Modeling of Stability in Miscible Fluid System

Dharmendra Kumar

A dissertation submitted for the partial fulfillment

of BS-MS dual degree in Science

Indian Institute of Science Education and Research Mohali

April 2019

Certificate of Examination

This is to certify that the dissertation titled **"Modeling of Stability in Miscible Fluid System "** submitted by **Mr. Dharmendra Kumar** (Reg. No. **MS14112**) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Prof. Kapil H. Paranjape

Dr. Shane D'mello

Dr. Lingaraj Sahu (Supervisor)

Dr. Manoranjan Mishra (Co-Supervisor)

Dated: April 25, 2019

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Manoranjan Mishra at IIT Ropar and as Local Guide Dr. Lingaraj Sahu at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

Dharmendra Kumar
(Candidate)

Dated: April 25, 2019

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Lingaraj Sahu
(Supervisor)

Dr. Manoranjan Mishra
(Co-Supervisor)

Acknowledgement

I would like to thanks from the bottom of my heart to all those who supported me during the preparation this thesis. It is my privilege to had the opportunity to work with Dr. Manoranjan Mishra and Dr. Lingaraj Sahu.

I also would like to thank Anoop Bhaiya and Vandeeta Di(PhD Student at IIT Ropar) for their generous support and special thanks to one of my friend Mr. Sanjay Kapoor for his support and giving me his important time. Last but not least, I have immense indebtedness towards my parents, sisters,brother, all family members, and well wishers for their support over the years.

A list of person who truely care for me... Mummy and Papa.

To

My Parents

Smt. Sanju Devi

and

Shri Devnandan Prasad

Abstract

The goal of this thesis is to study the modeling of stability in miscible fluid system. In general, displacing fluid is less viscous than displaced fluid there form a unstable interface pattern between these two fluids in a porous media called Viscous Fingering. However in inverse case more viscous displacing the others the interface is stable and there is no pattern form. Chouke was the first who analyse the mathematical linear stability of displacement for two immiscible fluid by considering surface tension to act at the interface and found there is a cutoff wave number of the stability and when applying their theory to miscible case, there is no surface tension and diffusion this shows that the growth constant increases with wave number with no bound and this is physically unrealistic. Introduction of diffusion makes any base state profile time dependent. To determine the stability of time dependent flow there are following methods.

1. The quasi-steady-state approximation in which we freeze the time and determine the growth constant.

2. The Self-similar QSSA

3. nonmodal analysis

So in this problem we get coupled partial differential equation which we reduce into ordinary differential equation therefore we finally get the system of first order differential equation, which can be written as $\frac{dX}{dt} = A(t)X$. Where matrix A(t) determine the stability of the system. In case of normal matrix we get the exponential time dependent solution but in case of non-normal matrix it fails to predict the stability appropriately. Therefore to determine the non-normality of A(t) we define two

quantity Numerical Abscissa and Spectral Abscissa. We freeze at different different times and calculate the these two quantity. In case of normal matrix both Numerical Abscissa and Spectral Abscissa will be equal. Therefore at infinite time give the same results in both case modal analysis and nonmodal analysis, but at finite time it does not give the true information of stability in modal analysis of non-normal matrix. However in Nonmodal analysis it gives true information about stability.

Contents

Chapter 1

Prerequisites

In this chapter we are defining some physical definition. Since in this whole thesis we are mainly focus or study about stability of the system so we first defined stability and different types of stability. We also use some mathematical formula and methods like Cnetral difference formula we use this formula in 3rd chapter SSQSSA to discretize the ODEs. In chapter 2nd we get the 4th order ODEs(ordinary differential equation),to solve this ODES we use shooting method to get appropriate solutions. So further we describe in detail about Shooting method. We define about the Condition number. The role of condition number is very important to get information about matrix behaviour how it changes by perturbing the matrix. We use some theorem like Singular value decomposition (SVD), this can be used in Nonmodal analysis to get simpler form of propagator matrix.

1.1 Introduction

Definition 1.1.1. Viscous fingering In general, displacing fluid is less viscous than displaced fluid there form a unstable interface pattern between these two fluids in a porous media called Viscous Fingering. However in inverse case more viscous displacing the others the interface is stable and there is no pattern form.

Definition 1.1.2. Linearly stable (Infinitesimal disturbance) : When system is Stable to small disturbance called Linearly stable.

Definition 1.1.3. Non-linearly unstable : When system is Unstable to sufficiently large disturbance called Non-linearly unstable.

The method of linear stability analysis consists of introducing Sinusoidal disturbances on a basis state (also called background or initial state), which is the flow whose stability is being investigated. For example consider

$$v(x,t) = v^*(y)e^{ikx+imz-\sigma t}$$

where u^* is a complex amplitude and $\sigma = \sigma_r + i\sigma_i$.
For various σ_r systems behave differently;

$$\sigma_r < 0 : \quad stable,$$

$$\sigma_r > 0 : \quad unstable,$$

$$\sigma_r = 0 : \quad neutrally - stable.$$

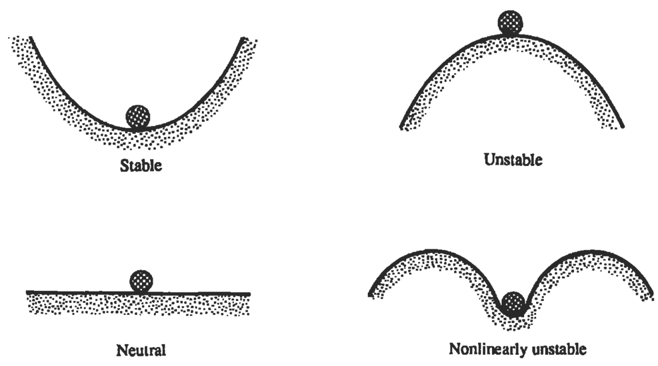

Figure 1.1: stable and unstable system

All things we defined above, for more detail here we can find [2] of chapter 12 (Instability). To study the stability of system we need various kind of concepts and methods like singular value decomposition (SVD) , condition number, central difference formula, shooting method etc.

1.2 Singular value Decomposition (SVD)

The singular value decomposition is a factorization of a matrix $A \in M_n(\mathbb{C})$. It is the generalization of the eigen-decomposition of a positive semi-definite normal matrix i.e its eigenvalue are non-negative.

Theorem 1.2.1. SVD : Let $A \in M_n(\mathbb{C})$ a matrix.The square root of the eigenvalues of A^*A, (where A^* is the conjugate transpose) is called Singular value.
Singular value decomposition (SVD) of a matrix A is given by;$A = UDV^*$, Where U and V are unitary matrices and D is diagonal matrices whose diagonal entries are singular values.

$$\text{For example : A} = \begin{bmatrix} 5 & 5 \\ -1 & -7 \end{bmatrix},$$

$$A^*A = \begin{bmatrix} 5 & -1 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 5 & 5 \\ -1 & -7 \end{bmatrix} = \begin{bmatrix} 26 & 18 \\ 18 & 79 \end{bmatrix}; \text{ eigenvalue of } A^*A \text{ is 20 and 80.}$$

$$\text{Hence } A^*A - 20I = \begin{bmatrix} 6 & 18 \\ 18 & 54 \end{bmatrix} \text{ therefore v} = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

$$\text{and } A^*A - 80I = \begin{bmatrix} -54 & 18 \\ 18 & -6 \end{bmatrix} \text{ therefore w} = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

$$\therefore V = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}, D = \begin{bmatrix} \sqrt{20} & 0 \\ 0 & \sqrt{80} \end{bmatrix}$$

$\because AV = UD;$

$$\therefore \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} = U \begin{bmatrix} \sqrt{20} & 0 \\ 0 & \sqrt{80} \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

1.3 Condition number[1]

For normal matrix condition number is the ratio of the largest to smallest singular value in the singular value decomposition of a matrix. A system is said to ill-condition if condition number is too large. If the system of equation given $Ax = b$;

$$A(x + \delta x) = b + \delta b$$
$$\frac{||\delta x||}{||\delta b||} * \frac{||b||}{||x||} \leq ||A|| * ||A^{-1}|| = K$$

called Condition number which denote what is the fractional change in solution to fractional change in b matrix. $(A + \delta A) * (x + \delta x) = b$

$$\frac{||\delta x||}{||\delta x + x||} * \frac{||A||}{||\delta A||} \leq ||A|| * ||A^{-1}||$$

this is also Condition Number which denote the fractional change in solution related to fractional change in matrix A

Since if A be a unitary matrix i.e $A^*A = AA^* = I$ then Condition number of unitary matrix is 1.

since in the whole thesis we are solving differential equation. So here is the some basic idea about differential equation ant its types.

• Differential equation : An equation that consist of derivatives is called differential equation. Differential equation are of two types :

1. Ordinary differential equation : A differential equation with one independent variable is called differential equation. Example : $3\frac{\mathrm{dy}}{\mathrm{dx}} + 5y^2 = 3x, y(0) = 5$.

2. Partial differential equation : A differential equation with more than one independent variable is called the partial differential equation. Example : $3\frac{\partial^2 y}{\partial x^2} + 2\frac{\partial^2 y}{\partial t^2} = x^2 + t^2$.

• Boundary Value Problem (BVP) : Suppose $y''(x) = f(x, y(x), y'(x))$ with $y(x_0) = y_0, y(x_1) = y_1$ boundary value problem.

• Initial Value Problem (IVP) : Let $y''(x) = f(x, y(x), y'(x))$ with $y(x_0) = y_0, y'(x_0) = a$ implies $y(x; a)$ denote the (IVP).

1.4 Central difference formula for derivatives

This formula is usually use to solve boundary value problem. Let $\phi : \mathbb{R} \to \mathbb{R}$ is a function and ϕ' is derivative of ϕ, ϕ is double derivative of ϕ.
$x_{i+1} = x_i + h$ and $x \in \mathbb{R}$

$$\phi'(x_i) = \frac{1}{2h}(\phi(x_{i+1}) - \phi(x_{i-1})) \quad for \quad i = 1, 2, 3.... \tag{1.1}$$

Where h is step size.

$$\phi''(x_i) = \frac{1}{h^2}(\phi(x_{i+1}) - 2\phi(x_i) + \phi(x_{i-1})) \quad for \quad i = 1, 2, 3.... \tag{1.2}$$

The boundary conditions played a vital role in finding the solution of a BVP and analyzing the dynamics of the solution.

