Theta functions in one variable

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A dissertation submitted for the partial fulfillment of BS-MS dual degree in Science

Supervised by

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April 2019

Certificate of Examination

This is to certify that the thesis titled "Theta function in one variable" submitted by Jaideep Mali(MS14135) for the partial fulfillment of BS-MS dual degree program at IISER Mohali has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this thesis has been carried out by me under the able guidance of Prof. Kapil Hari Paranjape at the Indian Institute of Science Education and Research, Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other institute or university. Wherever contributions of others are involved, every effort has been made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bona fide record of work done by me and all sources listed within have been detailed in the bibliography.

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Dated: April 15, 2019

In my capacity as the supervisor of the candidate's work during the project, I duly certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Kapil Hari Paranjape

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Acknowledgements

First, I would like to express my sincere gratitude to Prof. Paranjape for providing me the opportunity to work on such a fascinating topic and helping me throughout this endeavor of exploring various mathematical dimensions. This project not only imparted rigorousness in me but also gave me the flavour of doing Research oriented science and trying out various possibilities in the process.

Second, I would like to thank my parents for their constant guidance, encouragement and support. Then definitely, friends finds the most important mention in the list owing to their constant support, to all the fun I had with them, sharing all lows and highs with them.

Jaideep Mali

Abstract

Theta functions are special functions in several complex variables. Theory of theta functions holds significance in many areas of mathematics including number theory, algebraic geometry, among others. Most common form of theta functions appear in theories of elliptic curves and elliptic functions.

During my dissertation project, my aim was to study the Riemann theta function $\theta(z, t)$ for $z \in \mathbb{C}$, $t \in \mathbb{H}$, and analyzing its behaviour with respect to variable z. Restricting the variables to reals, we also realize the function as a fundamental periodic solution to heat equation when x lies on a circle($\mathbb{S}^1$). Apart from the standard theta function, we can construct its variants also (with rational characteristics).Constructively, using them, we try to show how these functions can be used to embed the torus ($\mathbb{C}/\mathbb{Z} + \mathbb{Z}.t$) inside a complex projective space and in particular, take the case of projective 3-space ($\mathbb{P}^3$). Then we show using theta relations, how the equation for the image curve can be found and in the process, we obtain many Riemann theta relations.

Next, I try to define doubly periodic meromorphic functions (elliptic functions) on the elliptic curve $\mathbb{E}_t$ using the variants of theta functions.

Contents

Chapter 1

Preliminaries

1.1 Theta function $\theta(z, t)$:

The holomorphic theta function $\theta(z, t)$ in two variables can be defined as:

$$\theta(z,t) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 t} \cdot e^{2\pi i n z} \tag{1.1}$$

where $z \in \mathbb{C}$, $t \in \mathbb{H}$ (upper half plane).

1.2 Lattice λ_t

[DS05]Lattice λ_t generated by 1 and t (this is taken as we see some periodic properties of the function w.r.t two transformations $z \to z + 1$ and $z \to z + t$).
Generally a lattice λ in $\mathbb{C}$ is a set λ = $\mathbb{Z}\omega_1 \bigoplus \mathbb{Z}\omega_2$ such that $\{\omega_1, \omega_2\}$ are $\mathbb{R}$-linear independent vectors forming basis of $\mathbb{C}$ over the field $\mathbb{R}$.

1.3 Complex torus ($\mathbb{C}/\lambda_t$)

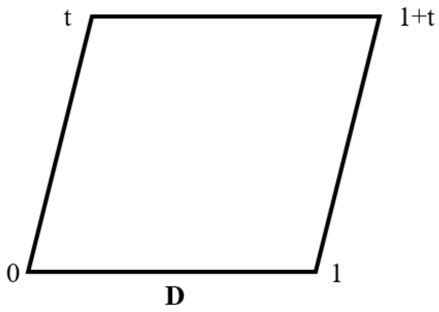
[DS05] It can be defined as the complex plane quotient out by a lattice.

$$\mathbb{C}/\lambda_t = z + \lambda_t : z \in \mathbb{C}$$

Geometrically, we can define it as a parallelogram spanned by basis vectors with its opposite sides identified. It is a compact Riemann surface.

1.4 Fundamental domain D for the complex torus ($\mathbb{C}/\lambda_t$)

Complex torus ($\mathbb{E}_t = \mathbb{C}/\lambda_t$) where λ_t = $\mathbb{Z} + \mathbb{Z}.t.$ Any $z \in \mathbb{C}$ upto translation has unique representation in D.



1.5 Projective n-space $\mathbb{P}^n$:

[ST92] Projective n-space can be defined as a set of equivalence classes of n+1 tuples.

$$\mathbb{P}^n = \{[a_0, a_1, a_2, ...a_n] \; s.t. \; a_0, ..., a_n \; not \; all \; zero\}/\equiv$$

, where $[a_0, a_1, ...a_n] \equiv [a'_0, a'_1, ...a'_n]$ if $a_0 = ta'_0, a_1 = ta'_1, .., a_n = ta'_n$ for $t \neq 0$.
If we take the case of n=2, we get,

$$\mathbb{P}^2 = \{[a, b, c] \; with \; a, b, c \; not \; all \; zero\}/\equiv$$

,where [a,b,c] $\equiv$ [a',b',c'] if $a = ta', b = tb', c = tc'$ for $t \neq 0$.
For n=1, we have $\mathbb{P}^1$ which is actually the extended complex plane or riemann surface $\mathbb{C} \cup \{\infty\}$

1.6 Notations

- $\frac{1}{l}\mathbb{Z}/\mathbb{Z} = \{0, \frac{1}{l}, \frac{2}{l}, ...\frac{l-1}{l}\}$
- $\frac{1}{l}\mathbb{Z}/l\mathbb{Z} \times \frac{1}{l}\mathbb{Z}/l\mathbb{Z} \equiv (\mathbb{Z}/l\mathbb{Z})^2$
- $\frac{1}{l}\mathbb{Z}/l\mathbb{Z} = \{0, \frac{1}{l}, \frac{2}{l}, ...\frac{l^2-1}{l}\}$

Chapter 2

Definition and Periodicity of theta function in z

2.1 Definition:

As defined above theta function $\theta(z,t)$ is a holomorphic function in two variables

$$\theta(z,t) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 t} \cdot e^{2\pi i n z} \tag{2.1}$$

where $z \in \mathbb{C}$, $t \in \mathbb{H}$. Now, let us check the convergence of the above series.
For that matter, let $z = u + iv$ and $t = x + iy,\ y > 0$ (due to above restrictions).Then

$$\begin{aligned}
\mid e^{\pi i n^2 t} e^{2\pi i n z} \mid &= \mid e^{\pi i (x+iy) n^2} e^{2\pi i n (u+iv)} \mid \\
&= \mid e^{\pi i x n^2 - \pi y n^2} \mid\mid e^{2\pi i n u - 2\pi n v} \mid \\
&= \mid e^{\pi i x n^2} \mid\mid e^{-\pi y n^2} \mid\mid e^{2\pi i n u} \mid\mid e^{-2\pi n v} \mid \\
&= \mid e^{-\pi n (yn + 2v)} \mid
\end{aligned}$$

(as $\mid e^{2\pi i n u} \mid, \mid e^{\pi i x n^2} \mid = 1.$)
Now, for large n's, we have, $\mid n \mid \leq \pi n(yn + 2v)$, as $(y > 0)$. Thus we have,

$$\mid e^{\pi i n^2 t} e^{2\pi i n z} \mid \leq e^{-|n|}$$

Therefore, we can say that the series converges uniformly and absolutely on compact sets.
[Mum83]Indeed we can show that it converges rapidly if we provide the condition that
if $\mid Im(z) \mid < c(const)$ and Im$(t) > \delta$ then

$$\mid e^{\pi i n^2 t} e^{2\pi i n z} \mid < (e^{-\pi\delta})^{n^2} (e^{2\pi c})^n$$

Thus, if we choose n_α such that

$$(e^{-\pi\delta})^{n_\alpha}(e^{2\pi c}) < 1$$

then

$$| e^{\pi in^2t+2\pi nz} |< (e^{-\pi\delta})^{n(n-n_\alpha)}$$

showing the rapidly converging nature of the series.

