The study of Classical groups as linear algebraic groups

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A dissertation submitted for the partial fulfillment of BS-MS dual degree in Science

Indian Institute of Science Education and Research Mohali

April 2019

Certificate of Examination

This is to certify that the dissertation titled "The study of Classical groups as linear algebraic groups" submitted by Mr. Sachin Chandran (Reg. No. MS14098) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 26, 2019

Declaration

The work in this dissertation has been carried out by me under the guidance of Dr. Tanusree Khandai at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Tanusree Khandai
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Acknowledgement

First and foremost, I would like to thank my supervisor Dr. Tanusree Khandai for her never-ending support, motivating me to go forward whenever it was difficult, and helping me throughout this two-semester journey, both academically and emotionally. I thank my committee members Dr. Amit Kulshrestha and Dr. Krishnendu Gongopadhyay for their valuable inputs. My sincere thanks to the IISER, Mohali Mathematics department for their encouragement. I also thank Aswin G.S and Somak Maitra for their valuable suggestions.

I am forever grateful to my parents and my brother for their support and endless love. I am thankful to each and everyone of my friends, who were a constant source of motivation and made the five years in this institute a memorable one. Special mentions to Abin Antony, Balu N.V, Jain P.K, J Vaitheesh for their supercalifragilisticexpialidocious support and conversations. At last, I express my sincere gratitude to Leesa Joyce and Swetha U Lal for their unconditional love and support.

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Contents

Abstract

Hermann Weyl, in his famous book *'The Classical, their Invariants and Representations'*, coined the phrases "classical groups" to describe certain families of groups of linear transformations. Even years after its introduction, these groups continue to retain their importance in mathematics (especially in the subject of linear Lie groups) and has applications in both classical and modern physics.

We begin the thesis by introducing classical groups and study them from the point of view of linear algebraic groups. We then develop the theory in order to obtain basic results on their representations and lie algebras. We further move on to study the structure of classical groups. We show that for a classical group G, the subgroup of diagonal elements H is the maximal torus and every maximal torus is conjugate to H. We find the decomposition of $\mathfrak{g}$, the lie algebra of classical group G in terms of root and root spaces, under the adjoint action of H.

In the second part of the thesis, we look at the 'tensor product problem'. To put it into perspective, consider a finite dimensional simple lie algebra $\mathfrak{g}$. It is known that if W is a tensor product of finite dimensional irreducible modules of $\mathfrak{g}$, then W is completely reducible. The major goal of tensor product problem is to determine the irreducible $\mathfrak{g}$-modules of W along with their multiplicities. We look at one of the recent studies aimed at tackling this problem.

Chapter 1

Algebraic Geometry

In this chapter, we fix the notations and recall the definitions and results from algebraic geometry which are used as tools to understand the structure and representations theory of classical groups over complex field $\mathbb{C}$.

1.1 Affine algebraic set

1.1.1 Introduction :

Let V be a finite dimensional vector space over $\mathbb{C}$. Given a basis, $P(V)$, the set of polynomial on V, is defined as follows,

$$P(V) = \{f : V \to \mathbb{C} : f(\sum_{i=1}^{n} z_i e_i) = \sum_{|I| \leq k} a_I z^I\}$$

where $z_i \in \mathbb{C}$, $I = (i_1, , , i_n) \in \mathbb{N}^n$ and $z^I = z_1^{i_1} .... z_n^{i_n}$. $P(V)$ forms an commutative algebra under point wise multiplication of functions and is freely generated as an algebra by the coordinate functions $\{x_1, , , x_n\}$, where $x_i(\sum_{i=1}^{n} z_i e_i) = z_i$. Thus, $P(V) \cong \mathbb{C}[x_1, ..., x_n]$. A subset X of V is called an affine algebraic set if there exists $f_1, ..., f_m \in P(V)$ such that

$$X = \{v \in V : f_i(v) = 0, \ \forall\, 1 \leq i \leq m\}$$

The affine ring of X is defined to be the restrictions of polynomial function over V.

$$Aff(X) = \{f|_X : f \in P(V)\}$$

These functions are called regular functions on X. Let

$$I_X = \{f \in P(V) : f|_X = 0\}$$

Then I_X is an ideal of $P(V)$ and $Aff(X) \cong P(V)/I_X$.

Recall that a ring A is called noetherian if and only if every ideal of A is finitely generated and by Hilbert basis theorem, it is known that if A is noetherian, then the polynomial ring $A[x]$ is also noetherian. Since $\mathbb{C}$ is noetherian, by Hilbert basis theorem, $\mathbb{C}[x_1, ..., x_n]$ is noetherian, which implies $P(V)$ is noetherian. Thus we have the following result which shall be used frequently.

Lemma 1.1.1. *Let $I \subset P(V)$ be an ideal. There exists a finite set of polynomials $f_1, , , f_d$ such that every $g \in I$ can be written as*

$$g = g_1 f_1 + ... + g_d f_d$$

for some choice of $g_1, , , , g_d \in P(V)$.

1.1.2 Zariski topology

Let $X \subset V$ be an algebraic set. $Y \subset X$ is called Zariski closed in X if Y is an algebraic subset of X. For any non zero $f \in Aff(X)$, the principal open set of X defined by f is

$$X^f = \{x \in X : f(x) \neq 0\}$$

Lemma 1.1.2. *The Zariski closed sets of X gives X the structure of a topological space. The finite union of principal open set X^f, for $f \in Aff(X)$ and $f \neq 0$, are the non empty open sets of this topology.*

Proof. In order to show that it is a topological space, we have to check whether arbitrary intersection and finite union of algebraic sets are algebraic. Let Y_1 and Y_2 be algebraic set defined by $\{f_1, , f_n\}$ and $\{g_1, , , g_m\}$ respectively, then $Y_1 \cup Y_2$ is also an algebraic set defined by $\{f_i g_j : \ where \ 1 \leq n, \ 1 \leq m\}$. Let $\{Y_\alpha : \alpha \in I\}$ be an arbitrary collection of algebraic sets. Then, their intersection is the zero set of possibly infinitely many polynomials. But by Hilbert basis theorem, we get that this intersection is also defined by finite number of polynomials, thus its an algebraic set.
Now, complement of any algebraic set, by definition is the finite union of principal open set X^f. $\square$

Let X, Y be two algebraic sets and $f : X \to Y$ be a map. Given any complex valued function g on Y, we define $f^*(g)$ as

$$f^*(g)(x) = g(f(x)), \ \ x \in X$$

Then f is said to be a regular map if $f^*(Aff(Y)) \subset Aff(X)$.

Lemma 1.1.3. *A regular map f between two algebraic sets is continuous with respect to Zariski topology.*

Proof. Let $Z \subset Y$ be a closed set defined by $\{g_1, , , g_n\}$. Then, $f^{-1}(Z)$ is the zero set of $\{f^*(g_i)\}$, which implies $f^{-1}(Z)$ is a closed set. Thus, f is continuous. $\square$

1.1.3 Product of affine sets

Let $X \subset V$ and $Y \subset W$ be two affine algebraic sets. Let $f_1, , , f_n \in P(V)$ be the defining polynomial of X and $g_1, , , g_m$ be the defining polynomials of Y. We can extend these function to $V \times W$ by setting $f_i(v, w) = f_i(w)$ and $g_i(v, w) = g_i(w)$. Therefore, $X \times Y$ is an algebraic set in the vector space $V \oplus W$ with $\{f_1, , , f_n, g_1, , g_m\}$ as the defining polynomials. By the universal mapping property of the tensor product, there exists a unique linear map $\mu : P(V) \otimes P(W) \to P(V \oplus W)$ such that

$$\mu(f' \otimes f'')(v, w)) = f'(v)f''(w)$$

Lemma 1.1.4. *The map μ induces a vector space isomorphism*

$$\Phi : Aff(X) \otimes Aff(Y) \to Aff(X \times Y).$$

Proof. We have the vector space isomorphism

$$\mu : P(V) \otimes P(W) \to P(V \oplus W)$$

Function in $Aff(X \times Y)$ are just restrictions of polynomials in $P(V \oplus W)$. Therefore, the map Φ is surjective. Now, it is left to show that Φ is injective.

Suppose $f = \sum_{i=1}^{r} f_i' f_i'' \in Aff(X) \otimes Aff(Y)$, such that $f \neq 0$. Without loss of generality, assume that functions $\{f_i''\}$ are linearly independent and $f_i' \neq 0$. Choose $g_i' \in P(V)$ and $g_i'' \in P(W)$ such that $f_i' \in g_i'|_X$ and $f_i'' = g_i''|_Y$. Set $g(v, w) = \sum g_i'(v) g_i''(w)$. Then $v(f) = g|_{X \times Y}$. Choose $x_0 \in X$ such that $f_1'(x_0) \neq 0$. Since $\{f_i''\}$ are linearly independent, we have

$$\sum_i f_i'(x_0) f_i'' \neq 0$$

in $Aff(Y)$.Thus, the map $y \to g(x_0, y)|_Y$ is nonzero. Hence, $\Phi(f) \neq 0$. This shows that Φ is injective.

$\square$

1.1.4 Principal open set

Let X be an algebraic set and let $f|_X \in Aff(X)$ with $f \neq 0$. We will give an algebraic set structure to the principal open set X^f in the following way. Define $\chi : X^f \to V \times \mathbb{C}$ by

$\chi(x) = (x, f(x)^{-1})$. Clearly this map is injective. Now, choose $\tilde{f} \in P(V)$ so that $\tilde{f}|_X = f$ and let $f_1, , , , f_n \in P(V)$ be the defining polynomials of X. Then

$$\chi(X^f) = \{(v,t) \in V \times \mathbb{C} : f_i(v) = 0, \ \forall\, i \ and \ \tilde{f}(v)t - 1 = 0\}$$

Thus, $\chi(X^f)$ gets an algebraic set structure. Now, we define the ring of regular function on X^f by taking the pull back of regular function on $\chi(X^f)$.

$$Aff(X^f) = \{g \circ \chi : g \in P(V \times \mathbb{C})\}$$

Notice that on $\chi(X^f)$, t has same restriction as f^{-1}. Therefore, the regular function on X^f are all of form $g(x_1, , , x_n, \tilde{f}^{-1})$.

1.1.5 Irreducible components

Let $X \subset V$ be a non empty closed subset. We say X is reducible if there exists non empty closed subsets $X_1 \neq X$ and $X_2 \neq X$ such that $X = X_1 \cup X_2$. If X is not reducible, then its called irreducible.

Lemma 1.1.5. *Let $X \subset V$ be an algebraic set, there exists finitely many irreducible closed subsets X_i's such that*

$$X = X_1 \cup X_2 \cup .... \cup X_n,$$

where $X_i \not\subset X_j$, for $i \neq j$. Moreover this decomposition is unique up to permutation of the indices and is called the incontractible decomposition of X. The subsets X_i's are called irreducible components of X.

Proof. We prove this using contradiction. Suppose the lemma does not hold for X. Then X has a decomposition into closed sets, $X = X_1 \cup X_2$, where $X_i \subset X$, such that the lemma does not hold for X_1 or X_2. Without loss of generality, we assume that the lemma does not hold for X_1. Then, X_1 will have a decomposition, $X_1 = X_3 \cup X_4$, such that the lemma does not hold. Continuing this procedure we get a infinite chain of closed subsets,

$$X \supset X_1 \supset .... \supset X_k......$$

This gives an infinite chain of increasing ideals in $P(V)$,

$$I_X \subset I_{X_1} \subset ....$$

which is a contradiction as $P(V)$ is noetherian. Thus the algebraic set X must have a finite decomposition into closed subsets. Any finite decomposition of X into closed subsets can be written in the form $X = X_1 \cup X_2 \cup .... \cup X_n$, where $X_i \not\subset X_j$, for $i \neq j$ by a deletion

process.
For the uniqueness part, suppose that X has two different incontractible decomposition,

$$X = X_1 \cup X_2 \cup .... \cup X_n$$

,

$$X = Y_1 \cup Y_2 \cup .... \cup Y_m$$

Then, $X_1 = X_1 \cap X = (X_1 \cap Y_1) \cup (X_1 \cap Y_2) \cup .... \cup (X_1 \cap Y_m)$. Since X_1 is a irreducible component i.e it is a maximal irreducible subset of X and $X_1 \subset X_1 \cap Y_i$, for some $1 \leq i \leq m$, we get that $X_1 \subset Y_i$, for some $1 \leq i \leq m$.
Similarly, $Y_i = (Y_1 \cap X_1) \cup (Y_1 \cap X_2) \cup .... \cup (Y_1 \cap X_n)$, which gives $Y_i \subset X_j$, for some $1 \leq j \leq n$. Therefore, $X_1 \subset Y_i \subset X_j$, which implies $X_1 = Y_i$. Hence, for $1 \leq i \leq n$, $X_i = Y_{\sigma(i)}$, where $\sigma \in S_m$. Now, staring with Y_i, repeating the same steps as before, we get that for $1 \leq j \leq m$, $Y_j = X_{\sigma(j)}$, where $\sigma \in S_n$. Thus $n = m$, proving the uniqueness of the decomposition. $\square$

Lemma 1.1.6. *Let V and W be two finite dimensional vector space. If $X \subset V$ and $Y \subset W$ be two irreducible algebraic sets, then $X \times Y$ is an irreducible algebraic set in $V \oplus W$.*

Proof. Suppose $X \times Y$ is not irreducible, then it can be written as union of two closed subsets, $X \times Y = Z_1 \cup Z_2$, where Z_1, Z_2 are closed subset of $X \times Y$. Notice that for $x \in X$, $x \times Y$ is irreducible, if not then it will be a contradiction to the irreducibility of Y. Therefore $x \times Y \subset Z_1$ or $x \times Y \subset Z_2$. This gives us a decomposition for X,

$$X_i = \{x \in X : x \times Y \subset Z_i\}$$

We now show that X_i's are closed , which will contradict the fact that X is irreducible. Let $\{f_a\}$ be the defining function of Z_i. Then we define

$$X_i^y = \{x \in X : f_a(x, y) = 0 \forall a\}$$

It is clear that X_i^y is closed subset and $X_i = \bigcap_{y \in Y} X_i^y$ is closed subset. Therefore $X = X_1 \cup X_2$, where X_1 and X_2 are closed subsets. This is a contradiction to the irreducibility of X, which shows that $X \times Y$ is irreducible. $\square$

Lemma 1.1.7. *Let X be an irreducible algebraic set, then $\overline{X}$ is also irreducible.*

Proof. It is enough to prove that every non empty open subsets of $\overline{X}$ has a non empty intersection. Take two non empty open subset of $\overline{X}$, say A and B. Then $A \cap X$ and $B \cap X$ are open in X. Since X is an irreducible set, $Y = (A \cap X) \cup (B \cap X)$ is non empty, which implies, $A \cap B$ is non empty. Therefore $\overline{X}$ is irreducible. $\square$

1.2 Tangent vector in an algebraic set

1.2.1 Tangent vector

Let X be an algebraic set. A tangent vector v at point $a \in X$ is defined as a linear function from $Aff(X) \to \mathbb{C}$ such that

$$v(fg) = v(f)g(a) + f(a)v(g) \quad ,\forall \ f, g \in Aff(X)$$

The collection of all tangent vectors at the point a is denoted as $T(X)_a$.Let v be a tangent vector, then by the derivation property, $v(1) = v(1.1) = v(1).1 + 1.v(1) = 2.v(1)$, $\Rightarrow$ $v(1) = 0$. Therefore, tangent vector acting on a constant gives 0.

Lemma 1.2.1. *Given an algebraic set X and $a \in X$, $m_a \subset Aff(X)$ denote the maximal ideal of functions which vanishes at a. Then, there exists a natural isomorphism between $T(X)_a$ and $(\ m_a/m_a^2)\ ^*$.*

Proof. Take $v \in T(X)_a$. Then, for any $f \in Aff(X)$, $v(f) = v(f - f(a))$ and $(f - f(a)) \in m_a$. Therefore the action of v on $Aff(X)$ is defined entirely by its restricted action on m_a. For $g \in m_a^2$, $g = hk$, where $h, k \in m_a$, $v(g) = v(hk) = v(h)k(a) + h(a)v(k) = 0$, $\Rightarrow$ $v(m_a^2) = 0$.Thus ,$v \in T(X)_a$ naturally defines a $v' \in (\ m_a/m_a^2)\ ^*$.
Conversely, for $v' \in (\ m_a/m_a^2)\ ^*$, we define a tangent vector at a as follows, $v(f) = v'(f - f(a))$. Only thing remaining to show is that the defined v is indeed a tangent vector. Consider $f, g \in Aff(X)$, then $(f - f(a))(g - g(a)) \in m_a^2$.

$$v((f - f(a))(g - g(a))) = v(fg) - v(f)g(a) - f(a)v(g) + v(f(a)g(a))$$

$$0 = v(fg) - v(f)g(a) - f(a)v(g)$$
$$v(fg) = v(f)g(a) + f(a)v(g)$$

Hence $v \in T(X)_a$. □

Note: 1. Consider $X = \mathbb{C}^n$, then $Aff(X) = \mathbb{C}[x_1, x_2, ..., x_n]$. Let $f \in Aff(X)$, $u, a \in X$, then $D_u f(a)$ denote the directional derivative of f in the direction of the vector u, which forms a tangent vector at a. Thus, $v(f) = D_u f(a)$ and suppose $u = (u_1, u_2, ..., u_n)$, $a = (a_1, a_2, ..., a_n)$ then we can write $v(f)$ as,

$$\begin{aligned} v(f) &= D_u f(a) \\ &= \sum_{i=1}^{n} \frac{u_i}{|u|} \frac{\partial}{\partial x_i} f(a) \end{aligned}$$

Notice that $\frac{\partial}{\partial x_i}|_a \in T(X)_a$. We will now show that in fact $\frac{\partial}{\partial x_i}|_a$, $1 \leq i \leq n$ forms the basis of $T(X)_a$. Take $f \in m_a$, then the Taylor series expansion of f at a is,

$$f(x) = f(a) + \sum_{i=1}^{n}(x_i - a_i)(\frac{\partial}{\partial x_i}f(a)) + \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - a_i)(x_j - a_j)(\frac{\partial^2}{\partial x_i \partial x_j}f(a)) + ...$$
$$= \sum_{i=1}^{n}(x_i - a_i)(\frac{\partial}{\partial x_i}f(a)) + \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - a_i)(x_j - a_j)(\frac{\partial^2}{\partial x_i \partial x_j}f(a)) + ...$$

So, $f \in m_a$ is generated by $(x_i - a_i)$, for $1 \leq i \leq n$. Thus, $(x_i - a_i)$, for $1 \leq i \leq n$ spans m_a/m_a^2, also $(x_i - a_i)$ forms a orthogonal set, which implies $(x_i - a_i)$, for $1 \leq i \leq n$ forms the basis of m_a/m_a^2. Now, using the result $T(X)_a \cong (\ m_a/m_a^2)\ ^*$, we see that $\frac{\partial}{\partial x_i}|_a \in (\ m_a/m_a^2)\ ^*$ and further $\{\frac{\partial}{\partial x_i}|_a\}$ forms dual basis to $\{(x_i - a_i)\}$, $\Rightarrow \{\frac{\partial}{\partial x_i}|_a\}$ forms the basis of $T(X)_a$.

