Combinatorial Species

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Dissertation

submitted for the partial fulfilment of

BS-MS dual degree in science

Department of Mathematical Science

Indian Institute of Science Education and Research
Mohali

April,2019

Certificate of Examination

This is to certify that the dissertation titled **“ Combinatorial Species ”,** submitted by **Renu Meena**[Reg No Ms14126] for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has carried out by me under the guidance of Dr. Chetan Tukaram Balwe at the Indian Institute of science Education and Research Mohali.This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute.Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgment of collaborative research and discussions.This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Chetan Tukaram Balwe

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Abstract

The combinatorial theory of species was introduced by Joyal in 1986.We can understand the use of generating series for both labeled and unlabeled structures from this theory.The theory of combinatorial species is an abstract,systematic method for analysing discrete structures in terms of generating function.
First section covers some basic information about combinatorial species, some examples and generating series for labeled and unlabeled structures is defined.Concluding that cycle index series contain more information then exponential and type generating series.In second section defined that species of structure can be combined to form new species by using set theoretical constructions.Resulting a variety of combinatorial operations on species including addition, multiplication, substitution etc.....
In 3rd section first we defined virtual species and explain the species logarithm Ω .finally there is an exposition of Γ and quotient species and calculate the cycle index series for Γ and quotient species.Further more we want to compute the S_2 cycle index $Z_{BC}^{S_2}$ and also enumeration for species of point determining bipartite graphs.

Acknowledgements

I wish to warmly thank all the people who helped me during the preparation of this work.I pay my greatest Gratitude to Dr. Chetan Tukaram Balwe for his guidance throughout the work.I dedicate this work to my Parents Mr.Ramswaroop Meena and Mrs.Sajana Devi.I thank my friends Sakshi Singhania and Sukhpal for constant support.I finally thank IISER Mohali for its facilities.

Contents

Chapter 1

Species Theory

1.1 Introduction

This chapter contains the basic concepts of the combinatorial theory of species of structures. A species is a way of thinking about a set of combinatorial structures. Naturally speaking, a species is a function that sends a set of labels to a set of structures. Species theory allows us to manipulate such structures in ways we would not be able to otherwise. [**1**]

1.1.1 Notion

A structure S is a construction γ which one performs on a set U. It consists of a pair

$$S = (\gamma, U)$$

Example: Here is a detailed description of the species C of oriented cycle. For a finite set $U = \{x, 4, y, a, 7, 8\}$ we denote by $C[U]$ the set of all structure of oriented cycle on U . Let $c \in C[U]$ is a structure of oriented cycle on U is a pair (U, γ). Where $\gamma = \{(4, y), (y, a), (a, x), (x, 7), (7, 8), (8, 4)\}$ is set of ordered pair of element of set U.

1.2 Definition and Example

1.2.1 Definition

A species is a rule F:

- F assigns to a finite set U the set F[U] of F structure along α
- F assigns to a bijection $\alpha :$ U $\rightarrow$V the bijection F[α] : F[U] $\rightarrow$F[V] called the Transport of F structure along α
- F preserves identity maps: $F[Id_u] = Id_F[u]$
- F preserves composition maps:F[$\alpha \circ \beta$] =$F[\alpha] \circ F[\beta]$

An element $S \in F[U]$ is called an F structure on U.
The function $F[\alpha]$ is called the transport of F structures along α .

1.2.2 Observation

Let [n]={0,1,2,.....,n-1} F be a species and denote F[[n]]= F[n]

1.2.3 Conclusion

The cardinality of F[U] depends only on the cardinality of U, not on nature of element.

1.2.4 Example

Given a set of vertices U, we define the species of simple graph G[U] as:
$G[U] = \{(U, E) \mid \binom{U}{2} \supseteq E\}$
Where $\binom{U}{2}$ is the set of unordered pairs of distinct elements of

U.Here U is set of vertices and E is set of edges. Given {1,2,3} as the set of labels, we obtain the set of all possible graphs on three labels, $G[\{1,2,3\}]$ are $\{(1),(2),(3)\}$, $\{(1,2),(2,3),(3,1)\}$, $\{(1,2),(3)\}$,$\{(1,3),(2)\}$, $\{(2,3),(1)\}$, $\{(1,3),(3,2)\}$, $\{(1,2),(2,3)\}$,$\{(1,2),(1,3)\}$.

1.2.5 Definition

Consider two F structures $s \in F[U]$ and $t \in F[V]$. A bijection σ : U→V is called an isomorphism of s to t if $s = \sigma.t$

1.2.6 Example

Consider the rooted tree S=(γ,U) , Whose underlying set is $U = \{a,b,c,d,e\}$ and $\gamma = \{d,(a,(c,(b,e,f)))\}$ and via bijection σ : U →V replace each element of U by $\sigma(u)\ \forall u \in U$. The bijection σ allows the transport of the rooted tree S onto a corresponding rooted tree T=(α,V) on the set $V = \{x,3,4,v,5,u\}$ and $\alpha = \{\sigma(d),(\sigma(a),(\sigma(c),(\sigma(b),\sigma(e),\sigma(f))))\} = \{x,(3,(4,(v,5,u)))\}$. We can say that the rooted tree T has been obtained by transporting the rooted tree S along the bijection σ .