2.2 Periodicity in z

The above can be considered as a Fourier series of a function in z, periodic w.r.t $z \to z+1$,

$$\theta(z,t) = \sum_{n\in\mathbb{Z}} a_n(t)e^{(2\pi inz)}, \quad a_n(t) = e^{(\pi in^2t)}$$

By simple computation,

$$\theta(z+1,t) = \theta(z,t) \tag{2.2}$$

Now let's try to probe the periodic behaviour w.r.t $z \to z+t$,
thus we have,

$$\begin{aligned}
\theta(z+t,t) &= \sum_{n\in\mathbb{Z}} e^{(\pi in^2t+2\pi in(z+t))} \\
&= \sum_{n\in\mathbb{Z}} e^{(\pi i(n+1)^2t-\pi it+2\pi inz)} \\
&= \sum_{n\in\mathbb{Z}} e^{(\pi il^2t-\pi it+2\pi ilz-2\pi iz)} \; where \; l = n+1 \\
&= e^{(-\pi it-2\pi iz)}.\theta(z,t)
\end{aligned} \tag{2.3}$$

Thus, from (2.2) and (2.3) we find that $\theta(z,t)$ has periodic behaviour w.r.t the lattice λ_t generated by 1 and t.
Now,combining the two periodicities, we get an elegant result:

$$\theta(z+at+b,t) = e^{(-\pi ia^2t-2\pi iaz)}.\theta(z,t) \tag{2.4}$$

2.3 Converse behaviour

[Mum83]Now suppose if we look for an entire function g(z) displaying the simplest possible quasi-periodic behaviour w.r.t the lattice λ_t. Then by Liouville's theorem, g cannot actually be periodic in 1 and t, as that would yield a bounded entire function. So let's consider the slightly more general possibilities:

$$g(z+1) = g(z) \; and \;\; g(z+t) = e^{(az+b)}.g(z)$$

Let's think g in terms of Fourier series as:

$$\sum_{n\in\mathbb{Z}} a_n e^{(2\pi inz)}, \qquad a_n \in \mathbb{C}$$

Now writing g(z+t+1) in terms of g(z), we get:

$$g(z+t+1) = g(z+t) = e^{az+b}.g(z) \tag{2.5}$$

and other way we can write it as:

$$g(z+t+1) = e^{a(z+1)+b}.g(z+1)$$

$$= e^{a}.e^{az+b}.g(z) \tag{2.6}$$

thus equating (2.5) and (2.6), we get a = $2\pi ik$ for $k \in \mathbb{Z}$. Now substituting the Fourier series into (2.6), we get that,

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} a_n e^{2\pi int} e^{2\pi inz} \\
&= g(z+t) \\
&= e^{2\pi ikz+b} g(z) \\
&= \sum_{n\in\mathbb{Z}} a_n e^{2\pi i(n+k)z} e^{b} \\
&= \sum_{n\in\mathbb{Z}} a_{n-k} \; e^{b} \; e^{2\pi inz}
\end{aligned}$$

Or other way, we can say that we have, for all $n \in \mathbb{Z}$,

$$a_n = a_{n-k}.e^{(b-2\pi int)} \tag{2.7}$$

Now, let's consider the scenario case by case:

First, if k = 0, we shall have $a_n \neq 0$ for at most one n, because in that case we have

$$a_n(1 - e^{(b-2\pi int)}) = 0$$

$$\implies b = 2\pi int + 2\pi im, \qquad m \in \mathbb{Z}$$
$$\implies n = (\frac{b - 2\pi im}{2\pi it})$$

and for this we would have the possibility that g(z) = $e^{(2\pi iz)}$.
Second, if $k \neq 0$, we get a recursive relation for solving a_{n+ks} in terms of a_n $\forall s$.
For example, if k = -1, we find after solving the equation:

$$a_n = a_0 e^{(-nb+\pi in(n-1)t)} \qquad \forall n \in \mathbb{Z}$$

this implies that,

$$g(z) = a_0 \sum_{n\in\mathbb{Z}} e^{(-nb-\pi int)} e^{(\pi in^2t+2\pi inz)}$$
$$= a_0 \, \theta(-z - \frac{1}{2}t - \frac{b}{2\pi i}, t)$$

If $k > 0$, the relation (2.7) clearly leads to rapidly growing co-efficients a_n (due to successive multiplication by exponential terms) and hence we obtain no such entire functions g(z).
If we consider $k < -1$, then by solving the recursive relation, we end up getting $\mid k \mid$ - dimensional vector space of possibilities for g(z).(For instance, if k = -2, then we obtain the function $g(z)$ in terms of a_0 and a_1.)
Therefore, we can conclude that $\theta(z, t)$ is the most general entire function with such quasi periodic behaviour.

Chapter 3

Fundamental periodic solution to heat equation on a Circle

3.1 Heat equation

It is a parabolic partial differential equation describing the distribution of heat in a given region over a particular time period.
For a function $u(x, y, z, t)$, the heat equation is

$$\frac{\partial u}{\partial t} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

α is a constant. In our case, we restrict the variables z, t to the case $z = x \in \mathbb{R}$ and $t = it', \quad t' \in \mathbb{R}^+$. Then,

$$\theta(x, it) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} e^{2\pi i n x}$$

$$= 1 + 2\sum_{n\in\mathbb{N}} e^{-\pi n^2 t} cos(2\pi n x) \tag{3.1}$$

Thus obtaining a real valued function in two variables (x, t). Now let's check few properties of above equation:

1. Periodicity: $\theta(x + 1, it) = \theta(x, it)$

2. Heat equation:

$$\frac{\partial}{\partial t}(\theta(x, it)) = 2\sum_{n\in\mathbb{N}}(-\pi n^2)e^{-\pi n^2 t} cos(2\pi n x)$$

$$\frac{\partial^2}{\partial x^2}(\theta(x, it)) = 2\sum_{n\in\mathbb{N}}(-4\pi^2 n^2)e^{-\pi n^2 t} cos(2\pi n x)$$

$$\implies \frac{\partial}{\partial t}(\theta(x,it)) = \frac{1}{4\pi}\frac{\partial^2}{\partial x^2}(\theta(x,it))$$

This way we characterise the function $\theta(x,it)$ as a unique solution to heat equation. [A17] Now, we examine the limiting behaviour of $\theta(x,it)$ as $t \to 0$, for that purpose we integrate it against a test periodic function

$$g(x) = \sum_{l\in\mathbb{Z}} a_l e^{(2\pi i l x)}$$

Integrating we get,

$$\begin{aligned}
\int_0^1 \theta(x,it)g(x)dx &= \int_0^1 \sum_{n,l\in\mathbb{Z}} a_l e^{(-\pi n^2 t)}.e^{(2\pi i(n+l)x)}dx \\
&= \sum_{n,l\in\mathbb{Z}} a_l e^{(-\pi n^2 t)}.\int_0^1 e^{(2\pi i(n+l)x)}dx \\
&= \sum_n a_n e^{(-\pi n^2 t)}
\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}
&\lim_{t\to 0}\int_0^1 \theta(x,it)g(x)dx \\
&\quad = \lim_{t\to 0}\sum_n a_n exp(-\pi n^2 t) \\
&\quad = \sum_n a_n \\
&\quad = g(0)
\end{aligned}$$

. Thus, we can see that $\theta(x,it)$ converges, as a distribution, at all integral points, to the sum of delta functions as $t \to 0$.

