Bd^m}(\Delta_{n-1})$ is $n-1$. Now assume that $u_1 = 0$. In this case, the polyhedral cell complex $\mathbf{Bd^m}(\Delta_{n-1})$ is obtained as in the earlier case, but now we delete all the faces containing the vertices of $n-1$-simplex $\Delta_{n-1}$, i.e. barycenters corresponding to the subsets $A \subseteq [n]$ with $|A| = 1$. The dimension of $\mathbf{Bd^m}(\Delta_{n-1})$ in the case $u_1 = 0$ can be any number from $0$ to $n-1$ depending on $\mathbf{m}$.

Let $f_i(\mathbf{Bd^m}(\Delta_{n-1}))$ be the number of $i$-dimensional faces of $\mathbf{Bd^m}(\Delta_{n-1})$. Clearly,

$f_0(\mathbf{Bd^m}(\Delta_{n-1}))$ equals the zeroth Betti number $\beta_0(I(\mathbf{u})^{[\mathbf{u_n+c-1}]})$. We now proceed to count the number of edges of the polyhedral cell complex $\mathbf{Bd^m}(\Delta_{n-1})$. Firstly, we consider the case $u_1 \geq 1$. For any two vertices of $\mathbf{Bd^m}(\Delta_{n-1})$ corresponding to the subsets $A, B \subseteq [n]$ with $|A| = s_\nu+1$ and $|B| = s_\omega+1$ for $0 \leq \nu < \omega < l$, there is an edge between these two vertices if and only if $A \subseteq B$. Also if $|A| = |B| = s_\alpha+1$, then there is an edge between these vertices if $|A \cup B| = s_\alpha + 2$ and $s_{\alpha+1} - s_\alpha \geq 2$. Now counting these subsets, we obtain a combinatorial formula

$$f_1(\mathbf{Bd^m}(\Delta_{n-1})) = \sum_{0\leq\nu<\omega<l} \binom{n}{s_\omega+1}\binom{s_\omega+1}{s_\nu+1} + \sum_{\substack{0\leq\alpha<l,\\ s_{\alpha+1}-s_\alpha\geq 2}} \frac{(s_\alpha+1)(s_\alpha+2)}{2}\binom{n}{s_\alpha+2}. \quad (2.3.3)$$

If $u_1 = 0$, then deleting all the edges containing the vertices of $n-1$-simplex $\Delta_{n-1}$, the combinatorial formula takes the form

$$f_1(\mathbf{Bd^m}(\Delta_{n-1})) = \sum_{0<\nu<\omega<l} \binom{n}{s_\omega+1}\binom{s_\omega+1}{s_\nu+1} + \sum_{\substack{0<\alpha<l,\\ s_{\alpha+1}-s_\alpha\geq 2}} \frac{(s_\alpha+1)(s_\alpha+2)}{2}\binom{n}{s_\alpha+2}.$$

A combinatorial formula for the higher dimensional faces of the polyhedral cell complex $\mathbf{Bd^m}(\Delta_{n-1})$ are quite cumbersome. But if $\mathbf{m} = (m_1, 1, \ldots, 1)$, then $f_i(\mathbf{Bd^m}(\Delta_{n-1}))$ can be easily calculated.

**Theorem 2.3.8.** *Let* $\mathbf{m} = (m_1, 1, \ldots, 1)$. *Then*

$$f_{i-1}(\mathbf{Bd^m}(\Delta_{n-1})) = \beta_{i-1}(I(\mathbf{u})^{[\mathbf{u_n+c-1}]}) \ \forall i \geq 1.$$

**Proof.** Firstly we consider the case $u_1 \geq 1$. We know that $(i-1)$-faces of the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ correspond to a chain of nonempty subsets of $[n]$ of length $i$. Since the barycenters corresponding to the subsets $A \subset [n]$ with

$1 < |A| \leq m_1$ are missing, an $(i-1)$-face of the polyhedral complex $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ corresponds to a chain

$$A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_t$$

of subsets of $[n]$ such that either all $A_i$'s represent vertices of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ or $1 < |A_1| \leq m_1 < |A_2|$. In the former case, $t = i$, while in the latter case, $t = i - |A_1| + 1$ as it represents the $(i-1)$-face spanned by vertices of $\Delta_{n-1}$, corresponding to singleton subsets of $A_1$, and the barycenters $A_2, \ldots, A_t$. Let $|A_i| = j_i$. Then $J = \{j_1, j_2, \ldots, j_t\}$ is a dual $\mathbf{m_u}$-isolated subset with $dwt_{\mathbf{m_u}}(J) = j_1 + (t-1) - 1 = i-1$, and every $(i-1)$-face of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ arises in this way. In this case, all the faces of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ are simplicial and thus $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ is a $(n-1)$-dimensional simplicial complex.

Let $f_{i-1}^J$ be the number of $(i-1)$-faces of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ associated to a dual $\mathbf{m_u}$-isolated subset $J \in \mathcal{I}^*_{\mathbf{m_u}}(\langle i-1 \rangle)$ with dual weight $i-1$. Then $f_{i-1}^J = \prod_{\alpha=1}^t \binom{j_{\alpha+1}}{j_\alpha}$, where $j_{t+1} = n$. For $\mathbf{m}$(or $\mathbf{m_u}$) $= (\mathrm{m}_1, 1, \ldots, 1)$, using Corollary 2.3.6 $\beta_{i-1}^J = \prod_{\alpha=1}^t \binom{j_{\alpha+1}}{j_\alpha}$, because either $j_{\alpha+1} - j_\alpha - 1 = 0$ or $s_{i_\alpha - 1} - j_\alpha = 0$. Thus

$$f_{i-1}(\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})) = \sum_{J \in \mathcal{I}^*_{\mathbf{m_u}}(\langle i-1 \rangle)} f_{i-1}^J = \sum_{J \in \mathcal{I}^*_{\mathbf{m_u}}(\langle i-1 \rangle)} \beta_{i-1}^J = \beta_{i-1}(I(\mathbf{u})^{[\mathbf{u_n} + \mathbf{c} - \mathbf{1}]}).$$

If $u_1 = 0$, then vertices of the $n-1$-simplex $\Delta_{n-1}$ are no longer vertices of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$. Thus an $(i-1)$-face of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ corresponds to a chain

$$A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_t$$

of subsets of $[n]$ with $t = i$ and $|A_1| > m_1$. Clearly, maximal such chain has length $n - m_1$ and hence the dimension of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ is $n - m_1 - 1$. In this case, we have

$$f_{i-1}(\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})) = \sum_{J \in \widetilde{\mathcal{I}}^*_{\mathbf{m_u}}(\langle i-1 \rangle)} f_{i-1}^J = \sum_{J \in \widetilde{\mathcal{I}}^*_{\mathbf{m_u}}(\langle i-1 \rangle)} \beta_{i-1}^J = \beta_{i-1}(I(\mathbf{u})^{[\mathbf{u_n} + \mathbf{c} - \mathbf{1}]}).$$

This completes the proof. $\square$

A vertex of $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ corresponds to a nonempty subset $A \subseteq [n]$, and it is naturally labeled with the monomial $(\prod_{i \in A} x_i)^{u_n - u_{|A|} + c}$. Thus $\mathbf{Bd}^{\mathbf{m}}(\Delta_{n-1})$ is a labeled polyhedral cell complex. The free complex associated to the labeled polyhedral cell complex $\mathbf{Bd^m}(\Delta_{n-1})$ gives a cellular free resolution of the quotient $R/I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$. We now investigate minimality of the cellular resolution supported by $\mathbf{Bd^m}(\Delta_{n-1})$ .

**Theorem 2.3.9.** *The cellular resolution supported by* $\mathbf{Bd^m}(\Delta_{n-1})$ *is the minimal resolution of* $R/I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ *if and only if* $m_\alpha = 1$ *for* $2 \leq \alpha \leq l$.

**Proof.** Suppose the free resolution supported by the labeled polyhedral cell complex $\mathbf{Bd^m}(\Delta_{n-1})$ minimally resolves $R/I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$. Then we have $\beta_1(I(\mathbf{u})^{[\mathbf{u_n+c-1}]})$ $= f_1(\mathbf{Bd^m}(\Delta_{n-1}))$, the number of edges of $\mathbf{Bd^m}(\Delta_{n-1})$. Using Equations 2.3.2 and 2.3.3, and the similar equations for the case $u_1 = 0$, we see that there are at most one $\alpha$ with $s_{\alpha+1} - s_\alpha \geq 2$; namely $\alpha = 0$ if $u_1 \geq 1$ and no such $\alpha$ if $u_1 = 0$. Thus in either case, $m_{\alpha+1} = s_{\alpha+1} - s_\alpha = 1$ for $\alpha \geq 1$. This proves the direct part.

Conversely, let $m_\alpha = 1$ for $\alpha \geq 2$. Then $\beta_{i-1}(I(\mathbf{u})^{[\mathbf{u_n+c-1}]}) = f_{i-1}(\mathbf{Bd^m}(\Delta_{n-1}))$ $\forall i \geq 1$, in view of Theorem 2.3.8. Thus the cellular free resolution supported by the labeled polyhedral cell complex $\mathbf{Bd^m}(\Delta_{n-1})$ is minimal. $\square$

**Remark 2.3.10.** In [17], it is proved that the cellular resolution supported by the multipermutohedron $P(\mathbf{u})$ is the minimal resolution of the quotient $R/I(\mathbf{u})$ if and only if $m_\alpha = 1$ for $2 \leq \alpha \leq l$. In spite of the identical resemblance, in view of the Remark 2.3.5, the Theorem 2.3.9 about the minimal resolution of the quotient $R/I(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ can not simply be deduced from the minimal resolution of $R/I(\mathbf{u})$.

At the end of this chapter, we give some examples of cellular free resolutions of the Alexander duals of multipermutohedron ideals.

**Example 2.3.11.** Let $\mathbf{u} = (a, a, a, b)$, $0 < a < b$. Consider $\mathbf{b} + \mathbf{c} - \mathbf{1} = (b + c - 1, b + c - 1, b + c - 1, b + c - 1)$, $c \geq 1$. Then the Alexander dual of $I(a, a, a, b)$ with respect to $\mathbf{b} + \mathbf{c} - \mathbf{1}$ is $I(a, a, a, b)^{[\mathbf{b+c-1}]} = \langle x^{b-a+c}, y^{b-a+c}, z^{b-a+c}, t^{b+a-c}, (xyzt)^c \rangle$ is an ideal in the polynomial ring $R = k[x, y, z, t]$. The polyhedral cell complex $\mathbf{Bd^m}(\Delta_3)$ is obtained by subdividing a 3-simplex into four regions by choosing the fifth vertex as the centroid and joining it with the four vertices of the 3-simplex

(Figure 2.1). Thus $\mathbf{Bd^m}(\Delta_3)$ is a simplicial complex with five vertices, ten edges and ten 2-dimensional faces and four 3-dimensional faces. Clearly, the labels on the vertices of the 3-simplex are $x^{b-a+c}, y^{b-a+c}, z^{b-a+c}, t^{b-a+c}$ while the label on the centroid is $(xyzt)^c$. The dual $\mathbf{m_u}$-isolated subsets are $J_0 = \{1\}, \bar{J}_0 = \{4\}, J_1 = \{2\}, \bar{J}_1 = \{1,4\}$, $J_2 = \{3\}, \bar{J}_2 = \{2,4\}$, and $\bar{J}_3 = \{3,4\}$ where $J_i$ (or $\bar{J}_i$) has dual $\mathbf{m_u}$-weight $i$. Using Corollary 2.3.6, we have $\beta_0^{J_0} = 4, \beta_0^{\bar{J}_0} = 1, \beta_1^{J_1} = 6, \beta_1^{\bar{J}_1} = 4$,$\beta_2^{J_2} = 4, \beta_2^{\bar{J}_2} = 6$ and $\beta_3(J_3) = 4$. Thus the Betti numbers of $I(a,a,a,b)^{[\mathbf{b+c-1}]}$ are $\beta_0 = 5, \beta_1 = 10$, $\beta_2 = 10$ and $\beta_3 = 4$. Thus the free complex associated with the labeled simplicial complex $\mathbf{Bd^m}(\Delta_3)$ is the minimal resolution of the Alexander dual $I(a,a,a,b)^{[\mathbf{b+c-1}]}$.

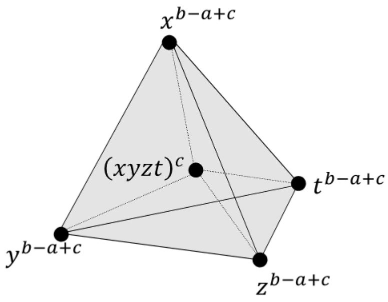



Figure 2.1: Polyhedral cell complex $\mathbf{Bd^m}(\Delta_3)$

**Example 2.3.12.** Let $\mathbf{u} = (a,b,b)$, $0 < a < b$. Consider $\mathbf{b+c-1} = (b+c-1, b+c-1, b+c-1)$, $c \geq 1$. Then the Alexander dual is $I(a,b,b)^{[\mathbf{b+c-1}]} = \langle x^{b-a+c}, y^{b-a+c}, z^{b-a+c}, (xy)^c, (xz)^c, (yz)^c \rangle$. The polyhedral cell complex $\mathbf{Bd^m}(\Delta_2)$ is obtained by subdividing a 2-simplex into four triangular regions by choosing three more vertices as the barycenters of the three edges of the 2-simplex and joining the barycenters with each other (Figure 2.2(i)). Thus $\mathbf{Bd^m}(\Delta_2)$ is a simplicial complex having six vertices, nine edges and four triangular faces. Clearly, labels on the vertices of the 2-simplex are $x^{b-a+c}, y^{b-a+c}, z^{b-a+c}$, while the labels on the barycenters of the three edges are $(xy)^c, (xz)^c, (yz)^c$. The dual $\mathbf{m_u}$-isolated subsets in this case are $J_0 = \{1\}, \bar{J_0} = \{2\}, J_1 = \{1,2\}, \bar{J_1} = \{3\}$ and $J_2 = \{1,3\}$, where $J_i$ (or $\bar{J_i}$) has dual $\mathbf{m_u}$-weight $i$. We have $\beta_0^{J_0} = 3, \beta_0^{\bar{J_0}} = 3, \beta_1^{J_1} = 6, \beta_1^{\bar{J_1}} = 2$ and $\beta_2^{J_2} = 3$. Thus the Betti numbers of $I(a,a,b)^{[\mathbf{b+c-1}]}$ are $\beta_0 = 6, \beta_1 = 8$ and $\beta_2 = 3$. Since the number of edges of $\mathbf{Bd^m}(\Delta_2)$ is $9 > 8 = \beta_1$, the free complex associated with this labeled simplicial complex is a non-minimal resolution of the Alexander dual $I(a,b,b)^{[\mathbf{b+c-1}]}$.

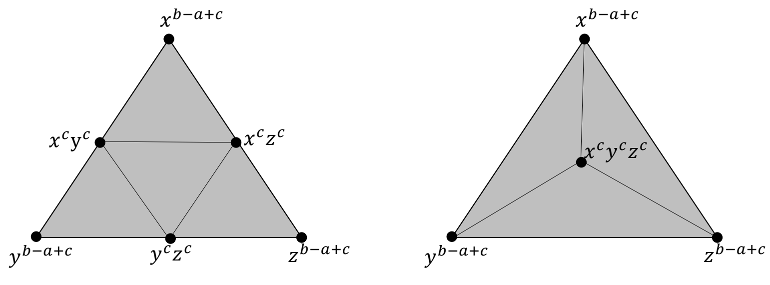



Figure 2.2: Polyhedral cell complexes for Example 2.3.12 and Example 2.3.13.

**Example 2.3.13.** Let $\mathbf{u} = (a,a,b)$, $0 < a < b$. Consider $\mathbf{b+c-1} = (b+c-1, b+c-1, b+c-1)$, $c \geq 1$. Then the Alexander dual $I(a,a,b)^{[\mathbf{b+c-1}]} = \langle x^{b-a+c}, y^{b-a+c}, z^{b-a+c}, (xyz)^c \rangle$ is an ideal in the polynomial ring $R = k[x,y,z]$. The polyhedral cell complex $\mathbf{Bd^m}(\Delta_2)$ is obtained by subdividing a 2-simplex into three triangular regions by choosing the fourth vertex as the centroid and joining it with the three vertices of the 2-simplex (Figure 2.2(ii)). Thus $\mathbf{Bd^m}(\Delta_2)$ is a simplicial complex with four vertices, six edges and three triangular faces. Clearly, the labels on the vertices of the 2-simplex are $x^{b-a+c}, y^{b-a+c}, z^{b-a+c}$ while the label on the centroid is $(xyz)^c$. The dual $\mathbf{m_u}$-isolated subsets are $J_0 = \{1\}, \bar{J_0} = \{3\}, J_1 = \{2\}, \bar{J_1} = \{1,3\}$ and $J_2 = \{2,3\}$, where $J_i$ (or $\bar{J_i}$) has dual $\mathbf{m_u}$-weight $i$. Using Corollary 2.3.6, we have $\beta_0^{J_0} = 3, \beta_0^{\bar{J_0}} = 1, \beta_1^{J_1} = 3, \beta_1^{\bar{J_1}} = 3$ and $\beta_2^{J_2} = 3$. Thus the Betti numbers of $I(a,a,b)^{[\mathbf{b+c-1}]}$ are $\beta_0 = 4, \beta_1 = 6$ and $\beta_2 = 3$. Thus the free complex associated with the labeled simplicial complex $\mathbf{Bd^m}(\Delta_2)$ is the minimal resolution of the Alexander dual $I(a, a, b)^{[\mathbf{b}+\mathbf{c}-\mathbf{1}]}$, which is also indicated by Theorem 2.3.9.

# Chapter 3

# Sum of Two Multipermutohedron Ideals

In this chapter, we describe the multigraded Betti numbers of the sum of two multipermutohedron ideals. We have used the reduced Mayer-Vietoris sequence to calculate their multigraded Betti numbers. Alexander duals of multipermutohedron ideals have been discussed in the second chapter. Again using the reduced Mayer-Vietoris sequence, we also describe the multigraded Betti numbers of the Alexander dual of the sum of two multipermutohedron ideals.

## 3.1 Betti numbers of the sum of two multipermutohedron ideals

We shall use the same notations as in the Introduction. Let $J$ denotes the $\mathbf{m}_\mathbf{u}$-isolated subset and $\mathcal{I}_{\mathbf{m}_\mathbf{u}}$ be set of all $\mathbf{m}_\mathbf{u}$-isolated subsets of $[n]$ (Definition 1.2.3). In view of Theorem 1.2.4, we recall the following facts about multipermutohedron ideals.

- For a multipermutohedron ideal $I(\mathbf{u})$, the upper Koszul simplicial complex $K^{\mathbf{b}(J)}(I(\mathbf{u}))$ is the join of the skeletons of simplices and

$$wt_{\mathbf{m}_\mathbf{u}}(J) - 1 = \dim K^{\mathbf{b}(J)}(I(\mathbf{u})).$$

- The multigraded Betti numbers $\beta_{i,\mathbf{b}}(I(\mathbf{u}))$ are given as follows. For $J = \{j_1, j_2, \ldots, j_t\} \in \mathcal{I}_{\mathbf{m_u}}$,

$$\beta_{i,\mathbf{b}(J)}(I(\mathbf{u})) = \Big[\prod_{\alpha=1}^{t} \binom{j_\alpha - j_{\alpha-1} - 1}{s_{i_\alpha - 1} - j_{\alpha-1}}\Big]\delta_{i,wt_{\mathbf{m_u}}(J)},$$

where $J \cap (s_{i_\alpha -1}, s_{i_\alpha}] = \{j_\alpha\}$ and $\delta_{i,q}$ is the Kronecker delta. If $\pi$ is a permutation of $\mathbf{b}(J)$, then $\beta_{i,\pi\mathbf{b}(J)}(I(\mathbf{u})) = \beta_{i,\mathbf{b}(J)}(I(\mathbf{u}))$.

