Cosmological perturbations and a quantum gravity motivated modified gravity model

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Certificate of Examination

This is to certify that the dissertation titled **"Cosmological perturbations and a quantum gravity motivated modified gravity model"** submitted by **Mr. Vikramaditya Mondal (Reg. No. MS14078)** for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Prof. J. S. Bagla	Dr. Vishal Bhardwaj	Dr. Kinjalk Lochan (Supervisor)
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Declaration

The work presented in this dissertation has been carried out by me under the guidance of **Dr. Kinjalk Lochan** at the **Indian Institute of Science Education and Research Mohali**.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Vikramaditya Mondal
(Candidate)

Dated: April 25, 2019

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Kinjalk Lochan
(Supervisor)

To my parents and my brother

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Abstract

We studied the cosmological linear perturbation theory. To learn how the perturbation variables evolve we derived a set of coupled Boltzmann-Einstein equations for the perturbation variables. These equations cannot be solved analytically all at once. We studied the leading order solutions in four approximations: super-horizon modes, sub-horizon modes, modes which crossed horizon at early and late times. The solutions are sensitive to the initial conditions. We also discussed a modified theory of gravity for the early universe motivated from the works in quantum gravity. Under such modified theory we again constructed the modified equations for dynamics and kinetics between the components of our universe. This modified theory suggest an inclusion of a pre-inflationary era. We propose for future exploration that one needs to examine how the initial conditions set at the end of the pre-inflationary era translates to the initial conditions at the end of inflation which are the standard initial conditions for the linear perturbation theory and see if there is any signature of the quantum gravity effect at the late time evolution of perturbation variables.

Chapter 1

Introduction

Cosmological perturbation theory: To a "zero-th order" approximation, the universe is considered to be homogeneous and isotropic, where all the components of the universe, such as photons, baryons, dark matter, neutrinos are in their corresponding equilibrium distribution and in their presence the geometry of spacetime evolves following the dynamical Einstein field equations of general relativity. The homogeneous, isotropic and expanding universe is characterized by a single variable of time, the scale factor, which appears in the metric. Given the energy-momentum content of the universe, Einstein field equations predict how the scale factor evolves in time. However, had our universe been perfectly homogeneous and isotropic from the beginning, no structure such as galaxies, clusters etc. would have ever formed and much less, by extension of the same argument, there would have been no existence of human beings. Therefore, in cosmological perturbation theory, a deviation from the homogeneous and isotropic universe is considered. In the linear perturbation theory we only deal with small perturbation quantities such that their second or higher order terms can be neglected. Small perturbations in the energy-momentum tensor of the components of the universe create deviations in the metric from the standard FLRW background and vice versa. Another issue is that, in perturbation theory, the components of the universe are not going to remain in their equilibrium distributions forever. Their distribution is going to change with time due the interactions between different species. For instance, photons and electrons interact with each other via Compton scattering and electron and proton interact with each other via Coulomb scattering, whereas both neutrinos and dark matter do not take part in collisions. To incorporate these

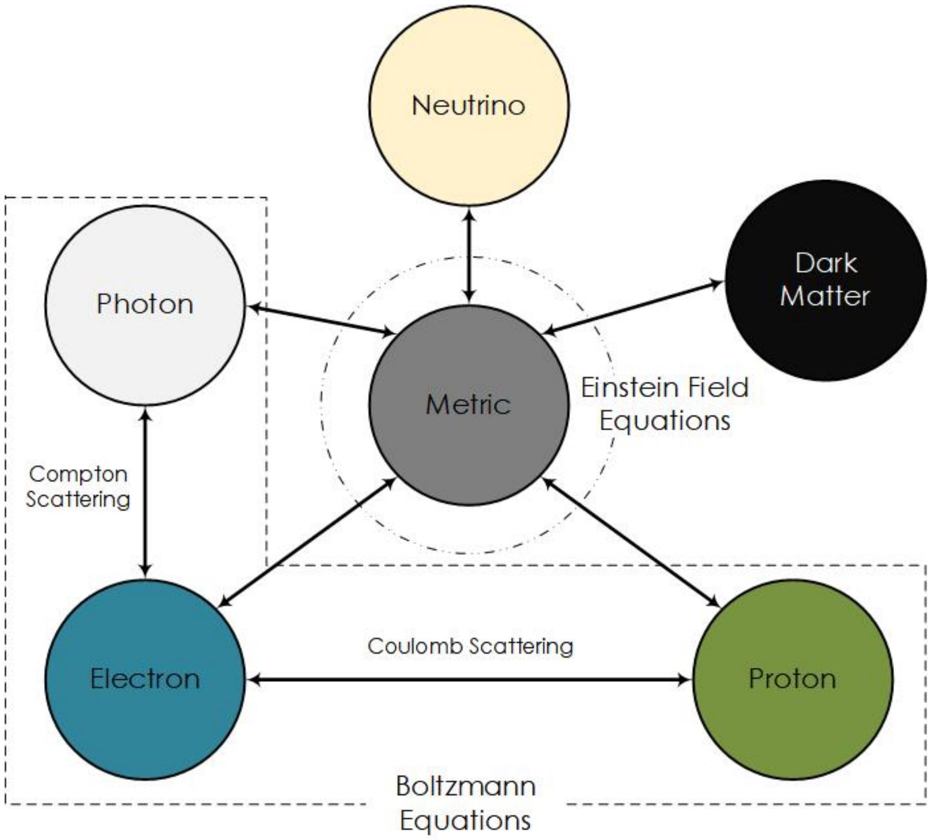

Figure 1.1: The interaction between different components of the universe

interactions we should use the framework of Boltzmann equations from the statistical mechanics. As the components are interacting with themselves they are also dynamically coupled to the metric of spacetime. The kinetics between the components is described by the corresponding Boltzmann equations and the dynamics between the components and spactime is given by the Einstein field equations. Thus, to study the perturbations one constructs a set of coupled Einstein-Boltzmann equations involving all the perturbation variables. The solution of the Einstein-Boltzmann equations would then give us the time evolution of the corresponding perturbation variables. This is the overall scheme that should be followed to study the cosmological perturbations. However, the Einstein-Boltzmann equations cannot be solved analytically all at once. To get the analytical solutions one uses approximations corresponding to super-horizon and sub-horizon modes and also modes which crosses horizon at radiation and matter dominated era respectively. In the following few results about these modes are mentioned. For super-horizon modes, at late times deep into the matter dominated era, the gravitational potential falls to $\frac{9}{10}$ of its initial value. In fact, modes of all sizes come to a constant value in the matter dominated era. Modes of gravitational potential, which crosses the horizon in the radiation dominated era itself, falls rapidly before crossing the epoch of equality, oscillates around zero value with decreasing amplitude and then settles down to zero value in the matter dominated era. The matter perturbations in the radiation dominated era grow logarithmically. The evolution of sub-horizon, cold dark matter perturbation is given by Meszaros equation. It has a growing solution and another decaying solution. At late times the the growing solution grows as y, whereas the decaying solution falls as $y^{-\frac{3}{2}}$, with y being, the scale factor over the scale factor at the epoch of equality.

Another significant perturbation variable is the temperature anisotropy, $\frac{\Delta T}{T}$, in the CMB photons. These photons which are free streaming to us from the last scattering surface almost have an isotropic temperature distribution with small deviations of the order $\frac{\Delta T}{T} \sim 10^{-5}$. The temperature anisotropy, a function of space, time and incoming photon directions, can be decomposed into an infinite set of Legendre modes, with the higher modes speaking of anisotropies at smaller angular scales in the sky. Among these only monopole and dipole are significant, in the limit when photon is tightly coupled with the baryons. The monopole and dipole satisfy differential equations of the form of a forced harmonic oscillator causing them to oscillate rapidly over comoving time. As the photon-baryon interaction is not efficient enough to make them behave as a single fluid, existence of a small but non-vanishing quadrupole moment damps the high k-modes in monopole and dipole. To detail the above analysis we follow the treatment presented in [1].

It can be seen from these analysis that the late time values of many perturbation variables are sensitive to the initial conditions. The initial conditions that are used in the linear perturbation theory are motivated from standard dynamics given by the Einstein field equations. If at very early times, due to the presence of quantum gravity effects, the dynamical and kinetical equations as well, were to be modified, so would the the initial conditions and their relationship with the late time values of perturbation variables. Keeping this proposal in mind we study a particular modification to the Einstein's gravity and construct the corresponding set of modified Boltzmann-Einstein equations.

Quantum gravity motivated theory of modified gravity: According to [2], and subsequent works [3] and [4], dark energy can be understood to be emerging from a violation of energy-momentum conservation. In an effective low energy theory of a discrete spacetime, where the spacetime is assumed to be smooth and continuous, one does not fully account for the exchange of energy and momentum from the matter degrees of freedom to the underlying discrete strata of spacetime. Thus, in the effective low energy limit energy and momentum coservation is violated. However, in the standard dynamcial equations of general relativity such violation is prohibited, as the energy-momentum conservation is enforced by the Bianchi identity of the Riemann curvature tensor. To make such violation compatible with gravitational dynamics, Einstein field equations are modified to a set of traceless dynamical equations called the equations of Unimodular Gravity. We argue that these equations describe a very early universe where the quantum gravity effects were dominant and also they suggest a pre-inflationary era, at the end of which the universe entered in a De Sitter or constant curvature or exponentially expanding phase and then the standard dynamics of general relativity took over as the quantum gravity effects depleted.

Chapter 2

Boltzmann Equation

In this chapter we derive the Boltzmann equation which quantifies the rate of change of the distribution function of a particular species in the components of the universe as a result of collisions and the expansion of the universe. The equations derived in this chapter will be directly used in the subsequent chapters while dealing with perturbation variables. The discussion in this chapter has its resemblance with the presentation of the corresponding topics in [5].

2.1 Single particle phase space representation

The Phase space of a system consisting of N particles in 3-dimensional physical space is $6N$ dimensional. Now in the case of inelastic collision between the particles, the number of particles may change and so may the dimension of the phase space. This is problematic. Thus we fix the phase space to be a 6-dimensional space, collection of points representing all possible position and momentum that a single particle can exhibit. A single point in this space is denoted by $(\vec{r}, \vec{p})$. In this space, the microstate of a system consisting of N particles is represented by N points. We define a quantity $f(\vec{r}, \vec{p}, t)$, which quantifies the particle density in phase space around a point $(\vec{r}, \vec{p})$ at any given time t. The number of particles within an infinitesimal volume of phase space around the same point is, thus, $d^3\vec{r}d^3\vec{p}f(\vec{r}, \vec{p}, t)$. The function $f(\vec{r}, \vec{p}, t)$ changes with time, if the system is not in equilibrium. The number density of particles in position space is given by,

$$n(\vec{r},t) = \int d^3\vec{p} \quad f(\vec{r},\vec{p},t). \tag{2.1}$$

In statistical mechanics of quantum particles, each states occupies a phase space volume $(2\pi\hbar)^3$. We can also define a quantity,

$$\nu(\vec{r},\vec{p},t) = (2\pi\hbar)^3 f(\vec{r},\vec{p},t), \tag{2.2}$$

which can be interpreted as the average number of particles occupying, at time t, a state with position and momentum $\vec{r}$ and $\vec{p}$, with corresponding uncertainties characterized by the phase space volume. If the system is homogeneous and isotropic and is in equilibrium, then $\nu = \nu(|\vec{p}|)$ and thus, by the virtue of dispersion relation can be expressed as a function of energy. This expression should be same as the BE or FD distribution.

2.2 Collisionless and general Boltzmann equation

Since a time t to $t + dt$, in absence of collison, the position and momentum of a particle will change as follows,

$$\begin{aligned} \vec{r'} &= \vec{r} + \frac{\vec{p}}{m}dt, \\ \vec{p'} &= \vec{p} + \vec{F}dt. \end{aligned} \tag{2.3}$$

A the particles inside an infinitesimal volume $d^3\vec{r}d^3\vec{p}$ will come to lie inside a new volume element $d^3\vec{r'}d^3\vec{p'}$. As there is no collision the number of particles must be conserved. Thus we can write,

$$d^3\vec{r}d^3\vec{p}f(\vec{r},\vec{p},t) = d^3\vec{r'}d^3\vec{p'}f(\vec{r'},\vec{p'},t). \tag{2.4}$$

Liouville's theorem tells us that up to first order in dt, the dynamics keeps the volume element preserved, then it follows,

$$d^3\vec{r}d^3\vec{p}f(\vec{r},\vec{p},t) = f(\vec{r} + \frac{\vec{p}}{m}dt, \vec{p} + \vec{F}dt, t + dt). \tag{2.5}$$

Taylor expanding the RHS of the equation around the point $(\vec{r}, \vec{p})$ and keeping the terms up to order of dt we get,

$$\frac{\partial f}{\partial t} + \frac{\vec{p}}{m} \cdot \frac{\partial f}{\partial \vec{r}} + \vec{F} \cdot \frac{\partial f}{\partial \vec{p}} = 0. \tag{2.6}$$

This is collisionless Boltzmann equation. In a more general situation when collision is involved, we can write,

$$\frac{\partial f}{\partial t} = \left.\frac{\partial f}{\partial t}\right|_{\text{free-stream}} + \left.\frac{\partial f}{\partial t}\right|_{\text{collision}}. \tag{2.7}$$

Using collisionless Boltzmann equation we can write,

$$\begin{aligned} \left.\frac{\partial f}{\partial t}\right|_{\text{collision}} &= \frac{\partial f}{\partial t} - \left.\frac{\partial f}{\partial t}\right|_{\text{free-stream}} \\ &= \frac{\partial f}{\partial t} + \frac{\vec{p}}{m} \cdot \frac{\partial f}{\partial \vec{r}} + \vec{F} \cdot \frac{\partial f}{\partial \vec{p}}. \end{aligned} \tag{2.8}$$

This is the general Boltzmann equation.

2.3 Calculation of the collision term

Considering only elastic collisions, where particle number is conserved, we can distinguish between two types of collision. First, a particle with momentum $\vec{p}$ in initial state produces particles with other momenta in the final state. Second, a particle with momentum $\vec{p}$ is produced in the collision of particles with other momenta. The single particle distribution particle $f(\vec{r}, \vec{p}, t)$ decreases in the first kind of process and decrease in the second kind of process. Thus we can write,

$$\left.\frac{\partial f}{\partial t}\right|_{\text{collision}} = -\left.\frac{\partial f}{\partial t}\right|_{\text{coll}}^{\text{first}} + \left.\frac{\partial f}{\partial t}\right|_{\text{coll}}^{\text{second}} \tag{2.9}$$

To calculate these two terms, let's consider a binary elastic collision. In such a collision the following conservation equations hold,

$$\begin{aligned} \vec{p}_1 + \vec{p}_2 &= \vec{p'}_1 + \vec{p'}_2 \\ \epsilon(\vec{p}_1) + \epsilon(\vec{p}_2) &= \epsilon(\vec{p'}_1) + \epsilon(\vec{p'}_2), \end{aligned} \tag{2.10}$$

where primed and unprimed quantities correspond to before and after collision respectively. In an isotropic system ϵ is only dependent upon the magnitude of momentum; thus, the above four equations eat up four degrees of freedom leaving only two among total six degrees of freedom. The conclusion is that specifying only two parameters are sufficient to describe a binary elastic collision. In CM frame, by definition, the initial total momentum is zero and by conservation of momentum, so is the final total momentum. Thus only one of the two initial momenta and one of the two final momenta are actually free. As energy depends only on the magnitude of the momentum, the energy conservation equation fixes the magnitude of the final momenta. The only undetermined parameters describe how the line of final momenta orients itself with the line of initial momenta. In 3-dimensional momentum space this corresponds to the freedom of choosing two angles, polar angle θ and azimuthal angle ϕ. Thus a set of two undetermined angles (θ, ϕ) describe a collision. Moreover, in a central force motion, the scattering occurs in a plane, in such case the angle ϕ can be set to zero and θ is sufficient to describe the collision (along with the magnitude of the initial momentum of one particle). This is known as the scattering angle.

Now, the first collision term is evaluated at $(\vec{r}_1, \vec{p}_1, t)$; as the final momenta are not fully determined by the initial momenta, the final momenta have to be integrated out and also as the second particle can come to collide with the first particle at location $\vec{r}_1$ with any possible momentum $\vec{p}_2$, this has to integrated out too. The integrand should be proportional to $f(\vec{r}_1, \vec{p}_1, t)$, because the collision cannot occur if there is no particle at the location $\vec{r}_1$ with momentum $\vec{p}_1$ at time t, i.e, if $f(\vec{r}_1, \vec{p}_1, t)$ is zero. The integrand should also be proportional to $f(\vec{r}_1, \vec{p}_2, t)$ for the same reason. The integrand should also be proportional to a quantity that represents the scattering rate. Thus Boltzmann heuristically used the following ansatz,

$$\left.\frac{\partial f}{\partial t}\right|_{\text{coll}}^{\text{first}} = \int d^3\vec{p}_2 \int d^3\vec{p'}_1 \int d^3\vec{p'}_2 \quad f(\vec{r}_1, \vec{p}_1, t) f(\vec{r}_1, \vec{p}_2, t) w(\vec{p'}_1, \vec{p'}_2 | \vec{p}_1, \vec{p}_2). \tag{2.11}$$

A similar argument goes for the second collision term and the whole Boltzmann equation becomes,

$$\frac{\partial f}{\partial t} + \frac{\vec{p}_1}{m} \cdot \frac{\partial f}{\partial \vec{r}_1} + \vec{F}_1 \cdot \frac{\partial f}{\partial \vec{p}_1}$$

$$= \int d^3\vec{p}_2 \int d^3\vec{p'}_1 \int d^3\vec{p'}_2 \quad \Big[f(\vec{r}_1, \vec{p'}_1, t) f(\vec{r}_1, \vec{p'}_2, t) w(\vec{p}_1, \vec{p}_2 | \vec{p'}_1, \vec{p'}_2)$$
$$- f(\vec{r}_1, \vec{p}_1, t) f(\vec{r}_1, \vec{p}_2, t) w(\vec{p'}_1, \vec{p'}_2 | \vec{p}_1, \vec{p}_2) \Big] . \quad (2.12)$$

2.4 Properties of the scattering factor

The scattering factor should manifest the fact that the energy and momentum are conserved in a collision. Thus the scattering factor must have the following form,

$$w(\vec{p'}_1, \vec{p'}_2 | \vec{p}_1, \vec{p}_2) = \delta^{(3)}(\vec{p'}_1 + \vec{p'}_2 - \vec{p}_1 - \vec{p}_1) \delta(\epsilon(\vec{p'}_1) + \epsilon(\vec{p'}_2) - \epsilon(\vec{p}_1) - \epsilon(\vec{p}_2)) \tilde{w}(\vec{p'}_1, \vec{p'}_2 | \vec{p}_1, \vec{p}_2). \quad (2.13)$$

If two identical particles are colliding and if they remain identical after the collision, then the following relations hold,

$$w(\vec{p'}_1, \vec{p'}_2 | \vec{p}_1, \vec{p}_2) = w(\vec{p'}_2, \vec{p'}_1 | \vec{p}_1, \vec{p}_2) = w(\vec{p'}_1, \vec{p'}_2 | \vec{p}_2, \vec{p}_1). \quad (2.14)$$

This relation is only true for identical particles. If the system exhibits space-inversion symmetry then,

$$w(\vec{p'}_1, \vec{p'}_2 | \vec{p}_1, \vec{p}_2) = w(-\vec{p'}_1, -\vec{p'}_2 | -\vec{p}_1, -\vec{p}_2). \quad (2.15)$$

If the system exhibits time-reversal symmetry then,

$$w(\vec{p'}_1, \vec{p'}_2 | \vec{p}_1, \vec{p}_2) = w(-\vec{p}_1, -\vec{p}_2 | -\vec{p'}_1, -\vec{p'}_2). \quad (2.16)$$

Combining the above two equations we get, for a system that exhibits time-reversal and space-inversion symmetry simultaneously,

$$w(\vec{p'}_1, \vec{p'}_2 | \vec{p}_1, \vec{p}_2) = w(\vec{p}_1, \vec{p}_2 | \vec{p'}_1, \vec{p'}_2). \quad (2.17)$$

Therefore, the classical, non-relativistic Boltzmann equation is,

$$\frac{\partial f}{\partial t} + \frac{\vec{p}_1}{m} \cdot \frac{\partial f}{\partial \vec{r}_1} + \vec{F}_1 \cdot \frac{\partial f}{\partial \vec{p}_1}$$
$$= \int d^3\vec{p}_2 \int d^3\vec{p'}_1 \int d^3\vec{p'}_2 \quad w(\vec{p'}_1, \vec{p'}_2 | \vec{p}_1, \vec{p}_2)$$
$$\Big[f(\vec{r}_1, \vec{p'}_1, t) f(\vec{r}_1, \vec{p'}_2, t) - f(\vec{r}_1, \vec{p}_1, t) f(\vec{r}_1, \vec{p}_2, t) \Big] . \quad (2.18)$$

Quantum scattering amplitude and probability: The differential probability of an initial state $|i\rangle$ to end up in a final state $|f\rangle$ after scattering, where the final state momenta lies within an infinitesimal volume of momentum space $d\Pi$, is [6],

$$dP = \frac{|\langle f|\hat{S}|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle} d\Pi. \tag{2.19}$$

Here, $d\Pi$ is the normalized volume element in momentum space, given by,

$$d\Pi = \prod_j \frac{V}{(2\pi)^3} d^3\vec{p}_j, \tag{2.20}$$

here, j denotes the final state particles and, V is the volume of space. In going from equally spaced discrete momenta in a finite box to a continuous momenta in a large box, one has to divide the elements by the density of the momentum states $\frac{(2\pi)^3}{V}$, that is,

$$\Delta\vec{p} \xrightarrow{\text{large box limit}} \frac{V}{(2\pi)^3} d^3\vec{p}. \tag{2.21}$$

Now, we shall consider the quantities $\langle i|i\rangle$ and $\langle f|f\rangle$. We know that a momentum state, $|\vec{p}\rangle = a^\dagger(\vec{p})|0\rangle$ (non-relativistic treatment). Thus,

$$\begin{aligned}\langle\vec{p}|\vec{p}\rangle &= \langle 0|a(\vec{p})a^\dagger(\vec{p})|0\rangle \\ &= (2\pi)^3\delta^{(3)}(0) = (2\pi)^3 \int \frac{d^3\vec{x}}{(2\pi)^3} = V\end{aligned}$$

Let the initial state be $|i\rangle = |\vec{p}_1\rangle|\vec{p}_2\rangle$, then $\langle i|i\rangle = V \cdot V$. Similarly, for final state $\langle f|f\rangle = \prod_j V$. It remains to calculate the transition probability $|\langle f|i\rangle|^2$. We define the transformation of the asymptotic state with an operator $\hat{S}$, such that,

$$\hat{S}|\text{in}, T = -\infty\rangle = |\text{out}, T = +\infty\rangle. \tag{2.22}$$

The completeness of the in and out asymptotic states implies,

$$\begin{aligned}\mathbb{1} &= \sum_{\text{out}} |\text{out}, T = +\infty\rangle\langle\text{out}, T = +\infty| \\ &= \sum_{\text{in}} \hat{S}|\text{in}, T = -\infty\rangle\langle\text{in}, T = -\infty|\hat{S}^\dagger \\ &= \hat{S}\hat{S}^\dagger.\end{aligned} \tag{2.23}$$

Therefore, $\hat{S}$ is unitary. As most the time the initial state remains unchanged $\hat{S}$ should include identity. Thus we define transfer matrix $\hat{T}_{i\to f}$, as follows,

$$\hat{S} = \mathbb{1} + i\hat{T}_{i\to f}. \tag{2.24}$$

This transfer matrix should impose the conservation of energy and momentum; this motivates the following definition,

$$\hat{T}_{i\to f} = (2\pi)^4 \delta^{(4)} \left(\sum p\right) \mathcal{M}, \tag{2.25}$$

here $(2\pi)^4$ is conventional. Now the amplitude for the transition is,

$$\begin{aligned}\langle f|\hat{S}|i\rangle &= \langle f, T = +\infty|\hat{S}|i, T = -\infty\rangle \\ &= \langle f|(\mathbb{1} + (2\pi)^4 \delta^{(4)} \left(\sum p\right) \mathcal{M})|i\rangle.\end{aligned} \tag{2.26}$$

If the initial final states are different then,

$$\begin{aligned}|\langle f|\hat{S}|i\rangle|^2 &= (2\pi)^8 \delta^{(4)} \left(\sum p\right) \delta^{(4)} \left(\sum p\right) |\langle f|\mathcal{M}|i\rangle|^2 \\ &= (2\pi)^8 \delta^{(4)} (0)\, \delta^{(4)} \left(\sum p\right) |\mathcal{M}|^2 \\ &= (2\pi)^4 VT \delta^{(4)} \left(\sum p\right) |\mathcal{M}|^2,\end{aligned} \tag{2.27}$$

here, V is the volume of the space and T is the total time of the process. Putting all the results together, we get,

$$\begin{aligned}dP &= \frac{(2\pi)^4 VT \delta^{(4)} \left(\sum p\right) |\mathcal{M}|^2}{V \cdot V \prod_j V} \prod_j \frac{V}{(2\pi)^3} d^3\vec{p}_j \\ &= \frac{T}{V} (2\pi)^4 \delta^{(4)} \left(\sum p\right) |\mathcal{M}|^2 \prod_j \frac{d^3\vec{p}_j}{(2\pi)^3}.\end{aligned} \tag{2.28}$$

Flux, across a surface of area A, of a beam with N_{inc} number of incident particles and velocity $\vec{v}$ equals to,

$$\Phi = \frac{\frac{N_{\text{inc}}}{V} |\vec{v}| t A}{At} = \frac{N_{\text{inc}}}{V} |\vec{v}|. \tag{2.29}$$

The number of scattered particles is equal to cross section times the flux of the beam that hits the target times the 'time' for scattering, that is, $N = \sigma T \Phi$ and the differential version of the equation is $dN = d\sigma T\Phi$. The differential probability is the ratio of number of scattered particles and the total number of incident particles, $dP = \frac{dN}{N_{\text{inc}}}$. Therefore,

$$dP = d\sigma T \frac{\Phi}{N_{\text{inc}}} = d\sigma \frac{T}{V}|\vec{v}|. \tag{2.30}$$

Now, $d\sigma|\vec{v}|$ is the differential scattering rate and sits in the integrand of the classical Boltzmann equation, along with the distribution functions.

$$d\sigma|\vec{v}| = (2\pi)^4 \delta^{(4)} \left(\sum p\right) |\mathcal{M}|^2 \prod_j \frac{d^3\vec{p}_j}{(2\pi)^3}. \tag{2.31}$$

However this is a non-relativistic treatment.