Therefore, $\theta(x,it)$ may be seen as fundamental solution to heat equation when the variable $x \in \mathbb{S}^1$.

Chapter 4

Theta functions with characteristics or variants of theta functions

Apart from the standard theta function θ which we now denote as θ_{00}, there are other variants also called theta functions with rational characteristics a, b which are essential in comprehending, inter-alia, various geometric applications of θ, functional equation of $\theta(z, t)$, identities satisfied by it.

Let's work them out. For that, we fix t and introduce transformations as:

For every analytic function f(z) and $a, b \in \mathbb{R}$, take:

$$P_b f(z) = f(z + b)$$
$$Q_a f(z) = e^{(\pi i a^2 t + 2\pi i a z)} f(z + at)$$

The motivation for choosing such tranformations emanated from the periodic behaviour of θ functions.Now we have,

$$P_{b_1}(P_{b_2})f = P_{b_1+b_2} f \quad \text{and} \quad Q_{a_1}(Q_{a_2})f = Q_{a_1+a_2} f$$

$$\begin{aligned} P_b(Q_a)f &= Q_a f(z + b) \\ &= e^{(\pi i a^2 t + 2\pi i a(z+b))} f(z + b + at)) \end{aligned}$$

and similarly,

$$\begin{aligned} Q_a(P_b f) &= e^{(\pi i a^2 t + 2\pi i a z)} P_b f(z + at) \\ &= e^{(\pi i a^2 t + 2\pi i a z)} f(z + at + b) \end{aligned}$$

therefore,

$$P_b \circ Q_a = e^{(2\pi iab)}(Q_a \circ P_b) \tag{4.1}$$

Thus P_b and Q_a doesn't commute. The group of tranformations generated by Q_a's and P_b's is the three dimensional group:

$$G = \mathbb{C}_1 \rtimes (\mathbb{R} \times \mathbb{R}) \qquad where\ \mathbb{C}_1 = (z \in \mathbb{C}\ s.t\ \mid z \mid = 1)$$

Now, (λ, a, b): tranformation represents an element of group G. It's operation can be realised as:

$$(X_{(\lambda,a,b)}f)(z) = \lambda((Q_a \circ P_b)f)(z)$$

$$= \lambda e^{(\pi i a^2 t + 2\pi i a z)} f(z + at + b) \tag{4.2}$$

Group law on G can be given by:

$$(\lambda, a, b).(\lambda, a', b') = (\lambda\lambda' e^{(2\pi i b a')}, a + a', b + b')$$

So, we have

$$Z(G) = \mathbb{C}_1^* = [G, G]$$

this implies G is nilpotent group. Now, we take the discrete subgroup of G:

$$\tau = \{(1, a, b) \in G | a, b \in \mathbb{Z}\}$$

. Moreover we have from (4.2) and by the way of characterization of theta function in chapter 2, that upto scalars, θ is the unique entire function invariant under τ.
Now let $l \in \mathbb{Z}^+$ and we set $l\tau = (1, la, lb)$. Clearly $l\tau \subset \tau$ (from the definition). Let's consider another set
S_l = {entire functions f(z) invariant under $l\tau$ }. θ functions upto scalars definitely falls in S_l. Now we have the following lemma that enumerates all such entire functions.

Lemma 4.0.1. An entire function f(z) is in S_l if and only if

$$f(z) = \sum_{n \in \frac{1}{l}\mathbb{Z}} c_n\, e^{(\pi i n^2 t + 2\pi i n z)}$$

such that $c_n = c_m \quad if\ n - m \in l\mathbb{Z}$ and in particular, Dim(S_l) = l^2.[Mum83]

Proof:
As usual, we identify Q_a with $(1, a, 0) \in$ G and P_b with $(1, 0, b) \in G$ for $a, b \in \mathbb{R}$.

First, its given that $f \in S_l$, then by its invariance under $Q_l \in l\tau$, we have

$$f(z) = \sum_{n \in \frac{1}{l}\mathbb{Z}} c_n^* e^{2\pi i n z}$$

Then, if we write c_n^* = $c_n e^{\pi i n^2 t}$ and express its invariance under P_l, we get $c_{n+l} = c_n$ for all $n \in \mathbb{Z}$.

Conversely, if we consider such a function f(z), then by the way of characterization of functions under S_l, it is held that such f(z) lies in S_l.

Now, for $i \in \mathbb{N}$, let $\mu_i \subseteq \mathbb{C}_1^*$, group of i^{th} roots of unity. For $l \in \mathbb{N}$, let G_l be a finite group defined as:

$$\begin{aligned} G_l &= \{(\lambda, a, b) | \lambda \in \mu_{l^2}; a, b \in \frac{1}{l}\mathbb{Z}/l\mathbb{Z}\} \; (mod \; l\tau) \\ &= \mu_{l^2} \rtimes \left(\left(\frac{1}{l}\mathbb{Z}/l\mathbb{Z} \right) \times \left(\frac{1}{l}\mathbb{Z}/l\mathbb{Z} \right) \right) \end{aligned}$$

with $Z(G_l) = \mu_{l^2}$. Generators of G_l are $P_{1/l}$ and $Q_{1/l}$. Indeed like G, generators $P_{1/l}$ and $Q_{1/l}$ acts on S_l as following:

$$P_{1/l}\left(\sum_{n \in \frac{1}{l}\mathbb{Z}} c_n e^{(\pi i n^2 t + 2\pi i n z)}\right) = \sum_{n \in \frac{1}{l}\mathbb{Z}} c_n e^{(2\pi i n/l)} e^{(\pi i n^2 t + 2\pi i n z)}$$

Similarly, we have

$$Q_{1/l}\left(\sum_{n \in \frac{1}{l}\mathbb{Z}} c_n e^{(\pi i n^2 t + 2\pi i n z)}\right) = \sum_{n \in \frac{1}{l}\mathbb{Z}} c_{n-\frac{1}{l}} e^{(\pi i n^2 t + 2\pi i n z)}$$

Now, we have the following lemma:

Lemma 4.0.2. Group G_l acts irreducibly on S_l.

