Sampling Ultrasound Signal with Finite Rate of Innovation

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A dissertation submitted for the partial fulfillment of BS-MS dual degree in Science

Indian Institute of Science Education and Research Mohali

April 2019

Certificate of Examination

This is to certify that the dissertation titled "Sampling Ultrasound with Finite Rate of Innovation" submitted by Mr. Ishant Sandil (Reg. No. MS14142) for the partial ful- filment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Prof Abhishek Chaudhuri Prof Kamal P. Singh Prof S.K. Biswas

(Supervisor)

Dated: April 23, 2019

Declaration

The work in this dissertation has been carried out by me under the guidance of Dr. Samir Kumar Biswas at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Ishant Sandil

(Candidate)

Dated: April 23, 2019

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. S.K. Biswas

(Supervisor)

Acknowledgement

First of all, I would like to express my gratitude to my advisor, Professor S.K. Biswas, for his support, patience, and encouragement throughout the project.

Further, I would also like to thank Dr. Abhishek Chaudhuri and Dr. K.P. Singh for reading drafts of this dissertation and providing valuable comments.

Finally, I would like to thank my parents, my seniors Shreyan Ganguly, Pankaj Kharra, my friends Parikshit Parihar, Shivam Abhineet, Vishal Sharma, Shubham Bhojane, Abhimanyu Nowbagh, especially Ramandeep Singh Arora and Shashank Prakash, the whole IISER community for encouragement, help, and providing a beautiful environment to work.

Ishant Sandil

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Notation

$*$: Convolution operator

g(x) : Continuous-time input signal

h(x) : Impulse response of acquisition device

$\varphi(x)$: Sampling kernel

T : Sampling interval

K : The degrees of freedom of the input signal

$c_{m,k}$: Coefficients of 1-D sampling kernels

$\hat{\phi}(t-n)$: Dual function of$\varphi(x)$

Contents

Abstract

The real world signal are analogue in nature but the computation is done digital. The process that makes it possible is known as the sampling process. Without the sampling process we cannot store, use, reuse or modify the real world signals. Nyquist sampling theorem tells us about how the continuous signals can be converted to digital signals, It provides a good approximation of the original signal. This approach restricts the class of signals that can be sampled and perfectly reconstructed to bandlimited signals. During the past few years, a new framework has emerged that overcomes these limitations and extends sampling theory to a broader class of signals named signals with Finite Rate of Innovation (FRI).

In this work I have used Finite Rate of Innovation to reconstruct an undersampled ultrasound signal.The FRI technique allow us to sample signals that are non-bandlimited and cannot be generally sampled using classing sampling theory that is, Nyquist sampling rate. That is, if we want to sample and reconstruct a signal back perfectly it has to be sampled at 7-10 times faster. The sampling rate directly affects the cost of the system, duration of sampling the signal and error produce from sampling the signal at high rates

The advantage that FRI gives us that it uses few sample points compared to the Nyquist theorem and by doing this we reduce the complexity of the acquisition device. Low sampling rate means less number of samples so less data space is used. To have a high sampling rate you need costly devices so looking at a technique which can lower the cost of instrument is also needed. In using FRI it gives all the above advantages. The process of sampling has allowed us to manipulate, store and transmit vast amount of data with increasing convenience. However, in data-intensive and/or power-limited applications such as sensor networks, the information contained is normally far less than the data observed, therefore, efficient sampling techniques is vital and necessary in such applications

In this thesis we consider the sampling of FRI signals and extend the results in to Ultrasound signal which is sampled at sub-Nyquist rate. We then try to reconstruct the original signal from those samples.

Chapter 1

Introduction

1.1 Signal Processing

What is Signal?

Signal is something that conveys information. Information about the state or behaviour of the system. The type of signal related to my work are Continuous signal and Discrete-time Signal. Continuous Signal are continuous in both amplitude and time domain. Discrete-time signal are discrete in time domain. Discrete-time signals can represented as sequence of numbers and can be written as

$$g = g[n], \qquad -\infty < n < \infty, \tag{1.1}$$

where n is an integer. In case we have obtained the discrete-time signal from a continuous signal

$$g[n] = g(nT), \qquad -\infty < n < \infty, \tag{1.2}$$

1.2 Sampling

Sampling is the measurement of the signal taken at frequent intervals. Therefore sampling converts a continuous signal to a discrete signal. We sample a signal so it can be stored in, we can modify it or later it can be used for a different purpose. In sampling it is important to sample a signal such that it is a good representation of the original signal.

T is the *sampling period* i.e, after what interval of time are we measuring the value of the signal. Sampling period is important as we will see in the next section. The reciprocal of sampling period is *sampling frequency* which tells the number of samples taken in a unit time or the frequency of the samples.

Therefore, the acquisition device for converting a continuous-time signal to a discrete-time signal can be made up of a sampler which samples the signal($g(x)$ after an interval of T and gives us sampled or discrete signal $g[n] = g(nT)$.