- $\beta_{i,\mathbf{b}}(I(\mathbf{u})) = 0$ if $\mathbf{b} \neq \pi\mathbf{b}(J)$ for any $J \in \mathcal{I}_{\mathbf{m_u}}$ and any permutation $\pi$ of $\mathbf{b}(J)$.

Let $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ with $1 \leq u_1 \leq u_2 \leq \ldots \leq u_n$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{N}^n$ with $1 \leq v_1 \leq v_2 \leq \ldots \leq v_n$. Consider the multipermutohedron ideals $I(\mathbf{u})$ and I($\mathbf{v}$). Then their intersection $I(\mathbf{u}) \cap I(\mathbf{v})$ is again a multipermutohedron ideal $I(\mathbf{w})$, where $\mathbf{w} = \mathbf{u} \vee \mathbf{v}$ with $w_i = \max(u_i, v_i)$. Let

$$B_{\mathbf{u}} = \{\mathbf{b}(J) \in \mathbb{N}^n : J \in \mathcal{I}_{\mathbf{m_u}}\}.$$

Similarly, we define subsets $B_{\mathbf{v}}$ and $B_{\mathbf{w}}$ of $\mathbb{N}^n$ and let $B = B_{\mathbf{u}} \cup B_{\mathbf{v}} \cup B_{\mathbf{w}}$.

We see that the upper Koszul simplicial complexes of the sum $I(\mathbf{u}) + I(\mathbf{v})$ and the intersection $I(\mathbf{u}) \cap I(\mathbf{v}) = I(\mathbf{w})$ are given by

$$K^{\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = K^{\mathbf{b}}(I(\mathbf{u})) \cup K^{\mathbf{b}}(I(\mathbf{v})) \text{ and } K^{\mathbf{b}}(I(\mathbf{w})) = K^{\mathbf{b}}(I(\mathbf{u})) \cap K^{\mathbf{b}}(I(\mathbf{v})).$$

Suppose $\triangle_1$ and $\triangle_2$ be two simplicial complexes and let $\triangle = \triangle_1 \cup \triangle_2$ and $\Gamma = \triangle_1 \cap \triangle_2$, Then there is a long exact sequence of the form

$$\ldots \to \widetilde{H}_i(\Gamma; k) \xrightarrow{\partial} \widetilde{H}_i(\triangle_1; k) \oplus \widetilde{H}_i(\triangle_2, k) \xrightarrow{\partial} \widetilde{H}_i(\triangle, k) \xrightarrow{\delta} \widetilde{H}_{i-1}(\Gamma; k) \to \ldots,$$

where $\delta$ is the connecting homomorphism. This sequence is called the *reduced Mayer-Vietoris sequence* [24]. Since the multigraded Betti numbers of a monomial ideal are given in terms of the dimension of reduced homology groups of the upper Koszul simplicial complex with coefficients in the field $k$, the Mayer-Vietoris sequence can be used to compute $\widetilde{H}_i(K^{\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})); k)$. In fact, we can take $\triangle_1 = K^{\mathbf{b}}(I(\mathbf{u}))$, $\triangle_2 = K^{\mathbf{b}}(I(\mathbf{v}))$, then $\triangle = K^{\mathbf{b}}(I(\mathbf{u})) \cup K^{\mathbf{b}}(I(\mathbf{v})) = K^{\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v}))$

and $\Gamma = K^{\mathbf{b}}(I(\mathbf{u})) \cap K^{\mathbf{b}}(I(\mathbf{v})) = K^{\mathbf{b}}(I(\mathbf{w}))$.

As discussed earlier for any degree $\mathbf{b} \in \mathbf{N}^n$, there is at most one non-zero multigraded Betti number $\beta_{i,\mathbf{b}}(I(\mathbf{u}))$. Therefore many terms in the Mayer-Vietoris sequence are zero and it is possible to obtain the dimension $\dim_k \widetilde{H}_i(\triangle; k)$. On considering various possibilities for $\mathbf{b} \in \mathbb{N}^n$, we have the following theorem.

**Theorem 3.1.1.** *For $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{N}^n$ and $i \geq 0$, let $\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v}))$ be the $i^{th}$ multigraded Betti number of the sum of two multipermutohedron ideals $I(\mathbf{u})$ and $I(\mathbf{v})$ in degree $\mathbf{b}$. As $\beta_{i,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v})) = \beta_{i,\pi\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v}))$ for any permutation $\pi$ of $\mathbf{b}$ and $\beta_{i,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v})) = 0$ if $\mathbf{b} \neq \pi\mathbf{b}'$ for any $\mathbf{b}' \in B$ and $\pi$ any permutation of $\mathbf{b}'$. Thus it is enough to take $\mathbf{b} \in B$ and we have the following cases:*

**Case 1** : $\mathbf{b} \in B_{\mathbf{u}}, \mathbf{b} \notin B_{\mathbf{v}}, \mathbf{b} \notin B_{\mathbf{w}}$ *and* $wt_{\mathbf{m_u}}(\mathbf{b}) = q$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})) & \text{if } i = q, \\ 0 & \text{otherwise.} \end{cases}$$

**Case 2** : $\mathbf{b} \notin B_{\mathbf{u}}, \mathbf{b} \in B_{\mathbf{v}}, \mathbf{b} \notin B_{\mathbf{w}}$ *and* $wt_{\mathbf{m_v}}(\mathbf{b}) = q$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{v})) & \text{if } i = q, \\ 0 & \text{otherwise.} \end{cases}$$

**Case 3** : $\mathbf{b} \notin B_{\mathbf{u}}, \mathbf{b} \notin B_{\mathbf{v}}, \mathbf{b} \in B_{\mathbf{w}}$ *and* $wt_{\mathbf{m_w}}(\mathbf{b}) = q$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q+1, \\ 0 & \text{otherwise.} \end{cases}$$

**Case 4** : $\mathbf{b} \in B_{\mathbf{u}} \cap B_{\mathbf{v}} \cap B_{\mathbf{w}}$. *Then we have the following sub-cases:*

*i)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_v}}(\mathbf{b}) = q - p, wt_{\mathbf{m_w}}(\mathbf{b}) = q - p - 1$ *for* $p \geq 1$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})) & \text{if } i = q, \\ \beta_{i,\mathbf{b}}(I(\mathbf{v})) + \beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q - p, \\ 0 & \text{otherwise.} \end{cases}$$

*ii)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_v}}(\mathbf{b}) = q, wt_{\mathbf{m_w}}(\mathbf{b}) = q$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})) + \beta_{i,\mathbf{b}}(I(\mathbf{v})) - \beta_{i,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q, \\ 0 & \text{otherwise.} \end{cases}$$

*iii)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_v}}(\mathbf{b}) = q - p, wt_{\mathbf{m_w}}(\mathbf{b}) = q - (p + g)$, *where* $p \geq 1, g \geq 2$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})) & \text{if } i = q, \\ \beta_{i,\mathbf{b}}(I(\mathbf{v})) & \text{if } i = q - p, \\ \beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q - (p + g) + 1, \\ 0 & \text{otherwise.} \end{cases}$$

*iv)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q = wt_{\mathbf{m_v}}(\mathbf{b}), wt_{\mathbf{m_w}}(\mathbf{b}) = q - 1$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})) + \beta_{i,\mathbf{b}}(I(\mathbf{v})) + \beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q, \\ 0 & \text{otherwise.} \end{cases}$$

*v)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q = wt_{\mathbf{m_v}}(\mathbf{b}), wt_{\mathbf{m_w}}(\mathbf{b}) = q - p$, *where* $p \geq 2$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})) + \beta_{i,\mathbf{b}}(I(\mathbf{v})) & \text{if } i = q, \\ \beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q - p + 1, \\ 0 & \text{otherwise.} \end{cases}$$

*vi)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_v}}(\mathbf{b}) = wt_{\mathbf{m_w}}(\mathbf{b}) = q - p$, *where* $p \geq 1$. *Then*

$$\beta_{i,b}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,b}(I(\mathbf{u})) & \text{if } i = q, \\ \beta_{i,b}(I(\mathbf{v})) - \beta_{i,b}(I(\mathbf{w})) & \text{if } i = q - p, \\ 0 & \text{otherwise.} \end{cases}$$

**Case 5** :$\mathbf{b} \in B_{\mathbf{u}} \cap B_{\mathbf{w}}, \mathbf{b} \notin B_{\mathbf{v}}$. *Then we have the following sub-cases:*

*i)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_w}}(\mathbf{b}) = q$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})) - \beta_{i,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q, \\ 0 & \text{otherwise.} \end{cases}$$

*ii)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_w}}(\mathbf{b}) = q - 1$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})) + \beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q, \\ 0 & \text{otherwise.} \end{cases}$$

*iii)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_w}}(\mathbf{b}) = q - p$ *for* $p \geq 2$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})) & \text{if } i = q, \\ \beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q - p + 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Case 6** :$\mathbf{b} \in B_{\mathbf{v}} \cap B_{\mathbf{w}}, \mathbf{b} \notin B_{\mathbf{u}}$. *Then we have the following sub-cases:*

*i)* $wt_{\mathbf{m_v}}(\mathbf{b}) = q, wt_{\mathbf{m_w}}(\mathbf{b}) = q$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{v})) - \beta_{i,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q, \\ 0 & \text{otherwise.} \end{cases}$$

*ii)* $wt_{\mathbf{m_v}}(\mathbf{b}) = q, wt_{\mathbf{m_w}}(\mathbf{b}) = q - 1$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{v})) + \beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q, \\ 0 & \text{otherwise.} \end{cases}$$

*iii)* $wt_{\mathbf{m_v}}(\mathbf{b}) = q, wt_{\mathbf{m_w}}(\mathbf{b}) = q - p$ *for* $p \geq 2$. *Then*

$$\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{v})) & \text{if } i = q, \\ \beta_{i-1,\mathbf{b}}(I(\mathbf{w})) & \text{if } i = q - p + 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** It is enough to take $\mathbf{b} \in B = B_{\mathbf{u}} \cup B_{\mathbf{v}} \cup B_{\mathbf{w}}$. Then we the following cases:
**Case 1** : $\mathbf{b} \in B_{\mathbf{u}}, \mathbf{b} \notin B_{\mathbf{v}}, \mathbf{b} \notin B_{\mathbf{w}}$ and $wt_{\mathbf{m_u}}(\mathbf{b}) = q$. Then only relevant portion of Mayer-Vietoris sequence is $0 \to \widetilde{H}_{q-1}(\triangle_1; k) \to \widetilde{H}_{q-1}(\triangle; k) \to 0$. From this exact sequence, we get the desired Betti numbers.
**Case 2** : $\mathbf{b} \notin B_{\mathbf{u}}, \mathbf{b} \in B_{\mathbf{v}}, \mathbf{b} \notin B_{\mathbf{w}}$ and $wt_{\mathbf{m_v}}(\mathbf{b}) = q$. Then only relevant portion of Mayer-Vietoris sequence is $0 \to \widetilde{H}_{q-1}(\triangle_2; k) \to \widetilde{H}_{q-1}(\triangle; k) \to 0$. From this exact sequence, we get the desired Betti numbers.
**Case 3** : $\mathbf{b} \notin B_{\mathbf{u}}, \mathbf{b} \notin B_{\mathbf{v}}, \mathbf{b} \in B_{\mathbf{w}}$ and $wt_{\mathbf{m_w}}(\mathbf{b}) = q$. Then only relevant portion of Mayer-Vietoris sequence is $0 \to \widetilde{H}_{q}(\triangle; k) \to \widetilde{H}_{q-1}(\Gamma; k) \to 0$. From this exact sequence, we get the desired Betti numbers.
**Case 4** : $\mathbf{b} \in B_{\mathbf{u}} \cap B_{\mathbf{v}} \cap B_{\mathbf{w}}$. Then we have the following sub-cases:

*i)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_v}}(\mathbf{b}) = q - p, wt_{\mathbf{m_w}}(\mathbf{b}) = q - p - 1$ for $p \geq 1$. Then Mayer-Vietoris sequence gives us two exact sequences. $0 \to \widetilde{H}_{q-1}(\triangle_1; k) \to \widetilde{H}_{q-1}(\triangle; k) \to 0$ and $0 \to \widetilde{H}_{q-p-1}(\triangle_2; k) \to \widetilde{H}_{q-p-1}(\triangle; k) \to \widetilde{H}_{q-p-2}(\Gamma; k) \to 0$. From the above exact sequences, we obtain the desired Betti numbers.

*ii)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_v}}(\mathbf{b}) = q, wt_{\mathbf{m_w}}(\mathbf{b}) = q$. Then only relevant portion of Mayer-Vietoris sequence is $0 \to \widetilde{H}_{q-1}(\Gamma; k) \to \widetilde{H}_{q-1}(\triangle_1; k) \oplus \widetilde{H}_{q-1}(\triangle_2; k) \to \widetilde{H}_{q-1}(\triangle; k) \to 0$. From this exact sequence, we obtain the desired Betti numbers. Note that $\dim \triangle_1 = \dim \triangle_2 = q - 1$ and because $\dim \triangle = \max\{\dim \triangle_1, \dim \triangle_2\} = q - 1$, we have $\widetilde{H}_q(\triangle; k) = 0$.

*iii)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_v}}(\mathbf{b}) = q - p, wt_{\mathbf{m_w}}(\mathbf{b}) = q - (p + g)$, where $p \geq 1, g \geq 2$. Then only relevant portions of Mayer-Vietoris sequence are $0 \to \widetilde{H}_{q-1}(\triangle_1; k) \to \widetilde{H}_{q-1}(\triangle; k) \to 0$, $0 \longrightarrow \widetilde{H}_{q-p-1}(\triangle_2; k) \longrightarrow \widetilde{H}_{q-p-1}(\triangle; k) \longrightarrow 0$ and $0 \longrightarrow \widetilde{H}_{q-(p+g)}(\triangle; k) \longrightarrow \widetilde{H}_{q-(p+g+1)}(\Gamma; k) \longrightarrow 0$. These exact sequences gives us the desired Betti numbers.

*iv)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q = wt_{\mathbf{m_v}}(\mathbf{b}), wt_{\mathbf{m_w}}(\mathbf{b}) = q - 1$. Then only relevant portion of Mayer-Vietoris sequence is $0 \to \widetilde{H}_{q-1}(\triangle_1; k) \oplus \widetilde{H}_{q-1}(\triangle_2; k) \to \widetilde{H}_{q-1}(\triangle; k) \to \widetilde{H}_{q-2}(\Gamma; k) \to 0$. From this exact sequence, we obtain the desired Betti numbers.

*v)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q = wt_{\mathbf{m_v}}(\mathbf{b}), wt_{\mathbf{m_w}}(\mathbf{b}) = q - p$, where $p \geq 2$.
Then only relevant portions of Mayer-Vietoris sequence are $0 \to \widetilde{H}_{q-1}(\triangle_1; k) \oplus \widetilde{H}_{q-1}(\triangle_2; k) \to \widetilde{H}_{q-1}(\triangle; k) \to 0$ and $0 \to \widetilde{H}_{q-p}(\triangle; k) \to \widetilde{H}_{q-p-1}(\Gamma; k) \to 0$. These exact sequences gives us the desired Betti numbers.

*vi)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_v}}(\mathbf{b}) = wt_{\mathbf{m_w}}(\mathbf{b}) = q - p$ where $p \geq 1$. Then Mayer-Vietoris sequence reduces to $0 \to \widetilde{H}_{q-1}(\triangle_1; k) \to \widetilde{H}_{q-1}(\triangle; k) \to 0 \to \widetilde{H}_{q-p}(\triangle; k) \xrightarrow{\delta} \widetilde{H}_{q-p-1}(\Gamma; k) \to \widetilde{H}_{q-p-1}(\triangle_2; k) \to \widetilde{H}_{q-p-1}(\triangle; k) \to 0$, where $\delta$ is a connecting homomorphism.
**Claim** : The connecting homomorphism $\delta : \widetilde{H}_{q-p}(\triangle; k) \to \widetilde{H}_{q-p-1}(\Gamma; k)$ is zero.
Let $z \in \widetilde{H}_{q-p}(\triangle; k)$. Then $z = c_1 + c_2$ with $\partial z = 0$ and $c_i$ is a $k$-valued $(q-p)$-chain of $\triangle_i$; $i = 1, 2$. As $\partial(z) = 0$, we have $\partial(c_1) = -\partial(c_2)$. Therefore the connecting homomorphism is given by $\delta(z) = \partial(c_1) = -\partial(c_2)$. Since $\dim \triangle_2 = \dim K^{\mathbf{b}}(I(\mathbf{v})) = wt_{\mathbf{m_v}}(\mathbf{b}) - 1 = q - p - 1$, we have $c_2 = 0$. This proves the claim. Thus we have the exact sequences $0 \to \widetilde{H}_{q-1}(\triangle_1; k) \to \widetilde{H}_{q-1}(\triangle; k) \to 0$ and $0 \to \widetilde{H}_{q-p-1}(\Gamma; k) \to \widetilde{H}_{q-p-1}(\triangle_2; k) \to \widetilde{H}_{q-p-1}(\triangle; k) \to 0$. These exact sequences gives us the desired Betti numbers.

**Case 5** : $\mathbf{b} \in B_{\mathbf{u}} \cap B_{\mathbf{w}}, \mathbf{b} \notin B_{\mathbf{v}}$. Then we have the following sub-cases:

*i)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_w}}(\mathbf{b}) = q$. Then only relevant portion of Mayer-Vietoris sequence is $0 \to \widetilde{H}_{q-1}(\Gamma; k) \to \widetilde{H}_{q-1}(\triangle_1; k) \to \widetilde{H}_{q-1}(\triangle; k) \to 0$. From the above exact sequences, we obtain the desired Betti numbers. We prove that the connecting homomorphism $\delta : \widetilde{H}_q(\triangle; k) \to \widetilde{H}_{q-1}(\Gamma; k)$ is zero. Let $z \in \widetilde{H}_q(\triangle; k)$. Then $z = c_1 + c_2$ with $\partial z = 0$ and $c_i$ a $k$-valued $q$-chain of $\triangle_i$; $i = 1, 2$. As $\partial(z) = 0$, we have $\partial(c_1) = -\partial(c_2)$. Therefore the connecting homomorphism is given by $\delta(z) = \partial(c_1) = -\partial(c_2)$. Since $\dim \triangle_1 = \dim K^{\mathbf{b}}(I(\mathbf{u})) = wt_{\mathbf{m_u}}(\mathbf{b}) - 1 = q - 1$, we have $c_1 = 0$.

*ii)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_w}}(\mathbf{b}) = q - 1$. Then only relevant portion of Mayer-Vietoris sequence is $0 \to \widetilde{H}_{q-1}(\triangle_1; k) \to \widetilde{H}_{q-1}(\triangle; k) \to \widetilde{H}_{q-2}(\Gamma; k) \to 0$. From this exact sequence, we obtain the desired Betti numbers.

*iii)* $wt_{\mathbf{m_u}}(\mathbf{b}) = q, wt_{\mathbf{m_w}}(\mathbf{b}) = q-p$, where $p \geq 2$. Then only relevant portions of Mayer-Vietoris sequence are $0 \to \widetilde{H}_{q-1}(\triangle_1; k) \to \widetilde{H}_{q-1}(\triangle; k) \to 0$, $0 \longrightarrow \widetilde{H}_{q-p}(\triangle; k) \longrightarrow \widetilde{H}_{q-p-1}(\Gamma; k) \longrightarrow 0$. These exact sequences gives us the desired Betti numbers.

Proof of **Case 6** is as in the **Case 5**. □

We remark that $B_{\mathbf{u}} \cap B_{\mathbf{v}} \subseteq B_{\mathbf{w}}$ and all the above cases are relevant. Using Theorem 3.1.1 we shall illustrate the computation of Betti numbers of $I(\mathbf{u})+I(\mathbf{v})$ in the following example.

