Probing Quantumness via Weak Measurements

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Certificate of Examination

This is to certify that the dissertation titled "**Probing Quantumness via Weak Measurements**" submitted by Raman Choudhary (Reg. No. MS14151) for the partial fulfillment of BS-MS dual degree programme of the institute, has been examined by the thesis committee duly appointed by the institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Kavita Dorai	Dr. Sandeep Goyal	Prof. Arvind (Supervisor)

Dated: May 11, 2019

Declaration

The work presented in this dissertation has been carried out by me under guidance of Prof. Arvind at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Raman Choudhary
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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Arvind
(Supervisor)

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Dedicated to my mother, father, sister and John Stewart Bell.

Abstract

Two of the most important ideas that distinguish the quantum world from the classical one are Non-locality and Contextuality. Non-locality is concerned with two physical systems sharing correlations that can not in any way be described by a Classical(local) model. Contextuality on the other hand takes Quantum weirdness to a whole new level. This feature allows values of Observables to be pre-existing but only in a way such that these values have to change as we bring a different apparatus to measure the same Observable. These two attributes have become a signature of nonclassicality.
A literature review on these foundational features majorly concerns either establishing new scenarios to test them or using them as resources in applications. These scenarios or use as resources have been established most-frequently by demanding Strong measurements of Observables on the quantum systems. But the fact that the Quantum framework allows a new form of measurement called Weak Measurement(WM) opens up a new gateway to test these features as well as this new measurement form. WMs are a more general class of measurements whose special case is the typical textbook-introduced Strong measurements. In WMs, the outcomes can no longer be labelled by eigenvalues and state no longer collapses to orthogonal states.
This thesis is an attempt to explore the combination of WMs & these weird features. We found that there is a degree of weakness of measurements that we can afford, below which we loose these non-classical features.

Chapter 1

What are Weak Measurements?

1.1 Introduction

According to Quantum Theory(QT), Corresponding to any measurable property of the system is an Hermitian operator $\hat{A}$ acting on the state space of the system. In our introductory Quantum Mechanics courses - we are taught that after the measurement of an observable, the outcome we get is the eigenvalue of this operator and the output state of the system is the corresponding eigenstate. This kind of measurement is called a Strong or Projective measurement. But the same observable $\hat{A}$ can be measured by the same apparatus in a different way in which the outcomes we get are not the eigenvalues and the output states we get are not the eigenstates of this Hermitian operator. These measurements are called Weak measurements. Weak measurements(along with weak values)[1] were introduced by Aharonov, Albert and Vaidman in 1988. Both Strong & Weak measurements can be better understood by following the paradigm of measurement in Quantum theory called the Von-Neumann measurement scheme[2]- as discussed in the following section. Strong measurement is represented by measurement operators called Projectors$\{\Pi_i\}$ which satisfy an extra condition of Orthogonality(other than Completeness)-$\Pi_i \Pi_j = \delta_{ij} \Pi_i$- compared to the measurement operators for weak measurement which only satisfy the Completeness relation $\hat{M}_m{}^\dagger \hat{M}_m = I$.

1.2 Von Neumann measurement scheme

Under this scheme, Measurement is a two step process:

1. Coupling of the system with the auxiliary system called Ancilla[1]/Pointer.
2. Projectively measuring the Ancilla.

This scheme is based on certain physical assumptions:

- Both the system and pointer are to be treated Quantum mechanically.
- Interaction Hamiltonian(H_{int}) must be effective only during the Interaction time T during which we can ignore the system's Hamiltonian.
- H_{int} must couple $\hat{A}$ to something that yields an Observable change in the pointer.

Based on these assumptions Von Neumann posits the form of H_{int} to be:

$$H_{int} = g(t)\hat{A} \otimes \hat{P}_d \tag{1.1}$$

where g(t) is a function non-zero only for time 'T'. To see how this scheme describes any measurement, let us see an example where we measure $\hat{\sigma_z}$ in a Stern-Gerlach(SG) experiment on a spin-$\frac{1}{2}$ system initially in state $|\Psi_{in}\rangle = \alpha\,|+\rangle + \beta\,|-\rangle$ where $|\pm\rangle$ are eigenvectors of $\hat{\sigma_z}$ with eigenvalues ± 1. Let ancilla's initial state be a gaussian state $|\Phi(x)\rangle = N\int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}}\,|x\rangle\,dx$ where $N = (\pi\sigma^2)^{-1/4}$. Then according to this scheme initial state of system + ancilla before the interaction is:

$$|\psi\rangle = |\Psi_{in}\rangle \otimes |\Phi(x)\rangle \tag{1.2}$$

Under this H_{int} the composite system evolves according to the Unitary $U_{int} = e^{-\int_0^T iH_{int}dt}$ ($\hbar = 1$) which can be written as $U_{int} = e^{-iG\hat{A}\otimes\hat{P}_d}$ where $G = \int_0^T g(t)dt$ called the effective coupling(which we safely assume to be 1). The state after the interaction is given by:

$$U\,|\psi\rangle = \alpha\,|+\rangle\,|\Phi(x-1)\rangle + \beta\,|-\rangle\,|\Phi(x+1)\rangle \tag{1.3}$$

[1]What we require is a new state space which could be the position space of the same spin-$\frac{1}{2}$ system and not necessarily a new quantum system.

The ancilla is now measured projectively in the position basis $\{|x\rangle\}$ and let the outcome be x' ,then the state(unnormalized) of the Composite system after the measurement is:

$$|x'\rangle \langle x'| U |\psi\rangle = \alpha \langle x'|\Phi(x-1)\rangle |+\rangle |'x\rangle + \beta \langle x'|\Phi(x+1)\rangle |-\rangle |x'\rangle \tag{1.4}$$

Now if the σ is small enough such that the gaussians in Eqn. 1.3(RHS) have negligible overlapping, then x' lies only in one of the gaussians and we have only one of the terms surviving here in eqn. 1.4 .The final state in such a case is either $|+\rangle\, or\, |-\rangle$.These are called **Strong Measurements**. But in general the Gaussians may overlap (large σ- called Weak Coupling), then x' could belong to both of them and system's state $|\Psi_{fin}\rangle$ can be obtained from the measurement operator given by:

$$\hat{M}_{x'} = \langle x'| U |\Phi\rangle = Ne^{\frac{-(x'-1)^2}{2\sigma^2}} |+\rangle \langle +| + Ne^{\frac{-(x'+1)^2}{2}\sigma^2} |-\rangle \langle -| \tag{1.5}$$

This acts on $|\Psi_{in}\rangle$ to give:

$$|\Psi_{fin}\rangle = \hat{M}_{x'} |\Psi_{in}\rangle = e^{\frac{-(x'-1)^2}{2\sigma^2}} \alpha |+\rangle + e^{\frac{-(x'+1)^2}{2\sigma^2}} \beta |-\rangle \tag{1.6}$$

So for these overlapping Gaussians the final state is not an eigenstate of $\hat{A}$ but still a superposition of its eigenkets. This type of measurement is called **Weak Measurement(WM)**[2]. In weak measurements we have a continuum of output states possible where each one is slightly biased towards $|+\rangle\, or\, |-\rangle$ depending on the pointer's outcome. The Von-Neumann model thus helps us explain how the same Hermitian operator $\hat{A}$ can be measured in different ways.

1.2.1 Applications of weak measurements

1. State estimation using Weak measurements-can be used for finite ensemble sizes where weakly measured(hence weakly perturbed) systems could be re-used[3].

2. Efficiency improvement for certain protocols like remote state preparation(RSP)[4].

[2]Either assume G=1 and σ to be very large(approach 1) or we can say σ is small and $G \to 0$(approach 2). Both approaches are equivalent & define weak coupling b/w system and ancilla

1.3 Weak measurements with post-selection

It is imperative to realize that before this section we have talked about WMs wiithout requiring any sequential strong measurement being done on the system.But the following discussions require the presence of a strong measurement right after the weak interaction. Aharanov et al. in 1988 with their provocating title "How the result of a measurement of a component of the spin of a spin-$\frac{1}{2}$ Particle can turn out to be 100" introduced these measurements on an ensemble of pre & post selected systems and came up with a new concept of weak values. The following sub-sections help understand this concept.

1.3.1 What are pre/post selected ensembles?

Pre-selection means preparing an ensemble in a particular Quantum state, say, $|\Psi\rangle$. Post selection means measuring an Observable $\hat{B}$ on the system strongly and keeping a particular outcome, say,$|b_n\rangle$ and discarding all the other outcomes.

