Continuum Clockwork as a De-constructed Extra Dimension

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A dissertation submitted for the partial fulfilment of
BS-MS dual degree in Science

Under the guidance of
Dr. Ketan M. Patel

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Certificate of Examination

This is to certify that the dissertation titled **"Continuum Clockwork as a Deconstructed Extra Dimension"** submitted by **Ravneet S. Bedi** (Reg. No. MS14012) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: 26.04.2019

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Ketan M. Patel at the Indian Institute of Science Education and Research Mohali, Mohali and Physical Research Laboratory, Ahmedabad.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of the work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Ketan M. Patel
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Notations

Unless stated otherwise, the following conventions and notations are used

- Greek indices vary from 0 to 3 over the usual 4-D spacetime indices
- Capital Latin indices vary over 0 to 3 and 5, where 5 corresponds to the extra spatial dimension
- Natural units are used, i.e. $\hbar = c = 1$, unless otherwise stated
- the signature of the metric g_{MN} is (1,-1,-1,-1,-1)
- g refers to the determinant of g_{MN} and g^{MN} to the inverse of g_{MN}
- warped extra dimension refers to the metric:

$$ds^2 = g_{MN}dx^M dx^N = e^{-2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2 \tag{1}$$

  And Randall-Sundrum(RS) metric refers to the case $A(y) = ky$

- Dirac matrices are used in chiral representation:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma^\mu} & 0 \end{pmatrix} and \gamma^5 = \begin{pmatrix} \iota\mathbb{1} & 0 \\ 0 & -\iota\mathbb{1} \end{pmatrix} \tag{2}$$

  with $\sigma = (-\mathbb{1}, \sigma^i)$ and $\bar{\sigma} = (-\mathbb{1}, -\sigma^i)$ with σ^i being the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

- Square brackets over indices indicate anti-symmetrization, e.g.

$$A_{[\mu}A_{\nu]} = A_\mu A_\nu - A_\nu A_\mu \tag{4}$$

- The domain of extra dimesional coordinate is $[0, 2\pi R]$, which under $\mathcal{Z}_2$ orbifolding reduces to $[0, \pi R]$

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Abstract

There are many cases in the Standard Model in which there is a huge hierarchy between parameters with no explanation in the fundamental theory. Consider, for example, the range of fermion masses: varying from 0.511 MeV for an electron to around 173 GeV for the top quark. ”Naturally”, we would have expected their masses to be of the similar order since they arise from similar interactions in the Standard Model.

We also have a huge hierarchy between the Electroweak symmetry breaking scale($\sim$246 GeV) and the Planck scale($\sim 10^{16}$ TeV). The whole void between these two scales is not understood at all. And it would be nice if these two scales can somehow be linked to each other, so that the Physics at intermediate scales can also be understood. An elegant way to resolve these is to consider the possibility of extra dimensions, which forms a part of this project. Both large extra dimensions and warped extra dimensions are considered in this project

The way out of such hierarchies in the Standard Model is not just extra dimensions. One other way to resolve hierarchies is the clockwork mechanism, which is a 4-D mechanism involving certain type of interactions between different fields. The results of this mechanism are, in a certain sense, similar to extra dimensional theories. As such, in this project, this mechanism implemented on a large number of fields is compared to a five-dimensional theory with a certain metric and the extra coordinate discretized. There exists a correspondence between the two in the sense that the continuum limit of the discrete theory matches the de-constructed five dimensional free field theory. This correspondence is further explored in the project and certain limitations are found.

Chapter 1

Extra Dimesions

1.1 Introduction

There are two fundamental energy scales one encounters in Fundamental Physics. The first being the Electroweak scale $\sim$ 246 GeV which describes the energy scale at which the weak force and the electromagnetic forces are unified. And the second being the Planck scale$\sim$ 10^{19} GeV, which is the scale associated to Newton's gravitational constant and hence, naively, corresponds to the scale of gravity. Even though we have a good theoretical understanding of Physics at the Electroweak scale, we do not understand Physics at the Planck scale, which is essentially understood as an extension of the theories at the Electroweak scale. Even the Physics at the intermediate scales is only understood by extrapolation. The ratio between the two scales is huge ($\sim 10^{16}$), and indicates the difference in the scales at which different forces become effective. The fact that the scales of these fundamental forces are so different is often referred to as the hierarchy problem, something that indeed deserves an explanation.

The nature of the Electroweak scale(m_{EW}) is different from that of the Planck scale(M_{Pl}) in the sense that the former is an experimental scale whereas the latter is an extrapolation of the Classical Gravity. A simple way to resolve this is by introducing 'n' extra dimensions [Arkani-Hamed 98]. To accommodate that we do not observe them, these can be considered to be compact with small enough radius $\sim R$(as in Fig 1.1), just like a cylinder which when viewed from distances much greater than it's radius looks like a one-dimensional wire. In such a scenario, the gravitational potential is given by:

$$V(r) \sim \frac{m_1 m_2}{M_{Pl(4+n)}^{n+2} r^{n+1}}, (r \ll R) \quad (1.1)$$

and

$$V(r) \sim \frac{m_1 m_2}{M_{Pl(4+n)}^{n+2} R^n r}, (r \gg R) \quad (1.2)$$

Thus the effective 4-D M_{Pl} is given by

$$M_{Pl}^2 \sim M_{Pl(4+n)}^{n+2} R^n \quad (1.3)$$

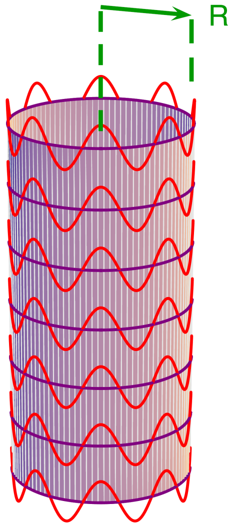

Figure 1.1: Illustration of an extra compact dimension [Ponton 13]

By varying n and R, $M_{Pl(4+n)}$ can be tuned to be close to M_{Pl}. For n=1, $R \sim 10^{13} cm$, which can be eliminated directly. But the case n=2 is interesting since it corresponds to radii, $R \sim 10^2 \mu m$. The current constraints are at $R \sim 37(44)\mu m$ [Kapner 07], [C. Patrignani 16],[Giudice 08] and hence extra dimensions, if any, are $n \geq 2$. Although gravity has been tested only upto these distances, the Standard Model fields have been tested at the Electroweak scale and at distances $\sim m_{EW}^{-1}$ accordingly. Thus all the Standard Model fields must be localized across the extra dimension. Some ways to do so are discussed in the following sections.

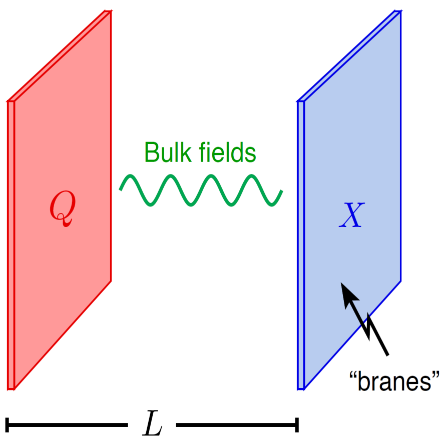

Figure 1.2: Extra dimensional models often consider SM fields to be localized to a brane or a finite width in 5-D[Ponton 13]

By bringing the two fundamental energy scales close to each other in the higher dimensional fundamental theory, the void between m_{EW} and M_{Pl} is killed and along with it the whole model building freedom to resolve issues related to neutrinos, flavor puzzles, etc. in Particle Physics. But the extra volume that these extra dimensions provide can help in building new mechanisms to resolve such problems[Arkani-Hamed 02].

But in the Arkani-Dimopolous-Dvali model, there still exists a hierarchy between the compactification scale($\sim 1/R$)and the Planck scale. There is an alternative approach to resolve the hierarchy by Randall-Sundrum[Randall 99a], in which they propose a background 5-D metric:

$$ds^2 = g_{MN}dx^M dx^N = e^{-2kr_c\phi}\eta_{\mu\nu}dx^\mu dx^\nu - r_c^2 d\phi^2; \tag{1.4}$$

where $y = r_c\phi$ corresponds to the extra dimension. The above metric can be shown to be a solution to the Einstein's equations in a certain scenario [Randall 99b]. Comparing the Einstein-Hilbert action in 5-D and the effective(flat) 4-D theory, we get:

$$M_{Pl}^2 = \frac{M_{Pl(4+1)}^3}{k}(1 - e^{-2kr_c\pi}) \tag{1.5}$$

Considering the (visible) fields to be confined to 4-D (a 3-brane), we have:

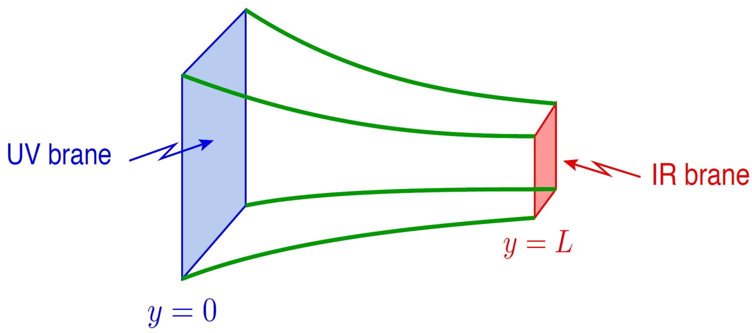

Figure 1.3: A depiction of the RS setup,in the sense that SM fields being at IR brane generate hierarchy[Ponton 13]

$$S_{vis} \supset \int d^4x e^{-4kr_c\pi}(\eta^{\mu\nu}e^{2kr_c\pi}D_\mu H^\dagger D_\nu H - \lambda(|H|^2 - v_0^2)^2) \tag{1.6}$$

which after a field redefinition gives $v = v_0 e^{-kr_c\pi}$, which further results in the same exponential suppression in mass parameters. In this way, M_{Pl} can be viewed as a fundamental scale with the "warping" factor resulting in the TeV scale, the additional advantage being that there is no hierarchy between M_{Pl} and the compactification scale.

