Potential Theory of Some Subordinate Brownian Motions

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Certificate of Examination

This is to certify that the dissertation titled "**Potential Theory of Some Subordinate Brownian Motions**" submitted by Harsh Verma (Reg. No. MS14061) for the partial fulfillment of BS-MS dual degree programme of the institute, has been examined by the thesis committee duly appointed by the institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Kapil Hari Paranjape	Dr. Shane D'Mello	Dr. Arun Kumar (Co-Supervisor)	Dr. Neeraja Sahasrabudhe (Supervisor)

Dated: April 26, 2019

Declaration

The work presented in this dissertation has been carried out by me under guidance of Dr. Arun Kumar at Indian Institute of Technology Ropar and Dr. Neeraja Sahasrabudhe at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bona-fide record of original work done by me and all sources listed within have detailed in the bibliography.

Harsh Verma
(Candidate)

Dated: April 26, 2019

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Arun Kumar
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Abstract

The aim of the thesis is to study the potential theory of some subordinate Brownian motions. More precisely, we establish the asymptotic behaviour of the Green function and the Lévy density of some subordinate Brownian motions. We study the tools and techniques used in [4] and use similar methods to prove the asymptotic behaviour of the Green function and the Lévy density of two new subordinated Brownian motions. We also try to compute the asymptotic behaviour using an alternative approach.

Contents

Chapter 1

Introduction

In recent years subordinated stochastic processes are getting increasing attention due to their wide applications in finance, statistical physics and biology. Subordinated stochastic processes are obtained by changing the time of some stochastic process called as parent or outer process by some other non-decreasing Lévy process called subordinator or the inner process. Note that subordination is a convenient way to develop stochastic models where it is required to retain some properties of the parent process and at the same time it is required to alter some characteristics. Some well known subordinated processes include variance gamma process [21, 22], normal inverse Gaussian process [23], fractional Laplace motion [24], multifractal models [25], Student processes [26], time-fractional Poisson process [27, 28, 29, 30] and space-fractional Poisson processes[31, 32] etc. In this work we deal with the potential theory of some subordinate Brownian motion where the parent process is Brownian motion and the subordinators are geometric stable subordinator, tempered stable subordinator and inverse Gaussian subordinator.

The term "potential theory" has its origin in the physics of 19th century, where a dominant belief was that the fundamental forces of nature were to be derived from potentials which satisfied Laplace's equation [33]. This theory has its origin in the two well known theories of physics namely Gravitational and Electromagnetic theory. The term potential function was first associated with the work done in moving a point charge from one point of space to other in the presence of another external charge particle. The basic potential function varies as $\frac{1}{d}$, where d is the distance between the particles and the dimension of the space is greater than or equal to 3. This function has a property that it satisfies Laplace equation and such functions are called harmonic functions. From a mathematical point of view, potential theory is basically the study of harmonic functions [35]. Potential theory has intimate connection with probability theory. One important connection is that the transition function of a Markov process can be used to define the Green function related to the potential theory. In this project, we study the potential theory of some subordinate stochastic processes by realizing them as Brownian motion subordinated with different subordinators. Precisely, we study the asymptotic behaviour of potential density and Lévy density associated with different subordinators and also Green function and Lévy density associated with the subordinate processes. In easy language, potential measure represents the average time stay of a subordinator in a Borel subset of real numbers and Lévy measure quantifies the density of the number of jumps per unit time of the subordinator. The Green function denoted by $G(x, y) = G(x - y)$ for the Markov process is the expected amount of time spent at y by the Markov process started at x (see e.g. [34, 36]). The potential measure may be of interest to an investor, who is concern for the average time the stock prices stay in a particular price range.

The content of this thesis work is organized as follows. In Chapter 1, we give an introduction of the thesis work. In Chapter 2, we state all the basic definitions and theorems which are used later in subsequent chapters. We have provided examples related to the definitions and also have proved some of the examples. In Chapter 3, we prove the results taken from [4] in more details. We use definitions and theorems from Chapter 2 to prove the results of section 3.1 of this chapter. Section 3.1 of this chapter deals with the asymptotic behaviour of potential density and Lévy density associated with the geometric stable subordinator which are used further to establish the asymptotic behaviour of Green function and Lévy density associated with the Geometric stable process in the next section. In Chapter 4, we use all the tools and techniques from chapter two and Chapter 3 to prove the similar results for two new subordinated processes namely Brownian motion time-changed by tempered stable subordinator and normal inverse Gaussian process which are known to have financial applications. We also use an alternative approach to prove these results. As per our knowledge, all the results mentioned in chapter four are new. Atlast, we talk about our future work.

Chapter 2

Preliminaries

2.1 Definitions

In this section we state all the important definitions. We first define Infinite divisible distributions. These distributions are very important since they play an important role in heavy-tail modelling of economic data, also, they are related to Lévy processes which we define next. The sequence of random variables of a Lévy process are all Infinite divisible random variables and with every Infinite divisible distribution we can uniquely define a Lévy process.

Definition 2.1.1 (Infinite Divisibility). *A real-valued random variable X with a cumulative distribution function F is said to be Infinite divisible if for each $n > 1$ there exist independent identically distributed random variables $X_1, X_2, ..., X_n$ with a distribution function F_n such that*

$$X \stackrel{d}{=} X_1 + X_2 + ... + X_n.$$

Example 2.1.1. *Normal distribution, Poisson distribution, χ^2 distribution are some examples of Infinite divisible random variables.*

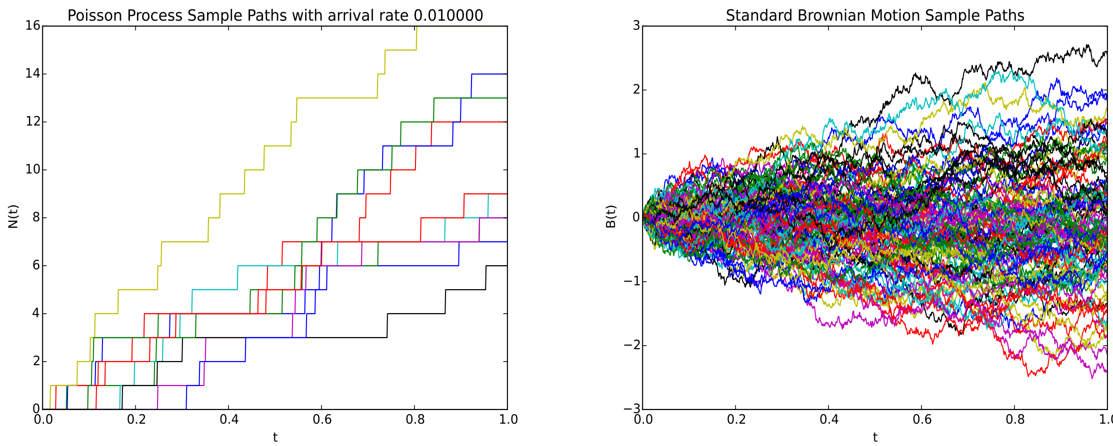
Definition 2.1.2 (Lévy Process). *A stochastic process $X = (X_t : t \geq 0)$ taking values in $\mathbb{R}^d$ is called a Lévy Process in $\mathbb{R}^d$ if*

1. $X_0 = 0$

2. *X has independent increments i.e for every $s, t > 0$, $X_{t+s} - X_t$ is independent of X_t.*

3. *For any $t, s > 0$, $X_{t+s} - X_t$ has the same distribution as X_s.*

Example 2.1.2. *Brownian motion, Poisson process, gamma process, inverse gamma process, inverse Gaussian process are some examples of Lévy processes.*



(a) Poisson Process

(b) Brownian Motion

Figure 2.1: Sample paths of Poisson process and Brownian motion and see appendix for the python code for the sample paths.

Definition 2.1.3 (Lévy Khintchine Representation)**.** *Lévy-Khintchine formula([8]): A probability law μ of a real valued random variable is infinitely divisible with characteristic exponent ψ,*

$$\int_R \exp(i\theta x)\,\mu(dx) = \exp(-\psi(\theta))$$

for $\theta \in \mathbb{R}$, if and only if there exists a triplet (a, b, l), $a \in \mathbb{R}$, $b \geq 0$ and l is measure concentrated on $\mathbb{R} - \{0\}$ satisfying $\int_{0^c}(1 \wedge x^2)l(dx) < \infty$, such that

$$\psi(\theta) = ia\theta + \frac{1}{2}b^2\theta^2 + \int_{\mathbb{R}} \left[1 - \exp(-i\theta x) + i\theta x 1_{|x|<1}\right] l(dx)$$

for every $\theta \in \mathbb{R}$. l is called the Lévy measure and it is unique.

Lévy process has the property that for $\forall\ t \geq 0$,

$$\mathbb{E}[\exp(i\theta X_t)] = \exp(-\psi_t(\theta)) = \exp(-t\psi(\theta)),$$

where $\psi(\theta) := \psi_1(\theta)$ is the characteristic exponent of X_1, which has an infinite divisible distribution. $\psi(\theta)$ is also called the characteristic exponent of Lévy process.

Next we show that the Lévy–Khintchine triplet of a drifted Brownian Motion is given by $(a, b, l) = (k, \sigma^2, 0)$.

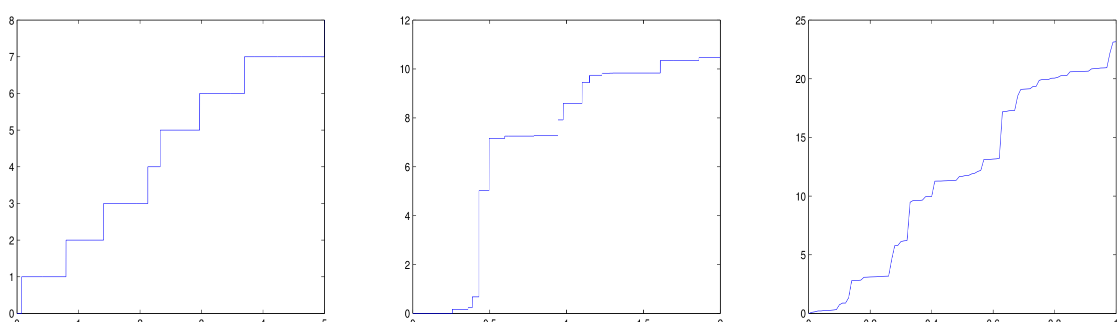
Let $X_t = kt + \sigma B_t, \forall t \geq 0$ be a drifted Brownian motion, where B_t is the standard Brownian motion. As we know B_t is same in distribution as $N(0, t)$, where $N(0, t)$ is a normal random variable with mean 0 and variance t, therefore, $kt + \sigma B_t$ will be distributed as $N(kt, \sigma^2 t)$. We know the exact form of characteristic function of a normal random variable, which gives $\psi(\theta) = \exp\left(ikt\theta - \frac{1}{2}\sigma^2\theta^2 t\right)$. Now after comparing this equation with the general form of Lévy–Khintchine formula for a Lévy process we get the Lévy–Khintchine triplet to be $(a, b, l) = (k, \sigma^2, 0)$.

Next, we define subordinators since they play a very important role in the theory of stochastic processes. Composition of a stochastic process with a subordinator is known as subordinated stochastic process and subordinated stochastic processes have applications in various areas of Mathematical finance.

Definition 2.1.4 (Subordinator)**.** *A subordinator is a 1-dimensional non-decreasing Lévy process. A subordinator $S = (S_t : t \geq 0)$ is characterized by its Laplace transform*

$$\mathbb{E}\left[(-\lambda S_t)\right] = \exp\left(-t\phi(\lambda)\right).$$

The function ϕ is called the Laplace exponent.



(a) Poisson Process (b) Gamma Process (c) Inverse Gaussian Process

Figure 2.2: Sample paths of Poisson, gamma and inverse Gaussian subordinators and see appendix for the python code for the sample paths.

Example 2.1.3. *Poisson process, gamma process, $\frac{\alpha}{2}$-geometric stable subordinator examples of subordinators.*

Next, we define slowly varying and regularly varying function. These two functions play an important role in Karamata's and de Haan's theory(for details see [1]). We use Karamata's and de Haan's Tauberian theorem([1]) to compute the asymptotic results in chapter 2 and 3 so it is useful to define slowly varying and regularly varying function.

Definition 2.1.5 (Slowly Varying Function)**.** *Let l be a positive measurable function, defined on some neighbourhood $[x, \infty)$ of infinity, and satisfying $\frac{l(\lambda x)}{l(x)} \to 1$ as $(x \to \infty)$ $\forall \lambda > 0$; then l is said to be slowly varying at infinity(in Karamata's sense)(for details see page 6, [1]).*

Example 2.1.4. *$f(x) = c$, where c is a constant, $f(x) = [\log(x)]^{-2}$ are some of the examples of slowly varying functions.*

Proposition 2.1.1. *$f(x) = [\log(x)]^{-2}$ is a slowly varying function at infinity.*

Proof. For any $\lambda > 0$ consider,

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} &= \lim_{x\to\infty} \frac{[\log(\lambda x)]^{-2}}{[\log(x)]^{-2}} \\
&= \lim_{x\to\infty} \left[\frac{\log(x)}{\log(\lambda x)}\right]^2 \\
&= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{\lambda x}\lambda} \text{ (Using L'Hospital's rule)} \\
&= 1
\end{aligned}$$

Thus, $f(x) = [\log(x)]^{-2}$ is slowly varying at infinity. $\square$

Definition 2.1.6 (Regularly Varying Function)**.** *A measurable(Baire) function $f > 0$ satisfying $\frac{f(\lambda x)}{f(x)} \to \lambda^\rho$ as $(x \to \infty)$ $\forall\ \lambda > 0$ is called regularly varying[Baire regularly varying] of index ρ; we write $f \in R_\rho$. Since f is Baire measurable therefore, $f(x) = x^\rho l(x)$, where $l(x)$ is slowly varying at infinity(for details see [1]).*

We next define Bernstein and complete Bernstein functions. These functions are usually related with Laplace exponent in probability theory. There is a chapter 2 which involves complete Bernstein function. To understand complete Bernstein function, we first define Bernstein function.

Definition 2.1.7 (Bernstein Function)**.** *Let $\phi : (0, \infty) \to [0, \infty)$ be a C^∞ function. Then ϕ is said to be a Bernstein function if $(-1)^n \phi^n \leq 0$ for all $n \in N, where \phi^n$ is the n^{th} derivative of ϕ(for details see [3] or [6]).*

Example 2.1.5. *$f(x) = x^\alpha \ \forall \ \alpha \in (0, 1]$, $f(x) = 1 - \exp(-x)$ are some of the examples Bernstein function.*

We now show $f(x) = x^\alpha \ \forall \ \alpha \in (0, 1]$ is a Bernstein function.