2. Take $X \subset \mathbb{C}^n$, then $Aff(X) = P(\mathbb{C}^n)/I_X$.For some $a \in X$, take $v \in T(X)_a, v : Aff(\wp(\mathbb{C}^n)/I_X) \to \mathbb{C}$, there exists a unique $\tilde{v} \in T(\mathbb{C}^n)_a, \tilde{v} : Aff(\wp(\mathbb{C}^n)) \to \mathbb{C}$, such that $\tilde{v}(f) = v(f + I_X)$ and $\tilde{v}(I_X) = 0$, for $f \in Aff(\wp(\mathbb{C}^n))$. Conversely, for some $\tilde{v} \in T(\mathbb{C}^n)_a$, it induces a $v \in Aff(X)$ defined by $v(f + I_X) = \tilde{v}(f)$. Therefore we get,

$$T(X)_a = \{\tilde{v} \in T(\mathbb{C}^n)_a : \tilde{v}(I_X) = 0\}$$

Suppose I_X is generated by polynomials $\{f_1, f_2, ..., f_m\}$. As we have already discussed $\tilde{v}$ can be expressed as $v = \sum\limits_{i=1}^{n} u_i \frac{\partial}{\partial x_i}|_a$ and $\tilde{v}(I_X) = 0 \Rightarrow \tilde{v}(f_i) = 0$, for $1 \leq i \leq m$. Hence we get,

$$\sum_{i=1}^{n} u_i \frac{\partial f_j(a)}{\partial x_i} = 0 \quad , \ for\ 1 \leq j \leq m,$$

where $u_i = \tilde{v}(x_i - u_i)$. So we get a system of m linear equations, i.e a $m \times n$ matrix, which is denoted by $J(a)$. Applying the rank-nullity theorem, we get $n = rank(J(a)) + nullity(J(a))$. Observe that $nullity(J(a))$ is exactly the $dim(T(X)_a)$. Therefore, $dim(T(X)_a) = n - rank(J(a))$.

Smoothness of an algebraic set: We first define

$$m(X) = \min_{x \in X} dim(T(X)_x)$$

Let $X_0 = \{x \in X : dim(T(X)_x) = m(X)\}$. The points of X_0 is called the smooth points of algebraic set. An algebraic set X is called smooth if $X = X_0$. For example, let $X = \mathbb{C}^n$ an algebraic set. Then, as we have seen $\forall x \in X, dim(T(X)_x) = dim(\mathbb{C}^n) = n$. Therefore, $X = \mathbb{C}^n$ is a smooth algebraic set.

1.2.2 Differential of a regular map

Let X, Y be two algebraic set and $\phi : X \to Y$ be a regular map.Then differential of ϕ at a point $a \in X$, denoted by $d\phi_a$ is a linear map from $T(X)_a$ to $T(Y)_{\phi(a)}$, defined as

$$d\phi_a : T(X)_a \to T(Y)_{\phi(a)}$$

$$(d\phi_a v)(f) = v(\phi^*(f)) \quad for\ v \in T(X)_a\ and\ f \in Aff(Y)$$

Since ϕ is a regular map, $\phi^*(Aff(Y)) \subset Aff(X)$, so $d\phi_a$ maps takes every $v \in T(X)_a$ to $d\phi_a(v)$ which acts on $Aff(Y)$. Now it remains to show that $d\phi_a(v)$ indeed belong to $T(Y)_{\phi(a)}$. Let $f, g \in Aff(Y)$,

$$\begin{aligned}(d\phi_a v)(fg) &= v(\phi^*(fg)) \\ &= v(\phi^*(f)\phi^*(g)) \quad (since\ \phi^*\ is\ an\ algebra\ homomorphism) \\ &= v(\phi^*(f))\phi^*(g)(a) + \phi^*(f)(a)v(\phi^*(g)) \\ &= v(\phi^*(f))(g)(\phi(a)) + (f)(\phi(a))v(\phi^*(g))\end{aligned}$$

Thus, $d\phi_a(v) \in T(Y)_{\phi(a)}$, implying $d\phi_a$ is a well defined linear map.

1.2.3 Vector field

Derivation of an algebra A is any linear map $D : A \to A$, which satisfies the Leibniz rule. If the algebra A is commutative and D, D' are derivations, then linear combination of D and D' also form a derivation, furthermore the commutator $[D, D'] = DD' - D'D$ also is a derivation. Therefore, the set of all derivations of a commutative algebra A forms a lie algebra, denoted by $Der(A)$. For an algebraic set X, a derivation on $Aff(X)$ is called a vector field on X and $vect(X)$ denote the lie algebra of all vector fields on X.

Chapter 2

Classical Groups

In this chapter we introduce the linear algebraic groups. We show that the class of matrix groups which H. Weyl referred to as the ”Classical groups” are in fact linear algebraic groups. We study some of the basic results on the Linear algebraic groups and the Lie algebras associated with them.

2.1 Linear algebraic group

2.1.1 Definitions and examples

Definition 1. *Let $GL(n, \mathbb{C})$ be the group of invertible $n \times n$ complex matrices and $M_n(\mathbb{C})$ denote the space of all $n \times n$ complex matrices. For $1 \leq i, j \leq n$, let x_{ij} denote the coordinate functions i.e for any $y \in M_n(\mathbb{C})$, $x_{ij}(y)$ gives the i, j entry of y. Then $P(M_n(\mathbb{C})) \cong \mathbb{C}[x_{11}, x_{12}, , , , x_{nn}]$. A subgroup $G \subset GL(n, \mathbb{C})$ is called a linear algebraic group if there exists a subset A of $P(M_n(\mathbb{C}))$ such that*

$$G = \{g \in GL(n, \mathbb{C}) : f_i(g) = 0, \forall f_i \in A\}$$

By Hilbert basis theorem, we get that any linear algebraic group is defined by finite set of polynomials.

Examples

1. Let $D_n \subset GL(n, \mathbb{C})$ denote the subgroup of diagonal matrices. The defining polynomials for D_n are $x_{ij} = 0$ for $i \neq j$. Then

$$D_n = \{g \in GL(n, \mathbb{C}) : x_{ij}(g) = 0, \ for \ i \neq j\}.$$

Hence D_n is a linear algebraic group.

2. **Special linear group** $SL(n, \mathbb{C})$**.**
$SL(n, \mathbb{C})$ is the group of invertible matrices with determinant one. Since determinant is polynomial map, define

$$SL(n, \mathbb{C}) = \{g \in GL(n, \mathbb{C}) : det(g) = 1\}$$

$SL(n, \mathbb{C})$ is a linear algebraic group.

3. **Orthogonal group** $O(n)$
Let B be a non degenerate symmetric bilinear form on $\mathbb{C}^n$, then we define the orthogonal group as,

$$O(\mathbb{C}^n, B) = \{g \in GL(n, \mathbb{C}) : B(x, y) = B(gx, gy), \ \forall x, y \in \mathbb{C}^n\}$$

From the above definition, we have

$$g \in O(\mathbb{C}^n, B) \iff g^t S g = S,$$

where S is the matrix corresponding the bilinear form B. This quadratic relation gives $O(\mathbb{C}^n, B)$ the linear algebraic group structure. For $g \in O(\mathbb{C}^n, B)$, we have $g^t S g = S$. Taking determinant, we get $det(S) = det(g^t) det(S) det(g)$. This implies, $det(g) = \pm 1$. We define

$$SO(\mathbb{C}^n, B) = \{g \in O(\mathbb{C}^n, B) : det(g) = 1\}$$

$SO(\mathbb{C}^n, B)$ is called the **special orthogonal group** relative to B.
A set of vectors $\{v_1, , , v_n\} \subset \mathbb{C}^n$ is called a B-orthonormal basis of $\mathbb{C}^n$ if $B(v_i, v_j) = \delta_{ij}$.

Lemma 2.1.1. *Let $\{v_1, , , v_n\}$ and $\{w_1, , , w_n\}$ be two B-orthonormal basis of $\mathbb{C}^n$. Let $g \in GL(n, \mathbb{C})$ such that $gv_i = w_i$, for $1 \le i \le n$, then $g \in O(\mathbb{C}^n, B)$.*

Proof. Since both are B-orthonormal basis, we get

$$B(gv_i, gv_j) = B(w_i, w_j) = B(v_i, v_j)$$

Thus, $B(gx, gy) = B(x, y)$, for every $x, y \in \mathbb{C}^n$, which implies $g \in O(\mathbb{C}^n, B)$.

□

Proposition 2.1.2. *Let B, B' be two non degenerate symmetric bilinear form on $\mathbb{C}^n$. Then, there exists $\gamma \in GL(n, \mathbb{C})$ such that $O(\mathbb{C}^n, B') = \gamma O(\mathbb{C}^n, B) \gamma^{-1}$.*

Proof. Consider a B-orthonormal basis $\{v_1, , , v_n\}$ and B'-orthonormal basis $\{w_1, , , w_n\}$ of $\mathbb{C}^n$. Let $\gamma \in GL(n, \mathbb{C})$ be such that $\gamma v_i = w_i$ for $1 \le i \le n$. Let S, S' be the matrix corresponding to the forms B and B' respectively. Then,

$$(S'\gamma v_i, \gamma v_j) = (\gamma^t S'\gamma v_i.v_j) = \delta_{ij} = (Sv_i, v_j)$$

Thus, $\gamma^t S'\gamma = S$. Now, for $g \in GL(n, \mathbb{C})$, set $h = \gamma g \gamma^t$. Then, we have

$$h^t S' h = (\gamma^{-1})^t g^t \gamma^t S' \gamma g \gamma^{-1} = (\gamma^{-1})^t g^t S g \gamma^{-1}$$

This implies $g \in O(\mathbb{C}^n, B)$ and hence $h \in O(\mathbb{C}^n, B')$. □

Definition 2. *When $n = 2l$, a basis $\{n_1, , , n_l, n_{-1}, , n_{-l}\}$ of $\mathbb{C}^n$ is called a B-isotropic basis if $B(v_i, v_j) = 0$, for $i \neq -j$ and $B(v_i, v_{-i}) = 1$, for all i and j. When $n = 2l + 1$, a basis $\{v_0, v_1, , v_l, v_{-1}, , v_{-l}\}$ of $\mathbb{C}^n$ that satisfies $B(v_i, v_j) = 0$, for $i \neq -j$ and $B(v_i, v_{-i}) = 1$, for all $i, j = 0, \pm 1, , , \pm l$ is called a B- isotropic basis.*

Remark : Given a non degenerate symmetric form on $\mathbb{C}^n$, we can always find a B-isotropic basis.

Lemma 2.1.3. *Suppose $\{v_i\}$ and $\{w_i\}$ be two B-isotropic bases of $\mathbb{C}^n$. If $g \in GL(n, \mathbb{C})$ such that $gv_i = w_i$, then $g \in O(\mathbb{C}^n, B)$.*

Proof. Same as the proof of Lemma 2.1.1. □

4. **Symplectic group** $Sp(\mathbb{C}^{2l}, \Omega)$

Let Ω be a non degenerate skew symmetric bilinear form on $\mathbb{C}^n$. We define the symplectic group relative to Ω as

$$Sp(\mathbb{C}^{2l}, \Omega) = \{g \in GL(n, \mathbb{C}) : \Omega(gx, gy) = \Omega(x, y),\ \forall x, y \in \mathbb{C}^n\}$$

Form the above definition, we get

$$g \in Sp(\mathbb{C}^{2l}, \Omega)\ if\ and\ only\ if\ g^t J g = J,$$

where J is the matrix corresponding to Ω. Thus, $Sp(\mathbb{C}^{2l}, \Omega)$ is an algebraic group.

Definition 3. *Given a basis $\{v_1, , v_l, v_{-1}, , v_{-l}\}$ of $\mathbb{C}^{2l}$, we call it an Ω- symplectic basis if $\Omega(v_i, v_j) = 0$, for $i \neq -j$ and $\Omega(v_i, v_{-i}) = 1$, for $1 \leq i \leq l$.*

Lemma 2.1.4. *Suppose $\{v_i\}$ and $\{w_i\}$ are two Ω-symplectic basis of $\mathbb{C}^{2l}$. Let $g \in GL(2l, \mathbb{C})$ such that $gv_i = w_i$, for all i, then $g \in Sp(\mathbb{C}^{2l}, \Omega)$.*

Proof. Same as the proof of Lemma 2.1.1. □

Proposition 2.1.5. *Let Ω and Ω' be two non degenerate skew symmetric form on $\mathbb{C}^{2l}$. Then, there exists $\gamma \in GL(2l, \mathbb{C})$ such that $Sp(\mathbb{C}^{2l}, \Omega') = \gamma Sp(\mathbb{C}^{2l}, \Omega)\gamma^{-1}$.*

Proof. Same argument as in Proposition 2.1.2. $\square$

The groups $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $O(n,\mathbb{C})$, $SO(n,\mathbb{C})$ and $Sp(2l,\mathbb{C})$ are called the classical groups.

2.1.2 Regular functions

$GL(n,\mathbb{C})$ is a principal open set in the $M_n(\mathbb{C})$ with respect to the Zariski topology. For $GL(n,\mathbb{C})$, the ring of regular function is defined as

$$Aff(GL(n,\mathbb{C})) = \mathbb{C}[x_{11}, x_{12},,, x_{nn}, (det)^{-1}]$$

Given $B \in End(\mathbb{C}^n)$, we define a linear functional on $End(\mathbb{C}^n)$ by,

$$f_B(A) = trace(AB),$$

for any $A \in End(\mathbb{C}^n)$. Notice that f_B is the linear combination of coordinate functions and hence it is a regular function. Any coordinate function x_{ij} can be written as a polynomial in f_B, where $B \in M_n(\mathbb{C})$. Thus, $Aff(GL(n,\mathbb{C}))$ is generated by f_B and $(det)^{-1}$, where B spans over $End(\mathbb{C}^n)$.
A complex valued function f on G is called regular if it is the restriction of a regular function on $GL(n,\mathbb{C})$. The set of regular functions $Aff(G)$, on G is a commutative algebra over $\mathbb{C}$ under point wise multiplication. Define

$$I_G = \{f \in Aff(GL(n,\mathbb{C})) : f(G) = 0\}$$

I_G is the ideal of $Aff(GL(n,\mathbb{C}))$, containing functions that vanishes over G. Then,

$$Aff(G) \cong Aff(GL(n,\mathbb{C}))/I_G$$

Let G and H be two algebraic group, then an abstract group homomorphism $\Pi : G \to H$ is said to be regular if $\Pi^*(Aff(H)) \subset Aff(G)$, where $\Pi^*(f)(g) = f(\Pi(g))$, for $\forall\ f \in Aff(H)$. We say G and H are isomorphic as algebraic group if there exists a regular homomorphism $\Pi : G \to H$, which has a regular inverse.

Theorem 2.1.6. *The multiplication map $\mu : G \times G \to G$ and inversion map $i : G \to G$ are regular. If $f \in Aff(G)$ then there exists an integer p and f_i' and $f_i'' \in Aff(G)$ for $i = 1,,, p$ such that*

$$f(gh) = \sum_{i=1}^{p} f_i'(g) f_i''(h),$$

for $g, h \in G$. Further, for a fixed $g \in G$, the maps $x \to L_g(x) = gx$ and $R_g(x) = xg$ from G to G are regular.

