Commutator subgroups of some generalized braid groups

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A thesis submitted for the partial fulfilment of the degree of Doctor of Philosophy

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dedicated

to

all my teachers

বিশ্বজোড়া পাঠশালা মোর,
সবার আমি ছাত্র
--- সুনির্মল বসু

English translation of the above Bengali verse:

" The whole world is my school,
I am student of everyone."

— by Sunirmal Basu

Declaration

The work presented in this thesis has been carried out by me under the guidance of Dr. Krishnendu Gongopadhyay at Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Date:

Place:

Soumya Dey

In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Krishnendu Gongopadhyay
(Supervisor)

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“ Whether you pushed me or pulled me, drained me or fueled me,
loved me or left me, hurt me or helped me, you are part of my growth
and no kidding, I thank you. ”

— copied from Facebook.

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Abstract of the thesis

Commutator subgroups of Artin's braid groups B_n are well studied by Gorin and Lin in their 1969 paper, where they obtained finite presentation for B_n' for each n. Later, in 1993, Savushkina gave a simpler presentation for B_n'.

The goal of this thesis is to understand the structure of the commutator subgroups of some of the generalizations of Artin's braid groups B_n, namely the welded braid groups WB_n, the generalized virtual braid groups GVB_n, the flat welded braid groups FWB_n, the flat virtual braid groups FVB_n, and the twin groups TW_n. As consequences of the above investigations we prove several algebraic and geometric properties of the above groups.

Contents

CHAPTER 1

Summary of the thesis

The *commutator subgroup* or *derived subgroup* of a group G is the subgroup G' generated by the elements of the form $x^{-1}y^{-1}xy$. The group G' is the smallest normal subgroup of G that abelianizes G, i.e. a quotient group G/N is abelian if and only if $G' \leq N$. Thus the group G' distinguishes the abelian factor groups of G from the non-abelian ones. The group G' is one measure to know how far is G from being abelian. The quotient G/G' is also isomorphic to $H_1(G, \mathbb{Z})$, the first homology group of G with integral coefficients. The commutator subgroup G' is also the smallest normal subgroup of G that is invariant under every automorphism of G. Given any group G, the structure of the commutator subgroup G' is thus a very crucial information about G. Applying this information, one may also try to get various geometric information, e.g. commutator width, of the group G.

The braid group on n strands, classically known as Artin's braid group, is a central object of investigation due to its appearances in several branches of mathematics; for details refer to the surveys [**Par09, BB05**]. The commutator subgroup B_n' of Artin's braid group on n strands B_n is well-studied. Gorin and Lin, in their 1969 paper [**GL69**], obtained a finite presentation for B_n' for each n. Simpler presentation for B_n' was obtained by Savushkina in [**Sav93**]. Several authors have investigated commutator subgroups of larger classes of spherical Artin groups, e.g. [**Zin75**], [**MR**], [**Ore12**]. Also refer [**GG11**].

In this thesis, we ask for the commutator subgroups of some of the generalized braid groups. There have been several generalizations of Artin's braid groups in the literature that appeared in different contexts. Many of these generalized braid groups are of importance in their own rights, and are topics of active research. For example, see [**Bel04**], [**BG12**], [**Dam17**], [**Ver06**], [**Ver14**] for some of the research directions on several avatars of the classical braid groups. In this context, it is a natural question to investigate structures of the commutator subgroups of these generalized braid groups.

In this thesis, we have investigated the commutator subgroups of four important classes of generalized braid groups, viz. the welded braid groups, the generalized virtual braid groups introduced by Fang, the flat quotients of the virtual (and welded) braid groups, and the twin groups or Grothendieck's cartographical groups. We have also applied the description of the commutator subgroups to extract some algebraic and geometric properties of the ambient groups.

In each of the above cases, we use the tool known as Reidemeister-Schreier method to obtain a presentation for the commutator subgroup. Exploiting this presentation in the respective cases we deduce several results about the commutator subgroups.

In the following we briefly describe the main results which are obtained in this thesis.

Welded Braid Groups. The *welded braid groups* WB_n are certain generalization of the Artin's braid groups. These groups have appeared in several contexts in the literature, often with different names, e.g. *loop braid groups*, *permutation braid groups*, *symmetric automorphisms of free groups*; see [**BWC07**], [**Col89**], [**FRR97**], [**Kau99**]. We refer to the recent survey article by Damiani [**Dam17**] for further details on different definitions and applications of these groups.

We investigate the commutator subgroup of WB_n in Chapter 3. We prove the following theorem.

Theorem 3.1: ([DG18a]) Let WB_n' denote the commutator subgroup of the welded braid group WB_n.

(i) WB_n' is a finitely generated group for all $n \geq 3$. For $n \geq 7$, the rank of WB_n' is at most $1 + 2(n-3)$, and for $3 \leq n \leq 6$, the rank is at most $4 + 2(n-3)$.

(ii) For $n \geq 5$, WB_n' is perfect.

Using the above theorem, we prove the following corollary.

Corollary 3.2: ([**DG18a**]) For any $n \geq 3$, WB_n' is Hopfian.

Another consequence of the above theorem is the following.

Corollary 3.3: ([**DG18a**]) For a free group F_k, the image of any nontrivial homomorphism $\phi : WB_n \to F_k$ is infinite cyclic.

Applying Roushon's results in [**Rou02, Rou04**], with part (ii) of the above theorem, we have the following.

Corollary 3.4: ([**DG18a**]) For $n \geq 5$, WB_n is adorable of degree 1, and for $n = 3, 4$, WB_n is not adorable.

Therefore, by Theorem 3.1 and Corollary 3.4 we immediately have the following.

Corollary 3.5: ([**DG18a**]) The group WB_n' is perfect if and only if $n \geq 5$.

This generalizes the fact that B_n is adorable of degree 1 for $n \geq 5$. It is easy to see that if $f : G \to H$ is a surjective homomorphism with G adorable, then H is also adorable and $doa(H) \leq doa(G)$, where $doa(G)$ denotes the degree of adorability, see [**Rou04**, Lemma 1.1]. It follows from [**BB09**, Proposition 8] that the commutator subgroup VB_n' of the virtual braid group VB_n is perfect for $n \geq 5$. Thus, for $n \geq 5$, VB_n is adorable of degree 1. The welded braid groups being quotients of these groups, are also adorable with degree ≤ 1. This gives a proof of the fact that WB_n' is perfect for $n \geq 5$. We have given a direct proof of this fact using the presentation for WB_n'. Further we have shown that WB_n' is not perfect for $n \leq 4$ to establish Corollary 3.5.

In a recent work of Zaremsky [**Zar18**], using techniques from Morse theory of complex symmetric graphs, the finite presentability of WB_n' for $n \geq 4$ is proved, see [**Zar18**, Theorem B]. The finite generation of WB_n' for $n \geq 3$ is also implicit in this work. However it would be an interesting problem to obtain an explicit finite presentation for WB_n', for $n \geq 4$, which remains open.

Fang's Generalized Virtual Braid Groups. The *virtual braid groups* VB_n are a generalization of the Artin's braid groups B_n, and an extension of the welded braid groups WB_n. It was introduced by L. H. Kauffman in [**Kau99**]. In [**Fan15**], Fang introduced *generalized virtual braid groups* GVB_n, which simultaneously generalize the notion of Artin's braid groups, as well as the virtual braid groups. Fang constructed this generalization as a group of symmetries behind quantum quasi-shuffle structures.

We have investigated commutator subgroup of GVB_n in Chapter 4. We prove the following.

Theorem 4.1: ([DG18c]) Let GVB_n' denote the commutator subgroup of GVB_n.

(i) GVB_n' is finitely generated for all $n \geq 4$. Further, for $n \geq 5$, rank of GVB_n' is at most $3n - 7$.

(ii) GVB_3' is not finitely generated.

(iii) GVB_n' is perfect if and only if $n \geq 5$.

Flat Welded (and Virtual) Braid Groups. We investigate the commutator subgroups of the *flat welded braid group on n strands* FWB_n and the *flat virtual braid groups on n strands* FVB_n in Chapter 5.

We deduce explicit finite presentations for FVB_n' and FWB_n' in **Theorem 5.3**. ([**DG18a**])

We also have the following observation.