In the following sections, the tools used in dealing with extra dimensions are illustrated for scalars, vectors and fermions.

1.2 Scalar fields

Consider a scalar field $\Phi(x^\mu, y)$[Tait 13], where x^μ are the coordinates corresponding to usual 4-d space-time and y is the coordinate corresponding to the compact extra dimension. With a flat metric, the action is:

$$S = \int \mathrm{d}^4x \int \mathrm{d}y \Big(\frac{1}{2}\partial^M \Phi \partial_M \Phi - V(\Phi)\Big) \tag{1.7}$$

where M varies over all the five coordinates.

$$\delta S = \int \mathrm{d}^4x \int \mathrm{d}y \Big(- \partial_M \partial^M \Phi - \frac{\partial V}{\partial \Phi}\Big)\delta\Phi - \Big[\int d^4x \partial_y \Phi \delta\Phi\Big]_0^{2\pi R} \tag{1.8}$$

Since the extra dimension considered is compact, the boundary term cannot necessarily be assumed to vanish and we need to choose the boundary conditions(B.C.) such that

$$\Big[\partial_y \Phi \delta\Phi\Big]_0^{2\pi R} = 0 \tag{1.9}$$

Very often, to visualize the boundary conditions or to reduce the fundamental domain of the theory, something called orbiflding is used

$$\Phi(Oy) = S\Phi(y) \tag{1.10}$$

Here O and S are representations of a discrete symmetry on the 5th coordinate and the wavefunction, respectively. For example, $\mathcal{S}^1/\mathcal{Z}_2$ (which corresponds to imposing $\mathcal{Z}_2$ over a circle) may be implemented as $\Phi(2\pi R - y) = \pm\Phi(y)$, and hence reduces the domain to $[0, \pi R]$.

Kaluza-Klein Decomposition

Finally we would like to look at the effective 4-D theory for which something called the Kaluza-Klein(KK) decomposition turns out to be very useful, which is similar to separation of variables

$$\Phi(x^\mu, y) = \sum_n f^{(n)}(y)\phi^{(n)}(x^\mu) \tag{1.11}$$

Looking at the effective theory for Φ , with V(Φ)=0, the action becomes:

$$S = \sum_{m,n} \int \mathrm{d}^4x \int_0^{2\pi R} \mathrm{d}y(f^{(m)} f^{(n)} \partial^\mu \phi^{(m)} \partial_\mu \phi^{(n)} - \partial_y f^{(m)} \partial_y f^{(n)} \phi^{(m)} \phi^{(n)}) \tag{1.12}$$

A convenient choice of $f^{(n)}$s is one for which

$$\int \mathrm{d}y f^{(m)} f^{(n)} = \delta_{m,n} \quad \& \quad \int dy \partial_y f^{(k)} \partial_y f^{(n)} = m_n^2 \delta_{k,n} \tag{1.13}$$

With the specific choice of periodic boundary conditions, this system of equations is solvable, giving constant solution for $f^{(0)}$ and sinusoidal solutions for other $f^{(n)}$;

$$S = \sum_n \int \mathrm{d}^4x (\partial^\mu \phi^{(n)} \partial_\mu \phi^{(n)} - m_n^2 \phi^{(n)} \phi^{(n)}) \tag{1.14}$$

thereby giving an effective theory which resembles an infinite tower of massive scalars in (1+3)-d with the mass spectrum $m_n = \frac{n}{R}$

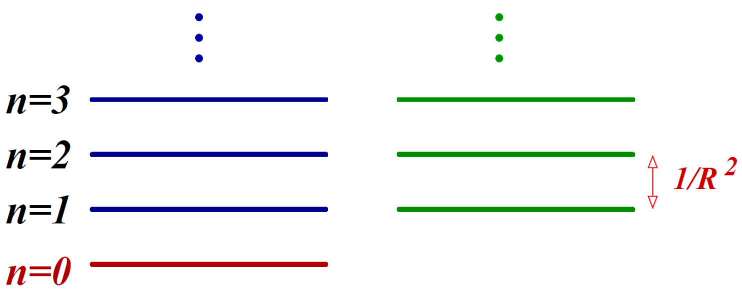

Figure 1.4: KK mass spectra with periodic B.C. [Perez-Lorenzana 05]

Consider a $\lambda_{5-D}\Phi^4$ interaction in (1.7), which would result in $\lambda_{4-D}\phi^4$ terms in the 4-d Lagrangian :

$$\lambda_{4-D} \sim \frac{\lambda_{5-D}}{2\pi R} \tag{1.15}$$

which gives a coupling smaller than the fundamental one by a factor equal to the size of the dimension. Thus we can see how working with an extra dimension may resolve hierarchies.

Equivalently, we could have approached (1.7) by finding the Euler-Lagrange equations, substituting (1.11) in them and then assuming that $\phi^{(n)}$ satisfy the 4-d equations of motion:

$$(\partial^\mu \partial_\mu + m_n^2)\phi^{(n)} = 0 \tag{1.16}$$

Consider this approach in the case of a warped metric, for which[Gherghetta 11]:

$$S = \frac{1}{2}\int d^5x\sqrt{g}(g^{MN}\partial_M\Phi\partial_N\Phi - m^2\Phi^2) \tag{1.17}$$

Using (1.11) and (1.16) in the Euler-Lagrange equations, we get

$$-e^{-2A}m_n^2 f^{(n)} - \partial_y(e^{-4A}\partial_y f^{(n)}) + e^{-4A}m^2 f^{(n)} = 0 \tag{1.18}$$

which gives exponential solution even for $m_n = 0$, which can help in addressing hierarchies as discussed later in the case of fermions.

1.3 Fermionic fields

The action for a fermionic field is given by:

$$S = \int \mathrm{d}^4x(\frac{\iota}{2}(\bar{\Psi}\gamma^\mu\partial_\mu\Psi - \partial_\mu\bar{\Psi}\gamma^\mu\Psi) - m\bar{\Psi}\Psi) = \int \mathrm{d}^4x(\iota\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi) \tag{1.19}$$

The spinor Ψ is a (0,1/2)+(1/2,0) representation of the Lorentz group and can be decomposed as $\Psi = \begin{pmatrix} \chi \\ \bar{\psi} \end{pmatrix}$, where χ is a (1/2,0) Weyl spinor and $\bar{\psi}$ a (0,1/2) Weyl spinor. These are the eigenstates to the projection operators $P = \frac{1}{2}(1 \pm \iota\gamma^5)$, and are called the left-handed and the right handed components of the Dirac spinor. The handed-ness/chirality does not change under Lorentz transformations since $[\gamma^5, \Sigma^{\mu\nu}] = 0$, where $\Sigma^{\mu\nu} = \frac{\iota}{4}[\gamma^\mu, \gamma^\nu]$ are generators of the Lorentz group. The significance of these representation lies in the fact that all the Standard Model fermions are Weyl spinors.