1.3 Associated series

For a species of structures F , there exist three power series that allow us to enumeration F structures. These power series are :

1.3.1 Exponential generating series

The theory of species allows us to count labeled structures using exponential generating functions.

Definition : The exponential generating series of a species of structures F is defined as:

$$F(x) = \sum_{n=0}^{\infty} (f_n) \frac{x^n}{n!}$$

Where $(f_n) = |F[n]|$ the number of F structure on a set of n elements (labeled structure).

Example : There are $f_n = n!$ linear ordering on a set of size n. Thus exponential generating function for the species L of linear orderings is

$$L(x) = \sum_{n=0}^{\infty} n! \frac{x^n}{n!}$$

$$L(x) = \frac{1}{1-x}$$

for species S of permutation $f_n = n!$

$$S(x) = \sum_{n=0}^{infty} n! \frac{x^n}{n!}$$

$$S(x) = \frac{1}{1-x}$$

1.3.2 Type generating series

The type generating series allows us to enumerate unlabeled f structures. In other words this can be thought of as isomorphism classes of labeled structures under permuting the labels.

Example : There are four unlabeled G structures four shapes on three vertices.
Wheres there are 8 labeled G structures on three vertices.(example 1.2.4)

Definition: Let $T(f_n)$ be the quotient set, denote F[n]/$\sim$, of isomorphism classes of F structures of order n. The type generating series of a species of structures F is the formal power series.

$$\widetilde{F}[U] = \sum_{n=0}^{\infty} \widetilde{f}_n x^n$$

Where $\widetilde{f}_n = |T(f_n)|$ is the unlabeled F structures of order n.

Example:

$$\widetilde{S}(x) = \prod_{k=0}^{\infty} \frac{1}{1-(x^k)}$$

$$\widetilde{L}(x) = \frac{1}{1-x}$$

1.3.3 Cycle index series

Definition : The cycle index series of a species structures F is the formal power series

$$Z_f(p_1, p_2, ...) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} (fixF[\sigma]) P_\sigma$$

Where S_n denotes the permutation group of [n] , $fixF[\sigma] = (F[\sigma])$, is the number of F structure on [n] fixed by f[σ] , P_σ is the monomial term $P_1^{\sigma_1} P_2^{\sigma_2} P_3^{\sigma_3} ....P_n^{\sigma_n}$ and σ_i is the number of i-cycles of σ.

Example : For n=2 G[{1,2}] contains only two graphs. the complete graph on two vertices (k_2) and its compliment (${k_2}^c$) . so applying any permutation of S_2 to these graphs leaves us with the original graphs. [**4**] Thus we have

$$\sum_{\sigma\in S_n}(fixG[\sigma])P_\sigma = 2{P_1}^2 + 2P_2$$

Note: This is a formal power series in a infinite number of variables $P_1, P_2, P_3....., P_n$.

1.4 The example of basic species

Examples are from [**1**, **3**]

1. (Simple graph structures) For each finite set U, let G[U] denote the set of all graphs having vertex set U. Then G is species of simple graphs.
2. (Connected simple graph structures) for each finite set U, let $G^c[U]$ denote the set of all connected graphs having vertex U.then G^c is species of connected graphs.
3. (Derangement structures Der) For each finite set U,let $Der[U]$ denote the set of all derangement of the set U. Then Der is species of derangements.
4. (Partition) For each finite set U, let Par[U] denote the set of all partition of U into nonempty subsets. If $\pi : U \to V$ is a bijection then π assign a bijection $par[\pi] : par[U] \to par[V]$ When if $U = \cup U_\alpha$ is partition of set U then par[U] carries this to the induced partition $V = \cup\pi(U_\alpha)$.
5. (Subset structure ρ) For each finite set U, $\rho[U]$ denote the collection of subsets of U. Then ρ is species of subset.

$$\rho[U] = \{S \mid S \subseteq U\}$$

6. (Permutations structures S) For each finite set U, let S[U] denote the set of all permutation of the set U. If$\pi : U \to V$ is a bijection of finite sets then every permutation $\sigma : U \to U$ of U determines a permutation of V, given by the formula $V \to \pi(\sigma(\pi^{-1}(V)))$. We therefore have

$$S[\pi] = \pi \circ \sigma \circ \pi^{-1}$$

7. (Linear order structures L) For each finite set U, let L[U] denote the set of all linear order of the set U. If $\pi : U \to V$ is a bijection of finite sets then π determines a bijection $L[\pi] : L[U] \to L[V]$, which carries an ordering $\{i_1 < i_2 < ....i_n\}$ to the induced ordering $\{\pi(i_1) < \pi(i_2) < ....\pi(i_n)\}$ of the set U . The L is a species , called the species of linear order.