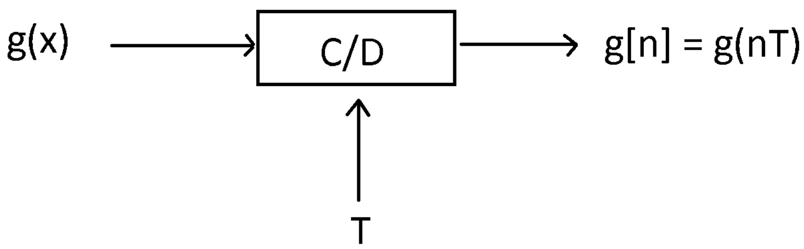

Figure 1.1: Block diagram representation of an ideal continuous-to-discrete-time(C/D) converter.

1.3 Nyquist Theorem

A continuous-time signal can be completely reconstructed from the knowledge of its values at points equally spaced in time. This surprising property follows from a result that is referred to as the sampling theorem.

If a function g(x) contains no frequencies higher than W cycles per second, it is completely determined by giving its ordinates at a series of points spaced 1/2W seconds apart.

Denoting f_s = 2W the sampling rate is measured in Hz, the nyquist interval $T = \frac{1}{f_s}$ correspond to the sampling period in seconds. In the ideal bandlimited case, we can obtain an unambiguous discrete-time representation of the signal by just storing its values every T seconds:

$$g[n] = g(t)|_{t=nT}$$

From samples x[n], we can perfectly reconstruct the original signal as follows:

$$g(x) = \sum_{n=1}^{\infty} g[n] sinc(\frac{x - nT}{T}), \tag{1.3}$$

where $sinc(x) = sin(\pi x)/\pi x$

This approach does not apply strictly to real world signals since it is well known that for a function to be bandlimited it must have infinite time duration. Moreover, in practice Shannon's reconstruction formula is rarely used due to the slow decay of the sinc function that is, it has infinite support. If g(x) is not bandlimited, prefiltering with an ideal lowpass filter (h(x) =$sinc(x/T)$ and reconstructing applying provides a lowpass approximation of g(x). However, it is an approximation, and perfect reconstruction of the original signal is not achieved. Moreover, the ideal lowpass filter is not realisable.

In classic way of sampling and reconstruction. We take an input signal which is usually continuous. The signal is sampled at regular intervals of times using a sampler. The to reconstruct the original signal from the samples that we have, the reconstruction is done using sinc function. The signal is reconstructed by multiplying each sample with sinc function. This gives us the continuous signal.

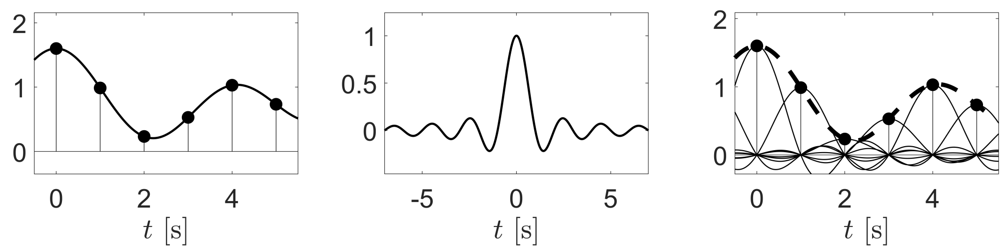

Figure 1.2: The bandlimited signal g(x) in (a) is sampled at regular intervals of time (black dots) leading to the discrete-time signal g[n]. The reconstruction is performed using the sinc function (b). The signal is reconstructed by weighting shifted versions of the sinc function with the discrete-time signal g[n].

1.4 Fourier Transform

These signal representations basically involve the decomposition of the signals in terms of sinusoidal (or complex exponential) components. With such a decomposition, a signal is said to be represented in the frequency domain. Representation of sequences using Fourier Transform, many sequences can be represented by a Fourier integral of the form

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\omega})e^{i\omega n}d\omega \tag{1.4}$$

$$X(e^{i\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{i\omega n} \tag{1.5}$$

1.5 Z-Transform

In this section we will see what a Z-transform is and what it does? It is an important concept in the annihilation filter. It is defined as:

$$X(z) = \sum_{n=-\inf}^{\infty} x[n]z^{-n} \tag{1.6}$$

This is an infinite sum equation and z is a complex variable. The sequence x[n] is converted to a continuous complex variable. This can be represented by

$$x[n] \leftrightarrow X(z)$$

Chapter 2

Finite Rate of Innovation

2.1 Introduction

Some classes of signals can be defined by a few number of parameters per unit time. The idea behind FRI is that if I can find these parameters I can construct the original signal back from these parameters. The number of parameters or degrees of freedom per unit time of signal is defined as Rate of Innovation. FRI technique allows us to even sample and reconstruct signals that are non-bandlimited e.g, Stream of Diracs. The number of samples or sampling rate depends on the number of parameters of that particular signal. FRI allow us to sample and reconstruct signals at lower rates than the classic Nyquist theory suggest i.e., twice the maximum frequency of the signal. But in practice it is 7-8 times the maximum frequency. But when we take the example of stream of Diracs which are an example of non-bandlimited signals we can see it is almost impossible to sample such kind of signal.