**Example 3.1.2.** Let $\mathbf{u} = (1,2,3,4,6,6)$ and $\mathbf{v} = (1,4,4,4,5,6)$ and $\mathbf{w} = (1,4,4,4,6,6)$. Writing $(a,b,b,c,c,c)$ compactly as $(a,b^2,c^3)$, it can be checked that

$$\begin{aligned}
B_{\mathbf{u}} &= \{(1,6^5),(2^2,6^4),(3^3,6^3),(4^4,6^2),(1,2,3,4,6^2),(1,2,3,6^3),(1,2,4^2,6^2),\\
&\quad (2^2,3,4,6^2),(1,3^2,4,6^2),(1,2,6^4),(1,3^2,6^3),(1,4^3,6^2),(2^2,3,6^3),(2^2,4^2,6^2),\\
&\quad (3^3,4,6^2),(6^6)\},\\
B_{\mathbf{v}} &= \{(1,6^5),(4^2,6^4),(4^3,6^3),(4^4,6^2),(1,4,6^4),(1,4^2,6^3),(1,4^3,6^2),(6^6)\}, \text{ and}\\
B_{\mathbf{w}} &= \{(1,6^5),(4^2,6^4),(4^3,6^3),(4^4,6^2),(5^5,6),(1,4,6^4),(1,4^2,6^3),(1,4^3,6^2),(1,5^4,6),\\
&\quad (4^2,5^3,6),(4^3,5^2,6),(4^4,5,6),(1,4,5^3,6),(1,4^2,5^2,6),(1,4^3,5,6),(6^6)\}.
\end{aligned}$$

If $\mathbf{b} \in B_{\mathbf{u}}$ only (*i.e.* $\mathbf{b} \notin B_{\mathbf{v}} \cup B_{\mathbf{w}}$), then $\beta_{i,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v})) = \beta_{i,\mathbf{b}}(I(\mathbf{u}))$. For instance $\mathbf{b} = (2^2,3,4,6^2)$, then $\beta_{1,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v})) = \beta_{1,\mathbf{b}}(I(\mathbf{u})) = 1$, where $wt_{m_{\mathbf{u}}}(\mathbf{b}) = 1$. and $\beta_{i,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v})) = 0$ for all $i \neq 1$. Similarly, we can calculate the Betti numbers of $I(\mathbf{u})+I(\mathbf{v})$ for $\mathbf{b} \in B_{\mathbf{v}}$ only (or $\mathbf{b} \in B_{\mathbf{w}}$ only). Also

$$\begin{aligned}
B_{\mathbf{u}} \cap B_{\mathbf{w}} &= \{(1,6^5),(4^4,6^2),(1,4^3,6^2),(6^6)\} = B_{\mathbf{u}} \cap B_{\mathbf{v}} \cap B_{\mathbf{w}} \text{ , and}\\
B_{\mathbf{v}} \cap B_{\mathbf{w}} &= \{(1,6^5),(4^2,6^2),(1,4^3,6^2),(6^6),(4^2,6^4),(4^3,6^3),(1,4,6^4),(1,4^2,6^3)\}.
\end{aligned}$$

If $\mathbf{b} \in B_{\mathbf{u}} \cap B_{\mathbf{v}} \cap B_{\mathbf{w}}$, for instance $\mathbf{b} = (4^4,6^2)$, then $wt_{m_{\mathbf{u}}}(\mathbf{b}) = 3, wt_{m_{\mathbf{v}}}(\mathbf{b}) = 2, wt_{m_{\mathbf{w}}}(\mathbf{b}) = 1$. In view of Theorem 3.1.1,$\beta_{3,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v})) = \beta_{3,\mathbf{b}}(I(\mathbf{u})) = 1$, $\beta_{2,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v})) = \beta_{2,\mathbf{b}}(I(\mathbf{v})) + \beta_{1,\mathbf{b}}(I(\mathbf{w})) = 3+3 = 6$ and $\beta_{i,\mathbf{b}}(I(\mathbf{u})+I(\mathbf{v})) = 0$

for all $i \neq 2, 3$. On the other hand, if $\mathbf{b} \in B_{\mathbf{v}} \cap B_{\mathbf{w}}$ but $\mathbf{b} \notin B_{\mathbf{u}}$, for instance $\mathbf{b} = (4^3, 6^3)$, then $wt_{m_{\mathbf{v}}}(\mathbf{b}) = 3, wt_{m_{\mathbf{w}}}(\mathbf{b}) = 2$. Again by Theorem 3.1.1, we have $\beta_{3,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = \beta_{3,\mathbf{b}}(I(\mathbf{v})) + \beta_{2,\mathbf{b}}(I(\mathbf{w})) = 2 + 4 = 6$ and $\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v})) = 0$ for all $i \neq 3$. Similarly, other Betti numbers can be calculated.

## 3.2 Betti numbers of an Alexander dual of the sum of two multipermutohedron ideals

Now we proceed to calculate the Betti numbers of an Alexander dual of the sum of two multipermutohedron ideals. For a monomial ideal $I$, the multigraded Betti numbers of $I$ in degree $\mathbf{b}$ are given by $\beta_{i,\mathbf{b}}(I) = \dim_k \widetilde{H}_{|Supp(\mathbf{b})|-i-2}(K_{\mathbf{b}}(I); k);\ \ i \geq 0$, where the support $Supp(\mathbf{b}) = \{i : b_i > 0\}$. Set $\mathbf{a} = \mathbf{w_n} + \mathbf{c} - \mathbf{1} = (w_n + c - 1, w_n + c - 1, \ldots, w_n + c - 1)$ for $c \geq 1$. Let $I(\mathbf{u})^{[\mathbf{a}]}$, $I(\mathbf{v})^{[\mathbf{a}]}$, $I(\mathbf{w})^{[\mathbf{a}]}$ be Alexander duals of $I(\mathbf{u})$, $I(\mathbf{v})$ and $I(\mathbf{w})$ with respect to $\mathbf{a}$.

It is easy to see that $I_1^{[\mathbf{a}]} + I_2^{[\mathbf{a}]} = (I_1 \cap I_2)^{[\mathbf{a}]}$ and $I_1^{[\mathbf{a}]} \cap I_2^{[\mathbf{a}]} = (I_1 + I_2)^{[\mathbf{a}]}$, for monomial ideals $I_1$ and $I_2$, whose minimal generators divides $\mathbf{x}^{\mathbf{a}}$. Also,

$$K_{\mathbf{b}}(I_1 + I_2) = K_{\mathbf{b}}(I_1) \cap K_{\mathbf{b}}(I_2) \text{ and } K_{\mathbf{b}}(I_1 \cap I_2) = K_{\mathbf{b}}(I_1) \cup K_{\mathbf{b}}(I_2).$$

Let $J$ be a dual $\mathbf{m_u}$-isolated subset of $[n]$ and $\mathcal{I}^*_{\mathbf{m_u}}$ be the set of all dual $\mathbf{m_u}$-isolated subsets of $[n]$ (Definition 2.3.2). Let $\tilde{B}_{\mathbf{u}} = \{\tilde{\mathbf{b}}(J) \in \mathbb{N}^n : J \in \mathcal{I}^*_{\mathbf{m_u}}\}$. Similarly, we define $\tilde{B}_{\mathbf{v}}$ and $\tilde{B}_{\mathbf{w}}$ and let $\tilde{B} = \tilde{B}_{\mathbf{u}} \cup \tilde{B}_{\mathbf{v}} \cup \tilde{B}_{\mathbf{w}}$. In view of Theorem 2.3.4, we have the following facts.

- The lower Koszul simplicial complex $K_{\tilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]})$ of an Alexander dual $I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ of a multipermutohedron ideal $I(\mathbf{u})$ is join of skeletons of simplices and

$$\dim K_{\tilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = j_t - dwt_{\mathbf{m_u}}(J) - 2.$$

- The multigraded Betti numbers of $I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ are given as follows. For

$J = \{j_1, j_2, \ldots, j_t\} \in \mathcal{I}^*_{\mathbf{m_u}}$,

$$\beta_{i-1,\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = \Big[\prod_{\alpha=1}^{t} \binom{j_\alpha - j_{\alpha-1} - 1}{s_{i_\alpha - 1} - j_{\alpha-1}}\Big] \delta_{i-1,dwt_{\mathbf{m_u}}(J)},$$

where $J \cap (s_{i_\alpha -1}, s_{i_\alpha}] = \{j_\alpha\}$. If $\pi$ is a permutation of $\widetilde{\mathbf{b}}(J)$, then

$$\beta_{i-1,\pi\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = \beta_{i-1,\widetilde{\mathbf{b}}(J)}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}).$$

- If $\mathbf{b} \neq \pi\widetilde{\mathbf{b}}(J)$ for any $J \in \mathcal{I}^*_{\mathbf{m_u}}(\langle i-1\rangle)$ and any permutation $\pi$ of $\widetilde{\mathbf{b}}(J)$, then

$$\beta_{i-1,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}) = 0.$$

If we take $\triangle_1 = K_{\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]})$, $\triangle_2 = K_{\mathbf{b}}(I(\mathbf{v})^{[\mathbf{a}]})$, then $\triangle = K_{\mathbf{b}}((I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]})$ and $\Gamma = K_{\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]})$. This again give rise to a reduced Mayer-Vietoris sequence of homology groups. Again, many terms in the reduced Mayer-Vietoris sequence are zero. Thus we can calculate the multigraded Betti numbers of $(I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}$.

**Theorem 3.2.1.** *For $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{N}^n$ and $i \geq 0$, let $\beta_{i,\mathbf{b}}(I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}$ be the $i^{th}$ multigraded Betti number of the Alexander dual of the sum of two multipermutohedron ideals $I(\mathbf{u})$ and $I(\mathbf{v})$ in degree $\mathbf{b}$. As $\beta_{i,\mathbf{b}}((I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}) = \beta_{i,\pi\mathbf{b}}((I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]})$ for any permutation $\pi$ of $\mathbf{b}$ and $\beta_{i,\mathbf{b}}((I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}) = 0$ if $\mathbf{b} \neq \pi\mathbf{b}'$, for $\mathbf{b}'$ any element of $\tilde{B}$ and $\pi$ any permutation of $\mathbf{b}'$. Thus, it is enough to take $\mathbf{b} \in \tilde{B}$ and we have the following cases:*

**Case 1** : $\mathbf{b} \in \tilde{B}_{\mathbf{u}}, \mathbf{b} \notin \tilde{B}_{\mathbf{v}}, \mathbf{b} \notin \tilde{B}_{\mathbf{w}}$ *and* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q$. *Then*

$$\beta_{i,\mathbf{b}}((I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]}) & \textit{if } i = q, \\ 0 & \textit{otherwise.} \end{cases}$$

**Case 2** : *For* $\mathbf{b} \notin \tilde{B}_{\mathbf{u}}, \mathbf{b} \in \tilde{B}_{\mathbf{v}}, \mathbf{b} \notin \tilde{B}_{\mathbf{w}}$ *and* $dwt_{\mathbf{m_v}}(\mathbf{b}) = q$. *Then*

$$\beta_{i,\mathbf{b}}((I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{v})^{[\mathbf{a}]}) & \textit{if } i = q, \\ 0 & \textit{otherwise.} \end{cases}$$

**Case 3** : *For* $\mathbf{b} \notin \tilde{B}_{\mathbf{u}}, \mathbf{b} \notin \tilde{B}_{\mathbf{v}}, \mathbf{b} \in \tilde{B}_{\mathbf{w}}$ *and* $dwt_{\mathbf{m_w}}(\mathbf{b}) = q$. *Then*

$$\beta_{i-1,\mathbf{b}}((I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]}) & \textit{if } i = q, \\ 0 & \textit{otherwise.} \end{cases}$$

**Case 4** : $\mathbf{b} \in \tilde{B}_{\mathbf{u}} \cap \tilde{B}_{\mathbf{v}} \cap \tilde{B}_{\mathbf{w}}$. *Then we have the following sub-cases:*

*i)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q, dwt_{\mathbf{m_v}}(\mathbf{b}) = q + p, dwt_{\mathbf{m_w}}(\mathbf{b}) = q + p + 1$ *for* $p \geq 1$. *Then*

$$\beta_{i,\mathbf{b}}((I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]}) & \textit{if } i = q, \\ \beta_{i,\mathbf{b}}(I(\mathbf{v})^{[\mathbf{a}]}) + \beta_{i+1,\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]}) & \textit{if } i = q + p, \\ 0 & \textit{otherwise.} \end{cases}$$

*ii)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q, dwt_{\mathbf{m_v}}(\mathbf{b}) = q, dwt_{\mathbf{m_w}}(\mathbf{b}) = q$. *Then*

$$\beta_{i,\mathbf{b}}((I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]}) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]}) + \beta_{i,\mathbf{b}}(I(\mathbf{v})^{[\mathbf{a}]}) - \beta_{i,\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]}) & \textit{if } i = q, \\ 0 & \textit{otherwise.} \end{cases}$$

*iii)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q, dwt_{\mathbf{m_v}}(\mathbf{b}) = q + p, dwt_{\mathbf{m_w}}(\mathbf{b}) = q + (p + g)$, *where* $p \geq 1, g \geq 2$. *Then*

$$\beta_{i,\mathbf{b}}((I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]}) & \textit{if } i = q, \\ \beta_{i,\mathbf{b}}(I(\mathbf{v})^{[\mathbf{a}]}) & \textit{if } i = q + p, \\ \beta_{i+1,\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]}) & \textit{if } i = q + p + g - 1, \\ 0 & \textit{otherwise.} \end{cases}$$

*iv)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q = dwt_{\mathbf{m_v}}(\mathbf{b}), dwt_{\mathbf{m_w}}(\mathbf{b}) = q + 1$. *Then*

$$\beta_{i,\mathbf{b}}((I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]}) + \beta_{i,\mathbf{b}}(I(\mathbf{v})^{[\mathbf{a}]}) + \beta_{i+1,\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]}) & \textit{if } i = q, \\ 0 & \textit{otherwise.} \end{cases}$$

*v)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q = dwt_{\mathbf{m_v}}(\mathbf{b}), dwt_{\mathbf{m_w}}(\mathbf{b}) = q + p,$ *where* $p \geq 2$. *Then*

$$\beta_{i,\mathbf{b}}((I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]}) + \beta_{i,\mathbf{b}}(I(\mathbf{v})^{[\mathbf{a}]}) & \text{if } i = q, \\ \beta_{i+1,\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]}) & \text{if } i = q + p - 1, \\ 0 & \text{otherwise.} \end{cases}$$

*vi)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q, dwt_{\mathbf{m_v}}(\mathbf{b}) = dwt_{\mathbf{m_w}}(\mathbf{b}) = q + p$ *where* $p \geq 1$. *Then*

$$\beta_{i,b}((I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}) = \begin{cases} \beta_{i,b}(I(\mathbf{u})^{[\mathbf{a}]}) & \text{if } i = q, \\ \beta_{i,b}(I(\mathbf{v})^{[\mathbf{a}]}) - \beta_{i,b}(I(\mathbf{w})^{[\mathbf{a}]}) & \text{if } i = q + p, \\ 0 & \text{otherwise.} \end{cases}$$

**Case 5** :$\mathbf{b} \in \tilde{B}_{\mathbf{u}} \cap \tilde{B}_{\mathbf{w}}, \mathbf{b} \notin \tilde{B}_{\mathbf{v}}$. *Then we have the following sub-cases:*

*i)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q, dwt_{\mathbf{m_w}}(\mathbf{b}) = q$. *Then*

$$\beta_{i,\mathbf{b}}((I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]}) - \beta_{i,\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]}) & \text{if } i = q, \\ 0 & \text{otherwise.} \end{cases}$$

*ii)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q, dwt_{\mathbf{m_w}}(\mathbf{b}) = q + 1$. *Then*

$$\beta_{i,\mathbf{b}}((I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]}) + \beta_{i+1,\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]}) & \text{if } i = q, \\ 0 & \text{otherwise.} \end{cases}$$

*iii)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q, dwt_{\mathbf{m_w}}(\mathbf{b}) = q + p$ *for* $p \geq 2$. *Then*

$$\beta_{i,\mathbf{b}}((I(\mathbf{u}) + I(\mathbf{v}))^{[\mathbf{a}]}) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]}) & \text{if } i = q, \\ \beta_{i+1,\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]}) & \text{if } i = q + p - 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Case 6** :$\mathbf{b} \in \tilde{B}_{\mathbf{v}} \cap \tilde{B}_{\mathbf{w}}, \mathbf{b} \notin \tilde{B}_{\mathbf{u}}$. *Then we have the following sub-cases:*

*i)* $dwt_{\mathbf{m_v}}(\mathbf{b}) = q, dwt_{\mathbf{m_w}}(\mathbf{b}) = q$. *Then*

$$\beta_{i,\mathbf{b}}((I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]}) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{v})^{[\mathbf{a}]}) - \beta_{i,\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]}) & \text{if } i=q, \\ 0 & \text{otherwise.} \end{cases}$$

*ii)* $dwt_{\mathbf{m_v}}(\mathbf{b}) = q, dwt_{\mathbf{m_w}}(\mathbf{b}) = q+1$. *Then*

$$\beta_{i,\mathbf{b}}((I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]}) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{v})^{[\mathbf{a}]}) + \beta_{i+1,\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]}) & \text{if } i=q, \\ 0 & \text{otherwise.} \end{cases}$$

*iii)* $dwt_{\mathbf{m_v}}(\mathbf{b}) = q, dwt_{\mathbf{m_w}}(\mathbf{b}) = q+p$ *for* $p \geq 2$. *Then*

$$\beta_{i,\mathbf{b}}((I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]}) = \begin{cases} \beta_{i,\mathbf{b}}(I(\mathbf{v})^{[\mathbf{a}]}) & \text{if } i=q, \\ \beta_{i+1,\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]}) & \text{if } i=q+p-1, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** It is enough to take $\mathbf{b} \in \tilde{B} = \tilde{B}_{\mathbf{u}} \cup \tilde{B}_{\mathbf{v}} \cup \tilde{B}_{\mathbf{w}}$. Then we the following cases:

**Case 1** : $\mathbf{b} \in \tilde{B}_{\mathbf{u}}, \mathbf{b} \notin \tilde{B}_{\mathbf{v}}, \mathbf{b} \notin \tilde{B}_{\mathbf{w}}$ and $dwt_{\mathbf{m_u}}(\mathbf{b}) = q$. Then only relevant portion of Mayer-Vietoris sequence is $0 \to \widetilde{H}_{j_t-q-2}(\triangle_1;k) \to \widetilde{H}_{j_t-q-2}(\triangle;k) \to 0$. From this exact sequence, we get the desired Betti numbers.

**Case 2** : $\mathbf{b} \notin B_{\mathbf{u}}, \mathbf{b} \in \tilde{B}_{\mathbf{v}}, \mathbf{b} \notin \tilde{B}_{\mathbf{w}}$ and $dwt_{\mathbf{m_v}}(\mathbf{b}) = q$. Then only relevant portion of Mayer-Vietoris sequence is $0 \to \widetilde{H}_{j_t-q-2}(\triangle_2;k) \to \widetilde{H}_{j_t-q-2}(\triangle;k) \to 0$. From this exact sequence, we get the desired Betti numbers.

**Case 3** : $\mathbf{b} \notin \tilde{B}_{\mathbf{u}}, \mathbf{b} \notin \tilde{B}_{\mathbf{v}}, \mathbf{b} \in \tilde{B}_{\mathbf{w}}$ and $dwt_{\mathbf{m_w}}(\mathbf{b}) = q$. Then only relevant portion of Mayer-Vietoris sequence is $0 \to \widetilde{H}_{j_t-(q-1)-2}(\triangle;k) \to \widetilde{H}_{j_t-q-2}(\Gamma;k) \to 0$. From this exact sequence, we get the desired Betti numbers.