8. The species E, of sets defined by $E[U] = \{U\}$ for each finite set U.

9. The species ε, of elements defined by $\varepsilon[U] = U$ Where the structures on U are the elements of U.

10. The species X for E, the species of singletons

$$X[U] = \begin{cases} \{U\} & \text{if } |U| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Species	E.G.S.	T.G.S.	C.I.S.
0	$0(x) = 0$	0	$Z_0(x_1, x_2, ...) = 0$
I	$I(x) = 1$	1	$Z_I(x_1, x_2, ...) = 1$
X	$X(x) = x$	x	$Z_X(x_1, x_2, ...) = x_1$
L	$L(x) = \frac{1}{1-x}$	$\frac{1}{1-x}$	$Z_L(x_1, x_2, ...) = \frac{1}{1-x_1}$
S	$S(x) = \frac{1}{1-x}$	$\prod_{k=0}^{infty} \frac{1}{1-(x^k)}$	$Z_S(x_1, x_2, ...) = \frac{1}{(1-x_1)(1-x_2)...}$
E	$E(x) = e^x$	$\frac{1}{1-x}$	$Z_E(x_1, x_2, ...) = exp(x_1 + \frac{x_2}{2} + \frac{x_3}{3}...)$
ϵ	$\epsilon(x) = x.e^x$	$\frac{x}{1-x}$	$Z_\epsilon(x_1, x_2, ...) = x_1.exp(x_1 + \frac{x_2}{2} + \frac{x_3}{3}...)$
C	$C(x) = -log(1-x)$	$\frac{x}{1-x}$	
ρ	$\rho(x) = e^{2x}$	$\frac{1}{(1-x)^2}$	

Combinatorial equality: When Two species of structures F and G are isomorphic.They said to be combinatorially equal .

Note: Consider the species of linear ordering (L) and the species of permutation (S). Their exponential generating series is same,but their type generating series and cycle index series are not equal. So they are not combinatorially equal.

We will end this section with a fact : if F and G two isomorphic species written $F \cong G$ then the generating series and type generating series are equal.

1.5 The cycle index series as a generalization

Theorem : For any species of structures F: [**1**]

$$a)F(x) = Z_F(x, 0, 0, 0.....)$$

$$b)\tilde{F}(x) = Z_F(x, x^2, x^3....)$$

Proof: a) Substituting $x_1 = x$ and $x_i = 0$ for all $i \geq 2$ in eq [**1**]

$$Z_F(x, o, o, o.....) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} (fixF[\sigma]) x^{\sigma_1} 0^{\sigma_2}.......$$

Now for each fired value of $n \geq 0$, $x^{\sigma_1} 0^{\sigma_2}.... = 0$ except if $\sigma_1 = n$ and $\sigma_i = 0$ for $i \geq 2$

$$Z_F(x, o, o, o.....) = \sum_{n=0}^{\infty} \frac{1}{n!} (fixF[Id_n]) x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} f_n x^n$$

$$= F(x)$$

Since all structures are fixed by transport along the identity.

Proof: b)

$$Z_F(x, x^2, x^3....) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} (fixF[\sigma]) x_1^{\sigma} x^{2\sigma_2}.......$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} (fixF[\sigma]) x^n$$

$$= |F[n]/\sim|x^n$$

$$= \tilde{F}(x)$$

Example:

$$Z_S(x,0,0,0.....) = \frac{1}{1-x} = S(x)$$

$$Z_S(x,x^2,x^3....) = \frac{1}{(1-x)(1-x^2)(1-x^3).....} = \tilde{S}(x)$$

Chapter 2

Algebraic operations on species

There are several important operations on species. We may build a species out of other species by defining operations such as addition and multiplication on species. We now make a sequence of definitions, each followed by explanation and examples that illuminate its underlying meaning [**1**] and [**4**].

2.1 Addition

Definition: Let F and G be species. Then their sum F+G is the species where

$$(f+G)[U] = F[U] + G[U]$$

Where $A \cup B$ disjoint union of A and B

$$(F+G)[\sigma](s) = \begin{cases} (F)[\sigma](s) & \text{if } s \in F[U] \\ (G)[\sigma](s) & \text{if } s \in G[U] \end{cases}$$

It is easily seen that addition of species is associative and commutative.

2.1.1 Proposition

- $(F+G)(x) = F(x) + G(x)$
- $(\widetilde{F+G})(x) = \widetilde{F}(x) + \widetilde{G}(x)$
- $Z_{F+G}(p_1, p_2, ...) = Z_F(p_1, p_2, ...) + Z_G(p_1, p_2, ...)$

The number of (F+G) structures on n elements is

$$|(F+G)[n]| = |(F)[n]| + |(G)[n]|$$

2.1.2 Example

1. If A is the species of trees, B is the species of forests, and B* is the species of disconnected forests, then $B = A + B^*$.
2. A simpler example is $1 + X$. Since 1 is the species of the empty set and X is the species of singleton sets, $1 + X$ is the species of sets of size at most 1.

2.2 Multiplication

Definition: Let F and G be species. Then their product $F \cdot G$ is the species such that:
$(F \cdot G)[U]$={(s,t) :there $s \in F[U_1]$ and $t \in G[U_2]$}
Where $U_1 \cup U_2 = U$ and $U_1 \cap U_2 = \phi$.
and $(F \cdot G)[\sigma](s,t) = (F[\sigma_1](s), G[\sigma_2](t))$ Where σ_1 and σ_2 are the restriction of σ to the label sets of s and t respectively.