2.5 Quantum Boltzmann equation

2.5.1 Non-relativistic equation

Due to uncertainty principle the concept of phase space in quantum mechanics is not well defined so Boltzmann equation cannot be strictly derived in this case, but the classical Boltzmann equation extended to quantum case by making the following changes. In quantum case f has to be changed to ν which is the BE or FD distribution over single particle energies. But for fermions if the unit cell in the phase space which represents a quantum state is completely filled, i.e. $\nu = 1$, there could be no transition of further fermion in that state due to Pauli's exclusion principle and therefore such contribution in collision term of the Boltzmann equation should vanish. This is achieved by introducing a factor $(1 - \nu)$ and known as Pauli blocking. An opposite effect is expected in the case of Bosons known as Bose enhancement. This can be seen in the QHO as its quanta behave like bosons. For QHO $a^\dagger|\nu\rangle = \sqrt{\nu+1}|\nu+1\rangle$. Thus, if there are already ν quanta, producing one more quanta involves $(1+\nu)$ as a factor in the probability. We should also include this factor. Putting these all together and $(\hbar = 1)$ the quantum Boltzmann equation becomes,

$$\begin{aligned}&\frac{\partial \nu_1}{\partial t} + \frac{\vec{p}_1}{m} \cdot \frac{\partial \nu_1}{\partial \vec{r}_1} + \vec{F}_1 \cdot \frac{\partial \nu_1}{\partial \vec{p}_1} \\ &= \int \frac{d^3\vec{p}_2}{(2\pi)^3} \int \frac{d^3\vec{p'}_1}{(2\pi)^3} \int \frac{d^3\vec{p'}_2}{(2\pi)^3} \quad (2\pi)^4 \delta^{(4)} \left(\sum p\right) |\mathcal{M}|^2\end{aligned}$$

$$[\nu_1'\nu_2'(1+\eta\nu_1)(1+\eta\nu_2) - \nu_1\nu_2(1+\eta\nu_1')(1+\eta\nu_2')]\,, \tag{2.32}$$

where, $\eta = \pm 1$ for BE and FD distribution functions. This is the non-relativistic quantum Boltzmann equation.

2.5.2 Relativistic equation

In relativistic case, the normalization factor $2E_{\vec{p}}$ has to be used in the definition of the states $|\vec{p}\rangle$ and this would lead to $\langle i|i\rangle = (2E_1V)(2E_2V)$ and similarly for final states, $\langle f|f\rangle = \prod_j (2E_jV)$. This would give the relativistic differential scattering rate to be,

$$dw(\vec{p'}_1, \vec{p'}_2|\vec{p}_1, \vec{p}_2) = \frac{1}{(2E_1)(2E_2)}(2\pi)^4\delta^{(4)}\left(\sum p\right)|\mathcal{M}|^2 \prod_j \frac{d^3\vec{p}_j}{(2\pi)^3 2E_j}. \tag{2.33}$$

Therefore, the relativistic quantum Boltzmann equation is,

$$\begin{aligned}
&\frac{\partial\nu_1}{\partial t} + \frac{\vec{p}_1}{m}\cdot\frac{\partial\nu_1}{\partial\vec{r}_1} + \vec{F}_1\cdot\frac{\partial\nu_1}{\partial\vec{p}_1} \\
&= \frac{1}{2E_1}\int\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\int\frac{d^3\vec{p'}_1}{(2\pi)^3 2E_1'}\int\frac{d^3\vec{p'}_2}{(2\pi)^3 2E_2'} \quad (2\pi)^4\delta^{(4)}\left(\sum p\right)|\mathcal{M}|^2 \\
&[\nu_1'\nu_2'(1+\eta\nu_1)(1+\eta\nu_2) - \nu_1\nu_2(1+\eta\nu_1')(1+\eta\nu_2')]\,.
\end{aligned} \tag{2.34}$$

2.5.3 Lorentz invariance of distribution functions

The density distribution functions are defined as,

$$f(\vec{r}, \vec{p}, t) = \sum_{i=1}^{N}\delta^{(3)}(\vec{r}-\vec{r}_i)\delta^{(3)}(\vec{p}-\vec{p}_i). \tag{2.35}$$

We define,

$$F(\vec{r}, \vec{p}, t) = \delta(p^2 - m^2)\Theta(p^0)f(\vec{r}, \vec{p}, t). \tag{2.36}$$

As both $\delta(p^2 - m^2)$ and $\Theta(p^0)$ are Lorentz invariant quantities then F and f should have same Lorentz transformation properties. Now,

$$\delta(p^2 - m^2)\Theta(p^0) = \frac{1}{2E_{\vec{p}}}\delta(p^0 - E_{\vec{p}}), \tag{2.37}$$

which implies for F,

$$F(\vec{r},\vec{p},t) = \frac{1}{2E_{\vec{p}}}\sum_{i=1}^{N}\delta^{(3)}(\vec{r}-\vec{r}_i)\delta^{(3)}(\vec{p}-\vec{p}_i)\delta(p^0-E_{\vec{p}})$$
$$= \frac{1}{2E_{\vec{p}}}\sum_{i=1}^{N}\int dt_i\delta(t-t_i)\delta^{(3)}(\vec{r}-\vec{r}_i(t_i))\delta^{(4)}(p^\mu-p_i^\mu(t_i)). \tag{2.38}$$

Using the relation $d\tau_i = \frac{mc^2}{E_{\vec{p}}}dt_i$ and dropping the particle index on time, we get,

$$F(\vec{r},\vec{p},t) = \frac{1}{2m}\sum_{i=1}^{N}\int d\tau\delta^{(4)}(x^\mu-x_i^\mu(\tau))\delta^{(4)}(p^\mu-p_i^\mu(\tau)). \tag{2.39}$$

Each term in the RHS are explicitly Lorentz invariant thus F and hence f and ν are also Lorentz invariant. However, the above treatment would not work for massless particles, as for them both the mass and proper time are inevitably zero. In such a case we choose a different parameterization (λ) of the curve traced out by these particles other than the proper time, such as, the four momenta of these particles are defined as follows,

$$p^\mu \equiv \frac{dx^\mu}{d\lambda}. \tag{2.40}$$

Now the same kind of argument as above can be pushed forward,

$$F(\vec{r},\vec{p},t) = \frac{1}{2p^0}\sum_{i=1}^{N}\delta^{(3)}(\vec{r}-\vec{r}_i)\delta^{(3)}(\vec{p}-\vec{p}_i)\delta(p^0-|\vec{p}|)$$
$$= \frac{1}{2p^0}\sum_{i=1}^{N}\int dt_i(\lambda)\delta(t-t_i(\lambda))\delta^{(3)}(\vec{r}-\vec{r}_i(\lambda))\delta^{(4)}(p^\mu-p_i^\mu(\lambda))$$
$$= \frac{1}{2p^0}\sum_{i=1}^{N}\int d\lambda\frac{dt_i}{d\lambda}\delta^{(4)}(x^\mu-x_i^\mu(\lambda))\delta^{(4)}(p^\mu-p_i^\mu(\lambda))$$
$$= \frac{1}{2p^0}\sum_{i=1}^{N}\int d\lambda p_i^0\delta^{(4)}(x^\mu-x_i^\mu(\lambda))\delta^{(4)}(p^\mu-p_i^\mu(\lambda)). \tag{2.41}$$

As there is a zeroth component of momenta inside the integral and one outside the integral in the denominator, upon Lorentz transformation, the transformation factor will cancel out and hence F is a Lorentz invariant quantity for massless particlea and therefore so is f.

2.6 Boltzmann equation in the expanding universe

When the universe expands the density distribution function, $\nu(\vec{r}, \vec{p}, t)$ changes due to the change in volume of the space. The density distribution function is inversely proportional to the volume being considered, i.e., $\nu \propto V^{-1}$. Now the following can be written using this statement,

$$\frac{1}{\nu}\dot{\nu} = \frac{-V^{-2}\dot{V}}{V^{-1}} = -\frac{\dot{V}}{V} = -\frac{3a^2\dot{a}}{a^3} = -3\frac{\dot{a}}{a} = -3H. \tag{2.42}$$

And hence, the rate of change in the density distribution function should be written as,

$$\left.\frac{\partial \nu}{\partial t}\right|_{\text{volume}} = -\frac{\dot{V}}{V}\nu = -3H\nu, \tag{2.43}$$

here, H is the Hubble parameter. As the is homogeneous and the whole universe in under consideration, there is no external force and hence free streaming terms in the Boltzmann equation are zero. The total contribution to the change in the density distribution should be from both change in volume of space and change due to collision, which can be written as,

$$\frac{\partial \nu}{\partial t} = \left.\frac{\partial \nu}{\partial t}\right|_{\text{volume}} + \left.\frac{\partial \nu}{\partial t}\right|_{\text{collisions}} \tag{2.44}$$

Thus the Boltzmann equation becomes,

$$\begin{aligned}\frac{\partial \nu_1}{\partial t} + 3H\nu_1 &= \left.\frac{\partial \nu_1}{\partial t}\right|_{\text{collision}} \\ &= \frac{1}{2E_1}\int \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\int \frac{d^3\vec{p'}_1}{(2\pi)^3 2E'_1}\int \frac{d^3\vec{p'}_2}{(2\pi)^3 2E'_2} \quad (2\pi)^4\delta^{(4)}\left(\sum p\right)|\mathcal{M}|^2 \\ &\left[\nu'_1\nu'_2(1+\eta\nu_1)(1+\eta\nu_2) - \nu_1\nu_2(1+\eta\nu'_1)(1+\eta\nu'_2)\right]. \end{aligned} \tag{2.45}$$

We can integrate with respect to momentum $\vec{p}_1$ to get the number density of particles in position space also using $n = \frac{1}{(2\pi)^3}\int d^3\vec{p}\nu$,

$$\frac{\partial n_1}{\partial t} + 3Hn_1 = \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\int \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\int \frac{d^3\vec{p'}_1}{(2\pi)^3 2E'_1}\int \frac{d^3\vec{p'}_2}{(2\pi)^3 2E'_2}$$

$$(2\pi)^4\delta^{(4)}\left(\sum p\right)|\mathcal{M}|^2\left[\nu_1'\nu_2'(1+\eta\nu_1)(1+\eta\nu_2)-\nu_1\nu_2(1+\eta\nu_1')(1+\eta\nu_2')\right]. \tag{2.46}$$

2.7 Saha Equilibrium

In the expanding universe, Boltzmann equation can be used to track the relative abundances of various species of particles. Let us consider a reaction between two species giving out two other species as product, i.e., $1+2 \leftrightarrow 3+4$. The number density of species 1 will change according to the Boltzmann equation derived above.

$$\frac{\partial n_1}{\partial t}+3Hn_1=\left(\prod_{i=1}^{4}\int\frac{d^3\vec{p}_i}{(2\pi)^3 2E_i}\right)(2\pi)^4\delta^{(4)}\left(\sum p\right)|\mathcal{M}|^2$$
$$\left[\nu_3\nu_4(1+\eta\nu_1)(1+\eta\nu_2)-\nu_1\nu_2(1+\eta\nu_3)(1+\eta\nu_4)\right]. \tag{2.47}$$

We also have similar equations for other species, namely, 2,3, and 4. If the scattering rate is much faster than the expansion rate of the universe, the species will be in a thermal or kinetic equilibrium. In such cases the distribution function will be of FD/BE type, with temperature set by the most populated and strongly interacting species. The reaction can also achieve chemical equilibrium in which case, the conservation of chemical potential has to be satisfied, i.e. $\mu_1+\mu_2=\mu_3+\mu_4$ must be true. If the reaction is out of equilibrium then one would require to solve differential equations concerning the chemical potentials. For many practical purposes a very interesting scenario of investigation is the limit where $(E-\mu)\gg k_B T$. In this limit the the FD/BE distribution functions reduces down to,

$$\frac{1}{e^{\frac{(E-\mu)}{T}}\pm 1}\to e^{\frac{\mu}{T}}e^{-\frac{E}{T}}. \tag{2.48}$$

The Bose enhancement and Pauli blocking factors also reduce to 1 in this particular limit. The conservation of energy implies, $E_1+E_2=E_3+E_4$. After all these simplifications the expression within the square bracket becomes,

$$e^{-\frac{E_1+E_2}{T}}\left(e^{\frac{\mu_3+\mu_4}{T}}-e^{\frac{\mu_1+\mu_2}{T}}\right). \tag{2.49}$$

Now, the number density of a particular species is given by,

$$n_i = g_i e^{\frac{\mu_i}{T}} \int \frac{d^3\vec{p}}{(2\pi)^3} e^{-\frac{E_i}{T}}. \tag{2.50}$$

Thus we can define the species dependent equilibrium density as,

$$n_i^{(0)} = g_i \int \frac{d^3\vec{p}}{(2\pi)^3} e^{-\frac{E_i}{T}} = \begin{cases} g_i \left(\frac{m_i T}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{m_i}{T}}, & m_i \gg |\vec{p}| \\ g_i \frac{T^3}{\pi^2}, & m_i \ll |\vec{p}| \end{cases} \tag{2.51}$$

The last equality was obtained by putting two different limits of the energy, which are $E = \sqrt{|\vec{p}|^2 + m^2} \approx m + \frac{|\vec{p}|^2}{2m}$, for non-relativistic case ($m_i \gg |\vec{p}|$) and $E = |\vec{p}|$, for relativistic case ($m_i \ll |\vec{p}|$). With all the simplifications considered above the Boltzmann equation becomes,

$$\frac{\partial n_1}{\partial t} + 3Hn_1 = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left\{ \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right\}, \tag{2.52}$$

where,

$$\langle \sigma v \rangle \equiv \frac{1}{n_1^{(0)} n_2^{(0)}} \left(\prod_{i=1}^{4} \int \frac{d^3\vec{p}_i}{(2\pi)^3 2E_i} \right) e^{-\frac{E_1+E_2}{T}} (2\pi)^4 \delta^{(4)} \left(\sum p \right) |\mathcal{M}|^2, \tag{2.53}$$

is called the thermally averaged cross section. As the universe expands, the gradual increment of the mean free path makes the interaction feeble. If the reaction rate is slower than the expansion rate of the universe, which is parameterized by Hubble constant, the reaction would fall out of equilibrium. But in the case where the reaction rate is much faster than the expansion rate of the universe, the reaction would establish equilibrium and hence would satisfy the following equation,

$$\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}. \tag{2.54}$$

The above equation is known as the Saha Equation.

Chapter 3

Einstein's Gravity

In this chapter, we shall first discuss few results from the standard model of cosmology and then go on discuss the cosmological perturbation theory. While discussing perturbation theory we shall derive the differential equations describing kinetics (Boltzmann equations) and dynamics (Einstein field equations) separately and then proceed to obtain the analytical solutions in approximated limit cases. Moreover, in this chapter, as the name suggests, the dynamics will be given by the standard Einstein field equations,

$$G_{\mu\nu} = (8\pi G)T_{\mu\nu}, \tag{3.1}$$

here, $G_{\mu\nu}$ is the Einstein tensor constructed from the Ricci tensor and Ricci scalar, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$; $T_{\mu\nu}$ is the energy-momentum tensor of the content of the universe and G is the Newton's universal gravitational constant.

3.1 FLRW background

We would first derive some important results for the homogeneous and isotropic universe or for the standard model of cosmology. The purpose for this is to put the results derived here, from Einstein's gravity, in contrast to the results that will be derived in the next chapter with modified gravity. Mathematically, the homogeneity and isotropy is the property of the metric that it satisfies translational and rotational invariance in three spatial space respectively. The geometry of a homogeneous and isotropic universe with zero spatial curvature is described by Friedmann-Lemaître-

Robertson-Walker metric and the metric in mathematical form reads,

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & a(t)^2 \end{pmatrix}, \tag{3.2}$$

with the scale factor $a(t)$, which is only a function of time and a dimensionless quantity. It is useful to derive the Christoffel symbols immediately from the metric. The Christoffel symbols are related to the metric in a metric compatible manifold as follows,

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(g_{\alpha\nu,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu} \right). \tag{3.3}$$

Upon computation, we can easily get the Christoffel symbols for the FLRW metric and the results are listed below,

$$\Gamma^{0}_{00} = 0; \quad \Gamma^{0}_{0i} = 0; \quad \Gamma^{0}_{ij} = a\frac{da}{dt}\delta_{ij}; \quad \Gamma^{i}_{0j} = \frac{1}{a}\frac{da}{dt}\delta^{i}_{j}; \quad \Gamma^{i}_{jk} = 0; \quad \Gamma^{i}_{00} = 0. \tag{3.4}$$

With the Christoffel symbols in hand we can immediately compute the geometric quantities such as the Ricci tensor and Ricci scalar as they will be useful in writing down the Einstein field equations. The definition of Ricci tensor in terms of Christoffel symbols is,

$$R_{\alpha\beta} = \partial_{\gamma}\Gamma^{\gamma}_{\alpha\beta} - \partial_{\beta}\Gamma^{\gamma}_{\alpha\gamma} + \Gamma^{\gamma}_{\alpha\beta}\Gamma^{\sigma}_{\gamma\sigma} - \Gamma^{\sigma}_{\alpha\gamma}\Gamma^{\gamma}_{\beta\sigma}. \tag{3.5}$$

Explicitly, the 00, ij and $0i$ components of the Ricci tensor are,

$$R_{00} = -3\frac{1}{a}\frac{d^2a}{dt^2}; \quad R_{ij} = \delta_{ij}\left[2\left(\frac{da}{dt}\right)^2 + a\frac{d^2a}{dt^2}\right]; \quad R_{0i} = 0. \tag{3.6}$$

The Ricci scalar is simply the trace of the Ricci tensor and takes the following form for our particular FLRW metric,

$$R = g^{\mu\nu}R_{\mu\nu} = 6\left[\frac{1}{a}\frac{d^2a}{dt^2} + \left(\frac{1}{a}\frac{da}{dt}\right)^2\right]. \tag{3.7}$$

Now, the 00 and ij components of the Einstein tensor can be calculated to be,

$$G_{00} = 3\left(\frac{1}{a}\frac{da}{dt}\right)^2; \quad G_{ij} = \delta_{ij}\left[-2a\frac{d^2a}{dt^2} - \left(\frac{da}{dt}\right)^2\right]. \tag{3.8}$$

We have now completed computing the left hand side of the Einstein field equations. For the right hand side, we need to know the energy-momentum tensor of the content of the universe. One usually uses a simplistic picture of perfect fluid to model the content of a homogeneous and isotropic universe. Perfect fluid is described by its rest frame energy density and isotropic pressure, whereas it has no viscosity, shear, stress or heat conduction. The energy-momentum tensor is written in mathematical form as follows,

$$T^{\mu}{}_{\nu} = \begin{pmatrix} -\rho(t) & 0 & 0 & 0 \\ 0 & P(t) & 0 & 0 \\ 0 & 0 & P(t) & 0 \\ 0 & 0 & 0 & P(t) \end{pmatrix}. \tag{3.9}$$

Now, the Bianchi identity enforces the conservation of energy and momentum that is the covariant derivative of the energy momentum tensor is zero by virtue of the dynamical equations (EFEs),

$$\begin{aligned} &\nabla_{\mu}T^{\mu}{}_{\nu} = 0 \\ &\implies \partial_{\mu}T^{\mu}{}_{\nu} + \Gamma^{\mu}_{\mu\alpha}T^{\alpha}{}_{\nu} - \Gamma^{\alpha}_{\nu\mu}T^{\mu}{}_{\alpha} = 0. \end{aligned} \tag{3.10}$$

Setting $\nu = 0$ in the above equation and expanding out the summations, we get,

$$\begin{aligned} &\partial_{\mu}T^{\mu}{}_{0} + \Gamma^{\mu}_{\mu\alpha}T^{\alpha}{}_{0} - \Gamma^{\alpha}_{0\mu}T^{\mu}{}_{\alpha} = 0 \\ &\implies \partial_0 T^0{}_0 + \Gamma^0_{00}T^0{}_0 + \Gamma^0_{0i}T^i{}_0 + \Gamma^i_{i0}T^0{}_0 + \Gamma^i_{ij}T^j{}_0 - \Gamma^0_{00}T^0{}_0 - \Gamma^0_{0i}T^i{}_0 - \Gamma^i_{00}T^0{}_i - \Gamma^i_{0j}T^j{}_i = 0 \\ &\implies -\frac{d\rho}{dt} + 3\frac{1}{a}\frac{da}{dt}(-\rho) - 3\frac{1}{a}\frac{da}{dt}P = 0. \end{aligned} \tag{3.11}$$

Now, using the equation of state $P = \omega\rho$ for a barotropic perfect fluid with a constant ω, the above equation becomes,

$$\frac{d\rho}{dt} + 3\frac{1}{a}\frac{da}{dt}(1+\omega)\rho = 0. \tag{3.12}$$

This equation is the continuity equation for the perfect fluid. The above equation can be expressed as a differential equation involving ρ and a,

$$\frac{d\rho}{\rho} = -3(1+\omega)\frac{da}{a}, \tag{3.13}$$

with the following solution,

$$\frac{\rho}{\rho_0} = \left(\frac{a}{a_0}\right)^{-3(1+\omega)}, \tag{3.14}$$

here, a_0 is the present day value of scale factor and ρ_0 is the corresponding value of the fluid's energy density. The above equation tells us how the energy density changes with the evolving scale factor. Now, we are ready to write down the Einstein field equations and solve them for a such that we know how the scale factor evolves in time when the universe is dominated by a single fluid with the equation of state as mentioned above. The 00-component of the Einstein field equations is,

$$\begin{aligned} G_{00} &= 8\pi G T_{00} \\ \implies 3\frac{1}{a^2}\left(\frac{da}{dt}\right)^2 &= 8\pi G\rho, \end{aligned} \tag{3.15}$$

whereas the ij-component is,

$$\begin{aligned} G_{ij} &= 8\pi G T_{ij} \\ \implies \delta_{ij}\left[-2a\frac{d^2a}{dt^2} - \left(\frac{da}{dt}\right)^2\right] &= 8\pi G\delta_{ij}a^2 P \\ \implies 2\frac{1}{a}\frac{d^2a}{dt^2} + \left(\frac{1}{a}\frac{da}{dt}\right)^2 &= -8\pi G P. \end{aligned} \tag{3.16}$$

Now, eliminating equation (3.15) from equation (3.16), we get,

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3P). \tag{3.17}$$

Equation (3.15) and (3.17) are called the Friedmann equations. Now, we can put the relation of energy density with the scale factor in equation (3.15) to solve for a,

$$\frac{1}{a^2}\left(\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3H_0^2}H_0^2\rho$$

$$
\begin{aligned}
&= H_0^2 \frac{\rho_0}{\rho_{\rm cr,0}} \left(\frac{a}{a_0}\right)^{-3(1+\omega)} \\
&= H_0^2 \Omega_{\omega,0} \left(\frac{a}{a_0}\right)^{-3(1+\omega)} \qquad \text{for a species with } P = \omega\rho. \qquad (3.18)
\end{aligned}
$$

Here, we have used the definition of critical energy density $\rho_{\rm cr} \equiv \frac{3H^2}{8\pi G}$, and $\rho_{\rm cr,0}$ denotes its value at the present time. Also, we have used the definition of density parameter for an ω-type fluid, $\Omega_\omega \equiv \frac{\rho}{\rho_{\rm cr}}$. Similarly, $\Omega_{\omega,0}$ denotes the parameter's value at present time. In an ω dominated universe and for $\omega \neq -1$, the solution of a in terms of time is the following,

$$
\begin{aligned}
&\frac{1}{a}\frac{da}{dt} = H_0 \sqrt{\Omega_{\omega,0}} \left(\frac{a}{a_0}\right)^{-\frac{3}{2}(1+\omega)} \\
&\implies \int_0^a \tilde{a}^{\frac{1}{2}(1+3\omega)} d\tilde{a} = H_0 \sqrt{\Omega_{\omega,0}} \left(\frac{1}{a_0}\right)^{-\frac{3}{2}(1+\omega)} \int_0^t dt' \\
&\implies a^{\frac{3}{2}(1+\omega)} = \frac{3}{2}(1+\omega) H_0 \sqrt{\Omega_{\omega,0}} \left(\frac{1}{a_0}\right)^{-\frac{3}{2}(1+\omega)} t \\
&\implies a(t) = \left[\frac{3}{2}(1+\omega) H_0 \sqrt{\Omega_{\omega,0}} \left(\frac{1}{a_0}\right)^{-\frac{3}{2}(1+\omega)} t\right]^{\frac{2}{3(1+\omega)}}. \qquad (3.19)
\end{aligned}
$$

For $\omega = -1$, which describes an interesting fluid with constant or scale factor independent energy density, we have the following solution,

$$
\begin{aligned}
&\frac{1}{a}\frac{da}{dt} = H_0 \sqrt{\Omega_{\omega,0}} \\
&\implies \int_a^{a_0} \frac{1}{\tilde{a}} d\tilde{a} = H_0 \sqrt{\Omega_{\omega,0}} \int_t^{t_0} dt \\
&\implies \ln \frac{a_0}{a} = H_0 \sqrt{\Omega_{\omega,0}} (t_0 - t) \\
&\implies a(t) = a_0 \exp\left[H_0 \sqrt{\Omega_{\omega,0}} (t - t_0)\right]. \qquad (3.20)
\end{aligned}
$$

Therefore, a universe dominated by a fluid with constant energy density grows exponentially over time. Such a fluid can be considered to be the dark energy which is responsible for the acceleration of the universe at late time. It is worth noting that the dependence of energy density on the scale factor and subsequently the dependence of scale factor on time comes from the continuity equation and the dynamical equations respectively. These relations may not be valid once we consider a different set of dynamical equations which will be the case in the next chapter.

3.2 Linear perturbations to the FLRW background

As we mentioned earlier that the homogeneous and isotropic picture of the universe is only a zero-th order approximation and we shall now move on and consider first order corrections to our zero-th order smooth universe. Therefore, we shall be adding small perturbations to the FLRW metric and the energy momentum tensor. After considering perturbed geometry and energy-momentum tensor we shall construct the equations corresponding to dynamics and kinetics. After that we shall derive some features about the approximated analytical solutions to those equations.