1.5 Methods to solve boundary value problem

The numerical methods for solving boundary value problem may broadly be classified into the following three types:

1. Shooting methods: These are initial value problem methods. This method solve a boundary value problem by reducing it to the solution of an IVP. Here, we add sufficient number of condition at one end point and adjust these conditions until the required conditions are satisfied at the other end.

2. Difference methods: The differential equation is replaced by a set of difference equations which are solve by direct or iterative methods.

3. Finite element methods: The differential equation is replaced by using approximate methods with the piece-wise polynomial solution.

Here we only talk in detail about Shooting method, this method is more appropriate than others and converse rapidly faster than the others.

1.6 Shooting Method

1.6.1 Procedure to solve second order BVP using shooting method

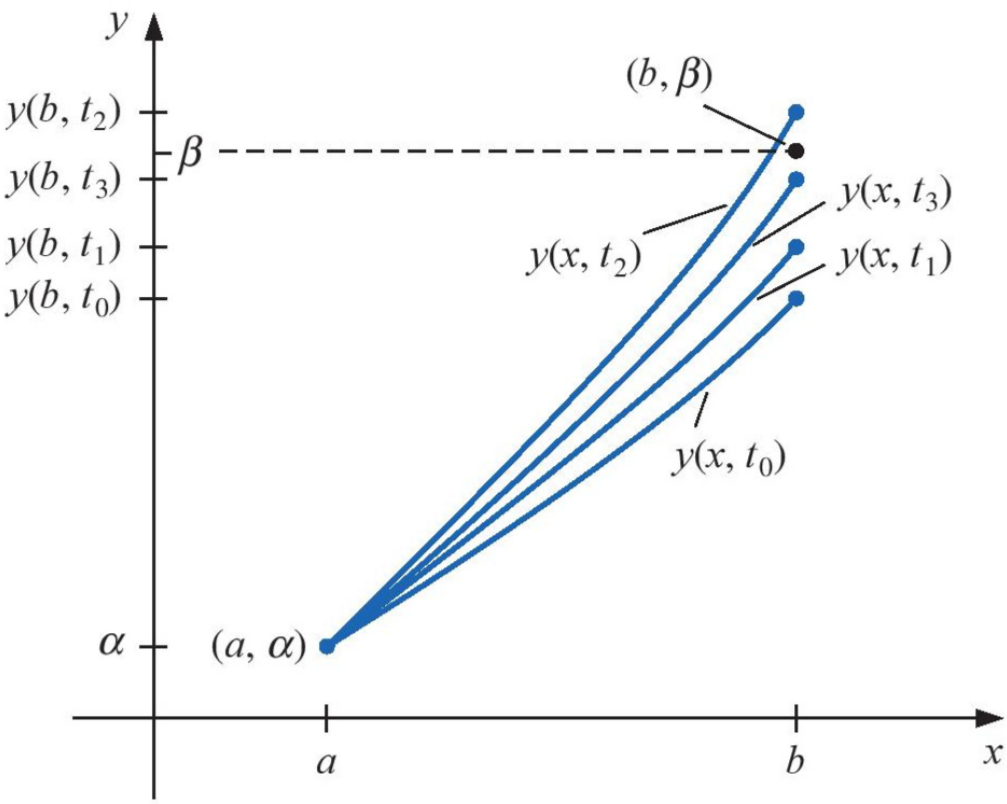

Figure 1.2:

The shooting technique to approximate the solution of non-linear second order BVP is that, the solution to the boundary value problem can be approximated by using the solutions to a sequence of initial-value problems involving a parameter t, known as shooting parameter. As the method superposition does not hold for non-linear ODEs, it is expected that we need more than two IVPs to solve $y'' = f(x, y, y'), x \in (a, b)$. Thus, to approximate the solution of boundary value problem, we need to covert it into corresponding initial value problem by taking shooting parameter t. Here, we will discuss it for three types of boundary conditions.

1. For first kind boundary condition (Dirichlet Boundary Condition): In this case, we need to solve the following IVP

$$y'' = f(x, y, y'); y(a) = \alpha; y_0(a) = t(shooting \quad parameter) \tag{1.3}$$

   the solution of the IVP will be of the form $y_t = y(x, t)$. Since this is approximated solution to boundary value problem, it should satisfy boundary condition at other end. So we need to choose the parameter, t, in a manner to ensure that

$$\phi_1(t) \equiv f(x, y, y'), y(b, t) - \beta = 0 \tag{1.4}$$

   Similarly, we can define it for Neumann and mixed type boundary conditions.

2. For Second kind of BVP (Neumann BC): In this case, we need to solve the following IVP

$$y'' = f(x, y, y'); y(a) = t; y'(a) = \alpha \tag{1.5}$$

   and solve up to $x = b$. Writing the solution of the IVP as $y_t = y(x, t)$, we need to choose the parameter s, in a manner to ensure that

$$\phi_2(t) \equiv y'(b, t) - \beta = 0 \tag{1.6}$$

3. For third kind of BVP (mixed BC): Here, the boundary conditions are $a_0 y(a) + a_1 y'(a) = \alpha$ and $b_0 y(b) + b_1 y'(b) = \beta$.

Thus, we can assume the value of $y(a)$ or $y'(a) = t$. Without loss of generality, let us assume $y'(a) = t$ then $a_0 y(a) + a_1 y0(a) = \alpha$ which gives $y(a) = \frac{\alpha - a_1 t}{a_0}$. In this case, we need to solve

$$y'' = f(x, y, y'); y(a) = \frac{\alpha - a_1 t}{a_0}, y'(a) = t \tag{1.7}$$

Writing the solution of the IVP as $y_t = y(x, t)$, we need to choose the parameter, t, in a manner to ensure that

$$\phi_3(t) \equiv b_0 y(b, t) + b_1 y'(b, t) - \beta = 0 : \tag{1.8}$$

Thus to solve the non-linear BVP, we need to find the root of non-linear equations, (1.4)-(1.6). Since $\phi(s)$ is an algebraic equation in s, we can use any root finding numerical method such as, Secant method or Newton- Raphson method. In our case we are using Newton-Raphson method.

Newton-Raphson Method for finding the root of $\phi(t) = 0$

$$t_{k+1} = t_k - \frac{\phi(t_k)}{\phi'(t_k)}$$

for $k = 1, 2, 3....$ Here t_0 is an initial approximation of Newton-Raphson method. The difficulty in Newton-Raphson method is to evaluate $\phi'(t)$.

Stopping criterion: $|\phi(t_{k+1})| < \epsilon$, $0 \leq \epsilon \ll 1$. Since in Newton Raphson formula at each iteration we have known every thing except $\phi'(t)$. So we need to find it.

How to calculate $\phi'(t)$? Let us calculate it for mixed kind boundary condition. As $y_t = y(x,t)) \Rightarrow y'_t = y'(x,t)$ and $y''_t = y''(x,t)$. Thus,

$$y'' = f(x, y_t, y'_t), y_t(a) = \frac{\alpha - a_1 t}{a_0} \quad and \quad y'_t(a) = t \tag{1.9}$$

and

$$\frac{\mathrm{d}\phi_3}{\mathrm{d}t} = b_0 \frac{\partial y_t}{\partial t} + b_1 \frac{\partial y'_t}{\partial t} \tag{1.10}$$

Hence for calculating the value of ϕ'_3, it is sufficient to calculate the value of $\frac{\partial y_t}{\partial t}$ and $\frac{\partial y'_t}{\partial t}$. Now, take partial derivative of Eq. (1.7) with respect to t, we have

$$\frac{\partial y''_t}{\partial t} = \frac{\partial f(x, y_t, y'_t)}{\partial t}$$

$$\Rightarrow \frac{\partial y''_t}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y_t}\frac{\partial y_t}{\partial t} + \frac{\partial f}{\partial y'_t}\frac{\partial y'_t}{\partial t} \tag{1.11}$$

Set, $v = \frac{\partial y_t}{\partial t}$. Then $v' = \frac{\partial y'_t}{\partial t}$ and $v'' = \frac{\partial y''_t}{\partial t}$. Then Eq.(1.9) can be rewritten as

$$v'' = \frac{\partial f}{\partial y_t} v + \frac{\partial f}{\partial y''_t} v', v(a) = \frac{-a_1}{a_0}, v'(a) = 1. \tag{1.12}$$

Thus IVP (1.10) can be solved step by step along with equation (1.7). When the computation of one cycle is completed $v(b)$ and $v'(b)$ will be found. Now, $\phi(t) = b_0 y(b,t) + b_1 y'(b,t) - \beta \Rightarrow \frac{\mathrm{d}\phi}{\mathrm{d}t} = b_0 \frac{\partial y_t}{\partial t} + b_1 \frac{\partial y'_t}{\partial t} \Rightarrow \frac{\mathrm{d}\phi}{\mathrm{d}t} = b_0 v(b) + b_1 v'(b)$. Thus we have finally the value of $\phi'(t)$. In case of Drichlet BC, we have $\frac{\mathrm{d}\phi}{\mathrm{d}t} = v(b)$.

Chapter 2

Stability of Miscible fluid system: using Quasi steady state approximation(QSSA)

MOTIVATION : Currently there are so many field of research in mixing in porous media that have many applications like in Oil extraction, chromatography etc. There are three method to analyse the stability of miscible fluid system in porous media. In this chapter we are discussing about QSSA. In this method we assume that the growth rate of the disturbance is much faster than the rate of change of the base state. We get the solution of having two time in the mathematical modeling, one is perturbation time and other is base state time. So we assume base state time is as frozen profile and we replace base state time t by constant t_0.

2.1 Basic formula

$$\nabla.V = 0 \tag{2.1}$$

$$\nabla p = -\mu V \tag{2.2}$$

$$\frac{Dc}{Dt} = D\nabla^2 c \tag{2.3}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V}.\nabla$.