2.2.1 Proposition

For any two species F and G series associated with the species $F \cdot G$

- $(F \cdot G)(x) = F(x) \cdot G(x)$

- $\widetilde{(F \cdot G)}(x) = (\widetilde{F}(x)) \cdot (\widetilde{G}(x))$
- $Z_{F \cdot G}(p_1, p_2, ...) = Z_F(p_1, p_2, ...) \cdot Z_G(p_1, p_2, ...)$

The number of $F \cdot G$ structures on n elements is $|(F \cdot G)[n]|$

$$= \sum_{i+j=n} \frac{n!}{i!j!} |F[i]||G[j]|$$

Note: That (FG)[U] is not same as (GF)[U] , but the species FG and GF are isomorphic . We usually identify species that are isomorphic.

2.2.2 Example

This is a typical example of the product of species of structures. let $|U| = n$ the generating series of species S ,of permutation on the finite set U is

$$S(x) = \sum_{n=0}^{\infty} |S[n]| \frac{x^n}{n!} = \sum_{n=0}^{\infty} n! \frac{x^n}{n!} = \frac{1}{1-x}$$

The generating series of species E, of set on U is

$$E(x) = \sum_{n=0}^{\infty} |E[n]| \frac{x^n}{n!} = \sum_{n=0}^{\infty} 1 \frac{x^n}{n!} = e^x$$

We can divide the structure of species S into two disjoint structures
(i) A set of fixed points.
(ii) A derangement of the elements.
We say that the species S of permutation is product of the species E of set with species Der of derangement. Here we use

$$S = E \cdot Der$$

as$F = E$,$G = Der$ and $F \cdot G = S$ so

$$\sum_{n=0}^{\infty} |S[n]| \frac{x^n}{n!} = \sum_{n\geq 0} \sum_{i+j=n} \frac{n!}{i!j!} |E[i]| \cdot |Der[j]| \frac{x^(i+j)}{n!}$$

$$\sum_{n=0}^{\infty} n! \frac{x^n}{n!} = \sum_{n\geq 0} \sum_{i+j=n} \frac{x^i}{i!} |E[i]| \cdot |Der[j]| \frac{x^j}{j!}$$

$$\frac{1}{1-x} = e^x \cdot Der(x)$$

$$Der(x) = \frac{e^{-1}}{1-x}$$

in this example we using definition of the generating series of species of structure to obtain the exponential generating function $Der(x) = \frac{e^{-1}}{1-x}$. Even though we can not make direct combinatorial sense out of $Der = S/E$.

2.3 Composition

Definition: Let F and G be species such that $G[\phi] = \phi$. We will define a species $F \circ G$ called the composition of F and G. Its structure set $(F \circ G)[U]$ is given by:

$$(F \circ G)[U] = \sum_{\pi partition of U} F[\pi] \times \Pi_{p\in\pi} G[p]$$

2.3.1 Proposition

- $(F \circ G)(x) = F(G(x))$
- $(\widetilde{F \circ G})(x) = (\widetilde{F}(\widetilde{G}(x), \widetilde{G}(x^2), \widetilde{G}(x^3).....)$
- $Z_{F\circ G}(p_1, p_2, ...) = Z_F(Z_G(p_1, p_2, ...), Z_G(p_2, p_3, ...)....)$

2.3.2 Example

Every permutation is a set of disjoint cycles we have the combinatorial equation.

$$S = E \circ C$$

2.3.3 Example

The endofunction $\wp$ can naturally be identified with a permutation of disjoint rooted trees. [**3**] so $End = S \circ A$

$$End(x) = S(A(x)$$

$$\sum_{n\geq 0} n^n \frac{x^n}{n!} = \frac{1}{1-A(x)}$$

Let me define one more species S_{Ver} if I is nonempty , then $S_{Ver}[I]$ is the collection of all triples (T,i,j) where i and j are element of I and T is a tree with vertex set I.
The exponential generating series of S_{Ver} and S_{End} is same so for every nonempty finite set I, the sets $S_{End}[I]$ and $S_{Ver}[I]$ have the same number of elements. so $|S_{End}[I]| = n^n = |S_{Ver}[I]|$ and $S_{Ver} = T_n \cdot n^2$

$$n^n = T_n \cdot n^2$$

$$n^{n-2} = T_n$$

T_n is the number of trees on n vertices.

Note: The species G of graphs is related to the species G^C of connected graphs by the combinatorial equation

$$G = E(G^C)$$

, Since every graph is an assembly of connected graphs.

2.4 Differentiation and pointing

2.4.1 Differentiation

Differentiation is most important in that it allows us to root or point a species. For instance, the species A of rooted tree can be expressed in terms of the species a of unrooted trees by the formula $A = X \cdot a'$

Definition: Given a species F, define its derivative F' to be the species such that if $* \notin U$ then

$$F'[U] = F[U \cup \{*\}]$$

and

$$F'[\sigma](s) = F[\sigma^+](s)$$

Where $\sigma^+(*) = *$ and $\sigma^+(*) = \sigma(x)$ if $x \in U$
Thus we take a derivative by additing a star to the label set and requiring that the star remain fixed by isomorphism. this definition equality $|F'[n]| = |F[n+1]|$ for all n.