3.2.1 Perturbed geometry

We introduce two scalar fields $\Psi(\vec{x},t)$ and $\Phi(\vec{x},t)$, which are dependent on both space and time, to the metric to describe a perturbed universe,

$$\begin{aligned} g_{00}(\vec{x},t) &= -1 - 2\Psi(\vec{x},t) \\ g_{0i}(\vec{x},t) &= 0 \\ g_{ij}(\vec{x},t) &= a(t)^2\delta_{ij}(1+2\Phi(\vec{x},t)). \end{aligned} \tag{3.21}$$

Here, the values of these fields are small or $|\Phi|, |\Psi| \ll 1$, in the sense that their second and higher order powers, whenever they appear, can be neglected. This particular choice of the metric is due to choosing a particular gauge, namely, the Conformal-Newtonian gauge and the choice is not unique. There can be other choices of gauge such as the synchronous gauge or the spatially flat slicing. However, dealing with perturbations in our discussion is much simpler in the Conformal-Newtonian gauge and as a result the other gauge choices remains out of the scope of this thesis. Having decided the metric we can now compute the Christoffel symbols, and up to first order they are,

$$\Gamma^0_{00} = \Psi_{,0}; \quad \Gamma^0_{0i} = \Psi_{,i}; \quad \Gamma^0_{ij} = \delta_{ij}a^2\left[H + 2H(\Phi-\Psi) + \Phi_{,0}\right]$$

$$\Gamma^i_{00} = \frac{1}{a^2}\Psi_{,i}; \quad \Gamma^i_{j0} = \delta_{ij}(H+\Phi_{,0}); \quad \Gamma^i_{jk} = \delta_{ij}\Phi_{,k} + \delta_{ik}\Phi_{,j} - \delta_{jk}\Phi_{,i}. \tag{3.22}$$

The 00, ij and $0i$ components of the Ricci tensor can be computed to be as follows,

$$R_{00} = \frac{1}{a^2}\nabla^2\Psi - 3\frac{1}{a}\frac{d^2a}{dt^2} - 3\Phi_{,00} + 3H(\Psi_{,0} - 2\Phi_{,0}) \tag{3.23}$$

$$R_{ij} = \delta_{ij}\left[\left(2a^2H^2 + a\frac{d^2a}{dt^2}\right)(1 + 2\Phi - 2\Psi) + a^2H(6\Phi_{,0} - \Psi_{,0}) + a^2\Phi_{,00} - \nabla^2\Phi\right] - \Phi_{,ij} - \Psi_{,ij} \tag{3.24}$$

$$R_{0i} = 2(H\Psi_{,i} - \Phi_{,0i}). \tag{3.25}$$

Now, the Ricci scalar is simply the trace of the Ricci tensor and reads,

$$R = 6(1 - 2\Psi)\left(H^2 + \frac{1}{a}\frac{d^2a}{dt^2}\right) - \frac{2}{a^2}\nabla^2\Psi + 6\Phi_{,00} - 6H(\Psi_{,0} - 4\Phi_{,0}) - \frac{4}{a^2}\nabla^2\Phi. \tag{3.26}$$

For the left hand side of the Einstein field equations we need to compute the 00, ij and $0i$ components of the Einstein tensor, and they are,

$$G_{00} = R_{00} - \frac{1}{2}g_{00}R = 3H^2 + 6H\Phi_{,0} - \frac{2}{a^2}\nabla^2\Phi \tag{3.27}$$

$$\begin{aligned} G_{ij} &= R_{ij} - \frac{1}{2}g_{ij}R \\ &= \delta_{ij}\left[\left(-a^2H^2 - 2a\frac{d^2a}{dt^2}\right)(1 + 2\Phi - 2\Psi) + 2a^2H(\Psi_{,0} - 3\Phi_{,0}) - 2a^2\Phi_{,00} + \nabla^2\Phi + \nabla^2\Psi\right] \\ &\quad - \Phi_{,ij} - \Psi_{,ij} \end{aligned} \tag{3.28}$$

$$G_{0i} = R_{0i} - \frac{1}{2}g_{0i}R = R_{0i}. \tag{3.29}$$

Now, for the left hand side of the Einstein field equations we need to consider the perturbed energy-momentum tensor and that is the next topic of discussion.

3.2.2 Perturbed energy-momentum tensor

For our purposes of discussion it suffices to consider photons and neutrinos as relativistic fluid and dark matter and baryons to be non-relativistic fluid. While dark matter and baryons can be modeled as perfect fluids with small perturbations around their homogeneous and isotropic energy density and pressure, computing the energy-momentum tensor for the radiations (photons and neutrinos) is much trickier. We shall discuss them separately.

Photons and neutrinos: The general relativistic expression for the energy-momentum tensor in terms of the distribution function is (for this definition of energy-momentum tensor as the second moment of the single particle distribution function and for the justification of the integration measure see [7]),

$$T^{\mu}{}_{\nu}(\vec{x},t) = \sum_{\text{all species } \alpha} g_\alpha \int \frac{d^3P}{(2\pi)^3}\sqrt{-g}\frac{P^\mu P_\nu}{P^0} f_\alpha(\vec{x},\vec{p},t). \tag{3.30}$$

Here, $P^\mu \equiv \frac{dx^\mu}{d\lambda}$ are the comoving momenta, λ being the parameter of photon's trajectory, i.e. $x^\mu = x^\mu(\lambda)$ and g_α is the degeneracy factor for the species α. Due to the masslessness of photons (and to an approximation neutrinos can also be considered massless, thus, also for nutrinos), the comoving momenta satisfy the following on-shell condition,

$$g_{\mu\nu}P^\mu P^\nu = 0. \tag{3.31}$$

After expanding out the summations, the above equation becomes,

$$\begin{aligned} -(P^0)^2 + g_{ij}P^iP^j &= 0 \\ \implies P^0 &= p, \end{aligned} \tag{3.32}$$

with, $p^2 = g_{ij}P^iP^j$. Now, let us assume that $P^i = C(p)\hat{p}^i$, where, $\hat{p}^i$ are the unit vectors along the photon's physical momenta p and satisfies $\delta_{ij}\hat{p}^i\hat{p}^j = 1$. We perturb the matter on the FLRW background, thus, we have,

$$\begin{aligned} p^2 &= g_{ij}P^iP^j \\ &= \delta_{ij}a^2C^2(p)\hat{p}^i\hat{p}^j \\ &= C^2(p)a^2. \end{aligned} \tag{3.33}$$

Therefore, $P^i = \frac{p}{a}\hat{p}^i$. Collecting all the expressions above, one can write the conversion between the components of comoving momenta and the magnitude of physical momenta as follows,

$$P^\mu = \left[p, \frac{p}{a}\hat{p}^i\right]. \tag{3.34}$$

And also,

$$P_\mu = g_{\mu\alpha}P^\alpha = \left[-p, ap\hat{p}_i\right], \tag{3.35}$$

here, $\hat{p}^i = \hat{p}_i$ and, $\hat{p}^i\hat{p}_i = 1$. Now, we can convert the comoving momenta in the integral measure to physical momenta,

$$d^3P = dP^1 \wedge dP^2 \wedge dP^3 = \frac{1}{a^3}dp^1 \wedge dp^2 \wedge dp^3 = \frac{1}{a^3}d^3p. \tag{3.36}$$

Here, we have called $p\hat{p}^i \equiv p^i$, which are the spatial components of physical momenta. This factor of $\frac{1}{a^3}$ cancels with the a^3 factor coming from $\sqrt{-g}$.

The perturbation variable corresponding to photon's energy-momentum tensor can be introduced as the temperature anisotropy in the photon distribution. If the photon distribution had been homogeneous and isotropic, the temperature would only be a function of time. Now, the photon temperature anisotropy can be introduced as follows,

$$f(\vec{x}, p, \hat{p}, t) = \left[\exp\left\{\frac{p}{T(t)[1 + \Theta(\vec{x}, \hat{p}, t)]}\right\} - 1\right]^{-1}. \tag{3.37}$$

So, $\Theta = \frac{\delta T}{T}$ measures the deviation from the average zero-th order temperature $T(t)$ and depends on the position vector, the direction of photon's momentum and time. Whereas, the zero order distribution function was,

$$f^{(0)} = \frac{1}{e^{p/T} - 1}. \tag{3.38}$$

As the deviation from the homogeneity and isotropy is small, we can, now, expand the distribution function (3.37) around its zero order form (up to first order correction),

$$\begin{aligned} f &\simeq f^{(0)} + \frac{\partial f^{(0)}}{\partial T}\delta T \\ &= f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta, \end{aligned} \tag{3.39}$$

as one observes $T\frac{\partial f^{(0)}}{\partial T} = -p\frac{\partial f^{(0)}}{\partial p}$. We also define, the monopole of the photon temperature anisotropy, which only depends on the position vector but not on the direction of the photon momenta, as follows,

$$\Theta_0(\vec{x},t) \equiv \frac{1}{4\pi}\int d\Omega' \Theta(\hat{p}',\vec{x},t), \tag{3.40}$$

here, $d\Omega$ is the differential solid angle and is defined as $\sin\theta d\theta d\phi$. Now, the 00-component of the energy-momentum tensor becomes, for photons or neutrinos, when we break the distribution function in its zero order and first order part,

$$\begin{aligned} T^0{}_0 &= -g\int \frac{d^3p}{(2\pi)^3} p\left[f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta\right] \\ &= -\rho_\gamma + \frac{g}{(2\pi)^3}\int_0^\infty dp p^4 \frac{\partial f^{(0)}}{\partial p}\int d\Omega\Theta \\ &= -\rho_\gamma - 4\frac{g}{2\pi^2}\int_0^\infty dp p^3 f^{(0)}\Theta_0, \quad \text{as } f^{(0)} \to 0 \text{ when } p \to \infty \\ &= -\rho_\gamma[1+4\Theta_0]. \end{aligned} \tag{3.41}$$

In the second equality of the above calculation we have broken the measure d^3p into its magnitude and angular part as $p^2 dp d\Omega$. In the third equality, we have integrated by parts. Similarly, the $0i$-component of the energy-momentum tensor is given by the following integration,

$$T^0{}_i = ga\int \frac{d^3p}{(2\pi)^3} p\hat{p}_i\left[f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta\right]. \tag{3.42}$$

For all purposes later, we shall be working in the Fourier space. Therein, naturally, a propagation vector $\vec{k}$ for the perturbations would arise. We define, a variable μ that expresses the cosine of angle between the photon's momentum and the perturbation's propagation,

$$\mu \equiv \frac{k^i\hat{p}_i}{k} = \frac{k_i\hat{p}^i}{k}. \tag{3.43}$$

With this definition and considering the propagation of the perturbations in the $\hat{z}$ direction, the integration over $\sin\theta d\theta$, can be converted into an integration with respect to μ, as $\sin\theta d\theta = d(\cos\theta)$, with limit from -1 to 1. Thus, for the integrations below we shall break the measure d^3p as $p^2 dp d\mu d\phi$. Now, to get a useful quantity, we can multiply $T^0{}_i$ with corresponding component of the propagation vector k^i and sum over all the directions. The result is the following,

$$
\begin{aligned}
T^0{}_i k^i &= ga \int \frac{d^3p}{(2\pi)^3} p\mu \left[f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta \right] \\
&= 2\pi gak \int_0^\infty \frac{dp}{(2\pi)^3} \int_{-1}^1 d\mu p^3 \mu \left[f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta \right] \\
&= -\frac{gak}{2\pi^2} \int_0^\infty dp p^4 \frac{\partial f^{(0)}}{\partial p} \int_{-1}^1 \frac{d\mu}{2}\mu\Theta \\
&= -\frac{4igak}{2\pi^2} \int_0^\infty dp p^3 f^{(0)} \Theta_1 \\
&= -4iak\rho_\gamma \Theta_1,
\end{aligned} \tag{3.44}
$$

above, we have defined,

$$
\Theta_1 \equiv \frac{1}{-i} \int_{-1}^1 \frac{d\mu}{2} \mu\Theta, \tag{3.45}
$$

which is called the dipole of the photon anisotropy. The ij-component of the energy-momentum tensor becomes,

$$
T^i{}_j = g \int \frac{d^3p}{(2\pi)^3} p\hat{p}^i \hat{p}_j \left[f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta \right]. \tag{3.46}
$$

It is convenient to work with longitudinal, traceless part of the above object so that the tensor perturbations decouple from the scalar ones. We use the projection operator $\hat{k}_i\hat{k}^j - \frac{1}{3}\delta_i^j$ to extract that part.

$$
\begin{aligned}
\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta_i^j \right) T^i{}_j &= g \int \frac{d^3p}{(2\pi)^3} p \left(\mu^2 - \frac{1}{3} \right) \left[f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta \right] \\
&= \frac{g}{2\pi^2} \int_0^\infty dp \int_{-1}^1 \frac{d\mu}{2} p^3 \frac{2}{3} \left(\frac{3}{2}\mu^2 - \frac{1}{2} \right) \left[f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta \right] \\
&= \frac{g}{2\pi^2} \int_0^\infty dp \int_{-1}^1 \frac{d\mu}{2} p^3 \frac{2}{3} \mathcal{P}_2(\mu) \left[f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta \right] \\
&= -\frac{g}{2\pi^2} \int_0^\infty dp p^4 \frac{\partial f^{(0)}}{\partial p} \int_{-1}^1 \frac{d\mu}{2} \frac{2}{3} \mathcal{P}_2(\mu)\Theta \\
&= -\frac{8}{3}\frac{g}{2\pi^2} \int_0^\infty dp p^3 f^{(0)} \Theta_2 \\
&= -\frac{8}{3}\rho_\gamma \Theta_2,
\end{aligned} \tag{3.47}
$$

here, we have defined,

$$
\Theta_2 \equiv \frac{1}{(-i)^2} \int_{-1}^1 \frac{d\mu}{2} \mathcal{P}_2(\mu)\Theta, \tag{3.48}
$$

and $\mathcal{P}_2(\mu)$ is the second Legendre polynomial. In the similar vein, we define, in general,

$$\Theta_l \equiv \frac{1}{(-i)^l} \int_{-1}^{1} \frac{d\mu}{2} \mathcal{P}_l(\mu)\Theta, \tag{3.49}$$

The above part of the energy-momentum tensor is known as the anisotrpic stress. With the help of the definition of Θ_l, we can decompose the photon anisotropies into infinite Legendre modes as follows,

$$\Theta = (-i)^l(2l+1)\sum_{l=0}^{\infty} \mathcal{P}_l(\mu)\Theta_l. \tag{3.50}$$

From the behaviour of the functions $\mathcal{P}_l(\mu)$, it can be interpreted that the Θ_ls with higher l speaks of anisotropy at the smaller angular scales and is known as the multipole moments of the photon anisotropy. Though we have only worked with the photon distributions, similar expressions work for neutrinos as well.

Dark matter and baryon: Dark matter and baryon are considered non-relativistic in our discussion. We can introduce small perturbation in the number density of the dark matter as,

$$n_{\text{dm}}(\vec{x},t) = n_{\text{dm}}^{(0)}(1+\delta(\vec{x},t)), \tag{3.51}$$

here, $n_{\text{dm}}^{(0)}$ is the average number density of the dark matter distribution. For non-relativistic particles the energy density is mass times the number density, and so, the denisty contrast δ can be written as,

$$\delta(\vec{x},t) = \frac{n_{\text{dm}} - n_{\text{dm}}^{(0)}}{n_{\text{dm}}^{(0)}} = \frac{\delta\rho_{\text{dm}}}{\rho_{\text{dm}}}. \tag{3.52}$$

In our discussion it suffices to consider both dark matter and baryon as pressureless and they are also assumed to have no anisotropic stresses. As above, similar relations hold for the baryon number desity (n_b), density contrast (δ_b) and energy density (ρ_b).

3.2.3 Kinetics: Boltzmann equations

We have derived the form of a general Boltzmann equation in the second chapter (see equation (2.8)). Schematically, the unintegrated Boltzmann equation reads,

$$\frac{df}{dt} = \mathcal{C}[f], \tag{3.53}$$

where $\mathcal{C}[f]$ contains all possible collision terms. As we are in a curved spacetime, computing $\frac{df}{dt}$ is not trivial. First we shall focus on the deriving the form of $\frac{df}{dt}$ and then later in this section we shall derive the collision terms for different types of interactions separately.

We are studying the kinetics of the components of the universe in a perturbed background for which the metric is given in (3.21). We start our discussion with photons (or relativistic particles such as neutrinos). Let the comoving coordinates of photons be denoted as $x^\mu(\lambda)$. Then the comoving momenta are,

$$P^\mu = \frac{dx^\mu}{d\lambda} \tag{3.54}$$

In principle, f is a function defined in 8-dimensional space ($f = f(x^\mu, P^\mu)$). However, masslessness of photons implies,

$$P^2 \equiv g_{\mu\nu} P^\mu P^\nu = 0. \tag{3.55}$$

Expanding out the summations and using the metric components, we get,

$$P^2 = g_{00}(P^0)^2 + p^2 = -(1+2\Psi)(P^0)^2 + p^2 = 0, \tag{3.56}$$

here, as before,

$$p^2 \equiv g_{ij} P^i P^j, \tag{3.57}$$

and is called the physical momentum (squared). Now, the zero-th component of the comoving momenta can be expressed in terms of the physical momentum as follows, ignoring the higher order terms in Ψ,

$$P^0 = p(1+2\Psi)^{-\frac{1}{2}} \approx p(1-\Psi). \tag{3.58}$$

With the new variable defined, f can be expressed as a function of time (t), position vector ($\vec{x}$), magnitude of physical momentum (p) and its directions ($\hat{p}^i$). Thus, the total derivative can be written as,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} + \frac{\partial f}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt}. \tag{3.59}$$

In the last term $\frac{\partial f}{\partial \hat{p}^i}$ is a first order term because in zero-th order of f there is no dependence on the direction of the photon momenta and also $\frac{d\hat{p}^i}{dt}$ is a first order term because the direction of photon travelling only changes in presence of the perturbations in the metric. Hence, as a whole the last term in the total derivative expression is a multiplication of two first order terms, i.e. is a second order term and can be dispensed with in our linear approximation. Now, we must derive expressions for $\frac{dx^i}{dt}$ and $\frac{dp}{dt}$.

$$\frac{dx^i}{dt} = \frac{dx^i}{d\lambda}\frac{d\lambda}{dt} = \frac{P^i}{P^0}. \tag{3.60}$$

We already have an expression for P^0. To express the P^is in terms of physical momenta we proceed as before, define,

$$P^i \equiv C(p)\hat{p}^i. \tag{3.61}$$

Putting this into the definition of physical momentum,

$$p^2 = g_{ij}P^iP^j = g_{ij}\hat{p}^i\hat{p}^jC^2 = a^2(1+2\Phi)\delta_{ij}\hat{p}^i\hat{p}^jC^2. \tag{3.62}$$

We know $\delta_{ij}\hat{p}^i\hat{p}^j = 1$, and hence,

$$C = \frac{p}{a}(1-\Phi), \tag{3.63}$$

up to the linear order. Thus, the spatial components of comoving momenta expressed in terms of the phyical momenta are as follows,

$$P^i = \frac{p}{a}(1-\Phi)\hat{p}^i. \tag{3.64}$$

Putting all these expressions together we have an expression for $\frac{dx^i}{dt}$,

$$\frac{dx^i}{dt} = \frac{P^i}{P^0} = \frac{\frac{p}{a}(1-\Phi)\hat{p}^i}{p(1-\Psi)} \approx \frac{\hat{p}^i}{a}(1-\Phi+\Psi). \tag{3.65}$$

Now, we need to compute the expression for $\frac{dp}{dt}$. We know that photons (or any particles, for that matter) travel along the geodesics. We can, thus, use the geodesic equation,

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}. \tag{3.66}$$

For $\mu = 0$ the above equation becomes,

$$\frac{dP^0}{d\lambda} = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta. \tag{3.67}$$

The 0-th component of the comoving momentum can be expressed in terms of the physical momentum, as we derived earlier,

$$\frac{dP^0}{d\lambda} = \frac{dP^0}{dt}\frac{dt}{d\lambda} = P^0\frac{dP^0}{dt} = p(1-\Psi)\frac{d}{dt}[p(1-\Psi)]. \tag{3.68}$$

With this, the geodesic equation can be written as follows,

$$\begin{aligned}
\frac{d}{dt}[p(1-\Psi)] &= -\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+\Psi) \\
\Longrightarrow \quad \frac{dp}{dt}(1-\Psi) &= p\frac{d\Psi}{dt} - \Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+\Psi) \\
\Longrightarrow \quad \frac{dp}{dt} &= p\left\{\frac{\partial\Psi}{\partial t} + \frac{\partial\Psi}{\partial x^i}\frac{dx^i}{dt}\right\}(1+\Psi) - \Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+2\Psi) \\
&= p\left\{\frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right\} - \Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+2\Psi),
\end{aligned} \tag{3.69}$$

here, we have written the total derivative of $\Psi(\vec{x}, t)$ with respect to time as $\frac{\partial\Psi}{\partial t} + \frac{\partial\Psi}{\partial x^i}\frac{dx^i}{dt}$, and also used the fact that up to linear order $(1+\Psi)(1-\Psi) = 1$. First let us focus on term involving the Christoffel symbol. Using the definition of the Christoffel symbols and the symmetric property of the metric tensor, we can manipulate the expression, algebraically, as follows,

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{1}{2}g^{0\nu}[g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}]\frac{P^\alpha P^\beta}{p}$$

$$= \frac{-1+2\Psi}{2}[2g_{0\alpha,\beta} - g_{\alpha\beta,0}]\frac{P^\alpha P^\beta}{p}$$
$$= \frac{-1+2\Psi}{2}\left[2\left(-2\frac{\partial\Psi}{\partial x^\beta}\right)\frac{P^0P^\beta}{p} - g_{\alpha\beta,0}\frac{P^\alpha P^\beta}{p}\right]. \qquad (3.70)$$

Now, again let us first look at second term inside the square brackets in the above expression,

$$-\frac{\partial g_{\alpha\beta}}{\partial t}\frac{P^\alpha P^\beta}{p} = -\frac{\partial g_{00}}{\partial t}\frac{P^0P^0}{p} - \frac{\partial g_{ij}}{\partial t}\frac{P^iP^j}{p}$$
$$= 2\frac{\partial\Psi}{\partial t}p - a^2\delta_{ij}\left[2\frac{\partial\Phi}{\partial t} + 2H(1+2\Phi)\right]\frac{P^iP^j}{p}$$
$$= 2\frac{\partial\Psi}{\partial t}p - p\left[2\frac{\partial\Phi}{\partial t} + 2H(1+2\Phi)\right](1-2\Phi), \qquad (3.71)$$

here, we kept everything up to the linear order and used the definition of the metric components to evaluate their time derivatives. For instance, we know $g_{ij} = a^2\delta_{ij}(1 + 2\Phi)$, thus,

$$\frac{\partial g_{ij}}{\partial t} = 2a^2\delta_{ij}\frac{\partial\Phi}{\partial t} + 2a\frac{da}{dt}\delta_{ij}(1+2\Phi)$$
$$= a^2\delta_{ij}\left[2\frac{\partial\Phi}{\partial t} + 2H(1+2\Phi)\right]. \qquad (3.72)$$

We have also used the definition of physical momentum to find the expression for $\delta_{ij}P^iP^j$,

$$p^2 = g_{ij}P^iP^j = a^2\delta_{ij}(1+2\Phi)P^iP^j$$
$$\implies \delta_{ij}P^iP^j = \frac{p^2}{a^2}(1-2\Phi). \qquad (3.73)$$

Using these results we can now find out the term with the Christoffel symbol,

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p} = \frac{-1+2\Psi}{2}\left[-4\frac{\partial\Psi}{\partial x^\beta}P^\beta + 2p\frac{\partial\Psi}{\partial t} - 2p\left\{\frac{\partial\Phi}{\partial t} + H(1+2\Phi)\right\}(1-2\Phi)\right]$$
$$= (-1+2\Psi)\left[-2\left\{\frac{\partial\Psi}{\partial t}p + \frac{\partial\Psi}{\partial x^i}\frac{p\hat{p}^i}{a}\right\} + p\frac{\partial\Psi}{\partial t} - p\left\{\frac{\partial\Phi}{\partial t} + H\right\}\right]$$
$$= (-1+2\Psi)\left[-p\frac{\partial\Psi}{\partial t} - 2\frac{\partial\Psi}{\partial x^i}\frac{p\hat{p}^i}{a} - p\left\{\frac{\partial\Phi}{\partial t} + H\right\}\right]. \qquad (3.74)$$

With these results we have the expression for $\frac{dp}{dt}$,

$$\frac{dp}{dt} = p\left\{\frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right\} - p\frac{\partial\Psi}{\partial t} - 2\frac{\partial\Psi}{\partial x^i}\frac{p\hat{p}^i}{a} - p\left\{\frac{\partial\Phi}{\partial t} + H\right\}, \tag{3.75}$$

which reduces to,

$$\frac{1}{p}\frac{dp}{dt} = -H - \frac{\partial\Phi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}. \tag{3.76}$$

The above expression is insightful as it describes the change in photons' physical momentum as they travel trough an expanding and perturbed universe. The first term after the equality says photons lose energy or get red shifted as the universe expands. Now, according to the sign convention we are using an overdense region corresponds to $\Phi > 0$, and $\Psi < 0$. Thus, the second term after the equality says photons lose energy in a gravitational potential well which is deepening with time $\left(\frac{\partial\Phi}{\partial t} > 0\right)$. And the third term says photons travelling into a potential well $\left(\frac{\partial\Psi}{\partial x^i} < 0\right)$ gain energy and conversely, get red shifted when emerging out of the well $\left(\frac{\partial\Psi}{\partial x^i} > 0\right)$.