1.3.1 In flat extra dimension

For a Dirac fermion in 5-D, the action is given by

$$S = \int \mathrm{d}^5x(\frac{\iota}{2}(\bar{\Psi}\Gamma^M\partial_M\Psi - \partial_M\bar{\Psi}\Gamma^M\Psi) - m\bar{\Psi}\Psi) \tag{1.20}$$

Γ^M follow the Clifford algebra: $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}$. And Γ^μ can be taken equal to γ^μ and $\Gamma^5 = \gamma^5$. But unlike 4-d, even though it may be defined, chirality is now not Lorentz invariant since $[\Gamma^5, \Sigma^{M5}] \neq 0$[Csaki 05][Quevedo 10]. However we may still separate the two chiral parts into $\psi(\psi = \bar{\psi}^\dagger)$ and $\chi(\bar{\chi} = \chi^\dagger)$ as before:

$$S = \int \mathrm{d}^5x(-\iota\bar{\chi}\bar{\sigma}^\mu\partial_\mu\chi - i\psi\sigma^\mu\partial_\mu\bar{\psi} + \frac{1}{2}(\psi\overleftrightarrow{\partial_5}\chi - \bar{\chi}\overleftrightarrow{\partial_5}\bar{\psi}) + m(\psi\chi + \bar{\chi}\bar{\psi})) \tag{1.21}$$

The finite boundary term in the action is given by

$$\delta S \supset \int d^5x \frac{1}{2}(\delta\psi \overleftrightarrow{\partial_5} \chi + \psi \overleftrightarrow{\partial_5} \delta\chi - \delta\bar{\chi} \overleftrightarrow{\partial_5} \bar{\psi} - \bar{\chi} \overleftrightarrow{\partial_5} \delta\bar{\psi}) \tag{1.22}$$

and thus a general B.C. that we impose must satisfy:

$$[-\delta\psi\chi + \psi\delta\chi + \delta\bar{\chi}\bar{\psi} - \bar{\chi}\delta\bar{\psi}]_0^L = 0 \tag{1.23}$$

We may choose an appropriate B.C. as per our needs, which serve as a tool for trivial extensions of the model under study[Ponton 13]. Here, e.g. we can do away with the chirality problem if

$$\psi|_{0,L} = 0 \implies (\partial_5 + m)\chi|_{0,L} = 0 \tag{1.24}$$

We can solve for the EOM:

$$-\iota\bar{\sigma}^\mu\partial_\mu\chi - \partial_5\bar{\psi} + m\bar{\psi} = 0; \qquad -\iota\sigma^\mu\partial_\mu\bar{\psi} + \partial_5\chi + m\chi = 0 \tag{1.25}$$

Just as before we can decompose each Weyl spinor into its KK modes, those of χ being g_n and f_n being the KK modes of $\bar{\psi}$. On substituting 4-d EOM,i.e. $-\iota\bar{\sigma}^\mu\partial_\mu\chi_n$ and $-\iota\sigma^\mu\partial_\mu\bar{\psi}_n$ by $-m_n\bar{\psi}_n$ and $-m_n\chi_n$ resp., we get the KK mode profiles. In particular, the zero(massless) modes are:

$$g_0 = be^{-my} \& f_0 = ce^{my} \tag{1.26}$$

and equation (1.24) gives c=0. As such we are only left with one of the chiral components of the massless(zero) mode. To be consistent with periodic B.C., additionally $\mathcal{Z}_2$ orbifolding may be imposed.

Kaplan-Tait model

An advantage of considering extra dimensional theories is that $\mathcal{O}(1)$ bulk mass(the mass in 5-D Lagrangian) can lead to exponential hierarchy in 4-D fermion masses, such as in the Kaplan-Tait model[Kaplan 01]. In this model, the Higgs is taken to be a 4-D field at one of the boundaries, and depending on the sign of the 5-D Yukawa couplings, different fermions may be localized at different boundaries. The profile (1.26) then implies exponential suppression of Yukawa coupling in the effective theory. As such this can explain the huge difference in fermion masses.

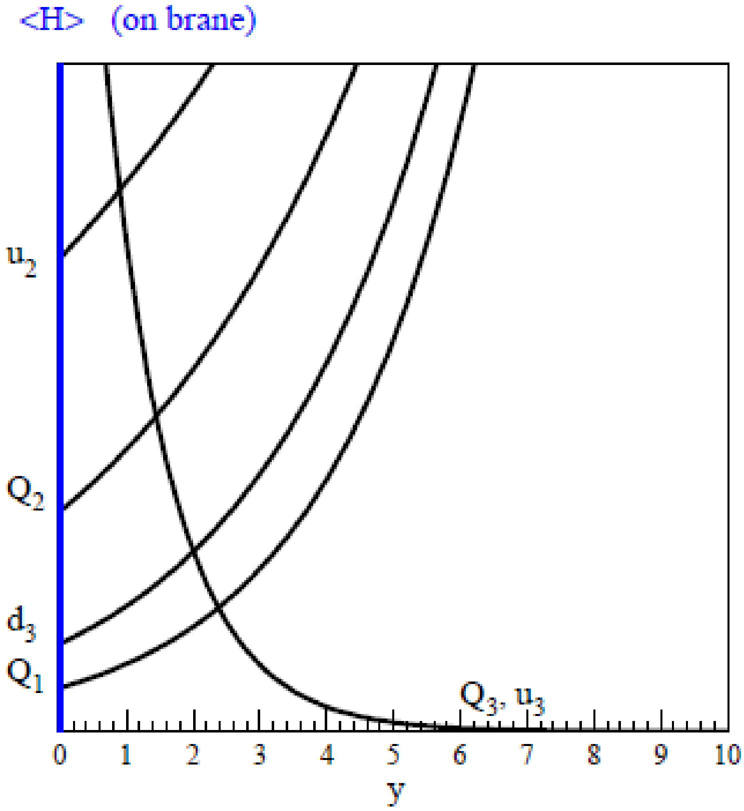

Figure 1.5: 0-mode profiles for some of the quarks, with Higgs at y=0 [Kaplan 01]

Consider, e.g., the following mass terms

$$\mathcal{L} = \delta(y) Y_{ij} \langle H \rangle \bar{q}_i d_j \tag{1.27}$$

which in the effective theory in 4-D would result in

$$\frac{m_{ij,4-D}}{v} = Y_{ij} g_0^{q_i}(0) g_0^{d_j}(0) \tag{1.28}$$

The factors g_0 result in a relative hierarchy if different fermions(generations) are localized at different boundaries, as depicted in the Figure 1.6.

Arkani-Hamed-Schmaltz Model

In this particular model[Arkani-Hamed 00] an auxiliary scalar field is introduced, which has a Yukawa type interaction with the SM fermions. The scalar acquires a postion dependent VEV such that the fermionic fields are localized across the 5th dimension. This also addresses fermion mass hierarchy from $\mathcal{O}(1)$ Yukawa interactions.

Consider the action:

$$S = \int \mathrm{d}^4x \int \mathrm{d}y \bar{\Psi}(\iota\gamma^M \partial_M + \Phi(y))\Psi \quad with \quad \langle\Phi\rangle = 2\mu^2 y \tag{1.29}$$

Decomposing Ψ as :

$$\Psi = \sum_n \langle y|Ln\rangle\, P_L\psi_n(x) + \sum_n \langle y|Rn\rangle\, P_R\psi_n(x); P_{R/L} = \frac{1 \pm \iota\gamma^5}{2} \tag{1.30}$$

where $|Ln\rangle$ and $|Rn\rangle$ are taken to be the eigenstates of $a^\dagger a$ and $aa^\dagger$ resp. with eigenvalues μ_n^2, where a is defined as $a = \partial_y + \langle\Phi\rangle$. As such we again get two Weyl fermions and a tower of Dirac fermions.

$$S = \int d^4x[\bar{\psi_L}\iota\gamma^\mu\partial_\mu\psi_L + \bar{\psi_R}\iota\gamma^\mu\partial_\mu\psi_R + \sum_{n=1}^{\infty}\bar{\psi_n}(\iota\gamma^\mu\partial_\mu + \mu_n)\psi_n)] \tag{1.31}$$

a and $a^\dagger$ can be shown to follow the commutation relation: $[a, a^\dagger] = 4\mu^2$, and hence are just like the creation and annihilation operators for a S.H.O. The zero modes can then be obtained by using $a\,|L, 0\rangle = 0$ and $a^\dagger\,|R, 0\rangle = 0$

$$\langle y|L, 0\rangle \equiv exp(-\int_0^y \langle\Phi\rangle\,(s)ds) \equiv (\langle y|R, 0\rangle)^{-1} \tag{1.32}$$

In this case only $\langle y|L, 0\rangle$ is normalizable and $\langle y|R, 0\rangle = 0$, and therefore only one Weyl spinor survives.

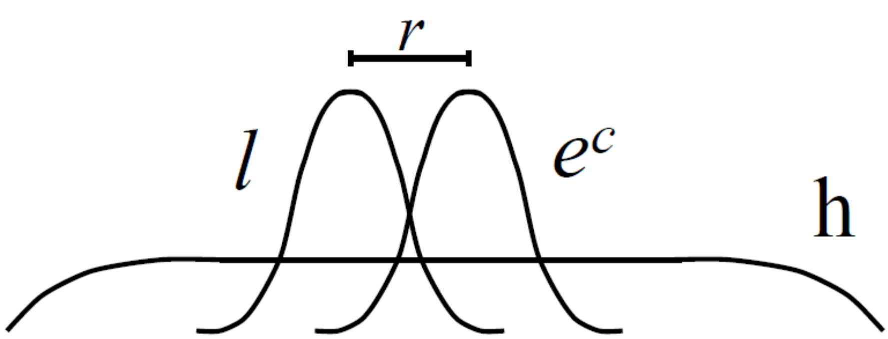

Figure 1.6: The overlap of different SM fermion profiles is in a tiny region resulting in hierarchies[Arkani-Hamed 00]

In a theory with multiple fermions, the zero mode of a fermion with bulk mass m gets localized at $y = \frac{m}{2\mu^2}$. In the effective 4-D theory, this then results in an exponential suppression in Standard Model type Yukawa interactions:

$$\int dy \frac{\sqrt{2}\mu}{\sqrt{\pi}} exp(-\mu^2 y^2) exp\Big(-\mu^2(y-\frac{m^2}{4\mu^4})^2\Big) \sim exp(\frac{-\mu^2 r^2}{2}) \tag{1.33}$$

1.3.2 In warped extra dimension

Consider a fermionic field in warped space with the action being given by

$$S = \int d^5x \sqrt{g} \frac{\iota}{2}(\bar{\Psi}\Gamma^M D_M \Psi - D_M \bar{\Psi}\Gamma^M \Psi) - m\bar{\Psi}\Psi) \tag{1.34}$$

Here the derivatives D_M and the gamma matrices Γ_M need to be re-defined such that $D\Psi$ transforms just like Ψ and $\{\Gamma^M, \Gamma^N\} = g^{MN}$ (discussed in Appendix A).