Proposition

For any species F,

- $F'(x) = \frac{d}{dx}F(x)$
- $\widetilde{F}'(x) = (\frac{\partial}{\partial x}Z_F)(x, x^2, x^3, ...)$
- $Z'_F(p_1, p_2, ...) = \frac{\partial}{\partial p_1}Z_F(p_1, p_2, ...)$

Example

L denote the species of linear orderings. We denote using the species of cyclic ordering by C. The generating series for C can be calculated using species differentiation. It turns out that the generating series of the derivative of cycle orderings in $C'(x) = L(x) = \frac{1}{1-x}$. That is, a C'-structure on a set U is a C-structure on the set $U \cup *$. Thus, we naturally imagine the derivative of a cylic ordering as a linear ordering, forgetting $*$, on the set U. Therefore, in terms of its generating series,

$$C(x) = \int_0^\infty \frac{dx}{1-x} = log\frac{1}{1-x}$$

$$L = C'$$

Note: The operation of differentiation can be iterated. For $F'' = (F')'$ We simply add successively two distinct elements $*_1$ and $*_2$

Remark: To underlying how the combinatorial differential calculus of species agree with the classical differential calculus of formal power series, we mention that the chain rule admits the combinatorial equivalent

$$(F \circ G)' = (F' \circ G) \cdot G'$$

Now we define the operation of pointing. The effect of pointing is to take an unrooted structure and turn it into a rooted structure.

2.4.2 Pointing

A typical $F^\bullet$ structure can be represented graphically by circling the pointed element.

Definition: Given a species F,the species of pointed F structures is the species $F^\bullet$, called F dot, is defined as follows;
An $F^\bullet$ structure on U is a pair $s = (f, u)$ where

1. f is an F structure on U,
2. $u \in U$

also

$$F^{\bullet}[U] = F[U] \times U$$
$$|F^{\bullet}[n]| = n|F[n]|$$

Proposition

Let F be a species of structures

- $F^{\bullet}(x) = x\frac{d}{dx}F(x)$
- $\widetilde{F^{\bullet}}(x) = x(\frac{\partial}{\partial x}Z_F)(x, x^2, x^3, ...)$
- $Z_F^{\bullet}(p_1, p_2, ...) = p_1\frac{\partial}{\partial p_1}Z_F(p_1, p_2, ...)$

Note: The operations of pointing and derivation are related by the combinatorial equation

$$F^{\bullet} = X \cdot F'$$

Example

Let us point the species a of tree . We obtain the species A of rooted trees

$$a^{\bullet} = A$$

because we have

$$|F^{\bullet}[n]| = n|F[n]|$$
$$|a^{\bullet}[n]| = n|a[n]| = |A[n]|$$
$$= |A[n]| = n \cdot n^{n-2} = n^{n-1}$$

A rooted tree is nothing more than a tree with a distinguished element. It is important to note that the distinguished element u of an $F^{\bullet}$ structure belongs to the underlying set U.

2.5 Cartesion product of species of structures

Definition : Let F and G be two species of structures , The species $F \times G$, called the cartesian product of F and G is defines as follow,
An $F \times G$ structure an a finite set U is a pair $s = (f, g)$ where

1. f is an F structure on U
2. g is an G structure on U

also

$$(F \times G)[U] = F[U] \times G[U]$$
$$|(F \times G)[n]| = |F[n]| \cdot |G[n]|$$

2.5.1 Proposition

Let F and G be two species of structures Then series associated to the species $F \times G$ satisfy the equalities

- $(F \times G)(x) = F(x) \times G(x)$
- $\widetilde{F \times G}(x) = (Z_F \times Z_G)(x, x^2, x^3, ...)$
- $Z_{F\times G}(p_1, p_2, ...) = Z_F(p_1, p_2, ...) \times Z_G(p_1, p_2, ...)$

2.5.2 Example

Consider the species C of oriented circles and the species a of trees , there is difference between an $(a \times C)$ structure and $(a \cdot C)$ a structure on a finite set . each of the structures a and c appearing in the formation of an $(a \times C)$ structure on U has underlying set U but for $(a \cdot C)$ structure (a, c) on U, the underlying sets U_1 and U_2 of a and c are disjoints.

2.6 Functorial composition

Definition Let F and G be two species of structures. The species $F\Box G$ called the functorial composite of F and G, is defined as follows:

$$(F\Box G)[U] = F[G[U]]$$

for any finite set U

$$|(F\Box G)[n]| = |F[G[n]]|$$

2.6.1 Proposition

For any two species F and G,

- $(F\Box G)(x) = F(x)\Box G(x)$
- $\widetilde{F\Box G}(x) = (Z_F\Box Z_G)(x, x^2, x^3, ...)$
- $Z_{F\Box G}(p_1, p_2, ...) = Z_F(p_1, p_2, ...)\Box Z_G(p_1, p_2, ...)$

2.6.2 Example

Using functorial composition of species we can express a variety of graph classes in terms of simple species of structure. For example the species of all simple graphs can be expressed as:

$$G = \rho\Box\rho^2$$

Where $\rho = E \cdot E$ species of subsets and $\rho^2 = E_2 \cdot E$ Structure on a set amounts to considering a pair of elements.

Chapter 3

Virtual species

We define the concept of a virtual species as a subtraction of two species. Addition ,multiplication work on virtual species as they do on regular species. We can find more details about section 3.1 and 3.2 in [**1**].