Gathering all the results together we, finally, have the expression for $\frac{df}{dt}$, the left hand side of the Boltzmann equation,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}\left[H + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right]. \tag{3.77}$$

From the above expression we can separate out the zero-th and first order parts. For zero-th order equation there is no collision term. Thus, the zero-th order equation becomes,

$$\left[\frac{df}{dt}\right]_{\text{zero-order}} = \frac{\partial f^{(0)}}{\partial t} - Hp\frac{\partial f^{(0)}}{\partial p} = 0. \tag{3.78}$$

We can convert the time derivative to a derivative with respect to temperature,

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial f^{(0)}}{\partial T}\frac{dT}{dt} = -\frac{p}{T}\frac{\partial f^{(0)}}{\partial p}\frac{dT}{dt}. \tag{3.79}$$

Putting that back into the zero-th order equation gives us,

$$\begin{aligned}\left[-\frac{dT/dt}{T} - \frac{da/dt}{a}\right]\frac{\partial f^{(0)}}{\partial p} &= 0 \\ \implies \frac{dT}{T} &= -\frac{da}{a}.\end{aligned} \tag{3.80}$$

Therefore, the zero-th order equation predicts that the temperature of the radiation falls as the universe expands,

$$T \propto \frac{1}{a}. \tag{3.81}$$

To get the first order equation we should break the distribution function into its zero-th and first order parts,

$$f \simeq f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta. \tag{3.82}$$

Putting the above expression into equation 3.77, we get,

$$\frac{df}{dt} = \frac{\partial f^{(0)}}{\partial t} - p\frac{\partial}{\partial t}\left[\frac{\partial f^{(0)}}{\partial p}\Theta\right] - p\frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}\frac{\partial f^{(0)}}{\partial p} - p\frac{\partial}{\partial p}\left[f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta\right] \left\{H + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right\}. \tag{3.83}$$

Collecting only the terms which are first order in perturbation variables gives,

$$\left[\frac{df}{dt}\right]_{\text{first order}} = -p\frac{\partial}{\partial t}\left[\frac{\partial f^{(0)}}{\partial p}\Theta\right] - p\frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}\frac{\partial f^{(0)}}{\partial p} - p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right] + Hp\Theta\frac{\partial}{\partial p}\left[p\frac{\partial f^{(0)}}{\partial p}\right]. \tag{3.84}$$

To simplify the above expression we look at the first term after the equal sign,

$$\begin{aligned} -p\frac{\partial}{\partial t}\left[\frac{\partial f^{(0)}}{\partial p}\Theta\right] &= -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial t} - p\Theta\frac{dT}{dt}\frac{\partial^2 f^(0)}{\partial T\partial p} \\ &= -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial\Theta}{\partial t} + p\Theta\underbrace{\frac{dT/dt}{T}}_{-H}\frac{\partial}{\partial p}\left[p\frac{\partial f^{(0)}}{\partial p}\right], \end{aligned} \tag{3.85}$$

here $\frac{dT/dt}{T} = -H$ is ensured by the zero-th order equation. Using this result, we finally have the left hand side of first order Boltzmann equation,

$$\left[\frac{df}{dt}\right]_{\text{first order}} = -p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial\Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i} + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right]. \tag{3.86}$$

We shall now discuss the collision term.

Collision term: Photons interact with electrons via Compton scattering. Consider the following scattering,

$$e^-(\vec{q}) + \gamma(\vec{p}) \leftrightarrow e^-(\vec{q'}) + \gamma(\vec{p'}), \tag{3.87}$$

the quantities within brackets indicate the momentum of the corresponding particles. Schematically, the collision term should look like the following,

$$\mathcal{C}[f(\vec{p})] = \sum_{\vec{q},\vec{q'},\vec{p'}} |\mathcal{M}|^2 \{f_e(\vec{q'})f(\vec{p'}) - f_e(\vec{q})f(\vec{p})\}, \tag{3.88}$$

here, $\mathcal{M}$ is the quantum amplitude for the scattering process and we shall borrow its expression from the quantum physics. We have neglected the Bose enhancement $1+f_\gamma$ and Pauli blocking $1-f_e$ terms because after electron-position annihilation the excess electron have a very small occupation number f_e and spontaneous emission is not important in the first order approximation. The full expression for the collision integral is as follows,

$$\begin{aligned}
\mathcal{C}[f(\vec{p})] &= \frac{1}{p}\int \frac{d^3\vec{q}}{(2\pi)^3 2E_e(\vec{q})}\int \frac{d^3\vec{q'}}{(2\pi)^3 2E_e(\vec{q'})}\int \frac{d^3\vec{p'}}{(2\pi)^3 2E(\vec{p'})}|\mathcal{M}|^2(2\pi)^4\delta^{(3)}(\vec{p}+\vec{q}-\vec{p'} \\
&\qquad - \vec{q'})\delta[E(p)+E_e(q)-E(p')-E_e(q')]\{f_e(\vec{q'})f(\vec{p'}) - f_e(\vec{q})f(\vec{p})\} \\
&= \frac{1}{p}\frac{(2\pi)^4}{(2\pi)^3 2m_e}\int \frac{d^3\vec{q}}{(2\pi)^3 2m_e}\int \frac{d^3\vec{p'}}{(2\pi)^3 2p'}|\mathcal{M}|^2\delta\left[p + m_e + \frac{q^2}{2m_e} - p' - m_e \right. \\
&\qquad \left. - \frac{(\vec{q}+\vec{p}-\vec{p'})^2}{2m_e}\right]\{f_e(\vec{q}+\vec{p}-\vec{p'})f(\vec{p'}) - f_e(\vec{q})f(\vec{p})\} \\
&= \frac{\pi}{4m_e^2 p}\int \frac{d^3\vec{q}}{(2\pi)^3}\int \frac{d^3\vec{p'}}{(2\pi)^3 p'}|\mathcal{M}|^2\delta\left[p + \frac{q^2}{2m_e} - p' - \frac{(\vec{q}+\vec{p}-\vec{p'})^2}{2m_e}\right] \\
&\qquad \{f_e(\vec{q}+\vec{p}-\vec{p'})f(\vec{p'}) - f_e(\vec{q})f(\vec{p})\},
\end{aligned} \tag{3.89}$$

here we have used the fact that the electrons are non-relativistic particles in our limit and then integrated over the momentum delta function. Also, the factor of $\frac{1}{p}$ in front of the collision integral comes from the fact that we are evaluating $\frac{df}{dt}$, instead of $\frac{df}{d\lambda}$. Now, we know $\frac{df}{d\lambda} = \frac{df}{dt}\frac{dt}{d\lambda} = \frac{df}{dt}p(1-\Psi)$. Thus the collision term should be scaled with $\frac{1}{p}(1+\Psi)$. However, as the collision term is already a first order quantity, only the factor $\frac{1}{p}$ is sufficient. In non-relativistic Compton scattering energy transfer from electron to photon is very small, and that enables us to approximate,

$$E_e(q) - E_e(\vec{q}+\vec{p}-\vec{p'}) = \frac{q^2}{2m_e} - \frac{(\vec{q}+\vec{p}-\vec{p'})^2}{2m_e} \simeq \frac{(\vec{p'}-\vec{p})\cdot\vec{q}}{m_e}. \tag{3.90}$$

Moreover, as the energy transfer is little, the delta function can be expanded as a Taylor series around $p-p'$. This is possible because the delta function is well behaved as long as its argument is not zero. The expansion looks like,

$$\begin{aligned}
&\delta\left[p + \frac{q^2}{2m_e} - p' - \frac{(\vec{q}+\vec{p}-\vec{p'})^2}{2m_e}\right] \\
&\simeq \delta(p-p') + (E_e(q') - E_e(q))\frac{\partial\delta(p+E_e(q)-p'-E_e(q'))}{\partial E_e(q')}|_{E_e(q')=E_e(q)} \\
&= \delta(p-p') + \frac{(\vec{p}-\vec{p'})\cdot\vec{q}}{m_e}\frac{\partial\delta(p-p')}{\partial p'},
\end{aligned} \tag{3.91}$$

in the last step we have used the mathematical identity $\frac{\partial f(x-y)}{\partial x} = -\frac{\partial f(x-y)}{\partial y}$. Again, as the energy transfer is little, $p-p'$ is small and the following electron distribution reduces to $f_e(\vec{q}+\vec{p}-\vec{p'}) \simeq f_e(\vec{q})$. With the help of these results, the collision integral becomes,

$$\mathcal{C}[f(\vec{p})] = \frac{\pi}{4m_e^2 p}\int\frac{d^3\vec{q}}{(2\pi)^3}f_e(\vec{q})\int\frac{d^3\vec{p'}}{(2\pi)^3 p'}|\mathcal{M}|^2\left\{\delta(p-p') + \frac{(\vec{p}-\vec{p'})\cdot\vec{q}}{m_e}\frac{\partial\delta(p-p')}{\partial p'}\right\} \\ \{f(\vec{p'}) - f(\vec{p})\}. \tag{3.92}$$

For further analysis we shall borrow the expression for the scattering amplitude. In [1], the following expression has been chosen, $|\mathcal{M}|^2 = 6\pi\sigma_T m_e^2(1+\cos^2[\hat{p}\cdot\hat{p}'])$, here, σ_T is the cross-section for the Thomson scattering. For the convenience of integration we break this quantity into two pieces, $|\mathcal{M}|^2 = 8\pi\sigma_T m_e^2 + 2\pi\sigma_T m_e^2[3\cos^2(\hat{p}\cdot\hat{p}') - 1]$. We first do the integration for the first part $8\pi\sigma_T m_e^2$,

$$\begin{aligned}
\mathcal{C}_1[f(\vec{p})] &= \frac{2\pi^2 n_e\sigma_T}{p}\int\frac{d^3\vec{p'}}{(2\pi)^3 p'}\left\{\delta(p-p') + (\vec{p}-\vec{p'})\cdot\vec{v_b}\frac{\partial\delta(p-p')}{\partial p'}\right\} \\
&\quad\left\{f^{(0)}(\vec{p'}) - f^{(0)}(\vec{p}) - p'\frac{\partial f^{(0)}}{\partial p'}\Theta(\hat{p'}) + p\frac{\partial f^{(0)}}{\partial p}\Theta(\hat{p})\right\} \\
&= \frac{n_e\sigma_T}{4\pi p}\int_0^\infty dp'p'\int d\Omega'\left[\delta(p-p')\left(-p'\frac{\partial f^{(0)}}{\partial p'}\Theta(\hat{p'}) + p\frac{\partial f^{(0)}}{\partial p}\Theta(\hat{p})\right)\right.
\end{aligned}$$

$$+(\vec{p}-\vec{p'})\cdot\vec{v_b}\frac{\partial\delta(p-p')}{\partial p'}(f^{(0)}(\vec{p'})-f^{(0)}(\vec{p}))\Bigg], \tag{3.93}$$

here we have broken the distribution functions into their zero-th and first order parts and kept all the terms up to the first order; used the definitions $\int\frac{d^3\vec{q}}{(2\pi)^3}f_e(\vec{q})\equiv n_e$ and $\int\frac{d^3\vec{q}}{(2\pi)^3}f_e(\vec{q})\frac{\vec{q}}{m_e}\equiv n_e\vec{v_b}$ (we are identifying electron fluid's average velocity with the baryon average velocity); used the property of delta function that $\delta(p-p')(f^{(0)}(\vec{p'})-f^{(0)}(\vec{p}))=0$. Moreover, the differential solid angle $d\Omega'$ is spanned by the unit vector $\hat{p}'$. In going ahead we shall use the definition of monopole of photon anisotropy and that leads to,

$$\begin{aligned}\mathcal{C}_1[f(\vec{p})]=\frac{n_e\sigma_T}{p}\int_0^\infty dp'p'\Bigg[\delta(p-p')\left(-p'\frac{\partial f^{(0)}}{\partial p'}\Theta_0+p\frac{\partial f^{(0)}}{\partial p}\Theta(\hat{p})\right)\\+\vec{p}\cdot\vec{v_b}\frac{\partial\delta(p-p')}{\partial p'}(f^{(0)}(\vec{p'})-f^{(0)}(\vec{p}))\Bigg]\end{aligned} \tag{3.94}$$

Integration by parts can be used for the last term inside the square brackets. Breaking, $\vec{p'}$ into its magnitude and direction part, $p'\cdot\vec{v_b}=p'\hat{p}'\cdot\vec{v_b}$, we have,

$$\begin{aligned}&\vec{p}\cdot\vec{v_b}\frac{\partial\delta(p-p')}{\partial p'}(f^{(0)}(\vec{p'})-f^{(0)}(\vec{p}))\\&=\frac{\partial}{\partial p'}\left[\vec{p}\cdot\vec{v_b}(f^{(0)}(\vec{p'})-f^{(0)}(\vec{p}))\right]-\delta(p-p')\frac{\partial}{\partial p'}\left[\vec{p}\cdot\vec{v_b}(f^{(0)}(\vec{p'})-f^{(0)}(\vec{p}))\right]\\&=-\delta(p-p')\vec{p}\cdot\vec{v_b}\frac{\partial f^{(0)}}{\partial p'}.\end{aligned} \tag{3.95}$$

Putting this expression back into the integral and integrating the delta function we have,

$$\begin{aligned}\mathcal{C}_1[f(\vec{p})]&=\frac{n_e\sigma_T}{p}\int_0^\infty dp'p'\left[\delta(p-p')\left(-p'\frac{\partial f^{(0)}}{\partial p'}\Theta_0+p\frac{\partial f^{(0)}}{\partial p}\Theta(\hat{p})\right)-\delta(p-p')p\hat{p}\cdot\vec{v_b}\frac{\partial f^{(0)}}{\partial p'}\right]\\&=-n_e\sigma_Tp\frac{\partial f^{(0)}}{\partial p}\left[\Theta_0-\Theta(\hat{p})+\hat{p}\cdot\vec{v_b}\right].\end{aligned} \tag{3.96}$$

Having done the integration for the first piece we now proceed to integrate the second piece corresponding to $2\pi\sigma_Tm_e^2[3\cos^2(\hat{p}\cdot\hat{p}')-1]$. Now,

$$\begin{aligned}&2\pi\sigma_Tm_e^2[3\cos^2(\hat{p}\cdot\hat{p}')-1]\\&=2\pi\sigma_Tm_e^2\cdot 2\mathcal{P}_2(\hat{p}\cdot\hat{p}')\end{aligned}$$

$$= 2\pi\sigma_T m_e^2 \frac{8\pi}{5} \sum_{m=-2}^{2} Y_{2m}(\hat{p}) Y_{2m}^*(\hat{p}'), \tag{3.97}$$

here we have used the definition of the second Legendre polynomial and used its relationship with the spherical harmonics,

$$\mathcal{P}_l(\hat{x}\cdot\hat{y}) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\hat{y}) Y_{lm}^*(\hat{x}). \tag{3.98}$$

The second part of the collision integral becomes,

$$\mathcal{C}_2[f(\vec{p})] = \frac{\pi}{4m_e^2 p} \int \frac{d^3\vec{q}}{(2\pi)^3} f_e(\vec{q}) \int \frac{d^3\vec{p'}}{(2\pi)^3 p'} \left(2\pi\sigma_T m_e^2 \frac{8\pi}{5} \sum_{m=-2}^{2} Y_{2m}(\hat{p}) Y_{2m}^*(\hat{p}') \right) \left\{ \delta(p-p') + \frac{(\vec{p}-\vec{p'})\cdot\vec{q}}{m_e} \frac{\partial\delta(p-p')}{\partial p'} \right\} \{f(\vec{p'}) - f(\vec{p})\}. \tag{3.99}$$

Now, we have the following expression for the spherical harmonics,

$$Y_{lm}(\hat{r}) \equiv (-1)^m i^l \left[\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{\frac{1}{2}} e^{im\phi} \mathcal{P}_l^m(\cos\theta), \tag{3.100}$$

here, $\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$ and $\cos\theta = \hat{r}\cdot\hat{z}$. In our case we have set the $\hat{z}$ direction along the direction of perturbation's propagation. Now, when we break the integral measures into the magnitude and angular parts, the spherical harmonics with $m \neq 0$ would not survive the integration over $d\phi$. For $m = 0$, $Y_{20}(\hat{p}) = -\sqrt{\frac{5}{4\pi}}\mathcal{P}_2(\hat{p}\cdot\hat{k}) = -\sqrt{\frac{5}{4\pi}}\mathcal{P}_2(\mu)$ and $Y_{20}(\hat{p}') = -\sqrt{\frac{5}{4\pi}}\mathcal{P}_2(\hat{p}'\cdot\hat{k}) = -\sqrt{\frac{5}{4\pi}}\mathcal{P}_2(\mu')$. Thus, the integral becomes,

$$\mathcal{C}_2[f(\vec{p})] = \frac{\pi^2 n_e \sigma_T}{p} \mathcal{P}_2(\mu) \int \frac{d^3\vec{p'}}{(2\pi)^3 p'} \mathcal{P}_2(\mu') \left\{ \delta(p-p') + (\vec{p}-\vec{p'})\cdot\vec{v_b} \frac{\partial\delta(p-p')}{\partial p'} \right\} \{f(\vec{p'}) - f(\vec{p})\}, \tag{3.101}$$

here, we have used the definitions $\int \frac{d^3\vec{q}}{(2\pi)^3} f_e(\vec{q}) \equiv n_e$ and $\int \frac{d^3\vec{q}}{(2\pi)^3} f_e(\vec{q}) \frac{\vec{q}}{m_e} \equiv n_e \vec{v_b}$. We can now break the distribution functions in their zero-th and first order parts. We have to remember the fact that the zero-th order terms do not depend on the angle of photon's momentum and hence, the integration of the second Legendre polynomial with the zero-th order term do not survive the angular integration. Also $\vec{v_b}$ is already a first order quantity and its multiplication with other first order quantities can be dropped. With these observations, the integral can be reduced to,

$$-\frac{n_e\sigma_T}{2p}\mathcal{P}_2(\mu)\int_0^\infty dp'p'\left(p'\frac{\partial f^{(0)}}{\partial p'}\right)\delta(p-p')\int_{-1}^{1}\frac{d\mu'}{2}\Theta(\mu')\mathcal{P}_2(\mu'). \tag{3.102}$$

Performing the delta function integration and using the definition of quadruple of photon anisotropy, we have,

$$\mathcal{C}_2[f] = p\frac{\partial f^{(0)}}{\partial p}\frac{n_e\sigma_T}{2}\mathcal{P}_2(\mu)\Theta_2. \tag{3.103}$$

Therefore, putting all the results together, we finally get the Boltzmann equation for photons,

$$\frac{\partial\Theta}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i}+\frac{\partial\Phi}{\partial t}+\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} = n_e\sigma_T\left[\Theta_0-\Theta(\hat{p})+\hat{p}\cdot\vec{v_b}-\frac{1}{2}\mathcal{P}_2\Theta_2\right]. \tag{3.104}$$

It is convenient to use the conformal time instead of the coordinate time. With the definition of conformal time, $d\eta = \frac{dt}{a(t)}$, and denoting derivative with respect conformal time with over dot, the Boltzmann equation takes the form,

$$\dot{\Theta}+\hat{p}^i\frac{\partial\Theta}{\partial x^i}+\dot{\Phi}+\hat{p}^i\frac{\partial\Psi}{\partial x^i} = n_e\sigma_T a\left[\Theta_0-\Theta(\hat{p})+\hat{p}\cdot\vec{v_b}-\frac{1}{2}\mathcal{P}_2\Theta_2\right]. \tag{3.105}$$

As we are working in a linear approximation, it is convenient to work in the Fourier space. The Fourier transformation of the photon temperature anisotropy reads,

$$\Theta(\vec{x},t) = \int\frac{d^3\vec{k}}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}\tilde{\Theta}(\vec{k},t). \tag{3.106}$$

The derivative with respect to spatial coordinates become,

$$\frac{\partial\Theta}{\partial x^i} = \int\frac{d^3\vec{k}}{(2\pi)^3}(ik_i)e^{i\vec{k}\cdot\vec{x}}\tilde{\Theta}(\vec{k},t). \tag{3.107}$$

The baryon velocity can be considered to be in the direction as the propagation vector and then for the corresponding Fourier transformed variable the following is true, $\tilde{\vec{v_b}}\cdot\hat{p} = \tilde{v_b}\hat{k}\cdot\hat{p} = \mu\tilde{v_b}$. With this observation we can compute the second term in the left hand side of the Boltzmann equation to be,

$$\hat{p^i}\frac{\partial \Theta}{\partial x^i} = \int \frac{d^3\vec{k}}{(2\pi)^3}(i\hat{p^i}k_i)e^{i\vec{k}\cdot\vec{x}}\tilde{\Theta}(\vec{k},t) = \int \frac{d^3\vec{k}}{(2\pi)^3}(ik\mu)e^{i\vec{k}\cdot\vec{x}}\tilde{\Theta}(\vec{k},t). \tag{3.108}$$

Moreover, we define a quantity called the optical depth as follows,

$$\tau(\eta) \equiv \int_{\eta}^{\eta_0} d\eta' n_e \sigma_T a, \tag{3.109}$$

with limits set in such a manner that its derivative is,

$$\dot{\tau} \equiv \frac{d\tau}{d\eta} = -n_e \sigma_T a. \tag{3.110}$$

Therefore, the Boltzmann equation in Fourier space, for each k modes reads,

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} = -\dot{\tau}\left[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{v}_b - \frac{1}{2}\mathcal{P}_2\tilde{\Theta}_2\right]. \tag{3.111}$$

In the early universe when Compton scattering was very effective, only the monopole and dipole had significant values. This can be argued as follows, if the scattering is very efficient the mean free path are much likely to be very small and all the photons that an observer would receiving would come from nearby over which distance the perturbations do not vary much in the early universe as most modes of interest lie outside the horizon early on. Therefore to that observer the sky appears uniform or the monopole is dominant. Other way of seeing this is to ignore the μv_b term coming from baryon drag velocity and Θ_2 term coming from the angular dependence of the Compton scattering in the above Boltzmann equation. Then we see that the, in the limit $k\eta \ll 1$, the Boltzmann equation has an attractor at $\Theta = \Theta_0$. Therefore, photon anisotropy, being provided with sufficient time to establish equilibrium, would drive to its monopole. But the drag velocity of the baryons are not exactly zero. As the photons are tightly coupled to the baryons, a drag velocity in their distribution introduces a dipole in the photon anisotropy. Next, if we calculate multiply the Boltzmann equation with $\mathcal{P}_2$ and then integrate over $\int_{-1}^{1}\frac{d\mu}{2}$ we get the following relation for Θ_2, in the limit $k\eta \ll 1$,

$$\dot{\Theta}_2 = -\frac{9}{10}\dot{\tau}\Theta_2. \tag{3.112}$$

The above equation has an attractor at $\Theta_2 = 0$ and hence, in equilibrium distribution the value of quadruple drives towards zero. Moreover, we shall argue later that in tightly coupled limit all the higher moments with $l > 2$ also have very small magnitude compared to monopole and dipole.

Boltzmann equation for cold dark matter: Deriving Boltzmann equation for cold dark matter is almost similar but with the distinction that dark matter is massive and has no collision with other components of the universe. For dark matter the trajectory can be parameterized with the proper time and the comoving momenta have the definition $P^\mu = m\frac{dx^\mu}{d\tau}$. Now, as the cold dark matter is massive,

$$P^2 \equiv g_{\mu\nu} P^\mu P^\nu = -m^2. \tag{3.113}$$

We can define the energy for cold dark amtter as, $E \equiv \sqrt{p^2 + m^2}$. And also with, $p^2 = g_{ij} P^i P^j$, the above equation becomes,

$$P^2 = g_{00}(P^0)^2 + p^2 = -(1 + 2\Psi)(P^0)^2 + p^2 = -m^2. \tag{3.114}$$

The 0-th component of comoving momentum is, now,

$$P^0 = \sqrt{\frac{p^2 + m^2}{1 + 2\Psi}} \approx E(1 - \Psi). \tag{3.115}$$

We get the spatial components proceeding as before, by defining,

$$P^i \equiv C(p)\hat{p}^i. \tag{3.116}$$

With the help of the definition for the physical momentum, we have,

$$p^2 = g_{ij} P^i P^j = a^2(1 + 2\Phi)\delta_{ij}\hat{p}^i\hat{p}^j C^2 = a^2(1 + 2\Phi)C^2 \tag{3.117}$$

Using, $\delta_{ij}\hat{p}^i\hat{p}^j = 1$, we determine C to be,

$$C = \frac{p}{a}(1 - \Phi). \tag{3.118}$$

Therefore, the comoving momenta are,

$$P^\mu = \left[E(1-\Psi), \frac{p}{a}(1-\Phi)\hat{p}^i\right]. \tag{3.119}$$

It is useful to compute $\frac{dx^i}{dt}$,

$$\frac{dx^i}{dt} = m\frac{dx^i}{d\tau}\frac{1}{m}\frac{d\tau}{dt} = \frac{P^i}{P^0} = \frac{\frac{p}{a}(1-\Phi)\hat{p}^i}{E(1-\Psi)} \approx \frac{p}{E}\frac{\hat{p}^i}{a}(1-\Phi+\Psi). \tag{3.120}$$

Similarly, as in the case for photons we express the distribution function for cold dark matter as a function of time, position vector, its energy and direction of physical momenta. Hence, the total derivative of the distribution function in the left hand side of the unintegrated Boltzmann equation for cold dark matter becomes,

$$\frac{df_{\rm dm}}{dt} = \frac{\partial f_{\rm dm}}{\partial t} + \frac{\partial f_{\rm dm}}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f_{\rm dm}}{\partial E}\frac{dE}{dt} + \frac{\partial f_{\rm dm}}{\partial \hat{p}^i}\frac{d\hat{p}^i}{dt}. \tag{3.121}$$

The last term here is also a second order term and can be dropped. To calculate $\frac{dE}{dt}$ we use the geodesic equation as before,

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}. \tag{3.122}$$

For $\mu = 0$ the geodesic equation reads,

$$m\frac{dP^0}{d\tau} = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta. \tag{3.123}$$

The left hand side of this equation can be written as,

$$m\frac{dP^0}{d\tau} = m\frac{dP^0}{dt}\frac{dt}{d\tau} = P^0\frac{dP^0}{dt} = E(1-\Psi)\frac{d}{dt}[E(1-\Psi)]. \tag{3.124}$$

Putting this into the geodesic equation,

$$\begin{aligned}
&\frac{d}{dt}[E(1-\Psi)] = -\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{E}(1+\Psi) \\
&\implies \frac{dE}{dt}(1-\Psi) = E\frac{d\Psi}{dt} - \Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{E}(1+\Psi) \\
&\implies \frac{dE}{dt} = E\left\{\frac{\partial\Psi}{\partial t} + \frac{\partial\Psi}{\partial x^i}\frac{dx^i}{dt}\right\}(1+\Psi) - \Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{E}(1+2\Psi)
\end{aligned}$$

$$= E\left\{\frac{\partial\Psi}{\partial t} + \frac{p}{E}\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right\} - \Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{E}(1+2\Psi) \tag{3.125}$$

Following the similar steps as before, the last term can be calculated to be,

$$\Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{E} = (-1+2\Psi)\left[-E\frac{\partial\Psi}{\partial t} - 2\frac{\partial\Psi}{\partial x^i}\frac{p\hat{p}^i}{a} - \frac{p^2}{E}\left\{\frac{\partial\Phi}{\partial t} + H\right\}\right]. \tag{3.126}$$

Putting the above results together the expression for $\frac{dE}{dt}$ becomes,

$$\begin{aligned}\frac{dE}{dt} &= E\left\{\frac{\partial\Psi}{\partial t} + \frac{p}{E}\frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right\} - E\frac{\partial\Psi}{\partial t} - 2\frac{\partial\Psi}{\partial x^i}\frac{p\hat{p}^i}{a} - \frac{p^2}{E}\left\{\frac{\partial\Phi}{\partial t} + H\right\} \\ &= -\frac{\partial\Psi}{\partial x^i}\frac{p\hat{p}^i}{a} - \frac{p^2}{E}\left\{\frac{\partial\Phi}{\partial t} + H\right\}.\end{aligned} \tag{3.127}$$