Just as before, we can obtain the EOM, the natural set of B.C., decompose into Kaluza-Klein modes and impose an appropriate B.C. so as to kill one of the chiral components of the massless KK mode [Gherghetta 11]. A particularly interesting feature is that in the Randall-Sundrum metric, the zero modes look like:

$$f^0_{L/R} \sim exp((2k \mp m)y) \tag{1.35}$$

Thus the effective kinetic term that we get from the action is:

$$S \supset \int d^4x e^{-4ky} e^{ky} e^{2(2ky-my)} \bar{\psi}^0_L \gamma^\mu \partial_\mu \psi^0_L \tag{1.36}$$

and there is an effective scaling by $exp\Big(\Big(\frac{1}{2}-\frac{m}{k}\Big)ky\Big)$. As such whether $c = \frac{m}{k} > 0.5$ or < 0.5 determines where it is localized. And the localization on different boundaries results in an exponential suppression of couplings. This is more effective than (1.28) since for slightly different bulk mass, the 0-mode can be localized on opposite branes, with the same sign for all Yukawa couplings.

1.4 Gauge theories

The case of a pure gauge theory is significantly different from that of scalars and fermions. It turns out that the massive KK modes of A_5 are not physical and the effective theory reduces to that of a Kaluza-Klein tower of massive 4-D gauge fields and a scalar(A_5). And in the low energy theory, a gauge field(A_M) in 5-D can be considered to contain a 4-D gauge

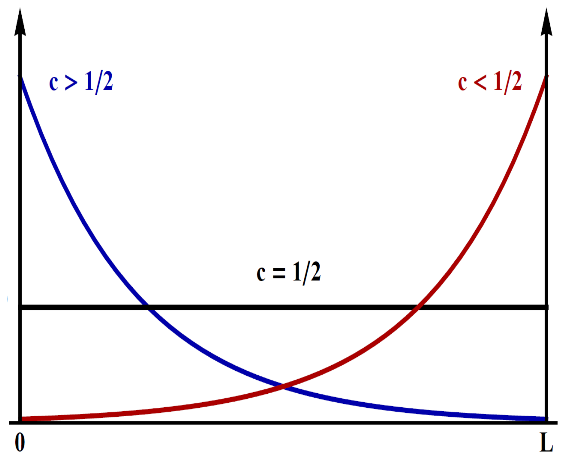

Figure 1.7: The profile of the massless mode for different values of m/k [Ponton 13]

field(A_μ) and a scalar(A_5).

For a local symmetry $\psi(x) \rightarrow V(x)\psi(x)$, with t^a being the generators of V(x), the action for a gauge field is given by[Peskin 95]

$$S = tr \int d^5x \Big(-\frac{1}{4} F_{MN} F^{MN} \Big) \tag{1.37}$$

where the covariant derivative and the field tensor are given by:

$$D_M = \partial_M - igA_M; \quad A_M = A_M^a t^a \tag{1.38}$$

$$F_{MN} = \frac{\iota}{g}[D_M, D_N] \tag{1.39}$$

and F_{MN} remains invariant under the transformation:

$$A'_M = \frac{\iota}{g}\Omega^{-1} D_M \Omega; \quad \Omega \subset V(x) \tag{1.40}$$

Now for both A_μ and A_5, KK decomposition can be done. But, when compactified on a circle, it turns out that with an appropriate choice of the gauge(1.41) only the massless mode survives[Sundrum 05].

$$\Omega \equiv \mathcal{P}exp(\iota g \int_0^\phi d\phi' R A_5(x^\mu, \phi')) exp(-\iota g A_5^{(0)}(x^\mu)\phi) \tag{1.41}$$

where $\mathcal{P}$ refers to path ordering.

Alternatively, to understand how such a gauge choice affects the quantization of the gauge field, instead of fixing the gauge we may add gauge breaking(fixing) terms to the Lagrangian just as in R_ξ gauges[Csaki 05], but so as to cancel the bulk and boundary mixing terms between A_μ and A_5. Consider, e.g. the mixing term between $A^\mu \& A^5$:

$$\int d^5x \partial_5 A^a_\mu \partial^\mu A^{5a} = \int d^5x \partial^\mu A^a_\mu \partial_5 A^{5a} - \int d^4x \partial^\mu A^a_\mu A^{5a}|^L_0 \tag{1.42}$$

The terms on right can be cancelled by a gauge fixing terms of the form, which also give restrictions on A_5.

$$S_{GF} = \frac{-1}{2\xi} \int d^5x (\partial^\mu A^a_\mu - \xi \partial_5 A^a_5)^2 + \frac{-1}{2\xi_b} \int d^4x (\partial_\mu A^{\mu a} \mp \xi_b A^a_5)^2|_{0,L} \tag{1.43}$$

Coming back to the previous approach, with the KK decomposition just being a Fourier series expansion, the action can be written as:

$$\begin{aligned} S &= tr \int d^x \int d\phi R\Big(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (D_\mu A^{(0)}_5)^2 + \frac{1}{2} (\partial_5 A_\mu)^2 \Big) \\ &= tr \int 2\pi R d^4x \Big(-\frac{1}{2} (\partial_{[\mu} A^{(0)}_{\nu]})^2 + \frac{1}{2} (\partial_\mu A^{(0)}_5)^2 + \sum_{n=1}^{\infty} \Big(-\frac{1}{2} |\partial_{[\mu} A^{(n)}_{\nu]}|^2 \\ &\qquad + \frac{n^2}{R^2} |A^{(n)}_\mu|^2 \Big) + \mathcal{O}(A^3) \Big) \end{aligned} \tag{1.44}$$

At low enough energies, we may as well just deal with the zero-mode:

$$S_{eff} \sim tr \int 2\pi R d^4x \Big(-\frac{1}{4} F^{(0)}_{\mu\nu} F^{(0)\mu\nu} + \frac{1}{2} (D_\mu A^{(0)}_5)^2 \Big) \tag{1.45}$$

Thus the effective theory is just that of a 4-D gauge field and a gauge charged scalar field, both of which are unified in the full theory.

Chapter 2

Clockwork mechanism

2.1 Introduction

In Particle Physics, we encounter several parameters that vary over several orders of magnitude with no dynamical explanation to their relative values. As such, it is natural to look for explanations for these hierarchies through some extensions of Standard Model. Clockwork is one such mechanism in which exponentially large interaction scales are generated from fundamentally comparable ones.

In general it may be defined as a quiver theory which gives rise to exponentially suppressed/enhanced couplings to a symmetry protected zero-mode from $\mathcal{O}(1)$ couplings in the fundamental theory[Giudice 17a]. The setup contains N+1 copies of a particle P with a symmetry $\mathcal{G}$ each. This symmetry keeps P massless: e.g. for a scalar it may be a Goldstone boson, for a photon the symmetry may be gauge invariance, etc. The overall system thus has the symmetry with a subset $\mathcal{G}^{N+1}$. The particles labelled with 'i' (varying from 0 to N) can be considered to be sites on a lattice. Then, N certain type of nearest neighbor interactions, between P_{i+1} and P_i are introduced which break N copiesof the symmetry $\mathcal{G}$. As such we are left with a single symmetry group $\mathcal{G}$ and hence a massless particle, which is a linear combination of P_is. With the interactions modelled in an appropriate fashion, this massless particle has an exponential profile across the (N+1) particles we began with. The other mass eigenstates, referred to as the clockwork gears, serve as an experimental test for this mechanism.