3.1 Virtual species

Virtual species allow us to give combinatorial meaning to the multiplicative inverse 1/E of species E of sets.

Definition: Let F and G be species . then we define the virtual species F-G as the element (F,G) in {(A,B):A and B are species}/∼, Where ∼ is the equivalence relation

$$(A, B) \sim (C, D) \Longleftrightarrow A + D = B + C$$

The additive inverse of F , denoted -F is given as -F=0-F

Note: That in this case that B is a subspecies of A, A-B may be thought of as the ordinary species of a structures that are not B structures .

Example: The combinatorial equation

$$G = G^c + G^d$$

allows us to define subtraction $G - G^c$ by setting

$$G - G^c = G^d$$

This combinatorial subtraction is possible because G^c is a subspecies of G.

Note: The set of virtual species is a commutative ring under addition and multiplication.

Example: The multiplicative inverse of the species E, consider the species E of sets

$$\frac{1}{E} = \frac{1}{1/1 + E_+}$$

$$= \sum_{n=0}^{\infty} (-1)^n (E_+^n)$$

we can only write this equation if the family is summable.
Every virtual species Φ can be written in reduce form.

$$\Phi = \Phi^+ - \Phi^-$$

3.1.1 Composition of virtual species

Definition: Let F,G,H and K be species of structures. one sets , for virtual species $\Phi = F - G$ and $\Psi = H - K$

1. $\Phi' = F' - G'$
2. $\Phi^\bullet = F^\bullet - G^\bullet$
3. $\Phi \times \Psi = (F \times H + G \times K) - (F \times K + G \times H)$
4. $\Phi \Box H = F \Box H - G \Box H$

Multisort species

The theory of species can be extended by considering structure constructed on sets containing several sorts of elements.

Example: The species of rooted tree constructed on a set having two sorts of element: leaves and internal vertex.
The transport are carried out along bijection preserving the sort of the elements. The transport of a rooted tree on two sorts (internal vertices and leaves).
The bijection $\sigma : U_1 + U_2 \to V_1 + V_2$ along which the transport is carried , send each internal vertex ($\in U_1$) to an internal vertex ($\in V_1$) and each leaf ($\in U_2$) to a leaf ($\in V_2$).

Definition: Let $m \geq 1$ be an integer . A species of m sorts is a rule F which

- Produces for each finite multi set$U = (U_1, U_2, ...U_m)$, a finite set $F[U_1, U_2, ...U_m]$.
- produces for each bijective multi function

  $$\sigma = (\sigma_1, \sigma_2, ...\sigma_m) : (U_1, U_2, ...U_m) \to (V_1, V_2, ...V_m)$$

  a function

  $$F[\sigma] = F[(\sigma_1, \sigma_2, ...\sigma_m)] : F[(U_1, U_2, ...U_m)] \to F[(V_1, V_2, ...V_m)]$$

Note the following combinatorial equation

$$F(mX, nY, ...) = F(X, Y) \times E(mX + nY....)$$

$$= F(X, Y) \times (E^m(X) \cdot E^n(Y).....)$$

Definition: The virtual species F(X-Y) on two sorts X,Y is defined by

$$F(X,Y) = F(X+Y) \times (E(X)E^{-1}(Y))$$

because we have this formula

$$F(mX,nY) = F(X+Y) \times (E^m(X)E^n(Y))$$

More generally for a virtual species

$$\Phi = \Phi(x) = F(x) - G(x)$$

$$\Phi(X-Y) = \Phi(X+Y) \times (E(X)E^{-1}(Y))$$
$$\Phi \circ \Psi = \Phi(X+Y) \times (E(X)E^{-1}(Y))$$

3.2 The species logarithm Ω

Let Ω be the com-positional inverse of ε_+, that is Ω is the virtual species that satisfies the equation

$$\varepsilon_+ \circ \Omega = \Omega \circ \varepsilon_+ = X$$

And Ω this species exist and it is unique .
The species $B = \Omega \circ A$ is known as the combinatorial logarithm of A.
This means that $A = \varepsilon_+ \circ B$ often $\Omega \circ A$ can be thought of as connected A structures.

Example: Lets take the species of graphs $\Omega \circ G = G^c$ this is true because arbitrary graphs are disjoint collections of connected graphs.

$$G = \varepsilon \circ G^c$$

and by left composing Ω

$$G^c = \Omega \circ G$$

in some cases there is no concept of connected A structures, and for some species , like linear order the species $\Omega \circ L$ is strictly virtual.

Definition: Let Φ be a virtual species . The species Φ^+ in the reduced form of Φ is called the positive part of Φ . if $\Phi^- \neq 0$ one says that Φ is strictly virtual. If $\Phi = \Phi^+$ one says that Φ is positive.
Example the species $\Omega \circ L$ is strictly virtual (has negative terms). $\Omega \circ G$ is positive (has no negative terms).