Since

$$f(x) = x^\alpha,$$

which implies

$$f'(x) = \alpha x^{\alpha-1}$$
$$f''(x) = (\alpha)(\alpha - 1)x^{\alpha-2}.$$

In general

$$f^n(x) = (\alpha)(\alpha - 1)(\alpha - 2)...(\alpha - (n - 1))x^{\alpha-n}.$$

Thus,

$$(-1)^n f^n(x) = (-1)^n(\alpha)(\alpha - 1)(\alpha - 2)...(\alpha - (n - 1))x^{\alpha-n}.$$

Now α and $x^{\alpha-n}$ both are greater than or equal to 0 and we can write $(\alpha - 1)(\alpha - 2)...(\alpha - (n - 1))$ as $(-1)^{n-1}k$ where k is a positive constant. Therefore, $(-1)^n f^n(x) = (-1)^{n+n-1}k\alpha x^{\alpha-n} = (-1)^{2n}(-1)k\alpha x^{\alpha-n} \leq 0$

Hence, $f(x) = x^\alpha \ \forall \ \alpha \in (0, 1]$ is a Bernstein function.

Remark 2.1.1. *ϕ is a Bernstein function if and only if it has the representation i.e, $\phi(x) = a + bx + \int_0^\infty [1 - \exp(-xt)] \mu(dt)$ where, $a, b \geq 0$, and μ is a σ finite measure on $(0, \infty)$. μ satiesfies $\int_0^\infty (1 \wedge t)\mu(dt) < \infty$(for details see [3] or [6]).*

Definition 2.1.8 (Complete Bernstein Function)**.** *A function $\phi : (0, \infty) \to (-\infty, \infty)$ is called a complete Bernstein function if there exists a Bernstein function ξ such that $\phi(x) = x^2 L\xi(x)$ for $x > 0$, where $L\xi$ is the Laplace transform of ξ(for details see [3] or [6]).*

Example 2.1.6. *$\phi(x) = x^\alpha \ \forall \ \alpha \in (0, 1]$, $f(x) = \frac{x}{x+1}, \phi(x) = \log(x + 1), \phi(x) = \log\left(1 + x^{\frac{\alpha}{2}}\right)$ are some examples of Complete Bernstein functions.*

Remark 2.1.2. *Note that every complete Bernstein function is a Bernstein function but converse is not true. For example-* $\phi(x) = 1 - \exp(-x)$ *is a Bernstein function but not a complete Bernstein function.*

Properties of complete Bernstein functions(for details see [3] or [6]):

1. The family of complete Bernstein functions is closed under compositions.

2. $\phi(x) : (0, \infty) \to (0, \infty)$, the following are equivalent,

   a. ϕ is a complete Bernstein function

   b. $x \to \frac{x}{\phi(x)}$ is a complete Bernstein function.

Proposition 2.1.2. *Laplace exponent of a* $\frac{\alpha}{2}$*- geometric stable subordinator,* $\phi(x) = \log\left(1 + x^{\frac{\alpha}{2}}\right)$ *is a complete Bernstein function.*

Proof. From discussion on page 12[13], we know that

$$\phi(x) = x^{\frac{\alpha}{2}} = \frac{\sin\left(\frac{\pi\alpha}{2}\right)}{\pi} \int_0^\infty \left(\frac{x}{x+t}\right) t^{\frac{\alpha}{2}-1} dt, \forall \alpha \in (0, 2]$$

and

$$\phi(x) = \log(x+1) = \int_0^\infty \left(\frac{x}{x+t}\right) t^{-1} 1_{(1,\infty)} dt.$$

Therefore, these two functions are complete Bernstein and hence, their composition $\phi(x) = \log\left(1 + x^{\frac{\alpha}{2}}\right)$ is also a complete Bernstein function(page12, [13]). □

2.2 Some Important Theorems

In this section we state some important theorems which will be used later to prove other theorems. We start with Tauberian theorems however, we are not going to prove these theorems since the proofs are beyond of the scope of this thesis. For more details and proof of these theorems please refer ([1], Theorems $1.7.1, 1.7.2, 1.7.1', 1.7.2b, 3.6.8$ and $3.7.3$). Tauberian theorems are useful when we want to know the asymptotic behaviour at infinity and zero of a real valued non-decreasing function but we do not know the exact form of it. However, we do know the exact form and asymptotic behaviour of its Laplace transform.

Theorem 2.2.1.

1. *(de Haan's Tauberian theorem)(Theorem 3.7.3,[1])Let* $U : (0, \infty) \to (0, \infty)$ *be an increasing function. If* l *is slowly varying at* ∞ *(resp at* $0+$*),* $c \geq 0$*, the following are equivalent:*

   a. *As* $x \to \infty$ *(resp* $x \to 0+$*)* $\frac{U(\lambda x) - U(x)}{l(x)} \to clog(\lambda), \forall \lambda > 0,$

   b. *As* $x \to \infty$ *(resp* $x \to 0+$*)* $\frac{LU\left(\frac{1}{\lambda x}\right) - LU\left(\frac{1}{x}\right)}{l(x)} \to clog(\lambda), \forall \lambda > 0.$

2. *(de Haan's monotone density theorem)(Theorem 3.6.8,[1]) Let* $U : (0, \infty) \to (0, \infty)$ *be an increasing function with* $dU(x) = u(x)dx$*, where* u *is monotone and non-negative and, let* l *be slowly varying at* ∞ *(resp at* $0+$*). Assume that* $c > 0$*. Then the following are equivalent:*

   a. *As* $x \to \infty$ *(resp* $x \to 0+$*)* $\frac{U(\lambda x) - U(x)}{l(x)} \to clog(\lambda), \forall \lambda > 0,$

   b. *As* $x \to \infty$ *(resp* $x \to 0+$*)* $u(x) \sim cx^{-1}l(x).$

3. *(Karamata Tauberian theorem)(Theorem 1.7.1,[1]) Let* U *be a non-decreasing right continuous function on* $\mathbb{R}$ *with* $U(x) = 0 \ \forall \ x < 0$*. If* l *slowly varies and* $c \geq$*,* $\rho \geq 0$*, the following are equivalent:*

   a. *As* $x \to \infty$ $U(x) \sim \frac{cx^{\rho}l(x)}{\Gamma(1+\rho)},$

   b. *As* $x \to 0$ $LU(x) \sim cx^{-\rho}l\left(\frac{1}{x}\right).$

4. *(Karamata monotone density theorem)(Theorem 1.7.2,[1])Let* $U(x) = \int_0^x u(y)dy$*. If* $U(x) \sim cx^{\rho}l(x)(x \to \infty)$ *where* $c \in \mathbb{R}$*,* $\rho \in \mathbb{R}$*,* l *be a slowly varying function, and* u *is ultimately monotone, then* $u(x) \sim c\rho x^{\rho-1}l(x)$*.*

Theorem 2.2.2. *Suppose that* $S = (S_t : t \geq 0)$ *is a subordinator whose Laplace exponent* $\phi(\lambda) = b\lambda + \int_0^{\infty} [1 - \exp(-\lambda t)] \mu(dt)$ *is a complete Bernstein function. Assume that* $b > 0$ *or* $\mu((0, \infty)) = \infty$*. Then the potential measure* U *has a density* u *which is completely monotone on* $(0, \infty)$*.*

The next chapter uses all the definitions and theorems stated in this chapter to prove some of the results taken from [4].

Chapter 3

Potential Theory of Geometric Stable Processes

In Potential Theory of Geometric Stable Processes, we study the asymptotic behaviour of potential density and Lévy density associated with different subordinators and also the Green function and Lévy density associated with the geometric stable processess. All the results mentioned in this chapter have been taken from [4].

In this thesis we use the following notation: If f and g are two function then $f \sim g$ means $\frac{f}{g}$ converges asymptotically to 1.

Definition 3.0.1. *(Potential measure and density) The potential measure of a subordinator S_t is defined by*

$$U(A) = \mathbb{E}\left[\int_0^\infty 1_{(S_t \in A)} dt\right],$$

where A is a Borel subset of $(0, \infty)$. The potential measure has a density which known as the potential density u of the subordinator.

We next define Stable distributions since it will be helpful to understand geometric stable distributions, which we define next. Stable distributions are known for their applications in many economic and physical systems [18]. They provide important practical models for asset returns however, geometric stable distributions are more practical models for asset return since we can also incorporate market crashes in the models.

Definition 3.0.2 (Stable Distribution)**.** *A random variable X is called stable, if for any a, $b \in \mathbb{R}^+$, $\exists$ $c \in \mathbb{R}^+$ and $d \in \mathbb{R}$, such that $aX_1 + bX_2 \stackrel{d}{=} cX + d$, where X_1 and X_2*

are independent copies of X and $X \stackrel{d}{=} Y$ means that X and Y have the same probability distributions.

Example 3.0.1. *Normal distribution, Cauchy distribution, Lévy distribution are some examples of stable distributions.*

Next, we show normal distribution is a stable distribution.

Let X be a normal random variable with mean μ and variance σ^2 and X_1 and X_2 be two i.i.d copies of X and $\phi_X(\theta) = \mathbb{E}\left[\exp(i\theta X)\right]$ be the characteristic function of random variable X. We know

$$\mathbb{E}\left[\exp(i\theta X)\right] = \exp\left(i\mu\theta - \frac{1}{2}\sigma^2\theta^2\right).$$

Now by the property of characteristic functions,

$$\begin{aligned}
\phi_{aX_1+bX_2}(\theta) &= \phi_{X_1}(a\theta)\phi_{X_2}(b\theta) \\
&= \exp\left(i\mu a\theta - \frac{1}{2}\sigma^2 a^2\theta^2\right)\exp\left(i\mu b\theta - \frac{1}{2}\sigma^2 b^2\theta^2\right) \\
&= \exp\left(i\mu(a+b)\theta - \frac{1}{2}\sigma^2\left(a^2+b^2\right)\theta^2\right).
\end{aligned}$$

If we take $c = \sqrt{a^2+b^2}$ and $d = (a+b-c)\mu$, then

$$\phi_{X_1}(a\theta)\phi_{X_2}(b\theta) = \exp(i\theta d)\exp(i\mu\theta c - \frac{1}{2}\sigma^2 c^2\theta^2) = \phi_{cX+d}(\theta).$$

Hence, for every a and b, $aX_1 + bX_2 \stackrel{d}{=} cX + d$ for some c and d.

A geometric stable distribution has a similar property as stable distribution, but here the number of elements in the sum is a geometrically distributed random variable.

Definition 3.0.3 (Geometric Stable Distribution)**.** *If $X_1, X_2, X_3, \ldots$ are i.i.d random variables taken from a geometric stable distribution, the limit of the sum $Y = a_{N_p}(X_1 + X_2 + X_3 + \cdots + X_{N_p}) + b_{N_p}$ approaches the distribution of X_is for some coefficient a_{N_p} and b_{N_p} as p approaches 0, where N_p is a geometrically distributed random variable with parameter p independent of X_is. Its characteristic function, which has the form:*

$$\phi(\theta, \alpha, \beta, \lambda, \mu) = (1 + \lambda^\alpha|\theta|^\alpha\omega - \iota\mu\theta)^{-1},$$

where

$$\omega = \begin{cases} 1 - i\tan(\frac{\pi\alpha}{2})\beta sign(\theta) & \text{if} \quad \alpha \neq 0 \\ 1 + i\frac{2}{\pi}\beta\log(|\theta|)sign(\theta) & \text{if} \quad \alpha = 0. \end{cases}$$

3.1 Geometric Stable Subordinator

The Laplace transform of the Geometric stable subordinator is given by

$$\mathbb{E}[(-\lambda S_t)] = \frac{1}{\left(1+\lambda^{\frac{\alpha}{2}}\right)^t}.$$

In this section, we first show the existence of the potential density u of the geometric stable subordinator. Next, we compute the asymptotic behaviour of potential density, Lévy density and transitional density associated with the $\frac{\alpha}{2}$-geometric stable subordinator using Tauberian theorems. We also prove a lemma taken from [4] since the lemma is useful to prove the results in the next section.

Now we use Theorem 2.2.2 twice to show the existence of potential density of potential measures of the geometric stable subordinator as it was done in [4].

As the case discussed in [4], $b = 0$ and for a $\frac{\alpha}{2}$-geometric stable subordinator $\phi(\lambda)$ $= \log(1+\lambda^{\frac{\alpha}{2}})$ and $\lim_{\lambda\to\infty} \phi(\lambda) \to \infty$. Because $[1-\exp(-\lambda t)]$ is a bounded function on $(0,\infty)$, $\lim_{\lambda\to\infty} \phi(\lambda) \to \infty$ forces $\mu\left((0,\infty)\right) = \infty$, therefore, by above theorem the potential measure U has a density u which is completely monotone on $(0,\infty)$.

Since from proposition 2.1.2 we know , $\phi(\lambda) = \log\left(1+\lambda^{\frac{\alpha}{2}}\right)$ is a complete Bernstein function therefore, $\psi(\lambda) = \frac{\lambda}{\phi(\lambda)}$ is also a complete Bernstein function. Let T be the subordinator with Laplace exponent ψ and V be the potential measure associated with it. Since $\lim_{\lambda\to\infty} \frac{\psi(\lambda)}{\lambda} = \lim_{\lambda\to\infty} \frac{1}{\phi(\lambda)} = 0$ and $\lim_{\lambda\to\infty} \psi(\lambda) = \infty$, therefore, the Lévy measure ν of T must satisfy $\nu(0,\infty) = \infty$. Hence, by theorem 2.2.2, the potential measure V of T has a density v which is completely monotone on $(0,\infty)$.

We now start proving the results. As we know from Theorem 2.2.2 that the potential measure of $\frac{\alpha}{2}$-geometric stable subordinator has a monotone density u, so now we will use Tauberian theorems to get the asymptotic behaviour of u.

Theorem 3.1.1. *For any $\alpha \in (0,2]$, we have*

1. $u(x) \sim \frac{2}{\alpha x \log(x)^2}, x \to 0^+,$
2. $u(x) \sim \frac{x^{\frac{\alpha}{2}-1}}{\Gamma(\frac{\alpha}{2})}, x \to +\infty.$

Proof.