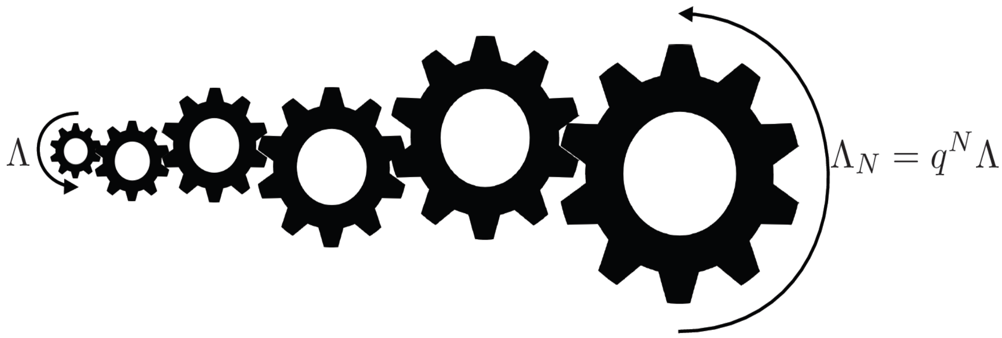

Figure 2.1: Clockwork mechanism resulting in an exponentially suppressed/enhanced scale [Giudice 17a]

2.2 Clockwork scalar

To implement clockwork mechanism for scalars, we may consider N+1 scalars, $U_j(x) = e^{\iota\pi_j/f}$ where j varies from 0 to N, with a U(1) symmetry each, which gets broken spontaneously at a scale f leading to a Goldstone boson. The symmetry may be explicitly broken by nearest neighbor interactions of the form $U_j^\dagger U_{j+1}^q$, which may be considered as explicit symmetry breaking terms. Alternatively, these terms may be considered to arise from spontaneous symmetry breaking which happens at a much lower scale than f leading to such terms. The way that these can be accommodated is to consider some auxiliary fields, with a certain charge(2.2) under the U(1)s, which acquire a VEV on spontaneous symmetry breaking.

$$\mathcal{L} = \frac{f^2}{2}\sum_{j=0}^{N} \partial_\mu U_j^\dagger \partial^\mu U_j + \frac{m^2 f^2}{2}\sum_{j=0}^{N-1}(U_j^\dagger U_{j+1}^q + h.c.) \tag{2.1}$$

where m^2 is the VEV of the j-th auxiliary field with charge

$$Q_i[(m^2)_j] = \delta_{ij} - q\delta_{i,j+1} \tag{2.2}$$

It may be noted that there is a symmetry of the Lagrangian (2.1) in which $U_j \to e^{\iota\phi/q^j} U_j$, which corresponds to the generator:

$$\mathcal{Q} = \sum_{j=0}^{N} \frac{Q_j}{q^j} \tag{2.3}$$

where Q_j is the generator of the U(1) corresponding to j-th particle $U_j(x)$. It can further be checked that the auxiliary fields have zero charge under $\mathcal{Q}$.

In terms of π_j,

$$\mathcal{L} \approx \frac{1}{2}\sum_{j=0}^{N} \partial_\mu \pi_j \partial^\mu \pi_j - \frac{m^2}{2}\sum_{j=0}^{N-1}(\pi_j - q\pi_{j+1})^2 \tag{2.4}$$

where apart from kinetic and mass terms we have an interaction term of the type $q\pi_j\pi_{j+1}$. This nearest neighbor interaction results in the mass squared matrix M_π^2:

$$M_\pi^2 = m^2 \begin{pmatrix} 1 & -q & 0 & . & . & . & 0 \\ -q & 1+q^2 & -q & . & . & . & 0 \\ 0 & -q & 1+q^2 & . & . & . & 0 \\ . & . & . & & & 1+q^2 & -q \\ 0 & 0 & . & . & . & -q & q^2 \end{pmatrix} \tag{2.5}$$

This matrix can be diagonalized as in Appendix B. The mass eigenvalues m_{ak}^2 and the mass eigenstates a_k (k=0,1,...N) are given by:

$$m_{a0} = 0; \quad m_{ak}^2 = m^2(1 + q^2 - 2qcos\frac{k\pi}{N+1}) \tag{2.6}$$

For the massive modes(k=1,2...,N), the profile is:

$$a_k = \sum_{j=0}^{N} \mathcal{N}_k \Big[qsin\frac{jk\pi}{N+1} - sin\frac{(j+1)k\pi}{N+1}\Big]\pi_j \tag{2.7}$$

and for the massless mode

$$a_0 = \sum_{j=0}^{N} \frac{\mathcal{N}_0}{q^j}\pi_j \tag{2.8}$$

The above exponential dependence is what makes clockwork an interesting tool. This can be used to obtain exponential hierarchies between coupling parameters without resorting to extra dimensions.

If we have an interaction of the form

$$\mathcal{L} \supset \frac{1}{g}\pi_N F_{\mu\nu}(x)\tilde{F^{\mu\nu}}(x) \tag{2.9}$$

then at low enough energies, only interactions with the massless mode are significant. With $\pi_N = \frac{\mathcal{N}_0}{q^N}a_0$+sinusoidal combinations of higher mass modes, the coupling scale is enhanced by q^N, i.e. $g \rightarrow \frac{q^N}{\mathcal{N}_0}g$ The clockwork mechanism was originally used in construction of axions by enhancing the effective scale of the theory [Kaplan 16]. Axions have interactions of the type (2.9) in Peccei-Quinn theory, which through the nature of effective couplings, resolves the strong CP problem[Peccei 08].

2.3 Clockwork fermion

In the case of fermions, the symmetry that can lead to masslessness is chirality. As such taking N+1 massless Dirac fermions ψ_j(j=0 to N), each with a left chiral part ψ_{Lj} and a right chiral part ψ_{Rj}, we may indeed implement clockwork through nearest neighbor interactions such as $\bar{\psi}_j\psi_{j+1}$.

First let us consider, for example, N left chiral and N+1 right chiral fermions, with each symmetry broken using parameters m_j and $(qm)_j$:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_{kin} - m\sum_{j=0}^{N-1}(\bar{\psi_{Lj}}\psi_{Rj} - q\bar{\psi_{Lj}}\psi_{R,j+1} + h.c.) \\ &= \mathcal{L}_{kin} - (\bar{\psi}_L M_\psi \psi_R + h.c.) \end{aligned} \tag{2.10}$$

The parameters m_j and $(qm)_j$ may agin be regarded as VEV of scalar fields with charges (1,-1) under $U(1)_{Lj} \times U(1)_{Rj}$ and $U(1)_{Lj} \times U(1)_{Rj+1}$ respectively, where $U(1)_{Lj}$ is the symmetry group of ψ_{Lj}(with charge 1) and $U(1)_{Rj}$ that of ψ_{Rj}.

The N$\times$(N+1) mass marix M_ψ is :

$$M_\psi = \begin{pmatrix} 1 & -q & 0 & . & . & . & 0 \\ 0 & 1 & -q & . & . & . & 0 \\ . & & . & & . & -q & 0 \\ 0 & 0 & & . & . & 1 & -q \end{pmatrix} \tag{2.11}$$

Solving for the singular vector of this matrix with singular value zero, we again get the same profile as for the zero mode of clockwork scalar. Rather, since $M_\psi^\dagger M_\psi = M_\pi^2$, the eigenvalues in that case are the singular values in this case. Denoting the left and right singular vectors as Ψ_L and Ψ_R, Ψ_R and ψ_R are related the same way as a and π as in (2.7)& (2.8) and for Ψ_L we have

$$\psi_L = U_L\Psi_L, \quad U_{Ljk} = \sqrt{\frac{2}{N+1}} sin\frac{jk\pi}{N+1} \tag{2.12}$$

A more natural choice of interactions would have been of the type $\bar{\psi}_j\psi_{j+1} = -(\bar{\psi}_{Lj}\psi_{Rj+1} + \bar{\psi}_{Rj}\psi_{Lj+1})$. But it may be noted that if both interaction terms are considered, then the zero mode (2.11) would not result in an exponential profile. Further only N left chiral fermions are used here instead of the general prescription of N+1 copies of it. Adding that extra fermion would not affect the profile of the zero mode and hence need not be considered.

The exponential profile of the zero mode as in (2.8) can have many applications such as fermion masses hierarchy, in neutrino physics, etc. For around N=25, with the model implemented for a right-handed neutrino, it can address the neutrino mass. It has been found that N=2 to 4 resolves the hierarchy between the Standard Model fermion masses and mixing[Patel 17].

2.4 Clockwork for Gauge fields

Clockwork for U(1) gauge theory is quite similar to that for the scalar. To obtain a clockwork photon, we may consider N+1 U(1) gauge groups with N complex scalars ϕ_j(j=0 to N-1).

$$\mathcal{L} = -\sum_{j=0}^{N} \frac{1}{4} F^j_{\mu\nu} F^{j\mu\nu} + \sum_{j=0}^{N-1} [|D_\mu \phi_j|^2 - \lambda(|\phi_j|^2 - f^2/2)^2] \tag{2.13}$$

The scalars are taken to be charged (1,-q) under $U(1)_j \times U(1)_{j+1}$ so that $D_\mu \phi_j = [\partial_\mu + \iota g(A^j_\mu - q A^{j+1}_\mu)]\phi_j$.