Proof: A representation $\phi : G \to GL(V)$ is irreducible if and only if the only G-invariant(G-stable subspace) of V are 0 and V i.e. no proper invariant subspace.
Here, we will show it by letting Y $\subset S_l$, a G_l - invariant subspace. Now,we take $g \neq 0 \in Y$, like,

$$g(z) = \sum_{n \in \frac{1}{l}\mathbb{Z}} c_n e^{(\pi i n^2 t + 2\pi i n z)}, \; c_{n_0} \neq 0. \tag{4.3}$$

Then, operating by powers of $P_{1/l}$ on g(z), we find in Y:

$$\sum_{0\leq r\leq l^2-1} e^{(-2\pi i n_0 r/l)}.(P_{r/l}g)(z)$$
$$= \sum_{n\in\frac{1}{l}\mathbb{Z}} c_n(\sum_{0\leq r\leq l^2-1} e^{(2\pi i(n-n_0)r/l)}.e^{(\pi i n^2 t+2\pi i n z)})$$
$$= l^2 c_{n_0}(\sum_{n\in n_0+l\mathbb{Z}} exp(\pi i n^2 t + 2\pi i n z)).$$

From (4.3), since $c_{n_0} \neq 0$, we get that Y contains the function of the kind:

$$\sum_{n\in n_0+l\mathbb{Z}} e^{(\pi i n^2 t+2\pi i n z)}.$$

Similarly, operating with $Q_{1/l}$, we find that Y contains similar functions for every $n_0 \in \frac{1}{l}\mathbb{Z}/l\mathbb{Z}$ and thus, $Y = S_l$. Therefore, we can conclude that G_l acts irreducibly on S_l.
Now comes the crucial point in the picture: Because of the irreducibility, the action of G_l on S_l determines the canonical basis of S_l.
A representation is irreducible if and only if for every non-zero $0 \neq v \in V$, the vectors $g.v_{g\in G}$ span V i.e. the canonical basis for S_l. In our case this helds as v are of the type of functions mentioned in Lemma(4.0.1).

Therefore, the standard basis of S_l is given by the so called theta functions $\theta_{a,b}$ with rational characteristics $a, b \in \frac{1}{l}\mathbb{Z}$, defined by: for $a, b \in \frac{1}{l}\mathbb{Z}$,

$$\theta_{a,b} = (P_b Q_a)\theta = e^{(2\pi i ab)}(Q_a P_b)$$

So, we have:

$$\theta_{a,b}(z,t) = e^{(\pi i a^2 t+2\pi i a(z+b))}\theta(z + at + b, t)$$
$$= \sum_{n\in\mathbb{Z}} e^{(\pi i(a^2+n^2)t+2\pi i n(z+at+b)+2\pi i a(z+b))}$$
$$= \sum_{n\in\mathbb{Z}} e^{\pi i(a+n)^2 t+2\pi i(n+a)(z+b))}$$

.

Now, let's sumarize the properties of $\theta_{a,b}$:

- $\theta_{0,0} = \theta$
- $P_{b_1}(\theta_{a,b}) = \theta_{a,b+b_1}$ for $a, b, b_1 \in \frac{1}{l}\mathbb{Z}$

- $Q_{a_1}(\theta_{a,b}) = e^{(-2\pi i a_1 b)}\theta_{a_1+a,b}$, for all $a, b, a_1 \in \frac{1}{l}\mathbb{Z}$
- $\theta_{a+s,b+r} = e^{(2\pi i a r)\theta_{a,b}}$ for all $s, r \in \mathbb{Z},\ a, b \in \frac{1}{l}\mathbb{Z}$

Thus, from the above properties, we can say that except for the exponential factor, $\theta_{a,b}$ seems merely a translate of θ. we can infer that $\theta_{a,b}$ upto a constant, depends only on $a, b \in \frac{1}{l}\mathbb{Z}/\mathbb{Z}$. From the Lemma(4.0.1) and the above fourier series expansion of $\theta_{a,b}$, it can be seen that a, b run through coset representatives of $(\frac{1}{l}\mathbb{Z}/\mathbb{Z})$ that's

$$a_i, b_i \in \left(0, \frac{1}{l}, \frac{2}{l}, \frac{3}{l}, ., ., .., \frac{l-1}{l}, 1\right) \times \left(0, \frac{1}{l}, \frac{2}{l}, \frac{3}{l}, ., ., .., \frac{l-1}{l}, 1\right)$$

, forming the basis of S_l i.e. $\theta_{a,b}$'s.

Chapter 5

Projective embedding of torus($\mathbb{C}/\mathbb{Z}+\mathbb{Z}t$) via variants of theta functions($\theta_{a,b}$):

This illustrates one of the most elegant geometric application of theta functions $\theta_{a,b}$ defined above.
Let E_t be the complex torus $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}t)$ (a compact riemann surface). Take $l > 1$. Let a_i, b_i be the coset representatives of $\left(\frac{1}{l}\mathbb{Z}/\mathbb{Z}\right) \times \left(\frac{1}{l}\mathbb{Z}/\mathbb{Z}\right)$ in $\left(\frac{1}{l}\mathbb{Z}\right)^2$(defined in last section.)
For simplicity, we denote,

$$\theta_i = \theta_{a_i,b_i}, \qquad 0 \le i \le l^2 - 1$$

Now, $\forall\ z \in \mathbb{C}$, consider the l^2 - tupple

$$(\theta_0(lz,t), \theta_1(lz,t), ..., \theta_{l^2-1}(lz,t))$$

modulo scalars(so that they represents the homogeneous co-ordinates of a point in projective space $\mathbb{P}_{\mathbb{C}}^{l^2-1}$). Now, due to periodicity of θ functions, we have the following relations:

1. $(\theta_0(z+l,t), ..., \theta_{l^2-1}(z+l,t))$ = $(\theta_0(z,t), ..., \theta_{l^2-1}(z,t))$ and
2. $(\theta_0(z+lt,t), ..., \theta_{l^2-1}(z+lt,t))$ = $e^{(-\pi i l^2 t - 2\pi i l z)}(\theta_0(z,t), ..., \theta_{l^2-1}(z,t))$

[Mum83]Thus, we define a holomorphic map from torus to projective space:

$$\phi_l : E_t \to \mathbb{P}^{l^2-1},$$
$$z \to (...., \theta_i(lz,t), ...).$$

Zeros of $\theta_{a,b}$ are obtained at points $z = (s+a+\frac{1}{2})t + (\frac{1}{2}+b+r), \quad s, r \in \mathbb{Z}$. Hence, θ_{00} has a zero at $\frac{1}{2}(1+t)$. In particular, there is no z, t for which they are all zero, thus $\theta_i's$ have no common zeros and thus by the definition of projective space, the above map ϕ_l is well defined.

Let us prove the following lemma to contemplate the above map:

Lemma 5.0.1. Every $f \in S_l$, $f \neq 0$, has exactly l^2 zeros (with multiplicities) in a fundamental domain for $\mathbb{C}/l\lambda_t$, where $\lambda_t = \mathbb{Z} + \mathbb{Z}t$.