1.3.2 What are weak values and how do we measure them?

One can model this measurement scenario via Von Neumann measurement scheme. Let us say we want to obtain Weak value of an Observable $\hat{\sigma_z}$ on an ensemble of spin-$\frac{1}{2}$ systems pre-selected in state $|\psi\rangle = \alpha\,|+z\rangle + \beta\,|-z\rangle$ and post selected in $|+x\rangle$.The first step is to 'weakly' couple the system with the pointer(ancilla), initially in the state $|\Phi(x)\rangle = N\int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}}\,|x\rangle\,dx$ ($N = (\pi\sigma^2)^{-1/4}$), under the Unitary $U_{int} = e^{-iG\hat{\sigma_z}\otimes\hat{P}_d}$:

$$|\psi\rangle \otimes |\Phi(x)\rangle \longmapsto U_{int}\,|\psi\rangle \otimes |\Phi(x)\rangle \tag{1.7}$$

The next step is to measure the Observable $\hat{\sigma_x}$ strongly on the system and choosing only state $|+x\rangle$. The state of the pointer(unnormalized) after post-selection is given by:

$$|\phi_{fin}\rangle = \langle +x|\,e^{-iG\hat{\sigma_z}\otimes\hat{P}_d}\,|\psi\rangle \otimes |\Phi(x)\rangle \tag{1.8}$$

$$|\phi_{fin}\rangle = \langle +x|\,(I - iG\hat{\sigma_z}\otimes\hat{P}_d - \frac{G^2\hat{\sigma_z}^2\otimes\hat{P}_d^{\,2}}{2!} + ...)\,|\psi\rangle \otimes |\Phi(x)\rangle \tag{1.9}$$

Since coupling b/w system and ancilla was assumed weak i.e. $G \to 0$ (using approach 2 mentioned in footnote 2 on Pg.3) we can neglect powers of G higher than 2. Therefore Eqn. 1.9 becomes:

$$|\phi_{fin}\rangle = \langle +x| \, (I - iG\hat{\sigma_z} \otimes \hat{P}_d) \, |\psi\rangle \otimes |\Phi(x)\rangle \tag{1.10}$$

This can we written as:

$$|\phi_{fin}\rangle = \langle +x|\psi\rangle \, (I - i\frac{G \langle +x| \, \hat{\sigma_z} \, |\psi\rangle \, \hat{P}_d}{\langle +x|\psi\rangle}) \, |\Phi(x)\rangle \tag{1.11}$$

since G is small we can re-exponentiate under this assumption leading to:

$$|\phi_{fin}\rangle = \langle +x|\psi\rangle \, e^{-iG{\sigma_z}^w \hat{P}_d} \, |\Phi(x)\rangle \tag{1.12}$$

where ${\sigma_z}^w = \frac{\langle +x|\hat{\sigma_z}|\psi\rangle}{\langle +x|\psi\rangle}$) is called the weak value of $\hat{\sigma}_z$ under the given pre and post selection. $\hat{P}_d$ is the momentum operator which is the generator of spatial translations and therefore Eqn. 1.12 (after normalization) becomes:

$$|\phi_{fin}\rangle = |\Phi(x - {\sigma_z}^w)\rangle \tag{1.13}$$

Clearly, ${\sigma_z}^w$ could be a complex quantity too and in that case Eqn.1.13 becomes a gaussian with the mean value equal to the Real part of ${\sigma_z}^w$. In that case the Imaginary part becomes the mean value of momentum-space state of the pointer and can be derived if we do the Fourier transform of Eqn 1.13.

1.3.3 Properties of weak values

1. Weak values can be complex too. Since Weak value of an observable $\hat{A}$ is given by $A^w = \frac{\langle \psi_f | \hat{A} | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle}$ whose denominator can be complex if the pre & post selection is done that way.

2. Weak value of the sum of two observables is the sum of weak values of the observables independently.

$$(A + B)^w = \frac{\langle \psi_f | \hat{A} + \hat{B} | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle} = \frac{\langle \psi_f | \hat{A} | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle} + \frac{\langle \psi_f | \hat{B} | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle} = A^w + B^w$$

3. Weak value can very well lie outside the eigenvalue spectrum $[min(a_i), max(a_i)]$ by choosing the pre & post selected states suitably.

4. Just like we calculate expectation value of an observable $<\hat{A}>=\sum_n P_n a_n$ Similarly in terms of weak values we have $<\hat{A}>=\sum_n P_n Re({A_n}^w)$ where $Re({A_n}^w)$ is the real part of $\frac{\langle a_n|\hat{A}|\psi_i\rangle}{\langle a_n|\psi_i\rangle}$.

1.3.4 Applications of Weak values

1. Amplification of signals and signal-to-noise ratios- Using almost orthogonal pre & post selected states can make leads to very high weak value which is utilized to increase the amplitudes of signals[5][6].

2. Doing Quantum Process Tomography[7].

3. Can be used to find Magnetic field gradient of an unknown Stern-Gerlach device.

4. Quantum State Tomography[8].

1.3.5 Protocol to obtain Weak value of any Observable $\hat{A}$

This is a five step protocol:

1. Take a pre-selected system in $|\psi_{in}\rangle$ and couple it weakly with a gaussian pointer.

2. Make a measurement of another Observable say $\hat{B}$ on this system and select the output $|b_n\rangle$ i.e. post selection.

3. Make a Projective measurement in the position basis $\{|x\rangle\}$ on the gaussian pointer in step 1.

4. Once we follow the above steps for the whole ensemble, the expectation value of the pointer position gives the real part of the Weak value of the Observable $\hat{A}$.

5. In a new ensemble make a projective measurement of momentum on the gaussian pointer instead of step 3 and what the mean momentum obtained is the complex part of A^w.

1.4 Summary

In this chapter we introduce two notions of Weak measurements. One with post-selection and other without post-selection. The concept of weak values appears only when we talk about post-selection. Most of my work presented in the following chapters is based on WM without post-selection. Although in third chapter we might see some use of weak values to look at Contextuality.

Chapter 2

Weak Measurements & Non-locality

2.1 Introduction

In 1935, A. Einstein, B. Podolsky and N. Rosen[9] or EPR launched an attack on the foundational aspect of Quantum theory by insisting that Quantum state is an incomplete description of the state of the system which needs to be completed by augmenting with it some variables called Hidden variables(HVs). In their urge to preserve locality and Realism , about which the formalism of Quantum theory seems silent, they came up with certain arguments("If without in any way.........physical reality") trying to convince people about the existence of these local hidden variables holding the real state of the system . But in 1964 Bell converted these arguments into an experimentally testable inequality called the Bell's Inequality[10]. This inequality predicts a bound on the maximum amount of correlations two parties can observe if the nature is fundamentally Local-Realistic. It is known that Quantum mechanics predicts violation of this inequality[11]-hence called a Non-local theory- by describing an experiment in which two parties make Strong Measurements on their systems and found the correlations to be more than Local HVT's prediction. We tweak with this scenario by allowing Alice and Bob to make weak measurements on their respective systems and see whether we still see any violation of the inequality or not.

2.2 Bell-CHSH inequality

$$|E_{HV}(\hat{a},\hat{b}) - E_{HV}(\hat{a},\hat{c})| \leq 1 + E_{HV}(\hat{b},\hat{c}) \tag{2.1}$$

Bell derived this inequality by making an assumption on his Local Hidden Variable theory(LHVT). The assumption is that whenever Alice and Bob point their SG apparatuses in same direction they obtain anti-correlations. Due to this physical assumption only the Singlet state $|\Psi\rangle = \frac{1}{\sqrt{2}}[|+n\rangle|-n\rangle - |-n\rangle|+n\rangle]$,where $\pm$n can be any direction in space , can be used to test this inequality since Quantum Mechanics follows this physical assumption only via this state.
In 1969, J. Clauser, M.Horne, A. Shimony & R. Holt[12] or CHSH improved upon Bell's Inequality by deriving a new inequality via a LHVT that does not require negative correlations when two parties align these devices in the same direction. This allows other entangled states too to be used to test this inequality.

$$|E_{HV}(\hat{a_1},\hat{b_1}) + E_{HV}(\hat{a_1},\hat{b_2}) + E_{HV}(\hat{a_2},\hat{b_1}) - E_{HV}(\hat{a_2},\hat{b_2})| \leq 2 \tag{2.2}$$

here $E_{HV}(\hat{a},\hat{b})$ denotes the correlation Alice and Bob observe when they point their apparatus in $\hat{a}$ and $\hat{b}$ directions respectively. Throughout this chapter we will consider the CHSH inequality(Eqn. 2.2) & refer to it as the Bell-CHSH inequality .

2.3 CHSH scenario & the Quantum violation

Consider two parties, Alice and Bob, who can perform two possible measurements of two outcomes each. Alice's measurements will be labelled by A_i(i = 0, 1) and can return possible results $a_i = \pm 1$. Similarly, Bob can choose measurements B_j(j = 0, 1) with possible outcomes $b_j = \pm 1$. A source emits particles towards Alice and Bob who make the aforementioned measurements as shown in Fig2.1.