Proof. Fix a basis for V and let x_{ij} be the coordinate functions with respect to this basis. We have $g^{-1} = (1/det(g))adjoint(g)$, where $adjoint(g)$ is the adjoint matrix of A. Since $adjoint(g)$ is a polynomial in the x_{ij}, thus we get that inversion is a regular function.
Let $g, h \in G$,

$$x_{ij}(gh) = \sum_r x_{ir}(g)x_{rj}(h)$$

By multiplicative property of determinant map, $(det)^{-1}(gh) = (det)^{-1}(g)(det)^{-1}(h)$. The coordinate function $\{x_{ij}\}$ and $(det)^{-1}$ generate $Aff(G)$ as an algebra, thus the property holds for any $f \in Aff(G)$.
Using the above result, we can write

$$\mu^*(f) = \sum_r f'_r f''_r$$

We have already seen that $Aff(G \times G) \cong Aff(G) \otimes Aff(G)$. Using this identification, we get $\mu^*(Aff(G)) \subset Aff(G \times G)$. Therefore, μ is a regular map. Similarly, we get

$$L_g^*(f) = \sum_r f'_r(g)f''_r, \quad R_g^*(f) = \sum_r f''_r(g)f'_r$$

Thus, L_g and R_g are regular functions. $\square$

2.1.3 Representation

A representation of a linear algebraic group G is a pair (ρ, V) , where $\rho : G \to GL(n, \mathbb{C})$ is an abstract group homomorphism and V is a complex vector space. A representation ρ is called regular if for a fixed basis of V, the component functions $g \to \rho_{ij}(g)$, for $1 \leq i, j \leq n$, are all regular functions. To put the regularity condition of ρ in a more convenient manner, for $B \in End(V)$, define a function on G as,

$$f_B^\rho(g) = trace(\rho(g)B) \quad , \quad for\ g \in G$$

Since trace function is a non degenerate bilinear function on $End(V)$, for any $\lambda \in End(V)^*$, there exists a unique $A_\lambda \in End(V)$ such that $\lambda(X) = trace(A_\lambda X)$ for any $X \in End(V)$. Therefore, $\rho_{ij}(g) = trace(A_{ij}g)$ for a unique $A_{ij} \in End(V)$. It thus follows that the representation (ρ, V) is regular if and only if f_B^ρ is a regular function for every $B \in End(V)$.
Set

$$E^\rho = \{f_B^\rho : B \in End(V)\}$$

The space E^ρ is called the space of representative functions associated with ρ.
An infinite dimensional representation (ϕ, V) of G is said to be locally regular if for any finite dimensional subspace $F \subset V$ there exists a finite dimensional G-invariant subspace

E such that $F \subset E$ and restriction of ϕ to E is a regular representation.

Examples

1. Let (ρ, V) be a regular representation of G, then we define the dual representation (ρ^*, V^*) as $\rho^*(g)v^* = v^* \circ \rho(g^{-1})$.

$$\begin{aligned}\langle v, \rho^*(g^{-1}v^*)\rangle &= \rho^*(g^{-1})v^*(v)\\ &= (\rho^*(g^{-1})v^*)^t(v)\\ &= ((\rho(g))^t v^*)^t v\\ &= (v^*)^t \rho(g)v\\ &= \langle \rho(g)v, v^*\rangle\end{aligned}$$

Therefore, the dual representation ρ^* is regular. E^{ρ^*} consists of functions of the form $g \to f(g^{-1})$, where $f \in E^{\rho}$.

2. Consider the representation $(R, Aff(G))$ (called the right translation representation) of G, defined by $R(g)f(x) = f(xg)$ for $g \in G$ and $f \in Aff(G)$. $R(g)$ is a example for locally regular representation. To see this, given $f \in Aff(G)$, there exists f_i' and f_i'' such that

$$R(g)f = \sum_{i=1}^{n} f_i''(g)f_i'$$

Let $W = span(f_i', f_i'' : 1 \leq i \leq n)$. Then W is invariant under the action of $R(g)$ for every $g \in G$ and is finite dimensional. Then $V(f) = span(R(g)f : \forall g \in G) \subset W$. Let F be any finite dimensional subspace of $Aff(G)$, and $\{f_1, f_2, ...f_s\}$ be the defining function of F. Then, set $E = \sum_{i=1}^{k} V(f_i)$.

Remark: Similarly we can define $L(g)f(y) = f(x^{-1}y)$, the left translation representation, which is also locally regular.

3. Let (ρ, V) and (σ, W) be two regular representations of G, then we define the tensor product representation $(\rho \otimes \sigma, V \otimes W)$ as ,

$$\rho \otimes \sigma(g)(v \otimes w) = \rho(g)v \otimes \sigma(w)$$

Clearly $\rho \otimes \sigma$ is regular as

$$\langle(\rho \otimes \sigma)(g)(v \otimes w), v^* \otimes w^*\rangle = \langle\rho(g)(v), v^*\rangle\langle\sigma(g)(w), w^*\rangle$$

Also,

$$E^{\rho\otimes\sigma} = span(E^{\rho}.E^{\sigma})$$

2.1.4 Connectedness

Theorem 2.1.7. *Let G be a linear algebraic group. Then G has a unique subgroup G^0 that is closed, irreducible and of finite index. Furthermore, G^0 is a normal subgroup of G and its cosets forms the irreducible and connected components of G.*

Proof. First we will show the existence of such a subgroup G^0, Since G is an algebraic set, it has a unique incontractible decomposition, i.e

$$G = X_1 \cup X_2 \cup ...... \cup X_r,$$

where each X_i is an irreducible component of G. We label the X_i's such that for $1 \leq i \leq p$, $1 \in X_i$ and for $i > p$, $1 \notin X_i$, where $1 \leq p \leq r$ is a natural number. Now we define a function $\mu : X_1 \times X_2 \times ...... \times X_p \to G$ such that $\mu(x_1, x_2, ..., x_r) = x_1.x_2....x_p$. Set $X_1 \times X_2 \times ...... \times X_p = X$. X is an irreducible set (as finite product of irreducible set is irreducible) and the the map μ is a regular map as multiplication is a regular map. Since regular map is continuous with respect to the Zariski topology, we see that the image of the irreducible set X is irreducible in G. Now observe that for $1 \leq i \leq p$, $X_i \subset X$ and each of the X_i's are maximal irreducible sets. This is only possible if $p = 1$ i.e $X = X_1$. Since irreducible components are closed with respect to the topology, we get $\overline{X} = \overline{X_1} = X_1 = X$. Thus X is a closed set.

For $x \in X$, $X \to x^{-1}X$ is a homeomorphism and $x^{-1}X$ contains identity element. Thus, $x^{-1}X = X$, which implies $x^{-1} \in X$, $\forall x \in X$. For $x, y \in X$, since $x^{-1} \in X$, $xX = X$, which implies $xy \in X$. This proves that X is subgroup of G. Moreover if $x \in G$, then $X \to xXx^{-1}$ is a homeomorphism and xXx^{-1} contains the identity. Thus $X = xXx^{-1}$, proving that X is a normal subgroup of G.

$G \to gG$, for $g \in G$, is a homeomorphism.

$$\Rightarrow G = gG = gX_1 \cup gX_2 \cup .... \cup gX_r$$

by uniqueness of incontractible decomposition of algebraic sets, we conclude that $gX_i = X_{\sigma(g)i}$, where $\sigma(g) \in S_r$. Furthermore $gX_1 = X_i$, if $g \in X_i$. Therefore the cosets of $X = X_1$ forms the irreducible components. Hence the index of X over G is finite(using the fact that a noetherian topological space can only have finitely many irreducible components). Since the irreducible components are disjoint and irreducibility implies connectedness, the irreducible components also form the connected components of G. Setting $G^0 = X$, we get the required subgroup.

Uniqueness of subgroup G^0: Suppose there exists another normal subgroup H of G with the given properties. Then the cosets of H will form the irreducible components of G, with H being the only component containing the identity, which implies subgroup $H = G^0$, proving the uniqueness of G^0. $\square$

Note: Each irreducible component X_i is a closed subset of G. Then, the complement of each X_i, $\overline{X_i}$ is the union of closed subsets, which implies $\overline{X_i}$ is closed in G. Hence X_i is also open in G. Thus the irreducible components X_i's are both closed and open in G.

Corollary 2.1.7.1. *A linear algebraic subgroup is connected if and only if it is irreducible.*

2.1.5 Subgroup and homomorphism

Definition 4. *Let G be an algebraic group. A closed subset H of G which is also a subgroup of G is called an algebraic subgroup of G.*

Lemma 2.1.8. *Any closed subgroup of G of finite index contains G^0.*

Proof. Let H be a closed subgroup of G with finite index, then $G \setminus H = \bigcup_{g \in G} gH$ i.e finite union of closed subsets. That implies $G \setminus H$ is closed , which implies H is open. Therefore $H \cap G^0$ is both closed and open. But since G^0 is connected, if $H \cap G^0 \subset G^0$, then it is a contradiction. Therefore, $H \cap G^0 = G^0$ and $G^0 \subseteq H$. $\square$

Lemma 2.1.9. *Let H be a subgroup of linear algebraic group G. Then $\overline{H}$ is an algebraic subgroup. Furthermore, if H contains a non empty open subset of $\overline{H}$, then H is an algebraic subgroup.*

Proof. First we show that $\overline{H}$ is a subgroup. The map $x \to x^{-1}$ is a homeomorphism, therefore $\overline{H}^{-1} = \overline{H^{-1}} = \overline{H}$, since $H^{-1} = H$. For $x \in H$, $xH = H$, thus $x\overline{H} = \overline{xH} = \overline{H}$. Hence $H\overline{H} = \overline{H}$. Now for $x \in \overline{H}$, $Hx \subset \overline{H}$ which implies $\overline{H}x = \overline{Hx} \subset \overline{H}$. Hence $\overline{H}\overline{H} \subset \overline{H}$ and $\overline{H}$ is an subgroup.
Let U be a open subset of $\overline{H}$ such that $U \subset H$. For $x \in U$, $x^{-1}U$ is an open subset of $\overline{H}$ in H, such that $1 \in x^{-1}U$. Now we set $V = x^{-1}U$ and take $y \in \overline{H} \setminus H$. Then yV is an open neighbourhood of y in $\overline{H} \setminus H$. Thus, for every $y \in \overline{H} \setminus H$ we have an open neighbourhood of y in $\overline{H} \setminus H$, which implies $\overline{H} \setminus H$ is open. Hence H is closed. $\square$

Lemma 2.1.10. *Let $\phi : G \to H$ be a regular homomorphism of linear algebraic group. Then,*
1.$ker(\phi)$ is closed in H.
2.$Image(\phi)$ is closed in H.
3.$\pi(G^0) = \phi(G)^0$.

Proof. 1. For $x \in H$, singleton $\{x\}$ is closed in H. Since regular functions are continuous with respect to the Zariski topology, $ker(\phi) = \phi^{-1}(e)$.

$\Rightarrow ker(\phi)$ is closed.

2.$\phi(G)$ contains a non empty open subset of $\overline{\phi(G)}$, $\Rightarrow \phi(G)$ is closed.

3.Restricting ϕ to G^0, then by (2), $\phi(G^0)$ is closed. Since ϕ is a regular map, G^0 is irreducible, which implies $\phi(G^0)$ is irreducible. By the same argument, $\phi(G^0)$ is also connected. Therefore $\phi(G^0) \subseteq \phi(G)^0$. Since G^0 has finite index in G, $\phi(G^0)$ has finite index in $\phi(G)$. We have already seen that any closed subgroup of finite index contains G^0, thus $\phi(G)^0 \subset \phi(G^0)$, which gives $\phi(G^0) = \phi(G)^0$. □

2.1.6 Group structure on affine varieties

Theorem 2.1.11. *Let X be an affine algebraic set such that $(x, y) \to xy$ and $x \to x^{-1}$ are regular mappings. Then there exists a linear algebraic group G and a group isomorphism $\Phi : X \longrightarrow G$ such that Φ is also an isomorphism of affine algebraic sets.*

Proof. Let $\{f_1, , , f_n\}$ be the functions that generate $Aff(X)$. We have isomorphism from $Aff(X \times X)$ to $Aff(X) \otimes Aff(X)$. The map $\mu : X \times X \to X$ induces a map $\Delta : Aff(X) \to Aff(X) \otimes Aff(X)$ such that $\Delta(f_i) = \sum_{j=1}^{p} f'_{ij} \otimes f''_{ij}$. Then,

$$f_i(xy) = \sum_{j=1}^{p} f'_{ij} f''_{ij},$$

for $x, y \in X$. Set $W = Span\{f'_{ij} : 1 \leq i \leq n, 1 \leq j \leq p\}$ and $V = Span\{R(y)f_i : y \in X, 1 \leq i \leq n\}$, where $R(y)f(x) = f(xy)$ is the right translation representation. It is clear that $V \subset W$ and thus, $dimV < \infty$. By definition V is $R(y)$-invariant. We define the map $\Phi : X \to GL(V)$ by $\Phi(y) = R(y)|_V$. Φ is a group homomorphism. Now we shall show that Φ has following properties
1. Φ is injective.
If $R(y)f_i = f_i$ for all i, then $f_i(xy) = f_i(x)$ for every $x \in X$. Hence $y = 1$. Therefore $ker\Phi = \{1\}$ and Φ is injective.
2. Φ is a regular map.
Let $\delta_x \in V^*$, defined by $\delta_x f = f(x)$. Clearly $\{\delta_x\}_{x \in X}$ spans V^*. Choose $x_i \in X$ such that $\{\delta_{x_1}, , , , \delta_{x_m}\}$ forms the basis of V and $\{g_1, , , g_m\}$ be the dual basis. Then,

$$R(x)g_j = \sum_{i=1}^{m} c_{ij}(x)g_i,$$

for $x \in X$. Also

$$c_{ij}(x) = \langle R(x)g_j, \delta_{x_i}\rangle = g_j(x_i x)$$

Therefore the map $x \to c_{ij}(x)$ is a regular function, which implies Φ is a regular function.

3. $\Phi(X)$ is closed in $GL(V)$.
Since Φ is a regular homomorphism, $\Phi(X)$ is a closed subgroup of $GL(V)$. Set $G = \Phi(X)$, then $X \cong G$ as an abstract group.
4.Φ^{-1} is regular map.
Since Φ is regular $\Phi^*(Aff(G)) \subset Aff(X)$. Also, the set $\{f_1, , , f_n\} \subset \Phi^*(Aff(G))$. Thus, $Aff(X) \subset \Phi^*(Aff(G))$. Therefore $\Phi^*(Aff(G)) = Aff(X)$ and Φ^{-1} is regular.
This completes the proof. □

2.2 Lie algebra of algebraic groups

2.2.1 Left invariant vector field

A vector field Y on a linear algebraic group $G = GL(V)$ is said to be left invariant if

$$L(g)(Yf) = Y(L(g)f) \quad for\ f \in Aff(G)\ and\ g \in G,$$

where $L(g)f(x) = f(g^{-1}x)$ is the left representation of G on $Aff(G)$.
Now, for $A \in End(V)$, we define a linear transformation $X_A : Aff(G) \to Aff(G)$ such that

$$X_A f(x) = \frac{d}{dt} f(x(I + tA))|_{t=0} \quad where\ A \in End(V)$$

X_A has the following properties,

1. $X_A(fg) = X_A(f)g + fX_A(g)$, for $f, g \in Aff(G)$ i.e satisfies the Leibniz rule.
2. $L_g(X_A f(x))) = X_A(L_g f(x))$ for $g \in G$ and $f \in Aff(G)$

Thus we see that X_A is a left invariant vector field on G. In fact, we will prove in a while that every left invariant vector field on G is of the form X_A, where $A \in End(V)$. Now given X_A and X_B,

$$\begin{aligned} X_A X_B(fg) &= X_A(X_B(f)g + fX_B(g)) \\ &= X_A(X_B(f))g + X_B(f)X_A(g) + X_A(f)X_B(g) + fX_A(X_B(g)) \end{aligned}$$

$$\begin{aligned} X_B X_A(fg) &= X_B(X_A(f)g + fX_A(g)) \\ &= X_B(X_A(f))g + X_A(f)X_B(g) + X_B(f)X_A(g) + fX_B(X_A(g)) \end{aligned}$$

We see that $X_A X_B$ is not a derivation. But, subtracting one equation form other , we get

$$
\begin{aligned}
(X_A X_B - X_B X_A)(fg) &= X_A(X_B(f))g + fX_A(X_B(g)) \\
&\quad - X_B(X_A(f))g - fX_B(X_A(g)) \\
&= (X_A X_B - X_B X_A)(f)g + f(X_A X_B - X_B X_A)(g)
\end{aligned}
$$

Thus, the commutator defined as $[X_A, X_B] = X_A X_B - X_B X_A$ is a derivation and is also left invariant, i.e $[X_A, X_B]$ is a left invariant vector field on G.

Lemma 2.2.1. *$G = GL(V)$, where V is a finite dimensional vector space. Then, for $A, B \in End(V)$*

$$[X_A, X_B] = X_{[A,B]}$$

Moreover, any left invariant vector field on G is of the form X_A, for a unique $A \in End(V)$.