1. We know from discussion on page 4, [4] that the Laplace transform of potential measure of the $\frac{\alpha}{2}$-geometric stable subordinator is given by

$$\begin{aligned} LU(x) &= \frac{1}{\phi(x)} \\ &= \frac{1}{\log(1+x^{\frac{\alpha}{2}})}. \end{aligned}$$

Now

$$\begin{aligned} \lim_{x\to 0^+}\left[LU\left(\frac{1}{xt}\right) - LU\left(\frac{1}{x}\right)\right] &= \lim_{x\to 0^+}\left[\frac{1}{\phi(\frac{1}{xt})} - \frac{1}{\phi(\frac{1}{x})}\right] \\ &= \lim_{x\to 0^+}\left[\frac{1}{\log\left[1+\frac{1}{(xt)^{\frac{\alpha}{2}}}\right]} - \frac{1}{\log\left[1+\frac{1}{(x)^{\frac{\alpha}{2}}}\right]}\right] \\ &= \lim_{x\to 0^+}\left[\frac{1}{\log\left[\frac{1+(xt)^{\frac{\alpha}{2}}}{(xt)^{\frac{\alpha}{2}}}\right]} - \frac{1}{\log\left[\frac{1+(x)^{\frac{\alpha}{2}}}{(x)^{\frac{\alpha}{2}}}\right]}\right] \\ &= \lim_{x\to 0^+}\frac{\log\left[\frac{1+(x)^{\frac{\alpha}{2}}}{(x)^{\frac{\alpha}{2}}}\right] - \log\left[\frac{1+(xt)^{\frac{\alpha}{2}}}{(xt)^{\frac{\alpha}{2}}}\right]}{\log\left[\frac{1+(xt)^{\frac{\alpha}{2}}}{(xt)^{\frac{\alpha}{2}}}\right]\log\left[\frac{1+(x)^{\frac{\alpha}{2}}}{(x)^{\frac{\alpha}{2}}}\right]} \end{aligned}$$

$$\begin{aligned} &= \lim_{x\to 0^+}\frac{\log\left[\frac{(xt)^{\frac{\alpha}{2}}}{1+(xt)^{\frac{\alpha}{2}}}\frac{1+(x)^{\frac{\alpha}{2}}}{(x)^{\frac{\alpha}{2}}}\right]}{\left[\log\left[1+(xt)^{\frac{\alpha}{2}}\right] - \log\left[(xt)^{\frac{\alpha}{2}}\right]\right]\left[\log\left[1+(x)^{\frac{\alpha}{2}}\right] - \log\left[(x)^{\frac{\alpha}{2}}\right]\right]} \\ &= \frac{\log(t^{\frac{\alpha}{2}})}{\left[0-\frac{\alpha}{2}\lim_{x\to 0^+}\log(xt)\right]\left[0-\frac{\alpha}{2}\lim\limits_{x\to 0^+}\log(x)\right]} \\ &= \frac{\alpha}{2}\frac{\log t}{(\frac{\alpha}{2})^2\left[\lim_{x\to 0^+}\log(x)\right]\left[\lim\limits_{x\to 0^+}\log(x)\right]} \\ &= \frac{2\log(t)}{\alpha\left[\lim\limits_{x\to 0^+}\log(x)\right]^2}\left(\text{because as}\lim_{x\to 0^+}\log(xt)\sim\log(x)\right). \end{aligned}$$

Now if we take a slowly varying function $l(x) = [\log(x)]^{-2}$, then

$$\lim_{x\to 0^+}\frac{\left[LU(\frac{1}{xt}) - LU(\frac{1}{x})\right]}{[\log(x)]^{-2}} \to \frac{2}{\alpha}\log(t). \tag{3.1}$$

Applying Theorem 2.2.1(1), Chapter 2 in equation 3.1 we get

$$\lim_{x\to 0^+} \frac{[U(xt)-U(x)]}{[\log(x)]^{-2}} \to \frac{2}{\alpha}\log(t), \tag{3.2}$$

Now applying Theorem 2.2.1(2), Chapter 2 in equation 3.2 we get

$$u(x) \sim \frac{2}{\alpha x(\log(x))^2}, x \to 0^+.$$

2. We know that $LU(x) = \frac{1}{\log\left(1+x^{\frac{\alpha}{2}}\right)}$, now as $x \to 0^+, \log\left(1+x^{\frac{\alpha}{2}}\right) \sim x^{\frac{\alpha}{2}}$ therefore,

$$LU(x) \sim \frac{1}{x^{\frac{\alpha}{2}}} = x^{-\frac{\alpha}{2}} l(x), \tag{3.3}$$

where $l(x) = 1 \; \forall \; x \in (0,\infty)$ be a slowly varying function at 0^+. Applying Theorem 2.2.1(3), Chapter 2 in equation 3.3 we get

$$U(x) \sim \frac{x^{\frac{\alpha}{2}}}{\Gamma\left(1+\frac{\alpha}{2}\right)} \tag{3.4}$$

Now applying Theorem 2.2.1(4), Chapter 2 in equation 3.4 we get

$$\begin{aligned} u(x) = U'(x) &\sim \frac{\alpha x^{\frac{\alpha}{2}-1}}{2\Gamma\left(1+\frac{\alpha}{2}\right)} \\ &= \frac{x^{\frac{\alpha}{2}-1}}{\Gamma(\frac{\alpha}{2})}, \end{aligned}$$

(using the properties of gamma function).

□

Next, we compute the asymptotic behaviour of Lévy density of the $\frac{\alpha}{2}$-geometric stable subordinator. Before that we need to know the existence of the Lévy density.

Since the Laplace exponent of the $\frac{\alpha}{2}$-geometric stable subordinator, $\phi(x) = \log\left(1+x^{\frac{\alpha}{2}}\right)$ is a complete Bernstein function therefore, the Lévy measure has a completely monotone density $\mu(x)$ (for details see [3] or [6]).

Theorem 3.1.2. *For any $\alpha \in (0,2]$, we have*

$\mu(x) \sim \frac{\alpha}{2x}, x \to 0^+$.

Proof. Let $G_{\frac{\alpha}{2}}(x)$ be a function. If $G_{\frac{\alpha}{2}}(x) = \sum\limits_{n=0}^{\infty} \frac{x^{\frac{n\alpha}{2}}}{\Gamma\left(1+\frac{n\alpha}{2}\right)}$, then $G_{\frac{\alpha}{2}}(x)$ is a Mittag-Leffler function(for details see [7]). Since

$$\phi(\lambda) = \int_0^\infty \left[1 - \exp(-\lambda x)\right] \mu(dx) = \log\left(1 + \lambda^{\frac{\alpha}{2}}\right),$$

differentiating both sides we get

$$\phi'(\lambda) = \frac{d\phi}{d\lambda} = \frac{\alpha}{2} \frac{\lambda^{\frac{\alpha}{2}-1}}{1+\lambda^{\frac{\alpha}{2}}}. \tag{3.5}$$

Now

$$\begin{aligned}\int_0^\infty \exp(-\lambda x) G_{\frac{\alpha}{2}}(-x) dx &= \int_0^\infty \exp(-\lambda x) \sum_{n=0}^{\infty} \frac{(-x)^{\frac{n\alpha}{2}}}{\Gamma\left(1+\frac{n\alpha}{2}\right)} dx \\ &= \sum_{n=0}^{\infty} \int_0^\infty \exp(-\lambda x) \frac{(-x)^{\frac{n\alpha}{2}}}{\Gamma\left(1+\frac{n\alpha}{2}\right)} dx\end{aligned}$$

(by Dominated Convergence theorem).

Let $\lambda x = u$ which implies $dx = du$ thus,

$$\begin{aligned}\int_0^\infty \exp(-\lambda x) G_{\frac{\alpha}{2}}(-x) dx &= \sum_{n=0}^{\infty} \int_0^\infty \exp(-u) \frac{1}{\lambda} \left(\frac{-u}{\lambda}\right)^{\frac{n\alpha}{2}} \frac{1}{\Gamma\left(1+\frac{n\alpha}{2}\right)} du \\ &= \sum_{n=0}^{\infty} \frac{1}{\lambda} \left(\frac{-1}{\lambda}\right)^{\frac{n\alpha}{2}} \int_0^\infty \exp(-u) u^{\frac{n\alpha}{2}+1-1} \frac{1}{\Gamma\left(1+\frac{n\alpha}{2}\right)} du \\ &= \sum_{n=0}^{\infty} \frac{1}{\lambda} \left(-\frac{1}{\lambda}\right)^{\frac{n\alpha}{2}} \left(\text{using}, \Gamma(k) = \int_0^\infty \exp(-u) u^{k-1} du\right) \\ &= \frac{1}{\lambda} \sum_{n=0}^{\infty} (-1)^n \frac{1}{\lambda^{\frac{\alpha}{2}}} \\ &= \frac{1}{\lambda} \frac{1}{1+\frac{1}{\lambda^{\frac{\alpha}{2}}}} \\ &= \frac{\lambda^{\frac{\alpha}{2}-1}}{1+\lambda^{\frac{\alpha}{2}}}.\end{aligned}$$

Therefore,

$$\int_0^\infty \exp(-\lambda x) G_{\frac{\alpha}{2}}(-x) dx = \frac{\lambda^{\frac{\alpha}{2}-1}}{1+\lambda^{\frac{\alpha}{2}}}. \tag{3.6}$$

Now using equations 2.5 and 2.6 we get

$$\phi'(\lambda) = \frac{\alpha}{2}\int_0^\infty \exp(-\lambda x) G_{\frac{\alpha}{2}}(-x)dx.(3.7)$$

Also on differentiating $\phi(\lambda)= \int_0^\infty [1-\exp(-\lambda x)]\,\mu(dx)$ w.r.t to λ we get

$$\phi'(\lambda) = \int_0^\infty x\exp(-\lambda x)\mu(dx)(3.8)$$

On comparing equations 3.7 and 3.8 we get

$$\mu(x) = \frac{\alpha}{2}\frac{G_{\frac{\alpha}{2}}(-x)}{x}. \tag{3.9}$$

Now let

$$\begin{aligned} G(x) &= 1 - G_{\frac{\alpha}{2}}(-x) \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{\frac{n\alpha}{2}}}{\Gamma\left(1+\frac{n\alpha}{2}\right)}. \end{aligned} \tag{3.10}$$

Then

$$\begin{aligned} LG(\lambda) := &\int_0^\infty \exp(-\lambda x)G(dx) \\ &= \lambda\int_0^\infty \exp(-\lambda x)\left[1-G_{\frac{\alpha}{2}}(-x)\right]dx \\ &= \lambda\int_0^\infty \exp(-\lambda x)dx + \lambda\int_0^\infty \exp(-\lambda x)G_{\frac{\alpha}{2}}(-x)dx \\ &= 1-\frac{\lambda^{\frac{\alpha}{2}}}{1+\lambda^{\frac{\alpha}{2}}} \\ &= \frac{1}{1+\lambda^{\frac{\alpha}{2}}} \\ &= \exp\left[-\log(1+\lambda^{\frac{\alpha}{2}})\right] \\ &= \mathbb{E}\left[\exp^{-\lambda S_1}\right]. \end{aligned}$$

Therefore, G is the distribution function of S_1 and from equations 3.9 and 3.10 we get

$$\mu(x) = \frac{\alpha}{2x}\left[1-G(x)\right].$$

Hence,

$$\lim_{x\to 0}\mu(x) = \frac{\alpha}{2x}\left[1-\lim_{x\to 0}G(x)\right] = \frac{\alpha}{2x}.$$

$\square$

Theorem 3.1.3. *For any $\alpha \in (0,2)$, we have*

$\mu(x) \sim \frac{\alpha}{2x^{\frac{\alpha}{2}+1}\Gamma\left(1-\frac{\alpha}{2}\right)}, x \to \infty.$

Proof. Since the potential measure $V(x)$ of the subordinator T has a Laplace exponent $\psi(\lambda) = \frac{\lambda}{\log\left(1+\lambda^{\frac{\alpha}{2}}\right)}$. Therefore,

$$LV(\lambda) = \frac{1}{\psi(\lambda)} = \frac{\log\left(1+\lambda^{\frac{\alpha}{2}}\right)}{\lambda} \sim \lambda^{\frac{\alpha}{2}-1}, \lambda \to 0^+. \tag{3.11}$$

Then applying Theorem 2.2.1(3), Chapter 2 in equation 3.11 we get

$$V(x) \sim \frac{x^{1-\frac{\alpha}{2}}}{\Gamma\left(2-\frac{\alpha}{2}\right)}, x \to \infty$$

Now using Theorem 2.2.1(4), Chapter 2 to get the asymptotic behaviour of the density of the potential measure at infinity,

$$v(x) \sim \left(1-\frac{\alpha}{2}\right)\frac{1}{x^{\frac{\alpha}{2}}}\Gamma\left(2-\frac{\alpha}{2}\right) = \frac{1}{x^{\frac{\alpha}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)}, x \to \infty.$$

By corollary 2.4.8 of [5], we know that $\mu\left((x,\infty)\right) = v(x)$, $x > 0$ which implies

$$\mu((x,\infty)) \sim \frac{1}{x^{\frac{\alpha}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)}, x \to \infty. \tag{3.12}$$

Now applying Theorem 2.2.1(4), Chapter 2 in equation 3.12 we get

$$\mu(x) \sim \frac{\alpha}{2x^{\frac{\alpha}{2}+1}\Gamma\left(1-\frac{\alpha}{2}\right)}, x \to \infty.$$

□

We know(see for instance [11]) that the distribution G of S_1 is absolutely continuous and has a decreasing density $g_{\frac{\alpha}{2}}(x)$ on $(0,\infty)$. The exact form of the density $g_{\frac{\alpha}{2}}(x)$ is known(page 7, [4]) for $\alpha = 2$ but not for $\alpha \in (0,2)$. In the next theorem, we will compute the asymptotic behaviour of $g_{\frac{\alpha}{2}}(x)$ by using Tauberian theorems again.

Theorem 3.1.4. *For any $\alpha \in (0,2)$, we have*

1. $g_{\frac{\alpha}{2}}(x) \sim \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}x^{\frac{\alpha}{2}-1}, x \to 0+,$
2. $g_{\frac{\alpha}{2}}(x) \sim 2\pi\Gamma\left(1+\frac{\alpha}{2}\right)\sin\left(\frac{\pi\alpha}{4}\right)x^{-\frac{\alpha}{2}-1}, x \to \infty.$

Proof.

1. From theorem 3.1.2, we know that the Laplace transform of the distribution G of S_1, is $LG(\lambda) = \frac{1}{1+\lambda^{\frac{\alpha}{2}}}$. Using Theorem 2.2.1(3), Chapter 2 we get
$$G(x) \sim \frac{x^{\frac{\alpha}{2}}}{\Gamma\left(1+\frac{\alpha}{2}\right)}, x \to 0+.$$
Now using Theorem 2.2.1(4), Chapter 2 we get
$$\begin{aligned} g_{\frac{\alpha}{2}}(x) &\sim \frac{\alpha x^{\frac{\alpha}{2}-1}}{2\Gamma\left(1+\frac{\alpha}{2}\right)}, x \to 0+ \\ &= \frac{x^{\frac{\alpha}{2}-1}}{\Gamma\left(\frac{\alpha}{2}\right)}, x \to 0+, \end{aligned}$$
(Using the property of Gamma functions).