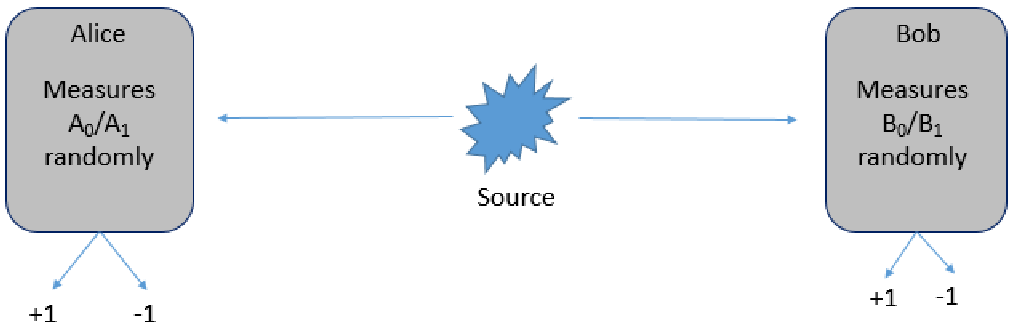

FIGURE 2.1: Random Dichotomic measurements by Alice and Bob

As mentioned above The Bell-CHSH inequality imposes a constraint on the correlations attainable by any local hidden-variable theory, and can be expressed as:

$$|E_{HV}(A_0, B_0) + E_{HV}(A_1, B_0) + E_{HV}(A_0, B_1) - E_{HV}(A_1, B_1)| \leq 2 \tag{2.3}$$

where the correlation $E_{HV}(A_i, B_j) = p(a_i = b_j|A_i, B_j) - p(a_i \neq b_j|A_i, B_j)$ with $p(a_i = b_j|A_i, B_j)$ being the probability of measurement outcomes being equal when settings A_i and B_j are chosen.
Let us describe the experiment that violates this inequality via Quantum mechanics. Say the source is emitting particles in the singlet state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ and let Alice choose between $\hat{A}_1 = \hat{\sigma_x}$ or $\hat{A}_0 = \hat{\sigma_z}$ and Bob between $\hat{B}_1 = \frac{\hat{\sigma_x} - \hat{\sigma_z}}{\sqrt{2}}$ or $\hat{B}_0 = -\frac{\hat{\sigma_x} + \hat{\sigma_z}}{\sqrt{2}}$ observables randomly. The expectation value of an observable $\hat{A} \otimes \hat{B}$ for the singlet state is given by $-\vec{a}.\vec{b}$ with $\vec{a}$ and $\vec{b}$ being measurement directions of Alice & Bob respectively i.e. $\langle\Psi|\, \hat{A} \otimes \hat{B} |\Psi\rangle$[1] $= -\vec{a}.\vec{b}$. Checking correlations for the respective pairs we get:

$$E(\hat{A}_0, \hat{B}_0) = \frac{1}{\sqrt{2}}; E(\hat{A}_0, \hat{B}_1) = \frac{1}{\sqrt{2}} E(\hat{A}_1, \hat{B}_0) = \frac{1}{\sqrt{2}}; E(\hat{A}_1, \hat{B}_1) = -\frac{1}{\sqrt{2}}$$

This leads to violation of the LHVT prediction:

$$E_{QM}(A_0, B_0) + E_{QM}(A_1, B_0) + E_{QM}(A_0, B_1) - E_{QM}(A_1, B_1) = \frac{4}{\sqrt{2}} = 2\sqrt{2} > 2$$

[1]This formula only holds true when measurements are Strong.

This result shows that a Local-Realistic theory can never be the fundamental theory of nature. This is why Quantum mechanics is called a Non-Local theory.

2.4 Weak measurements & CHSH scenario

In the above scenario Alice and Bob were making Strong measurements on their systems. We ask - given that Weak measurements perturb the system slightly, and offer lesser correlations compared to strong measurements. Still, is there a cut-off for the strength of measurement that shows the violation of Eqn. 2.3? To explore this question we created two scenarios:

1. Bob makes weak measurements of her observables $\hat{B}_0 \& \hat{B}_1$ whereas Alice still makes strong measurements of $\hat{B}_0 \& \hat{B}_1$.
2. Both Alice and Bob make weak measurements of their respective Observables.

2.4.1 Measurement strength parameter- G

We use Von Neumann's measurement scheme to describe our measurements. Let the pointer(ancilla) state chosen by Bob be in a Gaussian state: $|\Phi(x)\rangle = N \int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}} |x\rangle \, dx$ where $N = (\pi\sigma^2)^{-1/4}$. Consider the parameter(used as in [13]):

$$G = \int_{-1}^{+1} |\phi(x)|^2 dx$$

where $|\phi(x)|^2 = N^2 e^{\frac{-x^2}{\sigma^2}}$. G characterizes the strength of measurement and is a function of σ only. We know from Eqn. 1.3 that after the interaction of his system with the apparatus, The pointers begin to overlap depending on σ and hence on G. We now define that whenever $G \mapsto 1$, we call Bob's measurement Strong & otherwise it will be called a Weak measurement. Figures(2.2& 2.3) pictorially depict the reason to call G- the measurement strength parameter.

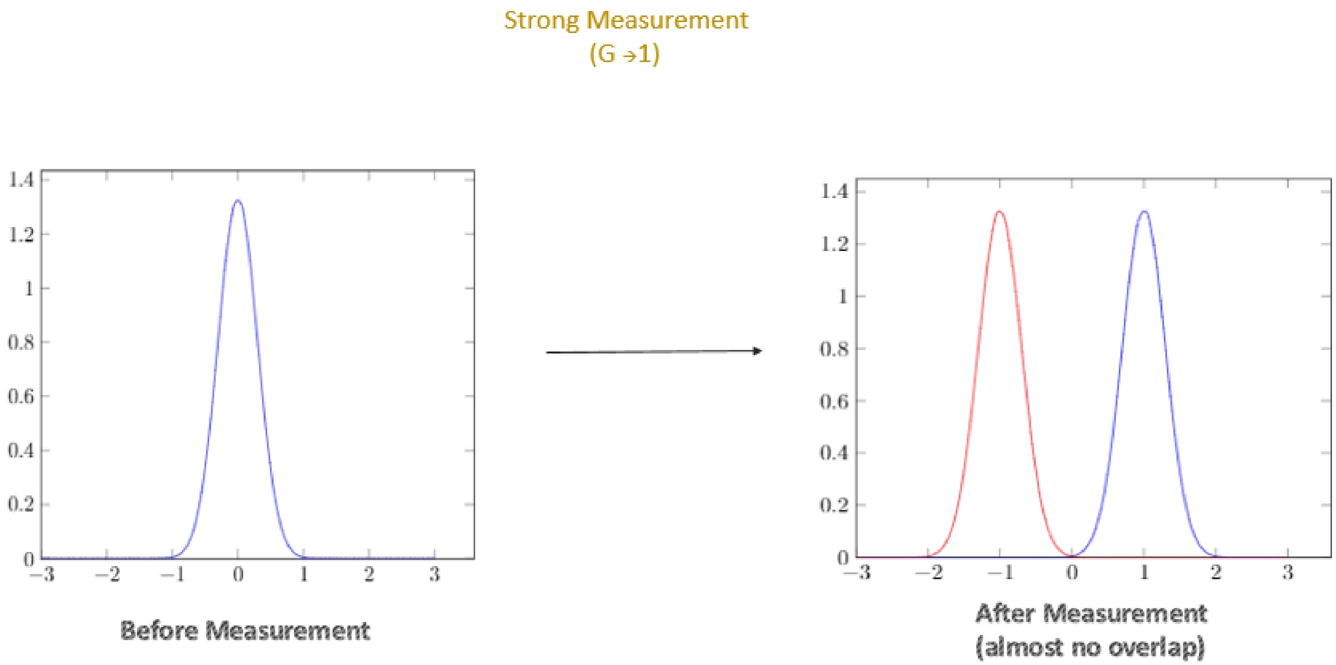

FIGURE 2.2: G=0.99957 or $\sigma = 0.3$

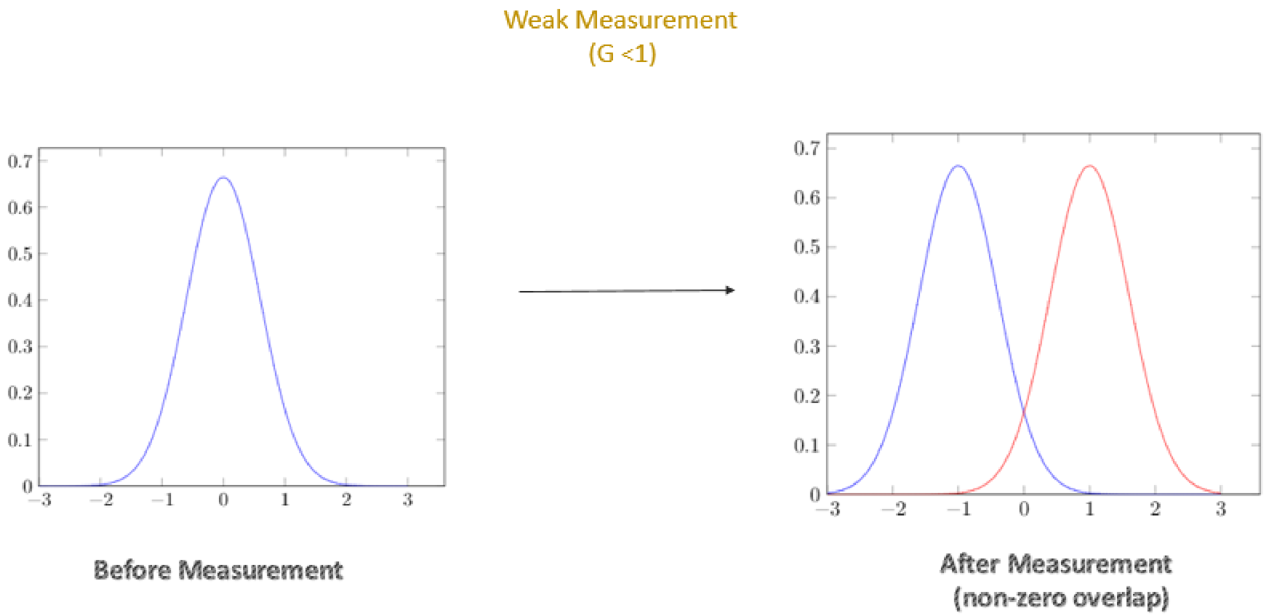

FIGURE 2.3: G=0.95221 or $\sigma = 0.9$

2.4.2 Calculating Correlations

Scenario 1:

Consider two parties Alice and Bob where the latter one makes Weak measurements. Initially the state is $|\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.The Correlation can be evaluated by :

$$E_{QM}(A_0, B_0) = P_{++} + P_{--} + P_{+-} - P_{-+} \tag{2.4}$$

where P_{++} is the probability of obtaining $++$ as outcomes on the two sides respectively .Calculating the correlation for P_{-+} firstly; Since Alice Strongly measures her system and assuming she got -1 as output. Therefore the system reaching Bob is in $|+z\rangle$ state on which $\hat{B}_0$ is to be weakly measured. Now Bob couples his pointer with the system as follows:

$$|+z\rangle \otimes |\phi(x)\rangle \longmapsto c_1 |+\rangle |\phi(x-1)\rangle + c_2 |-\rangle |\phi(x+1)\rangle = |\psi_f\rangle \tag{2.5}$$

where c_1, c_2 are amplitudes of $|+z\rangle$ in $\hat{B}_0$'s basis $|\pm\rangle$. Now making a projective measurement on the pointer can land us randomly on the + or - side of origin. Therefore:

$$P_{-+} = \int_0^\infty P(x|-z)P(-z)dx \tag{2.6}$$

where $P(x|-z)$ is the conditional probability of Bob getting b/w $x$and $x+dx$ when Alice gets -z(-1) as output. P(-z) is the probability of Alice obtaining -z(-1).The conditional probability is given by:

$$P(x|-z) = \int_0^\infty Tr(|x\rangle \langle x| \rho_p) \tag{2.7}$$

where ρ_p is pointer's reduced density matrix calculated by tracing the system out from state $|\psi_f\rangle$. After solving for the above relations we obtain:

$$P_{-+} = \int_0^\infty P(x|-z)P(-z)dx = \frac{1}{8}[2 - \sqrt{2}G] \tag{2.8}$$

Similarly by changing limit and Alice's outcome we obtain other terms in Eqn.(2.7)'s RHS as follows:

$$P_{+-} = \tfrac{1}{8}[2 - \sqrt{2}G]$$

$$P_{--} = \frac{1}{8}[2+\sqrt{2}G]$$
$$P_{++} = \frac{1}{8}[2+\sqrt{2}G]$$

This makes $E_{QM}(A_0, B_0) = \frac{G}{\sqrt{2}}$. Calculating other correlations:

$$E_{QM}(A_0, B_1) = \frac{G}{\sqrt{2}}$$
$$E_{QM}(A_1, B_0) = \frac{G}{\sqrt{2}}$$
$$E_{QM}(A_1, B_1) = -\frac{G}{\sqrt{2}}$$

Adding all these makes:

$$I_{CHSH} = E_{QM}(A_0, B_0) + E_{QM}(A_0, B_1) + E_{QM}(A_1, B_0) - E_{QM}(A_1, B_1) = 2\sqrt{2}G. \quad (2.9)$$

This can violate the Bell-CHSH inequality only if $I_{CHSH} > 2 \Rightarrow G > \frac{1}{\sqrt{2}} = 0.707$.

Scenario-2:
Consider two parties Alice and Bob again but this time both of them make Weak measurements. Initially the state of particles coming out from the source is $|\Psi\rangle = \frac{1}{\sqrt{2}}(|-1\rangle|+1\rangle - |+1\rangle|-1\rangle)$. Again our task is to calculate the correlations b/w them. We start off by calculating RHS of Eqn.2.4 and the other calculations will follow a similar pattern.

Alice's Stats:
Let Alice measure $\hat{\sigma_z}$ weakly by coupling her particle with the a pointer in the state $|\Phi_A(z)\rangle = N\int_{-\infty}^{\infty} e^{\frac{-z^2}{2\sigma_A{}^2}}|z\rangle\, dz$ where $N = (\pi\sigma_A{}^2)^{-1/4}$. Statistics for Alice are governed by the density matrix $\rho_A = \frac{1}{2}(|+1\rangle\langle+1| + |-1\rangle\langle-1|)$. After coupling her particle with her pointer(strength parameter G_A) the state becomes:

$$\rho_A \otimes |\phi\rangle\langle\phi| \quad (2.10)$$

Now probability of obtaining z on the position measurement on pointer is given by:

$$P(z) = \int Tr(|z\rangle\langle z|\rho_p)dz = \frac{1}{2}(\phi^2(z-1) + \phi^2(z+1))dz \quad (2.11)$$

here ρ_p is the reduced density operator of pointer obtained by tracing over the system. After Alice's turn now Bob makes his measurement. The state of the particles after

Alice's measurement is given by:

$$|\Psi_f\rangle = (\hat{M}_z \otimes I)\,|\Psi\rangle \longmapsto a\,|-1\rangle\,|+1\rangle - b\,|+1\rangle\,|-1\rangle \tag{2.12}$$

here $\hat{M}_z = \langle z|\,U\,|\phi\rangle$ & $a = \frac{e^{\frac{-(z+1)^2}{2\sigma^2}}}{e^{\frac{-(z+1)^2}{\sigma^2}}+e^{\frac{-(z-1)^2}{\sigma^2}}}$, $b = \frac{e^{\frac{-(z-1)^2}{2\sigma^2}}}{e^{\frac{-(z+1)^2}{\sigma^2}}+e^{\frac{-(z-1)^2}{\sigma^2}}}$.

Bob's Stats:

For Bob, the statistics are governed by ρ_B obtained by tracing Alice's system from $|\Psi_f\rangle\,\langle\Psi_f|$. Now to measure $\hat{B}_0$, Bob couples his system with the pointer in the state[2] $|\Phi_B(x)\rangle = N\int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma_B{}^2}}\,|x\rangle\,dx$ where $N = (\pi\sigma_B{}^2)^{-1/4}$. His measurement strength parameter is G_B. Clearly the probability of Bob's outcomes depend on what Alice's pointer output is since a & b are functions of z. The Conditional Probability($P(x|z)$) of obtaining outcome x on the pointer is given by:

$$P(x|z) = \int Tr(|x\rangle\,\langle x|\,\rho_p)dx \tag{2.13}$$

here ρ_p is obtained by tracing out the system from $\rho_B \otimes |\phi\rangle\,\langle\phi|$. Now to calculate P_{+-} we have:

$$P_{+-} = \int_0^{\infty}\int_{-\infty}^{0} P(x|z)P(z)dxdz \tag{2.14}$$

here z is from $(0,\infty)$ and x from $(-\infty,0)$.Solving this integral gives us $P_{+-} = \frac{1}{8}[2 - \sqrt{2}G_A G_B]$ Similarly we obtain other values:

$$P_{+-} = \tfrac{1}{8}[2-\sqrt{2}G_A G_B]$$
$$P_{--} = \tfrac{1}{8}[2+\sqrt{2}G_A G_B]$$
$$P_{++} = \tfrac{1}{8}[2+\sqrt{2}G_A G_B]$$

This makes $E_{QM}(A_0, B_0) = \frac{G_A G_B}{\sqrt{2}}$. Calculating other correlations:

$$E_{QM}(A_0, B_1) = \tfrac{G_A G_B}{\sqrt{2}}$$
$$E_{QM}(A_1, B_0) = \tfrac{G_A G_B}{\sqrt{2}}$$
$$E_{QM}(A_1, B_1) = -\tfrac{G_A G_B}{\sqrt{2}}$$

[2] x here actually means distance from origin along -$(\hat{x}+\hat{z})$

Adding all these makes:

$$I_{CHSH} = E_{QM}(A_0, B_0) + E_{QM}(A_0, B_1) + E_{QM}(A_1, B_0) - E_{QM}(A_1, B_1) = 2\sqrt{2}G_A G_B. \tag{2.15}$$

This can violate the Bell-CHSH inequality only if $I_{CHSH} > 2 \Rightarrow G_A G_B > \frac{1}{\sqrt{2}} = 0.707$.

2.4.3 Ensemble analysis

For Scenario-1, To see how Weak measurements compare with Strong measurements in terms of uncertainty and Bell-CHSH violation when we have a finite ensemble size (Data for the same is provided in Appendix A).