Proposition 5.1: ([**DG18a**]) The flat virtual braid group FVB_n and the flat welded braid group FWB_n are adorable groups of degree 1 for $n \geq 5$; i.e. commutator subgroups of these groups are perfect for $n \geq 5$.

Twin Groups or Grothendieck's Cartographical Groups. In [**Kho97**], Khovanov investigated the doodle groups, and introduced the *twin group on n arcs*, denoted by TW_n. The role of this group in the theory of 'doodles' on a closed oriented surface is similar to the role of Artin's braid groups in the theory of knots and links. Khovanov proved that the closure of a twin is a doodle on the (2 dimensional) sphere; see [**Kho97**] for details.

For $m \geq 1$, the group TW_{m+2} is isomorphic to Grothendieck's m-dimensional cartographical group $\mathcal{C}_m$; hence this group is of importance in Grothendieck's theory of 'dessins d'enfant'. Voevodsky used this group in [**Voe90**] as a generalization of the 2-dimensional cartographical group. It is a standard fact in this theory that the conjugacy classes of the 2-dimensional cartographical group $\mathcal{C}_2$ can be identified with combinatorial maps on connected surfaces, not necessarily orientable or without boundary, see [**JS94**] for more details. In [**Vin83a, Vin83b**], Vince looked at the group $\mathcal{C}_m$ as 'combinatorial maps' and investigated certain topological and combinatorial structures associated to this group.

We investigate the commutator subgroup of the group TW_n in Chapter 6. It's easy to see that TW_n' is finitely presented. But, in general, it is a difficult problem to obtain a finite presentation for a finitely presented group, and sometimes it is algorithmically impossible as well, see [**BW11**]. So, knowing that TW_n' is finitely presented is not enough to have a clear understanding about the structure of the group. We obtain an explicit finite presentation for TW_n'. We prove the following theorem.

Theorem 6.1: ([**DG18b**]) For $m \geq 1$, TW_{m+2}' has the following presentation:

Generators: $\quad \beta_p(j), \qquad 0 \leq p < j \leq m.$

Defining relations: For all $l \geq 3,\ 1 \leq k \leq j,\ j+2 \leq t \leq m$,

$$\beta_{j-k}(j)\ \beta_{t-(j+l)}(t) = \beta_{t-(j+l)}(t)\ \beta_{j-k}(j),$$

$$\beta_{t-k}(t) = \beta_{j-k}(j)^{-1}\ \beta_{t-(j+1)}(t)\ \beta_{j-k}(j).$$

Even if a group is finitely generated, it is a nontrivial problem to compute its rank. As an application of Theorem 6.1, we obtain the rank of TW_n' as follows.

Theorem 6.2: ([DG18b]) For $m \geq 1$, the group TW_{m+2}' has rank $2m-1$.

In [**PV16**], Panov and Verëvkin constructed classifying spaces for the commutator subgroups of right-angled Coxeter groups and have given a general formula for the rank of such groups, see [**PV16**, Theorem 4.5]. However, the number of minimal generators given in [**PV16**] is in general form and involves the rank of the zeroth homology groups of certain subcomplexes of the underlying classifying space. In our case, the rank obtained is in terms of the number of 'arcs' of the twin group, or the 'dimension' of the cartographical group; and thus it is more explicit.

The following is a consequence of Theorem 6.1 and Theorem 6.2.

Corollary 6.3: ([DG18b]) For $m \geq 1$, the quotient group TW_{m+2}'/TW_{m+2}'', is isomorphic to the free abelian group of rank $2m-1$, i.e. the group $\bigoplus_{i=1}^{2m-1} \mathbb{Z}$. In particular, TW_{m+2}' is not perfect for any $m \geq 1$.

We characterize freeness of TW_n' in the following corollary.

Corollary 6.4: ([DG18b]) TW_{m+2}' is a free group if and only if $m \leq 3$. The group TW_3' is infinite cyclic. The groups TW_4' and TW_5' are free groups of rank 3 and 5 respectively.

As applications of the above results, we also derive few geometric properties of the ambient group TW_n.

We prove the following characterization for word-hyperbolicity of TW_n.

Corollary 6.5: ([DG18b]) The group TW_{m+2} is word-hyperbolic if and only if $m \leq 3$.

We prove the following corollary.

Corollary 6.6: ([**DG18b**]) The group TW_{m+2} does not contain a surface group if and only if $m \leq 3$.

We also prove the following.

Corollary 6.7: ([**DG18b**]) The automorphism group of TW_{m+2} is finitely presented for $m \leq 3$.

CHAPTER 2

Preliminaries

1. Presentation for a group

In this section we will recall the definition of a presentation for a group and discuss some examples.

Let G be a group. A subset S of G is said to be a *set of generators* for G if for any element $g \in G$, g is equal to a product of some of the elements of $S \cup S^{-1}$. Here $S^{-1} = \{s^{-1} \mid s \in S\} \subset G$ and an element may occur multiple times in the product. A product of the form $s_1^{\epsilon_1} s_2^{\epsilon_2} \dots s_k^{\epsilon_k}$, where $s_1, s_2, \dots, s_k \in S$ and $\epsilon_1, \epsilon_2, \dots, \epsilon_k \in \{1, -1\}$, is called a *word* in S. Note that, two distinct words in S may represent the same element in G.

Note that a group can have many different sets of generators. For example, consider the symmetric group on 3 letters S_3. Clearly S_3 itself is a set of generators for S_3. The reader may check that each of the following 4 different subsets of S_3 is a set of generators for S_3:

$\{ (1\ 2),\ (1\ 2\ 3) \}$, $\{ (1\ 2),\ (2\ 3) \}$, $\{ (1\ 2),\ (1\ 3) \}$, $\{ (1\ 2),\ (1\ 3),\ (2\ 3) \}$.

A group G is called *finitely generated* if there exists a finite subset S of G such that S is a set of generators for G. If none of the sets of generators for G is finite then G is called *infinitely generated.*

The *rank* of a finitely generated group G is the smallest non-negative integer n such that G has a set of generators S with cardinality of S being n.

Suppose that, a group G and a set of generators for G, say S, are given. We may consider the free group on the set S, denoted as $F(S)$. Clearly, the inclusion map $\phi : S \to G$ i.e. $\phi(s) = s$, $\forall s \in S$, extends to an epimorphism $\Phi : F(S) \to G$. Hence, if we denote the kernel of Φ by N, we have: $G \cong F(S)/N$. Suppose that, for a set $R \subset F(S)$, N is the normal closure of R in $F(S)$. Then the expression $\langle\ S \mid R\ \rangle$ is called a *presentation* for G. Also in this case, the set R is called a *set of* *defining relators* for G with respect to the set of generators S. For defining relators $r_\mu \in R$ we have corresponding *defining relations* $r_\mu = 1$.

Note that, a group can have many different presentations. Further, for a given set of generators for a group, there can be many different sets of defining relators, giving distinct presentations for the group. For example, $\langle\ a,\ b \mid a^3,\ b^2,\ abab^{-1}\ \rangle$ is a presentation for S_3, where $a = (\ 1\ 2\ 3\ )$, $b = (\ 1\ 2\ )$. Also, for the same set of generators $\{\ a,\ b\ \}$ we can construct $\langle\ a,\ b \mid ab^2a^2,\ a^2(b^2a^3)^4a,\ b^3a^4ba\ \rangle$ which is another presentation for S_3. For $c = (\ 1\ 3\ )$, $d = (\ 2\ 3\ )$, we have $\langle\ c,\ d \mid c^2,\ d^2,\ (cd)^3\ \rangle$ as yet another presentation for S_3.

For a group G, if there exists a presentation $\langle\ S \mid R\ \rangle$ for G with both S and R being finite sets, then we say that G is *finitely presented*.

2. Reidemeister-Schreier method

We shall discuss about the technical tool called the *Reidemeister-Schreier method* developed by Kurt Reidemeister and Otto Schreier, which has been extensively used in the later chapters. For details on the history of this method, refer [**CM82**]. If a presentation for a group G is given, this efficient algorithm provides a way to find a presentation for a subgroup of G. It is well known, as Schreier's lemma, that a finite index subgroup of a finitely generated group is finitely generated. The fact that a finite index subgroup of a finitely presented group is finitely presented, can be proved using the Reidemeister-Schreier method. But, in general, the property of being finitely generated (and also being finitely presented) is not a subgroup-closed property.