2. It is known that if $Y = (Y_t : t \geq 0)$ is a Lévy process with characteristic function Φ and an exponential random variable χ with parameter 1 independent of Y. Then $Z = (Y(\chi))$ is a geometrically infinite divisible random variable. The characteristic function of this random variable is $\frac{1}{\log(1+|\Phi|)}$.

   Therefore,
$$g_{\frac{\alpha}{2}}(x) = \int_0^\infty \exp(-t) q_{\frac{\alpha}{2}}(t,x)dt,$$
(where $q_{\frac{\alpha}{2}}(t,x)$ is the transition density of $\frac{\alpha}{2}$-stable subordinator)
$$= \int_0^\infty \exp(-t) t^{\frac{-2}{\alpha}} q_{\frac{\alpha}{2}}\left(1, \frac{x}{t^{\frac{2}{\alpha}}}\right),$$
(using the scaling property of transition density page88-89, [15]).

   We know(see for instance [14]) that
$$q_{\frac{\alpha}{2}}(1,x) \sim 2\pi\Gamma\left(1+\frac{\alpha}{2}\right)\sin\left(\frac{\pi\alpha}{4}\right)x^{-\frac{\alpha}{2}-1}, x \to \infty$$
and
$$q_{\frac{\alpha}{2}}(1,x) \leq d\left(1 \wedge x^{\frac{-\alpha}{2}-1}\right),$$
where d is a positive constant.

Now using Dominated convergence theorem we get

$$g_{\frac{\alpha}{2}}(x) \sim \int_0^\infty \exp(-t) t^{\frac{2}{\alpha}+1} 2\pi\Gamma\left(1+\frac{\alpha}{2}\right) \sin\left(\frac{\pi\alpha}{4}\right) x^{-\frac{2}{\alpha}-1} dt, x \to \infty,$$

$$g_{\frac{\alpha}{2}}(x) \sim 2\pi\Gamma\left(1+\frac{\alpha}{2}\right) \sin\left(\frac{\pi\alpha}{4}\right) x^{-\frac{\alpha}{2}-1}, x \to \infty.$$

□

Now we proof the lemma taken from [4], this lemma will be used in the next chapter to establish the asymptotic behaviour of Green function and Lévy density associated with the Geometric stable process. Before proving the lemma, we will define an auxiliary function. Let l be a slowly varying function at infinity and $\beta > 0$, then define

$$f_{l,\beta}(y,t) = \begin{cases} \frac{l\left(\frac{1}{y}\right)}{l\left(\frac{4t}{y}\right)} & \text{if } \quad y < \frac{t}{\beta} \\ 0 & \text{if } \quad y \geq \frac{t}{\beta}. \end{cases}$$

Lemma 3.1.5. *Suppose that $w : (0, \infty) \to (0, \infty)$ be a decreasing function satisfying the following two assumptions:*

1. *There exist a constant $k_0 > 0$ and a continuous function $l : (0, \infty) \to (0, \infty)$ slowly varying at $+\infty$ such that*
$$w(t) \sim \frac{k_0}{tl\left(\frac{1}{t}\right)}, t \to 0^+,$$

2. *If $d = 1$ or 2, then there exist a constant $k_\infty > 0$ and $\gamma < \frac{d}{2}$ such that*
$$w(t) \sim k_\infty t^{\gamma-1}, t \to +\infty.$$

Let $g : (0, \infty) \to (0, \infty)$ be a function such that

$$\int_0^\infty t^{\frac{d}{2}-1} \exp(-t) g(t) dt < \infty$$

If there is $\beta > 0$ such that $f_{l,\beta}(y,t) \leq g(t)$ for all $y, t > 0$, then

$$H(x) := \int_0^\infty (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{(|x|)^2}{4t}\right) w(t) dt \sim \frac{k_0 \Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}}} \frac{1}{|x|^d l\left(\frac{1}{(|x|)^2}\right)}, |x| \to 0.$$

Proof. The assumptions of the lemma clearly shows that $H(x) < \infty$.

Let $\frac{|x|^2}{4t} = u$ which implies $-\frac{|x|^2}{4t^2}dt = du$. Therefore,

$$\int_0^\infty (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{4t}\right) w(t)dt = \int_\infty^0 (4\pi)^{-\frac{d}{2}} \exp(-u) \left(\frac{|x|^2}{4u}\right)^{-\frac{d}{2}} w\left(\frac{|x|^2}{4u}\right) \frac{-|x|^2}{4u^2} du$$
$$= \frac{1}{4\pi^{\frac{d}{2}}|x|^{d-2}} \int_0^\infty u^{\frac{d}{2}-2} \exp(-u) w\left(\frac{|x|^2}{4u}\right) du$$
$$= \frac{1}{4\pi^{\frac{d}{2}}|x|^{d-2}} \left[\int_0^{\beta|x|^2} u^{\frac{d}{2}-2} \exp(-u) w\left(\frac{|x|^2}{4u}\right) du + \int_{\beta|x|^2}^\infty u^{\frac{d}{2}-2} \exp(-u) w\left(\frac{|x|^2}{4u}\right) du\right],$$

(breaking the integral into two parts),

$$= \frac{1}{4\pi^{\frac{d}{2}}|x|^{d-2}}(H_1 + H_2)$$

When $d = 1$ or 2, then by using the assumption (2), we know that $w\left(\frac{|x|^2}{4u}\right) \leq k_1 u^{\gamma-1}$ for some positive constant k_1 and $\forall\ \frac{|x|^2}{4u} \geq \frac{1}{4\beta}$. Therefore,

$$H_1 \leq \int_0^{\beta|x|^2} u^{\frac{d}{2}-2} \exp(-u) k_1 \left(\frac{|x|^2}{4u}\right)^{\gamma-1} du$$

$$\leq k_2|x|^{2\gamma-2} \int_0^{\beta|x|^2} u^{\frac{d}{2}-\gamma-1} dt = k_3|x|^{d-2},$$

and thus,

$$\lim_{|x|\to 0} \frac{1}{4\pi^{\frac{d}{2}}|x|^{d-2}\frac{1}{|x|^d l\left(\frac{1}{|x|^2}\right)}} H_1 \leq \lim_{|x|\to 0} \frac{k_2|x|^{d-2}}{4\pi^{\frac{d}{2}}|x|^{d-2}\frac{1}{|x|^d l\left(\frac{1}{|x|^2}\right)}} = 0$$

Now for $d \geq 3$, we have $w\left(\frac{|x|^2}{4u}\right) \leq w\left(\frac{1}{4\beta}\right)$ for $\frac{|x|^2}{4u} \geq \frac{1}{4\beta}$. Therefore,

$$H_1 \leq \int_0^{\beta|x|^2} u^{\frac{d}{2}-2} \exp(-u) w\left(\frac{1}{4\beta}\right) du$$

$$\leq w\left(\frac{1}{4\beta}\right) \int_0^{\beta|x|^2} u^{\frac{d}{2}-2} du = k_4|x|^{d-2}$$

Again following the same steps as for $d = 1$ or 2 we get

$$\lim_{|x|\to 0} \frac{1}{4\pi^{\frac{d}{2}}|x|^{d-2}\frac{1}{|x|^d l\left(\frac{1}{|x|^2}\right)}} H_1 = 0 \tag{3.13}$$

Now consider H_2:

$$\frac{1}{4\pi^{\frac{d}{2}}|x|^{d-2}} H_2 = \frac{1}{4\pi^{\frac{d}{2}}|x|^{d-2}} \int_{\beta|x|^2}^\infty u^{\frac{d}{2}-2} \exp(-u) w\left(\frac{|x|^2}{4u}\right) du$$

$$= \frac{1}{\pi^{\frac{d}{2}}|x|^d l\left(\frac{1}{|x|^2}\right)} \int_{\beta|x|^2}^{\infty} u^{\frac{d}{2}-1} \exp(-u) \frac{w\left(\frac{|x|^2}{4u}\right)}{\frac{1}{\frac{|x|^2}{4u} l\left(\frac{|x|^2}{4u}\right)}} \frac{l\left(\frac{1}{|x|^2}\right)}{l\left(\frac{4t}{|x|^2}\right)} du$$

From assumption (1), we know that

$$\frac{w\left(\frac{|x|^2}{4u}\right)}{\frac{1}{\frac{|x|^2}{4u} l\left(\frac{|x|^2}{4u}\right)}} < k$$

for some constant k, $\forall$ u and x such that $\frac{|x|^2}{4u} \leq \frac{1}{4\beta}$. As we know l is slowly varying at infinity therefore,

$$\lim_{|x| \to 0} \frac{l\left(\frac{1}{|x|^2}\right)}{l\left(\frac{4t}{|x|^2}\right)} = 1, \forall t > 0.$$

Because

$$\frac{l\left(\frac{1}{|x|^2}\right)}{l\left(\frac{4t}{|x|^2}\right)} = f_{l,\beta}(|x|^2, u),$$

therefore,

$$u^{\frac{d}{2}-1} \exp(-u) \frac{w\left(\frac{|x|^2}{4u}\right)}{\frac{1}{\frac{|x|^2}{4u} l\left(\frac{|x|^2}{4u}\right)}} \frac{l\left(\frac{1}{|x|^2}\right)}{l\left(\frac{4t}{|x|^2}\right)} = u^{\frac{d}{2}-1} \exp(-u) \frac{w\left(\frac{|x|^2}{4u}\right)}{\frac{1}{\frac{|x|^2}{4u} l\left(\frac{|x|^2}{4u}\right)}} f_{l,\beta}(|x|^2, u)$$

$$\leq k u^{\frac{d}{2}-2} \exp(-u) g(u),$$

[Using the assumption that $f_{l,\beta}(|x|^2, u) \leq g(u)$] and

$$\int_0^{\infty} k u^{\frac{d}{2}-2} \exp(-u) g(u) < \infty,$$

(again from the assumption).

Thus, by Dominated Convergence theorem we get

$$\begin{aligned}
\lim_{|x| \to 0} \int_{\beta|x|^2}^{\infty} u^{\frac{d}{2}-1} \exp(-u) \frac{w\left(\frac{|x|^2}{4u}\right)}{\frac{1}{\frac{|x|^2}{4u} l\left(\frac{|x|^2}{4u}\right)}} \frac{l\left(\frac{1}{|x|^2}\right)}{l\left(\frac{4t}{|x|^2}\right)} du &= \int_{\beta|x|^2}^{\infty} u^{\frac{d}{2}-1} \exp(-u) \lim_{|x| \to 0} \frac{w\left(\frac{|x|^2}{4u}\right)}{\frac{1}{\frac{|x|^2}{4u} l\left(\frac{|x|^2}{4u}\right)}} \frac{l\left(\frac{1}{|x|^2}\right)}{l\left(\frac{4t}{|x|^2}\right)} du \\
&= k \int_0^{\infty} u^{\frac{d}{2}-1} \exp(-u) du \\
&= k\Gamma\left(\frac{d}{2}\right).
\end{aligned}$$

Therefore,

$$\lim_{|x|\to 0} \frac{\frac{1}{4\pi^{\frac{d}{2}}|x|^{d-2}} H_2}{\frac{1}{\pi^{\frac{d}{2}}|x|^{d} l\left(\frac{1}{|x|^2}\right)}} = k_0 \Gamma\left(\frac{d}{2}\right). \tag{3.14}$$

Now using equations 3.13 and 3.14 we get

$$\lim_{|x|\to 0} \frac{H(x)}{|x|^d l\left(\frac{1}{|x|^2}\right)} = \frac{k_0 \Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}}}$$

Hence,

$$\int_0^\infty (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{(|x|)^2}{4t}\right) w(t) dt \sim \frac{k_0 \Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}}} \frac{1}{|x|^d l\left(\frac{1}{(|x|)^2}\right)}, |x| \to 0.$$

□

3.2 Geometric Stable Processes

Let $\alpha \in (0, 2]$. A Lévy process $X = (X_t, P_x)$ is called a geometric strictly α-stable process if its characteristic exponent $\psi(\theta) = \log\left(\mathbb{E}_x\left[(i\theta\left(X_1 - X_0\right)\right]\right)$ is given by $\psi(\theta) = -\log\left(1 + \phi(\theta)\right), \theta \in \mathbb{R}^d$, with $\exp(\phi)$ being the characteristic function of some strictly α-stable distribution. Let $S = (S_t, t \geq 0)$ be a geometric $\frac{\alpha}{2}$-stable subordinator and Y be a d dimensional Brownian motion. Assuming Y and S to be independent, the symmetric geometric stable process X can be obtained by $X_t = Y(S_t)$.

In [4] the author is mainly interested in the rotationally invariant geometric strictly α-stable process in $\mathbb{R}^d$, that is, in the case when,

$$\psi(\theta) = \log\left(1 + |\theta|^\alpha\right), \in \mathbb{R}^d.$$

We next define α-potential of a function f associated with a standard process(for details please see page45, [19]]) X. For $\alpha = 0$, the 0-potential operator has a density which is known as Green function associated with the standard process.

Definition 3.2.1. *(α-Potential of a Function f or Potential Operator)(page 69, [19]) Let X be a standard process, f be a real valued function on $\mathbb{R}$ and $\alpha \geq 0$. Then the α-potential $U^\alpha f$ of a function f is given by*

$$U^\alpha f(x) = \mathbb{E}^x\left[\int_0^\infty \exp(-\alpha t) f(X_t) dt\right],$$

where the process is starting at x.

Now we define kernels as they are useful to transform a measurable function.

Definition 3.2.2. *(Kernel)(page 303, [20]) Let* Ω *be a subset of* $\mathbb{R}^n$ *then a function*

$$k : \Omega \times \Omega \to \mathbb{R}$$

is called as kernel. It can be used to transform a measurable function

$$f : \Omega \to \mathbb{R}$$

to a new function

$$g : \Omega \to \mathbb{R}$$

by putting

$$g(x) = \int_0^\infty k(x, y)f(y)dy, x \in \Omega,$$

provided that the integral is defined.

Next, we define Green function and Lévy density associated with the subordinated process. When the kernel is a Gaussian kernel then it transforms the potential density and Lévy density associated with the subordinator into Green function and Lévy density respectively associated with the subordinated process.

Let $Y = (Y_t, t \geq 0)$ be a d-dimensional Brownian motion with transitional density given by

$$p_2(t, x, y) = (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right), x, y \in \mathbb{R}^d, t > 0,$$

where $p_2(t, x, y)$ is the Gaussian kernel.