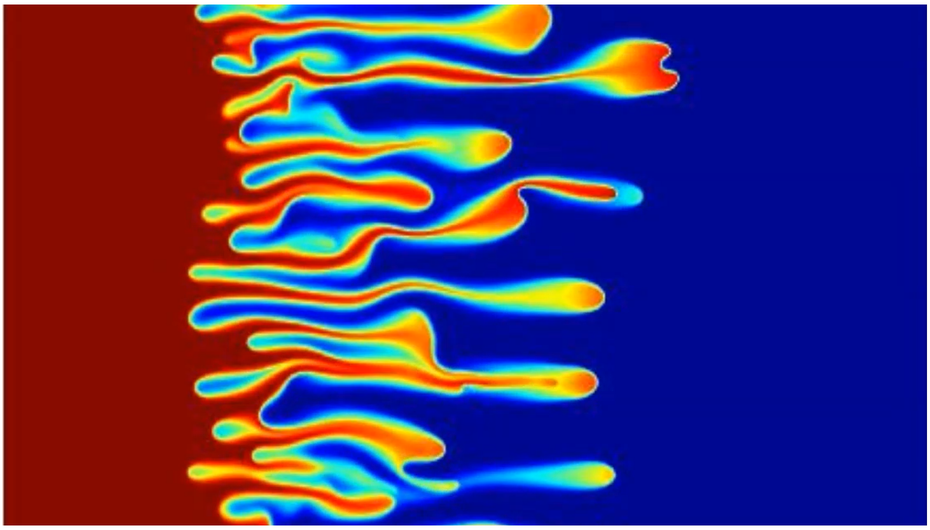

Figure 2.1: Suppose red fluid has viscosity $\mu_1$blue fluid has viscosity μ_2. Here $\mu_2 > \mu_1$. Therefore velocity of red fluid will be more so it will displace the blue fluid that is why at interface there make a pattern called viscous fingering.

In the above equation, c is the concentration of solvent, μ is the viscosity of the fluid divided by the permeability of the medium. Equation(2.1) is equation of continuity, equation(2.2) is Dercy's law while equation(2.3) is the Diffusion equation. Here is six unknown but we have only five equations therefore further we assume that viscosity is the function of concentration.

$$\mu = \mu(c) \tag{2.4}$$

Since the fluid is moving with constant velocity U, for convenient change it into a moving frame reference. $x = x_1 - Ut$.
Therefore Eqs. (2.1)-(2.3) can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2.5}$$

Where u, v, w are velocity in x, y, z direction respectively.

$$\frac{\partial p}{\partial x} = -\mu(u + U) = -\mu u - \mu U \tag{2.6}$$

$$\frac{\partial p}{\partial y} = -\mu v \tag{2.7}$$

$$\frac{\partial p}{\partial z} = -\mu w \tag{2.8}$$

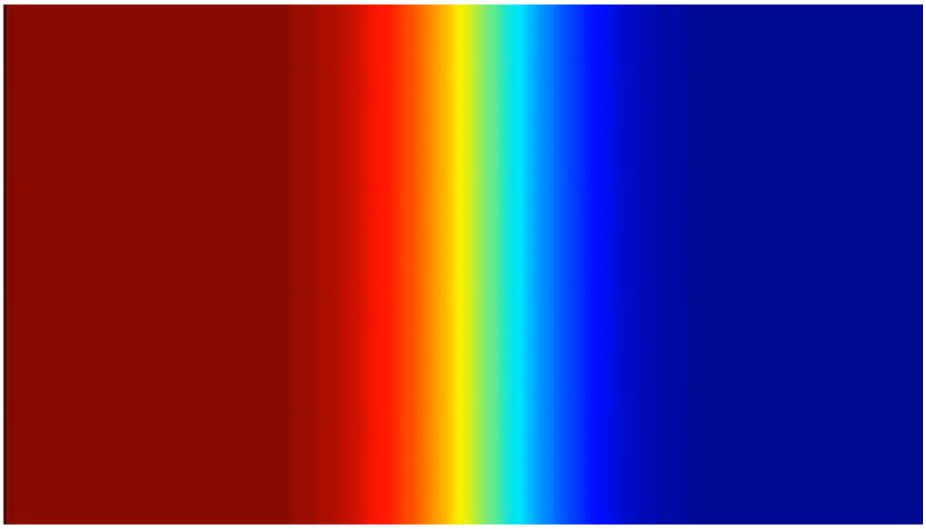

Figure 2.2: $\mu_2 < \mu_1$ here fluid will be only defuse due to concentration difference

Since $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V}.\nabla$.

Further we only look for system of 2-dimension, therefore we have ;

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial}{\partial y} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right) \tag{2.9}$$

2.2 Scaling

Since this experiment done in a very minor scale and there is no characteristic length and time so we scale length and time by $\frac{D}{U}$ and $\frac{D}{U^2}$ respectively. We also scales viscosity by the viscosity of the displacing fluids μ_1 and take $\mu_1 D$ as characteristic pressure. With these diffusive scales the dimensionless equations contain no parameters, thus the only parameter will be that entering the dimensionless viscosity-concentration relation. After doing the scaling we get following Equations.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.10}$$

$$\frac{\partial p}{\partial x} = -\mu u - \mu \tag{2.11}$$

$$\frac{\partial p}{\partial y} = -\mu v \tag{2.12}$$

$$\frac{\partial p}{\partial z} = -\mu w \tag{2.13}$$

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial}{\partial y} = (\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}). \tag{2.14}$$

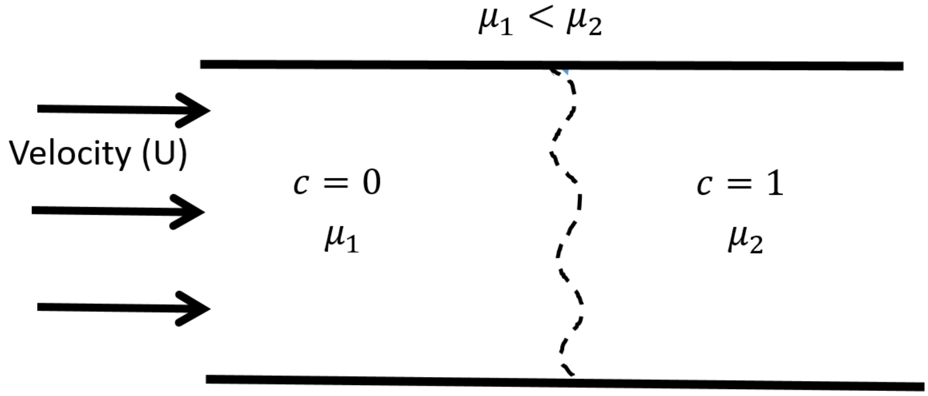

Figure 2.3: Figure show that there is system with the base uniform flow in the x_1-direction of two fluids with different viscosity μ_1 and μ_2 where $\mu_2 > \mu_1$ of concentration $c_1 = 0$ and $c_2 = 1$ with initial velocity U, Permeability k, Dispersion tensor D. Assuming flowing fluid is neutrally buoyant and incompressible.The medium homogeneous with a constant permeability and dispersion is isotropic.

2.3 Base-state solution

$$u = v = w = 0 \tag{2.15}$$

$$\mu_0 = \mu_0(c_0) = \mu_0(x, t) \tag{2.16}$$

Hence base state is time dependent. The time dependent of concentration is due to dispersion effect and since viscosity is dependent of concentration therefore viscosity is also time dependent. Concentration at bease state will be ;

$$c_0(x, t) = \frac{1}{2}[1 + \text{erf}(x/2\sqrt{t})], \tag{2.17}$$

where $\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \text{d}t$ is the error function.

2.4 Stability Analysis

Since we are interested in stability of this system so we do the small perturbation in the system. $(u, v, w, c, \mu, p) = (u, v, w, c, \mu, p)_{(base-state)} + (u, v, w, c, \mu, p)'$ Since the coefficient of equation which we get after perturbation are independent of y and z, so we can decompose the perturbation into Fourier component in the y , z components.

$$(u', c') = (\Phi, \Psi)e^{ik_y y}e^{ik_z z} \tag{2.18}$$

where k_y, k_z are the disturbance wave-number and Φ,Ψ are the function of x and t.

$$k^2 = k_y^2 + k_z^2 \tag{2.19}$$

Since we are considering 2-D system $k^2 = k_y^2$. Till now whatever we got the equation solve these using elimination method we get two coupled partial differential equations. Here below the mathematical explanation, how we get the perturb concentration.

$$(u_0, v_0, w_0, p_0, \mu_0) + (u', v', w', c', \mu').$$

$$u \to \epsilon u', \quad v \to \epsilon v', \quad c \to c_0 + \epsilon c' \text{ and } p \to p_0 + \epsilon p'.$$

$$\text{Since } \nabla . v = 0,$$

$$\begin{aligned} \frac{\partial \epsilon u'}{\partial x} + \frac{\partial \epsilon v'}{\partial y} &= 0 \\ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} &= 0. \end{aligned}$$

$$\begin{aligned} \text{Also } \frac{\partial p}{\partial x} &= \frac{\partial (p_0 + \epsilon p'}{\partial x} \\ &= -(\mu_0 + \mu'\epsilon) - (\mu_0 + \mu')u', \\ \text{implying } \frac{\partial p'}{\partial x}\epsilon &= -\mu'\epsilon - \mu_0 u'\epsilon - \mu' u'\epsilon^2. \end{aligned}$$

Since $p_0(x,t) = -\int^x \mu_0(x',t)dx', thus \frac{\partial p_0}{\partial x} = -\mu_0(x,t),$ and hence $\frac{\partial p'}{\partial x} = -\mu' - \mu_0 u'$.