**Case 4** : $\mathbf{b} \in \tilde{B}_{\mathbf{u}} \cap \tilde{B}_{\mathbf{v}} \cap \tilde{B}_{\mathbf{w}}$. Then we have the following sub-cases:

*i)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q, dwt_{\mathbf{m_v}}(\mathbf{b}) = q+p, dwt_{\mathbf{m_w}}(\mathbf{b}) = q+p+1$ for $p \geq 1$. Then Mayer-Vietoris sequence gives us two exact sequences. $0 \to \widetilde{H}_{j_t-q-2}(\triangle_1;k) \to \widetilde{H}_{j_t-q-2}(\triangle;k) \to 0$ and $0 \longrightarrow \widetilde{H}_{j_t-(q+p)-2}(\triangle_2;k) \longrightarrow \widetilde{H}_{j_t-(q+p)-2}(\triangle;k) \longrightarrow \widetilde{H}_{j_t-(q+p+1)-2}(\Gamma;k) \to 0$. From the above exact sequences, we obtain the desired Betti numbers.

*ii)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q, dwt_{\mathbf{m_v}}(\mathbf{b}) = q, dwt_{\mathbf{m_w}}(\mathbf{b}) = q$. Then only relevant portion of

Mayer-Vietoris sequence is $0 \to \widetilde{H}_{j_t-q-2}(\Gamma;k) \to \widetilde{H}_{j_t-q-2}(\triangle_1;k) \oplus \widetilde{H}_{j_t-q-2}(\triangle_2;k)$ $\to \widetilde{H}_{j_t-q-2}(\triangle;k) \to 0$. From this exact sequence, we obtain the desired Betti numbers. Note that $\dim\triangle_1 = \dim\triangle_2 = j_t - q - 2$ and because $\dim\triangle = \max\{\dim\triangle_1, \dim\triangle_2\} = j_t - q - 2$, we have $\widetilde{H}_{j_t-(q-1)-2}(\triangle;k) = 0$.

*iii)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q, dwt_{\mathbf{m_v}}(\mathbf{b}) = q+p, dwt_{\mathbf{m_w}}(\mathbf{b}) = q+(p+g)$, where $p \geq 1, g \geq 2$. In this case the only relevant portions of Mayer-Vietoris sequence are $0 \longrightarrow \widetilde{H}_{j_t-q-2}(\triangle_1;k) \longrightarrow \widetilde{H}_{j_t-q-2}(\triangle;k) \longrightarrow 0$, $0 \longrightarrow \widetilde{H}_{j_t-(q+p)-2}(\triangle_2;k) \longrightarrow \widetilde{H}_{j_t-(q+p)-2}(\triangle;k) \longrightarrow 0$ and $0 \longrightarrow \widetilde{H}_{j_t-(q+p+g-1)-2}(\triangle;k) \longrightarrow \widetilde{H}_{j_t-(q+p+g)-2}(\Gamma;k)$ $\longrightarrow 0$. These exact sequences gives us the desired Betti numbers.

*iv)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q = dwt_{\mathbf{m_v}}(\mathbf{b}), dwt_{\mathbf{m_w}}(\mathbf{b}) = q+1$. Then only relevant portion of Mayer-Vietoris sequence is $0 \to \widetilde{H}_{j_t-q-2}(\triangle_1;k) \oplus \widetilde{H}_{j_t-q-2}(\triangle_2;k) \to \widetilde{H}_{j_t-q-2}(\triangle;k) \to \widetilde{H}_{j_t-(q+1)-2}(\Gamma;k) \to 0$. From this exact sequence, we obtain the desired Betti numbers.

*v)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q = dwt_{\mathbf{m_v}}(\mathbf{b}), dwt_{\mathbf{m_w}}(\mathbf{b}) = q+p$, where $p \geq 2$.
Then only relevant portions of Mayer-Vietoris sequence are $0 \to \widetilde{H}_{j_t-q-2}(\triangle_1;k)$ $\oplus \widetilde{H}_{j_t-q-2}(\triangle_2;k) \to \widetilde{H}_{j_t-q-2}(\triangle;k) \to 0$ and $0 \longrightarrow \widetilde{H}_{j_t-(q+p-1)-2}(\triangle;k) \longrightarrow \widetilde{H}_{j_t-(q+p)-2}(\Gamma;k) \longrightarrow 0$. These exact sequences gives us the desired Betti numbers.

*vi)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q, dwt_{\mathbf{m_v}}(\mathbf{b}) = dwt_{\mathbf{m_w}}(\mathbf{b}) = q+p$ where $p \geq 1$. Then Mayer-Vietoris sequence reduces to $0 \longrightarrow \widetilde{H}_{j_t-q-2}(\triangle_1;k) \longrightarrow \widetilde{H}_{j_t-q-2}(\triangle;k) \to 0 \longrightarrow \widetilde{H}_{j_t-(q+p-1)-2}(\triangle;k) \xrightarrow{\delta} \widetilde{H}_{j_t-(q+p)-2}(\Gamma;k) \to \widetilde{H}_{j_t-(q+p)-2}(\triangle_2;k) \longrightarrow \widetilde{H}_{j_t-(q+p)-2}(\triangle;k) \to 0$, where $\delta$ is a connecting homomorphism.
**Claim** : $\delta : \widetilde{H}_{j_t-(q+p-1)-2}(\triangle;k) \to \widetilde{H}_{j_t-(q+p)-2}(\Gamma;k)$ is a zero connecting homomorphism.
Let $z \in \widetilde{H}_{j_t-(q+p-1)-2}(\triangle;k)$. Then $z = c_1 + c_2$ with $\partial z = 0$ and $c_i$ is a $k$-valued $(j_t-(q+p-1)-2)$-chain of $\triangle_i$; $i = 1,2$. As $\partial(z) = 0$, we have $\partial(c_1) = -\partial(c_2)$. Therefore the connecting homomorphism is given by $\delta(z) = \partial(c_1) = -\partial(c_2)$. Since $\dim\triangle_2 = \dim K_{\mathbf{b}}(I(\mathbf{v})^{[\mathbf{a}]}) = j_t-(q+p)-2$, we have $c_2 = 0$. This proves the claim. Thus we have the exact sequences

$0 \to \widetilde{H}_{j_t-q-2}(\triangle_1;k) \to \widetilde{H}_{j_t-q-2}(\triangle;k) \to 0$ and $0 \to \widetilde{H}_{j_t-(q+p)-2}(\Gamma;k) \to \widetilde{H}_{j_t-(q+p)-2}(\triangle_2;k) \to \widetilde{H}_{j_t-(q+p)-2}(\triangle;k) \to 0$. These exact sequences gives us the desired Betti numbers.

**Case 5** : $\mathbf{b} \in \tilde{B}_{\mathbf{u}} \cap \tilde{B}_{\mathbf{w}}, \mathbf{b} \notin \tilde{B}_{\mathbf{v}}$. Then we have the following sub-cases:

*i)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q, dwt_{\mathbf{m_w}}(\mathbf{b}) = q$. Then only relevant portion of Mayer-Vietoris sequence is $0 \to \widetilde{H}_{j_t-q-2}(\Gamma;k) \to \widetilde{H}_{j_t-q-2}(\triangle_1;k) \to \widetilde{H}_{j_t-q-2}(\triangle;k) \to 0$. From the above exact sequences, we obtain the desired Betti numbers. We prove that the connecting homomorphism $\delta : \widetilde{H}_{j_t-(q-1)-2}(\triangle;k) \to \widetilde{H}_{j_t-q-2}(\Gamma;k)$ is zero. Let $z \in \widetilde{H}_{j_t-(q-1)-2}(\triangle;k)$. Then $z = c_1+c_2$ with $\partial z = 0$ and $c_i$ a $k$-valued $(j_t-(q-1)-2)$-chain of $\triangle_i;\ i = 1,2$. As $\partial(z) = 0$, we have $\partial(c_1) = -\partial(c_2)$. Therefore the connecting homomorphism is given by $\delta(z) = \partial(c_1) = -\partial(c_2)$. Since $\dim \triangle_1 = \dim K_{\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]}) = j_t - q - 2$, we have $c_1 = 0$.

*ii)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q, dwt_{\mathbf{m_w}}(\mathbf{b}) = q+1$. In this case the only relevant portion of Mayer-Vietoris sequence is $0 \longrightarrow \widetilde{H}_{j_t-q-2}(\triangle_1;k) \longrightarrow \widetilde{H}_{j_t-q-2}(\triangle;k) \longrightarrow \widetilde{H}_{j_t-(q+1)-2}(\Gamma;k) \to 0$. From this exact sequence, we obtain the desired Betti numbers.

*iii)* $dwt_{\mathbf{m_u}}(\mathbf{b}) = q, dwt_{\mathbf{m_w}}(\mathbf{b}) = q+p$, where $p \geq 2$. Then only relevant portions of Mayer-Vietoris sequence are $0 \to \widetilde{H}_{j_t-q-2}(\triangle_1;k) \to \widetilde{H}_{j_t-q-2}(\triangle;k) \to 0$, $0 \to \widetilde{H}_{j_t-(q+p-1)-2}(\triangle;k) \to \widetilde{H}_{j_t-(q+p)-2}(\Gamma;k) \to 0$. These exact sequences gives us the desired Betti numbers.

Proof of **Case 6** is as in the **Case 5**. □

**Example 3.2.2.** Let $\mathbf{u} = (1,2,4,4,4)$ and $\mathbf{v} = (2,2,3,3,4)$ and $\mathbf{w} = (2,2,4,4,4)$.

It can be checked that

$$\begin{aligned}
\tilde{B}_{\mathbf{u}} &= \{(4,0,0,0,0),(3,3,0,0,0),(1,1,1,0,0),(1,1,1,1,0),\\
&\quad (1,1,1,1,1),(4,3,0,0,0),(4,1,1,0,0),(4,1,1,1,0),\\
&\quad (4,1,1,1,1),(3,3,1,0,0),(3,3,1,1,0),(3,3,1,1,1),\\
&\quad (4,3,1,0,0),(4,3,1,1,0),(4,3,1,1,1)\},\\
\tilde{B}_{\mathbf{v}} &= \{(3,0,0,0,0),(3,3,0,0,0),(2,2,2,0,0),(2,2,2,2,0),\\
&\quad (1,1,1,1,1),(3,2,2,0,0),(3,2,2,2,0),(3,1,1,1,1),\\
&\quad (3,3,2,0,0),(3,3,2,2,0),(3,3,1,1,1),(2,2,2,1,1),\\
&\quad (2,2,2,2,1),(3,2,2,1,1),(3,2,2,2,1),(3,3,2,1,1),\\
&\quad (3,3,2,2,1)\},\\
\tilde{B}_{\mathbf{w}} &= \{(3,0,0,0,0),(3,3,0,0,0),(1,1,1,0,0),(1,1,1,1,0),\\
&\quad (1,1,1,1,1),(3,1,1,0,0),(3,1,1,1,0),(3,1,1,1,1),\\
&\quad (3,3,1,0,0),(3,3,1,1,0),(3,3,1,1,1)\},\\
\tilde{B}_{\mathbf{u}} \cap \tilde{B}_{\mathbf{w}} &= \{(3,3,0,0,0),(1,1,1,0,0),(1,1,1,1,0),(1,1,1,1,1),\\
&\quad (3,3,1,0,0),(3,3,1,1,0),(3,3,1,1,1)\},\\
\tilde{B}_{\mathbf{v}} \cap \tilde{B}_{\mathbf{w}} &= \{(3,0,0,0,0),(3,3,0,0,0),(1,1,1,1,1),(3,1,1,1,1),\\
&\quad (3,3,1,1,1)\},\\
\tilde{B}_{\mathbf{u}} \cap \tilde{B}_{\mathbf{v}} \cap \tilde{B}_{\mathbf{w}} &= \{(3,3,0,0,0),(1,1,1,1,1),(3,3,1,1,1)\}.
\end{aligned}$$

If $\mathbf{b} \in \tilde{B}_{\mathbf{u}}$ only (*i.e.* $\mathbf{b} \notin \tilde{B}_{\mathbf{v}} \cup \tilde{B}_{\mathbf{w}}$), then $\beta_{i,\mathbf{b}}((I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]}) = \beta_{i,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]})$, where $dwt_{\mathbf{m_u}} = i$. For instance $\mathbf{b} = (4,3,0,0,0)$, then $\beta_{1,\mathbf{b}}((I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]})) = \beta_{1,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]}) = 1$, where $dwt_{m_{\mathbf{u}}}(\mathbf{b}) = 1$. and $\beta_{i,\mathbf{b}}((I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]}) = 0$ for all $i \neq 1$. Similarly, we can calculate the Betti numbers of $(I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]}$ for $\mathbf{b} \in \tilde{B}_{\mathbf{v}}$ only (or $\mathbf{b} \in \tilde{B}_{\mathbf{w}}$ only). If $\mathbf{b} \in \tilde{B}_{\mathbf{u}} \cap \tilde{B}_{\mathbf{v}} \cap \tilde{B}_{\mathbf{w}}$, for instance $\mathbf{b} = (3,3,1,1,1)$, then $dwt_{m_{\mathbf{u}}}(\mathbf{b}) = 3, dwt_{m_{\mathbf{v}}}(\mathbf{b}) = 2, dwt_{m_{\mathbf{w}}}(\mathbf{b}) = 4$. In view of Theorem 3.2.1, we have $\beta_{2,\mathbf{b}}((I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]}) = \beta_{2,\mathbf{b}}(I(\mathbf{v})^{[\mathbf{a}]}) = 1$, $\beta_{3,\mathbf{b}}((I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]}) = \beta_{3,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]}) + \beta_{4,\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]}) = 1+1 = 2$ and $\beta_{i,\mathbf{b}}((I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]}) = 0$ for all $i \neq 2,3$. If $\mathbf{b} \in \tilde{B}_{\mathbf{v}} \cap \tilde{B}_{\mathbf{w}}$ but $\mathbf{b} \notin \tilde{B}_{\mathbf{u}}$, for instance $\mathbf{b} = (3,1,1,1,1)$, then $dwt_{m_{\mathbf{v}}}(\mathbf{b}) = 1, dwt_{m_{\mathbf{w}}}(\mathbf{b}) = 3$.

Again by Theorem 3.1.1, we have $\beta_{1,\mathbf{b}}((I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]}) = \beta_{1,\mathbf{b}}(I(\mathbf{v})^{[\mathbf{a}]}) = 1$ and $\beta_{2,\mathbf{b}}((I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]}) = \beta_{3,\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]}) = 3$. If $\mathbf{b} \in \tilde{B}_{\mathbf{u}} \cap \tilde{B}_{\mathbf{w}}$ but $\mathbf{b} \notin \tilde{B}_{\mathbf{v}}$, for instance $\mathbf{b} = (3,3,1,1,0)$, then $dwt_{m_{\mathbf{u}}}(\mathbf{b}) = 2, dwt_{m_{\mathbf{w}}}(\mathbf{b}) = 3$. Again by Theorem 3.1.1, we have $\beta_{2,\mathbf{b}}((I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]}) = \beta_{2,\mathbf{b}}(I(\mathbf{u})^{[\mathbf{a}]}) + \beta_{3,\mathbf{b}}(I(\mathbf{w})^{[\mathbf{a}]}) = 1+1 = 2$.

As $u_1 \geq 1$, $v_1 \geq 1$, so $R/I((\mathbf{u}))+I(\mathbf{v}))^{[\mathbf{a}]}, R/I(\mathbf{u})^{[\mathbf{a}]}, R/I(\mathbf{v})^{[\mathbf{a}]}$ are Artinian $k$-algebras and $\dim_k(R/(I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]})$ is given by the following lemma.

**Lemma 3.2.3.** *For an Artinian $k$-algebra $R/(I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]}$, we have*

$$\begin{aligned}\dim_k(R/(I(\mathbf{u})+I(\mathbf{v}))^{[\mathbf{a}]}) &= \dim_k(R/(I(\mathbf{u}))^{[\mathbf{a}]}) + \dim_k(R/(I(\mathbf{v}))^{[\mathbf{a}]}) \\ &\quad - \dim_k(R/(I(\mathbf{u}) \cap I(\mathbf{v}))^{[\mathbf{a}]}).\end{aligned}$$

**Proof.** For ideals $I_1$ and $I_2$ in $R$, there exist an exact sequence of $R$-modules

$$0 \to R/(I_1 \cap I_2) \to R/I_1 \oplus R/I_2 \to R/(I_1+I_2) \to 0.$$

Now taking $I_1 = I(\mathbf{u})^{[\mathbf{a}]}$ and $I_2 = I(\mathbf{v})^{[\mathbf{a}]}$, the desired result follows. □

**Remark 3.2.4.** It is easy to check that Modular law holds for monomial ideals. In other words, if $I$, $J$ and $K$ are monomial ideals in a polynomial ring $R = k[x_1, x_2, \ldots, x_n]$, then $I \cap (J+K) = I \cap J + I \cap K$. Thus if $I(\mathbf{u}_i)$ are multipermutohedron ideals such that $R/I(\mathbf{u}_i)^{[\mathbf{a}]}$ are Artinian $k$-algebras for $1 \leq i \leq l$, then using the inclusion-exclusion principle we have the following formula in general.

$$\dim_k\left(\frac{R}{(\sum_{i=1}^{l} I(\mathbf{u}_i))^{[\mathbf{a}]}}\right) = \sum_{i=1}^{l} \dim_k\left(\frac{R}{I(\mathbf{u}_i)^{[\mathbf{a}]}}\right) - \sum_{1\leq i<j\leq l} \dim_k\left(\frac{R}{(I(\mathbf{u}_i) \cap I(\mathbf{u}_j))^{[\mathbf{a}]}}\right)$$

$$+ \ldots + (-1)^{l-1} \dim_k\left(\frac{R}{(I(\mathbf{u}_1) \cap I(\mathbf{u}_2) \cap \ldots \cap I(\mathbf{u}_l))^{[\mathbf{a}]}}\right).$$

It is clear that the intersection of multipermutohedron ideals is again a multipermutohedron ideal. Each term $\dim_k\left(\frac{R}{(\bigcap_{\alpha=1}^{j} I(\mathbf{u}_{i_\alpha}))^{[\mathbf{a}]}}\right)$ in the RHS of the above expression can be calculated using the Steck determinant formula (Theorem 3.1.1). Thus the quantity $\dim_k\left(\frac{R}{(\sum_{i=1}^{l} I(\mathbf{u}_i))^{[\mathbf{a}]}}\right)$ in LHS can also be calculated.

# Chapter 4

# Split-Multipermutohedron Ideals

In this chapter, we introduce a variant of multipermutohedron ideals called split-multipermutohedron ideals. We study the Alxander duals of these ideals. We also calculate the Betti numbers of split-multipermutohedron ideals and their Alexander duals.

## 4.1 Betti numbers of split-multipermutohedron ideals

Let $n = r + s$ with $r, s \geq 1$, be a positive integer and let $\mathfrak{S}_n$ be the set of all permutations of $\{1, 2, \ldots, n\}$. Consider the subset $H$ of $\mathfrak{S}_n$ consisting of permutations of the type $(\sigma_1, \sigma_2)$, where $\sigma_1$ is a permutation of $\{1, 2, \ldots, r\}$ and $\sigma_2$ is a permutation of $\{r+1, \ldots, r+s = n\}$. Clearly $H$ is a subgroup isomorphic to $\mathfrak{S}_r \times \mathfrak{S}_s$. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{N}^n$ with $0 \leq u_1 \leq u_2 \leq \ldots \leq u_n$. The convex hull of the points $\sigma\mathbf{u}$ for $\sigma \in H$ is a polytope, which is just the product of multipermutohedrons $P(\mathbf{v}) \times P(\mathbf{w})$, where $\mathbf{v} = (u_1, \ldots, u_r)$ and $\mathbf{w} = (u_{r+1}, \ldots, u_n) = (w_1, w_2, \ldots, w_s)$.