Definition 3.2.3. *(Green function for Markov process) The Green function for the Markov process is defined by*

$$G(x, y) = \int_0^\infty p(t, x, y)dt,$$

where $p(t, x, y)$ *is the transition function of the Markov process. It is the expected amount of time spent at* y *by the process started at* x *([36]).*

Definition 3.2.4. *(Green function and Lévy density)(page 8, [4]) The potential operator*

$$Gf(x) := \mathbb{E}^x\left[\int_0^\infty f(X_t)dt\right]$$

of X has a density $G(x,y)=G(y-x)$ with

$$G(x)=\int_0^\infty p_2(t,0,x)u(t)dt,$$

where $u(t)$ is the potential density of the subordinator S and $G(x)$ is called the Green function of X.

The Lévy density of X is given by

$$J(x)=\int_0^\infty p_2(t,0,x)\mu(t)dt,$$

where $\mu(t)$ is the Lévy density of S.

In this section we mainly deal with the Lévy density and the Green function of the geometric stable process. We use theorems and lemma from the previous chapter to prove the results of this section.

Theorem 3.2.1. *For any $\alpha\in(0,2]$, we have*

1. $G(x)\sim\dfrac{\Gamma\left(\frac{d}{2}\right)}{2\alpha\pi^{\frac{d}{2}}|x|^d\left[\log\left(\frac{1}{|x|}\right)\right]^2}, |x|\to 0,$

2. $G(x)\sim\dfrac{1}{\pi^{\frac{d}{2}}2^\alpha}\dfrac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}|x|^{\alpha-d}, |x|\to\infty.$

Proof.

1. Since
$$G(x)=\int_0^\infty (4\pi t)^{-\frac{d}{2}}\exp\left(-\frac{(|x|)^2}{4t}\right)u(t)dt$$
therefore, we can apply lemma 3.1.5 with $w(t)=u(t)$.

   We know from Theorem 3.1.1(1) that $u(t){\sim}\frac{2}{\alpha t\log(t)^2}$, as $x{\to}\, 0^+$. Taking $k_0=\frac{2}{\alpha}$ and $l(t)=\log^2(t)$. Also Theorem 3.1.1(2), we know that $u(x)\sim\frac{x^{\frac{\alpha}{2}-1}}{\Gamma\left(\frac{\alpha}{2}\right)}, x\to+\infty$, so $\gamma=\frac{\alpha}{2}<\frac{d}{2}$. Choose $\beta=\frac{1}{2}$ and
$$f_{l,1/2}(y,t)=\begin{cases}\frac{\log^2(y)}{\log^2\left(\frac{y}{4t}\right)} & \text{if} \quad y<2t\\ 0 & \text{if} \quad y\geq 0.\end{cases}$$

Define

$$g(t) = \begin{cases} \frac{\log^2(2t)}{\log^2(2)} & \text{if} \quad t < \frac{1}{4} \\ 1 & \text{if} \quad t \geq \frac{1}{4}. \end{cases}$$

Now to apply the lemma, we will first show

$$f_{l,1/2}(y,t) = \frac{\log^2(y)}{\log^2\left(\frac{y}{4t}\right)} \leq g(t) \forall t > 0.$$

For $0 < t < \frac{1}{4}$ and $0 < y < 2t$,both $\log^2(y)$ and $\log^2(\frac{y}{4t})$ increasing are functions but former is always greater than the latter therefore, $f_{l,1/2}(y,t) = \frac{\log^2(y)}{\log^2\left(\frac{y}{4t}\right)}$ is an increasing function thus,

$$\sup\left\{\frac{\log^2(y)}{\log^2\left(\frac{y}{4t}\right)} : 0 < y < 2t\right\} = f_{l,1/2}(2t,t) = \frac{\log^2(2t)}{\log^2(2)}.$$

We know that $\sup\left\{\frac{\log^2(y)}{\log^2(\frac{y}{4t})} : 0 < y < 2t\right\} \geq \frac{\log^2(y)}{\log^2\left(\frac{y}{4t}\right)}$ by the definition of supremum and thus,

$\mathrm{f}_{l,1/2}(y,t) \leq g(t) \ \forall \ 0 < t < \frac{1}{4}$.

For $t = \frac{1}{4}$,

$$f_{l,1/2}\left(y, \frac{1}{4}\right) = 1 = g\left(\frac{1}{4}\right).$$

For $t > \frac{1}{4}$ and $0 < y < 1$, $f_{l,1/2}(y,t)$ is a decreasing function for because $\log(y) > \log(\frac{y}{k}) \ \forall \ y > 0$ and $k > 1$ but $\log^2(y) < \log^2(\frac{y}{k}) \ \forall \ 0 < y < 1$ and $k > 1$. Therefore,

$$\sup\left\{\frac{\log^2(y)}{\log^2\left(\frac{y}{4t}\right)} : 0 < y < min\{2t, 1\}\right\} = \lim_{y \to 0} \frac{\log^2(y)}{\log^2\left(\frac{y}{4t}\right)} = 1,$$

which implies

$$f_{l,1/2}(y,t) \leq g(t), \forall t \geq \frac{1}{4},$$

thus,

$$f_{l,1/2}(y,t) \leq g(t), \forall y, t > 0.$$

Since $t^{\frac{d}{2}-1}\exp(-t)\frac{\log^2(2t)}{\log^2(2)}$ is a bounded by an integrable function therefore,

$$\int_0^\infty t^{\frac{d}{2}-1}\exp(-t)\frac{\log^2(2t)}{\log^2(2)}dt < \infty.$$

Hence,

$$G(x) \sim \frac{\Gamma(\frac{d}{2})}{2\alpha\pi^{\frac{d}{2}}|x|^d\left[\log\left(\frac{1}{|x|}\right)\right]^2}, |x| \to 0.$$

2. As we know from Theorem 3.1.1(2) that $u(t) \sim \frac{t^{\frac{\alpha}{2}-1}}{\Gamma(\frac{\alpha}{2})}, t \to +\infty$, which implies $\exists$ a constant t_0 such that $u(t) \leq t^{-1}$ $\forall$ $t_0 \in (0, t_0)$. Therefore, $\exists$ a constant D such that $u(t) \leq max\left(t^{-1}, t^{\frac{\alpha}{2}-1}\right)$.

Let $\frac{|x|^2}{4t} = y \implies -\frac{|x|^2}{4t^2}dt = dy$

$$\int_0^\infty (4\pi t)^{-\frac{d}{2}}\exp\left(-\frac{|x|^2}{4t}\right)u(t)dt = \int_\infty^0 (4\pi)^{-\frac{d}{2}}\exp(-y)\left(\frac{|x|^2}{4y}\right)^{-\frac{d}{2}} u\left(\frac{|x|^2}{4y}\right)\frac{-|x|^2}{4y^2}dy$$

$$= \frac{1}{4\pi^{\frac{d}{2}}|x|^{d-2}}\int_0^\infty y^{\frac{d}{2}-2}\exp(-y)u\left(\frac{|x|^2}{4y}\right)dy$$

$$= \frac{1}{4\pi^{\frac{d}{2}}\Gamma\left(\frac{\alpha}{2}\right)|x|^{d-\alpha}}\int_0^\infty y^{\frac{d}{2}-2}\exp(-y)\frac{u\left(\frac{|x|^2}{4y}\right)}{\frac{1}{\Gamma(\frac{\alpha}{2})}\left(\frac{|x|^2}{4y}\right)^{\frac{\alpha}{2}-1}}\left(\frac{1}{4y}\right)^{\frac{\alpha}{2}-1}dy$$

$$= \frac{1}{2^\alpha\pi^{\frac{d}{2}}\Gamma\left(\frac{\alpha}{2}\right)|x|^{d-\alpha}}\int_0^\infty y^{\frac{d}{2}-\frac{\alpha}{2}-1}\exp(-y)\frac{u\left(\frac{|x|^2}{4y}\right)}{\frac{1}{\Gamma(\frac{\alpha}{2})}\left(\frac{|x|^2}{4y}\right)^{\frac{\alpha}{2}-1}}dy$$

Now let $|x| \geq 2$ then

$$\frac{u\left(\frac{|x|^2}{4y}\right)}{\left(\frac{|x|^2}{4y}\right)^{\frac{\alpha}{2}-1}} \leq D\left[max\left(\left(\frac{|x|^2}{4y}\right)^{-\frac{\alpha}{2}}, 1\right)\right]$$

$$\leq D\left[max\left(\left(s^{\frac{\alpha}{2}},1\right)\right],\right.$$

(because $|x|\geq 2$). Therefore,

$$\int_0^\infty y^{\frac{d}{2}-\frac{\alpha}{2}-1}\exp(-y)\frac{u\left(\frac{|x|^2}{4y}\right)}{\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\left(\frac{|x|^2}{4y}\right)^{\frac{\alpha}{2}-1}}dy$$

$$\leq \int_0^\infty y^{\frac{d}{2}-\frac{\alpha}{2}-1}\exp(-y)\frac{D(max(s^{\frac{\alpha}{2}},1))}{\frac{1}{\Gamma(\frac{\alpha}{2})}}dy<\infty,$$

(because the integral is a gamma function).

Hence, applying Dominated Convergence theorem on the integral we get

$$\lim_{|x|\to\infty}\frac{1}{|x|^{\alpha-d}}\int_0^\infty (4\pi t)^{-\frac{d}{2}}\exp\left(-\frac{|x|^2}{4t}\right)u(t)dt$$

$$\begin{aligned}
&=\lim_{|x|\to\infty}\frac{1}{2^\alpha\pi^{\frac{d}{2}}\Gamma\left(\frac{\alpha}{2}\right)}\int_0^\infty y^{\frac{d}{2}-\frac{\alpha}{2}-1}\exp(-y)\frac{u\left(\frac{|x|^2}{4y}\right)}{\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\left(\frac{|x|^2}{4y}\right)^{\frac{\alpha}{2}-1}}dy\\
&=\frac{1}{2^\alpha\pi^{\frac{d}{2}}\Gamma\left(\frac{\alpha}{2}\right)}\int_0^\infty y^{\frac{d}{2}-\frac{\alpha}{2}-1}\exp(-y)\lim_{|x|\to\infty}\frac{u\left(\frac{|x|^2}{4y}\right)}{\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\left(\frac{|x|^2}{4y}\right)^{\frac{\alpha}{2}-1}}dy\\
&=\frac{1}{2^\alpha\pi^{\frac{d}{2}}\Gamma\left(\frac{\alpha}{2}\right)}\int_0^\infty y^{\frac{d}{2}-\frac{\alpha}{2}-1}\exp(-y)dy\\
&=\frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^\alpha\pi^{\frac{d}{2}}\Gamma\left(\frac{\alpha}{2}\right)}.
\end{aligned}$$

Hence,

$$G(x)\sim\frac{1}{\pi^{\frac{d}{2}}2^\alpha}\frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}|x|^{\alpha-d},|x|\to\infty.$$

$\square$

Theorem 3.2.2. *For every* $\alpha\in(0,2]$, *we have*

$$J(x)\sim\frac{\alpha\Gamma\left(\frac{d}{2}\right)}{2|x|^d},|x|\to 0.$$

Proof. We know from Definition 2.1.13, Theorem 3.1.2 and Theorem 3.1.3 that

$$J(x)=\int_0^\infty (4\pi t)^{-\frac{d}{2}}\exp\left(-\frac{(|x|)^2}{4t}\right)\mu(t)dt,$$

$$\mu(t) \sim \frac{\alpha}{2t}, t \to 0^+$$

and

$$\mu(t) \sim \frac{\alpha}{2t^{\frac{\alpha}{2}+1}\Gamma\left(1-\frac{\alpha}{2}\right)}, t \to \infty.$$

Therefore, applying lemma 3.1.5 with $w(t) = \mu(t)$, $\gamma = -\frac{\alpha}{2}$, $c_0 = \frac{\alpha}{2}$ and $l(t) = 1$. Choosing $\beta = \frac{1}{2}$ then $f_{l,1/2}(y,t) = 1$, $y < 2t$. Let $g(t) = 1$ then $f_{l,1/2}(y,t) = 1 \leq g(t)$ $\forall$ $y, t > 0$.

Now

$$\int_0^\infty t^{\frac{d}{2}-1}\exp(-t)g(t)dt = \int_0^\infty t^{\frac{d}{2}-1}\exp(-t)dt < \infty,$$

(because it is a gamma function). Hence,

$$J(x) \sim \frac{\alpha\Gamma\left(\frac{d}{2}\right)}{2|x|^d}, |x| \to 0.$$

$\square$

Theorem 3.2.3. *For every $\alpha \in (0,2)$, we have*

$J(x) \sim \frac{\alpha}{2^{\alpha+1}\pi^{\frac{d}{2}}}\frac{\Gamma\left(\frac{d+\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)}|x|^{-d-\alpha}, |x| \to \infty.$

Proof. We know that

$$\mu(t) \sim \frac{\alpha}{2t}, t \to 0^+$$

and

$$\mu(t) \sim \frac{\alpha}{2t^{\frac{\alpha}{2}+1}\Gamma\left(1-\frac{\alpha}{2}\right)}, t \to \infty,$$

therefore, $\exists$ a positive constant D such that $\mu(t) \leq Cmax\left(t^{-1}, t^{-\frac{\alpha}{2}-1}\right)$. Let $\frac{|x|^2}{4t} = y$ which implies $-\frac{|x|^2}{4t^2}dt = dy$. Therefore,

$$\int_0^\infty (4\pi t)^{-\frac{d}{2}}\exp\left(-\frac{|x|^2}{4t}\right)\mu(t)dt$$

$$= \int_\infty^0 (4\pi)^{-\frac{d}{2}}\exp(-y)\left(\frac{|x|^2}{4y}\right)^{-\frac{d}{2}}\mu\left(\frac{|x|^2}{4y}\right)\frac{-|x|^2}{4y^2}dy$$

$$= \frac{1}{4\pi^{\frac{d}{2}}|x|^{d-2}}\int_0^\infty y^{\frac{d}{2}-2}\exp(-y)\mu\left(\frac{|x|^2}{4y}\right)dy$$

$$= \frac{\alpha}{8\pi^{\frac{d}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)|x|^{-d-\alpha}} \int_0^\infty y^{\frac{d}{2}-2}\exp(-y) \frac{\mu\left(\frac{|x|^2}{4y}\right)}{\frac{\alpha}{2\Gamma\left(1-\frac{\alpha}{2}\right)}\left(\frac{|x|^2}{4y}\right.}^{-\frac{\alpha}{2}-1} \left(\frac{1}{4y}\right)^{-\frac{\alpha}{2}-1} dy$$

$$= \frac{\alpha}{2^{\alpha+1}\pi^{\frac{d}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)|x|^{-d-\alpha}} \int_0^\infty y^{\frac{d}{2}+\frac{\alpha}{2}-1}\exp(-y) \frac{\mu\left(\frac{|x|^2}{4y}\right)}{\frac{\alpha}{2\Gamma\left(1-\frac{\alpha}{2}\right)}\left(\frac{|x|^2}{4y}\right)^{-\frac{\alpha}{2}-1}} dy.$$