2.2 Classes of Signals

Here we will look at the different type or classes of signals that can be sampled uniquely. We consider a known form $f(t)$ and the form of the signal is can be written as

$$g(x) = \sum_{n \epsilon z} a_n f(x - x_n) \tag{2.1}$$

Here a_n and x_n are the amplitudes or the weight of the Diracs and time-instants or the location of the Diracs. There are n number of a_n and n number of t_n therefore there are $2n$ number of parameters and thus the sampling rate is 2n.

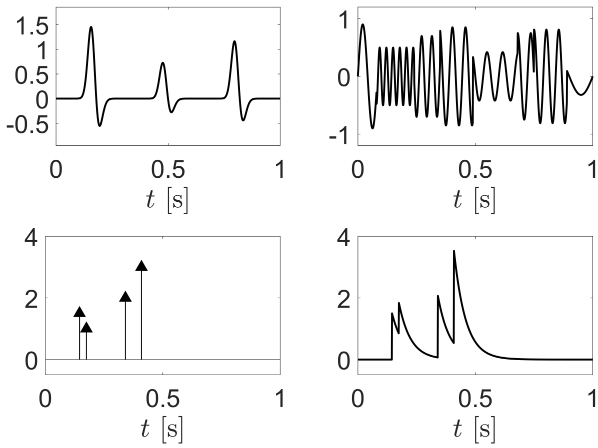

Figure 2.1: Examples of signals with FRI. When the shape of the pulse is known the signal depends only on the amplitude and location of such pulses.

The FRI technique requires a suitable sampling Kernel and an annihilation filter for the reconstruction. There are a few sampling kernels like the sinc filter(used in the original FRI paper) but the problem with such kind of filter is that it has an infinite support which makes the reconstruction process unstable. Therefore we would be looking at the class of kernels that have compact support e.g., Polynomial and exponential reproducing Kernels.

The annihilation filter or the Prony's method is a technique use to extract valuable information from the uniformly sampled signal.

Chapter 3

Sampling Kernels

In this section we will discuss the type of Kernels that can be used and we'll see the difference between them on their reproducing ability.

3.1 Infinite support Kernel

The original paper on FRI used the sinc Kernel and Gaussian for reconstruction of the signal. However as we discussed the problem with them is there infinite support which lead to unstable algorithms.

3.2 Polynomial and Exponential reproducing Kernels

A kernel $\varphi(t)$ is said to be a Polynomial reproducing kernel if it satisfies the following condition

$$\sum_{n\epsilon z} c_{m,n}\varphi(t) = x^m, \qquad m = 0, 1, 2..., P \tag{3.1}$$

that is sum of different weighted functions of $\varphi(t)$ gives a polynomial. This type of Kernel is compact in support

Similarly, A kernel $\varphi(t)$ is said to be a Polynomial reproducing kernel if it satisfies the following condition

$$\sum_{n\epsilon z} c_{m,n}\varphi(t) = e^{\alpha_m}t, \qquad m = 0, 1, 2..., P \tag{3.2}$$

that is sum of different weighted functions of $\varphi(t)$ gives a $e^{\alpha_m}t$. The Polynomial and exponential reproducing kernel will happen only if they satisfy Strang-Fix conditions.

3.3 Strang-Fix conditions

A function $\varphi(t)$ is said to be a Polynomial reproducing kernel that is

$$\sum_{n\epsilon z} c_{m,n}\varphi(t) = x^m, \qquad m = 0, 1, 2..., P \tag{3.3}$$

if and only if $\hat{\varphi(0)} \neq$ and $\hat{\varphi(0)}^m(2\pi l) \qquad for \begin{cases} l \in Z\ 0, \\ m = 0, 1, 2..., P. \end{cases}$

The superscript (m) at $\hat{\varphi(\omega)}$ stands for the mth derivative and nothing is mentioned then we follow the convention $\hat{\varphi(0)}(\omega) = \hat{\varphi(\omega)}$.

There is a whole lot of amount of information available and from that we know classes of family that satisfies these conditions. And Basis spline is a family of functions that satisfy these conditions. They were initially developed for designing vehicles and ultimately found their way in Signal Processing also. Generally, a spline is a piecewise polynomial function by which we mean that it is made up of different polynomial function or different type of polynomial function constitutes it. The extreme points of the spline refers to knots.