Definition 2.1. Let G be a group and H be a subgroup of G. Suppose that, $\langle\ S \mid R\ \rangle$ be a presentation for G. A set Λ consisting of some words in the generators from S is called a *Schreier set of coset representatives* for H in G if

(i) every right coset of H in G contains exactly one word from Λ, and

(ii) for each word in Λ any initial segment of that word is also in Λ.

THEOREM 2.2 (Schreier, 1927). *For every $H \leq G$, a Schreier set of coset representatives for H in G exists.*

We describe the Reidemeister-Schreier method as an algorithm. Let $H \leq G$, and $\langle\ S \mid R\ \rangle$ be a presentation for G. In order to deduce a presentation for H we proceed as follows.

Step 1: Find Λ, a Schreier set of coset representatives for H in G. Although the existence is guaranteed by the above theorem of Schreier, in general, it is not easy to actually find such a Schreier set.

Step 2: Deduce a set of generators $\{\ S_{\lambda,a} \mid \lambda \in \Lambda,\ a \in S\ \}$ for H defined by:

$$S_{\lambda,a} = (\lambda a)(\overline{\lambda a})^{-1},$$

where for any $x \in G$, $\overline{x} \in \Lambda$ denotes the unique element in $\Lambda \cap Hx$.

Step 3: Compute the defining relators $\tau(\lambda\ r_\mu\ \lambda^{-1})$ for H, for all $\lambda \in \Lambda$, and all the defining relators $r_\mu \in R$, where τ, called the *rewriting process*, is defined as follows. For a word $a_{i_1}^{\epsilon_1} \dots a_{i_p}^{\epsilon_p}$ in the generators from S, with $\epsilon_j = 1$ or -1 for $1 \leq j \leq p$,

$$\tau(\ a_{i_1}^{\epsilon_1}\ \dots\ a_{i_p}^{\epsilon_p}\ ) \ := \ S_{K_{i_1},a_{i_1}}^{\epsilon_1}\ \dots\ S_{K_{i_p},a_{i_p}}^{\epsilon_p},$$

$$\text{where } \ K_{i_j} = \begin{cases} \overline{a_{i_1}^{\epsilon_1} \dots a_{i_{j-1}}^{\epsilon_{j-1}}} & \text{if } \epsilon_j = 1, \\ \\ \overline{a_{i_1}^{\epsilon_1} \dots a_{i_j}^{\epsilon_j}} & \text{if } \epsilon_j = -1. \end{cases}$$

The above algorithm will produce a presentation $\langle\ \overline{S} \mid \overline{R}\ \rangle$ for H where we have $\overline{S} = \{\ S_{\lambda,a} \mid \lambda \in \Lambda,\ a \in S\ \}$ and $\overline{R} = \{\ \tau(\lambda\ r_\mu\ \lambda^{-1}) \mid \lambda \in \Lambda,\ r_\mu \in R\ \}$.

Let us take a simple example to understand the above algorithm. Consider the symmetric group S_3 along with the presentation $\langle\ a,\ b \mid a^3,\ b^2,\ abab^{-1}\ \rangle$. We will apply Reidemeister-Schreier algorithm on this presentation in order to deduce a presentation for the alternating group $A_3 \leq S_3$, which is the commutator subgroup of S_3. So we have $G = S_3,\ H = S_3'$. It's easy to see that G/H, i.e. the abelianization of G has 2 elements.

Step 1: Consider the set $\Lambda = \{\ 1,\ b\ \}$. We can easily check that Λ is a Schreier set of coset representatives for H in G.

Step 2: We calculate the generators $S_{\lambda,t}$ for all $\lambda \in \Lambda,\ t \in S = \{\ a,\ b\ \}$. Note that we have, $\overline{a} = 1,\ \overline{b} = b,\ \overline{b\ a} = b$ in Λ. So we get the following.
$S_{1,a} = 1\ .\ a\ .\ (\ \overline{1\ .\ a}\ )^{-1} = a\ .\ (\overline{a})^{-1} = a,\ \ S_{1,b} = 1\ .\ b\ .\ (\ \overline{1\ .\ b}\ )^{-1} = b\ .\ (\overline{b})^{-1} = 1,$
$S_{b,a} = b\ .\ a\ .\ (\ \overline{b\ .\ a}\ )^{-1} = bab^{-1},\ \ S_{b,b} = b\ .\ b\ .\ (\ \overline{b\ .\ b}\ )^{-1} = 1\ .\ (\ \overline{1}\ )^{-1} = 1.$

Step 3: We now calculate the defining relations for S_3' as follows.
For $\lambda = 1$, we get:

$$\tau(a^3) = \tau(a.a.a) = S_{1,a}\ S_{\overline{a},a}\ S_{\overline{a^2},a} = S_{1,a}^3$$

$$\tau(b^2) = \tau(b.b) = S_{1,b}\ S_{\overline{b},b} = 1$$

$$\tau(abab^{-1}) = S_{1,a}\ S_{\overline{a},b}\ S_{\overline{ab},a}\ S^{-1}_{\overline{abab^{-1}},\ b} = S_{1,a}\ S_{1,b}\ S_{b,a}\ S_{1,b}^{-1} = S_{1,a}\ S_{b,a}$$

For $\lambda = b$, we get:

$$\tau(ba^3b^{-1}) = \tau(b.a.a.a.b^{-1}) = S_{1,b}\ S_{\overline{b},a}\ S_{\overline{ba},a}\ S_{\overline{ba^2},a}\ S^{-1}_{\overline{ba^3b^{-1}},\ b} = S_{b,a}^3$$

$$\tau(bb^2b^{-1}) = \tau(b.b.b.b^{-1}) = S_{1,b}\ S_{\overline{b},b}\ S_{\overline{b^2},b}\ S^{-1}_{\overline{b.b.b.b^{-1}},\ b} = 1$$

$$\tau(babab^{-1}b^{-1}) = S_{1,b}\ S_{b,a}\ S_{\overline{ba},b}\ S_{\overline{bab},a}\ S^{-1}_{\overline{babab^{-1}},\ b}\ S^{-1}_{\overline{babab^{-1}b^{-1}},\ b} = S_{b,a}\ S_{1,a}$$

Thus, we have deduced a presentation for A_3, i.e. S_3', as follows:

$$\langle\ S_{1,a},\ S_{b,a}\ |\ S_{1,a}^3,\ S_{b,a}^3,\ S_{1,a}\ S_{b,a},\ S_{b,a}\ S_{1,a}\ \rangle \tag{2.1}$$

Here is an important observation to make. Suppose we start with a finite presentation $\langle\ S\ |\ R\ \rangle$ for a group G. It may well so happen that there is a finitely generated subgroup H of G which is of infinite index in G; and hence the Schreier set of coset representatives Λ for H in G is infinite. As a result the Reidemeister-Schreier method might produce a presentation $\langle\ \overline{S}\ |\ \overline{R}\ \rangle$ for H with $\overline{S}$ being an infinite set. So, the Reidemeister-Schreier method, in general, does not conclude anything about the finite generation of the subgroup.

To further investigate about finite generation and finite presentability of the subgroup H, the Tietze transformations are quite helpful techniques, which are discussed in the following section.

3. The Tietze transformations

Introduced by Heinrich Franz Friedrich Tietze in a paper in 1908, Tietze transformations are efficient techniques to transform a given presentation for a group to another, often simpler, presentation for the same group.

Let $\langle\ S \mid R\ \rangle$ be a presentation for G. So, by definition, we have $G \cong F(S)/N$ where N is the normal closure of R in $F(S)$. We define the 4 types of Tietze transformations $T_1,\ T_2,\ T_3,\ T_4$ as follows.

(T_1): Adding defining relator: If a word $u \in F(S)$ is not in R, but it can be derived from the defining relators in R, i.e. $u \in N$, then inserting u into the set of defining relators will produce an equivalent presentation $\langle\ S \mid R \cup \{u\}\ \rangle$ for G.

(T_2): Removing defining relator: If a defining relator $r \in R$ can be derived from other defining relators in R, i.e. r belongs to the normal closure of $R - \{r\}$ in $F(S)$, then r can be removed from R to form an equivalent presentation $\langle\ S \mid R - \{r\}\ \rangle$ for G.