Proof. We know that $Aff(GL(V))$ is generated by restriction of polynomial to $GL(V)$ along with $(det)^{-1}$. Define function $f_M : GL(V) \to \mathbb{C}$, where $M \in End(V)$, such that $f_M(g) = trace(gM)$. The function f_M along with $(det)^{-1}$ also generate the algebra $Aff(GL(V))$. Consider a left invariant vector field Y on G.

$$(Y det^{-1})(g) = -det^{-2}(Y det)(g)$$

Since determinant is a polynomial function, the action of Y on $Aff(G)$ is entirely defined by its action on the function f_M, as M ranges over $End(V)$.
For $A, C \in End(V)$, we take the vector field X_A and the function f_C,

$$X_A(f_C(x)) = \frac{d}{dt} trace(x(I + tA)C)|_{t=0}$$

Since $trace$ is a continous function, we take the derivative inside the trace function, we get

$$X_A(f_C(x)) = f_{AC}(x)$$

Therefore, for $A, B, C \in End(v)$, we get

$$
\begin{aligned}
[X_A, X_B](f_C) &= X_A X_B(f_C) - X_B X_A(f_C) \\
&= f_{ABC} - f_{BAC} \\
&= X_{[A,B]} f_C
\end{aligned}
$$

So, we have proved the first part of the lemma. Now to prove the second part, we define a linear functional, which takes $B \to Y(f_B(I))$, where Y is any vector field on G and $B \in End(V)$. Then there exists an unique $A \in End(V)$ such that $Y(f_C)(I) = trace(AB)$.

Then, for a left invariant vector field Y and $g \in G$

$$\begin{aligned} Y(f_C)(g) &= (L(g^{-1})(Y(f_C)))(I) \\ &= (Y(L(g^{-1})f_C))(I) \\ &= (X_A(L(g^{-1})f_C))(I) \\ &= (L(g^{-1})(X_A(f_C)))(I) \\ &= (X_A(f_C))(g) \end{aligned}$$

This holds true for every $C \in End(V)$, thus we have $Y = X_A$. $\square$

Definition 5. *(Lie algebra) A vector space G is caled a lie algebra if it has a bilinear form, $G \times G \to \mathbb{C}$, which takes $(x, y) \to [x, y]$, with the following properties,*

1. For $x, y \in G$, $[x, y] = -[y, x]$.
2. For $x, y, z \in G$, it satisfies the jacobi identity, i.e $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$.

For a linear algebraic group $G \subseteq GL(V)$, we define a Lie algebra associated with G, denoted by $Lie(G)$,

$$Lie(G) = \{A \in End(V) : X_A(I_G) \subseteq I_G\}$$

Few things that we can readily infer from the definition is that $Lie(G)$ is a subset of $End(V)$ and $Lie(GL(V)) = End(V)$, since $I_{GL(V)} = 0$. In fact, we will now show that $Lie(G)$ forms a subspace of $End(V)$.

Lemma 2.2.2. *Given any linear algebraic group $G \subset GL(V)$, the corresponding lie algebra of G, $Lie(G)$ is a lie subgroup of $End(V)$.*

Proof. Take $A, B \in Lie(G)$. we have already seen that X_A depends linearly on A, hence $X_{\alpha A+\beta B} = \alpha X_A + \beta X_B$, where $\alpha.\beta \in \mathbb{C}$.

$$X_{\alpha A+\beta B}(I_G) = \alpha X_A(I_G) + \beta X_B(I_G) \subset I_G$$

Therefore, for $A, B \in Lie(G)$, $\alpha A + \beta B \in Lie(G)$. Also, $X_A X_B$ leaves I_G invariant, thus $X_{[A,B]} = [X_A, X_B] = X_A X_B - X_B X_A$ also leaves I_G invariant. Therefore, $[A, B] \in Lie(G)$, whenever $A, B \in Lie(G)$, which shows that $Lie(G)$ is indeed a lie subspace of $End(V)$. $\square$

Lemma 2.2.3. *Let G be a linear algebraic group. For every $g \in G$, the map $Lie(G) \to T(G)_g$ is an isomorphism. Hence we get $dim(Lie(G)) = dimG$ and G is a smooth algebraic set.*

Proof. For any $g \in G$, define the map $A \to X_A(g)$. First we will show the injectivity of the map. Suppose $X_A(g) = 0$, then

$$\begin{aligned}(X_Af)(x) &= (X_Af)(xg^{-1}g)\\ &= L_{gx^{-1}}(X_Af)(g)\\ &= X_A(L_{gx^{-1}}f)(g)\\ &= 0\end{aligned}$$

We get $(X_Af) = 0\ \forall f \in Aff(G)$, then $(X_Af_B)(I) = 0, \forall B \in End(V)$, which implies $trace(AB) = 0, \forall B \in End(V)$.Therefore, $A = 0$, which shows the injectivity.
Now, to show the surjectivity, it is enough to show it in case of $g = I$ i.e the surjectivity of map $Lie(G) \to T(G)_I$, the same will follow for every $g \in G$ due to the left invariance of X_A. Take a $v \in T(G)_I$, we define a linear functional $B \to v(f_B)$. Then there exists an unique $A \in End(V)$ such that $v(f_B) = trace(AB)$. Thus we get $v(f_B) = (X_Af_B)(I)$. By the derivation property of v, we get

$$\begin{aligned}v(f_Bf_C) &= v(f_B)f_C(I) + f_B(I)v(f_C)\\ &= trace(AB)trace(C) + trace(B)trace(AC)\\ &= X_A(f_B)f_C(I) + f_B(I)X_A(f_C)\\ &= X_A(f_Bf_C)(I)\end{aligned}$$

Since f_B generate $Aff(G)$ as B ranges over $End(V)$, we get that $v(f) = X_A(f)(I)$. Now we are left to prove that $A \in Lie(V)$. If $f \in I_G$, then $f(x) = 0, \forall x \in G$. $L_g(f)(x) = f(g^{-1}x) = 0, \forall x \in G$ (since $g^{-1}x \in G,\ \forall x \in G$), thus we get $L_g(f) \in I_G$.

$$\begin{aligned}X_A(f)(x) &= L_{x^{-1}}(X_Af)(I)\\ &= X_A(L_{x^{-1}}f)(I)\\ &= v(L_{x^{-1}}f)\\ &= 0\ \ (since\ v\ vanishes\ at\ I_G)\end{aligned}$$

Hence, $A \in Lie(G)$. □

2.2.2 Lie algebras of Classical groups

Lemma 2.2.4. *Let $G \subset GL(n, \mathbb{C})$ be an linear algebraic group and $z \to \phi(z)$ be a rational map from $\mathbb{C}$ to $M_n(\mathbb{C})$ such that $\phi(0) = I$ and $\phi(z) \in G$ for all expect finitely many non zero $z \in \mathbb{C}$. Then the matrix $A = \frac{d}{dz}\phi(z)|_{z=0}$ is in $Lie(G)$.*

Proof. Consider the curve $z \to I + zA$. $Det(I + zA) = 0$ is a polynomial in one variable, thus has finite number of solution. Therefore, $I + zA \in GL(n, \mathbb{C})$ except for finitely many

$z \in \mathbb{C}$ (which is precisely the roots of equation $det(I + zA) = 0$). Taking the derivative at $z = 0$, we get its equal to A. Also the curves $z \to \phi(z)$ and $z \to I + zA$ have the same same value at $z = 0$, thus both the curve have same tangent vector at $z = 0$ and both are in $GL(n, \mathbb{C})$ for all but finitely many nonzero $z \in \mathbb{C}$. For $f \in I_G$ and $g \in G$,

$$X_A f(g) = \frac{d}{dz} f(g(I + zA))|_{z=0} = \frac{d}{dz} f(g\phi(z))|_{z=0} = 0$$

, since $g\phi(z) \in G$. Hence $A \in Lie(G)$. □

Special linear group: We now define the lie algebra of the linear algebraic group $G = SL(n, \mathbb{C})$. Define $\mathfrak{sl}(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) : trace(A) = 0\}$.
The function defined as $f(g) = det(g) - 1$ clearly belongs to I_G. Thus, for $A \in m_n(\mathbb{C})$,

$$\begin{aligned} X_A f(g) &= \frac{d}{dz} f(g(I + zA))|_{z=0} \\ &= \frac{d}{dz} det(g(I + zA))|_{z=0} \\ &= det(g) \frac{d}{dz} det(I + zA)|_{z=0} \end{aligned}$$

Let M be a differentiable map from $\mathbb{C}$ to space of $n \times n$ matrices, then we have the standard result for derivative of determinant,

$$\frac{d}{dz} det(M(z)) = det(M(z)) trace(M(z)^{-1} \frac{d}{dz} M(z)).$$

Using this result, we get

$$\begin{aligned} \frac{d}{dz} det(I + zA)|_{z=0} &= det(I + zA) trace((I + zA)^{-1} \frac{d}{dz}(I + zA))|_{z=0} \\ &= det(I + zA) trace((I + zA)^{-1} A)|_{z=0} \\ &= det(I) trace(I^{-1} A) \\ &= trace(A) \end{aligned}$$

Thus, if $A \in Lie(G)$, then $X_A f(g) = det(g) trace(A) = 0$, for every $g \in G$, which implies $trace(A) = 0$ and $Lie(G) \subseteq sl(n, \mathbb{C})$. Now, notice that for $i \neq j$,$I + zE_{ij} \in G$, for every $z \in \mathbb{C}$. Therefore by previous lemma $E_{ij} \in Lie(G)$. It follows that $[E_{ij}, E_{ji}] = E_{ii} - E_{jj} \in Lie(G)$. E_{ij} and $E_{ii} - E_{jj}$ for $i \neq j$ spans $\mathfrak{sl}(n, \mathbb{C})$, Thus,$sl(n, \mathbb{C}) \subseteq Lie(G)$, which further implies that $Lie(G) = \mathfrak{sl}(n, \mathbb{C})$.

Orthogonal and symplectic groups :
Consider a non degenerate bilinear form $B(x, y)$ on $\mathbb{C}^n$ with Γ the corresponding matrix of

the bilinear form. Define,

$$G_\Gamma = \{g \in GL(n, \mathbb{C} : g^t \Gamma g = \Gamma)$$

As we have already seen, depending on whether B is symmetric or skew-symmetric, G_Γ is the orthogonal or symplectic group relative to B. In order to study the lie algebra of G_Γ, define Cayley transform $c(A)$,

$$c(A) = (I + A)(I - A)^{-1} \quad for\ A \in M_n(\mathbb{C})\ and\ det(I - A) \neq 0$$

Lemma 2.2.5. *Suppose* $A \in M_n(\mathbb{C})$ *and* $det(I - A) \neq 0$, *then* $c(A) \in G_\Gamma$ *if and only if* $A^t\Gamma + \Gamma A = 0$.

Proof.

$$\begin{aligned}(I - A^t)c(A)^t\Gamma c(A)(I - A) &= (I - A^t)((I + A)(I - A)^{-1})^t\Gamma(I + A)(I - A)^{-1}(I - A) \\ &= (I + A)^t\Gamma(I + A) \\ &= \Gamma + \Gamma A + A^t\Gamma + A^t\Gamma A\end{aligned} \tag{2.1}$$

$$\begin{aligned}(I - A^t)\Gamma(I - A) &= \Gamma - A^t\Gamma - \Gamma A + A^t\Gamma A \\ &= \Gamma - (A^t\Gamma + \Gamma A) + A^t\Gamma A\end{aligned} \tag{2.2}$$

If $c(A) \in G_\Gamma$, then equation 2.1 and 2.2 are equal, which impies $(A^t\Gamma + \Gamma A) = 0$. Similarly, if $(A^t\Gamma + \Gamma A) = 0$, then from equation 2.1 and 2.2, we get that $c(A) \in G_\Gamma$.

□

Theorem 2.2.6. *The lie algebra of* G_Γ *consists of all* $A \in M_n(\mathbb{C})$ *such that* $A^t\Gamma + \Gamma A = 0$.

Proof. We will denote $Lie(G_\Gamma)$ as $\mathfrak{g}_\Gamma$. For $B \in M_n(\mathbb{C})$, define a function $\Psi \in Aff(GL(n, \mathbb{C}))$

$$\Psi_B(g) = trace((g^t\Gamma g - \Gamma)B)\ , \quad for\ g \in GL(n, \mathbb{C})$$

Notice that $\Psi_B(g)$ vanishes on G_Γ. Suppose $A \in \mathfrak{g}_\Gamma$.
$\Psi_B \in I_{G_\Gamma} \Rightarrow X_A\Psi_B \in I_{G_\Gamma}$

$$
\begin{aligned}
\Rightarrow 0 &= X_A \Psi_B(I) \\
&= \frac{d}{dz}\Psi_B(I(I+zA))|_{z=0} \\
&= \frac{d}{dz} trace(((I+zA^t)\Gamma(I+zA)-\Gamma)B)|_{z=0} \\
&= \frac{d}{dz} trace(((I+zA^t)\Gamma(I+zA))B)|_{z=0} \\
&= \frac{d}{dz} trace(((\Gamma+zA^t\Gamma)(I+zA))B)|_{z=0} \\
&= \frac{d}{dz} trace((\Gamma+z\Gamma A+zA^t\Gamma+z^2A^t\Gamma A)B)|_{z=0} \\
&= trace((\Gamma A+A^t\Gamma)B)
\end{aligned}
$$

Hence, we get $trace((\Gamma A + A^t\Gamma)B) = 0$ for every $B \in M_n(\mathbb{C})$, which implie $A^t\Gamma + \Gamma A = 0$.
Conversely, let A be such that $A^t\Gamma + \Gamma A = 0$. Consider the rational curve $z \to c(zA)$, from $\mathbb{C}\ to\ M_n(\mathbb{C})$. Since $zA^t\Gamma + \Gamma zA = 0$ and $det(I - zA) \neq 0$ for all except finitely many $z \in \mathbb{C}$, by the Lemma 2.2.5 $c(zA) \in G_\Gamma$ for every except finitely many $z \in \mathbb{C}$.

$$\Rightarrow \frac{d}{dz}c(zA)|_{z=0} \in \mathfrak{g}_\Gamma$$

$$
\begin{aligned}
\frac{d}{dz}c(zA)|_{z=0} &= \frac{d}{dz}(I+zA)(I-zA)^{-1}|_{z=0} \\
&= A(I-zA)^{-1} + A(I+zA)(I-zA)^{-2}|_{z=0} \quad = A + A \\
&= 2A
\end{aligned}
$$

Therefore, $2A \in g_\Gamma$, which implies $A \in \mathfrak{g}_\Gamma$. $\square$

Consider the symmetric bilinear form (x, y) where $\Gamma = I$, the identity matrix. Then $G_\Gamma = O(n, \mathbb{C})$, with elements g such that $g^t g = I$. The subgroup $SO(n, \mathbb{C})$ is open in $O(n, \mathbb{C})$, then it has the same lie algebra as $O(n, \mathbb{C})$. We denote this by $\mathfrak{so}(n, \mathbb{C})$. By Theorem 2.2.6,

$$\mathfrak{so}(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) : A^t = -A\}$$

Suppose $n = 2l$. Then we denote by s_0 the $l \times l$ matrix,

$$
s_0 = \begin{bmatrix}
0 & . & . & . & 1 \\
0 & . & . & 1 & 0 \\
. & . & . & . & . \\
. & . & . & . & . \\
1 & . & . & . & 0
\end{bmatrix}
$$

Set

$$J_+ = \begin{bmatrix} 0 & s_0 \\ s_0 & 0 \end{bmatrix}, \ J_- = \begin{bmatrix} 0 & s_0 \\ -s_0 & 0 \end{bmatrix}$$

Now, we define the bilinear forms,

$$B(x,y) = (x, J_+y), \ \Omega(x,y) = (x, J_-y) \ \ for \ x, y \in \mathbb{C}^n$$

Using Theorem 2.2.6 and with some matrix calculation, we get the following corollaries.

Corollary 2.2.6.1. *The lie algebra $\mathfrak{so}(\mathbb{C}^{2l}, B)$ of $SO(\mathbb{C}^{2l}, B)$ consists of all matrices*

$$A = \begin{bmatrix} a & b \\ c & -s_0 a^t s_0 \end{bmatrix},$$

where $a \in gl(n, \mathbb{C})$ and b, c are $l \times l$ matrices such that $b^t = -s_0 b s_0$ and $c^t = -s_0 c s_0$.

Corollary 2.2.6.2. *The lie algebra $\mathfrak{sp}(\mathbb{C}^{2l}, \Omega)$ of $Sp(\mathbb{C}^{2l}, \Omega)$ consists of all matrices*

$$A = \begin{bmatrix} a & b \\ c & -s_0 a^t s_0 \end{bmatrix},$$

where $a \in gl(n, \mathbb{C})$ and b, c are $l \times l$ matrices such that $b^t = s_0 b s_0$ and $c^t = s_0 c s_0$.

Now, for $n = 2l + 1$, we define symmetric bilinear form on $\mathbb{C}^n$ as $B(x, y) = (x, Sy)$ for $x, y \in \mathbb{C}^n$, and

$$S = \begin{bmatrix} 0 & 0 & s_0 \\ 0 & 1 & 0 \\ s_0 & 0 & 0 \end{bmatrix}$$

Corollary 2.2.6.3. *The lie algebra $\mathfrak{so}(\mathbb{C}^{2l+1}, B)$ of $SO(\mathbb{C}^{2l+1}, B)$ consists of all matrices*

$$A = \begin{bmatrix} a & w & b \\ u & 0 & -w^t s_0 \\ c & -s_0 u^t & -s_0 a^t s_0 \end{bmatrix},$$

where $a \in gl(n, \mathbb{C})$, b, c are $l \times l$ matrices such that $b^t = -s_0 b s_0$ and $c^t = -s_0 c s_0$, w is $l \times 1$ matrix and u is a $1 \times l$ matrix.