Consider the following function:

$$\beta_0(t) = \begin{cases} 1, 0 \leq t < 1, \\ 0, Otherwise, \end{cases}$$

This is the rectangular pulse or box function.

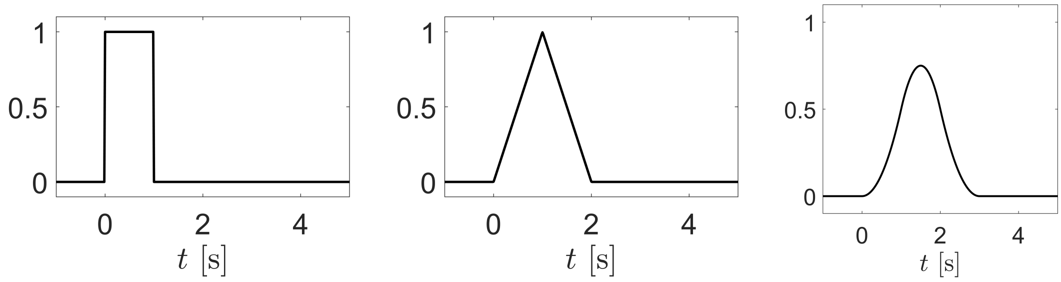

Figure 3.1: B-splines of orders P = 0,1,2. Note that the support of a B-spline of order P is [0,P + 1].

The Fourier Transform of this function can be written as

$$\hat{\beta}_0(t) = \frac{1 - e^{-i\omega}}{i\omega} \tag{3.4}$$

This is the classic sinc filter and from this filter we can create the rest of high degree polynomial reproducing function $\varphi(t)$. The degree of this polynomial spline is zero. Also, this function satisfies the Strang-fix conditions. Since it is a zero order reproducing spline it can produce constant functions as it can be visibly seen that it is a box function and if all the weights are same it will a constant value. Higher order B-spine are created by convolution of lower degree splines, that is,

$$\beta_p(t) = \beta_{p-1}(t) * \beta_0(t) \tag{3.5}$$

and we know that convolving functions in one domain that is in this case time-domain will multiply in other domain that is in this case frequency domain.

$$\hat{\beta_P}(\omega) = \hat{\beta}_{p-1}(\omega) * \hat{\beta}_0(\omega) = (\frac{1 - e^{-i\omega}}{i\omega})^{p+1} \tag{3.6}$$

These all satisfy the strang-fix conditions up to order P. As the name Polynomial reproducing Kernel, these functions can reproduce a specific polynomial t^m, we just have to find the values of the weight $c_{m,n}$ that are applied to $\varphi(t - n)$. The steps to compute these $c_{m,n}$ are given at the end of this chapter.

3.4 Strang-Fix conditions for Exponential reproducing case

As we saw in the case of polynomial reproducing kernel, the same can be extended to exponential reproducing kernel that is, withe different weighted functions $\varphi(t)$ we can reproduce exponential functions.

A function $\varphi(t)$ is said to be a Exponential reproducing kernel that is

$$\sum_{n\epsilon z} c_{m,n}\varphi(t-n) = x^r e^{i\omega_m t}, \tag{3.7}$$

if and only if

$$\hat{\varphi(\omega)} \neq 0 \quad \text{and} \quad \hat{\varphi(\omega)}^r(\omega_m + 2\pi l) = 0,$$

$$for \quad l\epsilon\ Z0, r = 0, 1, ...R_m \quad and \quad m = 0, 1, ...P$$

where the different values of r have some positive integer values. Similarly, as in the case of polynomial here also exists a family that is suitable for exponential reproduction, these are exponential B-Spline or E-Splines.

Consider the following function

$$\beta_\alpha(t) = \begin{cases} e^{\alpha t}, 0 \leq t < 1, \\ 0, Otherwise, \end{cases}$$

and the Fourier transform of this function is given by

$$\hat{\beta_\alpha}(\omega) = \frac{1 - e^{\alpha - i\omega}}{i\omega - \alpha} \tag{3.8}$$

This function too satisfies the generalized Strang-Fix conditions and is able to reproduce $e^{\alpha t}$. $\beta_\alpha(t)$ corresponds to the zero order E-Spline. As in the case of Polynomial spline, high-order splines are created by convolving zero order ones.

$$\beta_\alpha(t) = \beta_{i\omega_0}(t) * \beta_{i\omega_1}(t) * \beta_{i\omega_2}(t) * ... * \beta_{i\omega_p}(t), \tag{3.9}$$

and $\alpha = (i\omega_0, i\omega_1, ..., i\omega_P)$. And the generalized fourier transform for this can be written as

$$\hat{}_{\alpha}\omega = \prod_{m=0}^{P} \frac{1 - e^{-i\omega_m - \omega}}{i(\omega - \omega_m)} \tag{3.10}$$