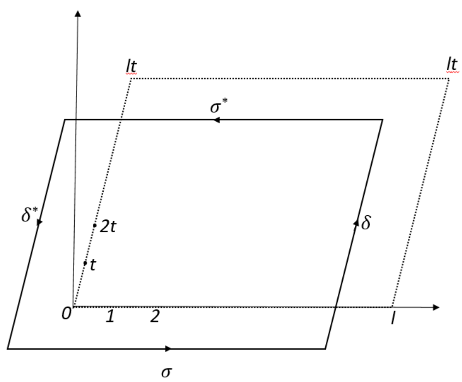
Proof: First, the dim(S_l) = l^2. Every $f \in S_l$ is of the kind

$$f(z) = \sum_{n \in \frac{1}{l}\mathbb{Z}} c_n \, e^{(\pi i n^2 t + 2\pi i n z)}$$

such that $c_n = c_m \quad if \; n - m \in l\mathbb{Z}$. Now, we calculate the zeros of $f \in S_l$ by contour integration via invoking the theorem that says:
[McQ03] If f is analytic inside and on the curve C except for a possibly finite number of poles and if f is non-zero on the curve C then we have:

(Zeros of f - Poles of f) = $\frac{1}{2\pi i} \int_C \frac{f'}{f} dz$

In our case, as f is an entire function, it will have no poles. So as per the above theorem, we choose a parallelogram missing the zeros of f as shown:



As there are no poles, we have:

$$Zeros\ of\ f = \frac{1}{2\pi i} \int_C \frac{f'}{f} dz$$

Since, we have the doubly-periodic behaviour of the function f, i.e., $f(z + l) = f(z)$ and $f(z + lt) = c.e^{-2\pi i l z}.f(z)$, where c = constant, we get:

$$\int_\delta + \int_{\delta^*} = (2\pi i) l^2 \; and \; \int_\sigma + \int_{\sigma^*} = 0.$$

Now, as we know by definition that $\theta(z,t)$ is even in z and we also compute the following:

$$\begin{aligned}\theta_{\frac{1}{2},\frac{1}{2}}(-z,t) &= \sum_{m\in\mathbb{Z}} e^{((\pi i(m+\frac{1}{2})^2)t+2\pi i(m+\frac{1}{2})(-z+\frac{1}{2}))} \\ &= \sum_{n\in\mathbb{Z}} e^{((\pi i(-n-\frac{1}{2})^2)t+2\pi i(-n-\frac{1}{2})(-z+\frac{1}{2}))}, where\ n=-1-m \\ &= \sum_{n\in\mathbb{Z}} e^{((\pi i(n+\frac{1}{2})^2)t+2\pi i(n+\frac{1}{2})(z+\frac{1}{2})-2\pi i(n+\frac{1}{2}))} \\ &= -\theta_{\frac{1}{2},\frac{1}{2}}(z,t)\end{aligned}$$

This proves that $\theta_{\frac{1}{2},\frac{1}{2}}$ is an odd function and thus is 0 at z=0. Now,since the above sum gives l^2 zeros (mod $l\lambda_t$), $\theta_{a,b}$ has the zeros as given.

Next, we show that the map ϕ_l is equivariant for the action of $G_l/Z(G_l)$.

If X and Y are both G-sets(action of G:g) for the same group G, then $f : X \to Y$ is said to be equivariant if $f(g.x) = g.f(x)$ for all $g \in G$ and $x \in X$. We can show it using the commutative diagram as:

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Y \end{array}$$

For our case, we consider the action of the group $G_l/Z(G_l)$, that's $\left(\frac{1}{l}\mathbb{Z}/l\mathbb{Z}\right)^2$ on the torus E_t and $\mathbb{P}^{l^2-1}$.
First, let's consider the action on E_t, so let $a, b \in \frac{1}{l}\mathbb{Z}$

$$\left(\frac{1}{l}\mathbb{Z}/l\mathbb{Z}\right)^2 \curvearrowright E_t$$

$$\left(\frac{1}{l}\mathbb{Z}/l\mathbb{Z}\right)^2 \times E_t \to E_t$$

$$z \to z + (az+b)/l$$

Now, the second one, i.e., the action on $\mathbb{P}^{l^2-1}$.
Here, if $X_{(1,a,b)}v_i = \sum_{0\le j\le l^2-1} c_{ij}v_j$. Precisely, as deduced from the last section, θ_i's are the basis vectors of V_l and $(1,a,b) \in G_l$

Action of $G_l/Z(G_l)$ on $\mathbb{P}^{l^2-1}$ can be given by:

$$G_l/Z(G_l) \curvearrowright \mathbb{P}^{l^2-1}$$

$$(z_0, z_1, ..., z_{l^2-1}) \to (\sum_j c_{0,j}z_j, \sum_j c_{1,j}z_j, ..., \sum_j c_{l^2-1,j}z_j)$$

Now, having defined the two actions, we are in the position to prove equivariance relation. So we have,

$$\phi_l(z + (at+b)/l) = (...., \theta_i(lz + at + b, t), ...) \ \in \mathbb{P}^{l^2-1}$$

$$= (....., X_{(1,a,b)}\theta_i(lz, t), ...)$$
$$= (....., \sum_j c_{ij}\theta_j(lz, t), ...)$$

Diagramatically, we can represent it as:

$$\begin{array}{ccc} E_t & \xrightarrow{r} & E_t \\ {\scriptstyle\phi_l}\downarrow & & \downarrow{\scriptstyle\phi_l} \\ \mathbb{P}^{l^2-1} & \xrightarrow{r} & \mathbb{P}^{l^2-1} \end{array}$$

Here, r represents action of $(\frac{1}{l}\mathbb{Z}/l\mathbb{Z})^2$. Therefore, we can say that ϕ_l is an equivariance. Moreover, action, $(\frac{1}{l}\mathbb{Z}/l\mathbb{Z})^2 \cong (\mathbb{Z}/l^2\mathbb{Z}) \ \curvearrowright \mathbb{P}^{l^2-1}$ is irreducible(..from Lemma (4.0.2), as elements of $\mathbb{P}^{l^2-1}$ are theta functions). Thus, there are no proper invariant subspace.

Now, we are left to prove the most important part of this construction i.e., ϕ_l is an embedding.

We are given a holomorphic map $\phi_l : \ E_t \to \mathbb{P}^{l^2-1}$.

Embedding can be defined as a homeomorphism onto its image $\phi_l(E_t)$. Explicitly, it is an injective continous map which yields homeomorphism between E_t and $\phi_l(E_t)$.In particular, image of an embedding $\phi_l(E_t)$ is a complex submanifold.

We will prove the embedding by contradiction.[Mum83]

Suppose, we take that $\phi_l(z_1)$ = $\phi_l(z_2)$, $z_1 \neq z_2$ in E_t.
Then, translating by some $(at+b)/l$ where $a, b \in \frac{1}{l}\mathbb{Z}$, we find another pair z_1', z_2' such that $\phi_l(z_1') = \phi_l(z_2')$.
Take $l^2 - 3$ points $y_1, y_2, ...y_{l^2-3}$, all points distinct modulo $(l\lambda_t)$. From above lemma(5.1) we know that $f \in S_l$ has l^2 $roots$ $(mod(l\lambda_t))$.

Then we seek an $f \in S_l, f \neq 0$, such that:

$$f(z_1) = f(z_1') = f(y_1) = f(y_1) = ... = f(y_{l^2-3}) \tag{5.1}$$

We could write this because, if we write $f = \sum_i \lambda_i \theta_i$, $\lambda_i \in \mathbb{C}$, we get $(l^2 - 1)$ linear equations in l^2 variables $\lambda_0, ..., \lambda_{l^2-1}$ in case of (5.1).
Since, $\phi_l(z_1) = \phi_l(z_2)$, we have $f(z_2) = 0$, or we can have, if d$\phi_l(z_1) = 0$, then f has double zero at z_1. Similarly, we get for $f(z_2') = 0$, or in other words, f has double zero at z_1'.
Thus, from this, we can infer that f has at least $(l^2 + 1)$ zeros in the torus $\mathbb{C}/l\lambda_t$ hence, contradicting the above lemma (5.0.1).