2.2.3 Differential of a representation

Let $G \subset GL(n, \mathbb{C})$ be an linear algebraic group and $\mathfrak{g}$ be its lie algebra. Let (ϕ, V) be regular representation of G. For $C \in End(V)$, we define $f_C(B) = trace(BC)$, linear functional on $End(V)$. Then the pull back $f_C \circ \pi$ of f_C will be a regular function on G.

Theorem 2.2.7. *Given a linear algebraic group G with regular representation (π, V), there exists a unique linear map $d\pi : \mathfrak{g} \to End(V)$ such that*

$$X_A(f_C \circ \pi)(I) = f_{d\pi(A)C}(I),$$

for all $A \in \mathfrak{g}$, $C \in End(V)$. This map is a lie algebra homomorphism,

$$d\pi([A, B]) = [d\pi(A), d\pi(B)] \;\; for\; A, B \in \mathfrak{g}$$

Furthermore, for $f \in Aff(GL(V))$ and $A \in \mathfrak{g}$,

$$X_A(f \circ \pi)(I) = (X_{d\pi(A)}f) \circ \pi$$

Proof. For a fixed a $A \in End(V)$, we define a linear functional on $End(V)$ by

$$C \to X_A(f_c \circ \pi)(I)$$

Then, there exists a unique $D \in End(V)$ such that

$$X_A(f_C \circ \pi) = trace(CD)$$

Set $d\pi(A) = D$. Clearly $d\pi$ is a linear map, and we get

$$(X_A(f_C \circ \pi))(I) = f_{d\pi(A)C}(I)$$

Now we are left to show that its a lie algebra homomorphism.
Notice that $L(g^{-1})(f_C \circ \pi) = f_{C\pi(g)} \circ \pi$. Hence, using the fact that X_A is left invariant, we get

$$\begin{aligned} X_A f_C(\pi(g)) &= (L(g^{-1})X_A(f_C \circ \pi))(I) \\ &= (X_A L(g^{-1}))(f_C \circ \pi)(I) \\ &= (X_A(f_{C\pi(g)} \circ \pi))(I) \\ &= f_{d\pi(A)C\pi(g)}(I) \\ &= f_{d\pi(A)C}(\pi(g)). \end{aligned}$$

Thus, we get $X_A(f_C \circ \pi) = f_{d\pi(A)C} \circ \pi$.
Now, for A, B and $C \in End(V)$, using the above result we get,

$$\begin{aligned} [X_A, X_B](f_C \circ \pi) &= X_A(f_{d\pi(B)C} \circ \pi) - X_B(f_{d\pi(A)C} \circ \pi) \\ &= f_{d\pi(A)d\pi(B)C} \circ \pi - f_{d\pi(B)d\pi(A)C} \circ \pi \\ &= f_{[d\pi(A), d\pi(B)]C} \circ \pi. \end{aligned}$$

But, we have already seen $[X_A, X_B] = X_{[A,B]}$, which gives

$$[X_A, X_B](f_C \circ \pi) = f_{d\pi([A,B])C} \circ \pi.$$

Evaluating at I, we get $trace([d\pi(A), d\pi(B)]C) = trace(d\pi([A,B])C)$ for all $C \in End(V)$. Thus, we get the $d\pi$ is a lie algebra homomorphism.
Now to proves the final result. For a fixed $g \in G$ and for $f \in Aff(GL(V))$, we define a linear functional by

$$f \to (X_{d\pi(A)}f)(\pi(g)) - X_A(f \circ \pi)(g).$$

Notice that the above defined function is a tangent vector to $GL(V)$ at $\pi(g)$. Then, by definition of $d\pi(A)$, it vanishes on f_C, which gives us the required result. $\square$

Theorem 2.2.8. *Suppose G is a linear algebraic group with Lie algebra $\mathfrak{g}$. Let (π, V) be a regular representation of G.*
1. Suppose $W \subset V$ is a linear subspace such that $\pi(g)W \subset W$ for all $g \in G$. Then $d\pi(A)W \subset W$ for all $A \in \mathfrak{g}$.
2. Assume that G is connected. If $W \subset V$ is a linear subspace such that $d\pi(X)W \subset W$ for all $X \in \mathfrak{g}$ then $\pi(g)W \subset W$ for all $g \in G$.

Proof. 1. For $v \in V$ and $u^* \in V^*$, we define a linear function f_{v,u^*} on G by $f_{v,u^*}(g) = \langle u^*, \pi(g)v \rangle$. Let C be a rank one linear transformation on V defined by $C(g) = \langle v^*, r \rangle v$ for $r \in V$. Then, we have

$$f_{v,v^*}(g) = trace(\pi(g)C) = f_C \circ \pi(g)$$

For $A \in \mathfrak{g}$,

$$X_A f_{v,v^*}(g) = X_A f_C \circ \pi(g) = f_{d\pi(A)C} \circ \pi(g) = trace(\pi(g)d\pi(A)C) = f_{d\pi(A)v,v^*},$$

for all $g \in G$. Let $W \subset V$ be a G invariant subspace of V. Define

$$W^\perp = \{v^* \in V^* : \langle v^*, w \rangle \ \forall w \in W\}.$$

Then, $f_{w,v^*} = 0$ for all $w \in W, \ v^* \in W^\perp$ and $g \in G$. That implies $f_{w,v^*} \in I_G$ for $w \in W$ and $v^* \in V^*$. Thus, $X_A f_{w,v^*} \in I_G$ i,e

$$X_A(f_{w,v^*})(g) = f_{d\pi(A)w,v^*}(g) = 0$$

Set $g = I$, then we have $\langle v^*, d\pi(A)w \rangle = 0$ for all $w \in W$ and $v^* \in W^\perp$. Therefore, $d\pi(A)w \in W$, which proves the result.
2. Let $dimW = d$. Consider the dth exterior product $\wedge^d V \subset \otimes^d V$. This forms a G-module.

Let the basis of W be $\{w_1, , , w_d\}$ and set $\xi = w_1 \wedge .. \wedge w_d$. Then $\wedge^d W = \mathbb{C}\xi$. Hence W is a is G-invariant subspace if and only if $\mathbb{C}\xi$ is a G- invariant subspace.
Let W be invariant under $\mathfrak{g}$. Then for every $A \in \mathfrak{g}$, $A.\xi = \lambda(A)\xi$, where $\lambda \in \mathfrak{g}^*$. Let $\mu \in (\wedge^d V)^*$ such that $\mu(\xi) = 0$. If we prove $\mu(g.\xi) = 0$ for every $g \in G$, then we are done. Define $f(g) = \mu(g.\xi)$ for $g \in G$. Then $f \in Aff(G)$.

$$X_A f(g) = \frac{d}{dt}\mu(g(I + tA).\xi)|_{t=0} = \lambda(A)f(g), \ \ \forall\, A \in \mathfrak{g}$$

Define

$$\xi_\lambda = \{\phi \in Aff(G) : X_A(\phi) = \lambda(A)\phi, \ \ \forall\, A \in \}$$

Claim: $dim\xi_\lambda \leq 1$.
Suppose $\phi, 0 \neq \psi \in \xi_\lambda$. Now since G is connected, it is irreducible which implies $Aff(G)$ does not have any zero divisors. Therefore, ϕ/ψ is well defined rational function on G and

$$X_A(\phi/\psi) = (\lambda(A)\psi - \phi\lambda(A))/\psi^2 = 0,$$

for all $A \in \mathfrak{g}$. The vector fields $\{X_A : A \in \mathfrak{g}\}$ spans the tangent space $T(G)_g$ for all $g \in G$. Thus, $D(\phi/\psi) = 0$ for any derivation of quotient field of $Aff(G)$. Thus, we get ϕ/ψ is a constant function i.e $dim\xi_\lambda \leq 1$.
If $dim\xi_\lambda = 0$, then $f = 0$ and we are done. If $dim\xi_\lambda = 1$, take $0 \neq \psi \in \xi_\lambda$. For any $x \in G$, the function $y \to \psi(xy)$ also belongs to ξ_λ. Then $\psi(xy) = c(x)\psi(y)$, for $c(x) \in \mathbb{C}$. Set $y = 1$, we get $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$, that implies $\psi(g) \neq 0$ for all $g \in G$ (since we took $\psi \neq 0$). Then, $f = a\psi$, for some $a \in \mathbb{C}$. But $f(1) = 0$, which implies $a = 0$ and hence $f = 0$.
This completes the proof. □

Theorem 2.2.9. *Let G and H be linear algebraic groups. If $\pi : G \to H$ is a regular homomorphism, then $d\pi : (Lie(G)) \subset Lie(H)$ and $\rho : H \to K$ is another regular homomorphism, then $d(\rho \circ \pi) = d\rho \circ d\pi$.*

Proof. Let $f \in I_H$ and $A \in Lie(G)$. We know that $X_A(f \circ \pi) = (X_{d\pi(A)f})$. Then, for $h \in H$

$$\begin{aligned} X_A(f \circ \pi)(h) &= (X_{d\pi(A)f})(h) \\ &= L(h^{-1})(X_{d\pi(A)f})(I) \\ &= X_{d\pi(A)}(L(h^{-1})f)(I) \\ &= X_A((L(h^{-1})f) \circ \pi)(I) \end{aligned}$$

$L(h^{-1})f \in I_H$, thus $(L(h^{-1})) \circ \pi(g) = 0$ for all $g \in G$. Thus, we get $X_A(f \circ \pi)(h) = 0\, \forall\, h \in H$, which implies $d\pi(Lie(G)) \subset Lie(H)$.
For the second part of the theorem, set $\sigma = \rho \circ \pi$. Then $d\sigma(Lie(G)) \subset Lie(K)$. For

$A \in Lie(G)$ and $f \in Aff(K)$,

$$\begin{aligned}(X_{d\sigma(A)}f) &= X_A(f \circ \sigma)\\ &= X_A(f \circ \rho \circ \pi)\\ &= (X_{d\rho d\pi(A)}f)\end{aligned}$$

Thus, we get $X_{d\sigma(A)} = X_{d\rho d\pi(A)}$. Therefore, we get $d\sigma = d\rho \circ d\pi$. $\square$

2.2.4 The adjoint representation

Lemma 2.2.10. *Let $A \in Lie(G)$ and $g \in G$. Then $gAg^{-1} \in Lie(G)$.*

Proof. $R(g)f(y) = f(yg)$ for $y, g \in GL(n, \mathbb{C})$ and $f \in Aff(GL(n, \mathbb{C}))$. For $A \in M_n(\mathbb{C})$,

$$\begin{aligned}(R(g)X_AR(g^{-1})f)(y) &= (X_AR(g^{-1})f)(yg)\\ &= \frac{d}{dz}R(g^{-1})f(yg(I + zA))|_{z=0}\\ &= \frac{d}{dz}f(yg(I + zA)g^{-1})|_{z=0}\\ &= \frac{d}{dz}f(g(I + zgAg^{-1}))|_{z=0}\\ &= X_{gAg^{-1}}f(y)\end{aligned}$$

Suppose $A \in Lie(G)$ and $f \in I_G$, then $R(g^{-1})f \in I_G$. Thus, $X_AR(g^{-1})f \in I_G$, which implies $X_{gAg^{-1}}f \in I_G$. Therefore,we get $gAg^{-1} \in Lie(G)$. $\square$

Definition 6. *Define $Ad(g)A = aAg^{-1}$ for $g \in G$ and $A \in Lie(G)$. Then by previous lemma, $Ad(g) : Lie(G) \to Lie(G)$. The representation $(Ad, Lie(G))$ is called the adjoint representation of G. For $A, B \in Lie(G)$, $Ad(g)[A, B] = gABg^{-1} - gBAg^{-1} = gAg^{-1}gBg^{-1} - gBg^{-1}gAg^{-1} = [Ad(g)A, Ad(g)B]$ Thus, $Ad(g)$ is a lie algebra automorphism.*

2.2.5 Nilpotent and unipotent matrix

A matrix $A \in M_n(\mathbb{C})$ is called nilpotent if $A^n = 0$ for some positive integer n. A matrix $u \in M_n(\mathbb{C})$ is called unipotent if $(u - I)$ is nilpotent. Determinant of a unipotent element is always 1.
Let $A \in M_n(\mathbb{C})$ be nilpotent. Then, $A^n = 0$. We define exponential A as,

$$exp\ A = \sum_{k=0}^{n-1} \frac{1}{k!}A^k = I + Y,$$

where $Y = A + \frac{1}{2!}A^2 + ... + \frac{1}{(n-1)!}A^{n-1}$. Notice that Y is nilpotent. Thus, we get $exp\ A$ is unipotent. Conversely, let $u \in GL(n, \mathbb{C})$ be unipotent, then $u = I + Y$, where Y is nilpotent. We define $log\ u$ as,

$$log\ u = \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} Y^k.$$

Now, we intend to show that the nilpotent elements of $M_n(\mathbb{C})$ is in one to one correspondence with unipotent elements of $GL(n, \mathbb{C})$. For a nilpotent $A \in M_n(\mathbb{C})$ and $z \in \mathbb{C}$, we define the function

$$\phi(z) = log(exp\ zA)$$

$\phi(z)$ is a polynomial function as zA is nilpotent and $\phi(0) = 0$. Also

$$\frac{d}{dz}\phi(z) = A, \ \ for\ all\ z \in \mathbb{C}.$$

Thus, we get $log(exp\ zA) = zA$ for all $z \in \mathbb{C}$. With the same argument, we get $exp(log\ (I+zY)) = I + zY$. Thus, the exponential map is a bijective map from nilpotent elements in $M_n(\mathbb{C})$ to unipotent elements in $GL(n, \mathbb{C})$, with log as its inverse.

Lemma 2.2.11. *(Taylor's formula) Suppose $A \in M_n(\mathbb{C})$ is nilpotent and f is a regular function on $GL(n, \mathbb{C})$. Then there exists an integer k so that $(X_A)^k f = 0$, and*

$$f(exp\ A) = \sum_{m=0}^{k-1} \frac{1}{m!}(X_A)^m f(I).$$

Proof. Since $(exp\ zA)$ is a polynomial, the function $z \to f(exp\ zA)$ is also a polynomial in $z \in \mathbb{C}$. Then there exists a positive integer k such that

$$(\frac{d}{dz})^k f(exp\ zA) = 0.$$

Claim : $\frac{d}{dz}f(exp\ zA) = (X_A f)(exp\ zA)$ for $z \in \mathbb{C}$.
$\frac{d}{dz}$ and X_A are derivations, thus it is sufficient to check if the relation is satisfied by the generating set of regular functions. Recall that the set of functions $\{f_B\}$ as B ranges over $M_n(\mathbb{C})$ and $(det)^{-1}$ generate the regular functions on $GL(n, \mathbb{C})$. Take $f = f_B$ for some $B \in M_n(\mathbb{C})$, then

$$\begin{aligned} \frac{d}{dz}f_B(exp\ zA) &= \frac{d}{dz}trace((exp\ zA)B) \\ &= trace((exp\ zA)AB) \\ &= f_AB(exp\ zA) \\ &= X_A f_B(exp\ zA) \end{aligned}$$

Now, take $f = (det)^{-1}$. $exp\ zA$ is unipotent, thus $f(exp\ zA) = 1$. Also, $X_A f = -trace(A)f = 0$. Thus, $f = (det)^{-1}$ satisfies the relation.

□

Theorem 2.2.12. *Let $G \subset GL(n, \mathbb{C})$ be a linear algbraic group.*
1. Let $A \in M_n(\mathbb{C})$ be a nilpotent matrix. Then $A \in Lie(G)$ if and only if $exp\ A \in G$.
2. Suppose $A \in Lie(G)$ is a nilpotent matrix and (ρ, V) is a regular representation of G. Then $d\rho(A)$ is a nilpotent transformation on V and $\rho(exp\ A) = exp\ d\rho(A)$.

Proof. 1. Take $f \in I_G$. If $A \in Lie(G)$, then $X_A f \in I_G$. Thus, $(X_A)^n f \in I_G$ for all non negative integers n, which implies $(X_A)^n f(I) = 0$ for all $n \geq 0$. Therefore, by Taylor's formula (Lemma 2.2.11) $f(exp\ A) = 0$, implies $exp\ A \in G$. For the converse part, suppose $exp\ A \in G$. Then, define the polynomial map $z \to f(exp\ zA)$. Notice that f vanishes for all integral values of z i.e f has infinite solutions. Since f is a polynomial, it should then vanish for all values of $z \in \mathbb{C}$. Thus, by Taylor's formula, we get $X_A f(I) = 0$, which implies $A \in Lie(G)$.
2. Take $B \in End(V)$. Then, as we have already seen that f_B^ρ is a regular function on G. By Taylor's formula, there exists a positive integer such that

$$(X_A)^k f_B^\rho(I) = 0$$

Thus,

$$(X_A)^k f_B^\rho(I) = trace(d\rho(A)^k B) = 0.$$

This is satisfied for every $B \in End(V)$, hence $d\rho(A) = 0$. Now, applying Taylor's formula to function f_B^ρ, we get

$$\begin{aligned} trace(B\rho(exp\ A)) &= \sum_{m \geq 0} \frac{1}{m!} X_A^m f_B^\rho(I) \\ &= \sum_{m \geq 0} \frac{1}{m!} trace(d\rho(A)^m B) \\ &= trace(Bexp\ d\rho(A)) \end{aligned}$$

This holds for all $B \in End(V)$, hence $\rho(exp\ A) = exp\ d\rho(A)$. □

Chapter 3

Basic structure of Classical group

In this chapter we study the structure of the classical groups G and the associated lie algebra $\mathfrak{g} = Lie(G)$. We begin by introducing maximal torus of G. We show that for a classical group G, the subgroup of diagonal matrices H, is a maximal torus and any maximal torus of G is G-conjugate to H. Finally, we give the structure theorem for $Lie(G)$. By considering the adjoint action of H and $Lie(H) = \mathfrak{h}$ on the lie algebra $Lie(G)$, we obtain the root space decomposition of $Lie(G)$.