The last equation implies that

$$\begin{aligned} \frac{\partial p'}{\partial y} = \frac{\partial (p_0 + \epsilon p')}{\partial y} &= -(\mu_0 + \epsilon\mu')\epsilon v' \\ \frac{\partial p_0}{\partial y} + \epsilon \frac{\partial p'}{\partial y} &= -\mu_0 \epsilon v' - \epsilon^2 \mu' v' \\ \frac{\partial p'}{\partial y} &= -\mu_0 v'. \end{aligned}$$

$$\text{Now, } \frac{\partial(c_0+\epsilon c')}{\partial t} + \epsilon u' \frac{\partial(c_0+\epsilon c')}{\partial x} + \epsilon v' \frac{\partial(c_0+\epsilon c')}{\partial y} = \frac{\partial^2(c_0+\epsilon c')}{\partial x^2} + \frac{\partial^2(c_0+\epsilon c')}{\partial y^2}$$

$$\epsilon\frac{\partial c'}{\partial t} + \frac{\partial c_0}{\partial t} + \epsilon u' \frac{\partial c_0}{\partial x} + \epsilon^2 u' \frac{\partial c'}{\partial x} + \epsilon v' \frac{\partial c_0}{\partial y} + \epsilon^2 v' \frac{\partial c'}{\partial y} = \frac{\partial^2 c_0}{\partial x^2} + \epsilon\frac{\partial^2 c'}{\partial x^2} + \frac{\partial^2 c_0}{\partial y^2} + \epsilon\frac{\partial^2 c'}{\partial y^2},$$

since $c_0(x,t)$ is independent of y, thus $\dfrac{\partial c'}{\partial t} + u' \dfrac{\partial c_0}{\partial x} = \dfrac{\partial^2 c'}{\partial x^2} + \dfrac{\partial^2 c'}{\partial y^2}.$

Since the coefficient of the equation are the independent of y so we can decompose the perturbation into fourier components in y-direction.

$$(u', c') = (\Phi, \Psi)\ \exp(ik_y y)$$

$$\text{implying } u' = \Phi\ \exp(ik_y y), \text{ and}$$

$$c' = \Psi\ \exp(ik_y y)$$

$$\text{Since } \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \text{ and}$$

$$\frac{\partial p}{\partial x} = -\mu - \mu_0 u.$$

$$\text{Thus } \frac{\partial}{\partial x}(\frac{\partial p}{\partial x}) = -\frac{\partial \mu}{\partial y} - u\frac{\partial \mu_0}{\partial y} - \mu_0\frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial x}(\frac{\partial p}{\partial y}) = -v\frac{\partial \mu_0}{\partial x} - \mu_0\frac{\partial v}{\partial x} \text{ (because of exact diffrentiation)}$$

$$-v\frac{\partial \mu_0}{\partial x} - \mu_0\frac{\partial v}{\partial x} = -\mu_0\frac{\partial u}{\partial y} - \frac{\partial \mu}{\partial y} \qquad (*)$$

$$\frac{\partial(\Psi e^{iky})}{\partial t} + \Phi e^{iky}\frac{\partial c_0}{\partial x} = \frac{\partial^2(\Psi e^{iky})}{\partial x^2} + \frac{\partial^2(\Psi e^{iky})}{\partial y^2}$$

$$\frac{\partial \Psi}{\partial t} + \Phi\frac{\partial c_0}{\partial x} = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} - \Psi k^2$$

$$\frac{\partial \Psi}{\partial t} - \frac{\partial^2 \Psi}{\partial x^2} + \Psi k^2 = -\Phi\frac{\partial c_0}{\partial x}$$

$$(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + k^2)\Psi = -\Phi\frac{\partial c_0}{\partial x}. \qquad (**)$$

Differentiating equation * w.r.t y we get

$$\frac{\partial \mu_0}{\partial x}\frac{\partial u}{\partial x} + \mu_0\frac{\partial^2 u}{\partial x^2} = -\mu_0\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial \mu_0}{\partial c_0}\frac{\partial c_0}{\partial x}\frac{\partial u}{\partial x} + \frac{1}{R}\frac{\partial \mu_0}{\partial c_0}\frac{\partial^2 u}{\partial x^2} + \frac{1}{R}\frac{\partial \mu_0}{\partial c_0}\frac{\partial^2 u}{\partial y^2} = 0$$

$$R\frac{\partial c_0}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

From equation *, we have

$$-v\frac{\partial \mu_0}{\partial c_0}\frac{\partial c_0}{\partial x} - \mu_0\frac{\partial v}{\partial x} = -\mu_0\frac{\partial u}{\partial y} - \frac{\partial \mu}{\partial y}$$

$$-vR\mu_0\frac{\partial c_0}{\partial x} - \mu_0\frac{\partial v}{\partial x} = -\mu_0\frac{\partial u}{\partial y} - \frac{\partial \mu}{\partial y}\frac{1}{\mu_0}$$

$$vR\frac{\partial c_0}{\partial x} + \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} + R\frac{\partial c}{\partial y},$$

differentiating w.r.t y, we get

$$R\frac{\partial^2 c_0 v}{\partial y \partial x} + \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 u}{\partial y^2} + R\frac{\partial^2 c}{\partial y^2}.$$

$$\text{Put } \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

$$-R\frac{\partial c_0}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} + R\frac{\partial^2 c}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + R\frac{\partial c_0}{\partial x}\frac{\partial u}{\partial x} + R\frac{\partial^2 c}{\partial y^2} = 0.$$

Since $u = \Phi e^{ik_y y}$, and

$c = \Psi e^{ik_y y}$, we have

$$\frac{\partial^2 \Phi}{\partial x^2} - k^2\Psi + R\frac{\partial c_0}{\partial x}\frac{\partial \Phi}{\partial x} = k^2\Psi R$$

$$(\frac{\partial^2}{\partial x^2} - k^2 + R\frac{\partial c_0}{\partial x}\frac{\partial}{\partial x})\Phi = k^2 R\Psi \quad \left(\text{because } \frac{1}{\mu}\frac{\partial \mu_0}{\partial c_0} = R\right)$$

2.5 Quasi-steady-state approximation (QSSA)

We assume that growth rate of the disturbances to be much faster than the rate of change of the base state. So adopting the QSSA, we replace the time t in the coefficient of founded above coupled equation by constant t_0 and treat this as frozen profile. So we define,

$$(\Phi, \Psi)(x,t) = (\phi, \psi)(x, t_0)e^{\sigma(t_0)t} \tag{2.20}$$

where σ is the quasi-static growth constant. it depend on the wave-number k and varies with the time t_0. After QSSA we get the two second order ordinary differential equation for which to solve we need viscosity, concentration relation. In general this relationship is very complicated so for our convenience we assume viscosity varies exponentially with concentration.

$$\frac{1}{\mu}\frac{d\mu}{dc} = R. \tag{2.21}$$

where R is a parameter determined by the mobility ratio $\alpha = \frac{\mu_2}{\mu_1}$.

$$\alpha = e^R. \tag{2.22}$$

We replace the time t in the coefficient of equation by constant t_o define:

$$(\Phi, \Psi)(x,t) = (\phi, \Psi)(x, t_0)e^{\sigma(t_0)t}$$

.

i.e $\Phi = \phi \mathrm{e}^{\sigma(t_0)t}, \Psi = \psi e^{\sigma(t_0)t}$. Therefore equation is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{1}{\mu_0}\frac{\partial \mu_0}{\partial x}(x,t)\frac{\partial}{\partial x} - k\right)\Phi = k^2\frac{\partial \mu}{\partial c}\frac{\Psi}{\mu(x,t)} \tag{2.23}$$

and

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + k^2\right)\Psi = -\frac{\partial c_0}{\partial x}(x,t)\Phi. \tag{2.24}$$

After substitution of Φ and Ψ in equation (2.12) and (2.13) we get

$$((\frac{\partial^2}{\partial x^2})e^{\sigma(t_0)t} + \frac{1}{\mu_0}\frac{\mu_0}{\partial x}(x,t_0)\frac{\partial\sigma(t_0)t}{\partial x} - k^2)\phi = k^2\frac{\partial\mu}{\partial c}\psi\frac{\sigma(t_0)t}{\mu_0(x,t_0)}$$

$$\implies (\frac{2}{\partial x^2} + \frac{1}{\mu_0}\frac{\partial}{\partial x}\mu_0(x,t_0)\frac{\partial}{\partial x} - k^2)\phi = \frac{k^2\frac{\mu}{\partial c}\psi}{\mu_0(x,t_0)}$$

$$\text{and } (\frac{\partial}{t} - \frac{\partial^2}{\partial x^2} + k^2)\psi(e^{\sigma(t_0)t}) = -\frac{\partial c_0}{\partial x}\phi(e^{\sigma(t_0)t})$$

$$\implies (\sigma(t_0) - \frac{\partial^2}{\partial x^2} + k^2)\psi = -\frac{\partial c_0}{\partial x}(x,t_0)\phi$$

Since $\mu = e^{RC} \implies \frac{\partial\mu}{\partial C} = Re^{RC} = R\mu$

$$\implies (\sigma(t_0) - \frac{\partial^2}{\partial x^2} + k^2)(\frac{\partial^2}{\partial x^2} + \frac{1}{\mu_0}\frac{\partial\mu_0}{\partial x}\frac{\partial}{\partial x} - k^2)\phi = \frac{-k^2}{\mu_0(x,t_0)}(\frac{\partial\mu}{\partial C})\frac{\partial c_0(x,t_0)}{\partial x}\phi$$

$$\implies (-\sigma(t_0) + \frac{\partial^2}{\partial x^2} - k^2)(\frac{\partial^2}{\partial x^2} + \frac{1}{\mu_0} \times \frac{\partial\mu}{\partial c_0} \times \frac{\partial c_0}{\partial x}\frac{\partial}{\partial x} - k^2)\phi = Rk^2\frac{\partial c_0(x,t_0)}{\partial x}\phi$$

Because $\mu = \mu(c)$ and $\frac{1}{\mu}\frac{\partial\mu}{\partial C} = R$, we have

$$(\frac{\partial^2}{\partial x^2} - k^2 - \sigma(t_0))(\frac{\partial^2}{\partial x^2} + R\frac{\partial c_0}{\partial x}\frac{\partial}{\partial x} - k^2)\phi = Rk^2\frac{\partial c_0(x,t_0)}{\partial x}$$

$$\implies \frac{\partial^4\phi}{\partial x^4} + R\frac{\partial c_0}{\partial x}\frac{\partial^3 x}{\partial x^3} - 2k^2\frac{\partial^2\phi}{\partial x^2} - R\frac{\partial c_0}{\partial x}\frac{\partial\phi}{x}(k^2+\sigma(t_0)) - \sigma(t_0)\frac{\partial^2}{\phi}\partial x^2 = \phi(Rk^2\frac{\partial c_0(x,t_0)}{\partial x} - k^4 - \sigma k^2)$$

we get forth order ordinary differential equation with boundary condition; Disturbance decay goes to zero as x tends to $\mp\infty$.