Therefore, the unintegrated Boltzmann equation for the cold dark matter has the following form,

$$\frac{df_{\rm dm}}{dt} = \frac{\partial f_{\rm dm}}{\partial t} + \frac{p}{E}\frac{\hat{p}^i}{a}\frac{\partial f_{\rm dm}}{\partial x^i} - \frac{\partial f_{\rm dm}}{\partial E}\left[H\frac{p^2}{E} + \frac{p^2}{E}\frac{\partial\Phi}{\partial t} + \frac{p\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right] = 0. \tag{3.128}$$

To get the equation in terms of recognizable quatities we compute the zero-th and first moments of this equation. For the zero-th moment we integrate both sides of the equation over all mommenta,

$$\begin{aligned}\frac{\partial}{\partial t}\int\frac{d^3\vec{p}}{(2\pi)^3}f_{\rm dm} + \frac{1}{a}\frac{\partial}{\partial x^i}\int\frac{d^3\vec{p}}{(2\pi)^3}f_{\rm dm}\frac{p\hat{p}^i}{E} - \left[\frac{da/dt}{a} + \frac{\partial\Phi}{\partial t}\right]\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial f_{\rm dm}}{\partial E}\frac{p^2}{E} \\ -\frac{1}{a}\frac{\partial\Psi}{\partial x^i}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial f_{\rm dm}}{\partial E}\hat{p}^i p = 0.\end{aligned} \tag{3.129}$$

We can use the definitions for dark matter number density,

$$n_{\rm dm} = \int\frac{d^3\vec{p}}{(2\pi)^3}f_{\rm dm}, \tag{3.130}$$

and dark matter average velocity,

$$v^i \equiv \frac{1}{n_{\rm dm}}\int\frac{d^3\vec{p}}{(2\pi)^3}f_{\rm dm}\frac{p\hat{p}^i}{E}. \tag{3.131}$$

Now, the third integral in the zero-th moment equation becomes,

$$\begin{aligned}
\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial f_{\rm dm}}{\partial E}\frac{p^2}{E} &= \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial f_{\rm dm}}{\partial p}\frac{\partial p}{\partial E}\frac{p^2}{E} \\
&= \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial f_{\rm dm}}{\partial p}p \\
&= \frac{1}{2\pi^2}\int_0^\infty dp p^3 \frac{\partial f_{\rm dm}}{\partial p} \\
&= \frac{1}{2\pi^2}\left[p^3 f_{\rm dm}\Big|_0^\infty - \int_0^\infty dp 3p^2 f_{\rm dm}\right] \\
&= -3\int_0^\infty \frac{4\pi p^2 dp}{(2\pi)^3} f_{\rm dm} \\
&= -3n_{\rm dm}. \qquad (3.132)
\end{aligned}$$

In the fourth integral the integrand only survives the angular integral if it is of first order. As there is $\frac{\partial \Psi}{\partial x^i}$ multiplying with the integral, the overall term is of second order and can be dropped. The zero-th moment of the Boltzmann equation becomes,

$$\frac{\partial n_{\rm dm}}{\partial t} + \frac{1}{a}\frac{\partial (n_{\rm dm} v^i)}{\partial x^i} + 3\left[\frac{da/dt}{a} + \frac{\partial \Phi}{\partial t}\right] n_{\rm dm} = 0. \qquad (3.133)$$

From the above equation the zero-order equation can be extracted,

$$\begin{aligned}
&\frac{\partial n^{(0)}_{\rm dm}}{\partial t} + 3\frac{da/dt}{a} n^{(0)}_{\rm dm} = 0 \\
\implies &\frac{d}{dt}\left(n^{(0)}_{\rm dm} a^3\right) = 0 \\
\implies &n^{(0)}_{\rm dm} \propto a^{-3} \qquad (3.134)
\end{aligned}$$

Therefore, the number density of the cold dark matter falls as the third power of the scale factor. To get the first-order equation we set,

$$n_{\rm dm} = n^{(0)}_{\rm dm}(1 + \delta(\vec{x}, t)). \qquad (3.135)$$

With this definition, keeping everything up to first order, the equation for density contrast has the following form, First-order equation,

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial \Phi}{\partial t} = 0. \qquad (3.136)$$

To get the second moment of the Boltzmann equation we multiply both sides of the Boltzmann equation with $\frac{p}{E}\hat{p}^j$ and integrate over all momenta as before, this leads to,

$$\frac{\partial}{\partial t}\int \frac{d^3\vec{p}}{(2\pi)^3} f_{\rm dm}\frac{p}{E}\hat{p}^j + \frac{1}{a}\frac{\partial}{\partial x^i}\int \frac{d^3\vec{p}}{(2\pi)^3} f_{\rm dm}\frac{p^2}{E^2}\hat{p}^i\hat{p}^j - \left[\frac{da/dt}{a} + \frac{\partial\Phi}{\partial t}\right]\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial f_{\rm dm}}{\partial E}\frac{p^3}{E^2}\hat{p}^j$$
$$-\frac{1}{a}\frac{\partial\Psi}{\partial x^i}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial f_{\rm dm}}{\partial E}\frac{p^2}{E}\hat{p}^i\hat{p}^j = 0. \tag{3.137}$$

Roughly speaking, $\frac{p}{E}$ is of the order of the velocity of cold dark matter and it is a first order quantity. Thus, the second term in the above equation can be dropped as it contains second power of $\frac{p}{E}$. With the relation $\frac{p}{E}\frac{\partial}{\partial E} = \frac{\partial}{\partial p}$ the third integral becomes,

$$\begin{aligned}
\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial f_{\rm dm}}{\partial p}\frac{p^2\hat{p}^j}{E} &= \int \frac{d\Omega}{(2\pi)^3}\hat{p}^j\int_0^\infty dp\frac{p^4}{E}\frac{\partial f_{\rm dm}}{\partial p} \\
&= \int \frac{d\Omega}{(2\pi)^3}\hat{p}^j\int_0^\infty dp f_{\rm dm}\left(\frac{4p^3}{E} - \frac{p^5}{E^3}\right) \\
&= -4\int \frac{d^3\vec{p}}{(2\pi)^3} f_{\rm dm}\frac{p\hat{p}^j}{E} \\
&= -4n_{\rm dm}v^j,
\end{aligned} \tag{3.138}$$

here we have used the fact that $\frac{p^5}{E^3} = p^2\left(\frac{p}{E}\right)^3$ is negligible. The fourth integral,

$$\begin{aligned}
\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial f_{\rm dm}}{\partial E}\frac{p^2\hat{p}^i\hat{p}^j}{E} &= \int \frac{d\Omega}{(2\pi)^3}\hat{p}^i\hat{p}^j\int_0^\infty dp p^3\frac{\partial f_{\rm dm}}{\partial p} \\
&= -\int \frac{d\Omega}{(2\pi)^3}\hat{p}^i\hat{p}^j\int_0^\infty dp 3p^2 f_{\rm dm} \\
&= -\frac{\delta^{ij}}{3}\int \frac{d^3\vec{p}}{(2\pi)^3}3f_{\rm dm} \\
&= -\delta^{ij}n_{\rm dm},
\end{aligned} \tag{3.139}$$

here we have used the following identity,

$$\int d\Omega\hat{p}^i\hat{p}^j = \frac{1}{3}\int d\Omega\delta^{ij}\delta_{ij}\hat{p}^i\hat{p}^j = \frac{4\pi}{3}\delta^{ij}. \tag{3.140}$$

Therefore, the first moment of the Boltzmann equation is,

$$\frac{\partial}{\partial t}(n_{\rm dm}v^j) + 4\frac{da/dt}{a}n_{\rm dm}v^j + \frac{n_{\rm dm}}{a}\frac{\partial\Psi}{\partial x^j} = 0. \tag{3.141}$$

As $n_{\rm dm}$ is always multiplying a first order quantity in the above equation, it can be replaced by $n_{\rm dm^{(0)}}$ up to first order,

$$
\begin{aligned}
& n_{\rm dm}^{(0)}\frac{\partial v^j}{\partial t} + \frac{\partial n_{\rm dm}^{(0)}}{\partial t}v^j + 4\frac{da/dt}{a}n_{\rm dm}^{(0)}v^j + \frac{n_{\rm dm}^{(0)}}{a}\frac{\partial \Psi}{\partial x^j} = 0 \\
\Longrightarrow\ & n_{\rm dm}^{(0)}\frac{\partial v^j}{\partial t} - 3\frac{da/dt}{a}n_{\rm dm}^{(0)}v^j + 4\frac{da/dt}{a}n_{\rm dm}^{(0)}v^j + \frac{n_{\rm dm}^{(0)}}{a}\frac{\partial \Psi}{\partial x^j} = 0 \\
\Longrightarrow\ & \frac{\partial v^j}{\partial t} + \frac{da/dt}{a}v^j + \frac{1}{a}\frac{\partial \Psi}{\partial x^j} = 0.
\end{aligned} \tag{3.142}
$$

In the above manipulation we have used the zero-th order Boltzmann equation. Similarly, in the case for photons, the Fourier transformed equations with the definition, $\tilde{v}^i = \left(\frac{k^i}{k}\right)\tilde{v}$ are as follows,

$$
\dot{\tilde{\delta}} + ik\tilde{v} + 3\dot{\tilde{\Phi}} = 0, \tag{3.143}
$$

and,

$$
\dot{\tilde{v}} + \frac{\dot{a}}{a}\tilde{v} + ik\tilde{\Psi} = 0. \tag{3.144}
$$

Here also over dot denotes the derivative with respect to the conformal time.

Boltzmann equation for baryons: The derivation for the left hand side of the Boltzmann equation for the baryons works similarly to the case for cold dark matter, however, for the right hand side we now have Compton scattering between electrons and photons also Coulomb scattering between the electrons and protons. As the protons and electrons remain tightly coupled through the Coulomb scattering and also as the universe is considered to be electrically neutral, electrons' and protons' density contrasts can be identified with each other,

$$
\frac{\rho_e - \rho_e^{(0)}}{\rho_e^{(0)}} = \frac{\rho_p - \rho_p^{(0)}}{\rho_p^{(0)}} \equiv \delta_b. \tag{3.145}
$$

Similarly, due to the tight coupling their average velocities can also be identified, $\vec{v}_e = \vec{v}_p \equiv \vec{v}_b$. To compute the collision terms for Compton and Coulomb scattering, let us assign momenta to the corresponding particles. Say, $\vec{p}$ and $\vec{p}'$ are the incoming and outgoing photon momenta, respectively; $\vec{q}$ and $\vec{q}'$ are the incoming and outgoing momenta, respectively and finally the incoming and outgoing momenta for the proton are $\vec{Q}$ and $\vec{Q}'$. Thus, we have following two unintegrated Boltzmann equation for electrons and protons,

$$\frac{df_e(\vec{x},\vec{q},t)}{dt} = \langle C_{ep}\rangle_{QQ'q'} + \langle C_{e\gamma}\rangle_{pp'q'} \tag{3.146}$$

$$\frac{df_p(\vec{x},\vec{Q},t)}{dt} = \langle C_{ep}\rangle_{qq'Q'} \tag{3.147}$$

In the above notation, the integrand of the collision term has been defined as follows,

$$C_{e\gamma} \equiv (2\pi)^4\delta^{(4)}(p+q-p'-q')\frac{|\mathcal{M}|^2}{8E(p)E(p')E_e(q)E_e(q')}\left\{f_e(q')f_\gamma(p') - f_e(q)f_\gamma(p)\right\}, \tag{3.148}$$

and the angular brackets around them denotes the integration over all the momenta that are indicated in the subscripts,

$$\langle(\ldots)\rangle_{pp'q'} \equiv \int\frac{d^3\vec{p}}{(2\pi)^3}\int\frac{d^3\vec{p'}}{(2\pi)^3}\int\frac{d^3\vec{q'}}{(2\pi)^3}(\ldots). \tag{3.149}$$

To get the equation for δ_b we integrate the electron Boltzmann equation with respect to $\int\frac{d^3\vec{q}}{(2\pi)^3}$, and then the left hand side of this equation becomes similar to the left hand side of the corresponding equation for cold dark matter, thus we have,

$$\frac{\partial n_e}{\partial t} + \frac{1}{a}\frac{\partial(n_e v_b^i)}{\partial x^i} + 3\left[\frac{da/dt}{a} + \frac{\partial\Phi}{\partial t}\right]n_e = \langle C_{ep}\rangle_{QQ'q'q} + \langle C_{e\gamma}\rangle_{pp'q'q} = 0. \tag{3.150}$$

The right hand side of the equation is equal to zero because both the collision integral vanishes. The reason for this is, for instance, in the case of the first integral the integration measure is invariant under the exchange of variables $Q \leftrightarrow Q'$ and $q \leftrightarrow q'$ but the integrand is antisymmetric under these exchanges. Similar reasoning works for the second integration. Then the above equation is identical to that of cold dark matter and in Fourier space the equation for δ_b can be written as,

$$\dot{\tilde{\delta}}_b + ik\tilde{v}_b + 3\dot{\tilde{\Phi}} = 0. \tag{3.151}$$

For the equation corresponding to v_b, we multiply equation (3.146) and (3.147) with $\vec{q}$ and $\vec{Q}$, respectively, and integrate over these corresponding momenta. Thus, this step is similar with the case for cold dark matter but there we had multiplied by $\vec{p}/E$ and we can borrow the result for the left hand side of the dark matter's Boltzmann equation so long as we also multiply a factor of corresponding mass. Then the left hand side of the first equation has mass of an electron multiplied to it whereas the second one has a proton's mass. As proton is much heavier than electron, upon adding the two equations, we get,

$$m_p \frac{\partial}{\partial t}(n_b v_b^j) + 4\frac{da/dt}{a} m_p n_b v_b^j + \frac{m_p n_b}{a}\frac{\partial \Psi}{\partial x^j} = \langle C_{ep}(q^j + Q^j)\rangle_{QQ'q'q} + \langle C_{e\gamma} q^j\rangle_{pp'q'q}. \quad (3.152)$$

As n_b in the above equation is always multiplying a first order quantity, it can be replaced by $n_b^{(0)}$. Using the fact that $n_b^{(0)} \propto a^{-3}$ and dividing both sides of the above equation by $\rho_b = m_p n_b^{(0)}$, we arrive at,

$$\frac{\partial v_b^j}{\partial t} + \frac{da/dt}{a} v_b^j + \frac{1}{a}\frac{\partial \Psi}{\partial x^j} = \frac{1}{\rho_b}\langle C_{e\gamma} q^j\rangle_{pp'q'q}, \quad (3.153)$$

here, the first collision integral in equation (3.152) has been set to zero by virtue of momentum conservation. In Coulomb scattering the sum of the initial momenta of proton and electron remains invariant before and after the collision. Now, it remains to evaluate the second collision integral corresponding to the Compton scattering,

$$\langle C_{e\gamma} q^j\rangle_{pp'q'q} = \int \frac{d^3\vec{p}}{(2\pi)^3}\int \frac{d^3\vec{p'}}{(2\pi)^3}\int \frac{d^3\vec{q'}}{(2\pi)^3}\int \frac{d^3\vec{q}}{(2\pi)^3}\delta^{(4)}(p+q-p'-q')|\mathcal{M}|^2$$
$$q^j\left\{f_e(q')f_\gamma(p') - f_e(q)f_\gamma(p)\right\} \quad (3.154)$$

In the centre of mass frame, we have, $\vec{p} = -\vec{q}$, and hence,

$$\langle C_{e\gamma}\vec{q}\rangle_{pp'q'q} = -\langle C_{e\gamma}\vec{p}\rangle_{pp'q'q}. \quad (3.155)$$

Multiplying both sides of the equation (3.154) by $\hat{k}_j$, and using the fact that $\hat{k}_j \cdot q^j \implies \hat{k}\cdot\vec{p} = \mu p$, we have,

$$-\frac{\langle C_{e\gamma}\mu p\rangle_{pp'q'q}}{\rho_b} = \frac{n_e\sigma_T}{\rho_b}\int \frac{d^3\vec{p}}{(2\pi)^3} p^2 \frac{\partial f^{(0)}}{\partial p}\mu[\tilde{\Theta}_0 - \tilde{\Theta}(\mu) + \mu\tilde{v}_b - \frac{1}{2}\mathcal{P}_2(\mu)\Theta_2], \quad (3.156)$$

here, we have used our previous result from the collision integral for the Compton scattering. Now, breaking the integration measure into magnitude and angular part, we can perform the integral as follows,

$$
\begin{aligned}
-\frac{\langle C_{e\gamma}\vec{q}\rangle_{pp'q'q}}{\rho_b} &= \frac{n_e\sigma_T}{\rho_b}\int_0^\infty \frac{dp}{(2\pi)^3}p^4\frac{\partial f^{(0)}}{\partial p}\int_0^{2\pi}d\phi\int_{-1}^{1}d\mu\mu[\tilde{\Theta}_0-\tilde{\Theta}(\mu)+\mu\tilde{v}_b] \\
&= -\frac{n_e\sigma_T}{\rho_b}\int_0^\infty \frac{dp}{2\pi^2}4p^3f^{(0)}\left[i\int_{-1}^{1}\frac{d\mu}{2}i\mu\tilde{\Theta}(\mu)+\frac{1}{3}\tilde{v}_b\right] \\
&= -\frac{n_e\sigma_T}{\rho_b}\int_0^\infty \frac{dp4\pi p^2}{(2\pi)^3}4pf^{(0)}\left[i\int_{-1}^{1}\frac{d\mu}{2}i\mu\tilde{\Theta}(\mu)+\frac{1}{3}\tilde{v}_b\right] \\
&= -n_e\sigma_T\frac{4\rho_\gamma}{3\rho_b}[3i\tilde{\Theta}_1+\tilde{v}_b] \\
&= \frac{1}{a}\dot{\tau}\frac{4\rho_\gamma}{3\rho_b}[3i\tilde{\Theta}_1+\tilde{v}_b].
\end{aligned}
\tag{3.157}
$$

Therefore, in Fourier space and using conformal time the Boltzmann equation for baryon average velocity becomes,

$$
\dot{\tilde{v}}+\frac{\dot{a}}{a}\tilde{v}+ik\tilde{\Psi}=\dot{\tau}\frac{4\rho_\gamma}{3\rho_b}[3i\tilde{\Theta}_1+\tilde{v}_b]. \tag{3.158}
$$

As from now on we shall always be working in the Fourier space, we adopt the notation of dropping tilde for the Fourier transformed variables and keep the fact in mind that we are working in the Fourier space rather than in the real space. Having derived all the relevant Boltzmann equations we shall now move on to deriving the dynamical Einstein field equations for linear perturbation theory.

3.2.4 Dynamics: Einstein equations

As we are dealing with equations that are linear in the perturbation variables, it is convenient to work in the Fourier space. Because of the linearity, different Fourier modes do not mix with each other and we can write equations for each of those modes separately. In going form real space equations to the Fourier space equations for each k mode we replace the spatial derivatives (∂_i) by ik_i. In this way our expressions for various components of Einstein tensor become in the Fourier space,

$$
G_{00}=3H^2+6H\Phi_{,0}+\frac{2k^2}{a^2}\Phi \tag{3.159}
$$

$$
G_{ij}=\delta_{ij}\left[\left(-a^2H^2-2a\frac{d^2a}{dt^2}\right)(1+2\Phi-2\Psi)+2a^2H(\Psi_{,0}-3\Phi_{,0})-2a^2\Phi_{,00}-k^2\Phi-k^2\Psi\right]
$$

$$+ k_i k_j \Phi + k_i k_j \Psi \tag{3.160}$$

$$G_{0i} = 2ik_i(H\Psi - \Phi_{,0}). \tag{3.161}$$

Here, the zero-th derivatives are with respect to coordinate time. As we projected the longitudinal, traceless part out of the ij-components of the energy-momentum tensor to avoid contributions from the tensor perturbations, we should do the same for the ij-components of the Einstein tensor as well. When we apply the operator $\hat{k}_i \hat{k}^j - \frac{1}{3}\delta_i^j$ on $G^i{}_j$, the term proportional to δ_j^i vanish and we are left with,

$$\left(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta_i^j\right) G^i{}_j = \left(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta_i^j\right)\left(\frac{k^i k_j(\Phi + \Psi)}{a^2}\right) = \frac{2}{3a^2}k^2(\Phi + \Psi), \tag{3.162}$$

the factor of $1/a^2$ comes from the metric when we raise one index. Now, we have both right and left hand sides of the Einstein field equations ready. Using conformal time, the 00-component of the equation becomes,

$$\begin{aligned} G^0{}_0 &= 8\pi G T^0{}_0 \\ \implies k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} - \Psi\frac{\dot{a}}{a}\right) &= 4\pi G a^2[\rho_{\rm dm}\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0]. \end{aligned} \tag{3.163}$$

The longitudinal, traceless part of the ij-components become,

$$\begin{aligned} \left(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta_i^j\right) G^i{}_j &= 8\pi G\left(\hat{k}_i \hat{k}^j - \frac{1}{3}\delta_i^j\right) T^i{}_j \\ \implies k^2(\Phi + \Psi) &= -32\pi G a^2[\rho_\gamma\Theta_2 + \rho_\nu\mathcal{N}_2]. \end{aligned} \tag{3.164}$$

And, finally, the $0i$-components are,

$$\begin{aligned} G^0{}_i k^i &= 8\pi G T^0{}_i k^i \\ \implies \dot{\Phi} - aH\Psi &= \frac{4\pi G a^2}{ik}[\rho_{\rm dm}v + \rho_b v_b - 4i\rho_\gamma\Theta_1 - 4i\rho_\nu\mathcal{N}_1]. \end{aligned} \tag{3.165}$$

As there are only two independent variables (Φ and Ψ) in the metric the equation corresponding to $0i$-components add no new information to the other two equations and we shall use the last equation as convenient to the situation. With all the Boltzmann and Einstein field equations derived we have the set of coupled Boltzmann-Einstein equations ready to be solved and that is the discussion of our next section.

3.3 Solving Einstein-Boltzmann equations for inhomogeneities

In order to study how perturbations evolve we must solve the set of coupled Boltzmann-Einstein equations. For our purpose here we would ignore the baryon density and its density contrast in favour of the corresponding cold dark matter quantities because the baryon density is much smaller than that of cold dark matter. Also, we shall assume that for photon (and for neutrinos as well) only monopole and dipole are important until very late in the matter dominated era and thus, we shall only include equations for these two quantities ignoring all the higher poles. Moreover, as the quadruple or the anisotropic stress is being considered negligible, the equation (3.164) implies $\Phi = -\Psi$. With the above considerations we have the following equations to solve.

The Boltzmann equations,

$$\dot{\Theta}_{r,0} + k\Theta_{r,1} = -\dot{\Phi} \tag{3.166}$$

$$\dot{\Theta}_{r,1} - \frac{k}{3}\Theta_{r,0} = -\frac{k}{3}\Phi \tag{3.167}$$

$$\dot{\delta} + ikv = -3\dot{\Phi} \tag{3.168}$$

$$\dot{v} + \frac{\dot{a}}{a}v = ik\Phi. \tag{3.169}$$

The Einstein field equations,

$$k^2\Phi + 3\frac{\dot{a}}{a}\left(\dot{\Phi} + \Phi\frac{\dot{a}}{a}\right) = 4\pi Ga^2[\rho_{\mathrm{dm}}\delta + 4\rho_r\Theta_{r,0}] \tag{3.170}$$

$$\Phi = -\Psi. \tag{3.171}$$

For convenience, in particular situations we may also use an algebraic Einstein field equation which is obtained when we substitute equation (3.165) into equation (3.163),

$$k^2\Phi = 4\pi Ga^2\left[\rho_{\mathrm{dm}}\delta + 4\rho_r\Theta_{r,0} + \frac{3aH}{k}\left(i\rho_{\mathrm{dm}}v + 4\rho_r\Theta_{r,1}\right)\right]. \tag{3.172}$$

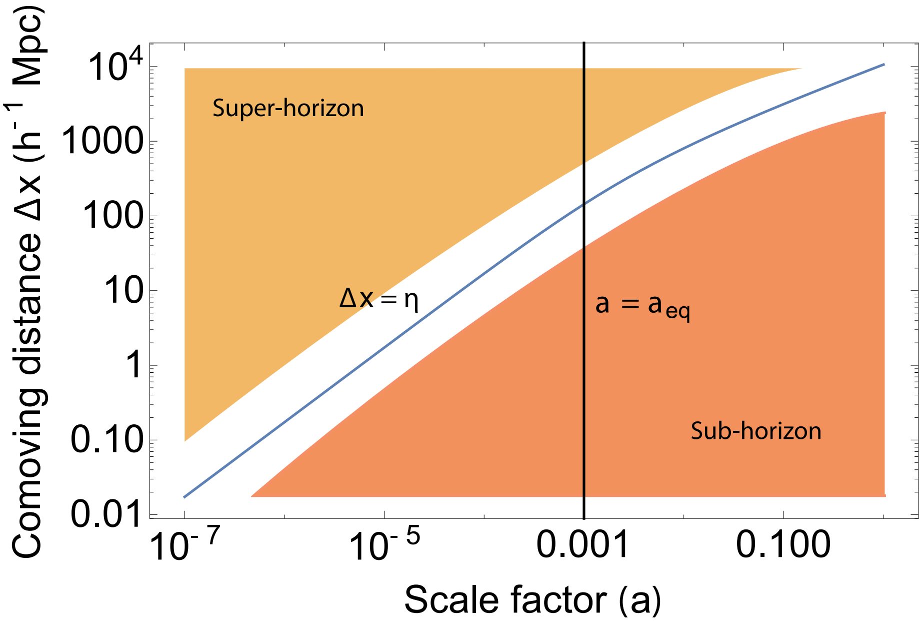

Figure 3.1: Super-horizon and sub-horizon region

In the above equations the subscript r means radiation which includes both photons and neutrinos. Now, it is not possible to solve all of the above equations analytically. To get leading order analytical solutions in various cases we must use approximations. We identify different regions in comparison with the comoving horizon size. If the universe was only radiation and matter dominated, the comoving horizon, in that case is given by the following expression,

$$\eta = \frac{2}{\Omega_m H_0^2}\left[\sqrt{a + a_{eq}} - \sqrt{a_{eq}}\right], \tag{3.173}$$

here, a_{eq} is the scale factor corresponding the epoch of equality. Now, we shall identify regions in the comoving distance scales which are always outside the horizon (super-horizon) and always inside the horizon (sub-horizon). This two regions are shaded in the figure 3.1. These two regions constitute our two primary regions of approximations. And also we shall look for the solutions corresponding to the modes which crosses the horizon at early times in the radiation dominated era or at late time in the matter dominated era. This two regions are shaded in the figure 3.2.