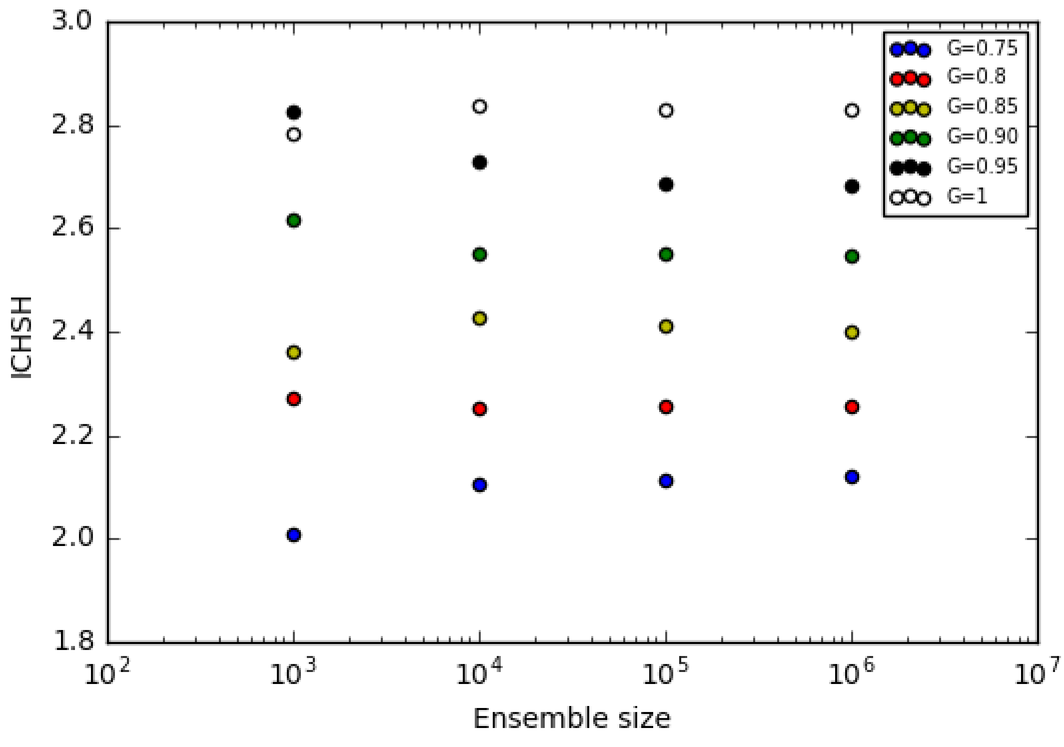

FIGURE 2.4: CHSH vs Ensemble size

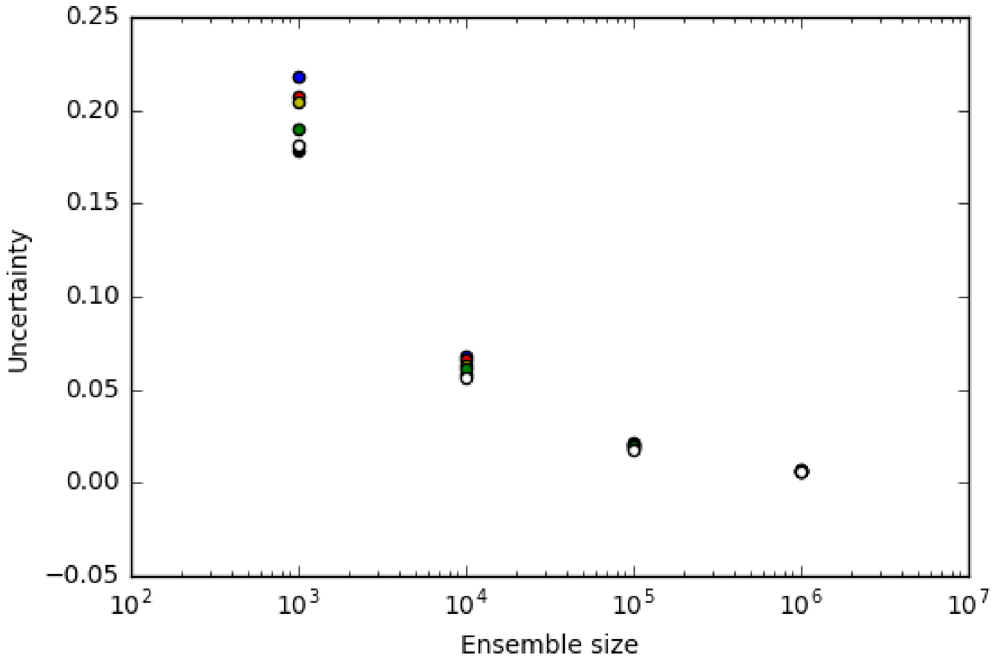

FIGURE 2.5: Uncertainty vs Ensemble size

2.5 Results

1. In scenario 1- where only one of the two parties makes weak measurements, we found a minimum bound on the strength of measurement that can allow non-local correlation i.e. $G > \frac{1}{\sqrt{2}}$.

2. In scenario 2- where both parties measure weakly, we found a minimum bound on the strength of both of the parties measurement to let them share non-local correlations i.e. $G_A G_B > \frac{1}{\sqrt{2}}$ and since both G_A & G_B are less than 1 $\Rightarrow$ for violation both parameters must be greater than $\frac{1}{\sqrt{2}}$ separately such that their product is also greater than $\frac{1}{\sqrt{2}}$. For e.g. if, say, $G_A = G_B = 0.8$ still $G_A G_B < \frac{1}{\sqrt{2}}$. Making $G_B > \frac{1}{\sqrt{2}}$ requires $G_A > \frac{1}{\sqrt{2}G_B}$ and vice-versa.

3. In the ensemble analysis we see that at a given Ensemble size as the strength of measurement increases the Violation increases and the Uncertainty decreases.
   As the Ensemble size increases the Uncertainty in the measurements decreases very rapidly.
   At a given measurement strength as the ensemble size increases there is no general pattern of Violation variation.

2.6 Additional Work

Other than the above-mentioned scenarios we also looked at various other cases where parties do Sequential measurements on their particles and see how much correlations do they obtain(for the singlet state only).

Scenario-1: Alice measures $\hat{A}_0$ weakly followed by measuring $\hat{A}_1$ strongly. Bob does the same thing by measuring $\hat{B}_0$ weakly and then $\hat{B}_1$ strongly. The correlation in this case that we got was $E = \frac{(G+F)^2}{\sqrt{2}}$ where $F = \int_{-\infty}^{\infty} \phi(x-1)\phi(x+1)dx$ i.e the overlap b/w shifted gaussians.

Scenario-2: Alice and Bob measure both of their observables weakly in sequence on their systems. The correlations obtained were $E = \frac{G^2(1+F)^2}{\sqrt{2}}$ where$F = \int_{-\infty}^{\infty} \phi(x-1)\phi(x+1)dx$.

But these scenarios can't be compared to the Bell-CHSH inequality since the inequality assumes hidden variable distribution corresponding to the initial combined state of particles we start with whereas the first distribution disturbs that state and therfore it seems there is no relation between the Bell-CHSH scenario and these cases.

Chapter 3

Weak Measurements & Contextuality

3.1 Introduction

The idea of Contextuality began with our Classical understanding of nature leading to question the fundamental randomness of measurement outcomes. Does the measurement apparatus reveal a pre-existing value to us or does it force an outcome randomly from a set of possibilities? In 1967, S. Kochen & E. Specker[14] proved that for Hilbert spaces(H_s) of dimension ≥ 3, this value pre-existence is not possible. They showed- if we assume that a system has a definite value of an Observable (in Hidden Variables) which exists independently of how we measure it(Independence of Context), then we arrive at a contradiction with Quantum mechanics. Since no such underlying Hidden Variable theory can explain Quantum mechanical predictions- we say Quantum mechanics is Contextual. Since then, many other proofs of Non-Contextuality have also appeared where for a small set of measurements, there exists a particular Quantum State that violates the Non-Contextuality condition expressed in the form of an inequality. Here we explore one such scenario called KCBS scenario[15] by replacing its Strong measurements with Weak measurements- to explore the degree of weakness we can have and still observe Contextuality in Quantum mechanics.
More Recently, Ravi Kunjwal[16] proved that under certain POVMs, Qubit(2-D H_s)

shows Contextuality[1]. By finding a 4-outcome POVM on a Qubit which violates LSW inequality. Here we look for a Physical Realization of these POVMs as Pojective measurements on a Higher dimensional Hilbert space and see if these Projective measurements follow any Contextuality(Non) relations or not.

3.2 Frameworks of Contextuality

3.2.1 Kochen-Specker(KS) Framework

Kochen-Specker proved that an Underlying Hidden Variable Theory(HVT) that assigns value to an observable, independently of what apparatus is being used to measure this observable is not an allowed theory. Their proof did not require any specific-state preparation to show the contradiction between QM and this HVT called Non-Contextual HVT(NCHVT). Such proofs are called **State-Independent** proofs of Contextuality .

There is another class of proofs which require very less number of observables to prove Contextuality of Quantum mechanics but they do so only for a specific-state preparation. Such proofs are called **State-Dependent** proofs of Contextuality. The state dependent proofs impose a bound on the measurement-statistics of a set of Observables . This allows experimental test of Contextuality of Quantum mechanics.

3.2.2 Rob Spekkens Framework

In 2005, Rob Spekkens[17] generalized the hitherto notion of Contextuality. Whenever an ensemble of systems is said to be prepared in the Quantum-Mechanical state $|\Psi\rangle$. Then in reality each of the systems is prepared in some state(λ) called the ontic state of that system. Therefore corresponding to $|\Psi\rangle$ we have a distribution of λs in the ensemble. Also we have a function $\xi_{M,k}(\lambda)$ which tells us the probability of obtaining outcome k if measurement M is performed on the system in ontic state λ. This function is called the response function. Now these λs are distributed with a probability

[1]This notion of Contextuality is slightly different from Kochen Specker one as is discussed in the following section under Spekkens Framework

distribution P(λ) such that they reproduce Quantum mechanical predictions for us.

$$P(k) = \int P(\lambda)\xi_{M,k}(\lambda)d\lambda = \langle\Psi|\,\Pi_k\,|\Psi\rangle \tag{3.1}$$

where P(k) is probability of outcome k.
To understand and define Ontological models better, Spekkens defined the following notions.