**Definition 4.1.1.** The monomial ideal $\mathcal{I} = \langle \mathbf{x}^{\sigma_1 \mathbf{v}} \mathbf{y}^{\sigma_2 \mathbf{w}} \ : (\sigma_1, \sigma_2) \in H \rangle$ in the polynomial ring $k[\mathbf{x}, \mathbf{y}] = k[x_1, x_2, \ldots, x_r, y_1, \ldots, y_s]$, with $y_j = x_{r+j}$ is called a *split-multipermutohedron ideal.*

We notice that the ideal $\mathcal{I} = I(\mathbf{v}) \otimes_k I(\mathbf{w})$, where $I(\mathbf{v})$ and $I(\mathbf{w})$ are the multipermutohedron ideals. Thus the split-multipermutohedron ideal $\mathcal{I}$ depends on the splitting of the vector $\mathbf{u} = (\mathbf{v}, \mathbf{w})$, but instead of denoting it by a cumbersome notation $\mathcal{I}_{(\mathbf{v},\mathbf{w})}$, we simply write $\mathcal{I}$ without causing any confusion. Let $R_1 = k[x_1, x_2, \ldots, x_r]$ and $R_2 = k[y_1, y_2, \ldots, y_s]$ be polynomials rings over a field $k$ and $R = R_1 \otimes_k R_2 = k[\mathbf{x}]\otimes_k k[\mathbf{y}] \cong k[\mathbf{x}, \mathbf{y}]$. Let $\mathbf{b} \in \mathbb{N}^n$, then we can write $\mathbf{b} = (\mathbf{b_v}, \mathbf{b_w})$, where $\mathbf{b_v} = (b_1, b_2, \ldots, b_r)$ and $\mathbf{b_w} = (b_{r+1}, b_{r+2}, \ldots, b_n)$.

**Lemma 4.1.2.** *The upper Koszul simplicial complex $K^{\mathbf{b}}(\mathcal{I})$ of $\mathcal{I}$ in degree $\mathbf{b} = (\mathbf{b_v}, \mathbf{b_w})$ is given by*

$$K^{\mathbf{b}}(\mathcal{I}) = K^{\mathbf{b_v}}(I(\mathbf{v})) * K^{\mathbf{b_w}}(I(\mathbf{w})),$$

*where $*$ denotes the simplicial join.*

**Proof.** For a square-free vector $\tau = (\tau_{\mathbf{v}}, \tau_{\mathbf{w}})$, we have

$$\begin{aligned} \tau \in K^{\mathbf{b}}(\mathcal{I}) &\Leftrightarrow \mathbf{x}^{(\mathbf{b_v}-\tau_{\mathbf{v}}, \mathbf{b_w}-\tau_{\mathbf{w}})} \in \mathcal{I} \\ &\Leftrightarrow \mathbf{x}^{\mathbf{b_v}-\tau_{\mathbf{v}}} \in I(\mathbf{v}) \text{ and } \mathbf{x}^{\mathbf{b_w}-\tau_{\mathbf{w}}} \in I(\mathbf{w}) \\ &\Leftrightarrow \tau_{\mathbf{v}} \in K^{\mathbf{b_v}}(I(\mathbf{v})) \text{ and } \tau_{\mathbf{w}} \in K^{\mathbf{b_w}}(I(\mathbf{w})) \\ &\Leftrightarrow \tau = (\tau_{\mathbf{v}}, \tau_{\mathbf{w}}) \in K^{\mathbf{b_v}}(I(\mathbf{v})) * K^{\mathbf{b_w}}(I(\mathbf{w})). \end{aligned}$$

This completes the proof. □

Let $\Sigma_1$ and $\Sigma_2$ be two simplicial complexes. Then the Künneth formula for the homology vector space of the join $\Sigma_1 * \Sigma_2$ is given by

$$\widetilde{H}_{i-1}(\Sigma_1 * \Sigma_2; k) = \bigoplus_{p+q=i-2} \widetilde{H}_p(\Sigma_1; k) \otimes \widetilde{H}_q(\Sigma_2; k). \tag{4.1.1}$$

We shall use this formula to calculate the multigraded Betti numbers of split-multipermutohedron ideals and their Alexander duals.

**Proposition 4.1.3.** *The multigraded Betti numbers of a split-multipermutohedron ideal $\mathcal{I}$ exist only in the multidegree $\mathbf{b}$ of the form $\mathbf{b} = (\mathbf{b_v}, \mathbf{b_w})$ and*

$$\beta_{i,\mathbf{b}}(\mathcal{I}) = \beta_{p,\mathbf{b_v}}(I(\mathbf{v}))\beta_{q,\mathbf{b_w}}(I(\mathbf{w})),$$

*where* $\mathbf{b_v} = \mathbf{b}(J)$, $\mathbf{b_w} = \mathbf{b}(J')$ *for* $J \in \mathcal{I}_{\mathbf{m_v}}$, $J' \in \mathcal{I}_{\mathbf{m_w}}$ *(as in Definition 1.2.3) and* $wt_{\mathbf{m_v}}(\mathbf{b_v}) = p$, $wt_{\mathbf{m_w}}(\mathbf{b_w}) = q$ *with* $p + q = i$.

**Proof.** In view of Theorem 3.5 in [17], it is sufficient to take $\mathbf{b} = (\mathbf{b_v}, \mathbf{b_w})$, where $\mathbf{b_v} = \mathbf{b}(J)$, $\mathbf{b_w} = \mathbf{b}(J')$ for $J \in \mathcal{I}_{\mathbf{m_v}}$, $J' \in \mathcal{I}_{\mathbf{m_w}}$. From the Lemma 4.1.2 we have $K^{\mathbf{b}}(\mathcal{I}) = K^{\mathbf{b_v}}(I(\mathbf{v})) * K^{\mathbf{b_w}}(I(\mathbf{w}))$. Thus using the Künneth formula 4.1.1 for the join of simplicial complexes, we have

$$\beta_{i,\mathbf{b}}(\mathcal{I}) = \sum_{p+q=i} \beta_{p,\mathbf{b_v}}(I(\mathbf{v}))\beta_{q,\mathbf{b_w}}(I(\mathbf{w})). \tag{4.1.2}$$

Further, in view of Theorem 3.5 in [17], we observe that

$$\beta_{p,\mathbf{b_v}}(I(\mathbf{v})) \neq 0 \Leftrightarrow p = wt_{\mathbf{m_v}}(\mathbf{b_v}).$$

Hence we have the desired result. □

**Remark 4.1.4.** From the last proposition, $i^{th}$ Betti number $\beta_i(\mathcal{I})$ can easily be calculated. We have

$$\begin{aligned}
\beta_i(\mathcal{I}) &= \sum_{\mathbf{b}} \beta_{i,\mathbf{b}}(\mathcal{I}) \\
&= \sum_{\mathbf{b}=(\mathbf{b_v},\mathbf{b_w})} \sum_{p+q=i} (\beta_{p,\mathbf{b_v}}(I(\mathbf{v}))\beta_{q,\mathbf{b_w}}(I(\mathbf{w}))) \qquad (4.1.2) \\
&= \sum_{p+q=i} \left(\sum_{\mathbf{b_v}} \beta_{p,\mathbf{b_v}}(I(\mathbf{v}))\right) \left(\sum_{\mathbf{b_w}} \beta_{q,\mathbf{b_w}}(I(\mathbf{w}))\right) \\
&= \sum_{p+q=i} \beta_p(I(\mathbf{v}))\beta_q(I(\mathbf{w})).
\end{aligned}$$

However, the above result on Betti numbers $\beta_i(\mathcal{I})$ of split-multipermutohedron ideals is true for a large class of ideals. Let $I_1$ and $I_2$ be ideals in polynomial rings $k[\mathbf{x}]$ and $k[\mathbf{y}]$, respectively. Suppose $\mathbb{F}_* \to I_1$ and $\mathbb{G}_* \to I_2$ are minimal free resolutions. Then it is known that the tensor complex $\mathbb{F}_* \otimes_k \mathbb{G}_*$ is the minimal free resolution of $I_1 \otimes_k I_2$. Hence formula for the $i^{th}$ Betti number of $\mathcal{I}$ holds in general for ideals of the form $I_1 \otimes_k I_2$.

## 4.2 Alexander duals of split multipermutohedron ideals

Let $\mathbf{v}$ and $\mathbf{w}$ be defined as above. Set $\mathbf{v_r} = (u_r, u_r, \ldots, u_r) \in \mathbb{R}^r$ and $\mathbf{w_s} = (u_n, u_n, \ldots, u_n) \in \mathbb{R}^s$.

**Lemma 4.2.1.** *Let* $\mathbf{a} = (\mathbf{v_r}, \mathbf{w_s}) = (u_r, u_r, \ldots, u_r, u_n, u_n, \ldots, u_n)$. *Then the Alexander dual of* $\mathcal{I}$ *with respect to* $\mathbf{a}$ *is given by*

$$\mathcal{I}^{[\mathbf{a}]} = I(\mathbf{v})^{[\mathbf{v_r}]} \otimes_k R_2 + R_1 \otimes_k I(\mathbf{w})^{[\mathbf{w_s}]}.$$

**Proof.** We recall that, for any $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ with $\mathbf{a} - \mathbf{b} \in \mathbb{N}^n$, the monomial $\mathbf{x^b} \notin \mathcal{I}$ if and only if $\mathbf{x^{a-b}} \in \mathcal{I}^{[\mathbf{a}]}$ (Proposition 1.1.26). Clearly $\mathbf{x^{a-b}}$ is a minimal generator of $\mathcal{I}^{[\mathbf{a}]}$ precisely when $\mathbf{b}$ is maximal. Thus a maximal $\mathbf{b} \preceq \mathbf{a}$ with $\mathbf{x^b} \notin \mathcal{I}$ is either $\mathbf{b} = (\mathbf{b_1}, \mathbf{w_s})$, where $\mathbf{b_1}$ is maximal with $\mathbf{x^{b_1}} \notin I(\mathbf{v})$ or $\mathbf{b} = (\mathbf{v_r}, \mathbf{b_2})$, where $\mathbf{b_2}$ is maximal with $\mathbf{x^{b_2}} \notin I(\mathbf{w})$. Hence

$$\mathcal{I}^{[\mathbf{a}]} = I(\mathbf{v})^{[\mathbf{v_r}]} \otimes_k R_2 + R_1 \otimes_k I(\mathbf{w})^{[\mathbf{w_s}]}.$$

This completes the proof. $\square$

**Lemma 4.2.2.** *The lower Koszul simplicial complex* $K_{\mathbf{b}}(\mathcal{I}^{[\mathbf{a}]})$ *of* $\mathcal{I}^{[\mathbf{a}]}$ *in degree* $\mathbf{b} = (\mathbf{b_v}, \mathbf{b_w})$ *is given by*

$$K_{\mathbf{b}}(\mathcal{I}^{[\mathbf{a}]}) = K_{\mathbf{b_v}}(I(\mathbf{v})^{[\mathbf{v_r}]}) * K_{\mathbf{b_w}}(I(\mathbf{w})^{[\mathbf{w_s}]}),$$

*where* $*$ *denotes the simplicial join.*

**Proof.** For a square-free vector $\tau = (\tau_{\mathbf{v}}, \tau_{\mathbf{w}})$, we have

$$\begin{aligned}
\tau \in K_{\mathbf{b}}(\mathcal{I}^{[\mathbf{a}]}) &\Leftrightarrow \mathbf{x}^{(\mathbf{b}'_{\mathbf{v}}+\tau_{\mathbf{v}}, \mathbf{b}'_{\mathbf{w}}+\tau_{\mathbf{w}})} \notin \mathcal{I}^{[\mathbf{a}]} \text{ (Definition 1.1.18)} \\
&\Leftrightarrow \mathbf{x}^{\mathbf{b}'_{\mathbf{v}}+\tau_{\mathbf{v}}} \notin I(\mathbf{v})^{[\mathbf{v_r}]} \text{ and } \mathbf{x}^{\mathbf{b}'_{\mathbf{w}}+\tau_{\mathbf{w}}} \notin I(\mathbf{w})^{[\mathbf{w_s}]} \\
&\Leftrightarrow \tau_{\mathbf{v}} \in K_{\mathbf{b_v}}(I(\mathbf{v})^{[\mathbf{v_r}]}) \text{ and } \tau_{\mathbf{w}} \in K_{\mathbf{b_w}}(I(\mathbf{w})^{[\mathbf{w_s}]}) \\
&\Leftrightarrow \tau = (\tau_{\mathbf{v}}, \tau_{\mathbf{w}}) \in K_{\mathbf{b_v}}(I(\mathbf{v})^{[\mathbf{v_r}]}) * K_{\mathbf{b_w}}(I(\mathbf{w})^{[\mathbf{w_s}]}).
\end{aligned}$$

This completes the proof. $\square$

**Theorem 4.2.3.** *The multigraded Betti numbers of $R/\mathcal{I}^{[\mathbf{a}]}$ exist only in the multi-degree $\mathbf{b}$ of the form $\mathbf{b} = (\mathbf{b_v}, \mathbf{b_w})$ and*

$$\beta_{i,\mathbf{b}}(R/\mathcal{I}^{[\mathbf{a}]}) = \beta_{p,\mathbf{b_v}}(R_1/I(\mathbf{v})^{[\mathbf{v_r}]})\beta_{q,\mathbf{b_w}}(R_2/I(\mathbf{w})^{[\mathbf{w_s}]}),$$

*where $\mathbf{b_v} = \widetilde{\mathbf{b}}(J)$, $\mathbf{b_w} = \widetilde{\mathbf{b}}(J')$ for $J \in \mathcal{I}^*_{\mathbf{m_v}}$, $J' \in \mathcal{I}^*_{\mathbf{m_w}}$ (as in Definition 2.3.2) and $p = dwt_{\mathbf{m_v}}(\mathbf{b_v})$, $q = dwt_{\mathbf{m_w}}(\mathbf{b_w})$ with $p + q = i$.*

**Proof.** In view of Theorem 2.3.4, it is sufficient to take $\mathbf{b} = (\mathbf{b_v}, \mathbf{b_w})$, where $\mathbf{b_v} = \widetilde{\mathbf{b}}(J)$, $\mathbf{b_w} = \widetilde{\mathbf{b}}(J')$ for $J \in \mathcal{I}^*_{\mathbf{m_v}}$, $J' \in \mathcal{I}^*_{\mathbf{m_w}}$. From the Lemma 4.2.2, we have $K_{\mathbf{b}}(\mathcal{I}^{[\mathbf{a}]}) = K_{\mathbf{b_v}}(I(\mathbf{v})^{[\mathbf{v_r}]}) * K_{\mathbf{b_w}}(I(\mathbf{w})^{[\mathbf{w_s}]})$. Thus using the Künneth formula 4.1.1, we have

$$\beta_{i,\mathbf{b}}(R/\mathcal{I}^{[\mathbf{a}]}) = \sum_{p+q=i} \beta_{p,\mathbf{b_v}}(R_1/I(\mathbf{v})^{[\mathbf{v_r}]})\beta_{q,\mathbf{b_w}}(R_2/I(\mathbf{w})^{[\mathbf{w_s}]}).$$

Further from Theorem 2.3.4, we observe that

$$\beta_{p,\mathbf{b_v}}(\mathcal{I}^{[\mathbf{a}]}) \neq 0 \Leftrightarrow p = dwt_{\mathbf{m_v}}(\mathbf{b_v}).$$

Hence we have the required result. $\square$

**Remark 4.2.4.** As in the case of split-multipermutohedron ideal, we can prove that

$$\beta_i(R/\mathcal{I}^{[\mathbf{a}]}) = \sum_{p+q=i} \beta_p(R_1/I(\mathbf{v})^{[\mathbf{v_r}]})\beta_q(R_2/I(\mathbf{w})^{[\mathbf{w_s}]}).$$

It follows from the above formula that if the cellular free complex supported on a polyhedral cell complex $\triangle_1$ gives us the (minimal) free resolution of $R_1/I(\mathbf{v})^{[\mathbf{v_r}]}$ and the cellular free complex supported on a polyhedral cell complex $\triangle_2$ gives us the (minimal) free resolution of $R_2/I(\mathbf{w})^{[\mathbf{w_s}]}$ then the cellular free complex supported on the join $\triangle_1 * \triangle_2$ gives us the (minimal) free resolution of $R/\mathcal{I}$. We illustrate this fact with the following example.

**Example 4.2.5.** Let $\mathbf{v} = (1, 2, 2)$ , $\mathbf{w} = (3, 4)$. Then $\mathbf{v_3} = (2, 2, 2), \mathbf{w_2} = (4, 4)$

and set $\mathbf{a} = (2,2,2,4,4)$. We see that

$$I(\mathbf{v})^{[\mathbf{v_3}]} = \langle x^2, y^2, z^2, xy, yz, zx \rangle,$$

and

$$I(\mathbf{w})^{[\mathbf{w_2}]} = \langle s^2, t^2, ts \rangle.$$

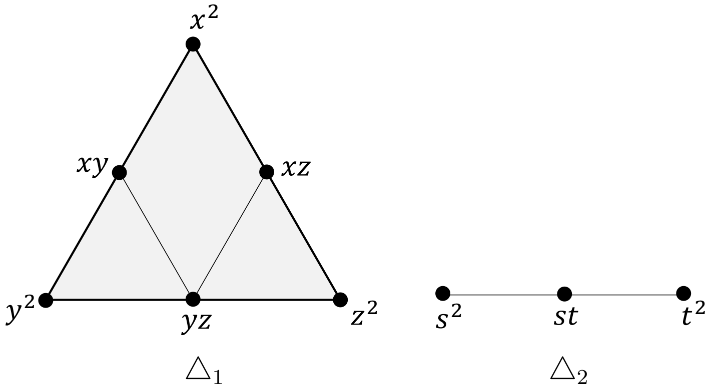



Figure 4.1: Polyhedral cell complexes $\triangle_1$ and $\triangle_2$.

Note $\beta_0(R_1/I(\mathbf{v})^{[\mathbf{v_3}]}) = 1$, $\beta_1(R_1/I(\mathbf{v})^{[\mathbf{v_3}]}) = 6$, $\beta_2(R_1/I(\mathbf{v})^{[\mathbf{v_3}]}) = 8$, $\beta_3(R_1/I(\mathbf{v})^{[\mathbf{v_3}]}) = 3$, and $\beta_0(R_2/I(\mathbf{w})^{[\mathbf{w_2}]}) = 1$, $\beta_1(R_2/I(\mathbf{w})^{[\mathbf{w_2}]}) = 3$, $\beta_2(R_2/I(\mathbf{w})^{[\mathbf{w_2}]}) = 2$.

As

$$\beta_i(R/\mathcal{I}^{[\mathbf{a}]}) = \sum_{p+q=i} \beta_p(R_1/I(\mathbf{v})^{[\mathbf{u_r}]})\beta_q(R_2/I(\mathbf{w})^{[\mathbf{w_s}]}).$$

Therefore, $\beta_0(R/\mathcal{I}^{[\mathbf{a}]}) = 1$, $\beta_1(R/\mathcal{I}^{[\mathbf{a}]}) = 9$, $\beta_2(R/\mathcal{I}^{[\mathbf{a}]}) = 28$, $\beta_3(R/\mathcal{I}^{[\mathbf{a}]}) = 39$, $\beta_4(R/\mathcal{I}^{[\mathbf{a}]}) = 25$, $\beta_5(R/\mathcal{I}^{[\mathbf{a}]}) = 6$. Clearly $\beta_i(\mathcal{I}^{[\mathbf{a}]}) = \mathcal{F}_i(\triangle_1 * \triangle_2)$. Thus the cellular free complex supported on the join $\triangle_1 * \triangle_2$ gives us the minimal free resolution of $R/\mathcal{I}$.

# Chapter 5

# Hypercubic Ideals

In the last and final chapter, we introduce and study a class of ideals called *hypercubic ideals*. Hypercubic ideals are certain variants of permutohedron ideals and these ideals have an interesting minimal property. The number of standard monomials in an Artinian quotient of an Alexander dual of a hypercubic ideal is obtained by counting $\lambda$-parking functions with certain restrictions. We obtained a combinatorial formula for the number of restricted $\lambda$-parking functions.