3.5 Function reproduction with splines

In this section we provide an efficient way of computing the c_{mn}involved in the reproduc- tion of polynomial functions. A polynomial reproducing kernel satisfies:

$$\sum_{n \epsilon z} c_{m,n}\varphi(t) = x^m, \qquad m = 0, 1, 2..., P$$

$$\sum_{n \in z} C_{m,n}\phi(t - n) = x^m. \tag{3.11}$$

Here we have expanded the polynomial t^m in set of orthogonal basis function $\phi(t-n)$. Coefficients $C_{m,n}$ could be found by taking the inner product with the dual of $\hat{\phi}(t-n)$, where dual is defined as,

$$\langle \phi(t - n)\hat{\phi}(t - n') \rangle = \int_{-\infty}^{\infty} \phi(t - n)\hat{\phi}(t - n')dt = \delta(n - n'). \tag{3.12}$$

Now to find $Cm'n'$, we take inner product with the above equation,

$$\sum_{n \in z} \int_{-\infty}^{\infty} C_{m,n}\phi(t - n)\hat{\phi}(t - n')dt = \int_{-\infty}^{\infty} t^m\hat{\phi}(t - n')dt. \tag{3.13}$$

$$\Rightarrow \sum_{n \in z} C_{m,n}\delta(n - n') = \int_{-\infty}^{\infty} t^m\hat{\phi}(t - n')dt. \tag{3.14}$$

In the above equation $C_{m,n}\delta(n - n')$ term is non-zero only when $n = n'$, hence the summation disappears and only surviving term is $C_{m,n'}$.

$$.C_{m,n'} = \int_{-\infty}^{\infty} t^m\hat{\phi}(t - n')dt. \tag{3.15}$$

Now changing the variable in integral as, $t \to t - n'$ and $n \to n'$. Therefore the above equation could be written as,

$$.C_{m,n} = \int_{-\infty}^{\infty} (t+n)^m \hat{\phi}(t) dt. \tag{3.16}$$

Now expanding $(t+n)^m$ binomially in the above equation we get,

$$.C_{m,n} = \int_{-\infty}^{\infty} \sum_{k=0}^{m} {}^{m}C_k n^{m-k} t^k \hat{\phi}(t) dt. \tag{3.17}$$

Now taking the summation out and seprating the constants from integral,

$$.C_{m,n} = \sum_{k=0}^{m} {}^{m}C_k n^{m-k} \int_{-\infty}^{\infty} t^k \hat{\phi}(t) dt. \tag{3.18}$$

Now the integral part is nothing but $C_{k,0}$ by definition. hence above equation could be written in terms of $C_{k,0}$ as ,

$$.C_{m,n} = \sum_{k=0}^{m} {}^{m}C_k n^{m-k} C_{k,0}. \tag{3.19}$$

Now plugging the value of $C_{m,n}$ to our intial equation,

$$t^m = \sum_{n \in z} \left(\sum_{k=0}^{m} {}^{m}C_k n^{m-k} C_{k,0} \right) \phi(t-n) \tag{3.20}$$

Now writing $\sum_{k=0}^{m} {}^{m}C_k n^{m-k} C_{k,0}$ as,

$$.\sum_{k=0}^{m} {}^{m}C_k n^{m-k} C_{k,0} = \sum_{k=0}^{m-1} {}^{m}C_k n^{m-k} C_{k,0} + C_{m,0} \tag{3.21}$$

Putting it on the above equation we get,

$$.t^m = \sum_{n \in z} \left(\sum_{k=0}^{m-1} {}^{m}C_k n^{m-k} C_{k,0} + C_{m,0} \right) \phi(t-n) \tag{3.22}$$

Seperating out $C_{m,0}$,

$$t^m = C_{m,0}\sum_{n\in z}\phi(t-n) + \sum_{n\in z}\left(\sum_{k=0}^{m-1} {}^mC_k n^{m-k} C_{k,0}\right)\phi(t-n) \tag{3.23}$$

Now using the above equation, we could get an expression for $C_{m,0}$,

$$C_{m,0} = \frac{t^m - \sum_{n\in z}\left(\sum_{k=0}^{m-1} {}^mC_k n^{m-k} C_{k,0}\right)\phi(t-n)}{\sum_{n\in z}\phi(t-n)} \tag{3.24}$$

Now, this gives a recursive set of expression for coefficients $C_{m,0}$ as, for m=0,

$$C_{0,0} = \frac{1}{\sum_{n\in z}\phi(t-n)}. \tag{3.25}$$

for m=1,

$$C_{1,0} = \frac{t - C_{0,0}\sum_{n\in z} n\phi(t-n)}{\sum_{n\in z}\phi(t-n)}. \tag{3.26}$$

for m=2,

$$C_{2,0} = \frac{t^2 - C_{0,0}\sum_{n\in z} n^2\phi(t-n) - 2C_{1,0}\sum_{n\in z} n\phi(t-n)}{\sum_{n\in z}\phi(t-n)}. \tag{3.27}$$

Similarly rest of the coefficients of form $C_{m,0}$ could be found.