3.3.1 Initial conditions

Before we go on to solve the set of Boltzmann-Einstein equations we should discuss the initial conditions. At very early time, when $\eta \to 0$ all modes of interest lie outside

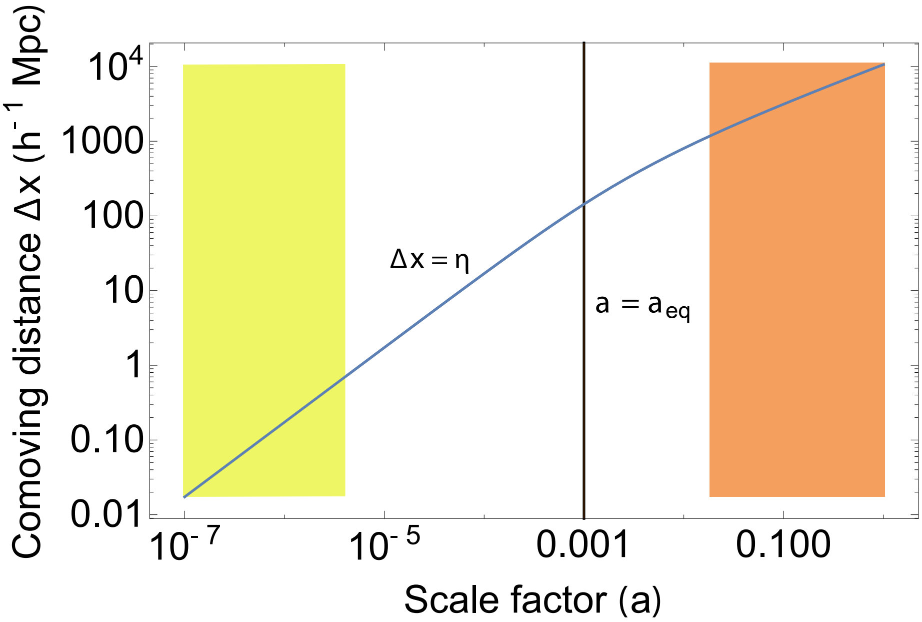

Figure 3.2: Early and late time horizon crossing

the horizon. In that case it is appropriate to use $k \to 0$, because the wavelengths ($\sim k^{-1}$) of the perturbations are so large and outside the horizon that they cannot be probed from inside. Also in this limit only Θ_0 is prominent among all the other moments because as the perturbations lie outside the horizon the sky for an hypothetical observer at that time appears almost homogeneous. Therefore, the Boltzmann equations reduce to,

$$\dot{\Theta}_0 + \dot{\Phi} = 0; \quad \dot{\delta} + 3\dot{\Phi} = 0$$
$$\dot{\mathcal{N}}_0 + \dot{\Phi} = 0; \quad \dot{\delta}_b + 3\dot{\Phi} = 0 \tag{3.174}$$

Here, we see that all the perturbation variables at early times are related to the gravitational potential. Then it suffices to figure out the initial condition for the potential. Θ_0 has the following solution with an integration constant,

$$\Theta_0 = -\Phi + \text{constant}. \tag{3.175}$$

Now, the Einstein field equation in the early time or in the radiation dominated era takes the following form,

$$3\frac{\dot{a}}{a}\left(\dot{\Phi} - \frac{\dot{a}}{a}\Psi\right) = 16\pi G a^2(\rho_\gamma \Theta_0 + \rho_\nu \mathcal{N}_0). \tag{3.176}$$

In radiation dominated era, we have, $a \propto \eta \implies \frac{\dot{a}}{a} = \frac{1}{\eta}$. Using this into the Einstein equation gives,

$$\begin{aligned}\frac{\dot{\Phi}}{\eta} - \frac{1}{\eta^2}\Psi &= \frac{16\pi G a^2}{3}\rho\left(\frac{\rho_\gamma}{\rho}\Theta_0 + \frac{\rho_\nu}{\rho}\mathcal{N}_0\right) \\ &= 2a^2H^2\left((1-f_\nu)\Theta_0 + f_\nu\mathcal{N}_0\right) \\ &= \frac{2}{\eta^2}\left((1-f_\nu)\Theta_0 + f_\nu\mathcal{N}_0\right),\end{aligned} \tag{3.177}$$

here, $\rho = \rho_\gamma + \rho_\nu$ is the total energy density and $f_\nu = \frac{\rho_\nu}{\rho}$ neutrino abundance fraction. We have also used the first Friedmann equation in the above manipulation. The above equation is re-written in the following form,

$$\dot{\Phi}\eta - \Psi = 2\left((1-f_\nu)\Theta_0 + f_\nu\mathcal{N}_0\right). \tag{3.178}$$

Using the fact that $\dot{\Theta}_0 = \dot{\mathcal{N}_0} = -\dot{\Phi}$ from the Boltzmann equations, we have,

$$\ddot{\Phi}\eta + \dot{\Phi} - \dot{\Psi} = -2\dot{\Phi}. \tag{3.179}$$

As the quadruple is negligible, $\Phi + \Psi = 0$, and hence,

$$\ddot{\Phi}\eta + 4\dot{\Phi} = 0. \tag{3.180}$$

We use the ansatz $\Phi = \eta^p$ as the solution of the above equation. This gives us the following algebraic equation in p

$$p(p-1) + 4p = 0, \tag{3.181}$$

with the solutions $p = 0$, and $p = -3$. As the second solution blows up in the $\eta \to 0$ limit we take $\Phi =$ constant as the solution. Now, we also have,

$$\Phi = 2\left((1-f_\nu)\Theta_0 + f_\nu\mathcal{N}_0\right). \tag{3.182}$$

Assuming, initially photon and neutrino monopoles were the same, i.e., $\Theta_0(k,\eta_i) = \mathcal{N}_0(k,\eta_i)$, we get,

$$\Phi(k,\eta_i) = 2\Theta_0(k,\eta_i). \tag{3.183}$$

Now, for the density contrast of dark matter we have from the Boltzmann equation,

$$\delta = 3\Theta_0 + \text{constant}. \tag{3.184}$$

In the model of adiabatic perturbations this constant is chosen to be zero. Due to this choice the matter to radiation number density ratio has a constant value,

$$\frac{n_{\rm dm}}{n_\gamma} = \frac{n_{\rm dm}^{(0)}}{n_\gamma^{(0)}}\left[\frac{1+\delta}{1+3\Theta_0}\right] \approx \frac{n_{\rm dm}^{(0)}}{n_\gamma^{(0)}}(1+\delta-3\Theta_0) = \frac{n_{\rm dm}^{(0)}}{n_\gamma^{(0)}}. \tag{3.185}$$

Therefore, the density contrast initially had the following relationship with the initial potential Φ_p,

$$\delta(k, \eta = 0) = \frac{3\Phi_p}{2}. \tag{3.186}$$

Finally, we can also see that the constant in the solution of Θ_0 in equation (3.175) has to be $\frac{3\Phi_p}{2}$ in order for the equation (3.183) to hold.

3.3.2 Large Scales

Our first focus will be to solve the above set of equations for the large scale modes which either always remain in the super-horizon scales or crosses horizon at late times during the matter dominated era.

Super-horizon solution

Before we dive into the solution, it is useful to derive and define few relations. We know that the radiation and matter density changes with the scale factor as follows,

$$\rho_r = \rho_{cr}\Omega_r a^{-4} \tag{3.187}$$

$$\rho_m = \rho_{cr}\Omega_m a^{-3}. \tag{3.188}$$

Then their ratio is,

$$\frac{\rho_m}{\rho_r} = \frac{\Omega_m}{\Omega_r}a. \tag{3.189}$$

At epoch of equality the matter and radiation density becomes equal to each other, i.e., at a_{eq}, the following relation is true $\rho_m = \rho_r$. Thus, $a_e q$ can be written as,

$$a_{eq} = \frac{\Omega_r}{\Omega_m}. \tag{3.190}$$

Let us define a new parameter y as follows,

$$\frac{\rho_m}{\rho_r} = \frac{a}{a_{eq}} \equiv y. \tag{3.191}$$

Now, the condition for super-horizon region is $k\eta \ll 1$. We take very small k modes such that for all time they are outside the horizon. At small ks, the v and $\Theta_{r,1}$ decouple from the evolution equations, leaving,

$$\dot{\Theta}_{r,0} = -\dot{\Phi} \tag{3.192}$$

$$\dot{\delta} = -3\dot{\Phi} \tag{3.193}$$

$$3\frac{\dot{a}}{a}\left(\dot{\Phi} + \frac{\dot{a}}{a}\Phi\right) = 4\pi G a^2[\rho_{\rm dm}\delta + 4\rho_r\Theta_{r,0}]. \tag{3.194}$$

We shall try to get a second order differential equation in terms of Φ and solve that. Eliminating $\dot{\Phi}$ first two of the above three equations, we have,

$$\dot{\delta} = 3\dot{\Theta}_{r,0} \tag{3.195}$$

$$\implies \delta - 3\Theta_{r,0} = \text{constant}. \tag{3.196}$$

We can set the constant to be zero which by demanding the following initial condition which corresponds to the adiabatic perturbation model, thus,

$$\delta = 3\Theta_{r,0}. \tag{3.197}$$

Now, in the Einstein equation we can factor out $\rho_{\rm dm}\delta$ and apply the definition of y,

$$3\frac{\dot{a}}{a}\left(\dot{\Phi} + \frac{\dot{a}}{a}\Phi\right) = 4\pi G a^2\rho_{\rm dm}\delta\left[1 + \frac{4}{3}\frac{\rho_r}{\rho_{\rm dm}}\right] = 4\pi G a^2\rho_{\rm dm}\delta\left[1 + \frac{4}{3y}\right]. \tag{3.198}$$

It is convenient to work with a differential equation in terms of y and for that purpose, we convert the comformal time derivatives into derivatives with respect to y,

$$\frac{d}{d\eta} = \frac{dy}{d\eta}\frac{d}{dy} = \left(\frac{d}{d\eta}\frac{a}{a_{eq}}\right)\frac{d}{dy} = aHy\frac{d}{dy}. \tag{3.199}$$

Above we have used the relation,

$$\dot{a} = a\frac{da}{dt} = a^2 H. \tag{3.200}$$

Now, $\rho_{\rm dm}$ can be expressed in terms of y as,

$$y = \frac{\rho_{\rm dm}}{\rho_r} \implies 1 + y = \frac{\rho}{\rho_r} \implies \frac{y\rho}{1+y} = \rho_{\rm dm}. \tag{3.201}$$

With the above expressions, the left hand side of Einstein equation becomes,

$$3\frac{\dot{a}}{a}\left(\dot{\Phi} + \frac{\dot{a}}{a}\Phi\right) = 3aH\left(aHy\Phi' + aH\Phi\right) = 3a^2H^2\left(y\Phi' + \Phi\right). \tag{3.202}$$

The prime denotes derivative with respect to y. Similarly, the right hand side takes the form,

$$4\pi G a^2 \rho_{\rm dm}\delta\left[1 + \frac{4}{3y}\right] = \frac{8\pi G}{3}\rho \cdot \frac{3}{2}\frac{y}{1+y}a^2\delta\frac{3y+4}{3y} = \frac{3}{2}a^2H^2\delta\frac{3y+4}{3(1+y)}, \tag{3.203}$$

here we have used the fact that from the zero-th order first Friedmann equation we have, $\frac{8\pi G}{3}\rho = H^2$. Putting the left and right hand sides together Einstein equation becomes,

$$y\Phi' + \Phi = \frac{3y+4}{6(1+y)}\delta. \tag{3.204}$$

To get a second order differential equation of Φ, we differentiate the above equation with respect to y,

$$\delta' = \frac{d}{dy}\left\{\frac{6(1+y)}{3y+4}\left[y\Phi' + \Phi\right]\right\}, \tag{3.205}$$

and then use the relation $\delta' = -3\Phi'$ to eliminate δ'. This leaves us with a second order differential equation of Φ,

$$\Phi'' + \frac{21y^2 + 54y + 32}{2y(1+y)(3y+4)}\Phi' + \frac{1}{y(1+y)(3y+4)}\Phi = 0. \qquad (3.206)$$

To simplify the above differential equation on introduces a new variable,

$$u \equiv \frac{y^3}{\sqrt{1+y}}\Phi. \qquad (3.207)$$

In terms of u the above differential takes the following simpler form,

$$u'' + u'\left[-\frac{2}{y} + \frac{3/2}{1+y} - \frac{3}{3y+4}\right] = 0. \qquad (3.208)$$

Integrating the above equation, we have,

$$\ln(u') = \ln A + 2\ln y - \frac{3}{2}\ln(1+y) + \ln(3y+4). \qquad (3.209)$$

We can exponentiate to get an expression for the u',

$$u' = A\frac{y^2(3y+4)}{(1+y)^{\frac{3}{2}}}. \qquad (3.210)$$

Integrating the above equation further, we get,

$$\frac{y^3}{\sqrt{1+y}}\Phi = A\int_0^y dy'\frac{y'^2(3y'+4)}{(1+y')^{\frac{3}{2}}}. \qquad (3.211)$$

We can determine the integration constant A considering the solution at early time. At early time Φ can be expanded to $\Phi(0) + y\Phi'|_{y=0} + \ldots$. Early on, in the limit $y \ll 1$, $y^3\Phi \to y^3\Phi(0)$. For small y, the integrand becomes $4y'^2$, and consequantly the integration is $\frac{4y^3}{3}$, thus for small y,

$$\Phi(0) = \frac{4A}{3} \implies A = \frac{3\Phi(0)}{4} \qquad (3.212)$$

Substituting $x = \sqrt{1+y}$, the integration can be completed, and the result reads,

$$\Phi = \frac{\Phi(0)}{10}\frac{1}{y^3}\left[16\sqrt{1+y} + 9y^3 + 2y^2 - 8y - 16\right]. \qquad (3.213)$$

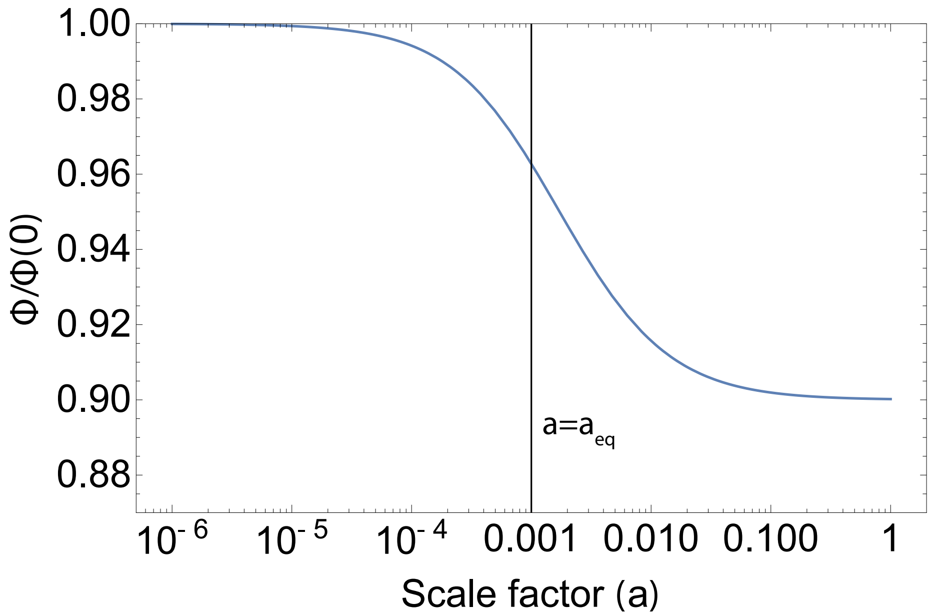

Figure 3.3: Evolution of large scale gravitational potential

At large y, once the universe has become matter-dominated, y^3 in the brackets lead, giving,

$$\Phi \rightarrow \frac{9}{10}\Phi(0). \tag{3.214}$$

Therefore, the late time value of the gravitational potential is sensitive to its initial value which we mentioned in the introductory remarks. The above function has been plotted in figure 3.3, and we see that the potential starts off as a constant and as it crosses the epoch of equality its value falls by almost 10% in the matter dominated era and asymptotically approaches a constant value.

Through horizon crossing

We are considering large scale modes of Φ which enter the horizon late in the matter dominated era. At late times, radiation is not important and the radiation perturbations do not govern the dynamics, thus, we can drop the equations concerning radiation perturbation. Also in this case the modes enter horizon, i.e., $k\eta \sim 1$, and hence, we cannot ignore the terms containing k as we did last time. The relevant Boltzmann equations for the cold dark matter are,

$$\dot{\delta} + ikv = 0, \tag{3.215}$$

$$\dot{v} + aHv = ik\Phi. \tag{3.216}$$

We shall be using the algebraic Einstein equation this time which can be manipulated to give,

$$\begin{aligned} k^2\Phi &= 4\pi G a^2 \left[\rho_{\rm dm}\delta + \frac{3aH}{k}\cdot i\rho_{\rm dm} v\right] \\ &= \frac{8\pi G}{3}\rho_{\rm dm}\frac{3}{2}a^2\left[\delta + \frac{3iaH}{k}v\right] \\ &= \frac{3}{2}a^2H^2\left[\delta + \frac{3iaH}{k}v\right]. \end{aligned} \tag{3.217}$$

From the knowledge that the super-horizon solution implies deep in the matter dominated era the gravitational potential reaches a constant value, one sets the initial condition for this problem as $\dot{\Phi} = 0$. Now, the task remains to see whether the above equations admit a solution Φ = constant, if so, then that is the solution picked up by the initial condition and merging of solutions of two different regimes. From the algebraic equation δ can be written as,

$$\delta = \frac{2k^2\Phi}{3a^2H^2} - \frac{3iaH}{k}v. \tag{3.218}$$

Using the fact that in matter dominated era, $H \propto a^{-\frac{3}{2}}$ and thus, $\frac{d(aH)}{d\eta} = -\frac{a^2H^2}{2}$ we evaluate the derivative of δ with respect to the conformal time,

$$\dot{\delta} = \frac{2k^2\dot{\Phi}}{3a^2H^2} + \frac{2k^2\Phi}{3aH} - \frac{3iaH}{k}\dot{v} + \frac{3ia^2H^2v}{2k}. \tag{3.219}$$

Putting this back into the equation (3.215), we have,

$$\frac{2k^2\dot{\Phi}}{3a^2H^2} + \frac{2k^2\Phi}{3aH} - \frac{3iaH}{k}\dot{v} + \frac{3ia^2H^2v}{2k} + ikv = 0. \tag{3.220}$$

Replacing $\dot{v}$ with v and Φ from equation (3.216),

$$\frac{2k^2\dot{\Phi}}{3a^2H^2} + \left(\frac{iv}{k} + \frac{2\Phi}{3aH}\right)\left[\frac{9a^2H^2}{2} + k^2\right] = 0. \tag{3.221}$$

If the second order differential equation is of the form $\alpha\ddot{\Phi}+\beta\dot{\Phi} = 0$, then Φ = constant is a solution. We differentiate the above equation with respect to conformal time and only examine the terms containing Φ,

$$\left(\frac{i\dot{v}}{k}+\frac{\Phi}{3}\right)\left(\frac{9a^2H^2}{2}+k^2\right)+\left(\frac{iv}{k}+\frac{2\Phi}{3aH}\right)\left(-\frac{9a^3H^3}{2}\right), \tag{3.222}$$

above we have used the fact that in matter dominated era $\frac{d(aH)^{-1}}{d\eta}=\frac{1}{2}$. Again, replacing $\dot{v}$ with v and Φ, we have,

$$-\left[\frac{iaHv}{k}+\frac{2\Phi}{3}\right](9a^2H^2+k^2) \tag{3.223}$$

The term in the square bracket is proportional to $\dot{\Phi}$. Therefore, there is no term proportional to Φ and constant potential is indeed a solution. Therefore the large scale modes of the gravitational potential which enter the horizon at late time remains constant.

3.3.3 Small scales

After discussing the large scale modes we shall now turn our attention to the small scale modes which either crosses the horizon at early times during the radiation dominated era or always remain in the sub-horizon scales.

Through horizon crossing

Small scale modes crosses the horizon in radiation dominated era itself. We assume that the matter perturbation starts off small in the radiation dominated era and do not take part in governing the dynamics of the universe. Thus, δ can be neglected in this limit. The monopole and dipole governs the dynamics and sets the evolution of the gravitational potential and once the potential is set it acts as a background in which the evolution of matter perturbations will take place. To begin with, we drop the cold dark matter equations and we are left with,

$$\dot{\Theta}_{r,0}+k\Theta_{r,1}=-\dot{\Phi}, \tag{3.224}$$

$$\dot{\Theta}_{r,1}-\frac{k}{3}\Theta_{r,0}=-\frac{k}{3}\Phi, \tag{3.225}$$

$$k^2\Phi=4\pi Ga^2\left[4\rho_r\Theta_{r,0}+\frac{3aH}{k}(4\rho_r\Theta_{r,1})\right]$$

$$
\begin{aligned}
&= \frac{8\pi G}{3}\rho_r \frac{3}{2} a^2 \cdot 4 \left[\Theta_{r,0} + \frac{3aH}{k}\Theta_{r,1}\right] \\
&= 6a^2H^2 \left[\Theta_{r,0} + \frac{3aH}{k}\Theta_{r,1}\right].
\end{aligned} \tag{3.226}
$$

In radiation era, we have the following expressions,

$$
H \propto t^{-1}, \quad \eta \propto a, \quad a \propto t^{\frac{1}{2}}, \quad aH \propto a^{-1}, \quad \text{and} \quad aH = \frac{1}{\eta}. \tag{3.227}
$$

Using the above facts, the Einstein equation becomes,

$$
\frac{k^2\eta^2\Phi}{6} - \frac{3}{k\eta}\Theta_{r,1} = \Theta_{r,0}. \tag{3.228}
$$

Taking derivative of the above equation with respect to conformal time, we, have,

$$
\dot{\Theta}_{r,0} = \frac{k^2\eta\Phi}{3} + \frac{k^2\eta^2}{6}\dot{\Phi} + \frac{3}{k\eta^2}\Theta_{r,1} - \frac{3}{k\eta}\dot{\Theta}_{r,1}. \tag{3.229}
$$

Using equation (3.224) in the avobe equation gives,

$$
\begin{aligned}
&\frac{k^2\eta\Phi}{3} + \frac{k^2\eta^2}{6}\dot{\Phi} + \frac{3}{k\eta^2}\Theta_{r,1} - \frac{3}{k\eta}\dot{\Theta}_{r,1} + k\Theta_{r,1} = -\dot{\Phi} \\
&\implies -\frac{3}{k\eta}\dot{\Theta}_{r,1} + k\Theta_{r,1}\left[1 + \frac{3}{k^2\eta^2}\right] = -\dot{\Phi}\left[1 + \frac{k^2\eta^2}{6}\right] - \Phi\frac{k^2\eta}{3}.
\end{aligned} \tag{3.230}
$$

On the other hand, from equation (3.225),

$$
\dot{\Theta}_{r,1} - \frac{k}{3}\left(\frac{k^2\eta^2}{6}\Phi - \frac{3}{k\eta}\Theta_{r,1}\right) = -\frac{k}{3}\Phi \tag{3.231}
$$

$$
\implies \dot{\Theta}_{r,1} + \frac{1}{\eta}\Theta_{r,1} = -\frac{k}{3}\Phi\left[1 - \frac{k^2\eta^2}{6}\right]. \tag{3.232}
$$

Eliminating $\dot{\Theta}_{r,1}$ from the above two equations, we have,

$$
\dot{\Phi} + \frac{1}{\eta}\Phi = -\frac{6}{k\eta^2}\Theta_{r,1} \tag{3.233}
$$

To get the second order differential equation in terms of Φ, we differentiate the above equation with respect to conformal time and substitute for $\dot{\Theta}_{r,1}$ and $\Theta_{r,1}$ from their above relations to get everything in terms of Φ and its derivatives,

$$\begin{aligned}\ddot{\Phi} - \frac{1}{\eta^2}\Phi + \frac{1}{\eta}\dot{\Phi} &= \frac{12}{k\eta^3}\Theta_{r,1} - \frac{6}{k\eta^2}\dot{\Theta}_{r,1} \\ &= \frac{12}{k\eta^3}\Theta_{r,1} - \frac{6}{k\eta^2}\left(-\frac{k}{3}\Phi\left[1 - \frac{k^2\eta^2}{6}\right] - \frac{1}{\eta}\Theta_{r,1}\right) \\ &= \frac{18}{k\eta^3}\Theta_{r,1} + \frac{2}{\eta^2}\Phi\left[1 - \frac{k^2\eta^2}{6}\right] \\ &= -\frac{3}{\eta}\dot{\Phi} - \frac{3}{\eta^2}\Phi + \frac{2\Phi}{\eta^2} - \frac{k^2}{3}\Phi. \end{aligned} \tag{3.234}$$

This leaves us with the following differential equation,

$$\ddot{\Phi} + \frac{4}{\eta}\dot{\Phi} + \frac{k^2}{3}\Phi = 0. \tag{3.235}$$

To solve the above equation, we define, $u \equiv \Phi\eta$, and the above equation in terms of u becomes,

$$\ddot{u} + \frac{2}{\eta}\dot{u} + \left[\frac{k^2}{3} - \frac{2}{\eta^2}\right]u = 0. \tag{3.236}$$

Substituting $x = \frac{k\eta}{\sqrt{3}}$, we have,

$$\frac{d^2u}{dx^2} + \frac{2}{x}\frac{du}{dx} + \left[1 - \frac{2}{x^2}\right]u = 0. \tag{3.237}$$

Solutions of the above equations are spherical Bessel function and spherical Neumann function of order 1. As Neumann function blows up for small η, that is rejected, leaving the solution to be,

$$\begin{aligned} u(x) &= A\frac{\sin x - x\cos x}{x^2} \\ \implies u\left(\frac{k\eta}{\sqrt{3}}\right) &= A\frac{\sin\left(\frac{k\eta}{\sqrt{3}}\right) - \left(\frac{k\eta}{\sqrt{3}}\right)\cos\left(\frac{k\eta}{\sqrt{3}}\right)}{\left(\frac{k\eta}{\sqrt{3}}\right)^2} \\ \implies \Phi &= A\frac{\sin\left(\frac{k\eta}{\sqrt{3}}\right) - \left(\frac{k\eta}{\sqrt{3}}\right)\cos\left(\frac{k\eta}{\sqrt{3}}\right)}{\left(\frac{k\eta}{\sqrt{3}}\right)^3}. \end{aligned} \tag{3.238}$$