Therefore, we have a injective homeomorphism onto its image $\phi_l(E_t)$.

We can comment even more by using Chow's theorem here to say that $\phi_l(E_t)$ is even an algebraic subvariety i.e., it can be defined by certain homogenous polynomials.
Chow's theorem says that any closed analytic subspace of complex projective space $\mathbb{P}^{l^2-1}$ is an algebraic subvariety that is, we can define it by certain homogeneous polynomials.This we will see in next chapter.

Chapter 6

Riemann theta Relations

Riemann theta relations are quadratic identities satisfied by theta functions ($\theta(z,t)$). Such identities can be obtained using an integral matrix $[P]_{n\times n}$ such that $P^tP = n^2 I_m$
Suppose for our consideration, we take,

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Clearly, this is a symmetric matrix with n =2 and m = 4, with matrix identity $P^tP = 4I_4$.

Definition 6.0.1. A homogenous polynomial of degree two in n-variables and coefficients over field $\mathbb{F}$ is called an n-ary quadratic form over $\mathbb{F}$.

We can use nxn symmetric matrix to express a quadratic form as follows:

$$\phi(x_1, ...x_n) = \sum_{1\leq i\leq n} a_{ij}x_ix_j = v^tAv = \phi(v)$$

where A = $\left(\frac{a_{ij}+aji}{2}\right)$ and v is a column vector. The above identity $P^tP = 4I_4$ can be shown equivalent to the identity between quadratic forms(homogeneous polynomials of degree 2):

To observe the equivalence, let's perform the computation:

$$(Pv)^t.(Pv)$$

$$= v^t(P^tP)v$$

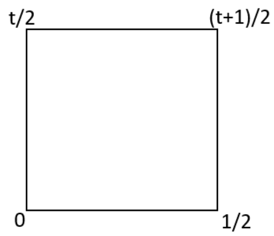
$$= 4(v^tv).$$

$$\left(\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\right)^t\left(\begin{bmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\right)=4\left[\mathrm{a}^2+b^2+c^2+d^2\right]$$

Thus, giving rise to identity between quadratic forms

$$((a+b+c+d)^2+(a+b-c-d)^2+(a-b+c-d)^2+(a-b-c+d)^2)=4(a^2+b^2+c^2+d^2)$$

For future purposes, we write $\theta(z,t) = \theta(z)$ and lattice $\lambda_t = \lambda$. [Mum83]Now, we take the lattice $\frac{1}{n}\lambda/\lambda$ in general and from above,in our case, we have $n = 2$.



Thus, for all choices of α in the chosen lattice($\frac{1}{2}\lambda/\lambda$), we perform the following operations: First,

A(0) := $\theta(a+0)\theta(b+0)\theta(c+0)\theta(d+0)$

$$=\sum_{p,q,r,s\in\mathbb{Z}} e^{(\pi i(\sum p^2)t+2\pi i(\sum ap))}$$

where $\sum p^2 = p^2+q^2+r^2+s^2$ and $\sum ap = ap+bq+cr+ds$
similarly the other ones,

A($\frac{1}{2}$) := $\theta(a+\frac{1}{2})\theta(b+\frac{1}{2})\theta(c+\frac{1}{2})\theta(d+\frac{1}{2})$

$$=\sum_{p,q,r,s\in\mathbb{Z}} e^{\left(\pi i(\sum p+\sum p^2)t+2\pi i(\sum ap)\right)}$$

A($\frac{t}{2}$) := $e^{(\pi i(t+\sum a))}\theta(a+\frac{t}{2})\theta(b+\frac{t}{2})\theta(c+\frac{t}{2})\theta(d+\frac{t}{2})$

$$=\sum_{p,q,r,s\in\mathbb{Z}} e^{\left(\pi i(\sum(p+\frac{1}{2})^2t)+2\pi i(\sum a(p+\frac{1}{2}))\right)}$$

$$\mathrm{A}(\tfrac{t+1}{2}) := e^{(\pi i(t+\sum a))}\theta(a+\tfrac{t+1}{2})\theta(b+\tfrac{t+1}{2})\theta(c+\tfrac{t+1}{2})\theta(d+\tfrac{t+}{2})$$

$$= \sum_{p,q,r,s\in\mathbb{Z}} e^{\left(\pi i(\sum p)+\pi i(\sum(p+\frac{1}{2})^2)+2\pi i(\sum a(p+\frac{1}{2}))\right)}$$

We can see that we have multiplied an extra exponential factor while deriving A($\frac{t}{2}$) and A($\frac{t+1}{2}$). Let's call it, e_α (shortly, we will know why we have put this extra exponential terms).
Now summing the above products as following, we obtain:

$$\sum_{\alpha\in\frac{1}{2}\lambda/\lambda} e_\alpha\theta(a+\alpha)\theta(b+\alpha)\theta(c+\alpha)\theta(d+\alpha)$$

$$= 2\sum_{p,q,r,s\in\frac{1}{2}\mathbb{Z}} e^{\left(\pi i(\sum p^2)t+2\pi i(\sum pa)\right)} \tag{6.1}$$

Here,we have the case of p, q, r, s all in $\mathbb{Z}$ or all of them in $\mathbb{Z}+\frac{1}{2}$. But in either of the case, we get their sum $p+q+r+s \in 2\mathbb{Z}$.

Now, let's reformulate the variables as

$$\begin{aligned} p_1 &= \frac{1}{2}(p+q+r+s), & a_1 &= \frac{1}{2}(a+b+c+d)\\ q_1 &= \frac{1}{2}(p+q-r-s), & b_1 &= \frac{1}{2}(a+b-c-d)\\ r_1 &= \frac{1}{2}(p-q+r-s), & c_1 &= \frac{1}{2}(a-b+c-d)\\ s_1 &= \frac{1}{2}(p-q-r+s), & d_1 &= \frac{1}{2}(a-b-c+d) \end{aligned}$$

Above, change of variables ensure that the new variables appear integers. Moreover, it also preserves the relations:

$$\sum p^2 = \sum p_1^2, \qquad \sum ap = \sum a_1p_1 \tag{6.2}$$

Now, putting (6.2) in (6.1), we obtain the following relation:

$$\sum_{\alpha\in\frac{1}{2}\lambda/\lambda} e_\alpha\theta(a+\alpha)\theta(b+\alpha)\theta(c+\alpha)\theta(d+\alpha)$$

$$= 2\sum_{p,q,r,s\in\frac{1}{2}\mathbb{Z}} e^{\left(\pi i(\sum p_1^2)t+2\pi i(\sum p_1a_1)\right)} \tag{6.3}$$

Thus, yielding our first Riemann theta identity,$R(1)$ (from the relation of A(0) and eqn

(6.3))

$$\sum_{\alpha\in\frac{1}{2}\lambda/\lambda} e_\alpha\theta(a+\alpha)\theta(b+\alpha)\theta(c+\alpha)\theta(d+\alpha) = 2\theta(a_1)\theta(b_1)\theta(c_1)\theta(d_1) \tag{6.4}$$

Had we took some other matrix satisfying $P_tP = n^2I_m$ and then chosen the lattice according to the value of n in ($\frac{1}{n}\lambda/\lambda$), we would have obtained another identity of order n, involving summation over all translates of θ with respect to $\alpha \in \frac{1}{n}\lambda/\lambda$.