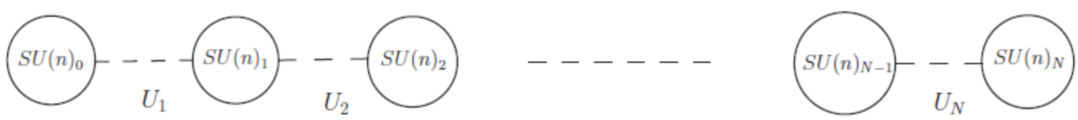

Figure 2.2: N+1 Gauge groups linked to each other by scalar fields[Bai 10]

As such after spontaneous symmetry breaking at the scale f, we obtain:

$$\mathcal{L} = -\sum_{j=0}^{N} \frac{1}{4} F^j_{\mu\nu} F^{j\mu\nu} + \frac{g^2 f^2}{2} \sum_{j=0}^{N-1} (A^j_\mu - q A^{j+1}_\mu)^2 \tag{2.14}$$

The above mass matrix is just the same as discussed in the case of scalar clockwork in Appendix B and thus has the same implications.

The massless mode has gauge invariance: $A_{(0)\mu} \to A_{(0)\mu} + \partial_\mu \alpha(x)$. In terms of the original fields, this becomes:$A_{j\mu} \to A_{j\mu} + \partial_\mu(\mathcal{N}_0 \alpha(x))/q^j$. Thus, if there is a scalar field χ that is charged under the gauge group:

$$|(\partial_\mu + \iota Q A_{k\mu})\chi|^2 \approx |(\partial_\mu + \iota Q \mathcal{N}_0 q^{-k} A_{(0)\mu})\chi|^2 \tag{2.15}$$

resulting in an exponentially suppressed charge.

An important feature of clockwork is that it is an Abelian phenomenon[Craig 17]. Consider a gauge field, for which $A^a_{j\mu} = c_j A^a_{(0)\mu} + \sum_k c_{jk} A^a_{(k)\mu}$. The kinetic terms of the full Lagrangian after a simple field redefinition can be written as:

$$\begin{aligned}\mathcal{L} &= -\sum \frac{1}{4g^2}(F^a_{j\mu\nu})^2 \\ &= -\sum \frac{1}{g^2}\left(\frac{1}{4}(\partial_{[\mu}A^a_{\nu]j})^2 + f^{abc}\partial_\mu A^a_{j\nu}A^{b\mu}_j A^{c\nu}_j + \frac{1}{4}f^{abc}f^{ars}A^b_{j\mu}A^c_{j\nu}A^{r\mu}_j A^{s\nu}_j\right)\end{aligned} \tag{2.16}$$

The terms corresponding to the massless gauge field are:

$$-\sum_j \frac{1}{g^2}\left(\frac{1}{4}c_j^2(\partial_{[\mu}A^a_{\nu](0)})^2 + c_j^3 f^{abc}\partial_\mu A^a_{(0)\nu}A^{b\mu}_{(0)}A^{c\nu}_{(0)} + \frac{1}{4}c_j^4 f^{abc}f^{ars}A^b_{(0)\mu}A^c_{(0)\nu}A^{r\mu}_{(0)}A^{s\nu}_{(0)}\right) \tag{2.17}$$

For the field $A_{(0)}$ to be gauge invariant with a gauge coupling $g_{(0)}$, we need that:

$$\frac{g^2}{g^2_{(0)}} = \sum c_j^2 = \sum c_j^3 = \sum c_j^4 \tag{2.18}$$

which holds for $c_j \in \{0, 1\}$, which in turn corresponds to the case $q = 1$ and thus no clockworking. That this is the only case in which (2.18) holds true is hard to show directly. For that, with the generators T^a_j(with $[T^a_i, T^b_j] = f^{abc}T^c_j\delta_{ij}$) corresponding to the symmetry group at site j, the requirement of gauge invariance on the massless mode implies(with generators $T^a_{(0)} = \sum a_j T^a_j$) :

$$\sum f^{abc}a_j^2 T^c_j = \sum f^{abc}a_j T^c_j \implies a_j^2 = a_j \tag{2.19}$$

which holds only for $a_j \in \{0, 1\}$. This implies that the unbroken symmetry is equally distributed among all the sites that have a component of it. Thus non-Abelian vector clockwork, although can be done, would not lead to a meaningful theory for the massless mode.

Chapter 3

Continuum Clockwork

3.1 Introduction

In the previous chapters, solutions of certain problems such as the fermion mass hierarchy through extra dimensions and through clockwork were discussed. In this chapter, possible relations between the two, in the sense that each particle(in 4-D clockwork) being associated to a site in the theory space, are explored. For this consider the $N \to \infty$ limit of the discrete clockwork. As such the mass spectrum is:

$$m_0^2 = 0; \quad m_n^2 = m^2(q-1)^2 + \left(\frac{m\pi}{N+1}\right) n^2 \tag{3.1}$$

This similarity to a flat extra dimension, combined with the exponential dependence of zero mode on the site, suggests that continuum clockwork can indeed be seen as an extra dimensional theory. In the following sections, the correspondence between the two is investigated.

3.2 Continuum limit of Clockwork

One way to see clockwork as an extra dimensional theory is to just take the limit $N \to \infty$ in the Lagrangian (2.4). This limit would then correspond to a discrete extra dimension with the index j of each particle P_j taking the role of the coordinate $y = ja$(j=0,1,...N), where a is the lattice spacing.[Choi 18][Craig 17]

Working on a circle with orbifolding, this corresponds to $(N + 1)a = \pi R$ and $a \to 0$.

For q^N to be finite we need $q \to 1$, rather $q \sim 1 + \tilde{m}a$. Even though it seems to lead to no clockworking, $q^j = (q^{1/a})^y$ is finite and hence $q \to 1$ does lead to clockworking in the continuum limit. Similarly from the mass spectrum, we need m/N to be finite, implying $m \to \infty$ and since the choice of a is upto us, we may set $ma = 1$.

As such on replacing the discrete sum by integral over y and a field redefinition (2.4) becomes:

$$\frac{1}{2}\int \mathrm{d}y\Big(\partial_\mu \pi \partial^\mu \pi - (\partial_y \pi + \tilde{m}\pi)^2\Big) \tag{3.2}$$

Thus clockwork can be obtained from a 5-D Lagrangian density but it clearly does not correspond to a covariant theory because of terms of the type $\pi\partial_y\pi$. As such let's take a digression from continuum limit of the clockwork and investigate which type of a metric, if any, can be considered as a viable 5-D theory.

3.3 De-constructing Dimensions

Instead of going from clockwork to extra dimension, we may try going from extra dimensions to clockwork. For this, consider discretizing an extra dimension with a metric given by:

$$ds^2 = X(y)dx^2 - Y(y)dy^2 \tag{3.3}$$

For a scalar field ϕ with the Lagrangian

$$S = \frac{1}{2}\int \mathrm{d}^5x\sqrt{g}(\partial_M\phi\partial^M\phi) \tag{3.4}$$

upon field redefinition so as to obtain canonical 4-D terms in flat metric, we obtain

$$S = \frac{1}{2}\int \mathrm{d}^5x\bigg((\partial_\mu\phi)^2 - \frac{X^2}{\sqrt{Y}}\Big(\partial_y\frac{\phi}{X^{1/2}Y^{1/4}}\Big)^2\bigg) \tag{3.5}$$

Now discretizing the y coordinate, i.e. setting $y = ja$, taking a sum instead of integral($\int \frac{\mathrm{d}y}{a} \to \sum$) and defining $f_j(x) = f(x, y)$:

$$S = \frac{1}{2}\int \mathrm{d}^4x\sum_{j=0}^{N}\left(\partial_\mu\phi_j\partial^\mu\phi_j - \sum_{j=0}^{N-1}\left(\frac{N^2X_j}{\pi^2R^2Y_j}\right)\left(\phi_j - \left(\frac{X_j^{1/2}Y_j^{1/4}}{X_{j+1}^{1/2}Y_{j+1}^{1/4}}\right)\phi_{j+1}\right)^2\right) \tag{3.6}$$

This is similar to the original clockwork Lagrangian. Comparing this Lagrangian with that in (2.4) indicates that indeed there exists a correspondence between clockwork and an extra dimension with a certain metric.

$$m^2 \equiv \frac{N^2 X_j}{\pi^2 R^2 Y_j} \qquad\qquad q \equiv \frac{X_j^{1/2} Y_j^{1/4}}{X_{j+1}^{1/2} Y_{j+1}^{1/4}} \tag{3.7}$$

For m^2 to be site independent, $X_j \propto Y_j$ and for q^N to be finite ($q^N \propto e^{k\pi R}$)

$$\implies X_j \propto Y_j \propto exp\left(\frac{4k\pi Rj}{3N}\right) \tag{3.8}$$

The setup that generates such a metric is discussed in [Cox 12].