## 5.1 Hypercubic ideals and their Alexander duals

Let $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$, $1 \leq u_1 < u_2 < \ldots < u_n$. For each non empty set $B \subseteq \mathfrak{S}_n$, we associate a monomial ideal $I_B = \langle \mathbf{x}^{\sigma\mathbf{u}} | \sigma \in B \rangle$. Clearly, $I_{\mathfrak{S}_n} = I(\mathbf{u})$ is a permutohedron ideal and the cellular free complex $\mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1}))$ supported on the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of a $(n-1)$-simplex is a minimal free resolution of an Alexander dual $I(\mathbf{u})^{[\mathbf{u_n}]}$. We asked a natural question. What is a minimal $B \subseteq \mathfrak{S}_n$ such that the cellular free complex associated to $\mathbf{Bd}(\Delta_{n-1})$ is a minimal free resolution of $I_B^{[\mathbf{u_n}]}$? In order to answer this question, we introduce a notion of *hypercubic ideals*. Firstly we consider a subset $W \subseteq \mathfrak{S}_n$ of $n$-permutations given by

$$W = \{\sigma \in \mathfrak{S}_n | \sigma(1) \text{ is arbitrary and } \sigma(j) = k \text{ for } j > 1 \text{ if either } \sigma(i) = k+1 \text{ or } \sigma(i) = k-1 \text{ for some } i < j\}.$$

**Definition 5.1.1.** The monomial ideal $I_W = \langle \mathbf{x}^{\sigma\mathbf{u}} | \sigma \in W \rangle$ in the polynomial ring $R = k[x_1, x_2, \ldots, x_n]$ is called a *hypercubic ideal.* We denote the hypercubic ideal $I_W$ by $J(\mathbf{u})$.

It is easy to see that $|W| = 2^{n-1}$ and the convex polytope spanned by $2^{n-1}$ points $\sigma\mathbf{u}, \sigma \in W$ is a $(n-1)$-dimensional hypercube and we denote it by $\mathcal{H}(\mathbf{u})$. A vertex $\sigma\mathbf{u}$ in the hypercube $\mathcal{H}(\mathbf{u})$ is naturally labeled with monomial $\mathbf{x}^{\sigma\mathbf{u}}$ for $\sigma \in W$. Thus the ideal $J(\mathbf{u})$ is the monomial ideal generated by vertex labels of the $(n-1)$-dimensional hypercube, which is the reason for calling it a hypercubic ideal.

The subset $W \subseteq \mathfrak{S}_n$ has been defined combinatorially as the set of permutations of $\{1, 2, \ldots, n\}$ such that apart from the leading term a number $k$ appears only if either $k+1$ or $k-1$ has already appeared. It is interesting to see that the hypercubic ideal $I_W = J(\mathbf{u})$ has the minimal property described earlier.

Now we proceed to systematically study hypercubic ideals. The following result is already known, although we have included its proof for the sake of completeness.

**Lemma 5.1.2.** *There is one to one correspondence between the set $W$ and the power set $\mathbf{P}[n-1]$ of $[n-1]$.*

**Proof.** Let $B =$ Set of $(n-1)$-tuples consisting of '0' and '1'. Clearly $|B| = 2^{n-1}$.
Define $f : W \to B$ as follows:
For $\sigma \in W, f(\sigma) \in B$ is given by

$$(f(\sigma))(i) = i^{th} \text{ coordinate of } f(\sigma) = \begin{cases} 1 & \text{if } \sigma(i) > \sigma(i+1), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{b} = (b_1, b_2, \ldots, b_{n-1}) \in B$ and $\sum_{i=0}^{n-1} b_i = l$. Then define $g : B \to W$ as follows:

$$(g(\mathbf{b}))(1) = l+1, \text{ and for } j \geq 2,\ (g(\mathbf{b}))(j) = \begin{cases} \max\limits_{1 \leq k \leq j-1} (g(\mathbf{b}))(k) + 1 & \text{if } b_{j-1} = 0, \\ \min\limits_{1 \leq k \leq j-1} (g(\mathbf{b}))(k) - 1 & \text{if } b_{j-1} = 1. \end{cases}$$

Clearly $g \circ f = 1_W$ and $f \circ g = 1_B$. □

**Facial Description of a Hypercube**

The Hasse diagram of the Boolean poset $(\mathbf{P}[n-1], \subseteq)$ constitutes vertices and edges of an $(n-1)$-dimensional hypercube. The faces of hypercube can be described by chains of subsets of $[n-1]$. A 0-dimensional face (vertex) of a hypercube corresponds to subsets of $[n-1]$. An 1-dimensional face (edge) corresponds to a chain $A_0 \subsetneq A_1 \subseteq [n-1]$ such that $|A_1| - |A_0| = 1$. Similarly, a 2-dimensional face corresponds to a chain $A_0 \subsetneq A_2 \subseteq [n-1]$ such that $|A_2| - |A_0| = 2$ and in general, a $d$-dimensional face corresponds to a chain $A_0 \subsetneq A_d \subseteq [n-1]$ such that $|A_d| - |A_0| = d$. Now we count the number of $d-$dimensional faces of a hypercube. As the number of ways of choosing $d$ elements out of $n-1$ elements is $\binom{n-1}{d}$ and total number of subsets of $[n-1-d]$ are $2^{n-1-d}$, so number of $d$-dimensional faces of a hypercube are exactly $2^{n-1-d}\binom{n-1}{d}$.

**Theorem 5.1.3.** *A hypercube $\mathcal{H}(\mathbf{u})$ supports the minimal cellular resolution of $J(\mathbf{u})$, where $J(\mathbf{u}) = \langle \mathbf{x}^{\sigma\mathbf{u}} \mid \sigma \in W \rangle$.*

**Proof.** A vertex $\sigma\mathbf{u}$ in the hypercube $\mathcal{H}(\mathbf{u})$ is naturally labeled with monomial $\mathbf{x}^{\sigma\mathbf{u}}$ for $\sigma \in W$ and monomial label on each face $F$ is a least common multiple of the monomial label on each of vertex $v \in F$. Thus $\mathcal{H}(\mathbf{u})$ is a labeled cell complex. It is clear that for any vector $\mathbf{b} \in \mathbb{N}^n$ either $\mathcal{H}(\mathbf{u})_{\preceq \mathbf{b}}$ is contractible or void. Also, the label of any face $F$ of hypercube is different from the label on any proper face of $F$. Thus in view of Theorem 1.1.33, we conclude that the cellular free complex $\mathbb{F}_*(\mathcal{H}(\mathbf{u}))$ associated to hypercube $\mathcal{H}(\mathbf{u})$ is a minimal free resolution of $J(\mathbf{u})$. $\square$

**Theorem 5.1.4.** *The minimal generators of Alexander dual $J(\mathbf{u})^{[\mathbf{u_n}]}$ of hypercubic ideal is given by*

$$J(\mathbf{u})^{[\mathbf{u_n}]} = \langle \prod_{j \in T} x_j^{\mu_{j,T}} \mid \emptyset \neq T = \{j_1, j_2, \ldots, j_t\} \subseteq [n]; j_1 < j_2 < \ldots < j_t \rangle,$$

*where $\mu_{j_1,T} = u_n - u_t + 1$ and $\mu_{j_i,T} = u_n - u_{t+j_i-i} + 1$ for $i \in \{2, 3, \ldots, t\}$.*

**Proof.** Let $T \subseteq [n]$ be a non-empty subset such that $T = \{j_1, j_2, \ldots, j_t\}$, where $j_1 < j_2 < \ldots < j_t$. Consider the vector

$$\mathbf{b}_T = \sum_{j \notin T} u_n e_j + (u_t - 1)e_{j_1} + \sum_{\alpha=2}^{t} (u_{t+j_\alpha-\alpha} - 1)e_{j_\alpha}. \tag{5.1.1}$$

Since $1 \leq j_1 < j_2 < \ldots < j_t \leq n$, we see that $j_2 - 2 \leq j_3 - 3 \leq \ldots \leq j_t - t$ and hence $t + j_\alpha - \alpha \leq j_t \leq n$ for $2 \leq \alpha \leq t$. Thus $\mathbf{b}_T \preceq \mathbf{u_n}$.

Claim: $\mathbf{x}^{\mathbf{b}_T} \notin J(\mathbf{u})$.

In order to prove the claim, we may write $T$ as a disjoint union of integer intervals. By integer interval $[a, b]$ for a, b $\in \mathbb{Z}$, we mean the set $[a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\}$. Let

$$\begin{aligned} T &= [j_1, j_{n_1}] \amalg [j_{n_1+1}, j_{n_2}] \amalg \ldots \amalg [j_{n_{r-1}+1}, j_{n_r}] \\ &= T_1 \amalg T_2 \amalg \ldots \amalg T_r, \end{aligned}$$

where $T_i = [j_{n_{i-1}+1}, j_{n_i}] = \{j_{n_{i-1}+1}, j_{n_{i-1}+2}, \ldots, j_{n_i}\}$ for $i = 1, 2, \ldots, r$; $n_0 = 0$ and $n_r = t$. Set $S = [1, n] - T$ and write $S$ also as a disjoint union of integer intervals i.e. $S = S_0 \amalg S_1 \amalg \ldots \amalg S_r$, where $S_\alpha = [j_{n_\alpha} + 1, j_{n_\alpha+1} - 1]; 0 \leq \alpha \leq r$. Clearly, $S_0 = \emptyset$ if and only if $j_1 = 1$ and $S_r = \emptyset$ if and only if $j_t = n$. Further, $S_\alpha \neq \emptyset$ for $1 \leq \alpha \leq r - 1$. Since $[1, n] - T = S$, we have

$$n = \sum_{i=0}^{r} |S_i| + t. \tag{5.1.2}$$

Also,$[1, j_{n_\alpha+1} - 1] = \left(\amalg_{i=1}^{\alpha} T_i\right) \amalg \left(\amalg_{i=0}^{\alpha} S_i\right)$ and $\amalg_{i=1}^{\alpha} T_i = \{j_1, j_2, \ldots, j_{n_\alpha}\}$ implies that

$$j_{n_\alpha+1} - 1 = n_\alpha + \sum_{i=0}^{\alpha} |S_i|. \tag{5.1.3}$$

We note that the vector $\mathbf{b}_T$ can also be expressed as

$$\mathbf{b}_T = \sum_{j \notin T} u_n e_j + (u_t - 1)e_{j_1} + \sum_{j_1 < j \leq j_{n_1}} (u_{t+j_1-1} - 1)e_j + \sum_{\alpha=2}^{r} \sum_{j \in T_\alpha} (u_{t+j_{n_\alpha}-n_\alpha} - 1)e_j.$$

Suppose, if possible $\mathbf{x}^{\mathbf{b}_T} \in J(\mathbf{u})$. Then there exists a vector $\mathbf{c} = \sum_{i=1}^{n} c_i e_i = \sigma\mathbf{u}$, $\sigma \in W$ such that $\mathbf{b}_T \succeq \mathbf{c}$. Since $\mathbf{c} = \sigma\mathbf{u}$ for some $\sigma \in W$, there exists some $i_0 \in [1, n]$ such that $c_{i_0} = u_n$. Also $j^{th}$ coordinate $(\mathbf{b}_T)_j$ of the vector $\mathbf{b}_T$ is $u_n$ if and only if $j \in S = [1, n] - T$. Thus $i_0 \in S_\alpha$ for some $0 \leq \alpha \leq r$. We now proceed to show that

$$\max_{j \in S_0} c_j \geq u_{n-|S_1|+|S_2|+\ldots+|S_\alpha|} \text{ if } \mathrm{S}_0 \neq \emptyset \tag{5.1.4}$$

or, in the other case

$$\max_{j \in S_1} c_j \geq u_{n-|S_2|+\ldots+|S_\alpha|} \qquad \text{if } \mathrm{S}_0 = \emptyset. \tag{5.1.5}$$

Suppose, if possible

$$\begin{aligned} \max_{j \in S_0} c_j &< u_{n-|S_1|+|S_2|+\ldots+|S_\alpha|} \text{ if } \mathrm{S}_0 \neq \emptyset \\ \text{or} \quad \max_{\mathrm{j} \in \mathrm{S}_1} \mathrm{c}_\mathrm{j} &< u_{n-|S_2|+\ldots+|S_\alpha|} \qquad \text{if } \mathrm{S}_0 = \emptyset. \end{aligned}$$

For any $\beta \leq \alpha$, in view of equations 5.1.2 and 5.1.3, we have

$$\begin{aligned} n - (|S_\beta| + |S_{\beta+1}| + \ldots + |S_\alpha|) &\geq n - \sum_{i=\beta}^{r} |S_i| \\ &= t + \sum_{i=0}^{\beta-1} |S_i| = t + j_{n_{\beta-1}+1} - 1 - n_{\beta-1}. \end{aligned}$$

Thus $u_{n-(|S_\beta|+\ldots+|S_\alpha|)} \geq u_{t+j_{n_{\beta-1}+1}-(n_{\beta-1}+1)} > u_{t+j_{n_{\beta-1}+1}-(n_{\beta-1}+1)} - 1$. Since the $j^{th}$ coordinate of $\mathbf{b}_T$ is $(\mathbf{b}_T)_j = u_{t+j_{n_{\beta-1}+1}-(n_{\beta-1}+1)} - 1$ for $j \in T_\beta, \beta \geq 1$ (except $j = j_1$) and $\mathbf{c} \preceq \mathbf{b}_T$, we have $c_j \leq u_{t+j_{n_{\beta-1}+1}-(n_{\beta-1}+1)} - 1 < u_{n-(|S_\beta|+\ldots+|S_\alpha|)}$ for $j \in T_\beta$ (except $j = j_1$). Further $c_{j_1} < u_t < u_{t+|S_0|} \leq u_{n-(|S_1|+\ldots+|S_\alpha|)}$. Thus

$$c_j < u_{n-(|S_\beta|+\ldots+|S_\alpha|)} \text{ for } j \in T_\beta, 1 \leq \beta \leq r. \tag{5.1.6}$$

Suppose $S_0 \neq \emptyset$ and $\max_{j \in S_0} c_j = u_{m_0}$. Then $u_{m_0} < u_{n-(|S_1|+\ldots+|S_\alpha|)}$. Also, using equation 5.1.6, we get $c_j < u_{n-(|S_1|+\ldots+|S_\alpha|)}$ for $j \in T_1$. Thus the maximum value of

$c_j$ for $j \in [1, j_{n_1}] = S_0 \cup T_1$ will remain less than $u_{n-(|S_1|+\ldots+|S_\alpha|)}$, i.e.

$$\max_{j\in[1,j_{n_1}]} c_j = u_{m_1} < u_{n-(|S_1|+\ldots+|S_\alpha|)}.$$

As $c \in \sigma\mathbf{u}$, for some $\sigma \in W$, we have $c_j = u_\alpha$ for $j > 1$ if and only if $c_k$ is either $u_{\alpha-1}$ or $u_{\alpha+1}$ for some $k < j$. If $c_{j_{n_1}+1} = u_{m_1+1}, c_{j_{n_1}+2} = u_{m_1+2}, \ldots, c_{j_{n_1+1}} - 1 = u_{m_1+(j_{n_1+1}-j_{n_1}-1)} = u_{m_1+|S_1|}$, then $\max_{j\in[1,j_{n_1+1}-1]} c_j = u_{m_1+|S_1|}$. Since the values of $c_j$ as $j$ varies over $S_1$ can not increase faster than the sequence $(u_{m_1+1}, \ldots, u_{m_1+|S_1|})$, we must have $\max_{j\in[1,j_{n_1+1}-1]} c_j = u_{m_1'}$, where $m_1' \leq m_1 + |S_1|$. Thus, we have

$$u_{m_1'} \leq u_{m_1+|S_1|} < u_{n-(|S_2|+\ldots|S_\alpha|)}.$$

Since $c_j < u_{n-(|S_2|+\ldots+|S_\alpha|)}$ for $j \in T_2$, $\max_{j\in[1,j_{n_2}]} c_j = u_{m_2} < u_{n-(|S_2|+\ldots+|S_\alpha|)}$.

Again, If $c_{j_{n_2}+1} = u_{m_2+1}, c_{j_{n_2}+2} = u_{m_2+2}, \ldots, c_{j_{n_2+1}} - 1 = u_{m_2+|S_2|}$, then we see that $\max_{j\in[1,j_{n_2+1}-1]} c_j = u_{m_2+|S_2|}$. Since the values of $c_j$ as $j$ varies over $S_2$ can not increase faster than the sequence $(u_{m_2+1}, \ldots, u_{m_2+|S_2|})$, we must have $\max_{j\in[1,j_{n_2+1}-1]} c_j = u_{m_2'}$, for some $m_2' \leq m_2 + |S_2|$. Thus

$$u_{m_2'} \leq u_{m_2+|S_2|} < u_{n-(|S_3|+\ldots|S_\alpha|)}.$$

Repeating this argument, we conclude that $\max_{j\in[1,j_{n_\alpha+1}-1]} c_j < u_n$, which is contrary to the fact that $c_{i_0} = u_n$ for $i_0 \in S_\alpha \subseteq [1, j_{n_\alpha+1} - 1]$. This proves that

$$\max_{j\in S_0} c_j \;\; \geq \;\; u_{n-|S_1|+\ldots+|S_\alpha|} \text{ if } \mathrm{S}_0 \neq \emptyset.$$

Similarly, we can prove the other case,

$$\max_{j\in S_1} c_j \;\; \geq \;\; u_{n-|S_2|+\ldots+|S_\alpha|} \text{ if } \mathrm{S}_0 = \emptyset.$$

For $S_0 \neq \emptyset$, we have $\max_{j\in S_0} c_j \geq u_{n-|S_1|+\ldots+|S_\alpha|} \geq u_{n-(\sum_{i=1}^r |S_i|)} = u_{t+|S_0|}$. This is possible only if $c_j \geq u_{t+1} \;\forall\; j \in S_0 = [1, j_1 - 1]$. As $c_{j_1} = u_t - 1$ and $c_j > u_t$ for $j < j_1$, we arrive at a contradiction. On the other hand, for $S_0 = \emptyset$, we have $\max_{j\in S_1} c_j \geq u_{n-|S_2|+\ldots+|S_\alpha|} \geq u_{n-(\sum_{i=2}^r |S_i|)} = u_{t+|S_1|}$. This shows that $c_j \geq u_{t+1} \;\forall\; j \in$

$S_1 = [j_1+1, j_{n_1}-1]$. Since $c_j \leq u_t - 1 < u_t \; \forall \, j \in T_1 = [1, j_1]$, we get a contradiction. Therefore, $\mathbf{x}^{\mathbf{b}_T} \notin J(\mathbf{u})$. This proves the claim.

We now proceed to show that $\mathbf{b}_T \preceq \mathbf{u_n}$ is maximal vector such that $\mathbf{x}^{\mathbf{b}_T} \notin J(\mathbf{u})$. In other words, we shall prove that $\mathbf{x}^{\mathbf{b}_T + e_j} \in J(\mathbf{u})$ for $j \in T$.