(T_3): Adding generator: If $a \in G$, i.e. $a = s_{i_1}^{\epsilon_1} \dots s_{i_p}^{\epsilon_p}$ for some elements $s_{i_1}, \dots, s_{i_p} \in S$ with $\epsilon_j = 1$ or -1 for $1 \leq j \leq p$, but $a \notin S$, then inserting a in the set of generators and inserting $a^{-1}\ s_{i_1}^{\epsilon_1} \dots\ s_{i_p}^{\epsilon_p}$ in the set of defining relators will produce an equivalent presentation $\langle\ S \cup \{a\} \mid R \cup \{a^{-1} s_{i_1}^{\epsilon_1} \dots s_{i_p}^{\epsilon_p}\}\ \rangle$ for G.

(T_4): Removing generator: If a generator $s \in S$ can be expressed in terms of some other generators using a defining relation, i.e. there is a defining relation which gives $s = s_{i_1}^{\epsilon_1} \dots s_{i_p}^{\epsilon_p}$, where $s,\ s_{i_1},\ \dots,\ s_{i_p} \in S$ with $\epsilon_j = 1$ or -1 for $1 \leq j \leq p$, and $s \notin \{\ s_{i_1},\ \dots,\ s_{i_p}\ \}$, then removing the corresponding defining relator from the set of defining relators after replacing s by $s_{i_1}^{\epsilon_1} \dots s_{i_p}^{\epsilon_p}$ wherever

s occurs in other relators and removing s from the set of generators will produce an equivalent presentation $\langle\ S - \{s\} \mid R'\ \rangle$ for G.

For example, consider the presentation (2.1) for A_3 that we deduced in the last section. Clearly the defining relator $S_{b,a}\ S_{1,a}$ gives the defining relation $S_{b,a} = S_{1,a}^{-1}$. Using this we can apply Tietze transformation T_4 on the presentation (2.1) through removing $S_{b,a}$ from the set of generators by replacing it with $S_{1,a}^{-1}$ in other relations. The obtained simpler presentation for A_3 is as follows.

$$\langle\ S_{1,a} \mid S_{1,a}^3,\ S_{1,a}^{-3}\ \rangle \tag{3.1}$$

Finally we apply Tietze transformation T_2 on the presentation (3.1) through removing $S_{1,a}^{-3}$ from the set of defining relators, as it can be derived from the defining relator $S_{1,a}^3$ by taking inverse. Hence we get a simpler presentation for A_n as follows.

$$\langle\ S_{1,a} \mid S_{1,a}^3\ \rangle \tag{3.2}$$

4. Adorability of groups

Motivated by the covering theory of aspherical 3-manifolds, Roushon defined the notion of an adorable group as follows.

Definition 2.3. A group G is called *adorable* if $G^i/G^{i+1} = 1$ for some i, where $G^i = [\ G^{i-1},\ G^{i-1}\ ]$ and $G^0 = G$ are the terms in the derived series of G. The smallest i for which the above property holds, is called the *degree of adorability* of G, denoted by $doa(G)$.

Clearly, we have $G^1 = [G, G] = G'$, the commutator subgroup of G.

Recall that, a group G is called *perfect* if $G' = G$.

Note that, a group G is adorable of degree 1 if and only if the commutator subgroup G' is perfect.

Examples of adorable groups include finite groups, simple groups and solvable groups. Examples of non-adorable groups include nonabelian free groups and fundamental groups of surfaces of genus greater than 1.

Let $G,\ H$ be any two groups and $f : G \to H$ be a surjective homomorphism with G adorable. Then H is also adorable and $doa(H) \leq doa(G)$.

For more details on adorability of groups, see [**Rou02, Rou04**].

CHAPTER 3

Welded Braid Groups

In this chapter, we investigate about the commutator subgroups of the welded braid groups.

1. Presentation for WB_n

The *welded braid group on n strands* WB_n is generated by a set of $2(n-1)$ generators: $\{\sigma_i, \rho_i,\ i = 1, 2, \ldots, n-1\}$ satisfying the following set of defining relations:

(1) *The braid relations*:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1;$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1};$$

(2) *The symmetric relations*:

$$\rho_i^2 = 1;$$

$$\rho_i \rho_j = \rho_j \rho_i, \text{ if } |i-j| > 1;$$

$$\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1};$$

(3) *The mixed relations*:

$$\sigma_i \rho_j = \rho_j \sigma_i, \text{ if } |i-j| > 1;$$

$$\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1};$$

(4) *The forbidden relations*:

$$\rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}.$$

2. Goal of the chapter

We prove the following:

THEOREM 3.1. *Let WB_n' denote the commutator subgroup of the welded braid group WB_n.*

(i) *WB_n' is a finitely generated group for all $n \geq 3$. For $n \geq 7$, the rank of WB_n' is at most $1 + 2(n-3)$, and for $3 \leq n \leq 6$, the rank is at most $4 + 2(n-3)$.*

(ii) *For $n \geq 5$, WB_n' is perfect.*

We note here that, $WB_2 \cong F_2 \rtimes S_2$. So, it's commutator subgroup WB_2' is infinitely generated.

Recall that a group G is called *Hopfian* if every epimorphism $G \to G$ is an isomorphism. In general, being Hopfian is not a subgroup-closed group property. Using Theorem 3.1, we prove the following.

COROLLARY 3.2. *For any $n \geq 3$, WB_n' is Hopfian.*

Another consequence of Theorem 3.1 is the following.

COROLLARY 3.3. *For a free group F_k, the image of any nontrivial homomorphism $\phi : WB_n \to F_k$ is infinite cyclic.*

We also prove the following.

COROLLARY 3.4. *For $n \geq 5$, WB_n is adorable of degree 1, and for $n = 3, 4$, WB_n is not adorable.*

Therefore, by Theorem 3.1 and Corollary 3.4 we immediately have the following.

COROLLARY 3.5. *The group WB_n' is perfect if and only if $n \geq 5$.*

We prove Theorem 3.1 by using the Reidemeister-Schreier method and Tietze transformations. In Section 3 we compute a set of generators for WB_n'. In Section 4 we deduce a set of defining relations for WB_n'. In Section 5 we simplify the obtained presentation. Proofs of Theorem 3.1 and corollaries are covered in Section 6 and Section 7.

3. A set of generators for WB'_n

In this section, we use Reidemeister-Schreier method to deduce a set of generators for WB'_n.

For $n \geq 3$, define the map ϕ:

$$1 \to WB'_n \to WB_n \xrightarrow{\phi} \mathbb{Z} \times \mathbb{Z}_2 \to 1$$

where, for $i = 1, \ldots, n-1$, $\phi(\sigma_i) = \overline{\sigma_1}$, $\phi(\rho_i) = \overline{\rho_1}$; here $\overline{\sigma_1}$ and $\overline{\rho_1}$ are the generators of $\mathbb{Z}$ and $\mathbb{Z}_2$ respectively when viewing it in the abelianization of WB_n.

Here, Image(ϕ) is isomorphic to the abelianization of WB_n, denoted as WB_n^{ab}. To prove this, we abelianize the above presentation of WB_n by inserting the relations $xy = yx$ in the presentation for all $x, y \in \{ \sigma_i, \rho_i \mid 1 \leq i \leq n-1 \}$. The resulting presentation is the following:

$$WB_n^{ab} = < \sigma_1, \rho_1 \mid \sigma_1\rho_1 = \rho_1\sigma_1,\ \rho_1^2 = 1 >$$

Clearly, WB_n^{ab} is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$. But as ϕ is onto, Image(ϕ) $= \mathbb{Z} \times \mathbb{Z}_2$. Hence, Image($\phi$) is isomorphic to WB_n^{ab}. Hence, ϕ defines the above short exact sequence and ϕ does have a section.