3.1 Semisimple and Unipotent elements

3.1.1 Semisimple and nilpotent elements

Definition 7. *(Maximal torus) An algebraic torus is an algebraic group T isomorphic to $(\mathfrak{C}^*)^l$, where l is the rank of T. Let G be a linear algebraic group, then a torus T is called maximal if it is not contained in any larger torus in G.*

Let G be one of the classical groups $GL(l, \mathbb{C})$, $SL(l+1, \mathbb{C})$, $Sp(\mathbb{C}^{2l}, \Omega)$, $SO(\mathbb{C}^{2l}, B)$, or $SO(\mathbb{C}^{2l+1}, B)$. Ω and B are the specific bilinear form defined in the subsection 2.2.2. Let H be the subgroup of diagonal matrices in G.

1. $G = Sl(l+1, \mathbb{C})$. Then

$$H = \{diag[x_1, ..., x_l, (x_1....x_l)^{-1}] : x_i \in \mathbb{C}^*\}.$$

and

$$Lie(H) = \{diag[a_1, ..., a_{l+1}] : a_i \in \mathbb{C}, \sum a_i = 0\}.$$

2. $G = Sp(\mathbb{C}^{2l}, \Omega)$ or $G = SO(\mathbb{C}^{2l}, B)$, then

$$H = \{diag[x_1, ..., x_l, x_l^{-1}, ..., x_1^{-1}] : x_i \in \mathbb{C}^*\}.$$

By Corollary 2.2.6.1 and 2.2.6.2, we have

$$Lie(H) = \{diag[a_1, ..., a_l, -a_l, ..., -a_1] : a_i \in \mathbb{C}\}.$$

3. $G = SO(\mathbb{C}^{2l+1}, B)$, then

$$H = \{diag[x_1, ..., x_l, 1, x_l^{-1}, ..., x_1^{-1}] : x_i \in \mathbb{C}^*\}.$$

By Corollary 2.2.6.3, we have

$$Lie(H) = \{diag[a_1, ..., a_l, 0, -a_l, ..., -a_1] : a_i \in \mathbb{C}\}.$$

Thus, in all the cases, H is isomorphic to an algebraic torus of rank l and

$$Aff(H) = \mathbb{C}[x_1, ..., x_l, x_1^{-1}, ..., x_l^{-1}]$$

Definition 8. *For an algebraic group K, a rational character of K is a regular homomorphism $\chi : K \to \mathbb{C}^*$. $\mathcal{X}(K)$ denote the set of all rational characters over K, and it have a abelian group structure under point wise multiplication.*

Lemma 3.1.1. *Let T be an algebraic torus of rank l. Then $\mathcal{X}(T)$ is isomorphic to $\mathbb{Z}^l$. Furthermore, $\mathcal{X}(T)$ is linearly independent as a set of functions over H.*

Proof. We can assume that $T = (\mathbb{C}^*)^l$. Now for $\lambda = [p_1, p_2, , , , p_l] \in \mathbb{Z}^l$ and $t = [t_1, t_2, , , , t_l]$, we define

$$t^\lambda = \prod_{i=1}^{l} t_i^{p_i}$$

Now, notice that the map $t \to t^\lambda$ is a rational character and we will denote it by χ_λ. The functions $t_i^{p_1}.t_2^{p_2}....t_l^{p_l}$ forms the basis of $Aff(T)$ and $t^{\lambda+\beta} = t^\lambda t^\beta$. Therefore, the map $\lambda \to \chi_\lambda$ is an injective homomorphism from $\mathbb{Z}^l$ to $\mathcal{X}(T)$. To show the surjectivity, we will show that every rational character over T is of the form χ_λ, for some λ. Consider a rational character χ. We define

$$\chi_k(t) = \chi(1, ..., t, ..., 1), \quad k = 1, 2, ..., l$$

where t occupies the k-th position. χ_k's are one dimensional regular representations of $\mathbb{C}^*$. Then $\chi_k(t) = t^{p_k}$ for some $p_k \in \mathbb{Z}$. Hence

$$\chi(t_1, ..., t_l) = \prod_{i=1}^{l} \chi_i(t_i) = \chi_\lambda(t_1, ..., t_l),$$

where $\lambda = [p_1, ..., p_l]$. Thus every rational character is of the form χ_λ for some $\lambda \in \mathbb{Z}^l$. $\square$

Theorem 3.1.2. *Let G be $GL(n,\mathbb{C}), SL(n,\mathbb{C}), SO(\mathbb{C}^n, B)$ or $Sp(\mathbb{C}^{2n}, \delta)$, and H is the subgroup of diagonal elements. Suppose $g \in G$ and $gh = hg \; \forall \; h \in H$, then $g \in H$.*

Proof. $G \subset GL(n,\mathbb{C})$. The action of H on the standard basis e_i for $\mathbb{C}^n$ is given by,

$$he_i = \theta_i(h)e_i \quad for \; 1 \leq i \leq n \; and \; h \in H$$

Suppose $v \in \mathbb{C}^n$ and $hv = \theta_i(h)v$ for some i and for all $h \in H$. We can write v as,

$$v = \sum_{i=1}^{n} \lambda_i e_i$$

Then, $hv = \sum_{j=1}^{n} \lambda_j \theta_j(h) e_j$. But,

$$hv = \theta_i(h)v = \sum_{j=1}^{n} \lambda_j \theta_i(h) e_j \;\; \forall \; h \in H.$$

Thus, $\lambda_j \theta_j(h) = \lambda_j \theta_i(h)$ for $1 \leq j \leq n$ and for all $h \in H$. But θ_i's are distinct, so we must have $\lambda_j = 0$ for $j \neq i$.
Assume that g commutes with H. Then,

$$hge_i = ghe_i = \theta_i(h)ge_i \;\; \forall \; h \in H, \; i = 1,,,,n$$

Thus, by the result we proved above, we have $ge_i = \lambda_i e_i$, i.e $g \in H$. □

Corollary 3.1.2.1. *Let G and H be as mentioned above. If $T \subset G$ be an abelian subgroup. If $H \subset T$, then $H = T$. In particular, H is a maximal torus in G.*

Theorem 3.1.3. *Let T be a torus. There exists an element $t \in T$ such that the subgroup generated by t is dense in T with respect to the Zariski topology.*

Proof. We can assume that $T = (\mathbb{C}^*)^l$. First we choose a $t \in T$ with coordinates $x_i(t) = t_i$ such that

$$\prod_{i=1}^{l} t_i^{p_i} \neq 1$$

for any $p_1, p_2, ..., p_l \in \mathbb{Z}$ with some $p_i \neq 0$ (we can simply choose elements which are algebraically independent over rationals).
Consider the subgroup $< t >$, generated by t. To show that $< t >$ is dense in T, we will show that for any $f \in Aff(T)$ if $f|_{<t>} = 0$, then f is identically equal to zero. Take $f \in \mathbb{C}[x_1, x_1^{-1},,, x_l, x_l^{-1}]$. Replacing f by $(x_1^{-1}...x_l^{-1})^r$, for a suitably large r, we can assume that $f \in \mathbb{C}[x_1,,,x_l]$. Then,

$$f(z) = \sum_{|K| \leq p} a_K z^K$$

Since we have $f(t^n) = 0 \; \forall \; n \in \mathbb{Z}$, we get

$$\sum_{|K| \leq p} a_K (t^K)^n = 0 \quad \forall \, n \in \mathbb{Z}$$

Claim: $\{t^K\}$ are all distinct.
Suppose not, then we have $t^k = t^m$ for $k \neq m$. Then, $t^{k-m} = 0$, which is a contradiction to the choice of t.
We enumerate the coefficients a_K as a_j and corresponding t_k as y_j. Then

$$\sum_{j=1}^{r} a_j (y_j)^n = 0 \quad for \; n = 0, 1, 2, , , r-1$$

Writing these equations in matrix form we get the coefficent matrix $V(y)$ as

$$\begin{bmatrix} y_1^{-1} & y_1^{-2} & . & . & y_1 & 1 \\ y_2^{-1} & y_2^{-2} & . & . & y_2 & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ y_r^{-1} & y_r^{-2} & . & . & y_r & 1 \end{bmatrix}$$

Now $det(V(y)) = \prod_{1 \leq i < j \leq r} (y_i - y_j)$. Since we have already proved that $y_i \neq y_j$ for $i \neq j$, the determinant in non zero. That implies that a_k's are zero for all k. $\square$

Definition 9. *(Semisimple element) Let G be a linear algebraic group over $\mathbb{C}$, then for $g \in G$, we say that g is semisimple if g is a diagonalizable matrix.*

Theorem 3.1.4. *Every semisimple element of G is G-conjugate to an element of H. Thus*

$$G_s = \bigcup_{g \in G} gHg^{-1}$$

Proof. We have $G \subset GL(n, \mathbb{C})$. Let $g \in G$ be a semisimple element. Then we have the eigen space decomposition,

$$\mathbb{C}^n = \oplus V_{\lambda_i}, \;\; gv = \lambda_i v \;\; for \; v \in V_{\lambda_i}$$

Let $\lambda_1, \lambda_2, , , \lambda_n$ be the eigen values of g (including the repeated eigen values) and let $v_1, v_2, , , , v_n$ be the corresponding eigen vectors. We will proceed to prove this theorem case by case.
For $G = GL(n, \mathbb{C})$ or $SL(n, \mathbb{C})$. Let $\{e_i\}$ be the standard basis of $\mathbb{C}^n$. We define $\mu v_i = e_i$. We can multiply v_1 by suitable constant so that $det\mu = 1$. Then, we have $\mu \in G$ such that $\mu g \mu^{-1} \in H$.

For orthogonal and symplectic group g preserves the bilinear form ω,

$$\omega(v,w)=\omega(gv,gw)=\lambda_i\lambda_j\omega(v,w)$$

where $v\in V_i$ and $w\in V_j$. That implies $\omega(v,w)=0$ when $\lambda_i\lambda_j\neq 1$. Since ω is a non degenerate bilinear form, we have $dimV_\lambda=dimV_{\lambda^{-1}}$, for $\lambda\in\mathbb{C}$. For $\lambda\neq\pm1$, we enumerate the eigen values of g as $\lambda_1,\lambda_2,,,,\lambda_{2r}$ such that $\lambda_i^{-1}=\lambda_{r+i}$. Set

$$W_i=V_{\lambda_i}\oplus V_{\lambda_i^{-1}}.$$

Then we have the following observations,
(1). $\mathbb{C}^n=V_1\oplus V_{-1}\oplus W_1\oplus....\oplus W_r$.
(2). restriction of ω to V_1, V_{-1} and W_i is non degenerate.
(3). $detg=(-1)^k$, where $k=dimV_{-1}$.
Take $G=Sp(\mathbb{C}^{2n},\delta)$. Then from (2), $dimV_1$ and $dimV_{-1}$ are even. We can find canonical symplectic basis for each of the subspaces in (1). The union of all this basis gives canonical basis for $\mathbb{C}^n$. We enumerate the basis as $v_1,,,v_n,v_{-1},,,v_{-n}$ such that

$$gv_i=\lambda_iv_i,\quad gv_{-i}=\lambda_i^{-1}v_{-i},\quad for\ i=1,2,...,n$$

Define μ such that $\mu v_i=e_i$ and $\mu v_{-i}=e_{2n+1-i}$. Then by Lemma 2.1.4, we have $\mu\in G$ and $\mu g\mu^{-1}\in H$.
Take G as the orthogonal group, $O(\mathbb{C}^n,B)$ or $SO(\mathbb{C}^n,B)$. Since $detg=1$, we have $dimV_{-1}$ is even from (3). Also $dimW_i$ is even. That implies $dimV_1$ is odd if and only if n is odd. Suppose $n=2l$, then $dimV_1=2r$. Like in the case of symplectic group, we get B isotropic basis for $\mathbb{C}^n$ and we enumerate the basis such that

$$gv_i=\lambda_iv_i\ \ gv_{-i}=\lambda_i^{-1}v_{-i}\ \ for\ i=1,2,...,l$$

Define μ such that $\mu v_i=e_i$ and $\mu v_{-i}=e_{n+1-i}$. By Lemma 2.1.1, we know that $\mu\in O(\mathbb{C}^n,B)$. We can interchange v_l and v_{-l}, if necessary to get $det\mu=1$. Thus,we have $\mu g\mu^{-1}\in H$.
For $n=2l+1$, we enumerate the basis such that

$$gv_0=v_0\ \ gv_i=\lambda_iv_i\ \ gv_{-i}=\lambda_i^{-1}v_{-i}\ \ for\ 1=1,2,...,l$$

Define μ such that $\mu v_0=v_0$, $\mu v_i=e_i$ and $\mu v_{-i}=e_{n+1-i}$. Then, by Lemma 2.1.1, $\mu\in O(\mathbb{C}^n,B)$. Replace μ by $-\mu$ if necessary so that $\mu\in SO(\mathbb{C}^n,B)$. Then, we have $\mu g\mu^{-1}\in H$. $\square$

3.1.2 Unipotent generators

For the group $G = SL(2, \mathbb{C})$, consider the subgroups consisting of unipotent elements of the form

$$N = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \quad \overline{N} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \quad a, b \in \mathbb{C}^*$$

Lemma 3.1.5. $SL(2, \mathbb{C})$ *is generated by* $N \cup \overline{N}$.

Proof. Elements of $SL(2, \mathbb{C})$ are of the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad ad - bc = 1$$

We will proceed in cases.
Case 1: When $a \neq 0$. Then we have $d = a^{-1} + a^{-1}cb$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a^{-1}c & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} 1 & a^{-1}b \\ c & 1 \end{bmatrix}$$

Case 2: When $a = 0$, then we have $c \neq 0$ and $b = -1/c$. Then

$$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} \begin{bmatrix} 1 & c^{-1}d \\ 0 & 1 \end{bmatrix}$$

If we show that $d(a) = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$ $for\ a \in \mathbb{C}^*$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is also generated by elements of N and $\overline{N}$, then we are done.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$d(a) = \begin{bmatrix} 0 & -a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a^{-1} - 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a - 1 & 1 \end{bmatrix}$$

Hence proved. ∎

Theorem 3.1.6. *Let* G *be* $SL(l+1, \mathbb{C}), SO(2l+1, \mathbb{C}), Sp(2l, \mathbb{C})$ *with* $l \geq 1$*, or* $SO(2l, \mathbb{C})$ *with* $l \geq 2$*. Then* G *is generated by its unipotent elements.*

Proof. We have $G \subset GL(n, \mathbb{C})$. Let G' be the subgroup generated by unipotent elements of the group G and H be the subgroup of diagonal elements.
Fact: Suppose $g \in G$ is unipotent i.e $(g - I)^n = 0$ for some $n \in \mathbb{Z}$, then any conjugate of

g is also unipotent.
For some $h \in G$,

$$\begin{aligned}(hgh^{-1} - I)^n &= (hgh^{-1} - hh^{-1})^n \\ &= h(g - I)^n h^{-1} \\ &= 0.\end{aligned}$$

By Jordan decomposition (cf. [H]), we know that any $g \in G$ can be expressed as product of a unipotent element and a semisimple element. Combining this with the fact that every semisimple element is the conjugate of diagonal element, it is enough to prove that the subgroup of diagonal elements $H \in G$ is generated by unipotent elements in order to prove that $G' = G$. We will proceed to prove this case by case using induction.

Case 1: $G = Sl(l + 1, \mathbb{C})$.
for $l = 1$, we have already showed that $SL(2, \mathbb{C})$ is generated by its unipotent elements. We assume the result holds true for $l = n - 1$.
For $l = n$, let H be the subgroup of diagonal elements of $SL(n, \mathbb{C})$ as mentioned above. For $h \in H$, let $h = diag[x_1, x_2, , , , x_n]$. Then, we write h as $h = h'h''$, where $h' = diag[x_1, x_1^{-1}, 1, 1, , , , 1]$ and $h'' = diag[1, x_1x_2, x_3, , , , x_n]$.
Let G_1 denote the subgroup of matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & I_{n-2} \end{bmatrix} \quad where\ a \in SL(2, \mathbb{C}).$$

We get $G_1 \cong SL(2, \mathbb{C})$.
Let G_2 denote the subgroup of matrices of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} \quad where\ b \in SL(n - 1, \mathbb{C}).$$

We get $G_2 \cong SL(n - 1, \mathbb{C})$.
Now $h' \in G_1$ and $h'' \in G_2$, thus by induction hypothesis h' and h'' and products of unipotent elements. Thus h is product of unipotent elements.
Therefore $H \subset G'$, which implies $G = G'$.

Case 2: $G = Sp(2l, \mathbb{C})$.
A skew symmetric bilinear form over complex is standard, thus it is enough to prove the result for any one skew symmetric bilinear form, as any other form will be equivalent to it.