In constrast to this, there is a major problem in comparing continuum clockwork fermion to a de-constructed extra dimension, rather the problem is in de-constructing an extra dimensional fermionic theory in any metric [Bai 10]. Consider a fermionic theory in a metric of the type (3.3) with $X(y) = Y(y) = exp(4ky/3)$. As discussed in Appendix A, $\Gamma^M = e^{-\frac{2ky}{3}}\gamma^M$ and the spin connections for the metric lead to

$$D_\mu = \partial_\mu + \iota\frac{k}{3}\begin{pmatrix} 0 & \sigma^\mu \\ -\bar{\sigma}^\mu & 0 \end{pmatrix}; \qquad D_5 = \partial_5 \tag{3.9}$$

The action is:

$$S = \int \mathrm{d}^5x \frac{\iota}{2}(\bar{\Psi}\Gamma^M D_M\Psi - D_M\bar{\Psi}\Gamma^M\Psi) \tag{3.10}$$

The contributions from the spin connections cancel and on decomposing into left and right chiral parts we are left with:

$$S = \int \mathrm{d}^4x \int \mathrm{d}y (e^{10ky/3})(e^{-2ky/3})\Big(\iota\bar{\Psi}\gamma^\mu\partial_\mu\Psi + \frac{1}{2}(\bar{\psi}_R\partial_5\psi_L - \partial_5\bar{\psi}_R\psi_L + \partial_5\bar{\psi}_L\psi_R - \bar{\psi}_L\partial_5\psi_R)\Big) \tag{3.11}$$

The problem lies in the different terms involving derivative w.r.t the 5th coordinate, which on discretizing (and appropriate redefinition to get canonical kinetic terms) lead to:

$$S \supset \int \mathrm{d}^4x \sum_j \frac{1}{a} e^{-4ka/3}(\bar{\psi}_{Lj}\psi_{Rj+1} - \bar{\psi}_{Lj+1}\psi_{Rj} + h.c.) \tag{3.12}$$

This is in contrast to the fermionic clockwork discussed in the previous chapter which had only half the terms appearing here. This is a rather typical feature of fermions on a lattice and can be resolved by adding an additional term[Rothe 87], which vanishes in the continuum limit (with the old field definitions), since $a \to 0$.

$$S_{add} = \eta a \int \mathrm{d}^5x \bar{\Psi}(\partial_5)^2\Psi \tag{3.13}$$

This when discretized and added to the previous terms we get:

$$S = \int \mathrm{d}^4x \sum_j \Big(\iota\bar{\Psi}_j\gamma^\mu\partial_\mu\Psi_j + \frac{2}{a}e^{-4ka/3}\Big(\Big(-\frac{1}{2}+\eta\Big)\bar{\psi}_{Lj}\psi_{Rj+1} + \Big(\frac{1}{2}+\eta\Big)\bar{\psi}_{Lj+1}\psi_{Rj} + h.c.\Big)$$
$$+\frac{2\eta}{a}(e^{-8ka/3}\bar{\psi}_{Lj+1}\psi_{Rj+1} + \bar{\psi}_{Lj}\psi_{Rj} + h.c.)\Big) \tag{3.14}$$

With the introduction of the new term, the redundant interactions can be removed by choosing $\eta = \pm\frac{1}{2}$. We are then left with standard clockwork Lagrangian for the fermionic case, with $m = 2/a$ and $q = e^{-4ka/3}$

3.4 Clockwork from geometry

To check that embedding the scalar field in such a metric works, we may compare the mass spectrum and Kaluza-Klein modes of a 5-D scalar in this metric. For this we need to find the equations of motion for (3.5) with $X(y) = Y(y) = exp(4ky/3)$. This gives the mass spectrum:

$$m_0^2 = 0; \quad m_n^2 = k^2 + \frac{n^2}{R^2} \tag{3.15}$$

And the profiles of KK modes being:

$$f_0(y) = \sqrt{\frac{k\pi R}{e^{2k\pi R} - 1}}; \quad f_n(y) = \frac{n}{m_n R}e^{-ky}\Big(\frac{kR}{n}sin\frac{ny}{R} + cos\frac{ny}{R}\Big) \tag{3.16}$$

Combined with the factors coming from $\sqrt{g}$ and g^{MN}, $e^{2ky}f_n^2\,\mathrm{d}(y/\pi R)$ turns out to be equivalent to the solutions in (2.8) and (2.7).

As such at the fundamental level of Lagrangian, the relation between continuum clockwork and an extra dimension has been established. But this does not imply that the two theories would lead to same phenomenological results. This is completely model dependent as illustrated in [Giudice 17b] for the axion-like coupling in (2.9). Consider, for example, an interaction term: [Craig 17]

$$\mathcal{S} \supset \int \mathrm{d}^5x\delta(y - y_0)\frac{1}{f_5^{3/2}}\phi F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x) \tag{3.17}$$

As such the effective interaction term in 4D Lagrangian will have the coupling constant $1/f_4 = f_0/f_5^{3/2}$, and the coupling is independent of the position of the brane where the interaction is localized.

Another thing that can be done is to add bulk mass terms to the 5-D Lagrangian with the clockwork metric. To begin with, consider a scalar

$$S = \frac{1}{2}\int \mathrm{d}^5x\sqrt{g}(g^{MN}\partial_M\Phi\partial_N\Phi - M^2\Phi^2) \tag{3.18}$$

in the metric

$$ds^2 = g_{MN}dx^M dx^N = e^{\frac{4ky}{3}}(\eta_{\mu\nu}dx^\mu dx^\nu - dy^2) \tag{3.19}$$

From the action, we can obtain the equations of motion and hence solve for the KK modes

$$\begin{aligned} S = \frac{1}{2}\int \mathrm{d}^5x(e^{2ky}(\partial_\mu\Phi\partial_\nu\Phi(\partial_y\Phi)^2) - e^{10ky/3}M^2\Phi^2) \\ \implies (m_n^2 + \partial_y^2 + 2k\partial_y - M^2 e^{\frac{4ky}{3}})f_n = 0 \end{aligned} \tag{3.20}$$

The solutions to these are:

$$\begin{aligned} f_0(y) = e^{-2ky}[c_1(sinh\theta - \theta cosh\theta) + c_2(cosh\theta + \theta sinh\theta)] \\ f_n(y) = e^{-ky}[c_1 I_{(-\phi)}(\theta)\Gamma(1-\phi) + c_2 I_{(\phi)}(\theta)\Gamma(1+\phi)] \\ \theta = \frac{3e^{2ky/3}m}{2k}; \qquad \phi = \frac{3\sqrt{k^2 - m_n^2}}{2k} \end{aligned} \tag{3.21}$$

where $I(x)$ is the modified Bessel function of first kind and $\Gamma(x)$ is the Gamma function. These results are significantly different from the previous scenario(even though still leading to exponential hierarchy) and hence put the correspondence into question.

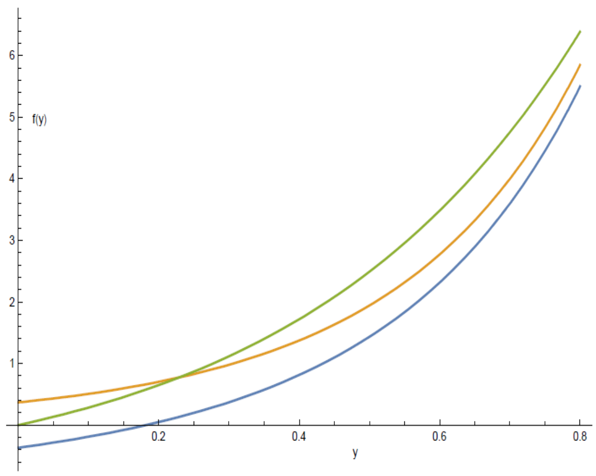

Figure 3.1: Comparison of the two zero modes with k=3/2 and m=1(blue and orange) and an exponential profile $e^{2.5x}$ (green)

Let's also look at a fermionic field in the clockwork geometry. Solving the action (1.20) for the equations of motion lead to KK modes. For the particular metric at hand, the contributions from the spin connections cancel and we are left with the Dirac equation $\imath\Gamma^M D_M \Psi = m\Psi$. Decomposing Ψ into its KK modes gives

$$\Psi = \sum_n \begin{pmatrix} \chi_n f_{+n} \\ \bar{\psi}_n f_{-n} \end{pmatrix} \tag{3.22}$$

$$\implies m_n f_{\mp n} \mp \frac{2k}{3} f_{\pm n} \mp \partial_5 f_{\pm n} = m e^{\frac{2ky}{3}} f_{\pm n} \tag{3.23}$$

The zero mode profiles are:

$$f_{\pm 0} \sim exp\left(\mp \frac{2k}{3} - \frac{3m}{2k} e^{2ky/3} \right) \tag{3.24}$$

which are again different from the exponential profiles obtained previously.

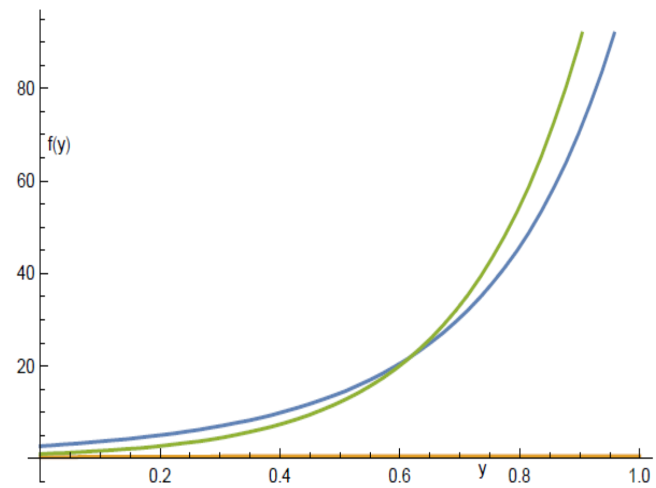

Figure 3.2: Comparison of the two zero modes with k=3/2 and m=1 (blue and orange) and an exponential profile e^{5x}(green)

Even though the zero-mode solutions would result in hierarchies just like clockwork, it would not result in a UV(high energy) complete theory in which all the modes match. Even for a massive field in flat geometry, we can get an exponential profile but it is not equivalent to clockwork.