Lemma 3.6. *WB'_n is generated by $\alpha_{m,\epsilon,i} = \sigma_1^m\rho_1^\epsilon\sigma_i\rho_1^\epsilon{\sigma_1^{-1}}\sigma_1^{-m}$ and $\beta_{m,\epsilon,i} = \sigma_1^m\rho_1^\epsilon\rho_i\rho_1\rho_1^\epsilon\sigma_1^{-m}$, where $m \in \mathbb{Z}$, $\epsilon \in \{0,1\}$, $1 \leq i \leq n-1$.*

Proof. Consider a Schreier set of coset representatives:

$$\Lambda = \{\sigma_1^m\rho_1^\epsilon \mid m \in \mathbb{Z}, \epsilon \in \{0,1\}\}.$$

The Reidemeister-Schreier method tells us that WB'_n is generated by the set

$$\{S_{\lambda,a} = (\lambda a)(\overline{\lambda a})^{-1} \mid \lambda \in \Lambda,\ a \in \{\sigma_i, \rho_i \mid i = 1, 2, \ldots, n-1\}\}.$$

Choose $\lambda = \sigma_1^m\rho_1^\epsilon$ from Λ.

For $a = \sigma_i$, $S_{\lambda,a} = \sigma_1^m\rho_1^\epsilon\sigma_i\rho_1^\epsilon{\sigma_1^{-1}}\sigma_1^{-m}$. For $a = \rho_i$, $S_{\lambda,a} = \sigma_1^m\rho_1^\epsilon\rho_i\rho_1\rho_1^\epsilon\sigma_1^{-m}$.

Hence, WB'_n is generated by the following elements:

$$\alpha_{m,\epsilon,i} = S_{\sigma_1^m\rho_1^\epsilon,\sigma_i} = \sigma_1^m\rho_1^\epsilon\sigma_i\rho_1^\epsilon{\sigma_1^{-1}}\sigma_1^{-m},$$

$$\beta_{m,\epsilon,i} = S_{\sigma_1^m\rho_1^\epsilon,\rho_i} = \sigma_1^m\rho_1^\epsilon\rho_i\rho_1\rho_1^\epsilon\sigma_1^{-m},$$

where $m \in \mathbb{Z}$, $\epsilon \in \{0,1\}$, $1 \leq i \leq n-1$. □

4. A set of defining relations for WB'_n

To obtain defining relations for WB'_n, following the Reidemeister-Schreier method, we apply re-writing process τ on $\lambda r_\mu \lambda^{-1}$, for all $\lambda \in \Lambda$, and r_μ the defining relators for WB_n:

$$r_1 = \sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1},\ |i-j|>1;$$
$$r_2 = \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1};$$
$$r_3 = \rho_i^2;$$
$$r_4 = \rho_i \rho_j \rho_i \rho_j,\ |i-j|>1;$$
$$r_5 = \rho_i \rho_{i+1} \rho_i \rho_{i+1} \rho_i \rho_{i+1};$$
$$r_6 = \sigma_i \rho_j \sigma_i^{-1} \rho_j,\ |i-j|>1;$$
$$r_7 = \rho_i \rho_{i+1} \sigma_i \rho_{i+1} \rho_i \sigma_{i+1}^{-1};$$
$$r_8 = \rho_i \sigma_{i+1} \sigma_i \rho_{i+1} \sigma_i^{-1} \sigma_{i+1}^{-1}.$$

We have the following lemma.

Lemma 3.7. *The generators $\alpha_{k,\mu,r}, \beta_{k,\mu,r}$, $k \in \mathbb{Z}$, $\mu \in \{0,1\}$, $1 \leq r \leq n-1$, of WB'_n satisfy the following set of defining relations:*

$$\alpha_{k,\mu,r}\ \alpha_{k+1,\mu,s}\ \alpha_{k+1,\mu,r}^{-1}\ \alpha_{k,\mu,s}^{-1} = 1,\ |r-s|>1; \tag{4.1}$$

$$\alpha_{k,\mu,r}\ \alpha_{k+1,\mu,r+1}\ \alpha_{k+2,\mu,r} = \alpha_{k,\mu,r+1}\ \alpha_{k+1,\mu,r}\ \alpha_{k+2,\mu,r+1}; \tag{4.2}$$

$$\beta_{k,\mu,r}\ \beta_{k,1-\mu,r} = 1; \tag{4.3}$$

$$(\beta_{k,\mu,r}\ \beta_{k,\mu,s})^2 = 1,\ |r-s|>1,\ r,s \geq 2; \tag{4.4}$$

$$(\beta_{k,\mu,r}\ \beta_{k,\mu,r+1})^3 = 1; \tag{4.5}$$

$$\alpha_{k,\mu,r}\ \beta_{k+1,1-\mu,s}\ \alpha_{k,1-\mu,r}^{-1}\ \beta_{k,\mu,s} = 1,\ |r-s|>1; \tag{4.6}$$

$$\alpha_{k,\mu,r}\ \beta_{k+1,\mu,r+1}\ \beta_{k+1,1-\mu,r}\ \alpha_{k,\mu,r+1}^{-1}\ \beta_{k,\mu,r}\ \beta_{k,1-\mu,r+1} = 1; \tag{4.7}$$

$$\alpha_{k,\mu,r+1}\ \alpha_{k+1,\mu,r}\ \beta_{k+2,\mu,r+1}\ \alpha_{k+1,1-\mu,r}^{-1}\ \alpha_{k,1-\mu,r+1}^{-1}\ \beta_{k,\mu,r} = 1; \tag{4.8}$$

$$\alpha_{k,0,1} = 1,\ k \in \mathbb{Z}; \tag{4.9}$$

(4.10) $$\alpha_{k,\mu,r} = \alpha_{0,0,r}\ ,\ \ k \in \mathbb{Z},\ \ \mu \in \{0,1\},\ \ r \geq 3;$$

(4.11) $$\beta_{k,\mu,1} = 1,\ \ k \in \mathbb{Z},\ \ \mu \in \{0,1\};$$

(4.12) $$\beta_{k,0,r} = \beta_{k,1,r}\ ,\ \ k \in \mathbb{Z},\ \ r \geq 3.$$

Proof. Note that, (4.9), (4.10), (4.11), (4.12) follow from the definitions of $\alpha_{k,\mu,r}$ and $\beta_{k,\mu,r}$.

By re-writing the conjugates of r_i (by elements of Λ) we get the relations: (4.1) $-$ (4.8).

Note that, $\tau(r_1) = \tau(\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1}) = S_{1,\sigma_i}S_{\sigma_1,\sigma_j}S^{-1}_{\sigma_1,\sigma_i}S^{-1}_{1,\sigma_j} = \alpha_{0,0,i}\ \alpha_{1,0,j}\ \alpha^{-1}_{1,0,i}\ \alpha^{-1}_{0,0,j}$.
So, this gives the relation: $\alpha_{0,0,i}\ \alpha_{1,0,j}\ \alpha^{-1}_{1,0,i}\ \alpha^{-1}_{0,0,j} = 1,\ |i-j| > 1$.
Then we have, $\tau(\rho_1 r_1 \rho_1) = \tau(\rho_1\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1}\rho_1) = S_{1,\rho_1}S_{\rho_1,\sigma_i}S_{\sigma_1\rho_1,\sigma_j}S^{-1}_{\sigma_1\rho_1,\sigma_i}S^{-1}_{\rho_1,\sigma_j}S_{\rho_1,\rho_1}$
$= \beta_{0,0,1}\ \alpha_{0,1,i}\ \alpha_{1,1,j}\ \alpha^{-1}_{1,1,i}\ \alpha^{-1}_{0,1,j}\ \beta_{0,1,1}$.
This gives the relation: $\alpha_{0,1,i}\ \alpha_{1,1,j}\ \alpha^{-1}_{1,1,i}\ \alpha^{-1}_{0,1,j},\ |i-j| > 1$ (using (4.11)).
Similarly, $\tau(\sigma_1^k r_1 \sigma_1^{-k}) = \tau(\sigma_1^k\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1}\sigma_1^{-k}) = S_{\sigma_1^k,\sigma_i}S_{\sigma_1^{k+1},\sigma_j}S^{-1}_{\sigma_1^{k+1},\sigma_i}S^{-1}_{\sigma_1^k,\sigma_j}$
$= \alpha_{k,0,i}\ \alpha_{k+1,0,j}\ \alpha^{-1}_{k+1,0,i}\ \alpha^{-1}_{k,0,j}$.
So, we have the relation: $\alpha_{k,0,i}\ \alpha_{k+1,0,j}\ \alpha^{-1}_{k+1,0,i}\ \alpha^{-1}_{k,0,j} = 1,\ |i-j| > 1$.
In a similar way, $\tau(\sigma_1^k\rho_1 r_1 \rho_1\sigma_1^{-k}) = \tau(\sigma_1^k\rho_1\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1}\rho_1\sigma_1^{-k})$
$= S_{\sigma_1^k,\rho_1}S_{\sigma_1^k\rho_1,\sigma_i}S_{\sigma_1^{k+1}\rho_1,\sigma_j}S^{-1}_{\sigma_1^{k+1}\rho_1,\sigma_i}S^{-1}_{\sigma_1^k\rho_1,\sigma_j}S_{\sigma_1^k,\rho_1} = \alpha_{k,1,i}\ \alpha_{k+1,1,j}\ \alpha^{-1}_{k+1,1,i}\ \alpha^{-1}_{k,1,j}$.
This gives the relation: $\alpha_{k,1,i}\ \alpha_{k+1,1,j}\ \alpha^{-1}_{k+1,1,i}\ \alpha^{-1}_{k,1,j} = 1,\ |i-j| > 1$.