$$\left(\frac{d^2}{dx^2} - k^2 - \sigma(t_0)\right)\left(\frac{d^2}{dx^2} + R\frac{dc_0}{dx}(x,t_0)\frac{d}{dx} - k^2\right)\phi = Rk^2\frac{dc_0}{dx}(x,t_0)\phi. \tag{2.25}$$

Solve this Forth order ODE analytically for the growth rate at $t = 0$ and numerically using forth order Runge-kutta method for $t > 0$. Since base state concentration is a step function since the derivative of step function is Dirac-delta function. Hence

$$\frac{dc_0}{dx} = \delta(x). \tag{2.26}$$

Finally we got the relation between growth constant σ and wave-number k at $t = 0$;

$$\sigma = \frac{1}{2}[(Rk - k^2) - k(k^2 + 2Rk)^{1/2}]. \tag{2.27}$$

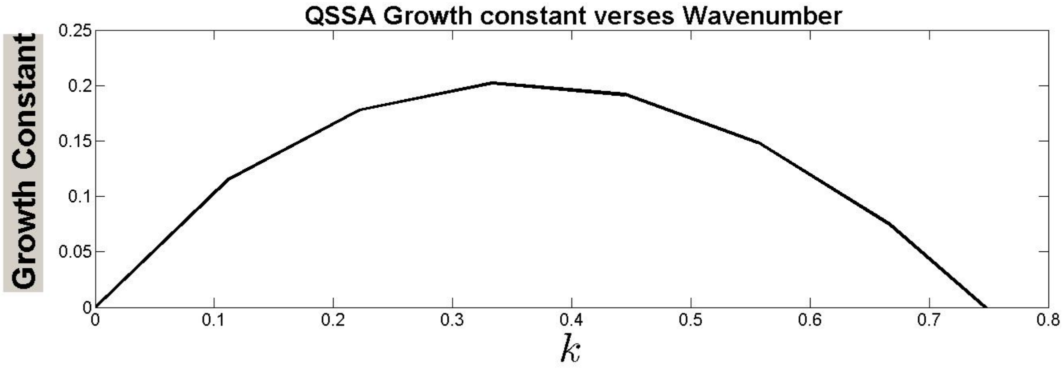

Figure 2.4: [3].

When we plot the disturbance concentration for $R = 3, k = 0$ at different time obtained from IVP in (x,t) coordinate system. It takes longer time to converge to dominant modes. So to solve these kind of problem we discuss SSQSSA which we discuss more about in next chapter.
when $\sigma = 0$ obtaining wave-number is called cutoff wave-number denoted as k_c.

$$k_c = R/4$$

and when growth constant is maximum obtaining wave-number is called dangerous mode $\sigma(max) \approx 0.0225R^2$.

2.6 Deriving Tan and Homsy dispersion relation analytically for step profile.

Since we have equation (2.14) with boundary condition disturbance decays is zero as $x \to \pm\infty$

$$\Big(\frac{d^2}{dx^2} - k^2 - \sigma(t_0)\Big)\Big(\frac{d^2}{dx^2} + R\frac{dc_0}{dx}(x,t_0)\frac{d}{dx} - k^2\Big)\phi = Rk^2\frac{dc_0}{dx}(x,t_0)\phi.$$

Analytically solution of equation (2.14) at $t = 0$.

$$c_0 = \begin{cases} 0, & x < 0 \\ 1, & x > 0. \end{cases}$$

Therefore

$$\frac{\mathrm{d}c_0}{\mathrm{d}x} = \delta(x). \tag{2.28}$$

Where

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0. \end{cases}$$

Therefore Eq.(2.14) at $x \neq 0$ becomes as;

$$\Big(\frac{d^2}{dx^2} - k^2 - \sigma\Big)\Big(\frac{d}{dx} - k^2\Big)\phi = 0. \tag{2.29}$$

Now applying boundary condition $\phi \to 0$ as $x \to \pm\infty$.

We get

$$\phi = c_1 e^{\rho x} + c_2 e^{-\rho x} + c_3 e^{kx} + c_4 e^{-kx}, \quad where \quad \rho^2 = \sigma + k^2.$$

Applying decaying boundary conditions,

$$\phi = \begin{cases} c_1 e^{\rho x} + c_3 e^{kx}, & x < 0 \\ c_2 e^{-\rho x} + c_4 e^{-kx}, & x > 0. \end{cases}$$

for convenience let $c_1 = A_1, c_2 = A_2, c_3 = B_1$ and $c_4 = B_2$.

Therefore

$$\phi = \begin{cases} A_1 e^{\rho x} + B_1 e^{kx}, & x < 0 \\ A_2 e^{-\rho x} + B_2 e^{-kx}, & x > 0. \end{cases} \tag{2.30}$$

To solve above problem we need four boundary conditions which are following;

now integrate Eq.(2.14) from 0+ to 0- we get using Eq.(2.17)

$$\int_{0-}^{0+} \Big(\frac{d^2}{dx^2} - k^2 - \sigma\Big)\Big(\frac{d^2}{dx^2} + R\delta(x)\frac{d}{dx} - k^2\Big)\phi dx = Rk^2 \int_{0-}^{0+} \delta(x)\phi dx.$$

Since disturbance ϕ and ψ are continuous at zero. Therefore by disturbance velocity continuity and disturbance concentration continuity around the interface,

$$\phi(0^+) = \phi(0^-)$$

and

$$\psi(0^+) = \psi(0^-)$$

also,

$$p'(0^+) = p'(0^-)$$

Since

$$\begin{aligned}
\frac{\partial p'}{\partial x} &= -\mu' - \mu_0 u' \\
&= -\frac{\mathrm{d}\mu}{\mathrm{d}c}|_{c_0} c' - \mu_0 u' \\
&= -\frac{\mathrm{d}\mu}{\mathrm{d}c}|_{c_0} \psi e^{ik+\sigma t} - \mu_0 \phi e^{ik+\sigma t} \\
&= -e^{ik+\sigma t}(\frac{\mathrm{d}\mu}{\mathrm{d}c}|_{c_0}\psi + \mu_0\phi) \\
&= -e^{ik+\sigma t}(\frac{\mu_0}{k^2}(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{\mu_0}\frac{\mathrm{d}\mu_0}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x} - k^2)\phi + \mu_0\phi).
\end{aligned} \tag{2.31}$$

Now integrating Eq.(2.31) both sides w.r.t x from 0^- to 0^+ we get,

$$p'(0^+) - p'(0^-) = \int_{0^-}^{0^+} -e^{ik+\sigma t}\left(\frac{\mu_0}{k^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{\mu_0}\frac{\mathrm{d}\mu_0}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x} - k^2\right)\phi + \mu_0\phi\right)dx$$

$$\Rightarrow 0 = -e^{ik+\sigma t}\int_{0^-}^{0^+}\left(\frac{\mu_0}{k^2}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \frac{1}{k^2}\frac{\mathrm{d}\mu_0}{\mathrm{d}x}\frac{\mathrm{d}\phi}{\mathrm{d}x} - \mu_0\phi + \mu_0\phi\right)dx$$

by using ILATE rule of integration.

$$\Rightarrow \mu_0(0^+)\frac{\mathrm{d}\phi}{\mathrm{d}x}|_{0^+} = \mu_0(0^-)\frac{\mathrm{d}\phi}{\mathrm{d}x}|_{0^-}$$

$$\Rightarrow \alpha\frac{\mathrm{d}\phi}{\mathrm{d}x}|_{0^+} = \frac{\mathrm{d}\phi}{\mathrm{d}x}|_{0^-} \quad where \quad \alpha = \frac{\mu_0(0^+)}{\mu_0(0^-)} = \frac{\mu_2}{\mu_1}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \frac{1}{\mu_0}\frac{\mathrm{d}\mu_0}{\mathrm{d}x}\frac{\mathrm{d}c}{\mathrm{d}c}\frac{\mathrm{d}\phi}{\mathrm{d}x} - k^2\phi = \frac{k^2}{\mu_0}(\frac{\mathrm{d}\mu}{\mathrm{d}c})\psi$$

$$\text{Since } R = \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}c} \Rightarrow \psi = \frac{1}{Rk^2}\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + R\frac{\mathrm{d}c_0}{\mathrm{d}x}\frac{\mathrm{d}\phi}{\mathrm{d}x} - k^2\phi\right).$$

$$\text{for } x \neq 0, \frac{\mathrm{d}c_0}{\mathrm{d}x} = 0 \text{ therefore } \psi = \frac{1}{Rk^2}\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - k^2\phi\right),$$

$$\text{if } x > 0, \phi = A_2e^{-\rho x} + B_2e^{-kx}.$$

$$\text{therefore, } \psi = \frac{1}{Rk^2}\left(A_2e^{-\rho x}\right)\left(\rho^2 - k^2\right)$$

$$\text{and if } x < 0, \phi = A_1e^{\rho x} + B_1e^{kx}.$$

$$\text{therefore, } \psi = \frac{1}{Rk^2}\left(A_1e^{\rho x}(\rho^2 - k^2)\right)$$

$$\text{since } \phi(0^+) = \phi(0^-) \Rightarrow A_1 + B_1 = A_2 + B_2$$

$$\text{and } \psi(0^+) = \psi(0^-) \Rightarrow A_1 = A_2$$

$$\text{similarly we get, } B_1 = B_2$$

$$\alpha\frac{\mathrm{d}\phi}{\mathrm{d}x}|_{0^+} = \frac{\mathrm{d}\phi}{\mathrm{d}x}|_{0^-} \Rightarrow \alpha\left(-\rho A_2 - B_2K\right) = \rho A_1 + B_1k$$

therefore either α = -1 or $\rho A_1 + B_1k = 0$, but $\alpha = \frac{\mu_2}{\mu_1} = e^R > 0$

$$\text{hence } \rho A_1 + B_1k = 0 \Rightarrow A_1 = -\frac{K}{\rho}B_1. \therefore \phi = \begin{cases} -\frac{k}{\rho}B_1e^{\rho x} + B_1e^{kx}, & x < 0 \\ -\frac{k}{\rho}B_1e^{-\rho x} + B_1e^{-kx}, & x > 0 \end{cases} \tag{2.32}$$