1. Context.
2. Operational Equivalence.

So Quantum mechanics tells us that only mutually commuting set of Observables can be measured simultaneously. Consider a set of 3 Observables A,B,C such that $[A, B] = 0 = [A, C]$ but $[B, C] \neq 0$. This tells us that there exists two different apparatuses via which A could be measured. One in which it is measured with B and in other with C. These two measurements apparatuses are called different **Contexts** for measuring A. Quantum mechanics says that these two devices give the same statistics for a given outcome k. For e.g. Consider two different arrangements measuring the vertical polarization of a photon.
1. M_1: A polaroid allowing vertically polarized light i.e. along $\hat{z}$ axis to pass through followed by a photodetector.
2. M_2: A Birefringent crystal oriented to separate light that is vertically polarized along $\hat{z}$ axis from horizontally polarized followed by a photodetector in the vertically polarized light.
Therefore these apparatuses M_1 and M_2 give us the same probability for any outcome k and are therefore called **Operationally Equivalent** set-ups. Changing the context i.e $M_1 \iff M_2$ does not change the equivalence class.

Non-contextual ontological model
A non-contextual hidden variable model is one wherein the response function does not change as we change between two measurement procedures that are operationally equivalent.
In other words a non-contextual model requires Ontic states to be measured in the

same way by two operationally equivalent measurement set-ups i.e.

$$\xi_{M,k}(\lambda) = \xi_{M',k}(\lambda) \tag{3.2}$$

Highlights of Spekkens framework:

1. His notion segregated the idea of Determinism(value pre-existence) from the question of Contextuality as opposed to Kochen-Spekken's framework where these two things are the same.

2. His notions allow an underlying noncontextual hidden variable model to itself be probabilistic i.e. Indeterministic Hidden variable model.

3. His notion also allows question of noncontextuality to be asked in case of Generalized measurements(POVMs) as opposed to Kochen-Spekker framework where only Hermitian observables (Strongly & Weakly both) are allowed to be measured.

4. His notion allows noncontextual hidden variable models for any operational theory and not only Quantum theory.

5. It talks about Measurement, Preparation & Transformation procedures whereas KS framework only talks about measurement (Non)Contextuality.

3.3 Weak measurements & KCBS scenario

The KCBS contextuality inequality is derived using KS framework. Its construction involves five projectors of the form $\Pi_i = |\phi_i\rangle\langle\phi_i| \in H_3$ given below[2]. The KCBS inequality is derived by a noncontextual model that assigns values 0 or 1 to these 5 projectors & imposes a maximum bound on the statistics of these projectors given by $\sum_{i=0}^{4} Tr(\rho\Pi_i) < 2$. It is a state dependent proof and the state $|\psi_{max}\rangle$ maximally violates this inequality and give us a Quantum bound of the inequality which is $\sqrt{5} > 2$. The explicit form of the state and the projectors is given in the following figure. In the above graph called the Exclusivity graph, the vertices represent Projectors and the edges represent Orthogonality.

[2]make sure to normalize the vectors

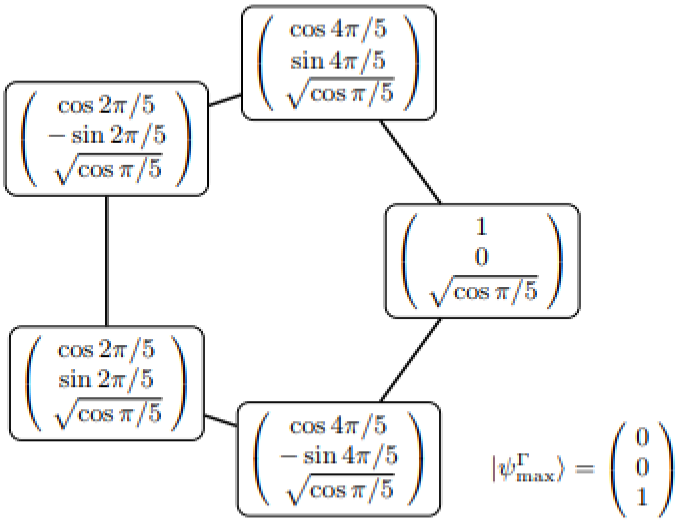

FIGURE 3.1: KCBS details

We tweak with this scenario by measuring these five projectors weakly to look for a bound on the weakness that is allowed which gives us this violation. To do this calculation of $\sum_{i=0}^{4} Tr(\rho\Pi_i)$ for our scenario. For each of the projectors Π_i we calculate the statistics[3] by Imagining measuring an Hermitian observable whose spectral decomposition contains two adjacent projectors and a new-found third projector orthogonal to the adjacent ones. for e.g. to measure statistics for Π_0 & Π_1 we find a third projector ${\Pi_2}'$ and these three uniquely identify an Hermitian observable $\hat{A}$. A weak measurement of this observable is performed from which we calculate the statistics for the clicks of Π_0 & Π_1. Similarly calculate it for other pairs of projectors.

In the weak measurement of $\hat{A}$ since we have three projectors we define the regions of pointer outcomes that represent the projectors say Π_0 is represented by $(-\infty, -1)$, ${\Pi_2}'$ is represented by (-1,1) & Π_1 by $(1, \infty)$. This is depicted in the following figure. To calculate the statistics for the Projectors:

First the system in the maximally violating state $|\psi_{max}\rangle$ is entangled with the pointer in the gaussian state $|\Phi(x)\rangle = N \int_{-\infty}^{\infty} e^{\frac{-x^2}{2\sigma^2}} |x\rangle\, dx$ where $N = (\pi\sigma^2)^{-1/4}$.

$$|\psi_{max}\rangle \otimes |\Phi(x)\rangle \longmapsto \alpha\, |\phi_0\rangle\, |\Phi(x-2)\rangle + \beta \left|\phi_2'\right\rangle |\Phi(x)\rangle + \gamma\, |\phi_1\rangle\, |\Phi(x+2)\rangle = |\xi\rangle \quad (3.3)$$

[3]The other way is to have a dichotomic measurement of $\{\Pi_i, I - \Pi_i\}$

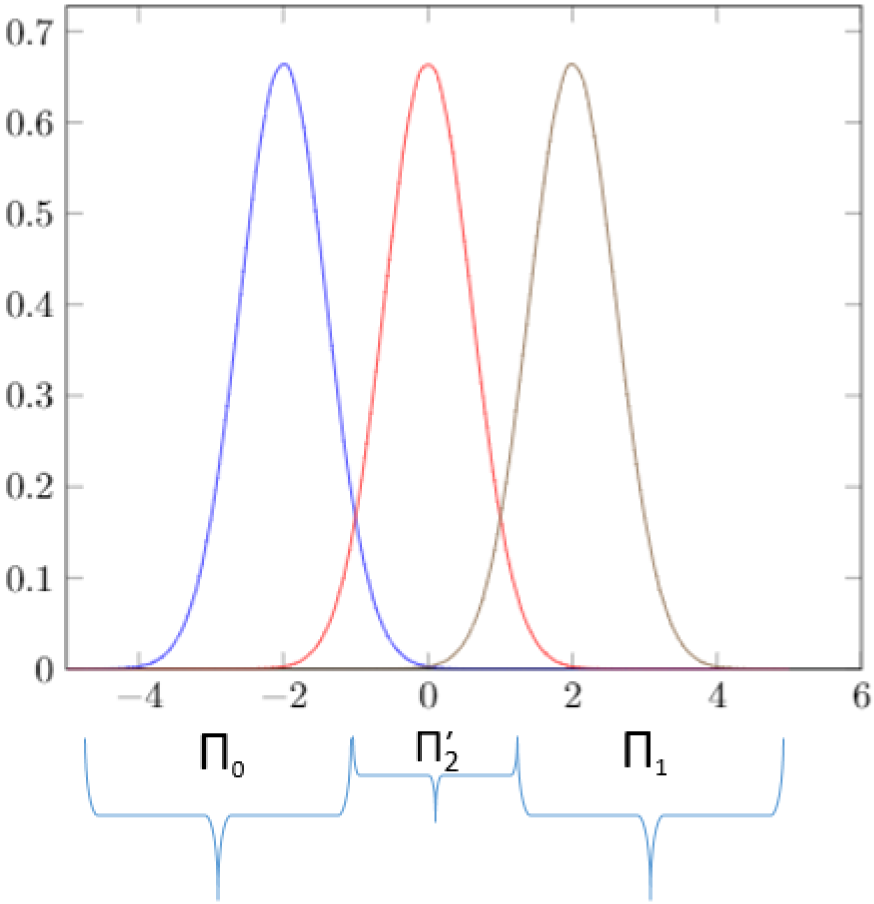

FIGURE 3.2: Regions of pointer outcomes corresponding to Projectors

where α, β, γ are amplitudes of $|\psi_{max}\rangle$ in basis of $\hat{A}$. Now we calculate the probability of our projector clicking by:

$$P(\Pi_0 = 1) = \int_{-\infty}^{1} Tr(|x\rangle\langle x|\,\rho_p)dx = \int_{-\infty}^{1} (|\alpha|^2\Phi^2(x-2)+|\beta|^2\Phi^2(x)+|\gamma|^2\Phi^2(x+2))dx \tag{3.4}$$

where ρ_p is reduced density operator of pointer by tracing out the system from $|\xi\rangle\langle\xi|$. Similarly all other $P(\Pi_i = 1)$ are calculated. The final plot of $\sum_{i=0}^{5} P(\Pi_i = 1)$ vs σ-The gaussian width deciding the strength of measurement is as plotted below.