Merging these 4 relations into one we get (4.1).

In a similar manner we re-write the conjugates of $r_2, r_3, \ldots r_8$ by elements of Λ and club them suitably to get the relations (4.2) $-$ (4.8).

So, we have a set of defining relations for WB_n', namely relations (4.1) $-$ (4.12) in the generators $\alpha_{k,\mu,r}$, $\beta_{k,\mu,r}$ for $k \in \mathbb{Z}$, $\mu \in \{0,1\}$, $1 \leq r \leq n-1$. Hence, Lemma 3.7 is proved. $\square$

5. Simplifying the presentation for WB_n'

Now, we will eliminate some of the generators and relations through Tietze transformations in order to get a finite set of generators for WB_n'.

LEMMA 3.8. *For $n \geq 3$, the group WB'_n is generated by finitely many elements, namely* $\alpha_{0,1,1},\ \alpha_{0,0,2},\ \alpha_{1,0,2},\ \beta_{0,0,2},\ \alpha_{0,0,r},\ \beta_{0,0,r},\ 3 \leq r \leq n-1$.

PROOF. From the relations (4.9), (4.11) it is evident that the generators $\alpha_{k,0,1}$ and $\beta_{k,\mu,1}$ are redundant and we can remove them from the set of generators.

Using (4.10) we can replace $\alpha_{k,\mu,r}$ by $\alpha_{0,0,r}$ for $r \geq 3$ and remove all $\alpha_{k,\mu,r}$ with either $k \neq 0$ or $\mu \neq 0$.

Using (4.12) we can remove $\beta_{k,1,r}$ by replacing the same with $\beta_{k,0,r}$ in all other relations.

From (4.7) we have $\alpha_{k,1,2} = \beta_{k,0,2}\alpha_{k,1,1}\beta_{k+1,1,2}$. We remove $\alpha_{k,1,2}$ by replacing this value in all other relations. After this replacement, from (4.8) we deduce:

$$\beta^{-1}_{k+1,1,2}\ \alpha^{-1}_{k,1,1}\ \beta^{-1}_{k,0,2}\ \alpha_{k,0,2}\ \beta_{k+2,0,2} = \alpha_{k+1,1,1}$$

From this relation, we can express $\alpha_{k,1,1}$ in terms of $\alpha_{0,1,1},\ \alpha_{k,0,2},\ \beta_{k,0,2},\ \beta_{k,1,2}$. We replace this value of $\alpha_{k,1,1}$ in all other relations and remove $\alpha_{k,1,1}$ for all $k \neq 0$.

For $\mu = 0,\ r = 1$, (4.7) becomes:

$$\beta_{k+1,0,2}\ \alpha^{-1}_{k,0,2}\ \beta_{k,1,2} = 1 \iff \beta_{k+1,0,2}\ \alpha^{-1}_{k,0,2}\ \beta^{-1}_{k,0,2} = 1 \text{ (using (4.3))}$$

Using this we have $\beta_{k,0,2} = \beta_{0,0,2}\ \alpha_{0,0,2}\ \alpha_{1,0,2} \ldots \alpha_{k-1,0,2}$ for $k \geq 1$ and we have $\beta_{k,0,2} = \beta_{0,0,2}\ \alpha^{-1}_{-1,0,2}\ \alpha^{-1}_{-2,0,2} \ldots \alpha^{-1}_{k,0,2}$ for $k \leq -1$.
As we have $\beta_{k,1,2} = \beta^{-1}_{k,0,2}$, we can express $\beta_{k,0,2}$ and $\beta_{k,1,2}$ in terms of $\alpha_{k,0,2},\ \beta_{0,0,2}$ in all the other relations and remove all $\beta_{k,1,2}$ and all $\beta_{k,0,2}$ except $\beta_{0,0,2}$.

For $\mu = 0,\ r = 1$, (4.2) becomes:

$$\alpha_{k+1,0,2} = \alpha_{k,0,2}\ \alpha_{k+2,0,2}$$

Using this we replace all $\alpha_{k,0,2}$ in terms of $\alpha_{0,0,2},\ \alpha_{1,0,2}$.

Lastly, if $n \geq 4$, using (4.6) we can remove $\beta_{k,0,r}$ for $k \neq 0,\ r \geq 3$.

Hence, we get a presentation of WB'_n with $4+2(n-3)$ generators $\alpha_{0,1,1},\ \alpha_{0,0,2},\ \alpha_{1,0,2},\ \beta_{0,0,2},\ \alpha_{0,0,r},\ \beta_{0,0,r},\ 3 \leq r \leq n-1$, and infinitely many defining relations. This proves finite generation of WB'_n for all $n \geq 3$. $\square$

Now, we treat the case $n \geq 7$. Notice that if $n \geq 7$, for every generator $\beta_{k,0,r}$ with $r \geq 3$, there is at least one $\beta_{k,0,s}$ with $s \geq 3$ and $|r-s| > 1$. This helps us to improve the number of generators of WB'_n for $n \geq 7$.

LEMMA 3.9. *For $n \geq 7$, WB'_n can be generated by $2(n-3)+1$ elements, namely $\beta_{0,0,2},\ \beta_{0,0,r},\ \alpha_{0,0,r}$ for $3 \leq r \leq n-1$.*

PROOF. We proceed with an alternative elimination process here.

As before, we eliminate $\alpha_{k,0,1},\ \beta_{k,\mu,1},\ \beta_{k,1,r}$ for all k, μ, $3 \leq r \leq n-1$ and $\alpha_{k,\mu,r}$ with either $k \neq 0$ or $\mu \neq 0$, using the relations (4.9), (4.10), (4.11), (4.12).

Note that, $\alpha_{k,0,2} = \beta_{k,1,2}\ \beta_{k+1,0,2}$ and $\alpha_{k,1,2} = \beta_{k,0,2}\ \alpha_{k,1,1}\ \beta_{k+1,1,2}$. Also note that, $\alpha_{k,1,1} = \beta_{k,0,r}\ \beta_{k+1,0,r}$.

At first, we replace $\alpha_{k,0,2}$ and $\alpha_{k,1,2}$ by $\beta_{k,1,2}\ \beta_{k+1,0,2}$ and $\beta_{k,0,2}\ \alpha_{k,1,1}\ \beta_{k+1,1,2}$ in all the above relations and remove these generators from the set of generators.

Next, we replace $\alpha_{k,1,1}$ by $\beta_{k,0,r}\ \beta_{k+1,0,r}$ (for every $3 \leq r \leq n-1$) in the current set of relations and remove these generators from the set of generators, and we have a new set of defining relations in the generators $\beta_{k,0,2},\ \beta_{k,1,2},\ \beta_{k,0,r},\ \alpha_{0,0,r}$, for all $k \in \mathbb{Z}$ and $3 \leq r \leq n-1$.