From Equation (38) we get,

$$\int_{0-}^{0+}\left(\frac{d^2}{dx^2}Rk^2\psi dx - \rho^2\left(\left(\frac{d^2}{dx^2} - k^2 - \sigma\right)\left(\frac{d^2}{dx^2} + R\delta(x)\frac{d}{dx} - k^2\right)\phi dx = Rk^2\int_{0-}^{0+}\delta(x)\phi dx\right.\right.$$

$$Rk^2\frac{\mathrm{d}\psi}{\mathrm{d}x}\Big|_{0^-}^{0^+} - \rho^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\Big|_{0^-}^{0^+} + R\frac{\mathrm{d}\phi(0)}{\mathrm{d}x} - k^2\int_{0^-}^{0^+}\phi dx\right) = Rk^2\phi(0)$$

$$\int_{0-}^{0+}\left(\frac{d^2}{dx^2} - k^2 - \sigma\right)\left(\frac{d^2}{dx^2} + R\delta(x)\frac{d}{dx} - k^2\right)\phi dx = Rk^2\int_{0-}^{0+}\delta(x)\phi dx.$$

$$\int_{0-}^{0+}\left(\frac{d^2}{dx^2}Rk^2\psi dx - \rho^2\left(\left(\frac{d^2}{dx^2} - k^2 - \sigma\right)\left(\frac{d^2}{dx^2} + R\delta(x)\frac{d}{dx} - k^2\right)\phi dx = Rk^2\int_{0-}^{0+}\delta(x)\phi dx\right.\right.$$

$$Rk^2\frac{\mathrm{d}\psi}{\mathrm{d}x}\Big|_{0^-}^{0^+} - \rho^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\Big|_{0^-}^{0^+} + R\frac{\mathrm{d}\phi(0)}{\mathrm{d}x} - k^2\int_{0^-}^{0^+}\phi dx\right) = Rk^2\phi(0)$$

Since ϕ is continuous i.e $\phi(0^+) = \phi(0) = \phi(0^-) = A_1 + B_1$,

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} \text{ is also continuous i.e } \frac{\mathrm{d}\phi}{\mathrm{d}x}(0^+) = \frac{\mathrm{d}\phi}{\mathrm{d}x}(0^-) = \frac{\mathrm{d}\phi}{\mathrm{d}x}(0) = -A_1\rho - B_1 k = 0$$

$$\frac{d\phi(0)}{dx} = 0$$

Assume that ϕ is bounded i.e $|\phi(x)| \leq M \forall x$.

$\implies \int_{0^+}^{0_-} \phi dx = \lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} \phi dx$

$\implies |\int_{0^+}^{0^-} \phi dx| = \lim_{\epsilon\to 0} |\int_{-\epsilon}^{\epsilon} \phi dx| \leq_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} |\phi dx| \leq \lim_{\epsilon\to 0} M \int_{-\epsilon}^{\epsilon} dx = \lim_{\epsilon\to 0} 2M\epsilon = 0$.

$\int_{0^-}^{0^+} \phi dx = 0$

$$Rk^2 \left(\frac{\mathrm{d}\psi}{\mathrm{d}x}|_{0^+} - \frac{\mathrm{d}\psi}{\mathrm{d}x}|_{0^-} \right) - \rho^2 \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}|_{0^+} - \frac{\mathrm{d}\phi}{\mathrm{d}x}|_{0^+} \right) = Rk^2 \left(A_1 + B_1 \right)$$

$$rK^2 \left(-\frac{\rho}{Rk^2} A_1(\rho^2 - k^2) - \frac{\rho}{Rk^2} A_2(\rho^2 - k^2) - \rho^2(A_1\rho - B_1 k A_2\rho - B_2 k) \right) = Rk^2(A_1 + B_1)$$

$$k(\rho - k)(Rk - 2\rho^2 - 2\rho k) = 0$$

since $k \neq 0, \quad \rho^2 = k^2 + \sigma \Rightarrow \rho - k \neq 0$

therefore we get the quadratic equation in σ; $2(k^2 + \sigma) - Rk = -2k\sqrt{k^2 + \sigma}$

squaring on both sides we get, $4\sigma^2 + 4k^2\sigma - 4Rk\sigma - (-R^2K^2 + 4Rk^3)$

$$\text{Hence, } \sigma = \frac{1}{2}\left(Rk - k^2 + k\sqrt{(k^2 + 2Rk)} \right)$$

Since ϕ is continuous i.e $\phi(0^+) = \phi(0) = \phi(0^-) = A_1 + B_1$,

$$Rk^2 \left(\frac{\mathrm{d}\psi}{\mathrm{d}x}|_{0^+} - \frac{\mathrm{d}\psi}{\mathrm{d}x}|_{0^-} \right) - \rho^2 \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}|_{0^+} - \frac{\mathrm{d}\phi}{\mathrm{d}x}|_{0^+} \right) = Rk^2 \left(A_1 + B_1 \right)$$

$$rK^2 \left(-\frac{\rho}{Rk^2} A_1(\rho^2 - k^2) - \frac{\rho}{Rk^2} A_2(\rho^2 - k^2) - \rho^2(A_1\rho - B_1 k A_2\rho - B_2 k) \right) = Rk^2(A_1 + B_1)$$

$$k(\rho - k)(Rk - 2\rho^2 - 2\rho k) = 0$$

since $k \neq 0, \quad \rho^2 = k^2 + \sigma \Rightarrow \rho - k \neq 0$

therefore we get the quadratic equation in σ; $2(k^2 + \sigma) - Rk = -2k\sqrt{k^2 + \sigma}$

squaring on both sides we get, $4\sigma^2 + 4k^2\sigma - 4Rk\sigma - (-R^2K^2 + 4Rk^3)$

$$\text{Hence, } \sigma = \frac{1}{2}\left(Rk - k^2 + k\sqrt{(k^2 + 2Rk)}\right)$$

Chapter 3

Stability of Miscible fluid system: using Self similar Quasi steady state approximation (SSQSSA)

In this chapter we discuss the method of numerical solutions of linearized equations.The linear perturbation equations in both (x,t) and (ξ, t)-coordinate systems are solved as an IVP and compare the obtained both results.

In QSSA there are two time variable one is perturbation time and another is base state time $c_0(x,t) = \frac{1}{2}[1 + \mathrm{erf}(x/2\sqrt{t})]$, where $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\mathrm{d}t$ is the error function. We transform the coordinates from (x, y, t) to (ξ, y, t). Where $\xi := x/\sqrt{t}$, so now we pretend base-state is time-independent. Transforming (x, y, t) to (ξ, y, t) we get

$$C_b(\xi) = \frac{1}{2}[1 + erf(\frac{\xi}{2})]. \tag{3.1}$$

Where $\xi = \frac{x}{\sqrt{t}}$ and ϕ_c is the perturb concentration.

$$\frac{\partial u'}{\partial x} = (\frac{\partial u'}{\partial \xi})(\frac{\partial \xi}{\partial x}) = \frac{1}{\sqrt{t}}\frac{\partial}{\partial x}\frac{\partial u'}{\partial \xi} \therefore \frac{1}{\sqrt{t}} + \frac{\partial v'}{\partial y} = 0; \tag{3.2}$$

$$\frac{1}{\sqrt{t}}\frac{\partial p'}{\partial \xi} = -\mu_b u' - \mu'; \frac{\partial p'}{\partial y} = -\mu_b v' \tag{3.3}$$

$$\because (\frac{\partial^2}{\partial x^2} + R\frac{\mathrm{d}C_0}{\mathrm{d}x}\frac{\partial}{\partial x} - k^2)\phi_u = k^2 R\phi_C$$

$$\frac{\partial c'}{\partial t} - \frac{\xi}{2t}\frac{\partial c'}{\partial \xi} + \frac{1}{\sqrt{t}}\frac{\mathrm{d}c_b}{\mathrm{d}\xi}u' = \frac{1}{t}\frac{\partial^2 c'}{\partial \xi^2} + \frac{\partial^2 c'}{\partial y^2} \tag{3.4}$$

since

$$(u', c')(\xi, y, t) = (\phi_u, \phi_c)(\xi, t)e^{iky}$$

$$\Rightarrow \frac{\partial \phi_c}{\partial t} - \frac{\xi}{2t}\frac{\partial \phi_c}{\partial \xi} + \frac{1}{\sqrt{t}}\frac{\mathrm{d}c_b}{\mathrm{d}\xi}\phi_u = \frac{1}{t}\frac{\partial^2 \phi_c}{\partial \xi} - k^2\phi_c \tag{3.5}$$

$\because x = \frac{\xi}{\sqrt{t}}$

$therefore \quad (\frac{\partial^2}{\partial \xi^2} + R\frac{\mathrm{d}c_b}{\mathrm{d}\xi} - k^2 t)\phi_u = tk^2 R\phi_c.$

With the help of above equation and equation (3.5) we eliminate ϕ_u so we get,

$$\frac{\partial \phi_c}{\partial t} = \frac{\xi}{2t}\frac{\partial \phi_c}{\partial \xi} + \frac{1}{t}\frac{\partial^2 \phi_c}{\partial \xi^2} - k^2\phi_c - \frac{1}{\sqrt{t}}\frac{\mathrm{d}c_b}{\mathrm{d}\xi}M_1^{-1}M_2$$

$\mathrm{M}_1 = (\frac{\partial^2}{\partial \xi^2} + R\frac{\mathrm{d}c_b}{\mathrm{d}\xi} - k^2 t);$

$$M_2 = tk^2 R;$$

$$M_3 = \frac{1}{t}(\frac{\xi}{2}\frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial \xi^2} - k^2 t);$$

$$M_4 = -\frac{1}{\sqrt{t}}\frac{\mathrm{d}c_b}{\mathrm{d}\xi}.$$

We get the system of ordinary differential equation

$$\frac{\mathrm{d}\phi_c(t)}{\mathrm{d}t} = A(t)\phi_c. \tag{3.6}$$

where $\phi_c(t) = \phi_c(\xi_i, t)$ and $A(t) = M_3 + M_4 M_1^{-1} M_2$ is the stability matrix of order n. By using Central difference formula we get;

on the left side of the equation (3.6) will be column matrix look like, $\begin{pmatrix} \frac{\mathrm{d}\phi_c(\xi_1,t)}{\mathrm{d}t} \\ \frac{\mathrm{d}\phi_c(\xi_2,t)}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}\phi_c(\xi_n,t)}{\mathrm{d}t} \end{pmatrix}$, where ξ_i are the grid points. h is the step size. Therefore the dimension of matrix will be number of grid points in x-axis substract by 2 i.e $n = length(\xi) - 2$, because we are not interested on boundary value. Dimension of matrix $A(t)$ is $n \times n$.