Now let $|x| \geq 2$, then

$$\frac{u\left(\frac{|x|^2}{4y}\right)}{\left(\frac{|x|^2}{4y}\right)^{-\frac{\alpha}{2}-1}} \leq D\left[max\left(\left(\frac{|x|^2}{4y}\right)^{\frac{\alpha}{2}}, 1\right)\right]$$

$$\int_0^\infty y^{\frac{d}{2}+\frac{\alpha}{2}-1}\exp(-y) \frac{\mu\left(\frac{|x|^2}{4y}\right)}{\frac{\alpha}{2\Gamma\left(1-\frac{\alpha}{2}\right)}\left(\frac{|x|^2}{4y}\right)^{-\frac{\alpha}{2}-1}} dy$$

$$\leq \int_0^\infty y^{\frac{d}{2}+\frac{\alpha}{2}-1}\exp(-y) \frac{D\left[max\left(\left(\frac{|x|^2}{4y}\right)^{\frac{\alpha}{2}}, 1\right)\right]}{\frac{\alpha}{2\Gamma\left(1-\frac{\alpha}{2}\right)}} dy < \infty,$$

(because the integral is a gamma function). Thus, on applying Dominated Convergence theorem we get

$$\lim_{|x|\to\infty} \frac{1}{|x|^{-\alpha-d}} \int_0^\infty (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{4t}\right)\mu(t)dt$$

$$= \lim_{|x|\to\infty} \frac{\alpha}{2^{\alpha+1}\pi^{\frac{d}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)} \int_0^\infty y^{\frac{d}{2}+\frac{\alpha}{2}-1}\exp(-y) \frac{\mu\left(\frac{|x|^2}{4y}\right)}{\frac{\alpha}{2\Gamma\left(1-\frac{\alpha}{2}\right)}\left(\frac{|x|^2}{4y}\right)^{-\frac{\alpha}{2}-1}} dy$$

$$= \frac{\alpha}{2^{\alpha+1}\pi^{\frac{d}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)} \int_0^\infty y^{\frac{d}{2}+\frac{\alpha}{2}-1}\exp(-y) \lim_{|x|\to\infty} \frac{\mu\left(\frac{|x|^2}{4y}\right)}{\frac{\alpha}{2\Gamma\left(1-\frac{\alpha}{2}\right)}\left(\frac{|x|^2}{4y}\right)^{-\frac{\alpha}{2}-1}} dy$$

$$= \frac{\alpha}{2^{\alpha+1}\pi^{\frac{d}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)} \int_0^\infty y^{\frac{d}{2}+\frac{\alpha}{2}-1}\exp(-y)dy$$

$$= \frac{\alpha\Gamma\left(\frac{d+\alpha}{2}\right)}{2^{\alpha+1}\pi^{\frac{d}{2}}\Gamma\left(1-\frac{\alpha}{2}\right)}.$$

Hence,

$$J(x) \sim \frac{\alpha}{2^{\alpha+1}\pi^{\frac{d}{2}}} \frac{\Gamma\left(\frac{d+\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)} |x|^{-d-\alpha}, |x| \to \infty.$$

$\square$

Theorem 3.2.4. *For $\alpha = 2$, we have*

$$J(x) \sim 2^{-\frac{d}{2}}\pi^{-\frac{d-1}{2}}\frac{\exp(-|x|)}{|x|^{\frac{d+1}{2}}}.$$

Proof. We know $\mu(t) = \frac{t^{-1}\exp(-t)}{2}$ therefore,

$$\begin{aligned} J(x) &= \frac{1}{2}\int_0^\infty (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{(|x|)^2}{4t}\right)\mu(t)dt \\ &= \frac{1}{2}\int_0^\infty (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{(|x|)^2}{4t}\right)t^{-1}\exp(-t)dt. \end{aligned}$$

Let $\frac{|x|^2}{t} = u$ which implies $-\frac{|x|^2}{t^2}dt = du$. Then

$$\begin{aligned} J(x) &= \int_\infty^0 \left(4\pi\frac{|x|^2}{u}\right)^{-\frac{d}{2}} \exp\left(-\frac{u}{4}\right)\left(\frac{|x|^2}{u}\right)^{-1} \exp\left(-\frac{|x|^2}{u}\right)\left(-\frac{|x|^2}{u^2}du\right) \\ &= 2^{-d-1}\pi^{-\frac{d}{2}}|x|^{-d}\int_0^\infty u^{\frac{d}{2}-1}\exp\left(-\frac{u}{4}-\frac{|x|^2}{u}\right)du. \end{aligned}$$

Let

$$\begin{aligned} H(p) &= H(|x|) \\ &= \int_0^\infty u^{\frac{d}{2}-1}\exp\left(-\frac{u}{4}-\frac{p}{u}\right)du \end{aligned}$$

$$= \exp(-p)\int_0^\infty \frac{(\sqrt{u})^d}{u}\exp\left(-\left(\frac{\sqrt{u}}{2}-\frac{p}{\sqrt{u}}\right)^2\right)du. \tag{3.15}$$

Also, let

$$\begin{aligned} y &= \frac{\sqrt{u}}{2} - \frac{p}{\sqrt{u}} \\ &= \frac{u-2p}{2\sqrt{u}} \end{aligned}$$

or

$$\sqrt{u} = \frac{u-2p}{2y} \tag{3.16}$$

which implies

$$dy = \left(\frac{1}{4\sqrt{u}} + \frac{p}{2u^{\frac{3}{2}}}\right)du$$

or

$$dy = \left(\frac{u+2p}{4u^{\frac{3}{2}}}\right) du$$

or

$$du = \left(\frac{4u^{\frac{3}{2}}}{u+2p}\right) dy \tag{3.17}$$

Since $y = \frac{u-2p}{2\sqrt{u}}$ therefore,

$$(u-2p)^2 = 4uy^2 \tag{3.18}$$

or

$$u^2 + 4p^2 - 4up - 4uy^2 = 0$$

Using quadratic equation formula we get

$$u = 2p + 2y^2 + 2y\sqrt{y^2+2p}.$$

or

$$\frac{u-2p}{2y} = y + \sqrt{y^2+2p} \tag{3.19}$$

Using equation 3.18 we get

$$(u+2p)^2 - 8up = 4uy^2$$

or

$$u + 2p = 2\sqrt{u}\sqrt{y^2+2p}$$

or

$$\frac{2\sqrt{u}}{u+2p} = \frac{1}{\sqrt{y^2+2p}} \tag{3.20}$$

From equations 3.16 and 3.19 we get

$$\sqrt{u} = y + \sqrt{y^2+2p} \tag{3.21}$$

Using equations 3.17 and 3.20 we get

$$du = \frac{2udy}{\sqrt{y^2+2p}}. \tag{3.22}$$

Using equations 3.15, 3.21 and 3.22 we get

$$
\begin{aligned}
H(p) &= \exp(-p)\int_{-\infty}^{\infty} \frac{\left(y+\sqrt{y^2+2p}\right)^d}{u} \frac{2u}{\sqrt{y^2+2p}} \exp(-y^2)dy \\
&= 2\exp(-p)\int_{-\infty}^{\infty} \frac{\left(y+\sqrt{y^2+2p}\right)^d}{\sqrt{y^2+2p}} \exp(-y^2)dy \\
&= 2\exp(-p)\int_{-\infty}^{\infty} \frac{y+\sqrt{y^2+2p}}{\sqrt{y^2+2p}} \left(y+\sqrt{y^2+2p}\right)^{d-1} \exp(-y^2)dy \\
&= 2\exp(-p)\int_{-\infty}^{\infty} \frac{y+\sqrt{y^2+2p}}{\sqrt{y^2+2p}} p^{\frac{d-1}{2}} \left(\frac{y}{\sqrt{p}}+\sqrt{\frac{y^2}{p}+2}\right)^{d-1} \exp(-y^2)dy
\end{aligned}
$$

Now using Dominated Convergence theorem we get

$$
\begin{aligned}
\lim_{p\to\infty} \frac{H(p)}{\exp(-p)p^{\frac{d-1}{2}}} &= 2\int_{-\infty}^{\infty} \lim_{p\to\infty} \frac{y+\sqrt{y^2+2p}}{\sqrt{y^2+2p}} \left(\frac{y}{\sqrt{p}}+\sqrt{\frac{y^2}{p}+2}\right)^{d-1} \exp(-y^2)dy \\
&= 2^{\frac{d}{2}+1}\int_{-\infty}^{\infty} \exp(-y^2)dy = 2^{\frac{d}{2}+1}\sqrt{\pi}.
\end{aligned}
$$

Therefore,

$$H(p) \sim 2^{\frac{d}{2}+1}\sqrt{\pi}\exp(-p)p^{\frac{d-1}{2}}, p \to \infty,$$

or

$$H(|x|) \sim 2^{\frac{d}{2}+1}\sqrt{\pi}\exp\left(-|x|\right)|x|^{\frac{d-1}{2}}, |x| \to \infty.$$

Hence,

$$J(x) \sim 2^{-d-1}\pi^{-\frac{d}{2}}|x|^{-d}2^{\frac{d}{2}+1}\sqrt{\pi}\exp(-|x|)|x|^{\frac{d-1}{2}}, |x| \to \infty,$$

or

$$J(x) \sim 2^{\frac{d}{2}}\pi^{-\frac{d-1}{2}}\frac{\exp(-|x|)}{|x|^{\frac{d+1}{2}}}, |x| \to \infty.$$

□

In this chapter, we mainly computed the asymptotic behaviour of potential density and Lévy density associated with the geometric stable subordinator and also the Green function and Lévy density associated with the geometric stable process. The tools and techniques used in this chapter to prove the results will be helpful to prove the same results in the next chapter.

Chapter 4

Potential Theory of Brownian Motion Time-Changed by Tempered Stable Subordinator and Normal Inverse Gaussian Process

In this chapter as per our knowledge, all the results mentioned are new and have never been proved before. Precisely, we will find the asymptotic behaviour of potential density and Lévy density associated with the two new subordinators and also the Green function and Lévy density associated with the for two new subordinated Brownian motions. We will try to use two approaches to prove the results. The first approach is same as what we have used in previous chapter or the author have used in [4]. The other approach is different. As we know Tauberian theorems are usually used for a non-decreasing function whose exact form is not known but its Laplace transform's exact form is known. So, in the second approach we will try to compute the exact inverse Laplace transform of the potential measure and then see its asymptotic behaviour. After knowing the exact form of potential measure, we will try to compute the asymptotic behaviour of Green function and similarly for Lévy density.

All the theorems in section 2.2 of chapter two are also valid for the two new processes mentioned in this chapter.

The first process is Brownian motion subordinated with tempered stable subordinator and the second is Brownian motion subordinated with inverse Gaussian process.

4.1 Tempered Stable Subordinator

The Laplace transform of the Geometric stable subordinator is given by

$$\mathbb{E}[(-\lambda S_t)] = \exp\left[-t\left((\lambda+\beta)^\gamma - \beta^\gamma\right)\right].$$

By the similar arguments of section 3.1, Chapter 3 to show the existence of the potential density of the geomteric stable subordinator, we can also show the existence of potential density of the tempered stable subordinator.

We now compute the asymptotic behaviour of the potential density of the tempered stable subordinator using the first approach.

Theorem 4.1.1. *For $\gamma \in (0,1)$ and $\beta > 0$, we have*

1. $u(x) \sim \frac{\gamma x^{\gamma-1}}{\Gamma(1+\gamma)}, x \to 0^+$,
2. $u(x) \sim \frac{1}{\gamma\beta^{\gamma-1}\Gamma(2)}, x \to \infty$.

Proof.

1. The Laplace exponent of the Tempered stable subordinator is given by

$$\phi(\lambda) = (\lambda+\beta)^\gamma - \beta^\gamma,$$

where $\gamma \in (0,1)$ and $\beta > 0$. Therefore, the Laplace transform of the potential measure U of tempered stable subordinator will be

$$LU(\lambda) = \frac{1}{\phi(\lambda)} = \frac{1}{(\lambda+\beta)^\gamma - \beta^\gamma}.$$

Now $(\lambda+\beta)^\gamma - \beta^\gamma = \beta^\gamma\left[\left(1+\frac{\lambda}{\beta}\right)^\gamma - 1\right]$, as $\lambda \to 0^+$ $\left(1+\frac{\lambda}{\beta}\right)^\gamma \sim 1 + \frac{\lambda\gamma}{\beta}$, therefore, as $\lambda \to 0^+$ $\phi(\lambda) \sim \beta^\gamma\left[\left(1+\frac{\lambda\gamma}{\beta}\right) - 1\right] = \gamma\beta^{\gamma-1}\lambda$.

Hence,

$$LU(\lambda) \sim \frac{1}{\gamma\beta^{\gamma-1}\lambda}, \lambda \to 0^+.$$

Therefore, by Theorem 2.2.1(3), Chapter 2 the potential measure

$$U(x) \sim \frac{1}{\gamma\beta^{\gamma-1}\Gamma(2)}x, x \to \infty.$$

Hence, by Theorem 2.2.1(4), Chapter 2 the potential density

$$u(x) \sim \frac{1}{\gamma\beta^{\gamma-1}\Gamma(2)}, x \to \infty.$$

2. As $\lambda \sim \infty$ $\phi(\lambda) \sim \lambda^{\gamma}$, therefore, $LU(\lambda) \sim \frac{1}{\lambda^{\gamma}}$. Thus, by Theorem 2.2.1(3), Chapter 2, we have

$$U(x) \sim \frac{x^{\gamma}}{\Gamma(1+\gamma)}, x \to \infty.$$

Therefore, by Theorem 2.2.1(4), Chapter 2, we have

$$u(x) \sim \frac{\gamma x^{\gamma-1}}{\Gamma(1+\gamma)}, x \to 0^{+}.$$

□

We have tried to use the second approach to find the asymptotic behaviour of potential density however, we were not successful. We tried to compute the exact form of the potential measure but we were finally stuck at a point where we were unable to integrate a function. We are still finding a way to compute the asymptotic behaviour of the complicated integral we got. We now show the steps we have tried to reach the desire result.