Let us assume $k \geq 0$. The case of $k < 0$ is similar. In the new set of relations, note that for $n \geq 7$, we have $\beta_{k+1,0,r} = \alpha^{-1}_{0,0,s}\ \beta_{k,0,r}\ \alpha_{0,0,s},\ |r-s| > 1,\ r, s \geq 3$. Hence, we have $\beta_{k,0,r} = \alpha^{-k}_{0,0,s}\ \beta_{0,0,r}\ \alpha^{k}_{0,0,s}$. Also, note that, $\beta_{0,1,2} = \beta^{-1}_{0,0,2}$.
We replace $\beta_{k,0,2},\ \beta_{k,1,2},\ \beta_{k,0,r}$ by $\alpha^{-k}_{0,0,l}\ \beta_{0,0,2}\ \alpha^{k}_{0,0,l},\ \alpha^{-k}_{0,0,l}\ \beta^{-1}_{0,0,2}\ \alpha^{k}_{0,0,l},\ \alpha^{-k}_{0,0,s}\ \beta_{0,0,r}\ \alpha^{k}_{0,0,s}$ in the current set of relations and remove $\beta_{0,1,2},\ \beta_{k,0,2},\ \beta_{k,1,2},\ \beta_{k,0,r}$, for all $k \neq 0$, from the set of generators.

This gives us a new set of defining relations in the $2(n-3)+1$ generators $\beta_{0,0,2},\ \beta_{0,0,r},\ \alpha_{0,0,r}$ for $3 \leq r \leq n-1$. This proves the lemma. $\square$

6. Perfectness of WB'_n

We have the following lemma.

LEMMA 3.10. *The group WB'_n is perfect for $n \geq 5$.*

Proof. For $n \geq 5$, we abelianize the above presentation of WB'_n by adding the extra relations $xy = yx$ for all x, y in the generating set.
After abelianizing, putting $r = 1,\ s = 3$ in (4.1) we get:

$$\alpha_{k,\mu,1}\ \alpha^{-1}_{k+1,\mu,1}\ \alpha_{k+1,\mu,3}\ \alpha^{-1}_{k,\mu,3} = 1$$

$$\Longleftrightarrow\ \alpha_{k,\mu,1} = \alpha_{k+1,\mu,1}\ \text{(using (4.10))}.$$

Hence, we have $\alpha_{k,1,1} = \alpha_{0,1,1},\ k \in \mathbb{Z}$. Note that we already have $\alpha_{k,0,1} = 1,\ k \in \mathbb{Z}$.
If we put $r = 2,\ s = 4$ (here we use $n \geq 5$) in (4.1) we get:

$$\alpha_{k,\mu,2}\ \alpha^{-1}_{k+1,\mu,2}\ \alpha_{k+1,\mu,4}\ \alpha^{-1}_{k,\mu,4} = 1$$

$$\Longleftrightarrow\ \alpha_{k,\mu,2} = \alpha_{k+1,\mu,2}\ \text{(using (4.10))}.$$

This gives us: $\alpha_{k,0,2} = \alpha_{0,0,2},\ \alpha_{k,1,2} = \alpha_{0,1,2},\ k \in \mathbb{Z}$.
Putting $r = 1,\ \mu = 0$ in (4.2) we get:

$$\alpha_{k,0,1}\ \alpha_{k+1,0,2}\ \alpha_{k+2,0,1} = \alpha_{k,0,2}\ \alpha_{k+1,0,1}\ \alpha_{k+2,0,2}$$

$$\Longleftrightarrow\ \alpha_{0,0,2} = \alpha^2_{0,0,2}\ \text{(using (4.9) and } \alpha_{k,0,2} = \alpha_{0,0,2}\ ,\ k \in \mathbb{Z}).$$

So, we have $\alpha_{0,0,2} = 1 \implies \alpha_{k,0,2} = 1,\ k \in \mathbb{Z}$.
For the case $r = 1,\ \mu = 1$ in (4.2) we have:

$$\alpha_{k,1,1}\ \alpha_{k+1,1,2}\ \alpha_{k+2,1,1} = \alpha_{k,1,2}\ \alpha_{k+1,1,1}\ \alpha_{k+2,1,2}$$

$$\Longleftrightarrow\ \alpha^2_{0,1,1}\ \alpha_{0,1,2} = \alpha_{0,1,1}\ \alpha^2_{0,1,2}\ \text{(using } \alpha_{k,1,1} = \alpha_{0,1,1}\ ,\ \alpha_{k,1,2} = \alpha_{0,1,2}\ ,\ k \in \mathbb{Z}).$$

Hence, we have $\alpha_{0,1,1} = \alpha_{0,1,2}$.
If we put $r = 2,\ \mu = 1$ in (4.2) we get:

$$\alpha_{k,1,2}\ \alpha_{k+1,1,3}\ \alpha_{k+2,1,2} = \alpha_{k,1,3}\ \alpha_{k+1,1,2}\ \alpha_{k+2,1,3}$$

$$\Longleftrightarrow\ \alpha^2_{0,1,2}\ \alpha_{0,0,3} = \alpha_{0,1,2}\ \alpha^2_{0,0,3}\ \text{(using (4.10) and } \alpha_{k,1,2} = \alpha_{0,1,2}\ ,\ k \in \mathbb{Z}).$$

This implies: $\alpha_{0,1,2} = \alpha_{0,0,3}$.
For the case $r = 2,\ \mu = 0$ in (4.2) we have:

$$\alpha_{k,0,2}\ \alpha_{k+1,0,3}\ \alpha_{k+2,0,2} = \alpha_{k,0,3}\ \alpha_{k+1,0,2}\ \alpha_{k+2,0,3}$$

$$\Longleftrightarrow\ \alpha_{0,0,3} = \alpha^2_{0,0,3}\ \text{(using (4.10) and } \alpha_{k,0,2} = 1,\ \forall k \in \mathbb{Z}).$$

So, we have $\alpha_{0,1,1} = \alpha_{0,1,2} = \alpha_{0,0,3} = 1$.
Putting $r \geq 3$ in (4.2) we get:

$$\alpha^2_{0,0,r}\ \alpha_{0,0,r+1} = \alpha_{0,0,r}\ \alpha^2_{0,0,r+1} \text{ (using (4.10))}.$$

This implies $\alpha_{0,0,r} = \alpha_{0,0,r+1},\ r \geq 3$ and hence we have $\alpha_{k,\mu,r} = 1$ for all k, μ, r.

Considering the case $r = 2,\ s \geq 4$ in (4.4) we get: $\beta^2_{k,\mu,2} = 1$, as $\beta^2_{k,\mu,s} = 1,\ s \geq 3$ (follows from (4.3) and (4.12)).
Also note that, putting $r = 1$ in (4.5) we have: $\beta^3_{k,\mu,2} = 1$.
Above two relations imply: $\beta_{k,\mu,2} = 1$.
Now if we put $r = 2$ in (4.5), we get: $\beta^3_{k,\mu,3} = 1$. But then we have $\beta^2_{k,\mu,3} = 1$, which imply $\beta_{k,\mu,3} = 1$.
Similarly, using (4.5) iteratively we deduce: $\beta_{k,\mu,r} = 1$ for all k, μ, r.

Hence, for $n \geq 5$, in the abelianization of WB'_n the generators $\alpha_{k,\mu,r}$ and $\beta_{k,\mu,r}$ become identity and hence the abelianization of WB'_n is trivial group.
This shows that for $n \geq 5$, WB'_n is perfect. □

Later, in Lemma 4.7, we prove that GVB'_n is perfect, for $n \geq 5$. So we have: $doa(GVB_n) \leq 1$, for $n \geq 5$. As WB_n is a homomorphic image of GVB_n, we have: $doa(WB_n) \leq doa(GVB_n) \leq 1$, for $n \geq 5$. Hence, WB'_n is perfect for $n \geq 5$. This is an alternative proof of Lemma 3.10 using Lemma 4.7.

7. Proof of Theorem 3.1 and corollaries

Theorem 3.1 follows from Lemma 3.8, Lemma 3.9 and Lemma 3.10.

We note here that, in the recent preprint [**BGN18**] it is proved that for $n \geq 4$, the rank of WB'_n is at most n.

7.0.1. *Proof of Corollary 3.2.* Recall (see for instance [**Dam17**, Corollary 4.3]) that WB_n is isomorphic to a subgroup of $Aut(F_n)$, and hence so also WB'_n. Since F_n is residually finite, using a result of Magnus [**Mag69**] it follows that $Aut(F_n)$ is also residually finite. Hence WB'_n as a subgroup of $Aut(F_n)$ is also residually finite. It is well-known that a finitely generated residually finite group is Hopfian.

Thus, using Theorem 3.1, WB_n' is Hopfian for all $n \geq 3$.