We take the bilinear form

$$\omega = \begin{bmatrix} 0 & s_0 \\ -s_0 & 0 \end{bmatrix} \quad where \;\; s_0 = \begin{bmatrix} 0 & . & . & . & 1 \\ 0 & . & . & 1 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & . & . & . & 0 \end{bmatrix}$$

First we intend to show that $Sp(\mathbb{C}^2, \omega) = SL(2, \mathbb{C})$. For that consider $v, w \in \mathbb{C}^2$. Let $v = (v_1, v_2)$ and $w = (w_1, w_2)$. Then

$$\begin{aligned} \omega(v, w) &= [v_1, v_2] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\ &= v_1 w_2 - v_2 w_2 \\ &= det \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \end{aligned}$$

Thus, we have $\omega(v, w) = det[v, w]$. Now for $g \in GL(2, \mathbb{C})$, $\omega(gv, gw) = det(g)\omega(v, w)$. That implies $g \in Sp(\mathbb{C}^2, \omega) \iff det(g) = 1$. Therefore, $Sp(\mathbb{C}^2, \omega) = SL(2, \mathbb{C})$.
Now we apply induction on l. For $l = 1$, we have already proved that $Sp(\mathbb{C}^2, \omega)$ is generated by unipotent elements. Therefore result holds for $l = 1$.
We assume result holds for $l = n - 1$. Now for $l = n$, we have to show that $Sp(\mathbb{C}^{2n}, \omega)$ is generated by unipotent elements. For $h \in H$, $h = diag[x_1, x_2, , , , x_n, x_n^{-1}, , , , x_1^{-1}]$. Then we can write $h = h'h''$, where $h' = diag[x_1, 1, , , , 1, x_1^{-1}]$ and $h'' = diag[1, x_2, x_3, , , , x_3^{-1}, x_2^{-1}, 1]$. Define $V_1 = span\{e_1, e_{2l}\}$ and $V_2 = span\{e_2, e_3, , , , e_{2l-2}, e_{2l-1}\}$. We see that $\mathbb{C}^{2l} = V_1 \oplus V_2$. Restriction of ω to V_1 and V_2 is non degenerate, thus we define

$$G_1 = \{g \in G |\; gV_1 = V_1 \; and \; g = I \; on \; V_2\}$$

$$G_2 = \{g \in G |\; g = I \; on \; V_1 \; and \; gV_2 = V_2\}$$

Now observe that $G_1 \cong Sp(\mathbb{C}^2, \omega)$, $G_2 \cong Sp(\mathbb{C}^{2l-2}, \omega)$ and $h' \in G_1$ and $h'' \in G_2$, Thus by induction hypothesis both h' and h'' are product of unipotent elements, which implies that $H \subseteq G'$.

Case 3: $G = SO(2l, \mathbb{C})$ or $G = SO(2l + 1, \mathbb{C})$ for $l \geq 1$.
Since symmetric bilinear form over complex is standard, we only need to show the result

for any one bilinear form. Consider the bilinear form over $\mathbb{C}^n$

$$\delta = \begin{bmatrix} 0 & s_0 \\ -s_0 & 0 \end{bmatrix}$$

or

$$\delta = \begin{bmatrix} 0 & 0 & s_0 \\ 0 & 1 & 0 \\ -s_0 & 0 & 0 \end{bmatrix}$$

depending upon the parity of dimension n. First, we will show that $SO(\mathbb{C}^3, \delta)$ is generated by unipotent elements.

Take $G = SO(\mathbb{C}^3, \delta)$. We will define a homomorphism $\rho : \tilde{G} = SL(2, \mathbb{C}) \to G$ such that ρ takes $\tilde{H}$ (subgroup of diagonal elements of $\tilde{G}$) onto H. Set $V = \{X \in M_2(\mathbb{C}) : trace(X) = 0\}$ and the action of $\tilde{G}$ on V is defined by $\rho(g)X = gXg^{-1}$ (the adjoint representation of $\tilde{G}$).

Consider the symmetric bilinear form $w(x, y) = \frac{1}{2}trace(xy)$. Notice that w is non degenerate and invariant under $\rho(g)$, thus we have w-isotropic basis for V,

$$v_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad v_1 = \begin{bmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \quad v_{-1} = \begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix}$$

We identify V with $\mathbb{C}^2$ with the following identifications,

$$v_1 \to e_1 \quad v_0 \to e_2 \quad v_{-1} \to e_3$$

With this identification we have $w = \delta$, which implies $\rho(\tilde{G}) \subset O(\mathbb{C}^3, \delta)$. But since $\tilde{G}$ is generated by unipotent elements and ρ maps unipotent elements to unipotent elements, we have $\rho(\tilde{G}) \subset SO(\mathbb{C}^3, \delta) = G$. Now for $h \in \tilde{H}$, we have $h = diag[x, x^{-1}]$.

$$\rho(h)v_0 = v_0 \quad \rho(h)v_1 = x^2 v_1 \quad \rho(h)v_{-1} = x^{-2}v_{-1}$$

Therefore, v_0, v_1 and v_{-1} are the eigen vectors of $\rho(h)$. Thus, for any $h \in \tilde{H}$, where $h = diag[x, x_{-1}]$, we have $h' \in H$ such that $h' = diag[x, 1, x^{-1}]$, which implies $\rho(\tilde{H}) = H$. Thus $H \subset G'$, which implies $SO(\mathbb{C}^3, \delta)$ is generated by unipotent elements. Thus $SO(3, \mathbb{C})$ is generated by unipotent elements.

Now the next step is to show that $G = SO(4, \mathbb{C})$ is generated by unipotent elements. We follow the same procedure as before. Take $G = SO(\mathbb{C}^4, \delta)$ and we will define a homomorphism $\rho : \tilde{G} \to G$, where $\tilde{G} = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$, such that ρ maps $\tilde{H}$ onto H.

Set $V = M_2(\mathbb{C})$ and we define the action of $\rho(a, b)$ on V as $\rho(a, b)X = aXb^{-1}$, for

$(a, b) \in \tilde{G}$. Define a symmetric bilinear form B on V given by,

$$B(X, Y) = det(X + Y) - det(X) - det(Y)$$

B is a non degenerate bilinear form over V , thus we have B-isotopic basis of V, given by

$$v_1 = E_{11}\ ,\ v_2 = E_{12}\ ,\ v_{-1} = E_{22}\ ,\ v_{-2} = -E_{21}$$

Upon identification with $\mathbb{C}^4$ we get that $B = \delta$ and $\rho(\tilde{G}) \subset G$. Following the same procedure are before, we get that $\rho(\tilde{H}) = H$, thus proving the theorem for $G = SO(4, \mathbb{C})$. Now we consider the group $G = SO(n, \mathbb{C})$ for $n \geq 5$. Notice that $SO(2l, \mathbb{C})$ can be embedded inside $SO(2l + 1, \mathbb{C})$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{bmatrix}$$

The diagonal elements of $SO(2l, \mathbb{C})$ is isomorphic to diagonal elements of $SO(2l + 1, \mathbb{C})$. Therefore it is sufficient to prove that diagonal elements of $SO(l, \mathbb{C})$ is generated by unipotent elements when l is even.

Consider $G = SO(2l, \mathbb{C})$ for $2l \geq 4$. We have already seen that for $l = 2$, $SO(4, \mathbb{C})$ is generated by its unipotent elements. Assume that the result holds for $l = n - 1 \geq 3$.

For $l = n$, we have

$$h = diag[x_1, x_2, , , , x_n, x_n^{-1}, , , , x_1^{-1}] \quad for\ h \in H$$

We can write $h = h'h''$, where

$$h' = diag[x_1, x_2, 1, , , , , , 1, x_2^{-1}, x_1^{-1}]$$

$$h'' = diag[1, 1, x_3, x_4, , , , , , x_4^{-1}, x_3^{-1}, 1, 1]$$

Define $V_1 = span\{e_1, e_2, e_{2n-1}, e_{2n}\}$ and $V_2 = span\{e_3, e_4, , , , e_{2n-3}, e_{2n-2}\}$. Then $\mathbb{C}^{2n} = V_1 \oplus V_2$, and restriction of B to V_1 and V_2 is non degenerated. Set

$$G_1 = \{g \in G : gV_1 = V_1\ and\ g = I\ on\ V_2\}$$

Then $G_1 \cong SO(4, \mathbb{C})$.Define $W_1 = span\{e_1, e_{2n}\}$ and $W_2 = span\{e_2, e_3, , , , e_{2n-2}, e_{2n-1}\}$. Then $\mathbb{C}^{2n} = W_1 \oplus W_2$, and restriction of B to W_1 and W_2 is non degenerated. Set

$$G_2 = \{g \in G : g = I\ on\ W_1\ and\ gW_2 = W_2\}.$$

Then $G_2 \cong SO(2n-2, \mathbb{C})$. Now $2n-2 \geq 4$ and $h' \in G_1$, $h'' \in G_2$. Then by induction hypothesis both h' and h'' is product of unipotent elements, which implies $H \subset G'$. This completes the proof. $\square$

Theorem 3.1.7. *The algebraic groups $GL(n, \mathbb{C}), SL(n, \mathbb{C}), Sp(n, \mathbb{C})$ and $SO(n, \mathbb{C})$ are connected with respect to Zariski topology.*

Proof. We have already seen that $GL(n, \mathbb{C})$ is connected as its a principal open set. In the case of $SL(1, \mathbb{C})$ and $SO(1, \mathbb{C})$, both groups are trivial. For $SO(2, \mathbb{C})$, we have already seen that its isomorphic to $GL(1, \mathbb{C})$, which shows that its connected. For the rest of the groups, by the previous theorem we know that it's generated by its unipotent elements. It is sufficient to show that Aff(G) does not have any zero divisors. Let $f \in Aff(G)$ such that $f \neq 0$. Let $h \in Aff(G)$ such that $fh = 0$. We need to show that then $h = 0$. Let $g \in G$, then we have $g = g_1 g_2 .... g_n$, where each g_i is an unipotent element. Since we know that unipotent elements and nilpotent elements are in bijection by the exponential map, we can write $g = \exp X_1 \exp X_2 .... \exp X_n$, where each X_i is a nilpotent element. Define

$$\phi(t) = \exp X_1 t \exp X_2 t ..... \exp X_n t$$

$t \to \phi(t)$ is a regular function from $\mathbb{C}$ to G(since X_i's are nilpotent). Now $f \circ \phi \neq 0$ and $(f \circ \phi)(h \circ \phi) = 0$. Now since $\mathbb{C}$ is irreducible, $Aff(\mathbb{C})$ cannot have any zero divisors. Thus, we get $h \circ \phi = 0$, i.e $h(g) = 0$ for every $g \in G$. So $h = 0$. $\square$

3.2 Adjoint representation

3.2.1 Roots with respect to a Maximal torus(we have mainly focused on the cases when $G = Gl(n, \mathbb{C})$) and $G = SL(n, \mathbb{C})$)

Let G be a connected classical groups of rank l and $\mathfrak{g}$ be the corresponding lie algebra. H be the subgroups of diagonal elements and $\mathfrak{h}$ be its lie algebra. Let $x_1, , , x_l$ be the coordinate functions of H.
We are trying to find a basis for $\mathfrak{h}^*$, the set of linear functionals on $\mathfrak{h}$.

1.$G = GL(l, \mathbb{C})$.
Let ζ_i denote the functional on H defined by $\langle \zeta_i, A \rangle = a_i$. Then, we can easily see that the set $\{\zeta_1, , , \zeta_l\}$ forms the basis for $\mathfrak{h}^*$.

2.$G = SL(l+1, \mathbb{C})$.
Restricting ζ_i defined above to $\mathfrak{h}$,we see that ζ_i is an element of $\mathfrak{h}^*$. Every element of $\mathfrak{h}^*$ can be written as $\sum_{i=1}^{l+1} \lambda_i \zeta_i$. Also $\sum_{i=1}^{l+1} \lambda_i = 0$ (since every element in $\mathfrak{h}$ is such that

sum of diagonal elements is zero). Thus, we get the $\{\zeta - \frac{1}{l+1}(\zeta_1 + ... + \zeta_{l+1})$ forms the basis.

Let $\mathcal{X}(H)$ be the group of rational characters over H. We have already seen that $\mathcal{X}(H)$ is isomorphic to $\mathbb{Z}^l$. We can see characters as one dimensional representations, thus it make sense to talk about differential of characters.

Define $P(G) = \{d\theta : \theta \in \mathcal{X}(H)\} \subset \mathfrak{h}^*$. Given $\lambda = \lambda_1\zeta_1 + ... + \lambda_l\zeta_l$, where $\lambda_i \in \mathbb{C}$, let e^λ be a rational character defined by the $\lambda = (\lambda_1, , , , \lambda_l)$. Then we claim that

$$de^\lambda(A) = \langle \lambda, A\rangle, \quad for\ A \in \mathfrak{h} \tag{3.1}$$

Proof. We know that $X_A = \sum_{i,j} a_{ij}X_{E_{ij}}$. Here we have $a_{ij} = 0$ for $i \neq j$ and $X_{E_{ii}} = \sum_{r=1}^n x_{ri}\partial/\partial x_{ri}$. Thus we get for $A \in \mathfrak{h}$, X_A is defined by

$$X_A = \sum_{i=1}^{l} \langle \zeta_i, A\rangle x_i \frac{\partial}{\partial x_i} \tag{3.2}$$

on $\mathbb{C}[x_1, x_1^{-1}, , , x_l, x_l^{-1}]$.

We know form differential of representation, $X_A(f_C \circ \pi)(I) = f_{d\pi(A)C}(I)$, where (π, V) is a regular representation and $f_C \in End(V)$. Apply this result,

$$X_A(f_C e^\lambda)(I) = f_{de^\lambda(A)C}(I), \ \ where\ f_C \in End(\mathbb{C})$$

putting $f_C = 1$

$$X_A(e^\lambda(I)) = f_{de^{\lambda(A)}}(I)$$

$$X_A(e^\lambda)(I) = de^\lambda(A) \tag{3.3}$$

from 3.2 and 3.3, we have

$$\begin{aligned} de^\lambda(A) &= X_A(x_1^{\lambda_1}....x_l^{\lambda_l})(I) \\ &= \sum_{i=1}^{l} \langle \zeta_i, A\rangle x_i \frac{\partial}{\partial x_i}(x_1^{\lambda_1}....x_l^{\lambda_l})(I) \\ &= \sum_{i=1}^{l} \langle \zeta_i, A\rangle \lambda_i \\ &= \langle \lambda, A\rangle \end{aligned}$$

which proves our claim. □

From 3.1, we see that $P(G) = \bigoplus_{i=1}^{l} \mathbb{Z}_{\zeta_i}$. Thus, $P(G)$ is a free abelian group of rank l in $\mathfrak{h}^*$, and is called **weight lattice** of G.

Adjoint action of H and $\mathfrak{h}$ on $\mathfrak{g}$.

For $\alpha \in \mathfrak{h}^*$, let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [A, X] = \langle \alpha, A \rangle X, \ for\ every\ A \in \mathfrak{h}\}$$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, then α is called the root and $\mathfrak{g}_\alpha$ is the root space.If α is a root then a nonzero element of $\mathfrak{g}_\alpha$ is called a root vector of α. Let ϕ denote the set of roots of $\mathfrak{g}$, and is called the root system of $\mathfrak{g}$. Suppose α be a root and X be a root vector of α, then for every $A \in \mathfrak{h}$,

$$\begin{aligned}
&= AX^t - X^tA \\
&= A^tX^t - X^tA^t \ \ (since\ A\ is\ a\ diagonal\ matrix) \\
&= -[A, X]^t \\
&= -\langle \alpha, A \rangle X^t
\end{aligned}$$

Thus, $-\alpha$ is also a root with X^t as root space.
Remark: The classical groups are closed under transposition, thus if $X \in G$, then $X^t \in G$.

Root system for Classical groups

1. $G = GL(n, \mathbb{C})$
$\mathfrak{g} = M_n(\mathbb{C})$.Let E_{ij} be the basis of $\mathfrak{g}$. For $A = diag[a_1, , , a_l] \in \mathfrak{h}$,

$$[A, E_{ij}] = (a_i - a_j)E_{ij} = \langle \zeta_i - \zeta_j, A \rangle E_{ij}$$

The roots are $\{\zeta_i - \zeta_j : 1 \leq i, j \leq l, i \neq j\}$ and the root space $g_\lambda = \mathbb{C}E_{ij}$, where $\lambda = \zeta_i - \zeta_j$.

2. $G = SL(l + 1, \mathbb{C})$
Similar as the above calculation, The roots are $\{\zeta_i - \zeta_j : 1 \leq i, j \leq l + 1, i \neq j\}$ and the root space $g_\lambda = \mathbb{C}E_{ij}$, where $\lambda = \zeta_i - \zeta_j$.

3.2.2 Structure theorem for $\mathfrak{g}$

Theorem 3.2.1. *Let G be a classical group and $H \subset G$ be a maximal torus. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the lie algebra of G and H respectively. Let ϕ denote the root system of $\mathfrak{h}$ on $\mathfrak{g}$,*
1. If $\alpha \in \phi$, then $\alpha \in P(G)$, $dim\mathfrak{g}_\alpha = 1$ and $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \phi} \mathfrak{g}$.
2. If $\alpha \in \phi$ and $c\alpha \in \phi$ for $c \in \phi$. then $c = \pm 1$.
3. The symmetric bilinear form $(X, Y) = trace(XY)$ on $\mathfrak{g}$ is invariant i.e

$$([X,Y],Z) = -(Y,[X,Z]),$$

for $X,Y,Z \in \mathfrak{g}$.
4. Let $\alpha, \beta \in \phi$ and $\alpha \neq -\beta$. Then $(\mathfrak{h}, \mathfrak{g}_\alpha) = 0$ and $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.
5. The form (X,Y) on $\mathfrak{g}$ is non degenerate.