Next, for deriving other Riemann theta relations, we reformulate theta functions(θ) using theta functions with rational characteristics $a, b \in \frac{1}{2}\mathbb{Z}$ as follows:

$$\theta_{0,0}(z,t) = \sum_{n\in\mathbb{Z}} e^{(\pi i n^2 t + 2\pi i n z)} = \theta(z,t)$$

$$\theta_{0,\frac{1}{2}}(z,t) = \sum_{n\in\mathbb{Z}} e^{(\pi i n^2 t + 2\pi i n(z+\frac{1}{2}))} = \theta((z+\frac{1}{2}),t)$$

$$\theta_{\frac{1}{2},0}(z,t) = \sum_{n\in\mathbb{Z}} e^{\left(\pi i(n+\frac{1}{2})^2 t + 2\pi i(n+\frac{1}{2})z\right)} = e^{(\frac{\pi i t}{4}+\pi i z)}\theta((z+\frac{1}{2}t),t)$$

$$\theta_{\frac{1}{2},\frac{1}{2}}(z,t) = \sum_{n\in\mathbb{Z}} e^{(\pi i(n+\frac{1}{2})^2 t + 2\pi i(n+\frac{1}{2})(z+\frac{1}{2}))} = e^{(\frac{\pi i t}{4}+\pi i(z+\frac{1}{2}))}\theta((z+\frac{1}{2}(t+1),t))$$

This explains the significance of exponential factors(e_α) taken initially. For convenience, we call the above functions as θ_{00}, θ_{01}, θ_{10}, θ_{11}.
By the above equations, it is clear that the first three are even functions and the fourth one is odd function i.e.,

$$\theta_{0,0}(-z,t) = \theta_{0,0}(z,t)$$

$$\theta_{0,1}(-z,t) = \theta_{0,1}(z,t)$$

$$\theta_{1,0}(-z,t) = \theta_{1,0}(z,t)$$

$$\theta_{1,1}(-z,t) = -\theta_{1,1}(z,t)$$

Therefore, $\theta_{1,1}(0,t) = 0$ at $z = 0$.(confirms the fact deduced in lemma(5.0.1)). Now, we can rewrite above Riemann's formula (R1) in terms of θ_{00}, θ_{01}, θ_{10}, θ_{11} and thus, obtaining our second Riemann's formula (R2):

$$\theta_{00}(a)\theta_{00}(b)\theta_{00}(c)\theta_{00}(d) + \theta_{01}(a)\theta_{01}(b)\theta_{01}(c)\theta_{01}(d)$$

$$+\theta_{10}(a)\theta_{10}(b)\theta_{10}(c)\theta_{10}(d) + \theta_{11}(a)\theta_{11}(b)\theta_{11}(c)\theta_{11}(d)$$

$$= 2\theta_{00}(a_1)\theta_{00}(b_1)\theta_{00}(c_1)\theta_{00}(d_1)$$

Furthermore, we can play with the above Riemann theta relations by making few modifications and can obtain many above such relations.
For instance, replacing $a+1$ for a, and this way accommodating the changing sign of $\theta_{1,0}$ and $\theta_{1,1}$, we get (R3)

$$\theta_{00}(a)\theta_{00}(b)\theta_{00}(c)\theta_{00}(d) + \theta_{01}(a)\theta_{01}(b)\theta_{01}(c)\theta_{01}(d)$$
$$+\theta_{10}(a)\theta_{10}(b)\theta_{10}(c)\theta_{10}(d) - \theta_{11}(a)\theta_{11}(b)\theta_{11}(c)\theta_{11}(d)$$
$$= 2\theta_{01}(a_1)\theta_{01}(b_1)\theta_{01}(c_1)\theta_{01}(d_1)$$

Then, replacing a with $a+t$ and multiplying by $e^{(\pi it+2\pi ia)}$, we get another formula, (R4).

$$\theta_{00}(a)\theta_{00}(b)\theta_{00}(c)\theta_{00}(d) - \theta_{01}(a)\theta_{01}(b)\theta_{01}(c)\theta_{01}(d)$$
$$+\theta_{10}(a)\theta_{10}(b)\theta_{10}(c)\theta_{10}(d) - \theta_{11}(a)\theta_{11}(b)\theta_{11}(c)\theta_{11}(d)$$
$$= 2\theta_{10}(a_1)\theta_{10}(b_1)\theta_{10}(c_1)\theta_{10}(d_1)$$

Similarly, replacing a with $a+t+1$ and multiplying by $e^{(\pi it+2\pi ia)}$, we get another formula, (R5).

$$\theta_{00}(a)\theta_{00}(b)\theta_{00}(c)\theta_{00}(d) - \theta_{01}(a)\theta_{01}(b)\theta_{01}(c)\theta_{01}(d)$$
$$-\theta_{10}(a)\theta_{10}(b)\theta_{10}(c)\theta_{10}(d) + \theta_{11}(a)\theta_{11}(b)\theta_{11}(c)\theta_{11}(d)$$
$$= 2\theta_{11}(a_1)\theta_{11}(b_1)\theta_{11}(c_1)\theta_{11}(d_1)$$

We can note the peculiarity exhibited by Riemann's formula here i.e., the signs between the terms of the above four equations are varied according to the entries in the matrix chosen and we get all four theta functions on the RHS by altering the LHS accordingly.Further, many more variants can be obtained by making similar substitutions.

Now, we can arrive at an important point by setting $a = b$, $c = d$ and $\theta_{11}(0)$ = 0 and then combining (R2) and (R5), we get:

$$\theta_{00}(a)^2\theta_{00}(c)^2 + \theta_{11}(a)^2\theta_{11}(c)^2 = \theta_{01}(a)^2\theta_{01}(c)^2 + \theta_{10}(a)^2\theta_{10}(c)^2$$
$$= \theta_{00}(a+c)\theta_{00}(a-c)\theta_{00}(o)^2$$

Similarly, we can tinker wih, like (R3 + R5) and (R4 + R5) to obtain:

$$\theta_{01}(a+c)\theta_{01}(a-c)\theta_{01}(0)^2 = \theta_{00}(a)^2\theta_{00}(c)^2 - \theta_{10}(a)^2\theta_{10}(c)^2$$
$$= \theta_{01}(a)^2\theta_{01}(c)^2 - \theta_{11}(a)^2\theta_{11}(c)^2$$

and next,

$$\theta_{10}(a+c)\theta_{10}(a-c)\theta_{10}(0)^2 = \theta_{00}(a)^2\theta_{00}(c)^2 - \theta_{01}(a)^2\theta_{01}(c)^2$$

$$= \theta_{10}(a)^2\theta_{10}(c)^2 - \theta_{11}(a)^2\theta_{11}(c)^2$$

Above formulas are called addition formulas for theta functions used for computing the co-ordinates of $\phi_2(a+c)$, $\phi_2(a-c)$ in terms of $\phi_2(a)$, $\phi_2(c)$ and $\phi_2(0)$. In our case, we have taken l = 2. Thus, we have:

$$\phi_2 : E_t \to \mathbb{P}^3 \tag{6.5}$$

$$a \to (\theta_0(la,t), \theta_1(la,t), \theta_2(la,t), \theta_3(la,t))$$

$$a \to (\theta_{00}(la,t), \theta_{01}(la,t), \theta_{10}(la,t), \theta_{11}(la,t))$$