Once we find $C_{m,0}$, $C_{m,n}$ could be found using,

$$.C_{m,n} = \sum_{k=0}^{m} {}^mC_k n^{m-k} C_{k,0}. \tag{3.28}$$

Hence we could get all the coefficients using the above equations.

Chapter 4

Sampling and Perfect Reconstruction of FRI Signals

Till here we have seen how to construct a filter. Now we will see how to use the properties of the polynomial and exponential reproducing kernels and see how to use those properties to create the signal. We have chosen our input signal as stream of Diracs because many signals can be converted to this shape after sampling

4.1 Reconstruction Algorithm

As we defined earlier the input signal is given by

$$g(x) = \sum_{n \epsilon z} a_n f(x - x_n) \tag{4.1}$$

it consists of n Diracs with amplitudes a_n located at x_n) locations. This signal is sampled using the setup. The sampling Kernel $\varphi(t)$ is used, it modifies the signal and the sampling rate is $2n$ as there are $2n$ unknown variables. The samples y_n are given by

$$y[n] = \left\langle g(x), \varphi(x/T - n) \right\rangle \tag{4.2}$$

In the original FRI paper they have shown how to reconstruct it using sinc and Gaussian filter. Still the form that we have obtained above we cannot separate the time-dependent part and the time-independent part. Therefore we will use the property of the filter that they can create polynomial of varying degree. That is, we will multiply the samples obtained with the coefficients $c_{m,n}$ and consider the case of weighted samples.

$$\tau_m = \sum_n c_{m,n} y_n \tag{4.3}$$

Now we substitute the value of y_n in above equation.

$$\tau_m = \sum_n g(x) \left\langle c_{m,n}, \varphi(x/T - n) \right\rangle \tag{4.4}$$

Here we have used the property of linearity of the inner product to move the sum operator inside the inner product. Now we will also replace the the second term defined by the filter that we are using and in this case it is polynomial reproducing kernel.

$$\tau_m = \int_{-\infty}^{\infty} g(x) x^m dx \tag{4.5}$$

$$= \int_{-\infty}^{\infty} \sum_{n=1}^{N} a_n f(x - x_n) x^m dx$$

$$= \int_{-\infty}^{\infty} \sum_{n=1}^{N} a_n \delta(x - x_n) x^m dx$$

$$= \sum_{n=1}^{N} a_n x_n^m, \quad n = 0, 1, ..., N \tag{4.6}$$

We can solve this for the case of a exponential reproducing kernel also, the moments of signal,τ_m obtained will be given by.

$$\tau_m = \int_{-\infty}^{\infty} g(x) e^{\alpha x} dx$$

$$= \int_{-\infty}^{\infty} \sum_{n=1}^{N} a_n f(x - x_n)(t) e^{\alpha x} dx$$

$$= \sum_{n=1}^{N} a_n e^{\alpha_m x_n} dx$$

$$= \sum_{n=1}^{N} \hat{a_n} u_n^m dx \tag{4.7}$$

where $\hat{a_n}$ is given by $a_n e^{\alpha_0 x_n}$ and u_n is given by $e^{\lambda x - k}$. The solution for this type of equation where one part in time-dependent and other is non time-dependent or in general case to extract valuable information from uniformly sampled signal. Annihilation filter or Prony's method gives the solution for this kind of problem.

4.2 Annihilation Filter Method

We begin by assuming n data samples x[1], x[2],...x[n]. These we have obtained as we saw earlier from the multiplication of different $c_{m,n}$ weights to data samples obtained after passing through the filter $\varphi(t)$. we consider here M-exponent discrete-time function

$$x[n] = \sum_{n=1}^{N} k_k z_k^{n-1}$$

We can express this above equation in a matrix form

$$\begin{bmatrix} z_1^0 & z_2^0 & ... & z_N^0 \\ z_1^1 & z_2^1 & ... & z_N^1 \\ . & . & ... & . \\ z_0^N & z_2^N & ... & z_N^N - 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ . \\ h_N \end{bmatrix} = \begin{bmatrix} x[1] \\ x[2] \\ . \\ x[N] \end{bmatrix}$$

This equation can be solved for the unknown value of the amplitudes. Now we propose a polynomial with roots z_k that is the value goes zero for the equation when we put z_k. This is why the name of the filter is annihilation filter. The new polynomial $F(z)$ can be written as

$$f(z) = \prod_{k=1}^{M} (z - z_k)$$

$$= (z - z_1)(z - z_k)...(z - z_k)$$

and if we further simplify this above equation and can be written as the sum:

$$f(z) = \sum_{m=0}^{M} a[m] z^{M-m}$$

$$= a[0]z^M + a[1]z^{M-1} + ... + a[M]$$

Now the shifting the index in the first equation that we wrote in this section from n to n-m and multiplying the parameter a[m] gives us:

$$a[m]x[n-m] = a[m] \sum_{k=1}^{M} h^k z_k^{n-m-1}$$

Modifying this equation

$$\sum_{m=0}^{M} a[m]x[n-m] = \sum_{k=1}^{M} h^k z_k^{n-m} \sum_{m=0}^{M} a[m] z_k^{M-m-1}$$