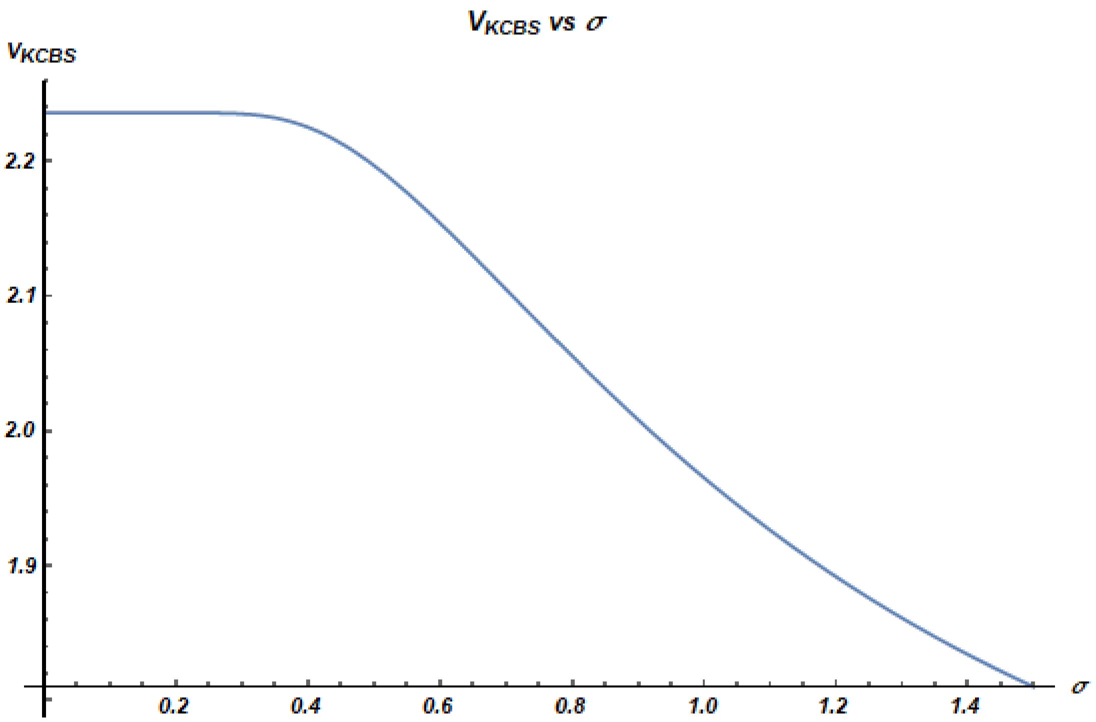

FIGURE 3.3: KCBS violation with weak measurements

3.4 Qubits and Contextuality

3.4.1 No contexts for Hermitian Operators

For measurements of Hermitian observables on a Qubit there is no more than one context i.e. In a 2-D H_s if we consider a set of Observables A,B,C(none equal to I) then if [A,B]=[A,C]=0 $\Rightarrow$ [B,C]=0. So all the mutually commuting Observables are simultaneously measurable[4] This is the reason that under KS framework Qubits can be described by a non-contextual model. But for Qubits a set of 3 POVMs can be constructed such that any two of them can be simultaneously measured. So a particular POVM can be measured in different contexts. Infact in an infinite number of ways.

[4]Except if one of the observables is I for e.g. $[I, \sigma_z] = 0 = [I, \sigma_x] \nRightarrow [\sigma_z, \sigma_x] = 0$ which is a trivial case as I only can take value 1, so every context assigns same value to it.

3.4.2 Contextuality under Generalized measurements(POVMs)

In 2011, Y.Liang, R.Spekkens & H. Wiseman (LSW)[18] found a non-contextuality inequality under Generalized measurements(POVMs) on Qubits. After deriving the inequality they conjectured that there exists no state and no set of POVMs- measurable mutually but not simultaneously- for Qubits that violate that inequality. But later in 2014 Ravi Kunjwal gave an explicit example of a state and a set of 3 POVMs that violate the LSW inequality.

LSW inequality

Consider three POVMs $M_k = \{E^k_+, E^k_-\}$ where k $\in$ {1,2,3} such that:

$$E^k_{\pm} = \frac{1-\eta}{2} + \eta\Pi^k_{\pm}, 0 \leq \eta \leq 1 \tag{3.5}$$

where $\Pi^{\pm}_k = \frac{I \pm \vec{\sigma}.\vec{\hat{n_k}}}{2}$ are Projectors. Therefore we call these measurements described by M_k as noisy measurements of these Observables defined by- $\Pi^{\pm}_k$. Consider a context M_i, M_j, the joint measurement of this context is represented by $G_{ij} \equiv \{G^{ij}_{++}, G^{ij}_{+-}, G^{ij}_{-+}, G^{ij}_{--}\}$. Now consider the quantity R_3 which is the average probability of anti-correlation on randomly choosing a context :

$$R_3 \equiv \frac{1}{3} \sum_{(ij)\in\{(12),(23),(13)\}} p(X_i \neq X_j | G_{ij}) \tag{3.6}$$

where X_i, X_j are measurement outcomes for M_i, M_j. So outcomes X_i, X_j in a joint measurement of M_i, M_j is represented by the effects $G^{ij}_{X_i X_j}$.

LSW showed that a non-contextual ontological model for these noisy observables has a bound:

$$R_3 \leq 1 - \frac{\eta}{3} \tag{3.7}$$

R. Kunjwal showed that this bound is violated by a set of observables measured in a plane at 120°(Trine-spin axes) to each other on a state lying along the normal to this plane on a bloch sphere. He found the explicit form of these joint POVMs. Here we look for a physical realization of these POVMs as Projective measurements on a higher dimensional Hilbert space and seek to explore Contextuality relations on these projectors(if any).

3.5 Physical Realization of POVMs

3.5.1 Tensor Product Extension

In this realization, a system on which the n-outcome POVM[5] is to be implemented is coupled with another system called Ancilla of dimension 'n'. Then a Projective measurement on the Ancilla is performed. Let us say we have a 3-outcome POVM {E1,E2,E3} to be implemented on a system initially in state $|\Psi\rangle$. Now we have to bring another system with hilbert space of dimension 3 initially in the state $|\phi\rangle_A$. The Composite system is now evolved under some Unitary operator $\hat{U}$ with the final entangled state given by:

$$|\Phi_f\rangle = \hat{U}\,|\Psi\rangle \otimes |\phi\rangle_A \tag{3.8}$$

Now consider a projective measurement of an Hermitian operator $\hat{A}$ with outcomes k($|k\rangle$) where k $\in$ {1,2,3}. Our task is to find the Unitary $\hat{U}$ given that the Ancilla(A) was initially in $|\phi\rangle_A$ and the measurement done on it, after the overall unitary evolution, was of observable $\hat{A}$. Now corresponding to each outcome k, the measurement operator(Krauss Operator) for the original system is given by $M_k = \langle k|\,U|\phi\rangle_A$ and therefore the POVM element corresponding to this is found by ${M_k}^\dagger M_k$. The Unitary $\hat{U}$ can be found by solving the following matrix equations $\forall$ k.

$${M_k}^\dagger M_k = \langle k|\,U|\phi\rangle_A = E_k \tag{3.9}$$

Finding this Unitary leads to $\langle\Psi|\,E_k\,|\Psi\rangle = p_k$, where p_k is probability of k^{th} outcome, which gives us the implementation of the POVM $\{E_1, E_2, E_3\}$.

Main Idea: The Tensor product implementation requires an external n-dimensional system for implementing an n-effect POVM.