Now, we further set $c = 0$, and above addition formula reduces to

$$\theta_{00}(a)^2\theta_{00}(0)^2 = \theta_{01}(a)^2\theta_{01}(0)^2 + \theta_{10}(a)^2\theta_{10}(0)^2 \tag{6.6}$$

and

$$\theta_{11}(a)^2\theta_{00}(0)^2 = \theta_{01}(a)^2\theta_{10}(0)^2 - \theta_{10}(a)^2\theta_{01}(0)^2 \tag{6.7}$$

Lastly, by setting $c = 0$, we obtain the jacobi identity, namely:

$$\theta_{00}(0)^4 = \theta_{01}(0)^4 + \theta_{10}(0)^4 \tag{6.8}$$

Thus the above two equations (6.6) and (6.7) are satisfied by the image $\phi_2(E_t)$ in $\mathbb{P}^3$. Moreover, we can conclude that $\phi_2(E_t)$ is indeed, the curve in $\mathbb{P}^3$ defined by the quadratic equations:

$$\theta_{00}(0)^2a_0^2 = \theta_{01}(0)^2a_1^2 + \theta_{10}(0)^2a_2^2$$

$$\theta_{00}(0)^2a_0^2 = \theta_{10}(0)^2a_1^2 - \theta_{01}(0)^2a_2^2$$

where $a_0 = \theta_{00}(0),\ a_1 = \theta_{01}(0),\ a_2 = \theta_{10}(0)$ and we know $\theta_{11}(0)$ = 0.

Chapter 7

Doubly Periodic meromorphic functions via theta functions

Definition 7.0.1. [SS10]A function on an open set D can be defined as meromorphic if there exists a sequence of points $\{z_0, z_1, ..\}$ having no limit points in D, and such that

- the function is holomorphic in D except at the above sequence of points, and
- f has poles at these points, not essential singularities

Thus, it is of the form $\frac{f(z)}{g(z)}$ where $f,\ g$ are holomorphic functions with $g \neq 0$.

Meromorphic function may have infinitely many poles and zeros.

Definition 7.0.2. Doubly Periodic function is a function defined over the complex plane and having two periods a, b linearly independent over $\mathbb{R}$ such that we have:

$$f(z+a) = f(z+b) = f(z) \qquad \forall z \in \mathbb{C}$$

Simply, it's a two dimensional extension of normal(single) periodic function.

Definition 7.0.3. Doubly Periodic meromorphic function can be defined as a doubly periodic function over $\mathbb{C}$ and holomorphic at all except some set of isolated poles. Elliptic functions are in particular, doubly periodic meromorphic functions.

We can write them as

$$\phi : \mathbb{C} \to \mathbb{C}$$

such that

$$\phi(z+\lambda) = \phi(z), \qquad \forall z \in \mathbb{C} \qquad and \qquad \lambda \in \lambda_t \, lattice.$$

Having defined the above functions, we are in a position to define meromorphic functions on elliptic curves (doubly periodic) by means of theta functions.

Approach I: As quotients of products of translates of $\theta(z)$ itself.

We know from chapter 5 that $\theta(z)$ in the parallelogram generated by 1 and t,has only one zero at $z = \frac{1}{2}(1+t)$.

Now we take the product as follows:

$$\prod_{1\leq i\leq k} \frac{\theta(z-a_i)}{\theta(z-b_i)}$$

where $a_1, a_2, ..., a_k, b_1, b_2, ..., b_k$ are such that $\sum a_i = \sum b_i$ (shortly, you will know why we took this).

Let's check the periodicity of above function w.r.t lattice λ_t generated by 1 and t. First w.r.t z $\to$ z+1. This gives us:

$$\prod_{1\leq i\leq k} \frac{\theta((z+1)-a_i)}{\theta((z+1)-b_i)} = \prod_{1\leq i\leq k} \frac{\theta(z-a_i)}{\theta(z-b_i)}$$

This clearly holds from the definition of theta functions.
Now consider w.r.t z $\to$ z+t,

$$\prod_{1\leq i\leq k} \frac{\theta((z+t)-a_i)}{\theta((z+t)-b_i)} = \prod_{1\leq i\leq k} \frac{exp(-\pi i t - 2\pi i(z+t-a_i))\theta(z-a_i,t)}{exp(-\pi i t - 2\pi i(z+t-b_i))\theta(z-b_i,t)}$$

$$= \prod_{1\leq i\leq k} \frac{exp(2\pi i a_i)exp(-\pi i t - 2\pi i(z+t))\theta(z-a_i,t)}{exp(2\pi i b_i)exp(-\pi i t - 2\pi i(z+t))\theta(z-b_i,t)}$$

$$= \frac{exp(2\pi i(\sum a_i)}{exp(2\pi i(\sum b_i)} \prod_{1\leq i\leq k} \frac{exp(-\pi i t - 2\pi i(z+t))\theta(z-a_i,t)}{exp(-\pi i t - 2\pi i(z+t))\theta(z-b_i,t)}$$

$$= \prod_{1\leq i\leq k} \frac{\theta(z-a_i)}{\theta(z-b_i)}$$

...(using the condition $\sum a_i = \sum b_i$ and periodic behaviour of θ w.r.t t).

Thus, we say that the above function is periodic w.r.t lattice λ_t. Above zeros of above function are at $a_i + \frac{1}{2}(1+t)$ and poles at $b_i + \frac{1}{2}(1+t)$.
Hence, this way we define a doubly periodic meromorphic function with isolated poles and zeros using translates of theta(θ) function.

Hence, this way we define a doubly periodic meromorphic function with isolated poles and zeros using translates of theta(θ) function.

[A17]Approach II: Sums of first logarithmic derivatives of translates of $\theta(z)$.

Meromorphic functions with simple poles on $\mathbb{P}^1_{\mathbb{C}}$ can be written as:

$$f(z) = \sum_{j \in \mathbb{Z}} \frac{\lambda_j}{(z - a_j)} + (const)$$

with poles at a_j and residue λ_i.

In the same vein, we consider a function with $a_1, a_2, ...a_k \in \mathbb{C}$ and $\lambda_1, \lambda_2, ..., \lambda_k \in \mathbb{C}$ such that $\sum_i \lambda_i = 0$:

$$f(z) = \sum_{j \in \mathbb{Z}} \lambda_j \frac{d}{dz} log(\theta(z - a_j)) + (const)$$

Again same as the previous one, above function is periodic w.r.t the lattice λ_t.

Hence, this defines a doubly periodic meromorphic function with simple poles at $a_j + \frac{1}{2}(1 + t)$ as θ has its zero at $\frac{1}{2}(1 + t)$.

We can also define meromorphic functions on elliptic curves (E_τ) in yet another ways as the second logarithmic derivatives of theta($\theta(z)$) (this is essentially Weierstrass p-function in terms of θ function). We can also relate these all these approaches using various Riemann theta identities and expansions.

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