3.2.1 Proposition

Let F be a species of structure satisfying the condition $F(0) = 1$. Then there exist a unique virtual species Γ satisfying the combinatorial equation

$$F = E(\Gamma)$$

Where E denote species of sets.

$$1 + F_+ = (1 + E_+) \circ \Gamma$$

$$F_+ = E_+ \circ \Gamma$$

$$\Gamma = E_+^{(-1)} \circ F_+$$

This suggests the terminology which consists of saying that Γ is the combinatorial logarithm of the species F.
$\Gamma = F^c$ The virtual species of connected F structures.
and from this equation $F = E(F^c)$ we know

$$F(x) = e^{F^c}(x)$$

so

$$F^c(x) = logF(x)$$

and we also define this earlier that $F^c = \Omega \circ F$ now we can see this

$$(\Omega \circ F)(x) = logF(x)$$

The species logarithm Ω We utilize the concept of virtual species and the combinatorial logarithm to build other species.

3.3 Γ and Quotient species

Our goal will be to compute the cycle index of the species F/Γ in terms of that of F and information about the Γ - action , so that enumerative data about the quotient species can be extracted. The following concepts of Γ - species and quotient species can be found in [**4**]

3.3.1 Γ species

Some species have structures that are best described as orbits of another species structures under some group action.
For that we have to describe how a group can act on the structures.

Definition: For a Γ a group a Γ - species F is a combinatorial species F together with an action of Γ on F structures by species isomorphism.

Example: Let G denote the species simple graphs . Let the group S_2 act on such graphs by sending each graphs G to itself via the identity and sending G to its compliment G^c via the group element (1,2).

As an intermediate step to the computation of the cycle index associated to this quotient species. We associate a cycle index to a Γ species F.

Theorem: For a Γ species F, define the Γ cycle index Z_F^Γ , for each $\gamma \in \Gamma$

$$Z_F^\Gamma(\gamma) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} (fix(\gamma \cdot F[\sigma])) P_\sigma$$

The algebraic relationship between ordinary species and their cycle indices generally extend without modification to the Γ species context.

3.3.2 Quotient species

Under the action by Γ , a Γ - species F pass to the quotient species

Definition: Let Γ be a group and F be a Γ -species. We say the Γ- orbit of an F structure s is the orbit of s under the group action Γ.

Definition: For F a Γ - species, define F/Γ the quotient species of F under the action of Γ to be the species of Γ orbits of F structure.

Example: By passing the S_2- species G through its quotient, we put a graph G and its compliment G^c into an equivalence class. Thus , we have effectively constructed the species of pairs of compliments as G/S_2.

3.3.3 Theorem

For a Γ species F , the ordinary cycle index of quotient species F/Γ is given

$$Z_{F/\Gamma} = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} Z_F^{\Gamma}(\gamma)$$

$$= \frac{1}{|\Gamma|}\sum_{n=0}^{\infty}\frac{1}{n!}\sum\nolimits_{n\geq 0,\sigma\in S_n,\gamma\in\Gamma}(fix(\gamma\cdot F[\sigma]))P_\sigma$$

. This formula comes from Burnside's lemma and also we can see the proof in [**5**].

Chapter 4

The species of bipartite blocks

The enumeration of point determinig bipartite graphs appear to be absent from the literature. with the help of species theory and Γ- species we can solve the problem.

4.1 Introduction

Through out this section we denote by BC the species of bicolored graphs and BP the species of bipartite graphs.

Definition: A bipartite graph is a graph whose vertices can be divided into two disjoint and independent sets U and V such that every edge connects a vertex in V to one in U. A bipartite graph is a graph that does not contain any odd length cycles.
we can denote a bipartite graph as
$G = (U, V, E)$ whose partition has the parts U and V, with E denoting the edges of the graph.

Definition: A bicolored graph is a graph of which each vertex has been assigned one of two colors so that each edge connects vertices of different colors.

The number of bicolored graphs on n vertices

$$b_n = \sum_{i+j=n} \binom{n}{i} 2^{ij}$$

4.2 Cycle index series for bicolored graph

Now we want to compute the S_2 cycle index $Z_{BC}^{S_2}$, For this we will compute separately $Z_{BC}^{S_2}(e)$ and $Z_{BC}^{S_2}(\tau)$. a proof of $Z_{BC}^{S_2}(e)$ and $Z_{BC}^{S_2}(\tau)$ can be found in [**5**]
To compute the cycle index of a species, we need to enumerate the fixed points of each $\sigma \in S_n$.
Recall the formula for the cycle index of a Γ species

$$Z_{F/\Gamma} = \frac{1}{|\Gamma|} \sum_{n=0}^{\infty} \frac{1}{n!} \sum\nolimits_{\sigma \in S_n} \gamma \in \Gamma(fix(\gamma \cdot F[\sigma])) P_\sigma$$

4.2.1 Computing $Z_{BC}^{S_2}(e)$

For each $n > 0$ and each permutation $\pi \in S_n$ we omit empty graph and define $B(\pi) = \pi$ and let $\lambda \vdash n$ .we wish to count bicolored graphs for which a chosen permutation π of cycle type λ is a color preserving automorphism.
Consider an edge connecting two cycles of length m and n, the length of its orbit under the permutation is lcm(m,n) number of orbits of edge between these two cycles is $mn/lcm(m,n) = gcd(m,n)$.
We may then construct any possible graph fixed by our permutation by making a choice of a subset of these cycles to fill with edges, so the total number of such graphs is $\Pi 2^{gcd(m,n)}$ for a fixed coloring.