3.5.2 Direct Sum Extension

The direct sum extension[19] is often a very less physically motivated implementation of POVMs. In this case the Hilbert space /system on which a POVM is to be implemented is assumed to be embedded in a higher dimensional Hilbert space. On this bigger Hilbert space we can find a projective measurement that gives us the outcomes

[5] For e.g. 3-outcome POVM has 3 elements E_k where $k \in \{1, 2, 3\}$ each one called 'Effect'

with the same probabilities as given by the POVM elements. Consider a 3-Outcome POVM measurement to be implemented on a qubit via the Direct-Sum Extension. The POVM is given by:

$$F_a = \frac{2}{3} \left|\uparrow_{\hat{n}_a}\right\rangle \left\langle\uparrow_{\hat{n}_a}\right| , a \in \{1, 2, 3\} \tag{3.10}$$

where $\hat{n}_1 + \hat{n}_2 + \hat{n}_3 = 0$. Let $\hat{n_1} = (0, 0, 1)$, $\hat{n_2} = (\frac{\sqrt{3}}{2}, 0, \frac{-1}{2})$, $\hat{n_3} = (\frac{-\sqrt{3}}{2}, 0, \frac{-1}{2})$ This POVM can be realized as an orthogonal measurement by looking at the qubit being embedded in a Qutrit. If the POVM elements are one-dimensional, then they can be written as :

$$F_a = \left|\tilde{\psi_a}\right\rangle \left\langle\tilde{\psi_a}\right| \tag{3.11}$$

where $\left|\tilde{\psi_a}\right\rangle$ is an unnormalized ket. In our case these take the following form:

$$\left|\tilde{\psi_1}\right\rangle, \left|\tilde{\psi_2}\right\rangle, \left|\tilde{\psi_3}\right\rangle = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix}, \begin{pmatrix} \sqrt{\frac{1}{6}} \\ \sqrt{\frac{1}{2}} \end{pmatrix}, \begin{pmatrix} -\sqrt{\frac{1}{6}} \\ \sqrt{\frac{1}{2}} \end{pmatrix} \tag{3.12}$$

Now once we have found these vectors, we can find 2 projectors on this extended Hilbert space is our implementation from these and the third can be found by using the property of normalization and orthogonality. In this case the 2 projectors are found by arranging the first elements of these column vectors in a new column(3x1) to give one projector and similarly the second elements give another projector. So the final forms are as follows:

$$\left|u_1\right\rangle, \left|u_2\right\rangle, \left|u_3\right\rangle = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ \sqrt{\frac{1}{3}} \end{pmatrix}, \begin{pmatrix} \sqrt{\frac{1}{6}} \\ \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{3}} \end{pmatrix}, \begin{pmatrix} -\sqrt{\frac{1}{6}} \\ \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{3}} \end{pmatrix}. \tag{3.13}$$

Main Idea: In this implementation, only a single Hilbert space or a single physical system is required to perform a projective measurement on a state prepared in only a subspace of the this Hilbert Space.

This Direct Sum Extension is what we use to implement[6] the joint POVMs described by Ravi Kunjwal to show a Qubit's Contextuality. The advantage of Direct Sum Extension is that it requires a smaller Hilbert space in general as compared to the

[6]Finding the Unitary via Tensor product extension requires us to find 8x8 matrix which is slightly complex with this POVM matrix

Tensor Product extension. This reduces Computational complexity since the size of Unitary operator to be found also reduces in the Direct Sum extension.

3.6 Realizing Kunjwal's POVMs

His POVMs have the following general form G^{ij}:

$$G^{ij}{}_{++} = \frac{1}{2}\{\frac{\alpha_{ij}}{2}I + \vec{\sigma}.\frac{1}{2}[\eta(\hat{n_i} + \hat{n_j}) - \vec{\alpha_{ij}}]\}. \tag{3.14}$$

$$G^{ij}{}_{+-} = \frac{1}{2}\{(1 - \frac{\alpha_{ij}}{2})I + \vec{\sigma}.\frac{1}{2}[\eta(\hat{n_i} - \hat{n_j}) + \vec{\alpha_{ij}}]\}. \tag{3.15}$$

$$G^{ij}{}_{++} = \frac{1}{2}\{(1 - \frac{\alpha_{ij}}{2})I + \vec{\sigma}.\frac{1}{2}[\eta(-\hat{n_i} + \hat{n_j}) + \vec{\alpha_{ij}}]\}. \tag{3.16}$$

$$G^{ij}{}_{--} = \frac{1}{2}\{\frac{\alpha_{ij}}{2}I + \vec{\sigma}.\frac{1}{2}[\eta(-\hat{n_i} - \hat{n_j}) - \vec{\alpha_{ij}}]\}. \tag{3.17}$$

taking η=0.69 & $\hat{n_1} = \hat{k}, \hat{n_2} = \frac{1}{\sqrt{2}}(\hat{i} - \hat{k})$, $\hat{n_2} = \frac{1}{\sqrt{2}}(-\hat{i} - \hat{k})$ & $\alpha_{ij} = 1 + \eta^2 \hat{n_i}.\hat{n_j}$ I implemented these 3 sets of joint POVMs(12,23,13) and since these are Rank-one POVMs I used the Eqn.3.11 and found the Projectors in a similar manner as described above. Since these are 4-outcome joint POVMs, The Hilbert space required to implement these is 4-dimensional(Indivisible) and therefore 4 projectors for each POVM G_{ij}.For Implementing G_{ij} we have the following matrices A_{ij} where each row represents a projector. In each of A_{ij}, all rows together define the Observable to be measured to implement the POVM G_{ij}.

$$A_{12} = \begin{pmatrix} 0.+0.526058\,i & 0.153602+0.283979\,i & -0.225305 & 0.41258-0.630904\,i \\ 0.+0.753832\,i & -0.107191-0.198173\,i & -0.322858 & -0.287917+0.440274\,i \\ 0.+0.225305\,i & 0.358641-0.663053\,i & 0.526058 & 0.322858 \\ 0.+0.322858\,i & -0.250276+0.462709\,i & 0.753832 & -0.225305 \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} 0.+0.526058\,i & 0.153602-0.283979\,i & -0.225305 & -0.41258-0.630904\,i \\ 0.+0.753832\,i & -0.107191+0.198173\,i & -0.322858 & 0.287917+0.440274\,i \\ 0.+0.225305\,i & -0.358641-0.663053\,i & 0.526058 & 0.322858 \\ 0.+0.322858\,i & 0.250276+0.462709\,i & 0.753832 & -0.225305 \end{pmatrix}$$

$$A_{23} = \begin{pmatrix} 0.-0.134117\,i & 0.602484 & -0.352451 & 0.703423 \\ 0.-0.556338\,i & -0.145242-0.537045\,i & -0.375015 & -0.169575+0.459981\,i \\ 0.-0.556338\,i & -0.145242+0.537045\,i & -0.375015 & -0.169575-0.459981\,i \\ 0.-0.602484\,i & 0.134117 & 0.771042 & 0.156587 \end{pmatrix}$$

FIGURE 3.4: Rows of each A_{ij} represent projectors (4-D H_s) to implement the corresponding POVM G_{ij}

Now, there exists a Non-Contextuality inequality developed by Nagali et. al[20] for the 4-dimensional Hilbert space such that the projectors must satisfy the following graphical representation. Under a non-contextual value assignment these must satisfy ζ.

$$\zeta = \sum_{i=0}^{9} P(\Pi_i = 1) \leq 3. \tag{3.18}$$

whereas the max QM bound is $\frac{7}{2}$. I checked the 12 projectors I have to look for the orthogonality relation but found that none of the projectors from a particular A_{ij} are orthogonal to the projectors of the other A_{il} or A_{ik}.

3.7 Results

1. In the KCBS scenario with Weak measurements we found that a the maximum width of pointer's Gaussian state that is allowed to violate the KCBS inequality is about 0.92.

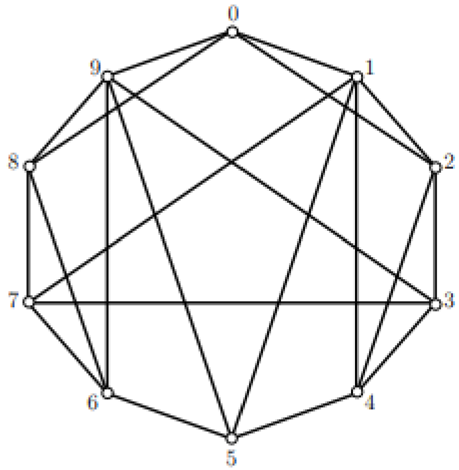

FIGURE 3.5: Orthogonality graph- The Vertices represent Projectors and the edges represent Orthogonality

2. In Kunjwal's POVM implementation scenario we proposed a good question but did not find any realtion between the POVMs violating the Qubit Non-Contextuality inequality(LSW) defined under Spekkens framework and the Nagali Inequality on a 4-D H_s derived under KS framework.

Appendix A

Ensemble Analysis DATA

ESIZE	UNCERTAINTY	STRENGTH	ICHSH	VIOLATION
1000	0.225445655	0.7	1.8	NO
10000	0.068925736	0.7	2.0288	NO
100000	0.02193701	0.7	1.99224	NO
1000000	0.006954603	0.7	1.97696	NO
1000	0.21815871	0.75	2.008	NO
10000	0.067996647	0.75	2.1056	YES
100000	0.021477135	0.75	2.11352	YES
1000000	0.006784524	0.75	2.119552	YES
1000	0.207522614	0.8	2.272	YES
10000	0.066077465	0.8	2.2544	YES
100000	0.020888773	0.8	2.2564	YES
1000000	0.006604017	0.8	2.257592	YES
1000	0.204040002	0.85	2.36	YES
10000	0.063565682	0.85	2.428	YES
100000	0.020193399	0.85	2.40944	YES
1000000	0.006402451	0.85	2.398352	YES
1000	0.190029562	0.9	2.616	YES
10000	0.061621095	0.9	2.5496	YES
100000	0.019483377	0.9	2.55144	YES
1000000	0.006169534	0.9	2.546408	YES
1000	0.178336477	0.95	2.824	YES
10000	0.058465683	0.95	2.7296	YES
100000	0.018753629	0.95	2.68464	YES
1000000	0.005933552	0.95	2.682944	YES
1000	0.181133099	1	2.784	YES
10000	0.056384892	1	2.8368	YES
100000	0.017877411	1	2.83016	YES
1000000	0.005653655	1	2.830016	YES

FIGURE A.1: Non-Locality Simulation outputs

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