As such clockworking a field in the continuum limit is equivalent to (3.19) but the metric (3.19) is not equivalent to clockwork. The correspondence between clockwork and geometry, however, is not limited to it. In a limited sense continuum clockwork scalar can be seen as a 5-D scalar embedded in flat spacetime. For this, consider a field redefinition in the Lagrangian (3.4) or equivalently look at (3.5) instead of (3.4). This gives:

$$S = \frac{1}{2}\int \mathrm{d}^5x\bigg((\partial_\mu\phi)^2 - e^{2ky}(\partial_5(e^{-ky}\phi))^2\bigg) \tag{3.25}$$

$$S = \frac{1}{2}\int \mathrm{d}^5x\bigg((\partial_\mu\phi)^2 - (\partial_5\phi)^2 - k^2\phi^2 + 2k\phi\partial_5\phi\bigg)$$

where the last term in the second equation can be written as $\partial_5(\phi)^2$, implying

$$\int \mathrm{d}y 2\phi\partial_5\phi = \phi^2(y=\pi R) - \phi^2(y=0) = 2\int \mathrm{d}y\phi^2(\delta(y-\pi R) - \delta(y-0)) \tag{3.26}$$

and as such

$$S = \frac{1}{2}\int \mathrm{d}^5x\bigg((\partial_\mu\phi)^2 - (\partial_5\phi)^2 - \phi^2(k^2 + 2k\delta(y) - 2k\delta(y-\pi R))\bigg) \tag{3.27}$$

which correspond to a scalar Lagrangian in flat space, with bulk and brane mass terms.

Conclusion

Even though it is not fundamentally important to explain the hierarchy in the scales and parameters in Particle Physics, it adds an aesthetic appeal to any physical theory. Some approaches to address the hierarchy problem were studied in this project: extra dimensions and clockwork. A common feature of such approaches is that they tend to generate an exponential profile which, in one way or another, explains the huge difference in fundamental parameters. For example, both the models for extra dimensions discussed here try to bring the Planck and Electroweak scale close through such a profile and also explain the fermion mass hierarchy through the same profile. In the case of clockwork too, it is such an exponential profile that is put to use in addressing phenomenological problems.

The similarity in the way both of these lead to such hierarchies along with the similarity in the effective theory in each case motivates one to examine the correspondence between the two. We began with the findings of [Giudice 17a], in which clockwork in the large N limit is related to a 5 dimensional theory with a certain space-time metric. We tried investigating whether the inverse result holds, i.e. whether a theory in that particular metric can be de-constructed to a clockwork like interaction. In particular, the metric fails to generate clockwork like features in the discrete theory for massive field. As already indicated by [Giudice 17b], such results are totally model dependent and cannot be generalized.

Appendix A

Spin connections

While dealing with fermions in curved space, the first thing that needs to be redefined is the set of Dirac matrices in the Clifford algebra. To do this, we need to consider a non-coordinate basis $e_{(a)} = e_a^M e_{(M)}$ where e_a^M are called the vierbeins [Weinberg 72] [Carroll 04] [Yepez 11], with the properties:

$$g_{MN} = e_M^{\ a} e_N^{\ b} \eta_{ab}; \quad e^M_{\ a} e_N^{\ a} = \delta_N^M; \quad e_M^{\ a} e^M_{\ b} = \delta_b^a \tag{A.1}$$

As such general covariance reduces to Lorentz covariance in this basis.

Since the Dirac matrices Γ^M must follow

$$\{\Gamma^M, \Gamma^N\} = 2g^{MN} \tag{A.2}$$

we can define

$$\Gamma^M = e_a^M \gamma^a \tag{A.3}$$

Additionally, in contrast to a scalar, the derivative of a spinor does not transform like the product of a vector times a spinor. Thus the derivative of a spinor needs to be redefined in a covariant form.

For our spin-1/2 field, we need that as $e^a_{\ M} \to \Lambda^a_{\ b} e^b_{\ M}$ and $\psi \to S(\Lambda)\psi$, the derivative of fermionic field must transform as:

$$D_a \psi \to \Lambda_a^{\ b} S(\Lambda) D_b \psi \tag{A.4}$$

Thus if $D_a = e_a{}^M(\partial_M + \omega_M)$, then

$$\omega_M \to S(\Lambda)\omega_M S^{-1}(\Lambda) - (\partial_M S(\Lambda))S^{-1}(\Lambda) \tag{A.5}$$

where $S(\Lambda) = exp(\frac{1}{2}\omega_{ab}\sigma^{ab})$ if for an infinitesimal Lorentz transformation $\Lambda = \mathbf{1} + \omega$

This suggests a choice of ω_M to be:

$$\omega_M = \frac{1}{2}\sigma^{ab} e_a{}^N \nabla_M e_{bN} \tag{A.6}$$

where ∇_M is the covariant derivative for a vector.

For a metric of the type (3.3), the non-zero Christoffel symbols are(no summation over repeated index):

$$\Gamma^5_{\mu\mu} = \frac{\partial_5 X}{2Y}\eta_{\mu\mu}; \quad \Gamma^5_{55} = \frac{\partial_5 Y}{2Y}; \quad \Gamma^\mu_{5\mu} = \frac{\partial_5 X}{2X}; \tag{A.7}$$

and the vierbeins are:

$$e^\mu{}_a = \delta^\mu{}_a \frac{1}{\sqrt{X}}; \quad e^5{}_a = \delta^5{}_a \frac{1}{\sqrt{Y}} \tag{A.8}$$

The spin connections

$$\omega_M = \frac{1}{2}\eta^{bb}(e_N{}^a e^P{}_b \Gamma^N_{MP} - e^P{}_b \partial_M e_P{}^b)\sigma_{ab} \tag{A.9}$$

in this case are:

$$\omega_\mu = \frac{\partial_5 X}{4\sqrt{XY}}\gamma_5\gamma_\mu; \quad \omega_5 = 0 \tag{A.10}$$

Appendix B

Diagonalization of tri-diagonal matrices

Consider the matrix encountered in (2.4):

$$M = \begin{pmatrix} 1 & -q & 0 & . \; . & . & 0 \\ -q & 1+q^2 & -q & . \; . & . & 0 \\ 0 & -q & 1+q^2 & . \; . & . & 0 \\ . & . & . & & 1+q^2 & -q \\ 0 & 0 & . & . \; . & -q & q^2 \end{pmatrix} \tag{B.1}$$

Diagonalizing such a matrix with arbitrary size might seem a difficult task at first but is rather straightforward. First let's simplify the situation by decomposing:

$$M = a\mathbf{1} + bT; \quad where \quad a = 1 + q^2; \; b = -q \tag{B.2}$$

so that if $M\lambda_j = m_j\lambda_j$, then $T\lambda_j = (\frac{1+q^2-m_j}{q})\lambda_j = t_j\lambda_j$. Thus to diagonalize M it's sufficient to diagonalize T.

And for T we have:

$$\begin{gathered} qv_1 + v_2 = tv_1 \\ v_{i-1} + v_{i+1} = tv_i \\ v_N + \frac{1}{q}v_{N+1} = tv_{N+1} \end{gathered} \tag{B.3}$$

where the particular eigenvector at hand χ with eigenvalue t is taken as:

$$\chi^T = \begin{pmatrix} v_1 & v_2 & . & . & . & v_{N+1} \end{pmatrix} \tag{B.4}$$

For $m_j = 0$, $t = (1+q^2)/q$ and (B.3) thus gives recursion relations

$$(qv_{i+1} - v_i) = (qv_i - v_{i-1}) \quad with \quad v_2 = v_1/q \tag{B.5}$$

These give $v_i = 1/q^i v_1$ and v_1 can be chosen so as to get a normalized eigenvector.

For $m_j \neq 0$, the second equation in (B.3) suggests an ansatz

$$v_j = Asinj\theta + Bcosj\theta \tag{B.6}$$

since $sin(j+1)\theta + sin(j-1)\theta = (2cos\theta)sin(j\theta)$. Putting this back in the rest of(B.3) gives:

$$Aqsin\theta = B(1 - qcos\theta); \quad sin(N+1)\theta(q^2 + 1 - 2qcos\theta) \tag{B.7}$$

The first equation fixes the coefficients upto a normalization and the second implies that θ is a multiple of $\pi/(N+1)$. Thus we have:

$$v_j = sinj\theta - qsin(j-1)\theta; \quad t = 2cos\theta; \quad \theta = \frac{k\pi}{N+1} \tag{B.8}$$

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