This right-hand summation in nothing but the polynomial defined by us $f(z)$ evaluated at each of its roots z_k yielding the result O

$$\sum_{m=0}^{M} a[m]x[n-m] = 0$$

4.3 Moments of signal

Thus using the annihilation filter we get

$$h_m * \tau_m$$

The filter $f(z)]$ is called the annihilating filter as it annihilates the observed signal τ_m. The zeroes of such a filter define the distinct locations u_n. To retrieve the locations we write the convolution equation in following form.

$$\tau.F = \begin{bmatrix} \tau_K & \tau_{K-1} & \cdots & \tau_0 \\ \tau_{K+1} & \tau_K & \cdots & \tau_1 \\ . & . & \cdots & . \\ \tau_{2K} & \tau_{2K-1} & \cdots & \tau_K \\ . & . & \cdots & . \\ \tau_M & \tau_{M-1} & \cdots & \tau_{M-K} \end{bmatrix} \times \begin{bmatrix} h_0 \\ h_1 \\ . \\ h_K \end{bmatrix} = 0$$

The size of the Toeplitz matrix is $(M-K+1) \times (K+1)$, the length of the column vector H is K+1 and M $\geq 2K-1$as at least 2K consecutive values of the sample τ_m are required. We convert the set of above equations to Yule-Walker system by assuming $h_0 = 1$

$$\begin{bmatrix} \tau_{K-1}K & \tau_{K-2} & \cdots & \tau_0 \\ \tau_K & \tau_K & \cdots & \tau_1 \\ . & . & \cdots & . \\ \tau_{2K} & \tau_{2K-1} & \cdots & \tau_K \\ . & . & \cdots & . \\ \tau_{M-1} & \tau_{M-1} & \cdots & \tau_{M-K} \end{bmatrix} \times \begin{bmatrix} h_1 \\ h_2 \\ . \\ h_K \end{bmatrix} = \begin{bmatrix} \tau_K \\ \tau_{K+1} \\ . \\ h_M \end{bmatrix}$$

Solving this will give the roots which gives the locations of the Diracs. The system of the equations above gives a unique solution for u_k since the filter coefficients h_m are unique for a given signal. After finding the locations u_k we can find the weights a_k from the power series expression. It can be written in the matix form such as.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ u_1 & u_2 & \cdots & u_K \\ u_1^2 & u_2^2 & \cdots & u_K^2 \\ . & . & \cdots & . \\ u_1^{K-1} & u_2^{K-1} & \cdots & u_K^{K-1} \end{bmatrix} \times \begin{bmatrix} a_1 \\ a_2 \\ . \\ a_K \end{bmatrix} = \begin{bmatrix} \tau_0 \\ \tau_1 \\ . \\ h_{K-1} \end{bmatrix}$$

The above system is also known as Vandermonde system and leads to a unique solution for the amplitude a_k and u_k. Here we conclude that perfect reconstruction of a stream of K Diracs is possible with any kernel able to reproduce polynomials and exponential.

4.4 Sampling with FRI

Now, we have defined everything that is required in FRI. We will assemble all the steps together and see it's working. The signal$g(x)$ is passed through through the filter$h(x)$ which modifies the signal $g(x)$ and filters the signal with unwanted frequencies to the predefined property of the filter. After that the modified signal is sampled regularly. The number of samples or the sampling rate is determined by the number of parameters in the signal. The moments of signal which are obtained after sampling are passed through an Annihilation filter. The working of Annihilation filter has been shown in the previous section. What is basically does is it itself depends on these moment of signals. It will work in such a way that it destroy these moments of signals and will give us the location of the Diracs. After finding the location of Diracs we can find the amplitude of these Diracs. and this is what FRI says a signal can be created if we know all its parameters and essentially we have find these parameters and reconstruct the signal back. Here, we have assumed a signal consisting of four Diracs. The

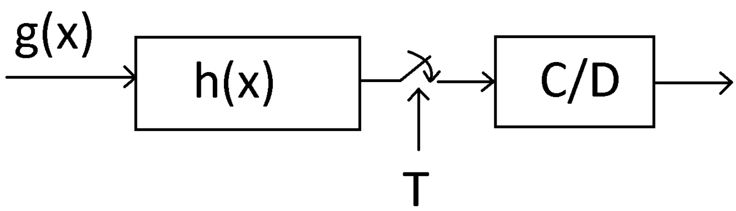

Figure 4.1: A typical sampling setup for 1-D FRI signals. Here, g(x) is the continuous-time input signal, h(x) the impulse response of the acquisition de- vice, $\varphi(x)$ the sampling kernel and T the sampling period.

idea is to find the locations and amplitudes of all the four Diracs. We pass the signal through the exponential reproducing function. The filter gives a continuous-time signal as the weight of Diracs are multiplied to the exponential reproducing function. This continuous time signal is sampled at regular intervals which gives us moment of samples, from this pass these moments of signals to a an annihilation filter to give us information about the location of Diracs and consequently about amplitudes. Thus allowing us to reconstruct the signal.