Proof. 1. Straight from the calculations we made for classical groups.

2. Straight from the calculations we made for classical groups.

3.
$$\begin{aligned}
([X,Y],Z) &= trace((XY-YX)Z) \\
&= trace(XYZ-YXZ) \\
&= trace(YZX-YXZ) \\
&= trace(Y(ZX-XZ)) \\
&= (Y,[Z,X]) \\
&= -(Y,[X,Z])
\end{aligned}$$

4. Take $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$ such that $\alpha \neq -\beta$. Then by (3), for $A \in \mathfrak{g}$, we have,

$$\begin{aligned}
0 &= ([A,X],Y) + (X,[A,Y]) \\
&= (\langle \alpha, A\rangle X, Y) + (X, \langle \beta, A\rangle Y) \\
&= \langle \alpha, A\rangle (X,Y) + \langle \beta, A\rangle (X,Y) \\
&= (\langle \alpha, A\rangle + \langle \beta, A\rangle)(X,Y) \\
&= \langle \alpha+\beta, A\rangle (X,Y)
\end{aligned}$$

Now, $\alpha+\beta \neq 0$. We can choose suitable $A \in \mathfrak{g}$ such that $\langle \alpha+\beta, A\rangle \neq 0$. This forces $(X,Y) = 0$.
Similarly, when we take $Y \in \mathfrak{h}$,

$$\begin{aligned}
0 &= ([A,X],Y) + (X,[A,Y]) \\
&= \langle \alpha, A\rangle (X,Y)
\end{aligned}$$

We can choose A suitably such that $\langle \alpha, A\rangle \neq 0$. This forces $(X,Y) = 0$.

5. We need to prove that restriction of trace form to $\mathfrak{h} \times \mathfrak{h}$ and $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$ are non degenerate for every $\alpha \in \phi$.
For $X, Y \in \mathfrak{h}$,

$$trace(XY) = \sum_{i=1}^{n} \zeta_i(X)\zeta_i(Y)$$

Thus trace form is non degenerate on $\mathfrak{h} \times \mathfrak{h}$.
Now for $\alpha \in \phi$, let $X_\alpha \in \mathfrak{g}_\alpha$. Then, $X_\alpha X_{-\alpha}$ is given by

$$X_{\zeta_i - \zeta_j} X_{\zeta_j - \zeta_i} = E_{ii},$$

for $1 \leq i < j \leq l + 1$.
Thus, the trace form is non degenerate on $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$. $\square$

Chapter 4

Intertwiners of arbitrary tensor product representations

Let $\mathfrak{g}$ be a simple finite dimensional lie algebra over $\mathbb{C}$. Let $\{V_i\}_{i=1}^n$ be a finite family of finite dimensional irreducible representations of $\mathfrak{g}$. Then $\otimes_{i=1}^n V_i$ is a finite dimensional $\mathfrak{g}$-module where, for $x \in \mathfrak{g}$ and $v_1 \otimes ..... \otimes v_n \in \otimes_{i=1}^n V_i$,

$$x.v_1 \otimes .... \otimes v_n = \sum_{i=1}^{n} v_1 \otimes .... \otimes x.v_i \otimes .... \otimes v_n.$$

By Weyl's complete reducibility theorem we know $\otimes_{i=1}^n V_i$ is completely reducible as a $\mathfrak{g}$-module. One of the engaging problems in this field is to determine the irreducible $\mathfrak{g}$-modules of $\otimes_{i=1}^n V_i$ or equivalently, to determine the dimension of $Hom_{\mathfrak{g}}(V, \otimes_{i=1}^n V_i)$, for any finite dimensional irreducible $\mathfrak{g}$-module V.
Many results along this direction have been proved [K]. Recently in [S], this problem has been studied using the representation theorey of the associated current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$.

4.1 Setup

Let $\mathfrak{g}$ be a finite dimensional simple lie algebra over $\mathbb{C}$ and $\mathbb{C}[t]$ be the polynomial algebra in one variable, then $\mathfrak{g} \otimes \mathbb{C}[t]$ is a lie algebra with the lie bracket defined as follows:

$$[x(P), y(Q)] = [x, y](PQ)\ ,\ \ for\ x, y \in \mathfrak{g}\ , P, Q \in \mathbb{C}[t],$$

where $x(P)$ denotes $x \otimes P$. This lie algebra is called the current algebra associated with $\mathfrak{g}$. Let $U(\mathfrak{g} \otimes \mathbb{C}[t])$ denote the universal enveloping algebra of $\mathfrak{g} \otimes \mathbb{C}[t]$. For $\theta = \sum_i x_1^i \otimes x_2^i \otimes .... \otimes x_k^i \in [\mathfrak{g}^{\otimes k}]^{\mathfrak{g}}$, and any $P_1, ..., P_k \in \mathbb{C}[t]$,

$$\theta(P_1, ..., P_k) = \sum_i x_1^i(P_1)....x_k^i(P_k)\ \in U(\mathfrak{g} \otimes \mathbb{C}[t]).$$

Lemma 4.1.1. *$\theta(P_1,,,P_k)$ commutes with $\mathfrak{g}$ i.e $[\mathfrak{g},\theta(P_1,,,,P_k)]=0$.*

Proof. Let $T(\mathfrak{g}\otimes\mathcal{A})$ denote the tensor algebra of $\mathfrak{g}\otimes\mathcal{A}$. Then we have the quotient map, $\pi: T(\mathfrak{g}\otimes\mathcal{A})\to U(\mathfrak{g}\otimes\mathcal{A})$. Consider $\tilde{\theta}(P_1,,,P_k)\in T(\mathfrak{g}\otimes\mathcal{A})$ defined by,

$$\tilde{\theta}(P_1,,,P_k)=\sum_i x_1^i(P_1)\otimes...\otimes x_k^i(P_k).$$

Now, for any $y\in\mathfrak{g}$,

$$[y,\tilde{\theta}(P_1,,,P_k)]=\sum_i\sum_{j=1}^k x_1^i(P_1)\otimes...\otimes[y,x_j^i](P_j)\otimes....\otimes x_k^i(P_k)$$

$\mathfrak{g}(P_1)\otimes....\otimes\mathfrak{g}(P_k)\cong\mathfrak{g}\otimes...\otimes\mathfrak{g}$ as a $\mathfrak{g}$-module and $[\mathfrak{g},\theta]=0$. Thus, we get

$$[y,\tilde{\theta}(P_1,,,P_k)]=0$$

Also, $\pi(\tilde{\theta}(P_1,,,P_k))=\theta(P_1,,,P_k)$. Therefore, we get $[\mathfrak{g},\theta(P_1,,,P_k)]=0$. □

Given a k-tuple of integers $(n_1,....,n_k)$, and $\theta\in[\mathfrak{g}^{\otimes k}]^{\mathfrak{g}}$, set

$$\theta(n_1,.....,n_k)=\theta(t^{n_1},.....,t^{n_k}).$$

Lemma 4.1.2. *For a finite dimensional simple lie algebra $\mathfrak{g}$, the subalgebra $[U(\mathfrak{g}[t])]^{\mathfrak{g}}$ of $U(\mathfrak{g}[t])$ is spanned by $\{\theta(n_1,,,n_k)\}$, where θ ranges over homogeneous elements in the basis of $[T(\mathfrak{g})]^{\mathfrak{g}}$ and for $k=deg\ \theta$, $(n_1,....,n_k)\in\mathbb{Z}_+^k$, such that $n_1\le n_2\le.....\le n_k$. Further, $[U(\mathfrak{g}[t])]^{\mathfrak{g}}$ is generated (as an algebra) by $\{\theta(n_1,,,n_k)\}$, where θ runs over homogeneous algebra generators of $[T(\mathfrak{g})]^{\mathfrak{g}}$ and for $k=deg\ \theta$, $(n_1,....,n_k)\in\mathbb{Z}_+^k$, such that $n_1\le n_2\le.....\le n_k$.*

Proof. We have the quotient map

$$\pi: T(\mathfrak{g}[t])\to U(\mathfrak{g}[t])$$

Notice that $\{t^n\}$, for $n\ge 0$ forms a homogeneous basis of $\mathbb{C}[t]$. Thus, $\mathfrak{g}[t]=\bigoplus_{n=0}^\infty\mathfrak{g}(n)$, where $\mathfrak{g}(n)=\mathfrak{g}\otimes t^n$. Thus,

$$\begin{aligned}T(\mathfrak{g}[t])&=\bigoplus_{k\ge0}\bigoplus_{n_1,,n_k\in\mathbb{Z}}\mathfrak{g}(n_1)\otimes..\otimes\mathfrak{g}(n_k)\\&\cong\bigoplus_{k\ge0}\bigoplus_{n_1,,n_k\in\mathbb{Z}}\mathfrak{g}^{\otimes^k}[n_1,,n_k],\ \ as\ \mathfrak{g}-modules\end{aligned}$$

where $\mathfrak{g}^{\otimes k}[n_1,.....,n_k]=\mathfrak{g}(n_1)\otimes......\otimes\mathfrak{g}(n_k)$. Now from the surjective homomorphism

π, we get the map

$$[T(\mathfrak{g}[t])]^{\mathfrak{g}} \to [U(\mathfrak{g}[t])]^{\mathfrak{g}}$$

Hence,

$$[T(\mathfrak{g}[t])]^{\mathfrak{g}} \cong \bigoplus_{k\geq 0} \bigoplus_{n_1,,n_k\in\mathbb{Z}} [\mathfrak{g}^{\otimes^k}]^{\mathfrak{g}}[n_1,,n_k]$$

Thus, we get that $\{\tilde{\theta}(n_1,,n_k)\}$ spans $[T(\mathfrak{g}[t])]^{\mathfrak{g}}$, where θ runs over homogeneous basis of $[T(\mathfrak{g})]^{\mathfrak{g}}$ and for $k = deg\ \theta$, n_i's are non negative integers ($\{\tilde{\theta}(n_1,,n_k)\}$ denote the element in $[T(\mathfrak{g}[t])]^{\mathfrak{g}}$ as in the proof of Lemma 4.2.1). From this surjective algebra homomorphism, we get that $\{\theta(n_1,,n_k)\}$ spans $[T(\mathfrak{g}[t])]^{\mathfrak{g}}$. Similarly we get that, $\{\theta(n_1,,,n_k)\}$ generate the algebra $[T(\mathfrak{g}[t])]^{\mathfrak{g}}$, as θ runs over the homogeneous algebra generators of $[T(\mathfrak{g})]^{\mathfrak{g}}$ and where $\{n_1,.....,n_k\} \in \mathbb{Z}_+^k$.

□

4.2 Main result

It is well known that there is a one-one correspondence between the finite-dimensional irreducible modules for a simple finite dimensional complex Lie algebra $\mathfrak{g}$ and the set of dominant integral weights $P^+(\mathfrak{g})$ of $\mathfrak{g}$.
For $\lambda \in P^+(\mathfrak{g})$, let $V(\lambda)$ be the corresponding finite dimensional irreducible $\mathfrak{g}$-module and for $(\lambda_1,.....,\lambda_k) \in (P^+(\mathfrak{g}))^k)$, let $V(\vec{\lambda}) = V(\lambda_1)\otimes......\otimes V(\lambda_k)$. Given $\vec{p} = (p_1,.....,p_k) \in \mathbb{C}^k$, one can consider $V(\vec{\lambda})$ as a $\mathfrak{g}[t]$-module by defining the $\mathfrak{g}[t]$ action on $V(\vec{\lambda})$ as follows:

$$x(P).(v_1 \otimes .... \otimes v_k) = \sum_{i=1}^{k} P(p_i) v_1 \otimes ... \otimes x.v_i \otimes ... \otimes v_k,$$

for $x \in \mathfrak{g}$, $P \in \mathbb{C}[t]$ and $v_i \in V_{\lambda_i}$. We denote such a $\mathfrak{g}[t]$-module by $V_{\vec{p}}(\vec{\lambda})$ and refer to them as evaluation modules for $\mathfrak{g}[t]$. It was proved in [C], that $V_{\vec{p}}(\vec{\lambda})$ is an irreducible $\mathfrak{g}[t]$-module if and only if $p_i \neq p_j$ for $i \neq j$. Using this fact, an alternative approach to tackle the tensor decomposition problem of the $\mathfrak{g}$-module $V(\vec{\lambda})$ was given in [S]. Let $V_{\vec{p}}(\vec{\lambda}) = \bigoplus_{\mu} V_{\vec{p}}(\vec{\lambda})[\mu]$ be the $\mathfrak{g}$-module decomposition of $V_{\vec{p}}(\vec{\lambda})$ into its isotypic components, where $V_{\vec{p}}(\vec{\lambda})[\mu]$ denote the $\mathfrak{g}$-isotypic component corresponding to $\mu \in P^+(\mathfrak{g})$. (Isotypic components of weight μ of a lie algebra module is the direct sum of all irreducible submodules which is isomorphic to highest weight module with weight μ.)
For $g \in \mathfrak{g}$, $u \in [U(\mathfrak{g}[t])]^{\mathfrak{g}}$

$$[g, u].(v) = g.(u.v) - u.(g.v), \quad for\ v \in V_{\vec{p}}(\vec{\lambda})$$

But, $[g, u] = 0$, Thus we get $g.(u.v) = u.(g.v)$, i.e the action of $\mathfrak{g}$ commutes with the action of $[U(\mathfrak{g}[t])]^{\mathfrak{g}}$ on $V_{\vec{p}}(\vec{\lambda})$. Thus, we get an action of $\mathfrak{g} \times [U(\mathfrak{g}[t])]^{\mathfrak{g}}$ on $V_{\vec{p}}(\vec{\lambda})$ stabilizing each isotypic components.

Theorem 4.2.1. *Let $\mathfrak{g}$ be a simple finite dimensional lie algebra. Then each isotypic components of $V_{\vec{p}}(\vec{\lambda})$, $V_{\vec{p}}(\vec{\lambda})[\mu]$, where μ is the highest weight, is an irreducible module of $\mathfrak{g} \times [U(\mathfrak{g}[t])]^{\mathfrak{g}}$.*

Proof. We will denote $V_{\vec{p}}(\vec{\lambda})$ by W and the isotypic component $V_{\vec{p}}(\vec{\lambda})[\mu]$ as $W[\mu]$. Choose a Borel subalgebra (cf. [H]) $\mathfrak{b}$ of $\mathfrak{g}$. Since $W[\mu]$ is $\mathfrak{g}$- module, we can see $W[\mu]$ as a $\mathfrak{b}$-module. Then, $W[\mu]$, as a $\mathfrak{b}$-module, decomposes into isotypic components. Let $W[\mu]^+$ be $\mathfrak{b}$-isotypic component. Then, $[U(\mathfrak{g}[t])]^{\mathfrak{g}}$ acts on $W[\mu]^+$. From the representation W, we have the ring homomorphism,

$$\psi : U(\mathfrak{g}[t]) \to End_{\mathbb{C}}(W)$$

Since W is an irreducible $\mathfrak{g}[t]$-module, by Burnside's theorem (cf. [L]), ψ is a surjective homomorphism. From ψ, we get the surjective homomorphism,

$$\psi^0 : [U(\mathfrak{g}[t])]^{\mathfrak{g}} \to End_{\mathfrak{g}}(W) \cong End_{\mathfrak{g}}(W[\mu]),$$

where $End_{\mathfrak{g}}(W)$ is the space of $\mathfrak{g}$-module endomorphisms of W. Taking the projection from $End_{\mathfrak{g}}(W)$ to $End_{\mathfrak{g}}(W[\mu])$, we get the surjective map,

$$\psi^0_\mu : [U(\mathfrak{g}[t])]^{\mathfrak{g}} \to End_{\mathfrak{g}}(W[\mu]) \cong End_{\mathbb{C}}(W[\mu]^+)$$

Since, the map is surjective, we get $W[\mu]^+$ is an irreducible module of $[U(\mathfrak{g}[t])]^{\mathfrak{g}}$. From this the theorem follows. □

From the above theorem we get that there is a correspondence between $Hom_{\mathfrak{g}}(V_{\vec{p}}(\vec{\lambda})[\mu], V_{\vec{p}}(\vec{\lambda}))$ and irreducible modules of $\mathfrak{g} \times [U(\mathfrak{g}[t])]^{\mathfrak{g}}$, where μ is the dominant integral weight of $\mathfrak{g}$, which gives an alternate approach to the study of the $\mathfrak{g}$-isotypic components of tensor products of finite dimensional $\mathfrak{g}$-modules.

Bibliography

[S] Shrawan Kumar, *A complete set of intertwiners for a arbitrary tensor product representations via current algebras*, Transformation Groups , Springer Science Business Media New York, 2017, no. 115-125.

[C] Chari, Vyjayanthi, Pressley, Andrew , *New unitary representations of loop groups*. Math. Ann.275, 1986, no. 1, 87–104.

[K] Shrawan Kumar, *Tensor Product Decomposition*, International Congress of Mathematicians, 2010.

[G] R. Goodman and N.R. Wallach, *Representations and invariants of the Classical Groups*, Cambridge University Press, 1998.

[H] J.E. Humphreys, *Introduction to Lie algebra and Representation theory*, Springer, 1980.

[L] Serge Lang, *Algebra*, Springer, 2002.