Theorem 4.1.2. *For $\gamma \in (0,1)$ and $\beta > 0$, we have*

$U(t) = \exp(-\beta t)\frac{\sin(\pi\gamma)}{\pi}\int_0^{\infty} \frac{\exp(-xt)x^{\gamma}}{[x^{2\gamma}+\beta^{2\gamma}-2\cos(\pi\gamma)x^{\gamma}]}dx.$

Proof. Let $\phi(s) = f(s+\beta) = (s+\beta)^{\gamma} - \beta^{\gamma}$ be the Laplace exponent of the tempered stable subordinator, where $f(s) = s^{\gamma} - \beta^{\gamma}$. Then using the properties of inverse Laplace transform we have

$$L^{-1}\left[LU(s) = \frac{1}{\phi(s)}\right](t) = \exp(-\beta t)L^{-1}\left[\frac{1}{f(s)}\right](t),$$

where $L^{-1}[LU(s)]$ is the inverse Laplace transform of the potential measure related to the tempered stable subordinator.

Now to find the inverse Laplace transform of the potential measure we will first find

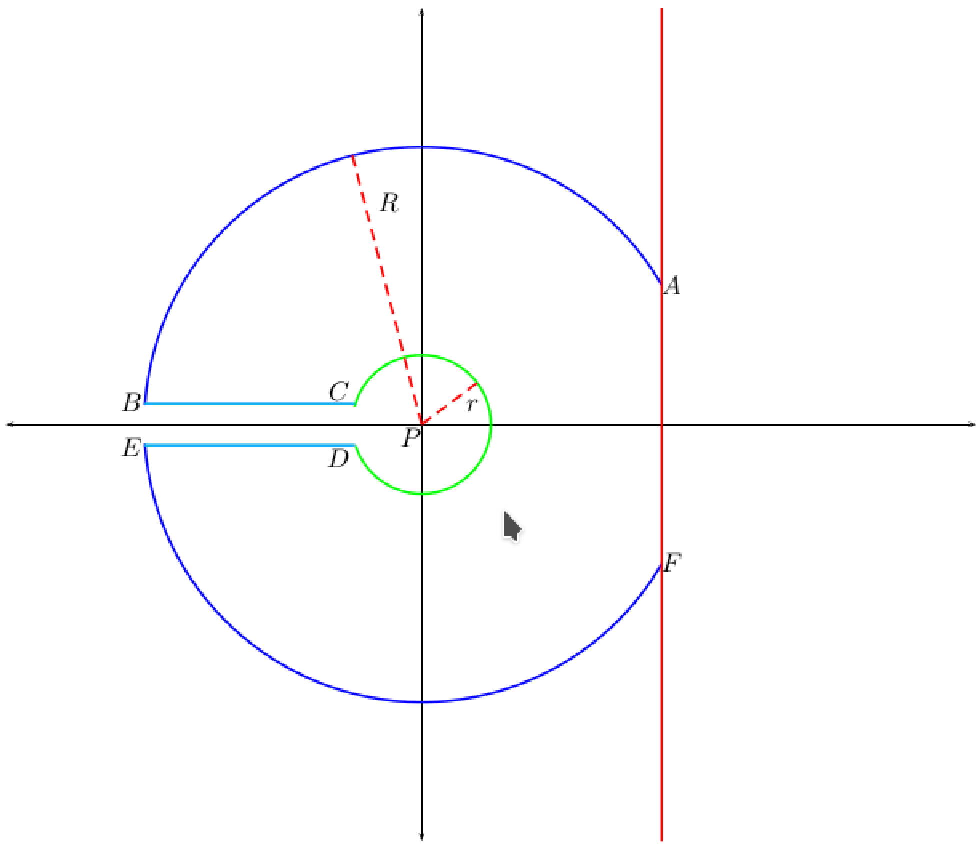


Figure 4.1: Contour C anti-clockwise, Figure 4.8, [2]

$L^{-1}\left[\frac{1}{f(s)}\right](t)$ using complex inversion formula([2]). Since $s = 0$ is a branch point of $\frac{1}{f(s)}$ so we take a branch cut along non-positive real line to make the function single valued as shown in the figure 4.1. Inside and on the contour the function is analytic so by Cauchy's theorem

$$\frac{1}{2\pi i}\int_C \frac{\exp(st)}{f(s)}ds = 0.$$

Now

$$\frac{1}{2\pi i}\int_C \frac{\exp(st)}{f(s)}ds = \tag{4.1}$$

$$\frac{1}{2\pi i}\int_{AB} \frac{\exp(st)}{f(s)}ds + \frac{1}{2\pi i}\int_{BC} \frac{\exp(st)}{f(s)}ds + \frac{1}{2\pi i}\int_{CD} \frac{\exp(st)}{f(s)}ds$$
$$+\frac{1}{2\pi i}\int_{DE} \frac{\exp(st)}{f(s)}ds + \frac{1}{2\pi i}\int_{EF} \frac{\exp(st)}{f(s)}ds + \frac{1}{2\pi i}\int_{FA} \frac{\exp(st)}{f(s)}ds = 0,$$

also

$$\int_{AB} \frac{\exp(st)}{f(s)}ds = \int_{CD} \frac{\exp(st)}{f(s)}ds = \int_{EF} \frac{\exp(st)}{f(s)}ds = 0,$$

(see [2] for details). We know from [2] that as $r \to 0$ and $R \to \infty$

$$\frac{1}{2\pi i}\int_{FA}\frac{\exp(st)}{f(s)}ds = L^{-1}\left[\frac{1}{f(s)}\right](t), \tag{4.2}$$

therefore, from equations 3.1 and 3.2 we get

$$L^{-1}\left[\frac{1}{f(s)}\right](t) = -\left(\frac{1}{2\pi i}\int_{BC}\frac{\exp(st)}{f(s)}ds + \frac{1}{2\pi i}\int_{DE}\frac{\exp(st)}{f(s)}ds\right). \tag{4.3}$$

Consider,

$$\int_{BC}\frac{\exp(st)}{f(s)}ds = \int_{BC}\frac{\exp(st)}{s^\gamma - \beta^\gamma}ds = \int_{-R}^{-r} -\frac{\exp(st)}{s^\gamma - \beta^\gamma}ds,$$

let $s = x\exp(i\pi)$ then $ds = -ds$, thus,

$$\int_{BC}\frac{\exp(st)}{s^\gamma - \beta^\gamma}ds = \int_{R}^{r} -\frac{\exp(-xt)}{s^\gamma - \beta^\gamma}dx.$$

Taking the limits $r \to 0$ and $R \to \infty$ on both side of the above equation we get

$$\int_{BC}\frac{\exp(st)}{s^\gamma - \beta^\gamma}ds = \int_{0}^{\infty}\frac{\exp(-xt)}{x^\gamma\exp(i\pi\gamma) - \beta^\gamma}dx. \tag{4.4}$$

Now

$$\int_{DE}\frac{\exp(st)}{f(s)}ds = \int_{DE}\frac{\exp(st)}{s^\gamma - \beta^\gamma}ds = \int_{-r}^{-R} -\frac{\exp(st)}{s^\gamma - \beta^\gamma}ds,$$

let $s = x\exp(-i\pi)$ then $ds = -ds$, thus,

$$\int_{DE}\frac{\exp(st)}{s^\gamma - \beta^\gamma}ds = \int_{r}^{R} -\frac{\exp(-xt)}{s^\gamma - \beta^\gamma}dx.$$

Again we take the limits $r \to 0$ and $R \to \infty$ on both side of the above equation we get

$$\int_{DE}\frac{\exp(st)}{s^\gamma - \beta^\gamma}ds = \int_{0}^{\infty} -\frac{\exp(-xt)}{x^\gamma\exp(-i\pi\gamma) - \beta^\gamma}dx. \tag{4.5}$$

From equations 3.3, 3.4 and 3.5 we get

$$\begin{aligned}
L^{-1}\left[\frac{1}{f(s)}\right](t) &= -\frac{1}{2\pi i}\left[\int_0^\infty \frac{\exp(-xt)}{x^\gamma\exp(i\pi\gamma) - \beta^\gamma}dx - \int_0^\infty \frac{\exp(-xt)}{x^\gamma\exp(-i\pi\gamma) - \beta^\gamma}dx\right] \\
&= -\frac{1}{2\pi i}\left[\int_0^\infty \exp(-xt)\left[\frac{1}{x^\gamma\exp(i\pi\gamma) - \beta^\gamma} - \frac{1}{x^\gamma\exp(-i\pi\gamma) - \beta^\gamma}\right]dx\right. \\
&= -\frac{1}{2\pi i}\int_0^\infty \exp(-xt)\frac{[x^\gamma\exp(-i\pi\gamma) - \beta^\gamma - x^\gamma\exp(i\pi\gamma) - \beta^\gamma]}{[x^\gamma\exp(i\pi\gamma) + \beta^\gamma]\,[x^\gamma\exp(-i\pi\gamma) - \beta^\gamma]}dx \\
&= -\frac{1}{2\pi i}\int_0^\infty \exp(-xt)\frac{[x^\gamma\,[\exp(-i\pi\gamma) - \exp(i\pi\gamma)]]}{[x^{2\gamma} + \beta^{2\gamma} - x^\gamma\,[\exp(-i\pi\gamma) + \exp(i\pi\gamma)]\,]dx}
\end{aligned}$$

$$= -\frac{1}{2\pi i}\int_0^\infty \exp(-xt)\frac{-2i\sin(\pi\gamma)x^\gamma}{[x^{2\gamma}+\beta^{2\gamma}-2\cos(\pi\gamma)x^\gamma]\,]dx}$$

$$= \frac{\sin(\pi\gamma)}{\pi}\int_0^\infty \frac{\exp(-xt)x^\gamma}{[x^{2\gamma}+\beta^{2\gamma}-2\cos(\pi\gamma)x^\gamma]}dx$$

Thus, the potential measure $U(t) = L^{-1}[LU(s)](t) =$

$$\exp(-\beta t)\frac{\sin(\pi\gamma)}{\pi}\int_0^\infty \frac{\exp(-xt)x^\gamma}{[x^{2\gamma}+\beta^{2\gamma}-2\cos(\pi\gamma)x^\gamma]}dx.$$

Unfortunately, we could not find a method to compute the above integral although we have tried to see the asymptotic behaviour(as $t \to \infty$) of this integral using Watson's lemma, still we were unsuccessful because the integral is too complicated. We are still finding a way to get the asymptotic behaviour of the above integral. □

4.2 Brownian Motion time-changed by Tempered Stable Subordinator

In this section we compute the asymptotic behaviour of the Green function and Lévy density associated with the Brownian motion time-changed by tempered stable subordinator.

Theorem 4.2.1. *For $\gamma \in (0,1)$ and $\beta > 0$, we have*

1. $G(x) \sim \frac{\Gamma\left(\frac{d-2\gamma}{2}\right)}{\pi^{\frac{d}{2}}4^\gamma\Gamma(\gamma)}|x|^{2\gamma-d}, |x| \to 0^+,$
2. $G(x) \sim \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}}\gamma\beta^{\gamma-1}\lambda}|x|^2, |x| \to \infty.$

Proof.

1. As $\lambda \to \infty$, $\phi(\lambda) \sim \lambda^\gamma$, then it follows directly from Theorem 3.1 of [6] that
$$G(x) \sim \frac{\Gamma\left(\frac{d-2\gamma}{2}\right)}{\pi^{\frac{d}{2}}4^\gamma\Gamma(\gamma)}|x|^{2\gamma-d}, |x| \to 0^+.$$
2. As $\lambda \to 0^+$, $\phi(\lambda) \sim \gamma\beta^{\gamma-1}\lambda$, then it follows directly from Theorem 3.3 of [6] that
$$G(x) \sim \frac{1}{\pi^{\frac{d}{2}}2^2\gamma\beta^{\gamma-1}\lambda}\frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{2}{2}\right)}|x|^2, |x| \to \infty$$
or
$$G(x) \sim \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{\frac{d}{2}}\gamma\beta^{\gamma-1}\lambda}|x|^2, |x| \to \infty.$$

□

We now find the asymptotic behaviour of the and Lévy density $J(x)$ of the subordinated process. We can not use the same method as in [4] because the asymptotic behaviour of Lévy density $\mu(x)$ of the Lévy measure μ associated with the tempered stable subordinator does not follow assumption 2 of lemma 3.1.5 so we use the other method. The exact form of Lévy density associated with the subordinator is already known([16]).

Theorem 4.2.2. *For $\gamma \in (0,1)$ and $\beta > 0$, we have*

1. $J(x) \sim \frac{4^{\gamma+1}}{\pi^{\frac{d}{2}}}\left(\gamma + \frac{d}{2}\right) c|x|^{-(2\gamma+d)}, |x| \to 0,$
2. $J(x) \sim \frac{2^{\frac{2\gamma-d+1}{2}}\beta^{\frac{2\gamma+d-1}{4}}}{\pi^{\frac{d-1}{2}}} c|x|^{-\frac{2\gamma+d-1}{2}} \exp\left(-\sqrt{\beta}|x|\right), |x| \to \infty.$

Proof. The Lévy density associated with the tempered stable subordinator is given by([16])

$$\mu(t) = \frac{c\exp(-\beta t)}{t^{\gamma+1}},$$

thus, the Lévy density associated with the subordinated process

$$\begin{aligned}
J(x) &= \int_0^\infty p_2(t,0,x)\mu(t)dt \\
&= \int_0^\infty (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{(|x|)^2}{4t}\right)\frac{c\exp(-\beta t)}{t^{\gamma+1}}dt \\
&= \frac{c}{2^d\pi^{\frac{d}{2}}}\int_0^\infty t^{-\left(\gamma+1+\frac{d}{2}\right)} \exp\left(-\frac{(|x|)^2}{4t} - \beta t\right) dt \\
&= \frac{c}{2^d\pi^{\frac{d}{2}}}\int_0^\infty t^{-\left(\gamma+1+\frac{d}{2}\right)} \exp\left[-\frac{1}{2}\left(\frac{(|x|)^2}{2t} + 2\beta t\right)\right] dt
\end{aligned}$$

Let $t = \frac{|x|y}{2\sqrt{\beta}}$ then $dt = \frac{|x|dy}{2\sqrt{\beta}}$ thus ,

$$\begin{aligned}
J(x) &= \frac{c}{2^d\pi^{\frac{d}{2}}}\int_0^\infty \left[\frac{|x|y}{2\sqrt{\beta}}\right]^{-\left(\gamma+1+\frac{d}{2}\right)} \exp\left[-\frac{1}{2}\left(\sqrt{\beta}|x|y + \frac{\sqrt{\beta}|x|y}{y}\right)\right]\frac{|x|dy}{2\sqrt{\beta}} \\
&= \frac{c}{2^d\pi^{\frac{d}{2}}}\left[\frac{|x|}{2\sqrt{\beta}}\right]^{-\left(\gamma+\frac{d}{2}\right)}\int_0^\infty y^{-\left(\gamma+1+\frac{d}{2}\right)} \exp\left[-\frac{1}{2}\sqrt{\beta}|x|\left(y+\frac{1}{y}\right)\right] dy \\
&= \frac{2^{\left(\gamma-\frac{d}{2}\right)}\beta^{\frac{1}{2}\left(\gamma+\frac{d}{2}\right)}}{\pi^{\frac{d}{2}}} c|x|^{-\left(\gamma+\frac{d}{2}\right)}\int_0^\infty y^{-\left(\gamma+1+\frac{d}{2}\right)} \exp\left[-\frac{1}{2}\sqrt{\beta}|x|\left(y+\frac{1}{y}\right)\right] dy \\
&= \frac{2^{\left(\gamma-\frac{d}{2}\right)}\beta^{\frac{1}{2}\left(\gamma+\frac{d}{2}\right)}}{\pi^{\frac{d}{2}}} c|x|^{-\left(\gamma+\frac{d}{2}\right)} 2K_{-\lambda}(\omega), \qquad (4.6)
\end{aligned}$$

where

$$K_{-\lambda}(\omega) = \frac{1}{2}\int_0^\infty y^{-\left(\gamma+1+\frac{d}{2}\right)} \exp\left[-\frac{1}{2}\sqrt{\beta}|x|\left(y+\frac{1}{y}\right)\right] dy,$$

$\lambda = \left(\gamma + \frac{d}{2}\right)$ and $\omega = \sqrt{\beta}|x|$. Now $K_{-\lambda}(\omega)$ is a modified Bessel function of third type and $K_{-\lambda}(\omega) = K_\lambda(\omega)$([17]).