To determine the constant we take early time limit. When $\eta \to 0$, the function $\frac{\sin\left(\frac{k\eta}{\sqrt{3}}\right) - \left(\frac{k\eta}{\sqrt{3}}\right)\cos\left(\frac{k\eta}{\sqrt{3}}\right)}{\left(\frac{k\eta}{\sqrt{3}}\right)^3} \to \frac{1}{3}$ and the gravitational potential reaches its initial value $\Phi \to$

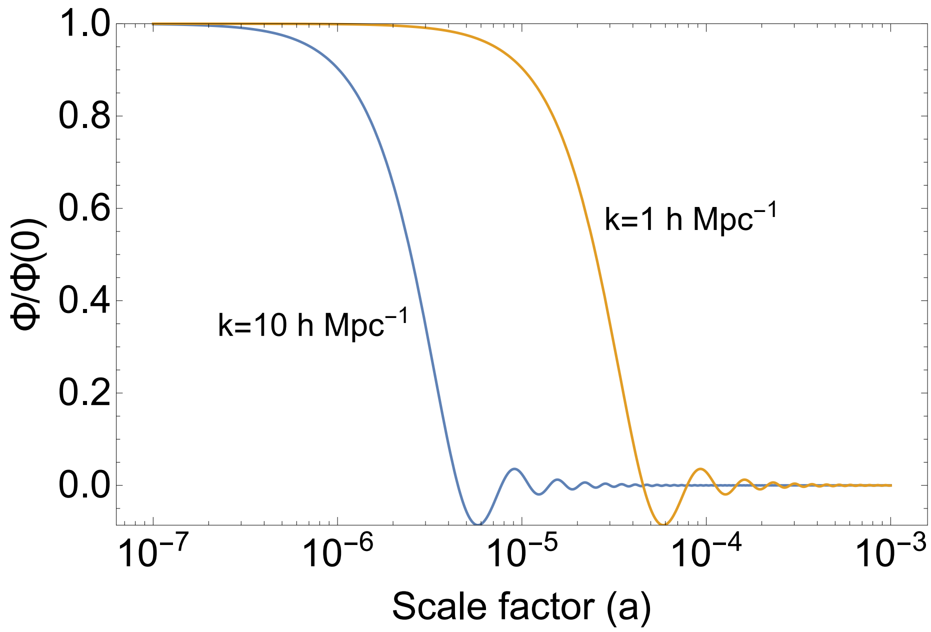

Figure 3.4: Evolution of small scale gravitational potential

Φ_p. Thus, we can set $A = 3\Phi_p$, and we have,

$$\Phi = 3\Phi_p \frac{\sin\left(\frac{k\eta}{\sqrt{3}}\right) - \left(\frac{k\eta}{\sqrt{3}}\right)\cos\left(\frac{k\eta}{\sqrt{3}}\right)}{\left(\frac{k\eta}{\sqrt{3}}\right)^3}. \tag{3.239}$$

The above profile for the small scale mode of the gravitational potential has been plotted in figure 3.4. The potential starts from a constant value and as it crosses the horizon it falls rapidly, oscillates around its zero value and settles at zero at late time. We shall ,now, see how the matter perturbations evolve, affected by the potential. The Boltzmann equations corresponding to matter perturbations are,

$$\dot{\delta} + ikv = -3\dot{\Phi}, \tag{3.240}$$

$$\dot{v} + \frac{\dot{a}}{a}v = ik\Phi. \tag{3.241}$$

Differentiating the first equation and substituting $\dot{v}$ from the second equation gives,

$$\begin{aligned} \ddot{\delta} + ik\left(-\frac{\dot{a}}{a}v + ik\Phi\right) &= -3\ddot{\Phi} \\ \implies \ddot{\delta} + \frac{1}{\eta}\dot{\delta} &= \underbrace{-3\ddot{\Phi} - \frac{3}{\eta}\dot{\Phi} + k^2\Phi}_{S(k,\eta)}, \end{aligned} \tag{3.242}$$

in the last step we have again substituted v from the first equation. This is a linear differential equation for δ with a source term $S(k,\eta)$. Let us look at the solution of the homogeneous equation first. For the homogeneous equation $\ddot{\delta} + \frac{1}{\eta}\dot{\delta} = 0$, two solutions are, $\delta =$ constant and $\delta = \ln[\eta] = \ln a$. We can construct a Green's function from the above two solutions, and it takes the form,

$$
\begin{aligned}
G(\eta,\eta') &= \frac{s_1(\eta)s_2(\eta') - s_1(\eta')s_2(\eta)}{\dot{s}_1(\eta')s_2(\eta') - s_1(\eta')\dot{s}_2(\eta')} \\
&= \frac{C\ln[\eta'] - C\ln[\eta]}{0 - C\frac{1}{\eta'}} \\
&= -\eta'\left(\ln[k\eta'] - \ln[k\eta]\right).
\end{aligned}
\tag{3.243}
$$

The full solution of the inhomogeneous differential equation is the linear combination of the two homogeneous equation and the particular solution obtained from Green's function,

$$
\delta(k,\eta) = C_1 + C_2\ln[\eta] - \int_0^{\eta} d\eta' S(k,\eta')\eta'\left(\ln[k\eta'] - \ln[k\eta]\right). \tag{3.244}
$$

We, now, have to determine the constants C_1 and C_2. At very early time, when $\eta \to 0$, the contribution from the integral is small. The initial condition that at early time $\delta(k,\eta=0) = \frac{3\Phi_p}{2}$ is a constant sets $C_2 = 0$ and $C_1 = \frac{3\Phi_p}{2}$. Now as we know from the profile of the potential that at late times after it has crossed the horizon it settles down to zero. Thus the source integral has dominant contribution around $k\eta \sim 1$. The part of the integral with $S(k,\eta')\ln[k\eta']$ then asymptotically settles to a constant value after the horizon crossing whereas the part with $S(k,\eta')\ln[k\eta]$ becomes proportional to $\ln[k\eta]$. Thus, we expect the following solution for the density contrast after it crosses the horizon, constituting of a constant part and a logarithmic growing part,

$$
\delta(k,\eta) = A\Phi_p \ln(Bk\eta). \tag{3.245}
$$

Now, from the above argument, the constant part should be,

$$
A\Phi_p\ln[B] = \frac{3\Phi_p}{2} - \int_0^{\infty} d\eta' S(k,\eta')\eta'\ln[k\eta'], \tag{3.246}
$$

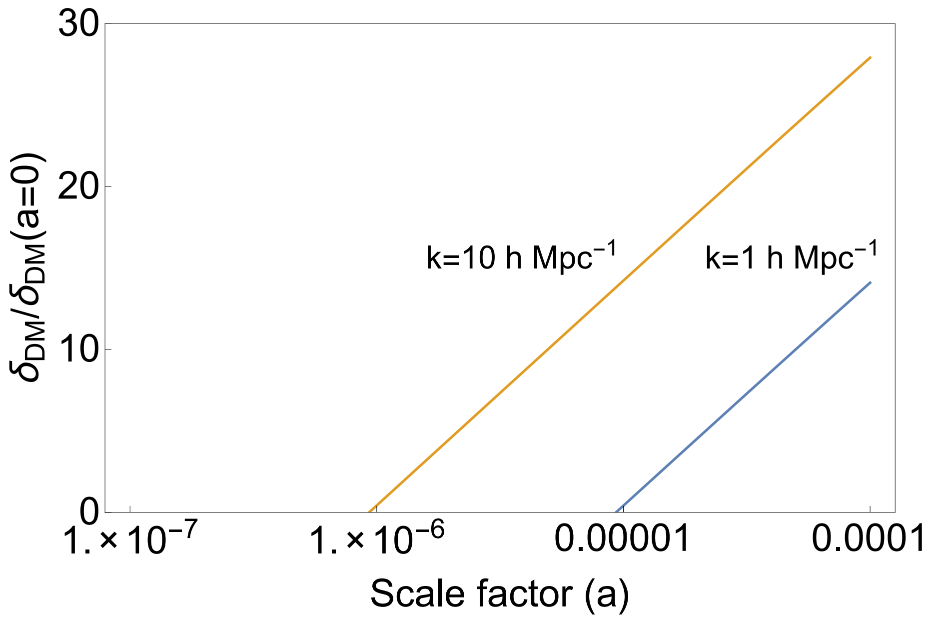

Figure 3.5: Evolution of matter perturbations in radiation dominated era

and the coefficient of the logarithmic term should be,

$$A\Phi_p = \int_0^\infty d\eta' S(k,\eta')\eta'. \tag{3.247}$$

The late time has been exaggerated to infinity to pick up the asymptotic contributions. Performing the above two integrals in Mathematica with the potential profile given in equation (3.239), we get the following values for the constants $A = 9$ and $B = 0.62$. The evolution of the matter perturbation has been plotted in the figure 3.5.

Sub-horizon solution

For small scale perturbations the cold dark matter density contrast δ grows logarithmically in the radiation dominated era and for $\Theta_{r,0}$, the growth is suppressed due to the existence of radiation pressure. Eventually, $\rho_{\rm dm}\delta$ becomes much larger than $\rho_r\Theta_{r,0}$ even though it may still be that $\rho_{\rm dm} < \rho_r$. In this limit, radiation perturbations can be neglected and we are left with the following set of equations,

$$\delta' + \frac{ikv}{aHy} = -3\Phi', \tag{3.248}$$

$$v' + \frac{v}{y} = \frac{ik\Phi}{aHy}, \tag{3.249}$$

$$k^2\Phi = \frac{3}{2}\frac{y}{1+y}a^2H^2\left[\delta + \frac{3iaHv}{k}\right] = \frac{3y}{2(1+y)}a^2H^2\delta. \tag{3.250}$$

In the above equations we have used the facts that $\frac{\dot{a}}{a} = \frac{1}{a}aHy\frac{da}{d\left(\frac{a}{a_{eq}}\right)} = aH$, and the zero-th order first Friedmann equation is $4\pi G\rho_{\rm dm} = H^2\frac{y}{1+y}\cdot\frac{3}{2}$, and for sub-horizon modes we have $\frac{aH}{k} \ll 1$. Now, the first Friedmann equation can again be written as,

$$H^2 = \frac{8\pi G}{3}(\rho_{\rm dm} + \rho_r) = \frac{8\pi G}{3}\rho_r(1+y). \tag{3.251}$$

Also the second Friedmann equation is written as,

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho_{\rm dm} + \rho_r + 3P_r) = -\frac{4\pi G}{3}\rho_r(2+y). \tag{3.252}$$

From the two Friedmann equations we get,

$$\frac{d^2a}{dt^2} = -\frac{1}{2}\frac{aH^2(2+y)}{1+y}. \tag{3.253}$$

For a later purpose we need to compute the expression for $\frac{d}{dy}\left(\frac{1}{aHy}\right)$, which is,

$$\frac{d}{dy}\left(\frac{1}{aHy}\right) = -\frac{1}{(aHy)^2}\frac{d}{dy}(aHy), \tag{3.254}$$

which, prompts us to compute the expression for $\frac{d}{dy}(aHy)$, and that is,

$$\begin{aligned}\frac{d}{dy}(aHy) = y\frac{d}{dy}aH + aH = y\frac{dt}{dy}\frac{d^2a}{dt^2} + aH &= y\frac{1}{\frac{1}{a_{eq}}\frac{da}{dt}}\left(-\frac{1}{2}\frac{aH^2(2+y)}{1+y}\right) + aH \\ &= aH\left(1 - \frac{2+y}{2(1+y)}\right) = \frac{aHy}{2(1+y)}\end{aligned} \tag{3.255}$$

And with the help of these results finally we have the expression for $\frac{d}{dy}\left(\frac{1}{aHy}\right)$, in the following form,

$$\frac{d}{dy}\left(\frac{1}{aHy}\right) = -\frac{1}{2aHy(1+y)}. \tag{3.256}$$

Now to get the second order differential equation in terms of the density contrast we differentiate equation (3.248) with respect to y and use the previous result in (3.256) to get,

$$\delta'' + \frac{ikv'}{aHy} - \frac{ikv}{2aHy(1+y)} = -3\Phi''. \tag{3.257}$$

We can eliminate v' using the equation (3.249),

$$\begin{aligned}
\delta'' + \frac{ik}{aHy}\left(-\frac{v}{y} + \frac{ik\Phi}{aHy}\right) - \frac{ikv}{2aHy(1+y)} &= -3\Phi'' \\
\implies \delta'' - \frac{ikv}{aHy}\left(\frac{1}{y} + \frac{1}{2(1+y)}\right) &= -3\Phi'' + \frac{k^2\Phi}{a^2H^2y^2} \\
&= \frac{k^2\Phi}{a^2H^2y^2}, \qquad \frac{k}{aH} \gg 1
\end{aligned} \tag{3.258}$$

As the value of the gravitational potential of all k modes falls after crossing the horizon, in the sub-horizon limit, potential can be neglected in favour of the density contrast in the equation (3.248) and that leaves for us, $\delta' = \frac{ikv}{aHy}$. Also from the Einstein equation we have, $\frac{k^2\Phi}{a^2H^2y^2} = \frac{3\delta}{2y(1+y)}$. Using these two expressions the second order differential equation for the density contrast becomes,

$$\delta'' + \frac{2+3y}{2y(1+y)}\delta' - \frac{3}{2y(1+y)}\delta = 0. \tag{3.259}$$

The above equation is called Meszaros equation. We take the ansatz that one of the solutions is a polynomial of y of order 1. That implies $\delta'' = 0$; and then for the first solution D_1, we have,

$$\begin{aligned}
\frac{2+3y}{2y(1+y)}D_1' - \frac{3}{2y(1+y)}D_1 &= 0 \\
\implies \frac{D_1'}{D_1} &= \frac{1}{\frac{2}{3}+y} \\
\ln D_1 = \ln\left(\frac{2}{3}+y\right) \\
\implies D_1 = y + \frac{2}{3}
\end{aligned} \tag{3.260}$$

Now, we find the second solution by defining, $u \equiv \frac{\delta}{D_1}$. In terms of u then the Meszaros equation takes the form,

$$(1+\frac{3y}{2})u'' + \frac{u'}{y(1+y)}\left[\frac{21}{4}y^2 + 6y + 1\right] = 0. \tag{3.261}$$

We integrate the above equation as follows,

$$\frac{du'}{u'} = -\frac{\frac{21}{4}y^2 + 6y + 1}{y(1+y)\left(1+\frac{3y}{2}\right)}dy$$

$$\Longrightarrow \frac{du'}{u'} = -\left(\frac{1}{y} + \frac{1}{2(1+y)} + \frac{3}{\left(1+\frac{3y}{2}\right)}\right) dy$$
$$\Longrightarrow \ln u' = -\ln y - \frac{1}{2}(1+y) - 2\ln(2+3y)$$
$$\Longrightarrow u' = \frac{1}{y\sqrt{1+y}(2+3y)^2}. \tag{3.262}$$

To get the solution for u, we need to compute the following integral,

$$\int \frac{1}{y\sqrt{1+y}(2+3y)^2} dy = -\frac{1}{4}\left[\ln\left|\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right| - \frac{2\sqrt{1+y}}{\frac{2}{3}+y}\right]. \tag{3.263}$$

Using the above result and the definition of u, we get the second solution as,

$$D_2 = uD_1 = -\frac{1}{4}\left[D_1 \ln\left|\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right| - 2\sqrt{1+y}\right]. \tag{3.264}$$

Now, the solution of the Meszaros equation will be a linear combination of the above two solutions. However, to determine the constant coefficients of the linear combination one requires to do the numerical fitting and we have left that discussion out.

3.4 Solving Einstein-Boltzmann equations for anisotropies

We have already seen the solutions of Einstein-Boltzmann equations for inhomogeneities in the previous section. Now shall turn our focus to the solutions for the anisotropies.

3.4.1 Large scale anisotropies

For super-horizon scales when $k\eta \ll 1$, we have the following solution for Θ_0, as we already calculated earlier,

$$\Theta_0(k,\eta) = -\Phi(k,\eta) + \frac{3\Phi_p}{2} \tag{3.265}$$

The CMB photons that reaches us today are free streaming to us from the time of recombination. As recombination takes place long time after the epoch of equality, now, for $y \gg 1$, we have $\Phi(k,\eta_*) \to \frac{9\Phi_p}{10}$. Thus at recombination the value of Θ_0 is given as,

$$\begin{aligned}\Theta_0(k,\eta_*) &= -\Phi(k,\eta_*) + \frac{3\Phi_p(k)}{2} \\ &= -\Phi(k,\eta_*) + \frac{5}{3}\Phi(k,\eta_*) \\ &= \frac{2}{3}\Phi(k,\eta_*). \end{aligned} \tag{3.266}$$

One defines the observed anisotropy as follows,

$$(\Theta_0 + \Psi)(k,\eta_*) \simeq \Theta_0 - \Phi. \tag{3.267}$$

Thus we have,

$$(\Theta_0 + \Psi)(k,\eta_*) = -\frac{1}{3}\Phi(k,\eta_*) = \frac{1}{3}\Psi(k,\eta_*). \tag{3.268}$$

From the analysis in the section regarding initial conditions we had $\dot{\delta} = -3\dot{\Phi}$ and the initial value of δ was set to be $\frac{3\Phi_p}{2}$. Now, integrating the large scale Boltzmann equation for δ, we have,

$$\begin{aligned}\delta(\eta_*) - \delta(\eta_i) &= -3(\Phi(\eta_*) - \Phi_p) \\ \implies \delta(\eta_*) &= \frac{3}{2}\Phi_p - 3(\Phi(\eta_*) - \Phi_p) \\ &= \frac{9}{2}\cdot\frac{10}{9}\Phi(\eta_*) - 3\Phi(\eta_*) \\ &= 2\Phi(\eta_*). \end{aligned} \tag{3.269}$$

Using this result we can express the observed large scale anisotropy as,

$$(\Theta_0 + \Psi)(k,\eta_*) = -\frac{1}{6}\delta(\eta_*). \tag{3.270}$$

The above equation relates the observed anisotropy at the recombination to the density perturbation of the dark matter at the recombination.

3.4.2 Tightly coupled limit of the Boltzmann equation

Before recombination all electrons were ionized and they were strongly coupled to the photons via Compton scattering and the scattering rate was much faster than the expansion rate of the universe. This limit corresponds to a very high optical depth,

$$\tau(\eta) \equiv \int_{\eta}^{\eta_0} d\eta' n_e \sigma_T a \gg 1. \tag{3.271}$$

We have the Boltzmann equation for photons, which reads,

$$\dot{\Theta} + ik\mu\Theta = -\dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 - \Theta + \mu v_b + \frac{1}{2}\mathcal{P}_2\Theta_2\right]. \tag{3.272}$$

For, $l > 2$, we multiply the Boltzmann equation with $\mathcal{P}_l(\mu)$ and integrate with respect to $\frac{1}{(-i)^l}\int_{-1}^{1}\frac{d\mu}{2}$. Using the orthogonality of the Legendre polynomials and the definition for Θ_l, we get,

$$\dot{\Theta}_l + \frac{k}{(-i)^{l+1}}\int_{-1}^{1}\frac{d\mu}{2}\mu\mathcal{P}_l(\mu)\Theta(\mu) = \dot{\tau}\Theta_l. \tag{3.273}$$

Now, we use the following mathematical identity for the Legendre polynomials,

$$\begin{aligned} (l+1)\mathcal{P}_{l+1}(\mu) &= (2l+1)\mu\mathcal{P}_l(\mu) - l\mathcal{P}_{l-1}(\mu) \\ \mu\mathcal{P}_l(\mu) &= \frac{(l+1)\mathcal{P}_{l+1}(\mu) + l\mathcal{P}_{l-1}(\mu)}{2l+1}. \end{aligned} \tag{3.274}$$

Using the above identity the integral in equation (3.273) becomes,

$$\begin{aligned} \int_{-1}^{1}\frac{d\mu}{2}\mu\mathcal{P}_l(\mu)\Theta(\mu) &= \int_{-1}^{1}\frac{d\mu}{2}\frac{(l+1)\mathcal{P}_{l+1}(\mu) + l\mathcal{P}_{l-1}(\mu)}{2l+1}\Theta(\mu) \\ &= \frac{(-i)^{l+1}(l+1)}{2l+1}\Theta_{l+1} + \frac{(-1)^{l-1}l}{2l+1}\Theta_{l-1}. \end{aligned} \tag{3.275}$$

Therefore, the Boltzmann equation finally takes the following form,

$$\dot{\Theta}_l + \frac{k(l+1)}{2l+1}\Theta_{l+1} - \frac{kl}{2l+1}\Theta_{l-1} = \dot{\tau}\Theta_l. \tag{3.276}$$

Let us take the term $\dot{\Theta}_l$ to the right hand side of the equation and then the right hand side can be written as,

$$-\dot{\Theta}_l + \dot{\tau}\Theta_l = \frac{d}{d\eta}\left(\tau\Theta_l - \Theta_l\right) - \tau\dot{\Theta}_l \tag{3.277}$$

As $\tau \gg 1$, the terms inside the parenthesis reduce to $\tau\Theta_l$. Hence, the right hand side of the Boltzmann equation beomes $\dot{\tau}\Theta_l$,

$$\frac{k(l+1)}{2l+1}\Theta_{l+1} - \frac{kl}{2l+1}\Theta_{l-1} = \dot{\tau}\Theta_l. \tag{3.278}$$

Neglecting the term corresponding to $l+1$ for the moment, we have,

$$\Theta_l = \frac{kl}{(2l+1)n_e\sigma_T a}\Theta_{l-1}. \tag{3.279}$$

As $n_e\sigma_T$ is proportional to the reaction rate for the Compton scattering its value is very large for the tightly coupled limit. Therefore, the value of Θ_{l-1} is much larger than that of Θ_l. Therefore, our scheme of throwing away the Θ_{l+1} is self consistent. In conclusion for $l > 2$ all the higher moments are subsequently suppressed compared to the previous moments.

For our discussion in this section we shall only consider the monopole and dipole of the photon temperature anisotropy and assume that all higher moments are negligible compared to them. With these approximations the relevant Boltzmann equations for the photons and the electrons (or baryons) are,

$$\dot{\Theta}_0 + ik\Theta_1 = -\dot{\Phi}, \tag{3.280}$$

$$\dot{\Theta}_1 - \frac{k}{3}\Theta_0 = \frac{k}{3}\Psi + \dot{\tau}\left[\Theta_1 + \frac{v_b}{3i}\right], \tag{3.281}$$

$$\dot{\delta}_b + ikv_b = -3\dot{\Phi}, \tag{3.282}$$

$$\dot{v}_b + \frac{\dot{a}}{a}v_b = -ik\Psi + \dot{\tau}\frac{4\rho_\gamma^{(0)}}{3\rho_b^{(0)}}\left[v_b + 3i\Theta_1\right]. \tag{3.283}$$

Defining $\frac{1}{R} \equiv \frac{4\rho_\gamma^{(0)}}{3\rho_b^{(0)}}$, we re-write the last equation in the following form,

$$v_b = -3i\Theta_1 + \frac{R}{\dot{\tau}}\left[\dot{v}_b + \frac{\dot{a}}{a}v_b + ik\Psi\right]. \tag{3.284}$$

As $\dot{\tau}$ is proportional to the reaction rate and is a much larger quantity than terms in the square brackets because individually all the perturbation variables are considered to have magnitude much less than 1, we can approximate $-3i\Theta_1$ as the zero-th order solution of v_b. To get the first order correction, we put this value back into the above equation and get,

$$v_b \simeq -3i\Theta_1 + \frac{R}{\dot{\tau}}\left[-3i\dot{\Theta}_1 - 3i\frac{\dot{a}}{a}\Theta_1 + ik\Psi\right]. \tag{3.285}$$

Putting this expression of v_b into equation (3.281), we get,

$$\begin{aligned}\dot{\Theta}_1 - \frac{k}{3}\Theta_0 &= \frac{k}{3}\Psi + \dot{\tau}\left[\Theta_1 - \frac{i}{3}\left(-3i\Theta_1 + \frac{R}{\dot{\tau}}\left(-3i\dot{\Theta}_1 - 3i\frac{\dot{a}}{a}\Theta_1 + ik\Psi\right)\right)\right] \\ &= \frac{k}{3}\Psi + \dot{\tau}\left[\frac{R}{\dot{\tau}}\left(-\dot{\Theta}_1 - \frac{\dot{a}}{a}\Theta_1 + \frac{k}{3}\Psi\right)\right] \\ &= -R\dot{\Theta}_1 - \frac{\dot{a}}{a}R\Theta_1 + \frac{k}{3}(1+R)\Psi. \end{aligned} \tag{3.286}$$

Rearrenging the terms we can write,

$$\dot{\Theta}_1 + \frac{\dot{a}}{a}\frac{R}{1+R}\Theta_1 - \frac{k}{3(1+R)}\Theta_0 = \frac{k}{3}\Psi. \tag{3.287}$$

Differentiating equation (3.280) with respect to conformal time $\dot{\Theta}_1$ and Θ_1 in favour of Θ_0, we get,

$$\begin{aligned} &\ddot{\Theta}_0 + k\dot{\Theta}_1 = -\ddot{\Phi} \\ &\implies \ddot{\Theta}_0 + k\left[\frac{k}{3}\Psi - \frac{\dot{a}}{a}\frac{R}{1+R}\Theta_1 + \frac{k}{3(1+R)}\Theta_0\right] = -\ddot{\Phi} \\ &\implies \ddot{\Theta}_0 + \frac{k^2}{3}\Psi - \frac{\dot{a}}{a}\frac{kR}{1+R}\left(\frac{1}{k}(-\dot{\Phi} - \dot{\Theta}_0)\right) + \frac{k^2}{3(1+R)}\Theta_0 = -\ddot{\Phi} \\ &\implies \ddot{\Theta}_0 + \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Theta}_0 + k^2\frac{1}{3(1+R)}\Theta_0 = -\frac{k^2}{3}\Psi - \frac{\dot{a}}{a}\frac{R}{1+R}\dot{\Phi} - \ddot{\Phi} \end{aligned} \tag{3.288}$$

In the above equation the derivatives of Θ_0 and Φ appears almost similarly and using the fact that $R \propto a$ we can write the above equation as,

$$\left\{\frac{d^2}{d\eta^2} + \frac{\dot{R}}{1+R}\frac{d}{d\eta} + k^2 c_s^2\right\}[\Theta_0 + \Phi] = \frac{k^2}{3}\left[\frac{1}{1+R}\Phi - \Psi\right], \tag{3.289}$$

here, $c_s \equiv \sqrt{\frac{1}{3(1+R)}}$. The above equation is of the form of a damped and forced harmonic oscillator. When R is small the speed of sound c_s is much larger and the friction term can be ignored in its favour and we also notice that the homogeneous equation has two solutions in this limit, $S_1(k,\eta) = \sin[kr_s(\eta)]$ and $S_2(k,\eta) = \cos[kr_s(\eta)]$, with the definition $r_s(\eta) \equiv \int_0^\eta d\eta' c_s(\eta')$. From these two solutions we can build the Green's function as before and that turns out to be $\frac{\sqrt{3}\sin[k(r_s - r_s')]}{k}$. Therefore, the general solution to the inhomogeneous equation is,

$$(\Theta_0 + \Phi)(\eta) = C_1 S_1(\eta) + C_2 S_2(\eta) + + \frac{k}{\sqrt{3}} \int_0^\eta d\eta' [\Phi(\eta') - \Psi(\eta')] \sin\left[k(r_s(\eta) - r_s(\eta'))\right], \tag{3.290}$$

here, we have ignored R so long as it does not appear in the argument of an oscillating function. In the limit $\eta \to 0$, both Θ_0 and Φ becomes constant and also the contribution from the integral is much less, then C_1 must vanish (as sine function goes to zero in this limit) and $C_2 = \Theta_0(0) + \Phi(0)$ (as cosine goes to 1). Therefore, the solution for the observed monopole can be written as,

$$\Theta_0(\eta) + \Phi(\eta) = [\Theta_0(0) + \Phi(0)] \cos(kr_s) + \frac{k}{\sqrt{3}} \int_0^\eta d\eta' [\Phi(\eta') - \Psi(\eta')] \sin\left[k(r_s(\eta) - r_s(\eta'))\right]. \tag{3.291}$$

From equation (3.281) the corresponding equation for the dipole,

$$\Theta_1(\eta) = [\Theta_0(0) + \Phi(0)] \sin(kr_s) - \frac{k}{\sqrt{3}} \int_0^\eta d\eta' [\Phi(\eta') - \Psi(\eta')] \cos\left[k(r_s(\eta) - r_s(\eta'))\right]. \tag{3.292}$$

Considering the contribution from the integrals to be small we see that the monopole and dipole oscillate out of phase with respect to each other. These two are the primary solutions of photon anisotropy from the analysis of Boltzmann-Einstein equations. However, the presence of a small but non-negligible quadruple moment dampens the oscillations for higher k modes, a phenomenon known as diffusion damping.