Right side will look like $A(\xi, t)\begin{pmatrix} \phi_c(\xi_1) \\ \phi_c(\xi_2) \\ \vdots \\ \phi_c(\xi_n) \end{pmatrix}$

$$M_1(i,j)(t) = \begin{cases} \frac{1}{h^2} \pm \frac{R}{4h^2}\left(c_0(j+1) - c_0(j-1)\right), & i = j \mp 1 \\ -\frac{2}{h^2} - k^2 t, & i = j \\ 0, & \text{otherwise,} \end{cases}$$

$$M_2(i,j)(t) = \begin{cases} k^2 R t, & i = j \\ 0, & \text{otherwise,} \end{cases}$$

$$M_3(i,j)(t) = \begin{cases} \frac{1}{th^2} \pm \frac{\xi_j}{4ht}, & i = j \mp 1 \\ -\frac{2}{th^2} - k^2, & i = j \\ 0, & \text{otherwise,} \end{cases}$$

and

$$M_4(i,j)(t) = \begin{cases} -\left(\frac{c_0(j+1) - c_0(j-1)}{2h\sqrt{t}}\right), & i = j \\ 0, & \text{otherwise.} \end{cases}$$

In this chapter what we get the matrix, we discuss about its normality and normality and change ODE system in terms of propagator matrix and analysis the transient energy growth and onset instabilty in the next chapter. Fig(3.1) is SSQSSA plot while Fig(3.2) QSSA plot. The first benefit of SSQSSA odal analysis is that it converge rapidly fast to the dominant eigenmode while in QSSA modal analysis it takes longer time to converge to dominant eigenmode. Second benefit is that if we see the plot

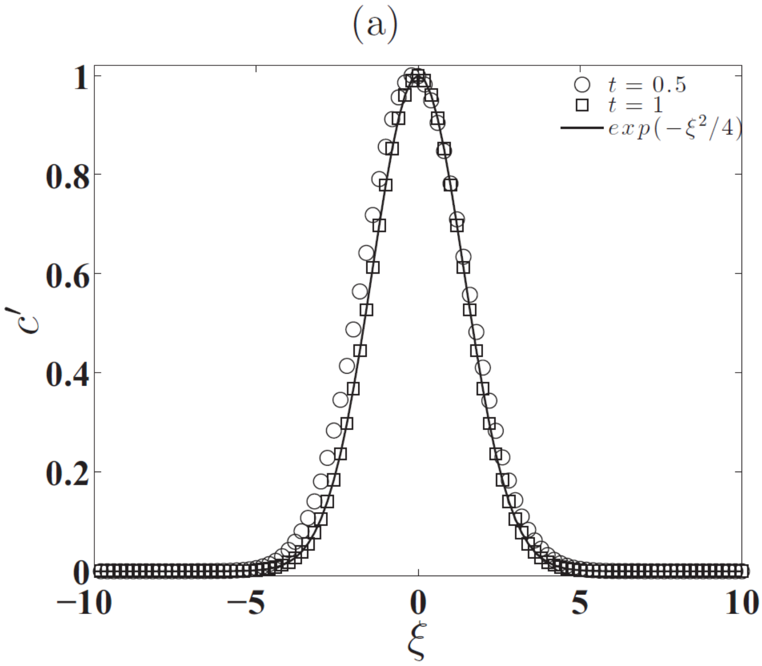

Figure 3.1: (ξ, t) coordinate system[4].

there is disturbance occur around the origin since we study the instability at the interface while in QSSA plot there are disturbance scattered all arround.

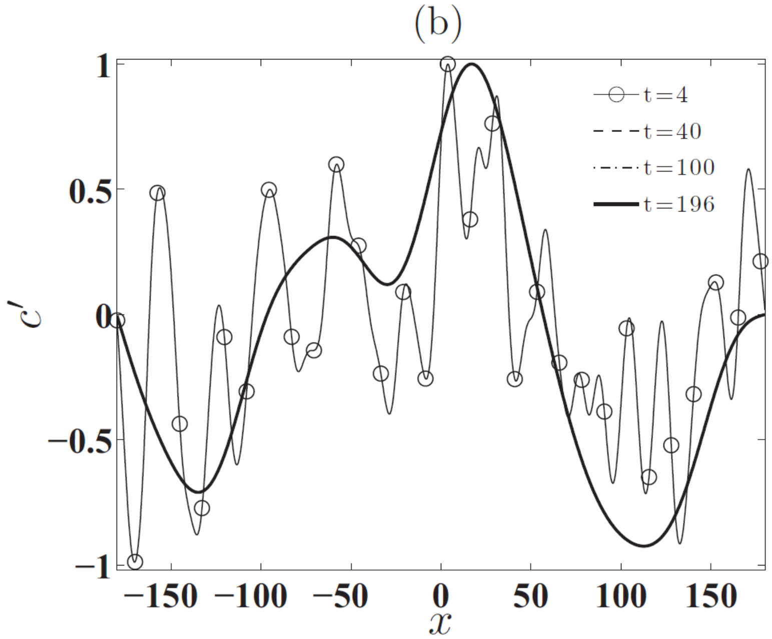

Figure 3.2: (x,t) coordinate system[4].

Chapter 4

Stability of Miscible fluid system: using Nonmodal linear stability analysis

In this chapter we discuss about the determination of linear stability of time dependent flow in porous media by Nonmodal analysis. The advantage of this method is as compare to SSQSSA, we have known initial condition but in SSQSSA method we were taking random initial condition. Therefore in Nonmodal analysis we get the solution more accurate than SSQSSA. Supoose $A(t_0)c' = \sigma(t_0)c'$ is eigenvalue problem. To analyse the behaviour of the system to the external excitation, we study about the Pseudospectra which play a very important role and it is a very helpful tool to analyse system behaviour. But study about Pseudospectra is a very big challenge so in this chapter just discussing the basic idea about the Pseudospectra.

4.1 Pseudospecra of Matrix

For each $\epsilon \geq 0$ ϵ-Pseudospectra [4] of a Matrix A is defined as

$$\Lambda_\epsilon(A) := z \in \mathbb{C} : ||(zI - A)^{-1}|| \geq \epsilon^{-1} \tag{4.1}$$

For Normal Matrices : $||(zI - A)^{-1}|| = \frac{1}{dis(z,\Lambda)}$.

In above equation $\Lambda(A)$ is equal to union of the closed ϵ-ball about the eigenvalue of

A. however in general it may be much larger.

For Non-Normal Matrices : The norm of $(zI - A)^{-1}$ is its largest singular value, i.e the inverse of the smallest singular value of $(zI - A)$. Therefore an equivalent definition of the Pseudospectra is

$$\Lambda_\epsilon(A) = z \in \mathbb{C} : \sigma(A) \leq \epsilon \tag{4.2}$$

for example : A = $\begin{bmatrix} -1 & 5 \\ 0 & -2 \end{bmatrix}$

since A is non-normal matrix, Figure is contour plot(in matlab using eigtool(A)) of pseudospectra of matrix A[5].

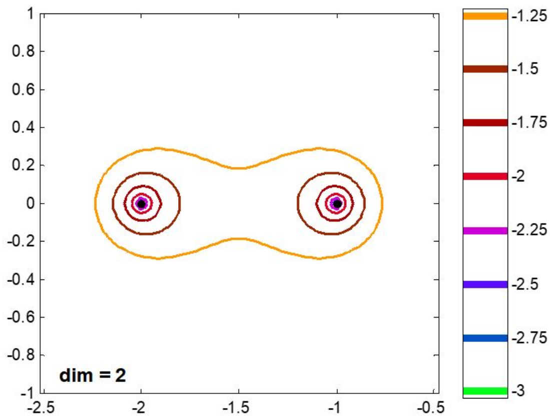

Figure 4.1:

Since we determine the instability of the system by the behaviour of the eigenvalue of the linearized stability matrix A(t), which assumes an exponential time dependence of A(t). While it fails to determine the instability appropriately. The temporal eigenmodes of time-dependent matrix A(t) (or non-autonomous operator). Therefore to quantify the Non-normality of matrix A(t), we freeze A(t) at different times and defined two quantities name as Spectral abscissa and Numerical abscissa. We define the Spectral abscissa $\alpha(A)$ of matrix $A(t)$ as

$$\alpha(A) := \max\{\Re(\lambda(A))\} \tag{4.3}$$

and Numerical abscissa $\eta(A)$ of matrix A is define as

$$\eta(A) := \sup_{z \in W(A)} \Re(z) = \sup\{\lambda(A + A^*)/2\} \tag{4.4}$$

where $\lambda(\cdot)$ represents the spectrum of the matrix and $\Re(\cdot)$ denotes the real part. $\eta(\mathcal{A})$ measures the maximum possible instantaneous growth rate corresponding to any initial condition as $t \to 0$. $W(A)$ is called Numerical Range or Field of Values of complex $n \times n$ matrix A define as

$$W(A) := \{\mathbf{x}^* A\mathbf{x} : \mathbf{x} \in \mathbb{C}^n, \| \mathbf{x} \|= 1\} \tag{4.5}$$

where the superscript $(*)$ denotes the transpose. The growth or decay of the initial energy can be studied from the numerical abscissa.
Since in case of Normal matrix the spectral abscissa and Numerical abscissa will be equal.

Proposition 4.1.1. [6] If A is normal matrix then $\eta(A) = \alpha(A)$.

Lemma 4.1.2. $T = 0 \Leftrightarrow < v, Tw >= 0$ for all $v, w \in V \Leftrightarrow < v, Tv >= 0$ for all $v \in V = (\mathbb{C}^n, <>)$.