Let $j \in T$. Then we construct a vector $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ as follows. For $i \in T - \{j\}$, $c_i = u_{rank(i)}$, where $rank(i)$ is the rank of the element '$i$' in the set $T - \{j\}$. In other words, if $A = \{a_1, a_2, \ldots, a_l\}$ with $a_1 < a_1 < \ldots < a_l$, then $rank(a_k) = l - k + 1; \; 1 \leq k \leq l$. Also, $c_j = (\mathbf{b}_T)_j + 1$ and the value of $c_i$ for $i \notin T$ are obtained by arranging remaining $u_\alpha' s$ in an increasing order. We illustrate the choice of vector $\mathbf{c}$ by an example. Let $n = 7$ and $T = \{2, 3, 5, 6\}$. Then $\mathbf{b}_T = (u_7, u_4 - 1, u_5 - 1, u_7, u_6 - 1, u_6 - 1, u_7)$. Let $j = 5$. Then $\mathbf{b}_T + e_5 = (u_7, u_4 - 1, u_5 - 1, u_7, u_6, u_6 - 1, u_7)$. Then $c_6 = u_1, c_3 = u_2, c_2 = u_3, c_5 = u_6, c_1 = u_4, c_4 = u_5, c_7 = u_7$. Let $T = \{j_1, j_2, \ldots, j_t\}$ and $j = j_k \in T$. Then

$$c_{j_k} = \begin{cases} u_t & ; k = 1, \\ u_{t+j_k-k} & ; k > 1. \end{cases}$$

and for $i \neq k$,

$$c_{j_i} = \begin{cases} u_{t-i+1} & ; i > k, \\ u_{t-i} & ; i < k. \end{cases}$$

As $(\mathbf{b}_T)_{j_i} = u_{t+j_i-i} - 1 \geq u_{t+j_i-i-1} \geq c_{j_i}$ and

$$(\mathbf{b}_T + e_{j_k})_{j_k} = \begin{cases} u_t & ; k = 1, \\ u_{t+j_k-k} & ; k > 1. \end{cases}$$

We deduce that $\mathbf{b}_T + e_j \succeq \mathbf{c}$. It is clear that $u_1, u_2, \ldots, u_{t-1}$ appears in the vector $\mathbf{c}$ in a decreasing order at '$t-1$' places $j_1, j_2, \ldots, j_{k-1}, j_{k+1}, \ldots, j_t$. Thus if $u_\alpha$ appears at the $i^{th}$ position in the vector $\mathbf{c}$ for $i > 1$, then certainly $u_{\alpha-1}$ appears in $\mathbf{c}$ before the $i^{th}$ position. This shows that $\mathbf{x}^{\mathbf{c}} \in J(\mathbf{u})$. Thus $\mathbf{x}^{\mathbf{b}_T + e_j} \in J(\mathbf{u})$. $\square$

Consider the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of an $(n-1)$-simplex $\Delta_{n-1}$. A vertex of $\mathbf{Bd}(\Delta_{n-1})$ corresponds to a non-empty subset $T \subseteq [n]$ and hence it is naturally labeled with the monomial $\prod_{j \in T} x_j^{\mu_{j,T}} = x_{j_1}^{\mu_{j_1,T}} x_{j_2}^{\mu_{j_2,T}} \ldots x_{j_t}^{\mu_{j_t,T}}$ which are minimal generators of $J(\mathbf{u})^{[\mathbf{u_n}]}$ and $\mu_{j,T}$ are defined in Theorem 5.1.4. An $(i-1)$-dimensional face of $\mathbf{Bd}(\Delta_{n-1})$ corresponds to a tuple $(A_1, A_2, \ldots, A_i)$ of non-empty subsets of $[n]$ with $\emptyset = A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_i$ and the monomial label on this $(i-1)$-face can be seen to be $\prod_{q=1}^{i}(\prod_{j\in A_q - A_{q-1}} x_j^{\mu_{j,\, A_q}})$. Thus $\mathbf{Bd}(\Delta_{n-1})$ is a labeled simplicial complex. If we set $X = \mathbf{Bd}(\Delta_{n-1})$, then we see that $X_{\preceq \mathbf{b}}$ is either contractible or void. Thus using Theorem 1.1.33, we see that the cellular free complex $\mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1}))$ associated to the first Barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ is a minimal free resolution of the quotient $R' = R/J(\mathbf{u})^{[\mathbf{u_n}]}$.

**Proposition 5.1.5.** *Let $K(\mathbf{u})$ be a monomial ideal such that $J(\mathbf{u}) \subsetneq K(\mathbf{u}) \subsetneq I(\mathbf{u})$. Then minimal generators of the Alexander dual $K(\mathbf{u})^{[\mathbf{u_n}]}$ is parameterized by non-empty subsets $T$ of $[n]$.*

**Proof.** Let $\emptyset \neq T \subseteq [n]$. Then in view of Lemma 2.1.1 and Theorem 5.1.4 there are vectors $\mathbf{b}_T \preceq \mathbf{u_n}$ and $\mathbf{b'_T} \preceq \mathbf{u_n}$ maximal with property that $\mathbf{x}^{\mathbf{b}_T} \notin I(\mathbf{u})$ and $\mathbf{x}^{\mathbf{b'_T}} \notin J(\mathbf{u})$. Now $\mathbf{x}^{\mathbf{b}_T} \notin I(\mathbf{u})$ implies that $\mathbf{x}^{\mathbf{b}_T} \notin K(\mathbf{u})$. If $\mathbf{b}_T \preceq \mathbf{u_n}$ is maximal such that $\mathbf{x}^{\mathbf{b}_T} \notin K(\mathbf{u})$, then using Theorem 1.1.26 we have $\mathbf{x}^{\mathbf{u_n}-\mathbf{b}_T} \in K(\mathbf{u})^{[\mathbf{u_n}]}$. Otherwise there exists a vector $\mathbf{w}_T$ with $\mathbf{u_n} \succcurlyeq \mathbf{w}_T \succ \mathbf{b}_T$, so that $\mathbf{w}_T$ is maximal with the property that $\mathbf{x}^{\mathbf{w}_T} \notin K(\mathbf{u})$. Further we notice that $\mathbf{w}_T \preceq \mathbf{b'_T}$. Otherwise $\mathbf{x}^{\mathbf{w}_T} \in J(\mathbf{u})$, which implies $\mathbf{x}^{\mathbf{w}_T} \in K(\mathbf{u})$, a contradiction. Hence we conclude that for all non-empty subsets $T$ of $[n]$ there is a vector $\mathbf{w}_T$ with $\mathbf{b}_T \preceq \mathbf{w}_T \preceq \mathbf{b'_T}$, which is maximal with the property that $\mathbf{x}^{\mathbf{w}_T} \notin K(\mathbf{u})$. In other words, $\mathbf{x}^{\mathbf{u_n}-\mathbf{w}_T}$ is a minimal generator of $K(\mathbf{u})^{[\mathbf{u_n}]}$. □

**Theorem 5.1.6.** *For a non empty subset $B$ of $\mathfrak{S}_n$, the ideal $I_W = J(\mathbf{u})$ has a property that if $B \supseteq W$, then the cellular free complex $\mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1}))$ supported on $\mathbf{Bd}(\Delta_{n-1})$ is a minimal free resolution of the quotient $R/I_B^{[\mathbf{u_n}]}$ and if $B \subsetneq W$, then the cellular resolution of the quotient $R/I_B^{[\mathbf{u_n}]}$ is not minimally supported on $\mathbf{Bd}(\Delta_{n-1})$.*

**Proof.** A proof of this theorem depends heavily on the proof of Theorem 5.1.4. Therefore, we continue to use notations and terminologies used in the proof of

Theorem 5.1.4. Let $J'(\mathbf{u}) = I_{W'} = \langle \mathbf{x}^{\sigma\mathbf{u}} | \sigma \in W' \rangle$, where $W' = W - \{\sigma\}$ for some $\sigma \in W$. For a non-empty subset $T \subseteq [n]$ such that either $1 \notin T$ but $2 \in T$ or $1 \in T$ but $2 \notin T$, we now define a vector $\mathbf{v}_T$ as follows.

$$\mathbf{v}_T = \sum_{j \notin T} u_n e_j + (u_{t+1} - 1)e_{j_1} + \sum_{\alpha=2}^{t} (u_{t+j_\alpha - \alpha} - 1)e_{j_\alpha},$$

where $\emptyset \neq T = \{j_1, j_2, \ldots, j_t\} \subseteq [n]$ with $j_1 < j_2 < \ldots < j_t$ and either $j_1 = 1$ but $j_2 > 2$ or $j_1 = 2$. We notice that there are exactly $2^{n-1}$ subsets $T \subseteq [n]$ of the above type and hence there are exactly $2^{n-1}$ associated vectors $\mathbf{v}_T$.

Claim: There exists a unique vector $\mathbf{c} = \sigma\mathbf{u}$ for $\sigma \in W$ such that $\mathbf{v}_T \succeq \mathbf{c}$ or, equivalently $\mathbf{x}^{\mathbf{c}}$ divides $\mathbf{x}^{\mathbf{v}_T}$. We now proceed to prove this claim.

*Case* 1. Take $T = \{j_1 = 2, j_2, \ldots, j_t\}$, where $j_1 < j_2 < \ldots < j_t$. Write $T = [j_1, j_{n_1}] \amalg [j_{n_1+1}, j_{n_2}] \amalg \ldots \amalg [j_{n_{r-1}+1}, j_{n_r}] = T_1 \amalg T_2 \amalg \ldots \amalg T_r$ and set $S = [1, n] - T = S_0 \amalg S_1 \amalg \ldots \amalg S_r$. Then $|S_0| = 1$. Set $T' = \{1\} \cup T$ and consider a vector $\mathbf{b}_{T'} = \sum_{j \notin T'} u_n e_j + (u_{t+1} - 1)e_1 + \sum_{\alpha=1}^{t} (u_{t+1+j_\alpha-(\alpha+1)} - 1)e_{j_\alpha}$ as in Theorem 5.1.4 (Equation 5.1.1). Clearly $\mathbf{v}_T \succ \mathbf{b}_{T'}$. Thus $\mathbf{x}^{\mathbf{v}_T} \in J(\mathbf{u})$. Since the ideal $J(\mathbf{u})$ is minimally generated by $\mathbf{x}^{\sigma\mathbf{u}}$; $\sigma \in W$, there exists $\mathbf{c} = \sigma\mathbf{u}$ with $\sigma \in W$ such that $\mathbf{v}_T \succeq \mathbf{c}$. Consider a vector $\mathbf{b}_T = \sum_{j \notin T} u_n e_j + (u_t - 1)e_{j_1} + \sum_{\alpha=2}^{t} (u_{t+j_\alpha-\alpha} - 1)e_{j_\alpha}$ as in Theorem 5.1.4 (Equation 5.1.1). Note that $\mathbf{v}_T$ and $\mathbf{b}_T$ differ only in one coordinate ($j = j_1$). Using the same technique as in Theorem 5.1.4 one can show that $\max\limits_{j \in S_0} c_j \geq u_{n-|S_1|+|S_2|+\ldots+|S_\alpha|}$ for some $1 \leq \alpha \leq r$. In fact,

$$\max_{j \in S_0} c_j \geq u_{n-|S_1|+|S_2|+\ldots+|S_\alpha|} \geq u_{n-|S_1|+|S_2|+\ldots+|S_r|} = u_{t+|S_0|}.$$

Thus $c_1 \geq u_{t+1}$. Also $c_2 \leq (\mathbf{v}_T)_2 = u_{t+1} - 1$. This implies $c_1 = u_{t+1}$ and $c_2 = u_t$. Now using equation 5.1.3, we have $t + j_{n_{\beta-1}+1} - (n_{\beta-1} + 1) = t + \sum_{i=0}^{\beta-1} |S_i|$ for $1 \leq \beta \leq r$. Thus $(\mathbf{v}_T)_j = u_{t+j_{n_{\beta-1}+1}-(n_{\beta-1}+1)} - 1 = u_{t+\sum_{i=0}^{\beta-1}|S_i|} - 1$; $j \in T_\beta$ (except $j \neq j_1$). Thus using the fact that $\mathbf{c} \preceq \mathbf{v}_T$, we have $c_j < u_{t+\sum_{i=0}^{\beta-1}|S_i|}$; $j \in T_\beta$. Since $c_{j_1} < u_{t+1}$, we have $c_j < u_{t+1}$; $j \in T_1$. This implies that $c_j = u_{t-j+2}$ for $j \in T_1$. Again using the same technique as in Theorem 5.1.4, one can show that $\max\limits_{j \in [1, j_{n_1+1}-1]} c_j \geq u_{n-(|S_2|+\ldots|S_\alpha|)} = u_{t+|S_0|+|S_1|}$. This implies that $c_j = u_{t+s+|S_0|}$, where

$j = j_{n_1} + s \in S_1$ and $s = 1, 2, \ldots, |S_1|$. Continuing in this manner we see that $\mathbf{c} = u_{t+1}e_1 + \sum_{h=0}^{r-1} \sum_{l=n_h+1}^{n_{h+1}} u_{t-l+1}e_{j_l} + \sum_{i=1}^{r} \sum_{s=1}^{|S_i|} (u_{t+|S_0|+|S_1|+\ldots+|S_{i-1}|+s})e_{j_{n_i}+s}$.

*Case* 2. Take $T = \{j_1 = 1, j_2, \ldots, j_t\}$, where $j_2 > 2$ and $j_1 > j_2 > \ldots > j_t$. Set $T'' = \{2\} \cup T$ and consider a vector $\mathbf{b}_{T''}$ as in Theorem 5.1.4 (Equation 5.1.1). Now proceeding as in Case 1, we get

$$\mathbf{c} = \sum_{h=0}^{r-1} \sum_{l=n_h+1}^{n_{h+1}} u_{t-l+1}e_{j_l} + \sum_{i=1}^{r} \sum_{s=1}^{|S_i|} (u_{t+|S_0|+|S_1|+\ldots+|S_{i-1}|+s})e_{j_{n_i}+s},$$

where $n_1 = 1, |S_0| = 0$.

From the above discussion, we conclude that to each vector $\mathbf{v}_T$ associated to $2^{n-1}$ possible subsets $T \subseteq [n]$, there exists a unique vector $\mathbf{c} = \sigma\mathbf{u}$, $\sigma \in W$. Since $J(\mathbf{u})$ is minimally generated by $\mathbf{x}^{\sigma\mathbf{u}}$, $\sigma \in W$. We see that minimal generators of $J(\mathbf{u})$ are in one-one correspondence with $\mathbf{v}_T$. Thus if we delete one of the generator of $J(\mathbf{u})$, there will be exactly one $\mathbf{v}_T$ such that $\mathbf{x}^{\mathbf{v}_T} \notin J'(\mathbf{u})$. Consider the case $T = \{j_1 = 2, j_2, \ldots, j_t\}$. Let $\mathbf{v}_T \succeq \mathbf{c}$ for a unique vector $\mathbf{c} = \sigma_1\mathbf{u}$, $\sigma_1 \in W$ and set $W' = W - \{\sigma_1\}$. Now in view of Theorem 5.1.4, we have $\mathbf{x}^{\mathbf{b}_{T'}+e_j} \in J(\mathbf{u})$ for $j \in T'$. We will take $j \in T$ (means $j \neq 1$), in this case also, $\mathbf{x}^{\mathbf{b}_{T'}+e_j} \in J(\mathbf{u})$. In other words $\mathbf{b}_{T'} + e_j \succeq \mathbf{d}$, for some $\mathbf{d} = \sigma\mathbf{u}$, $\sigma \in W$ with $j \in T$. Since $(\mathbf{b}_{T'} + e_j)_1 = u_{t+1} - 1$, for $j \in T$ and $c_1 = u_{t+1}$ (Case 1). We have $\mathbf{d} \neq \mathbf{c}$. Thus $\mathbf{d} = \sigma\mathbf{u}, \sigma \in W'$. Which further implies $\mathbf{x}^{\mathbf{b}_{T'}+e_j} \in J'(\mathbf{u}) = I_{W'}$, $j \in T$. Since $(\mathbf{v}_T)_j = (\mathbf{b}_{T'})_j$ for all $j \neq 1$ and $u_n = (\mathbf{v}_T)_1 > (\mathbf{b}_{T'})_1$. We conclude that $\mathbf{v}_T \preceq \mathbf{u_n}$ is maximal such that $\mathbf{x}^{\mathbf{v}_T} \notin J'(\mathbf{u}) = I_{W'}$. Thus $\mathbf{x}^{\mathbf{u_n}-\mathbf{v}_T}$ is a minimal generator of $J'(\mathbf{u})^{[\mathbf{u_n}]}$. Also $\mathbf{x}^{\mathbf{u_n}-\mathbf{b}_T} \in J'(\mathbf{u})^{[\mathbf{u_n}]}$ and $\mathbf{x}^{\mathbf{u_n}-\mathbf{b}_{T'}} \in J'(\mathbf{u})^{[\mathbf{u_n}]}$. Since $\mathbf{v}_T \succ \mathbf{b}_{T'}$ and $\mathbf{v}_T \succ \mathbf{b}_T$, the monomial $\mathbf{x}^{\mathbf{u_n}-\mathbf{v}_T}$ strictly divides $\mathbf{x}^{\mathbf{u_n}-\mathbf{b}_{T'}}$ and $\mathbf{x}^{\mathbf{u_n}-\mathbf{b}_T}$. Thus there will be no minimal generator of $J'(\mathbf{u})^{[\mathbf{u_n}]}$ corresponding to non-empty subset $T'$ of $[n]$. Similarly for $T = \{j_1 = 1, j_2, \ldots, j_t\}$; $j_2 > 2$, we have $(\mathbf{b}_{T''}+e_j)_2 = u_{t+1}-1$, for $j \in T$ and $c_2 = u_{t+1}$ (Case 2). Thus in this case also we have the same conclusion. Hence the cellular resolution of the quotient $R/I_{W'}^{[\mathbf{u_n}]}$ is not minimally supported on $\mathbf{Bd}(\Delta_{n-1})$.

Now if $B \supseteq W$, then in view of Proposition 5.1.5 the minimal generators of an Alexander dual $I_B^{[\mathbf{u_n}]}$ is parameterized by non-empty subsets $T$ of $[n]$. Thus the vertices of first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ of $(n-1)$-simplex can be naturally labeled with the minimal generators of $I_B^{[\mathbf{u_n}]}$. Now from the way $\mathbf{Bd}(\Delta_{n-1})$ is labeled and in view of Theorem 1.1.33, we conclude that the cellular free complex $\mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1}))$ supported on $\mathbf{Bd}(\Delta_{n-1})$ is a minimal free resolution of the quotient $R/I_B^{[\mathbf{u_n}]}$. □

We will illustrate the proof of the last theorem by analyzing the following example.

**Example 5.1.7.** Let $J(\mathbf{u}) = \langle xy^2z^3, x^2yz^3, x^2y^3z, x^3y^2z\rangle$. Then the Alexander dual of $J(\mathbf{u})$ with respect to $\mathbf{u_3}$ is $J(\mathbf{u})^{[\mathbf{u_3}]} = \langle x^3, y^3, z^3, x^2y^2, y^2z, x^2z, xyz\rangle$. We see that for $T = \{1\}$, we have $\mathbf{v}_T = (1,3,3) \succ (1,2,3)$. For $T = \{2\}$, we have $\mathbf{v}_T = (3,1,3) \succ (2,1,3)$. For $T = \{1,3\}$, we have $\mathbf{v}_T = (2,3,2) \succ (2,3,1)$. For $T = \{2,3\}$, we have $\mathbf{v}_T = (3,2,2) \succ (3,2,1)$. Now delete $xy^2z^3$ from $J(\mathbf{u})$ and consider $J_1'(\mathbf{u}) = \langle x^2yz^3, x^2y^3z, x^3y^2z\rangle$, then $J_1'(\mathbf{u})^{[\mathbf{u_3}]} = \langle x^2, y^3, z^3, y^2z, xyz\rangle$. Now delete $x^2y^1z^3$ from $J(\mathbf{u})$ and set $J_2'(\mathbf{u}) = \langle xy^2z^3, x^2y^3z, x^3y^2z\rangle$, then $J_2'(\mathbf{u})^{[\mathbf{u_3}]} = \langle x^3, y^2, z^3, x^2z, xyz\rangle$. Similarly if we delete $x^2y^3z$ from the ideal $J(\mathbf{u})$ and consider the ideal $J_3'(\mathbf{u}) = \langle xy^2z^3, x^2yz^3, x^3y^2z\rangle$, then the Alexander dual of $J_3'(\mathbf{u})$ is $J_3'(\mathbf{u})^{[\mathbf{u_3}]} = \langle x^3, y^3, z^3, x^2y^2, y^2z, xz\rangle$. Similarly if we delete $x^3y^2z^1$ from the ideal $J(\mathbf{u})$ and consider the ideal $J_4'(\mathbf{u}) = \langle xy^2z^3, x^2yz^3, x^2y^3z\rangle$, then $J_4'(\mathbf{u})^{[\mathbf{u_3}]} = \langle x^3, y^3, z^3, x^2y^2, x^2z, yz\rangle$.