1. As $|x| \to 0$, $\omega \to 0$ thus, using $A.7$ of [17] we get

$$K_\lambda(\omega) \sim \Gamma\left(\gamma + \frac{d}{2}\right) 2^{\gamma+\frac{d}{2}-1}(\sqrt{\beta}|x|)^{-\left(\gamma+\frac{d}{2}\right)}.$$

   Using equation 3.6 we get

$$J(x) \sim \frac{2^{\left(\gamma-\frac{d}{2}\right)}\beta^{\frac{1}{2}\left(\gamma+\frac{d}{2}\right)}}{\pi^{\frac{d}{2}}} c|x|^{-\left(\gamma+\frac{d}{2}\right)} 2\Gamma\left(\gamma+\frac{d}{2}\right) 2^{\gamma+\frac{d}{2}+1}(\sqrt{\beta}|x|)^{-\left(\gamma+\frac{d}{2}\right)}, |x| \to 0,$$

   or

$$J(x) \sim \frac{4^{\gamma+1}}{\pi^{\frac{d}{2}}}\left(\gamma+\frac{d}{2}\right) c|x|^{-(2\gamma+d)}, |x| \to 0.$$

2. As $|x| \to 0$, $\omega \to 0$ thus, from [17] we get

$$K_\lambda(\omega) \sim \sqrt{\frac{\pi}{2}} \exp\left(-\sqrt{\beta}|x|\right)\left(\sqrt{\beta}|x|\right)^{-\frac{1}{2}}.$$

   Using equation 3.6 we get

$$J(x) \sim \frac{2^{\left(\gamma-\frac{d}{2}\right)}\beta^{\frac{1}{2}\left(\gamma+\frac{d}{2}\right)}}{\pi^{\frac{d}{2}}} c|x|^{-\left(\gamma+\frac{d}{2}\right)} 2\sqrt{\frac{\pi}{2}} \exp\left(-\sqrt{\beta}|x|\right)\left(\sqrt{\beta}|x|\right)^{-\frac{1}{2}}, |x| \to \infty,$$

   or

$$J(x) \sim \frac{2^{\frac{2\gamma-d+1}{2}}\beta^{\frac{2\gamma+d-1}{4}}}{\pi^{\frac{d-1}{2}}} c|x|^{-\frac{2\gamma+d-1}{2}} \exp\left(-\sqrt{\beta}|x|\right), |x| \to \infty.$$

□

4.3 Inverse Gaussian Subordinator

The Laplace transform of the Geometric stable subordinator is given by

$$\mathbb{E}[(-\lambda S_t)] = \exp\left[-t\left(\delta\left(\sqrt{2\lambda+\gamma^2}-\gamma\right)\right)\right].$$

By the similar arguments in section 3.1, Chapter 3 to show the existence of the potential density of the geomteric stable subordinator, we can also show the existence of potential density of the inverse Gaussian subordinator.

In this section we again prove some of the previous results for the Inverse Gaussian subordinator using the approach mentioned in [4] using both the approaches.

Theorem 4.3.1. *For $\delta, \gamma > 0$, we have*

1. $u(x) \sim \frac{1}{2\delta\Gamma\left(\frac{3}{2}\right)\sqrt{x}}, x \to 0^+$,
2. $u(x) \sim \frac{\gamma}{\delta\Gamma(2)}, x \to \infty$.

Proof.

1. The Laplace exponent of the Inverse Gaussian subordinator is given by

$$\phi(\lambda) = \delta\left(\sqrt{2\lambda + \gamma^2} - \gamma\right),$$

where $\gamma \in (0,1)$ and $\beta > 0$. Therefore, the Laplace transform of the potential measure U of tempered stable subordinator will be

$$LU(\lambda) = \frac{1}{\delta\left(\sqrt{2\lambda + \gamma^2} - \gamma\right)}.$$

Since $\delta\left(\sqrt{2\lambda + \gamma^2} - \gamma\right) = \delta\gamma\left[\left(1 + \frac{2\lambda}{\gamma^2}\right)^{\frac{1}{2}} - 1\right]$ and as $\lambda \to 0^+$ $\left(1 + \frac{2\lambda}{\gamma^2}\right)^{\frac{1}{2}} \sim 1 + \frac{\lambda}{\gamma^2}$, therefore as $\lambda \to 0^+$, $\phi(\lambda) \sim \delta\gamma\left[\left(1 + \frac{\lambda}{\gamma^2}\right)^{\frac{1}{2}} - 1\right] = \frac{\delta}{\gamma}\lambda$, thus,

$$LU(\lambda) \sim \frac{\gamma}{\delta\lambda}, \lambda \to 0^+.$$

Therefore, by Theorem 2.2.1(3), Chapter 2 the potential measure

$$U(x) \sim \frac{\gamma}{\delta\Gamma(2)}x, x \to \infty,$$

and hence, by Theorem 2.2.1(4), Chapter 2 the potential density

$$u(x) \sim \frac{\gamma}{\delta\Gamma(2)}, x \to \infty.$$

2. As $\lambda \sim \infty$, $\phi(\lambda) \sim \delta\sqrt{2\lambda}$, therefore, $LU(\lambda) \sim \frac{1}{\delta\sqrt{2\lambda}}$. Thus, by Theorem 2.2.1(3), Chapter 1, we have

$$U(x) \sim \frac{1}{\delta\Gamma\left(\frac{3}{2}\right)}\sqrt{x}, x \to 0^+.$$

Therefore, by by Theorem 2.2.1(4), Chapter 2, we have

$$u(x) \sim \frac{1}{2\delta\Gamma\left(\frac{3}{2}\right)\sqrt{x}}, x \to 0^+.$$

$\square$

We still need to find a way to find the asymptotic behaviour of the potential measure of the inverse Gaussian subordinator. We now use the alternative approach to compute the exact form of the potential measure associated with the inverse Gaussian subordinator.

Theorem 4.3.2. *For $\delta, \gamma > 0$, we have*

$$U(t) = \tfrac{1}{\sqrt{2}\delta}\left[\tfrac{\exp\left(-\frac{\gamma^2 t}{2}\right)}{\sqrt{\pi t}} + \tfrac{\gamma}{\sqrt{2}} erf\left(-\tfrac{\gamma\sqrt{t}}{\sqrt{2}}\right)\right].$$

Proof. Let $\phi(s) = f(s+\frac{\gamma^2}{2}) = \sqrt{2}\delta\left(\sqrt{s+\frac{\gamma^2}{2}} - \frac{\gamma}{\sqrt{2}}\right)$ be the Laplace exponent of the inverse Gaussian subordinator, where $f(s) = \sqrt{2}\delta\left(\sqrt{s} - \frac{\gamma}{\sqrt{2}}\right)$. Then using the properties of inverse Laplace transform we get

$$L^{-1}\left[LU(s) = \frac{1}{\phi(s)}\right](t) = \exp\left(-\frac{\gamma^2 t}{2}\right) L^{-1}\left[\frac{1}{f(s)}\right](t), \tag{4.7}$$

where $L^{-1}[LU(s)]$ is the inverse Laplace transform of the potential measure related to the inverse Gaussian subordinator.

Now

$$\begin{aligned} L^{-1}\left[\frac{1}{f(s)}\right](t) &= L^{-1}\left[\frac{1}{\sqrt{2}\delta\left(\sqrt{s} - \frac{\gamma}{\sqrt{2}}\right)}\right](t) \\ &= \frac{1}{\sqrt{2}\delta}\left[\frac{1}{\sqrt{\pi t}} + \frac{\gamma}{\sqrt{2}}\exp\left(\frac{\gamma^2 t}{2}\right) erf\left(-\frac{\gamma\sqrt{t}}{\sqrt{2}}\right)\right] \end{aligned} \tag{4.8}$$

using formula 128 on page 16 of [37]. Thus from equations 4.7 and 4.8 we get

$$L^{-1}[LU(s)] = \frac{1}{\sqrt{2}\delta}\left[\frac{\exp\left(-\frac{\gamma^2 t}{2}\right)}{\sqrt{\pi t}} + \frac{\gamma}{\sqrt{2}} erf\left(-\frac{\gamma\sqrt{t}}{\sqrt{2}}\right)\right].$$

$\square$

The Lévy density associated with the inverse Gaussain subordinator is already known so Now we move to compute the asymptotic behaviour of Green function and Lévy density associated with the inverse Gaussian subordinator.

4.4 Normal Inverse Gaussian Process

The associated Lévy density of Normal inverse Gaussian process has a known exact form but not the associated Green function has so, we compute the asymptotic behaviour of Green function only using the first approach or the approach mentioned in [4].

Theorem 4.4.1. *For $\delta, \gamma > 0$, we have*

1. $G(x) \sim \frac{1}{2^{\frac{3}{2}}\pi^{\frac{d}{2}}\delta}\frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}|x|^{1-d}, |x| \to 0^+,$
2. $G(x) \sim \frac{\gamma}{4\pi^{\frac{d}{2}}\delta}\frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma(1)}|x|^{2-d}, |x| \to \infty.$

Proof.

1. As $\lambda \sim \infty$, $\phi(\lambda) \sim \delta\sqrt{2\lambda}$, then it follows directly from Theorem 3.1 of [6] that
$$G(x) \sim \frac{1}{2^{\frac{3}{2}}\pi^{\frac{d}{2}}\delta}\frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}|x|^{1-d}, |x| \to 0^+.$$
2. As $\lambda \to 0^+$, $\phi(\lambda) \sim \frac{\delta}{\gamma}\lambda$, then it follows directly from Theorem 3.3 of [6] that
$$G(x) \sim \frac{\gamma}{4\pi^{\frac{d}{2}}\delta}\frac{\Gamma\left(\frac{d-2}{2}\right)}{\Gamma(1)}|x|^{2-d}, |x| \to \infty.$$

$\square$

In this chapter, we mainly computed the asymptotic behaviour of potential density and Lévy density associated with the tempered stable subordinator and inverse Gaussian subordinator and also the asymptotic behaviour of Green function and Lévy density associated with the Brownian motion timed changed by tempered stable subordinator and inverse Gaussian process. We used two different approaches to prove the theorems in this chapter.

4.5 Future Work

We plan to rigorously study how harmonic, subharmonic and superharmonic functions of Classical Potential theory are related to martingale, submartinagle and supermartingale of Probability Theory and Stochastic processes. Further, we want to study how Transient Markov processes are related to Potential theory([19], [20]). Further, we are still trying to use the second approach to compute the asymptotic behaviour of the potential density associated with the tempered stable subordinator and inverse Gaussian subordinator and hence, for their associated Green function also. Precisely, we need to use Theorems 4.1.2, and 4.3.2 and Theorem 2.2.1(4), Chapter 2 to compute the asymptotic behaviour of the potential density associated with the tempered stable subordinator and inverse Gaussian subordinator.

It is a well known that marginals of a continuous-time Lévy process is always infinite divisible and we want to study the infinite divisibility of marginals of processes which are not Lévy and generally arise from subordination. Researchers have proved some non-Lévy processes to be non-infinite divisible and some to be divisible. We need more mathematical tools to prove the infinite divisibility of some non-Lévy processes and that is also one of the motivation behind studying the topic of the thesis. At the begining of the thesis work we thought by the end, we can prove infinite divisibility of some non-Lévy processes but due to time constrained we were unable to. However, hopefully in future we can prove infinite divisibility of some non-Lévy processes using the tools and techniques used in this thesis work. Brownian motion time changed with inverse stable subordinator, Fractional Brownian motion time changed with inverse stable subordinator, etc are some of the examples of non-Lévy processes which we will try to prove infinite divisible.

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Appendix

This section contain the python codes used to generate figure 2.1 and 2.2.

```
import numpy as np
import matplotlib.pyplot as plt
import math
from math import sin, pow, pi, log
import pandas as pd

def poipaths(N = 1000, lam = .01, paths =5):
for i in range(paths):
rng = np.arange(0,1, 1.0/N)
rvs = np.random.poisson(lam*1.0/N, N)
incr = list(rvs)
incr.insert(0,0.0)
rng = list(rng)
rng.append(1.0)
ar = np.array(incr)
cms = ar.cumsum()
ff = plt.plot(np.array(rng), cms)
plt.xlabel('t')
plt.ylabel('N(t)')
title = 'Poisson Process Sample Paths with arrival rate plt.title(title)
plt.show(ff)

def bmpaths(N = 1000, paths =5):
for i in range(paths):
rng = np.arange(0, 1, 1.0/N)
rvs = np.random.normal(0, 1, N)
incr = list(map(lambda x: x*math.sqrt(1.0/N), rvs))
incr.insert(0,0.0)
rng = list(rng)
rng.append(1.0)
ar = np.array(incr)
cms = ar.cumsum()
```

```
ff = plt.plot(np.array(rng), cms)
ub = list(map(lambda x: 3*math.sqrt(x), rng))
lb = list(map(lambda x: -3*math.sqrt(x), rng))
plt.plot(np.array(rng), ub)
plt.plot(np.array(rng), lb)
plt.xlabel('t')
plt.ylabel('B(t)')
plt.title('Standard Brownian Motion Sample Paths')
plt.show(ff)
```