Now we focus on the possible coloring of the graph which are compatible with a permutation of specified cycle type λ . Let $\lambda = U \cup V$ here U corresponds to the white cycles and V the black.

Then the total number of graphs fixed by such a permutation with a specified decomposition is

$$fix(U, V) = \Pi 2^{gcd(i,j)}$$

Now L_i,m_i and n_i are the multiplicities of the part i in the partition λ, U and V respectively.Then the L_i i cycles can be colored in $\frac{L_i!}{M_i!,N_i!}$ number of ways.

so in all there are

$$\Pi \frac{L_i!}{m_i!n_i!} = \frac{Z_\lambda}{Z_u Z_v}$$

$$fix(\lambda) = \frac{Z_\lambda}{Z_u Z_v} fix(u, v)$$

$$= \sum_{u \cup v = \lambda} \frac{Z_\lambda}{Z_u Z_v} \Pi_{i \in u j \in v} 2^{gcd(i,j)}$$

Now we can obtain formula for $Z_{BC}^{S_2}(e)$

$$= \sum_{n>0} \sum_{(u,v)(u \cup v \vdash n)} \frac{P_{u \cup v}}{z_u z_v} \Pi_{i,j} 2^{gcd(u_i, v_j)}$$

4.2.2 Computing $Z_{BC}^{S_2}(\Gamma)$

The nontrivial element of $\Gamma \in S_2$ acts on bicolored graphs by reversing all colors. we wish to count bicolored graph on [n] for $\Gamma.\pi$ is an automorphism. Which is to say that π itself is a color reversing automorphism.

Each cycle of vertices must be coloralternating and hence of even length. So the partition λ must have only even parts..
The total number of possible white black edges is 2mn, each of which has an orbit length of $lcm(2m, 2n) = 2lcm(m, n)$. The total number of orbits is $\frac{2mn}{2lcm(m,n)} = gcd(m, n)$. Then the number of orbits for a fixed coloring of permutation of cycle type 2λ is

$$\sum_i [\lambda_i/2] + \sum_{i<j} 2^{gcd(\lambda_i,\lambda_j)}$$

. Thus total number of possible graph for a given vertex coloring is

$$\Pi_i 2^{[\lambda_i/2]} + \Pi_{i<j} 2^{gcd(\lambda_i,\lambda_j)}$$

.

$$Z_{BC}^{S_2}(\Gamma) = \sum_{n>0,neven} \sum_{\lambda \vdash n/2} 2^{l(\lambda)} \frac{p_{2\lambda}}{Z_{2\lambda}} \Pi_i 2^{[\lambda_i/2]} + \Pi_{i<j} 2^{gcd(\lambda_i,\lambda_j)}$$

4.3 Enumerating point-determining bipartite graphs

In this section , we pass from bicolored to bipartite graphs by taking a quotient under the color reversing action of S_2 only in the connected case.
For this we have to define some notation. We will refer to species of bicolored graphs as BC, the species of bipartite graphs as BP, the species of point determining graphs as P, the species of connected point determining bipartite graphs as CPBP. The following concepts of PBP , BP and CPBP species can be found in [**4**]

4.3.1 Definition

Given a graph G, recall that the neighborhood of a vertex $V \in V(a)$ is the set of vertices to which V connected by an edge. A point determining graph is a graph where no two vertices share a neighborhood.

4.3.2 Lemma 1

This lemma proves that point determining bipartite graph is connected bipartite graph. We can write PBP as

$$PBP = BP \circ \Omega$$

from the species logarithm Ω.

Proof: Consider a point determining bipartite graph p. Every vertex in P has a unique neighborhood, but an arbitrary bipartite graph G can have many vertices sharing the same neighborhood. Therefore each vertex in P corresponds to a nonempty set of vertices in G, each of which has the same neighborhood as the original vertex . This clearly respects transports so

$$PBP = BP \circ \Omega$$

4.3.3 Lemma 2

This lemma states: A connected bipartite graph is an orbit of connected bicolored graphs under the action of S_2. We can write CBP as

$$CBP = CBC/S_2$$

Proof: By operate S_2 action on CBC we can flip the color of each vertex to the opposite color and which produces another bicolored graph and that action is an involution for any CBC graph G. Let I be the image of G under the S_2 action, and if we remove the colors from G and I then we will get two copies of same bipartite graph P.(if K is the number of connected component of a bipartite graph G, then G may be properly bicolored in 2^k ways.) So P has exactly 2 proper bi-coloring so the S_2 -orbit of G is exactly the set of bicolored graphs that produce P when we remove the colors. Thus we can associate P with the S_2 orbit of G. This clearly respects transports, so the result holds.

4.3.4 Lemma 3

$$PBP = (\varepsilon \circ ((\Omega \circ BC_+)/S_2)) \circ \Omega$$

Proof: We know $PBP = BP \circ \Omega$. since CBP is connected bipartite graphs, $BP = \varepsilon \circ CBP$ and also $CBP = CBC/S_2$ .finally a nonempty bicolored graph is a nonempty set of connected bicolored graphs, so by left composing Ω, we get $CBC = (\Omega \circ BC_+)$. From there we can get the desired result.

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