Proof:
We claim that $< v, Tw >= 0 \implies T = 0$.

Let $v = Tw$, therefore $< Tw, Tw >= 0 \implies Tw = 0$ for all $w \in V \implies T = 0$
Now,

$$T = 0 \Leftrightarrow < v, Tv >= 0 \text{ for all } v \in V.$$

If $v = \alpha w_1 + \beta w_2$, then $< v, Tv >= 0$ for all $v \in V$, which implies,

$$< \alpha w_1 + \beta w_2, \alpha T w_1 + \beta T w_2 >= 0.$$

Thus we get

$$< \alpha\alpha^* < w_1, Tw_1 > +\alpha\beta^* < w_1, Tw_2 > \\ + \alpha^*\beta^* < w_2, Tw_1 > +\beta\beta^* < w_2, Tw_2 >= 0.$$

$\Rightarrow \alpha\beta^* < w_1, Tw_2 > +\alpha^*\beta^* < w_2, Tw_1 >= 0.$
For $\alpha = 1, \beta = 1$,then $< w_1, Tw_2 > + < w_2, Tw_1 >= 0.$
For $\alpha = 1, \beta = i$,then $< w_1, Tw_2 > - < w_2, Tw_1 >= 0.$

$\Rightarrow 2 < w_1, Tw_2 >= 0. \Rightarrow < w_1, Tw_2 >= 0$

Now we will use

$$\|Ax\| = \|A^*x\|$$

$\Leftrightarrow A$ is normal

Since $\langle Av, w\rangle = \langle v, A^*w\rangle$

$\Rightarrow < Ax, Ax >=< A^*x, A^*x > \Rightarrow < x, A^*Ax >=< x, AA^*x > \ \Rightarrow < x, A^*A - AA^*)x >= 0$

By using lemma 4.1.2 we get

$$(A^*A - AA^*) = 0$$

$\Rightarrow$A$^*A = AA^*$

$\Rightarrow A$ is normal.

$\Rightarrow$(A - λI) is normal.

$\therefore \|(A - \lambda I)x\| = \|(A^* - \lambda^* I)x\| \Rightarrow (A^* - \lambda^* I)x = 0 \Rightarrow A^*x = \lambda^* x$

$\Rightarrow \lambda^*$ is an eigenvalue of A* and λ is an eigenvalue of A

$\therefore (A + A^*)x = (\lambda + \lambda^*)x$

$Hence, \quad \eta(A) = \sup\{\lambda(A + A^*)/2\} = \sup\{(\lambda + \lambda^*)/2 = \sup \mathbb{R}(\lambda) = \alpha(A)$

Since the main goal of doing Modal analysis is to study the Spectral abscissa and their eigenmodes. In fig(4.2) there is a plot of spectral abscissa and numerical abscissa for the stability matrix A at different times for a given wave number $k = 0.2$ and $R = 3$.

There is a matlab code for above plot.

```
R = 3;                % log-mobolity ratio
k = 0.2;              % Wave number
L = 100;              % Equal to length of domain
h = 0.2;               % Spatial size
xi = -L:h:L;            % Middle case domain
t = 0.0001:0.1:20;
%%
Nxi = length(xi);         % Number of grid points in x axis
n = Nxi-2;              % Number of Internal points
```

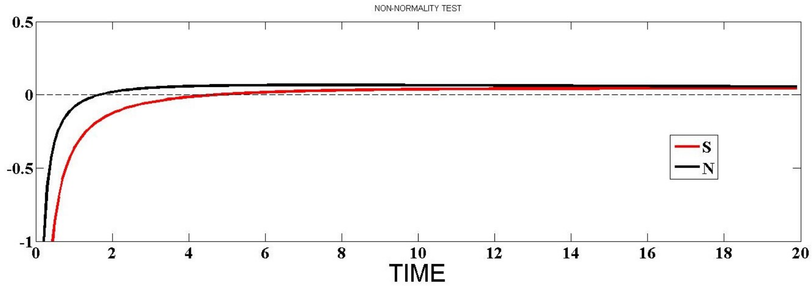

Figure 4.2: we plot the numerical abscissa and spectral abscissa verses time. figure shows that initially matrix is highly non-normal but with increasing time spectral abscissa approaching to numerical abscissa means it is approaching to normality.

```
%% Base states

cb = 0.5*(1+erf((xi)/2));

for ii=1:length(t)
    t0=t(ii);
M1 = zeros(n,n);
        M3 = zeros(n,n); M4 = zeros(n,n);
        %%
        kk = k.*k;
        hh = h*h;
        h4t = 4*h*t0;
        h1t = 1/hh./t0;
        h2 = 2*h;
        hh4 = R/4/hh;
        tsq = sqrt(t0);
        % Formulation of the diagonal elements
        for j = 1:n
            M1(j,j) = -2/hh - kk*t0;
        end
        % Formulate the off-diagonal elements
            M1(1,2) = 1/hh + hh4*(cb(3) - cb(1));
```

```
        for j=2:n-1
            M1(j,j+1) = 1/hh + hh4*(cb(j+2) - cb(j));
            M1(j,j-1) = 1/hh - hh4*(cb(j+2) - cb(j));
        end
          M1(n,n-1) = 1/hh - hh4*(cb(n+2) - cb(n));
        %% Formulation of the matrix M2
            M2 = t0*R*kk*eye(n);
        %% Formulate the matrix M3
        % Formulation of the diagonal elements
        for j = 1:n
            M3(j,j) = (-2*h1t - kk);
        end
        % Formulate the off-diagonal elements
            M3(1,2) = h1t + xi(2)/h4t ;
        for j = 2:n-1
            M3(j,j+1) = h1t + xi(j+1)/h4t ;
            M3(j,j-1) = h1t - xi(j+1)/h4t ;
        end
            M3(n,n-1) = h1t - xi(n+1)/h4t ;
        %% Fomulate the diagonal matrix M4
            M4(1,1) = -(cb(3) - cb(1))/h2/tsq;
        for j = 2:n-1
            M4(j,j) = -(cb(j+2) - cb(j))/h2/tsq;
        end
            M4(n,n) = -(cb(n+2) - cb(n))/h2/tsq;
        %% M = M3 + M4*inv(M1)*M2
        M = M3 + (M4*(M1\M2));
        a = max(real(eig(M)));
        b = max(eig(M+M'));
        n2 = b/2;
        s1(ii) = a;
        n1(ii) = n2;
end

%%
figure
hold on; box on;
```

```
plot(t,s1,t,n1)
```

4.2 Transient energy growth

Here we analysis the Transient growth energy and the onset instability.

Suppose equation (3.4) can be rewritten as

$$c'(t) := \Phi(t_0;t)c'_0 \tag{4.6}$$

with arbitrary initial condition $c'(t_0) = c'_0$ and $\Phi(t_0;t)$ is called propagator matrix, since it propagate the information forward from the initial time to time t. What we did earlier in SSQSSA there were random initial condition but here initial condition is $\Phi(t_0;t_0) = I$ where I is identity matrix of same order of matrix $A(t)$.

substitute the equation(4.7) into equation(3.4) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t_0;t) = A(t)\Phi(t_0;t). \tag{4.7}$$

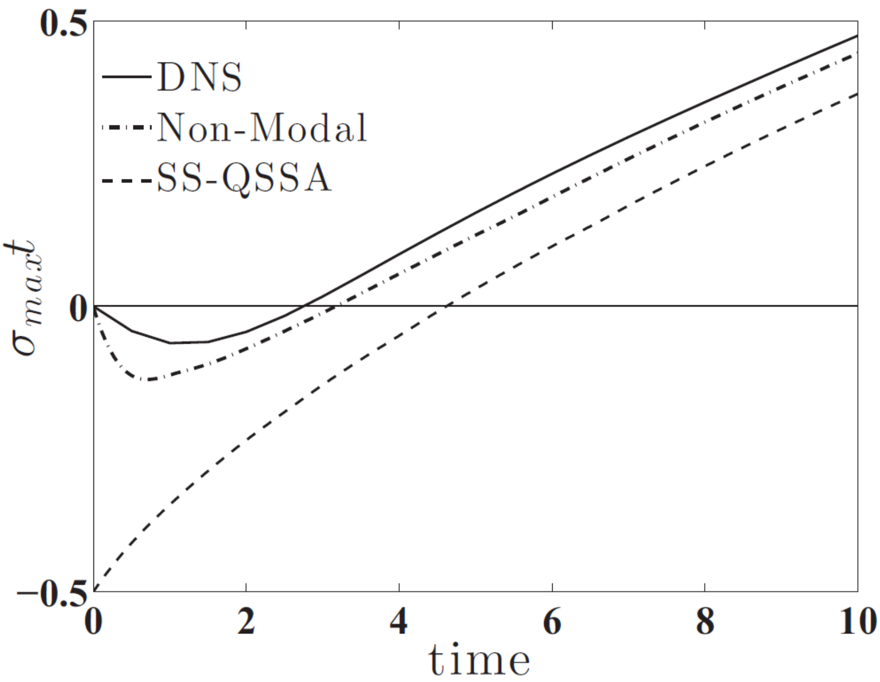

Figure 4.3: It shows that in SSQSSA with time growth rate is increasing which is not true but in Nonmodal analysis solution approaching to DNS solution[4].

Bibliography

[1] Kenneth Hoffman and Ray Kunze. Linear algebra. 1971. *Englewood Cliffs, New Jersey.*

[2] Pijush K Kundu, Ira M Cohen, and DW Dowling. Fluid mechanics 2nd, 2002.

[3] C. T. Tan and G. M. Homsy. Stability of miscible displacements in porous media: Rectilinear flow. *Phys Fluids*, 29(11):3549, 1986.

[4] S. Pramanik T. K. Hota and M. Mishra. Nonmodal linear stability analysis of miscible viscous fingering in porous media. *Discrete and Computational Geometry*, 92(05):3007(12), 2015.

[5] Lloyd N Trefethen and Mark Embree. *Spectra and pseudospectra: the behavior of nonnormal matrices and operators.* Princeton University Press, 2005.

[6] T. K. Hota. Nonmodal stability analysis of miscible displacement flows in porous media. *ph.D. Thesis at IIT Ropar*, 1:89, 2015.