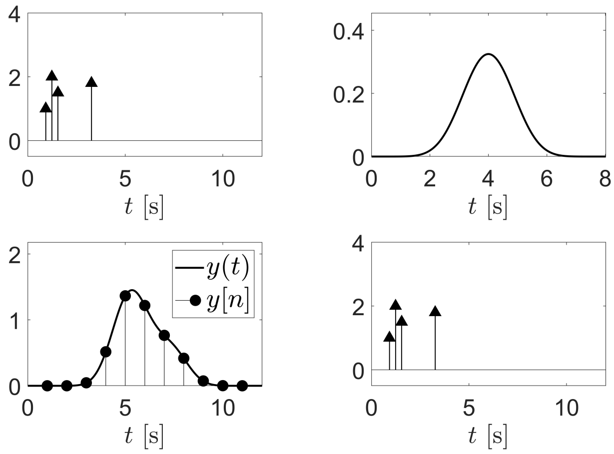

Figure 4.2: Sampling and perfect reconstruction of a stream of Diracs. (a) is the continuous-time stream of Diracs, (b) the sampling kernel h(t). is the continuous-time signal y(x) = g(x)h(x) and the corresponding discrete samples

Chapter 5

Summary & Conclusions

5.1 Concluding Remarks

Using FRI technique we nearly obtained the Ultrasound signal from the few samples we took. There was one considerable error noted in our that the reconstructed signal was stretched which it should not be.

Here, we took the Ultrasound signal and took few samples of this Ultrasound. Now our objective was to reconstruct the original signal from these few samples using FRI. The signal was passed through the FRI acquisition device. The exponential reproducing gives the continuous form of the signal that is, the reconstructed form. But still we want to see the accuracy of our reconstruction technique so we sample it at regular intervals and obtain the samples. Here what we observed the starting sample of the signal was reconstructed with zero error but as the other points were reconstructed too, the error in location and amplitude also increased, thus stretching our signal. Further improvement of this technique can help us improve the quality of signal at lower rates.

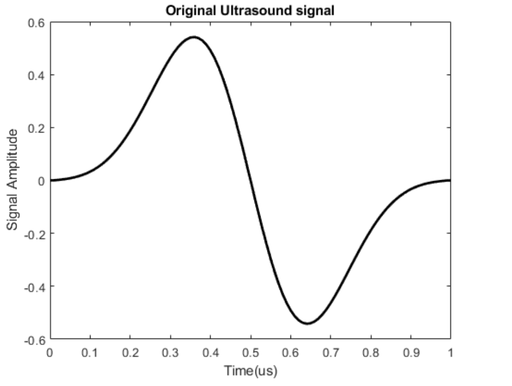

Figure 5.1: The original Ultrasound signal

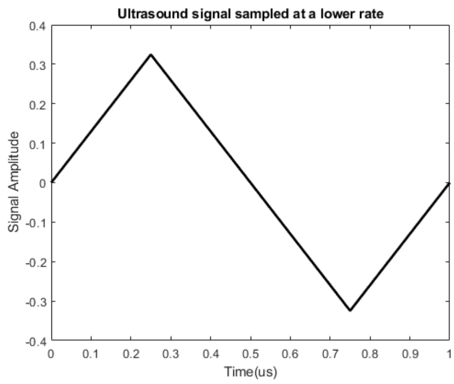

Figure 5.2: The Ultrasound signal which is sampled at a lower rate

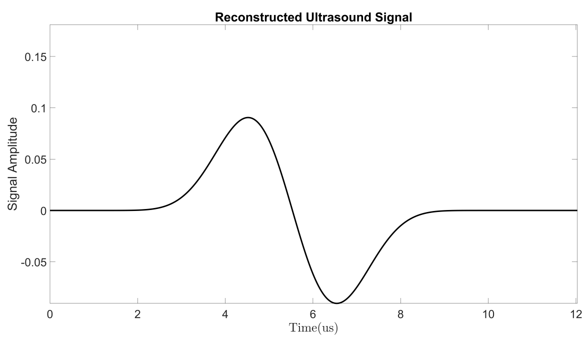

Figure 5.3: Reconstruction of Ultrasound signal using FRI technique

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