Chapter 4

Modified Gravity

As in the early universe the value of the scale factor was considerably small and the energy scale of the universe was much higher we expect quantum gravity effects to be dominant in the past. With this proposition in mind we review a particular effective low energy model of a theory of discrete spacetime and review some of its consequences in this chapter.

4.1 The modified gravity theory

We shall now briefly describe the modified gravity theory suggested in the works [2], and [3]. In these works, they have rejected the view that, for a quantum theory of gravity, in an already discretized spacetime the quantum matter should display its predetermined properties, rather they suggest the discreteness of the spacetime is relational to the matter degrees of freedom which can probe the discreteness. For instance, the geodesics in the general theory of relativity can only be realized in relation to the motion of the test particles. Similarly, the discreteness of the spacetime can only be realized through the interaction between the matter degrees of freedom and the spacetime. Thus matter degrees of freedom exchange energy and momentum with the underlying discrete structure of the spacetime. If, to an effective low energy limit, we consider the spacetime to be a smooth manifold, then we would not be taking into account the whole of the energy and momentum that is being exchanged between the matter degrees of freedom and the discrete spacetime. Therefore in such an effective low energy theory there should be violation of the principle of energy-momentum conservation. However, Einstein's theory of gravity cannot be the model for such an effective low energy theory because in this theory the energy-momentum conservation is enforced by the geometric Bianchi identity. One should look beyond this theory and choose a modified set of equations. The particular choice taken in the above mentioned works is the equations of Unimodular Gravity (UG), in which the dynamical equations do not enforce a conservation of energy-momentum. Another motivation for choosing this set of equations is that in UG the vacuum energy does not gravitate and might be plausible solution for the cosmological constant problem [8].

As given in [9], the equations for UG is achieved with the constraint, $\delta\sqrt{-g} = 0 \implies g^{\mu\nu}\delta g_{\mu\nu} = 0$. If the deformation of the manifold is given by a vector field ξ^μ, then, we have,

$$\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \xi^\alpha \nabla_\alpha g_{\mu\nu} + (\nabla_\mu \xi^\lambda) g_{\lambda\nu} + (\nabla_\nu \xi^\lambda) g_{\mu\lambda} = 2\nabla_{(\mu}\xi_{\nu)}. \tag{4.1}$$

Thus, the constraint is equivalent to, $g^{\mu\nu}\delta g_{\mu\nu} = 0 \implies \nabla_\mu \xi^\mu = 0$, having a divergence free deformation of the manifold, i.e., transformations in which the 4-volume is preserved. To achieve the UG dynamical equations we take into consideration the following action,

$$S = \int d^4x \frac{1}{16\pi G}\left[\sqrt{-g}R - \lambda(x)(\sqrt{-g} - \epsilon_0)\right] + S_m, \tag{4.2}$$

here, ϵ_0 is a constant tensor density and S_m is the matter action. Varying the action with respect to the metric and setting the variation to be zero, we get,

$$\delta S = \int d^4x \frac{1}{16\pi G}\left[G_{\mu\nu} + \frac{\lambda(x)}{2} g_{\mu\nu}\right]\sqrt{-g}\delta g^{\mu\nu} + \int d^4x \frac{\delta S_m}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = 0. \tag{4.3}$$

And the equation of motion now reads,

$$G_{\mu\nu} + \frac{\lambda(x)}{2} g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{4.4}$$

Also we have a constraint equation $\sqrt{-g} = \epsilon_0$, obtained by varying the action with respect the Lagrange's multiplier λ. Taking the trace of the equation of motion we have, $\lambda(x) = \frac{1}{2}(R + 8\pi G T)$ and recasting this relationship into the equation of motion again we achieve the traceless dynamical equation for UG, as given below,

$$R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = 8\pi G\left(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T\right). \tag{4.5}$$

The Bianchi identity no longer ensures the conservation of energy and momentum. One defines energy-momentum violation current as $(8\pi G)\nabla^\mu T_{\mu\nu} = J_\nu$. Now, the variation of matter action gives,

$$\begin{aligned}\delta S_m &= \int d^4x \frac{\delta S_m}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = -\int d^4x \frac{\sqrt{-g}}{2}T_{\mu\nu}\delta g^{\mu\nu} \qquad \left[T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}\right]\\ &= \int d^4x\sqrt{-g}T_{\mu\nu}\nabla^\mu\xi^\nu\\ &= -\frac{1}{(8\pi G)}\int d^4x\sqrt{-g} \quad j_\mu \ \xi^\mu. \end{aligned} \tag{4.6}$$

From the condition for volume preserving diffeomorphism, we have, $\nabla_\mu\xi^\mu = 0 \implies \xi^\mu = \epsilon^{\mu\nu\alpha\lambda}\nabla_\nu\omega_{\alpha\lambda}$, and with the help of this one obtains,

$$\begin{aligned}\delta S_m &= -\frac{1}{(8\pi G)}\int d^4x\sqrt{-g} \quad j_\mu \ \epsilon^{\mu\nu\alpha\lambda}\nabla_\nu\omega_{\alpha\lambda}\\ &= \frac{1}{(8\pi G)}\int d^4x\sqrt{-g} \quad \epsilon^{\mu\nu\alpha\lambda}\nabla_\nu j_\mu\omega_{\alpha\lambda}. \end{aligned} \tag{4.7}$$

If the matter action is invariant under volume preserving diffeomorphisms, then $\delta S_m = 0$. As $\omega_{\alpha\lambda}$, is arbitrary, one concludes,

$$\epsilon^{\mu\nu\alpha\lambda}\nabla_\nu j_\mu\omega_{\alpha\lambda} = 0 \implies dJ = 0 \implies j_\mu = \nabla_\mu Q. \tag{4.8}$$

Now, the UG equation can be written as,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{1}{4}g_{\mu\nu}R = 8\pi G\left(T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T\right). \tag{4.9}$$

Using Bianchi identity, we get,

$$\frac{1}{4}\nabla_\mu R = \nabla_\mu\left(Q - 8\pi G\cdot\frac{1}{4}T\right). \tag{4.10}$$

This is equivalent to, $\frac{1}{4}R = \left(\Lambda_0 + Q - 8\pi G\cdot\frac{1}{4}T\right)$ with an integration constant Λ_0.

One recasts the value of Ricci scalar in the UG equation to obtain,

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\left(\Lambda_0 + Q - 8\pi G \cdot \frac{1}{4} T\right) &= 8\pi G \left(T_{\mu\nu} - \frac{1}{4} g_{\mu\nu} T\right) \\
\implies R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\left[\Lambda_0 + \int_l J\right] &= 8\pi G T_{\mu\nu}. \qquad (4.11)
\end{aligned}
$$

We were after the above equation. It is worth noting that depending on whether the gradient of Q vanishes the UG equation includes both the possibility of having or not having violation of energy-momentum conservation. The UG equation is invariant under the transformation $T_{\mu\nu} \to T_{\mu\nu} + C g_{\mu\nu}$. Therefore, the vacuum fluctuations do not gravitate. We, now, write down a slightly more general form of the modified Einstein's Equation(s) as,

$$
G_{\mu\nu} + X_{\mu\nu} = (8\pi G) T_{\mu\nu}, \qquad (4.12)
$$

here, $X_{\mu\nu}$ is a function of geometric and matter quantities. This kind of modification violates the local energy-momentum conservation and gives rise to a four-current,

$$
\nabla^\mu X_{\mu\nu} = (8\pi G) \nabla^\mu T_{\mu\nu} \equiv J_\nu. \qquad (4.13)
$$

For the case of Unimodular Gravity, we set,

$$
X_{\mu\nu} = \frac{1}{4} g_{\mu\nu} \left(R + (8\pi G) T\right). \qquad (4.14)
$$

We shall now look at the effects of such modification of the dynamical equation.

4.2 FLRW background

With the same metric structure and the energy-momentum tensor given in equations (3.2) and (3.9), respectively, the energy-momentum conservation violation equation (4.13) becomes, (for $\nu = 0$),

$$
\begin{aligned}
\nabla_\mu T^\mu{}_0 &= \frac{1}{8\pi G} J_0(t) \\
\implies \frac{d\rho}{dt} + 3 \frac{1}{a} \frac{da}{dt} (1+\omega)\rho &= -\frac{1}{8\pi G} J_0(t). \qquad (4.15)
\end{aligned}
$$

The solution of this equation can be obtained to be the following,

$$\rho(t) = a^{-3(1+\omega)} \left[\left(-\frac{1}{8\pi G} \right) \int^t J_0(t') a(t')^{3(1+\omega)} dt' + C \right]. \tag{4.16}$$

For $\nu = i$, one gets,

$$\begin{aligned}
&\partial_\mu T^\mu{}_i + \Gamma^\mu_{\mu\alpha} T^\alpha{}_i - \Gamma^\alpha_{i\mu} T^\mu{}_\alpha = \frac{1}{8\pi G} J_i(t) \\
&\implies \partial_0 T^0{}_i + \partial_j T^j{}_i + \Gamma^0_{00} T^0{}_i + \Gamma^0_{0j} T^j{}_i + \Gamma^j_{j0} T^0{}_i + \Gamma^j_{jk} T^k{}_i - \Gamma^0_{i0} T^0{}_0 - \Gamma^0_{ij} T^j{}_0 \\
&\quad - \Gamma^j_{i0} T^0{}_j - \Gamma^j_{ik} T^k{}_j = \frac{1}{8\pi G} J_i(t) \\
&\implies J_i(t) = 0.
\end{aligned} \tag{4.17}$$

In the kind of modified theory being studied here, the form of the current J_μ depends on our choice. It can be put by hand. But if the FLRW background is being used the spatial components of J_μ cannot be set to nonzero values.

Now, it is time to calculate the modified field equations. From the modified field equations (4.5), we have, for 00-component,

$$\begin{aligned}
R_{00} - \frac{1}{4} g_{00} R &= (8\pi G) \left[T_{00} - \frac{1}{4} g_{00} T \right] \\
\implies \left(\frac{1}{a} \frac{da}{dt} \right)^2 - \frac{1}{a} \frac{d^2 a}{dt^2} &= 4\pi G (\rho + P)
\end{aligned} \tag{4.18}$$

The ij-component equations are exactly the same as above.

To get the relation how the scale factor evolves with time we take single fluid model with the equation of state $P = \omega\rho$, and we substitute equation (4.16) into the field equation,

$$\begin{aligned}
\left(\frac{1}{a} \frac{da}{dt} \right)^2 - \frac{1}{a} \frac{d^2 a}{dt^2} &= 4\pi G (1+\omega) \rho \\
&= 4\pi G (1+\omega) a^{-3(1+\omega)} \left[\left(-\frac{1}{8\pi G} \right) \int^t J_0(t') a(t')^{3(1+\omega)} dt' + C \right] \\
\implies -\frac{d}{dt} \left(\frac{1}{a} \frac{da}{dt} \right) &= 4\pi G (1+\omega) a^{-3(1+\omega)} \left[\left(-\frac{1}{8\pi G} \right) \int^t J_0(t') a(t')^{3(1+\omega)} dt' + C \right].
\end{aligned} \tag{4.19}$$

Using the definition of H, multiplying both sides of the equation by $a^{3(1+\omega)}$ and taking derivative with respect to time, we obtain the following differential equation,

$$\frac{d^2H}{dt^2} + 3(1+\omega)H\frac{dH}{dt} = \frac{1}{2}(1+\omega)J_0(t). \tag{4.20}$$

This is an inhomogeneous differential equation in H and we can solve it via Green's function method as earlier. Homogeneous equation has two solutions, $H_1 =$ constant, and,

$$H_2(t) = \frac{A\sqrt{\frac{2}{3}}\tanh\left[A\sqrt{\frac{3(1+\omega)}{2}}(t+B)\right]}{\sqrt{1+\omega}} \tag{4.21}$$

$H_2 = 0$ at $t = 0$, implies $B = 0$. As we mentioned earlier that the UG may or may not have a violation current activated, when the violation current is absent, the scale factor from H_2 can be derived to be,

$$a(t) = D\left(\cosh\left[A\sqrt{\frac{3(1+\omega)}{2}}t\right]\right)^{\frac{2}{3(1+\omega)}}. \tag{4.22}$$

It is interesting to see that at large coordinate time, the scale factor takes the form, $a(t\to\infty) = \frac{D}{2}e^{Ht}$, with $H = A\sqrt{\frac{2}{3(1+\omega)}}$. We may also say that the at late times H_2 approaches H_1. Therefore, after the modified dynamics plays itself out, the universe ends up in a constant curvature or De Sitter spacetime and the standard Einsteinian dynamics take over. If D is a sufficiently small constant, this modified dynamics can be considered to come from dominant quantum gravity effects at very early time and can be interpreted as a pre-inflationary era. The general solution for the Hubble parameter, when the violation current is active can be written as,

$$H(t) = C_1 + C_2H_2(t) + \frac{1}{2}(1+\omega)\int_0^t dt' G(t-t')J_0(t'), \tag{4.23}$$

with,

$$G(t-t') = \frac{(H_2(t') - H_2(t))}{A^2\,\mathrm{sech}^2\left[A\sqrt{\frac{3(1+\omega)}{2}}(t'+B)\right]}. \tag{4.24}$$

At very early time, the contribution from the source integral would be very small and if we demand that at that time H_2 is the solution (whereas H_1 is achieved at the end of the modified dynamics), then we can set $C_1 = 0$ and $C_2 = 1$.

4.3 Linear perturbations to the FLRW background

We shall now derive the equations corresponding to dynamics and kinetics for the Unimodular Gravity model.

4.3.1 Dynamics: modified Einstein equations

As we mentioned earlier the dynamical equations are,

$$G^{\mu}{}_{\nu} + X^{\mu}{}_{\nu} = (8\pi G)T^{\mu}{}_{\nu}, \tag{4.25}$$

with X^{μ}_{ν} set to,

$$X^{\mu}{}_{\nu} = \frac{1}{4}\delta^{\mu}{}_{\nu}\left(R + (8\pi G)T\right). \tag{4.26}$$

For photons and neutrinos $T = 0$. For baryons and dark matter, pressure is considered negligible compared to the energy density. Thus, $T = \rho_{\mathrm{dm}}(1+\delta) + \rho_b(1+\delta_b)$. Thus the different components of X^{μ}_{ν} takes the following forms,

$$X^{0}{}_{0} = \frac{3}{2}(1-2\Psi)\left(H^2 + \frac{1}{a}\frac{d^2a}{dt^2}\right) - \frac{1}{2a^2}\nabla^2\Psi + \frac{3}{2}\Phi_{,00} - \frac{3}{2}H(\Psi_{,0} - 4\Phi_{,0}) - \frac{1}{a^2}\nabla^2\Phi \\ -(2\pi G)[\rho_{\mathrm{dm}}(1+\delta) + \rho_b(1+\delta_b)], \tag{4.27}$$

$$\left(\hat{k}^i\hat{k}_j - \frac{1}{3}\delta^i_j\right)X^{i}{}_{j} = 0, \tag{4.28}$$

$$X^{0}{}_{i} = 0. \tag{4.29}$$

Therefore, the first order part of the 00 equation, $G^{0(1)}_{\ 0} + X^{0(1)}_{\ 0} = (8\pi G)T^{0(1)}_{\ 0}$, is,

$$\frac{3}{2}H\Psi_{,0} + \frac{k^2}{a^2}\Phi - 3\Psi\left(H^2 - \frac{1}{a}\frac{d^2a}{dt^2}\right) - \frac{k^2}{2a^2}\Psi - \frac{3}{2}\Phi_{,00} = 4\pi G\left[\frac{3}{2}(\rho_{\mathrm{dm}}\delta + \rho_b\delta_b) + 8\rho_\gamma\Theta_0 + 8\rho_\nu\mathcal{N}_0\right] \tag{4.30}$$

Converting the time derivatives to derivatives with respect to conformal time, which shall be denoted by over dots, we get,

$$k^2(2\Phi-\Psi)+3\frac{\dot{a}}{a}\left(\dot{\Phi}+\dot{\Psi}-4\frac{\dot{a}}{a}\Psi\right)+6\Psi\frac{\ddot{a}}{a}-3a\ddot{\Phi}=8\pi Ga^2\left[\frac{3}{2}(\rho_{\rm dm}\delta+\rho_b\delta_b)+8\rho_\gamma\Theta_0+8\rho_\nu\mathcal{N}_0\right]. \tag{4.31}$$

The traceless, longitudinal part of the space-space components of the modified equations (first order) is the same as before, $\left(\hat{k}^i\hat{k}_j-\frac{1}{3}\delta^i_j\right)\left[G^i{}_j+X^i{}_j=(8\pi G)T^i{}_j\right]$

$$k^2(\Phi+\Psi)=-32\pi Ga^2[\rho_\gamma\Theta_2+\rho_\nu\mathcal{N}_2]. \tag{4.32}$$

The time-space equation $\left(G^0{}_i+X^0{}_i\right)k^i=(8\pi G)T^0{}_ik^i$, in the first order, also remains same as before,

$$\dot{\Phi}-\frac{\dot{a}}{a}\Psi=\frac{4\pi Ga^2}{ik}[\rho_{\rm dm}v+\rho_bv_b-4i\rho_\gamma\Theta_1-4i\rho_\nu\mathcal{N}_1]. \tag{4.33}$$

4.3.2 Kinetics: modified Boltzmann equations

Having discussed the modified dynamical equations we shall, now, turn our attention to the modified equations for kinetics between the components of the universe. In our previous understanding the general Boltzmann equation was of the form, schematically,

$$\frac{df}{dt}=\mathcal{C}[f]. \tag{4.34}$$

This equation meant the number of particles in the phase space inside an infinitesimal volume $d^3\vec{r}d^3\vec{p}$ around a point $(\vec{r},\vec{p})$ did not change with time unless there were some collisions to bring in particles into the box or throw them out of the box. But in the scenario of the modified theory the local energy and momentum are not conserved. Particles can now disappear from the box or conjure up inside the box. Therefore, we must add an ad hoc modification term to our general Boltzmann equation to account for the violation of energy-momentum conservation. The exact form of the term must be decided from the purpose of the theory. We must also break this modification term into its zero-th order homogeneous part and first order perturbation part. For instance, the Boltzmann eqaution for the cold dark matter should now look like,

$$\frac{\partial n_{\rm dm}}{\partial t}+\frac{1}{a}\frac{\partial(n_{\rm dm}v^i)}{\partial x^i}+3\left[\frac{da/dt}{a}+\frac{\partial\Phi}{\partial t}\right]n_{\rm dm}=\frac{1}{m}\left(-\frac{1}{8\pi G}\right)\left[J_0^{(0),{\rm dm}}(t)+J_0^{(1),{\rm dm}}(\vec{x},t)\right], \tag{4.35}$$

here m is the dark matter mass. From the above equation the zero-order equation can be extracted,

$$\frac{\partial n^{(0)}_{\rm dm}}{\partial t}+3\frac{da/dt}{a}n^{(0)}_{\rm dm}=\frac{1}{m}\left(-\frac{1}{8\pi G}\right)\left[J_0^{(0),{\rm dm}}(t)\right]. \tag{4.36}$$

This is the form we should have expected from the equation (4.15). Now using the definition,

$$n_{\rm dm}=n^{(0)}_{\rm dm}(1+\delta(\vec{x},t)), \tag{4.37}$$

and following, keeping everything up to first order, and recasting the zero-th order equation, the equation for density contrast has the following form,

$$\frac{\partial\delta}{\partial t}+\frac{1}{a}\frac{\partial v^i}{\partial x^i}+3\frac{\partial\Phi}{\partial t}=\frac{1}{mn^{(0)}_{\rm dm}}\left(-\frac{1}{8\pi G}\right)\left[J_0^{(1),{\rm dm}}(\vec{x},t)-J_0^{(0),{\rm dm}}(t)\delta(\vec{x},t)\right]. \tag{4.38}$$

Similarly, other equations can also be modified using different ad hoc terms. For example the modified Boltzmann equation for photons can be suggested to have the following form in the Fourier space,

$$-p\frac{\partial f^{(0)}}{\partial p}\left[\dot{\Theta}+ik\mu\Theta+\dot{\Phi}+ik\mu\Psi\right]=p\frac{\partial f^{(0)}}{\partial p}\dot{\tau}\left[\Theta_0-\Theta+\mu v_b+\frac{1}{2}\mathcal{P}_2\Theta_2\right]+J^{\rm photon}(k,\mu,t). \tag{4.39}$$

Introducing a term like $J^{\rm photon}(k,\mu,t)$, now, challenges the idea that in the early universe or in tightly coupled limit the anisotropic stress and other higher moments of photon anisotropy are negligible. One must be careful in postulating the magnitude of such term in a specific theory. Having suggested the modifications to be made in the Boltzmann-Einstein set of equations in the framework of a modified theory of gravity we close our discussion.

Chapter 5

Conclusions & future outlook

In the preceding chapters we have introduced several perturbation variables and learned their evolution with time by solving the set of coupled Boltzmann-Einstein equations. We saw that the modes of all sizes of the gravitational potential falls in their magnitude as the universe becomes matter dominated. Whereas the matter perturbations grow with time which is needed for structure formation. The matter perturbation grows logarithmically in the radiation dominated era and then linearly in the matter dominated era. If the magnitude of the perturbation variable grows significantly, then the linear approximation would not hold and one must consider the non-linear perturbation theory. On the other hand as the radiation has non-negligible pressure, their evolution becomes a tug of war between gravitational pull and outward directed radiation pressure. As a result, the photon temperature anisotropies oscillate, and more specifically, we derived that the monopole and dipole oscillate out of phase with respect to each other. Another interesting aspect of the solutions is that, as we mentioned earlier, the solutions are sensitive to the initial conditions. The late time value of many perturbation variables are directly linked to their initial values. For example the late time value of the large scale gravitational potential is $\frac{9}{10}$-th of its initial value.

In the later part of this thesis, we considered a modified theory of gravity. In the early universe we expect the quantum gravity effects to be dominant and we have chosen a particular effective low energy theory which glosses over the details of the discreteness of the spacetime arising from its interaction with the matter degrees of freedom. As a result of this glossing over we do not account for all the energy-

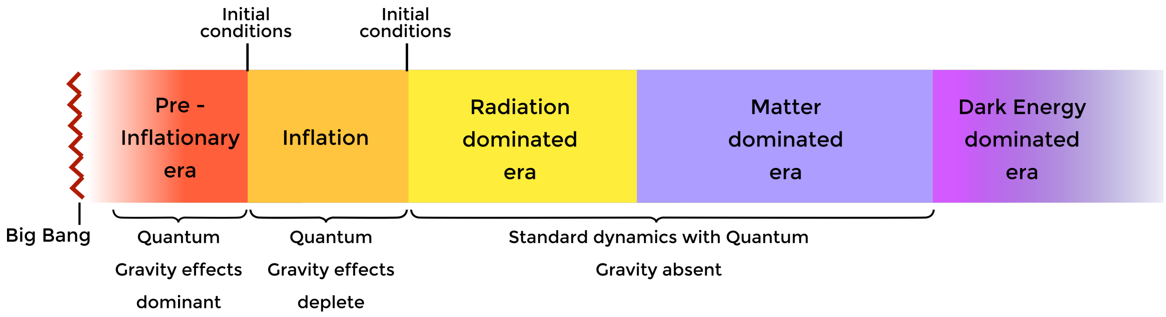

Figure 5.1: Pre-inflationary era and initial conditions

momentum exchanges between the discrete spacetime and matter and we should expect a violation of the energy-momentum conservation in the effective low energy limit of the theory. In such a theory we modified the dynamical equations of gravity to traceless UG equations. Dynamics with such modifications implies possibility of a pre-inflationary era after which the universe entered a De Sitter phase. As a part of the future exploration we propose that one should study the perturbation theory in this pre-inflationary era and see how the initial conditions set at the end of this phase translate to the initial conditions set in the aftermath of inflation in the standard cosmological theories (figure 5.1). To that end we have constructed the modified Einstein equations for the linear perturbation theory and then also suggested the changes to be made in the Boltzmann equations. The next logical step should be to solve these equations.

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