**Remark 5.1.8.** The minimality property of the hypercubic ideal $J(\mathbf{u})$ as described in Theorem 5.1.6 does not characterize these ideals. There are some ideals $I_B \neq J(\mathbf{u})$, where $B \subsetneq \mathfrak{S}_\mathfrak{n}$ also having the same minimality property. We illustrate it by the following example.

**Example 5.1.9.** Let $I_B = \langle xy^3z^2, x^2yz^3, x^2y^3z, x^3yz^2\rangle$. Then the Alexander dual of $I_B$ with respect to $\mathbf{u_3}$ is $I_B^{[\mathbf{u_3}]} = \langle x^3, y^3, z^3, x^2y, yz^2, x^2z^2, xyz\rangle$. If we delete $xy^3z^2$ from $I_B$ and set $I_{B_1} = \langle x^2yz^3, x^2y^3z, x^3yz^2\rangle$, then $I_{B_1}^{[\mathbf{u_3}]} = \langle x^2, y^3, z^3, yz^2, xyz\rangle$. If we delete $x^2y^1z^3$ from $I_B$ and set $I_{B_2} = \langle xy^3z^2, x^2y^3z, x^3yz^2\rangle$, then $I_{B_2}^{[\mathbf{u_3}]} = \langle x^3, y^3, z^3, xy, x^2z^2, yz^2\rangle$. If we delete $x^2y^3z$ from $I_B$ and consider the ideal $I_{B_3} = \langle xy^3z^2, x^2yz^3, x^3yz^2\rangle$, then $I_{B_3}^{[\mathbf{u_3}]} = \langle x^3, y^3, z^2, x^2y, xyz\rangle$. If we delete $x^3yz^2$ from

$I_B$ and consider the ideal $I_{B_4} = \langle xy^3z^2, x^2yz^3, x^2y^3z, \rangle$, then its Alexander dual is $I_{B_4}^{[\mathbf{u_3}]} = \langle x^3, y^3, z^3, x^2y, yz, x^2z^2 \rangle$.

**Question:** It will be an interesting question to characterize all minimal subsets $B \subseteq \mathfrak{S}_\mathfrak{n}$ such that the quotient $R/I_B^{[\mathbf{u_n}]}$ of an Alexander dual $I_B^{[\mathbf{u_n}]}$ has the minimal cellular resolution supported on the first barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$.

## 5.2 Restricted $\lambda$-parking functions

Now we proceed to study the standard monomials in an Artinian quotient $R/J(\mathbf{u})^{[\mathbf{u_n}]}$ of the Alexander dual of the hypercubic ideal $J(\mathbf{u})$. We recall that the standard monomials of an Artinian quotient $R/I(\mathbf{u})^{[\mathbf{u_n - c + 1}]}$ of the Alexander duals of multipermutohedron ideals corresponds bijectively to generalized parking functions. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. A sequence $(p_1, p_2, \ldots, p_n)$ of positive integers is said to be a $\lambda$-parking function of length '$n$' if its non-decreasing rearrangement $(q_1 \leq q_2 \leq \ldots \leq q_n)$ satisfies $q_i \leq \lambda_{n-i+1}$ $\forall$ $i$. We now introduce a notion of restricted $\lambda$-parking functions.

**Definition 5.2.1.** A sequence $(p_1, p_2, \ldots, p_n)$ of positive integers is called a *restricted $\lambda$-parking function* if there exists some permutation $\alpha \in \mathfrak{S}_\mathfrak{n}$ such that

$$p_{\alpha_i} - 1 < \mu_{\alpha_i, T_i},$$

where $\alpha(i) = \alpha_i$ and $T_i = [n] - \{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}\}$, for all $1 \leq i \leq n$. The expression $\mu_{\alpha_i, T_i}$ is as in the Theorem 5.1.4.

In the Definition 5.2.1, $\lambda = (u_n - u_1 + 1, u_n - u_2 + 1, \ldots, u_n - u_n + 1)$. Although, $\lambda$ is not explicit in the definition of restricted $\lambda$-parking functions, we shall see later that every restricted $\lambda$-parking function is indeed a $\lambda$-parking function. This fact is not obvious from the definition.

The following simple lemma will be used in characterizing the standard monomials in the Artinian quotient $R/J(\mathbf{u})^{[\mathbf{u_n}]}$.

**Lemma 5.2.2.** *Let $S = \{a_1, \ldots, a_{i_1}, a_{i_1+1}, \ldots, a_{i_2}, \ldots, a_{i_m}, \ldots, a_n\}$ be a subset of positive integers (arranged in an increasing order) and $T = \{a_{i_1}, a_{i_2}, \ldots, a_{i_m}\} \subset S$. Then $m - r \leq n - i_r$. In particular,*

$$m + a_{i_r} - r \leq n + a_{i_r} - i_r.$$

**Proof.** In order to prove the required inequality, we need only to show that $i_r - r \leq n - m$. Suppose if possible $i_r - r > n - m$. Then $i_r \geq n - m + (r + 1)$. Also $i_m \geq i_r + (m - r)$ implies that

$$\begin{aligned} i_m &\geq n - m + (r+1) + m - r \\ &= n + 1, \end{aligned}$$

which is a contradiction as $i_m \leq n$. $\square$

**Theorem 5.2.3.** *A monomial $\mathbf{x^p} = x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$ is a standard monomial in the Artinian $k$-algebra $R' = R/J(\mathbf{u})^{[\mathbf{u_n}]}$ if and only if $\mathbf{p} + \mathbf{1} = (p_1+1, p_2+1, \ldots, p_n+1)$ is a restricted $\lambda$-parking function .*

**Proof.** Let $\mathbf{x^{p-1}}$ be a standard monomial in Artinian $k$-algebra $R/J(\mathbf{u})^{[\mathbf{u_n}]}$. Thus $\mathbf{x^{p-1}} \notin J(\mathbf{u})^{[\mathbf{u_n}]}$. Therefore for every non-empty subset $T \subseteq [n]$ there exists some $\alpha \in T$ such that

$$p_\alpha - 1 < \mu_{\alpha,T}. \tag{5.2.1}$$

If $T_1 = \{1, 2, \ldots, n\}$, then there exists some $\alpha_1 \in T_1$ such that $p_{\alpha_1} - 1 < \mu_{\alpha_1,T_1}$. Now take $T_2 = T_1 - \{\alpha_1\}$, then there exists some $\alpha_2 \in T_2$ such that $p_{\alpha_2} - 1 < \mu_{\alpha_2,T_2}$. Continuing in this manner, by choosing $T_i = T_1 - \{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}\}$ for $i = 1, 2, \ldots, n$, we have the desired result. Conversely let $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ be a restricted $\lambda$-parking function. Let $T = \{j_1, j_2, \ldots, j_t\}$ be a non-empty subset of $[n]$, where $j_1 < j_2 < \ldots < j_t$.

**Claim:** $p_j - 1 < \mu_{j,T}$ for some $j \in T$.

Let $t = n - q$; $q \geq 0$. If $T = [n] - \{\alpha_1, \alpha_2, \ldots, \alpha_q\} = T_{q+1}$, then by definition there exists some $\alpha_{q+1} = j_s$ (say) $\in T_{q+1}$ such that $p_{j_s} - 1 < \mu_{j_s,T}$. Otherwise suppose $\{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r}\} \subseteq T;\ i_1 < i_2 < \ldots < i_r \leq q$. Let $s = \max\{i : T \subseteq T_i\}$. Then $T \subsetneq T_s$ but $T \nsubseteq T_{s+1}$. Thus $T_{s+1} = T_s - \{\alpha\}$ for some $\alpha \in T$. Clearly $\alpha = \alpha_{i_l}$ for some $l \in \{1, 2, \ldots, r\}$. Now from the definition of restricted $\lambda$-parking function it follows that $p_{\alpha_{i_l}} - 1 < \mu_{\alpha_{i_l}, T_s}$. Further using the Lemma 5.2.2, we have $\mu_{\alpha_{i_l}, T_s} \leq \mu_{\alpha_{i_l}, T}$. □

**Remark 5.2.4.**

1. We notice that $J(\mathbf{u}) \subsetneq I(\mathbf{u}) \Rightarrow I(\mathbf{u})^{[\mathbf{u_n}]} \subsetneq J(\mathbf{u})^{[\mathbf{u_n}]}$. Thus the restricted $\lambda$-parking functions are indeed $\lambda$-parking functions.

2. Let $G$ be a digraph on the set of vertices $0, 1, \ldots, n$. The vertex 0 will be the root of G. For a subset $I$ in $[n]$ and a vertex $i \in I$, let

   $$d_I(i) = \sum_{j \notin I} a_{ij},$$

   i.e., $d_I(i)$ is the number of edges from the vertex $i$ to a vertex outside of the subset $I$. A sequence $\mathbf{b} = (b_1, \ldots, b_n)$ of non-negative integers is a *G-parking function*, if for any nonempty subset $I \subseteq \{1, \ldots, n\}$, there exists $i \in I$ such that $b_i < d_I(i)$. Let $\mathcal{I}_G = \langle m_I \rangle$ be a monomial ideal in the polynomial ring $R = k[x_1, x_2, \ldots, x_n]$ generated by the monomials $m_I = \prod_{i \in I} {x_i}^{d_I(i)}$, where $I$ ranges over all nonempty subsets $I \subseteq [n]$. Clearly, a non-negative integer sequence $\mathbf{b} = (b_1, \ldots, b_n)$ is a $G$-parking function if and only if the monomial $\mathbf{x}^{\mathbf{b}} \notin \mathcal{A}_G = R/\mathcal{I}_G$. Postnikov and Shapiro [23] introduced the notion of $G$-parking functions and proved that $\dim_k \mathcal{A}_G$ is the number of spanning trees of $G$.

   It is important to note that $J(\mathbf{u})^{[\mathbf{u_n}]} \neq \mathcal{I}_G$ for any digraph $G$ on the set of vertices $0, 1, 2 \ldots, n$. For example, if $n = 3$ and $\mathbf{u} = (1, 2, 3)$, we have

   $$J(\mathbf{u})^{[\mathbf{u_3}]} = \langle x_1^3, x_2^3, x_3^3, x_1^2 x_2^2, x_1^2 x_3, x_2^2 x_3, x_1 x_2 x_3 \rangle.$$

   Suppose $J(\mathbf{u})^{[\mathbf{u_3}]} = \mathcal{I}_G$ for a digraph $G$. Then $d_{\{3\}}(3) = 3$ and $d_{\{1,3\}}(3) = 1$, which implies that there are two edges from vertex 3 to 1, but it is contrary to the fact that $d_{\{2,3\}}(3) = 1$. This shows that, the notion of restricted $\lambda$-parking functions is not a particular case of $G$-parking functions. Hence, it is interesting to calculate the number of restricted $\lambda$-parking functions.

Now we shall derive a combinatorial formula for counting restricted $\lambda$-parking functions of length $n$. Let ${\Lambda_n}'$ be the set of restricted $\lambda$-parking functions of length $n$. Let $X$ be the labeled polyhedral cell complex and $\mathbb{F}_*(X)$ be the associated cellular chain complex as described in the Introduction (equation 1.1.2). Then the multigraded Hilbert series of $R/I(X)$ is given by

$$\begin{aligned} H(R/I(X), \mathbf{x}) &= \sum_{i=0}^{\dim(X)+1} (-1)^i H(F_i, \mathbf{x}) \\ &= \sum_{i=0}^{\dim(X)+1} (-1)^i \sum_{\sigma \in \mathcal{F}_{i-1}} \frac{\mathbf{x}^{\nu(\sigma)}}{(1-x_1)(1-x_2)\ldots(1-x_n)}. \end{aligned}$$

As discussed earlier the free complex $\mathbb{F}_*(\mathbf{Bd}(\Delta_{n-1}))$ associated to the first Barycentric subdivision $\mathbf{Bd}(\Delta_{n-1})$ is a minimal free resolution of the ideal $J(\mathbf{u})^{[\mathbf{u_n}]}$. This resolution can be used to calculate the multigraded Hilbert series $H(R', \mathbf{x})$ of the quotient $R'$. We have

$$H(R', \mathbf{x}) = \frac{1}{\prod_{l=1}^{n}(1-x_l)} \sum_{i=0}^{n} (-1)^i \sum_{(A_1, A_2, \ldots, A_i) \in \mathcal{F}_{i-1}} \prod_{q=1}^{i} \Big( \prod_{j \in A_q - A_{q-1}} x_j^{\mu_{j,A_q}} \Big). \quad (5.2.2)$$

**Proposition 5.2.5.** *Let $J(\mathbf{u})$ be a hypercubic ideal and $J(\mathbf{u})^{[\mathbf{u_n}]}$ be its Alexander dual with respect to $\mathbf{u_n}$ and $R' = R/J(\mathbf{u})^{[\mathbf{u_n}]}$. Then the number of restricted $\lambda$-parking functions of length $n$ is given by*

$$|\Lambda_n'| = \sum_{i=1}^{n} (-1)^{n-i} \sum_{\emptyset \subsetneq A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_i = [n]} \prod_{q=1}^{i} \prod_{j \in A_q - A_{q-1}} \mu_{j,\, A_q},$$

*where $\mu_{j,\, A_q}$ is defined above.*

**Proof.** In view of Theorem 5.2.3, we see that

$$H(R', \mathbf{x}) = \sum_{\mathbf{p} \in \Lambda'_n} \mathbf{x}^{\mathbf{p}-\mathbf{1}}.$$

Thus $H(R', \mathbf{1}) = |\Lambda'_n|$. Now passing to the limit $x_i \to 1$ simultaneously for each $i$ in the rational expression for the Hilbert series $H(R', \mathbf{x})$, we obtain

$$H(R', 1) = \lim_{\substack{x_1 \to 1, \\ \ldots, \\ x_n \to 1}} H(R', \mathbf{x}) = \lim_{\substack{x_1 \to 1, \\ \ldots, \\ x_n \to 1}} \frac{Q(\mathbf{x})}{\prod_{l=1}^{n}(1 - x_l)},$$

where

$$Q(\mathbf{x}) = \sum_{i=0}^{n} (-1)^i \sum_{(A_1, A_2, \ldots, A_i) \in \mathcal{F}_{i-1}(\mathbf{Bd}(\Delta_{n-1}))} \prod_{q=1}^{i} \prod_{j \in A_q - A_{q-1}} x_j^{\mu_{j,\, A_q}}.$$

Now apply L'Hospital Rule, we see that

$$|\Lambda'_n| = \frac{1}{(-1)^n} \left. \frac{\partial^n Q(\mathbf{x})}{\partial x_1 \partial x_2 \ldots \partial x_n} \right|_{\mathbf{x}=\mathbf{1}}.$$

As in the partial derivative $\frac{\partial^n Q(\mathbf{x})}{\partial x_1 \partial x_2 \ldots \partial x_n}$, term corresponding to tuple $(A_1, A_2, \ldots, A_i)$ survives only if $|A_i| = n$. Thus we get the desired result. □

We illustrate the last theorem with the help of following example.

**Example 5.2.6.** Let $n = 3$ and $\mathbf{u} = (1, 2, 3)$. Then for different values of $i$, we have the following possibilities for the chains of type $\emptyset = A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_i = [n]$ (as shown in the table). Now using Proposition 5.2.5, the number of restricted $\lambda$-parking function of length 3 are given by,

$$\begin{aligned} |\Lambda'_3| &= \sum_{i=1}^{3} (-1)^{3-i} \sum_{\emptyset \subsetneq A_1 \subsetneq A_2 \subsetneq \ldots \subsetneq A_i = [3]} \prod_{q=1}^{i} \prod_{j \in A_q - A_{q-1}} \mu_{j,\, A_q} \\ &= 1 - (3.1.1 + 3.1.1 + 3.1.1 + 2.2.1 + 2.1.1 + 2.1.1) + (3.2.1 + 3.1.1 + 3.2.1 \\ &\quad +3.1.1 + 3.2.1 + 3.2.1) \\ &= 14. \end{aligned}$$

| $i$ | Chains |
|---|---|
| 1 | $\emptyset \subsetneq \{1,2,3\}$ |
| 2 | $\emptyset \subsetneq \{1\} \subsetneq \{1,2,3\}$<br>$\emptyset \subsetneq \{2\} \subsetneq \{1,2,3\}$<br>$\emptyset \subsetneq \{3\} \subsetneq \{1,2,3\}$<br>$\emptyset \subsetneq \{1,2\} \subsetneq \{1,2,3\}$<br>$\emptyset \subsetneq \{2,3\} \subsetneq \{1,2,3\}$<br>$\emptyset \subsetneq \{1,3\} \subsetneq \{1,2,3\}$ |
| 3 | $\emptyset \subsetneq \{1\} \subsetneq \{1,2\} \subsetneq \{1,2,3\}$<br>$\emptyset \subsetneq \{1\} \subsetneq \{1,3\} \subsetneq \{1,2,3\}$<br>$\emptyset \subsetneq \{2\} \subsetneq \{1,2\} \subsetneq \{1,2,3\}$<br>$\emptyset \subsetneq \{2\} \subsetneq \{2,3\} \subsetneq \{1,2,3\}$<br>$\emptyset \subsetneq \{3\} \subsetneq \{1,3\} \subsetneq \{1,2,3\}$<br>$\emptyset \subsetneq \{3\} \subsetneq \{2,3\} \subsetneq \{1,2,3\}$ |

Here $\lambda = (3,2,1)$. So in this case the $\lambda$-parking functions are indeed the ordinary parking functions. The parking functions in this case are as follows.

$$\begin{aligned}\Lambda_3 &= \{(1,2,3),(1,3,2),(2,3,1),(3,2,1),(3,1,2),(2,1,3),\\ &\quad (1,1,2),(1,2,1),(2,1,1),(1,1,3),(1,3,1),(3,1,1),\\ &\quad (1,2,2),(2,1,2),(2,2,1),(1,1,1)\}.\end{aligned}$$

Out of these $(3,1,2)$ and $(1,3,2)$ are not restricted $\lambda$-parking functions. This verifies the above example that the number of restricted $\lambda$-parking functions in this case is 14.

We have seen that the standard monomials in the quotient $R/I(1,2,\ldots,n)^{[n]}$ of the tree ideal correspond bijectively to the parking functions of length $n$. More generally, the standard monomials in the Artinian quotient $R/I(\mathbf{u})^{[\mathbf{u_n}]}$ of the Alexander dual of multipermutohedron ideal correspond bijectively to the $\lambda$-parking functions of length $n$. In the final chapter, we have characterized the standard monomials in the Artinian quotient $R/J(\mathbf{u})^{[\mathbf{u_n}]}$ in terms of restricted $\lambda$-parking functions of length $n$ and obtained a combinatorial formula for counting restricted $\lambda$-parking functions of length $n$. We end this chapter with the following questions.

1. It may be an interesting problem to characterize the standard monomials in the Artinian Quotient $R/(I(\mathbf{u})^l)^{[l\mathbf{u_n}]}$ of the Alexander dual of the $l^{th}$ power of

multipermutohedron ideal and obtain a combinatorial formula for the number of standard monomials in this quotient. In other words, what is the dimension $\dim_k\left(\frac{R}{(I(\mathbf{u})^l)^{[l\mathbf{u}_\mathbf{n}]}}\right)$? Does this number has any other combinatorial significance?

2. Betti numbers of all the higher powers of the maximal ideal $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$ in the polynomial ring $k[x_1, x_2, \ldots, x_n]$ are calculated using Eagon-Northcott complex or by Eliahou-Kervaire resolution. On the similar lines, one would like to calculate the Betti numbers of higher powers of multipermutohedron ideals and their Alexander duals.

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