7.0.2. *Proof of Corollary 3.3.* Suppose $\phi : WB_n \to F_k$ be a nontrivial homomorphism. By Theorem 3.1, WB_n' is finitely generated. Hence, $\phi(WB_n') = \phi(WB_n)'$ is finitely generated. But, $\phi(WB_n)$ is free group of finite rank. Hence, $\phi(WB_n)'$ is finitely generated only if rank of $\phi(WB_n)$ is at most 1. This proves Corollary 3.3.

7.0.3. *Proof of Corollary 3.4.* It follows from [**Bar03**] that there is a non-trivial homomorphism from the pure welded braid group, PWB_n, onto a free group. Hence PWB_n is not adorable. For $n = 3, 4$, WB_n/PWB_n is a finite solvable group. Hence, by [**Rou04**, Proposition 1.7], for $n = 3, 4$, WB_n is not adorable (in particular, WB_n' is not perfect for $n = 3, 4$). By part (ii) of Theorem 3.1 for $n \geq 5$, WB_n is adorable of degree 1. This proves Corollary 3.4.

CHAPTER 4

Generalized Virtual Braid Groups

In this chapter, we investigate the commutator subgroups of generalized virtual braid groups GVB_n, which are introduced by Fang in [**Fan15**].

1. Presentation for GVB_n

The group GVB_n is generated by a set of $2(n-1)$ generators: $\{\ \sigma_i,\ \ \rho_i \mid i = 1, 2, \dots, n-1\ \}$ satisfying the following set of defining relations:

(1) *The braid relations among* σ_i:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1;$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1};$$

(2) *The braid relations among* ρ_i:

$$\rho_i\rho_j = \rho_j\rho_i, \text{ if } |i-j| > 1;$$

$$\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1};$$

(3) *The mixed relations*:

$$\sigma_i\rho_j = \rho_j\sigma_i, \text{ if } |i-j| > 1;$$

$$\rho_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\rho_{i+1};$$

$$\rho_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\rho_i.$$

1.1. WB_n as a homomorphic image of GVB_n. We have a natural homomorphism from GVB_n (given by the above presentation) to WB_n (given by the presentation in the previous chapter). It's easy to check that the map sending $\sigma_i, \rho_i \in GVB_n$ to $\rho_i, \sigma_i \in WB_n$ respectively, extends to a surjective homomorphism. Hence WB_n is a quotient group of GVB_n.

1.2. Relationship of GVB_n with VB_n and B_n. Let S_n, B_n and VB_n denote the symmetric group on n letters, Artin's braid group on n strands, and virtual braid group on n strands, respectively. Recall the following.

1.2.1. *Presentation for S_n.* The symmetric group S_n is generated by a set of $(n-1)$ generators: $\{ \sigma_i \mid i = 1, 2, \ldots, n-1 \}$, satisfying the following set of defining relations:

$$\sigma_i^2 = 1;$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1;$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$$

1.2.2. *Presentation for B_n.* The Artin's braid group B_n is generated by a set of $(n-1)$ generators: $\{ \sigma_i \mid i = 1, 2, \ldots, n-1 \}$, satisfying the following set of defining relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1;$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$$

1.2.3. *Presentation for VB_n.* The virtual braid group VB_n is generated by a set of $2(n-1)$ generators: $\{ \sigma_i,\ \rho_i \mid i = 1, 2, \ldots, n-1 \}$, satisfying the following set of defining relations:

(1) *The symmetric relations among σ_i*:

$$\sigma_i^2 = 1;$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1;$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1};$$

(2) *The braid relations among ρ_i*:

$$\rho_i\rho_j = \rho_j\rho_i, \text{ if } |i-j| > 1;$$

$$\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1};$$

(3) *The mixed relations*:

$$\sigma_i\rho_j = \rho_j\sigma_i, \text{ if } |i-j| > 1;$$

$$\rho_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\rho_i.$$

The relationship among the groups S_n, B_n, VB_n and GVB_n is given by the following commutative diagram:

$$\begin{array}{ccc} GVB_n & \xrightarrow{\alpha} & B_n \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\ VB_n & \xrightarrow{\delta} & S_n. \end{array}$$

where α, β, γ, δ are quotient maps defined by normal subgroups generated by ρ_i, σ_i^2, σ_i^2, and ρ_i respectively.

2. Goal of the chapter

We prove the following:

THEOREM 4.1. *Let GVB_n' denote the commutator subgroup of the generalized virtual braid group GVB_n.*

(i) *GVB_n' is finitely generated for all $n \geq 4$. Further, for $n \geq 5$, the rank of GVB_n' is at most $3n-7$.*

(ii) *GVB_3' is not finitely generated.*

(iii) *GVB_n' is perfect if and only if $n \geq 5$.*

We prove Theorem 4.1 using the Reidemeister-Schreier method and the Tietze transformations. In Section 3 we compute a set of generators for GVB_n'. In Section 4 we deduce a set of defining relations for GVB_n'. In Section 5 we prove part (ii) of Theorem 4.1. Proof of part (i) of Theorem 4.1 is covered in Section 6 and Section 7. In Section 8 we prove part (iii) of Theorem 4.1.

3. A set of generators for GVB_n'

In this section, we use Reidemeister-Schreier method to deduce a set of generators for GVB_n'.

For $n \geq 3$, define the map ϕ:

$$1 \to GVB_n' \to GVB_n \xrightarrow{\phi} \mathbb{Z} \times \mathbb{Z} \to 1$$

where, for $i = 1, \ldots, n-1$, $\phi(\sigma_i) = \overline{\sigma_1}$, $\phi(\rho_i) = \overline{\rho_1}$; here $\overline{\sigma_1}$ and $\overline{\rho_1}$ are the generators of the 2 copies of $\mathbb{Z}$. Here, Image(ϕ) is isomorphic to the abelianization of GVB_n, denoted as GVB_n^{ab}. To verify this, we abelianize the above presentation of GVB_n by inserting the relations $xy = yx$ in the presentation for all $x, y \in \{ \sigma_i, \rho_i \mid 1 \leq i \leq n-1 \}$. The resulting presentation is the following:

$$GVB_n^{ab} = < \sigma_1, \rho_1 \mid \sigma_1\rho_1 = \rho_1\sigma_1 >$$

Clearly, GVB_n^{ab} is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. But as ϕ is onto, Image(ϕ) $= \mathbb{Z} \times \mathbb{Z}$. Hence, Image($\phi$) is isomorphic to GVB_n^{ab}. Hence, ϕ defines the above short exact sequence.

LEMMA 4.2. *GVB_n' is generated by the words $\alpha_{m,k,i} = \sigma_1^m\rho_1^k\sigma_i\rho_1^{-k}\sigma_1^{-1}\sigma_1^{-m}$ and $\beta_{m,k,i} = \sigma_1^m\rho_1^k\rho_i\rho_1^{-k}\rho_1^{-1}\sigma_1^{-m}$, where $m, k \in \mathbb{Z}$, $1 \leq i \leq n-1$.*

PROOF. Consider a Schreier set of coset representatives:

$$\Lambda = \{\sigma_1^m\rho_1^k \mid m, k \in \mathbb{Z}\}.$$

For $a \in GVB_n$, we denote by $\overline{a}$ the unique element in Λ which belongs to the coset corresponding to $\phi(a)$ in the quotient GVB_n/GVB_n'.

Reidemeister-Schreier method tells us that GVB_n' is generated by the set

$$\{S_{\lambda,a} = (\lambda a)(\overline{\lambda a})^{-1} \mid \lambda \in \Lambda,\ a \in \{\sigma_i, \rho_i |\ i = 1, 2, \ldots, n-1\}\}.$$

Choose $\lambda = \sigma_1^m\rho_1^k$ from Λ. For $a = \sigma_i$, $S_{\lambda,a} = \sigma_1^m\rho_1^k\sigma_i\rho_1^{-k}\sigma_1^{-1}\sigma_1^{-m}$. For $a = \rho_i$, $S_{\lambda,a} = \sigma_1^m\rho_1^k\rho_i\rho_1^{-k}\rho_1^{-1}\sigma_1^{-m}$.
Hence, GVB_n' is generated by the following elements:

$$\alpha_{m,k,i} = S_{\sigma_1^m\rho_1^k,\sigma_i} = \sigma_1^m\rho_1^k\sigma_i\rho_1^{-k}\sigma_1^{-1}\sigma_1^{-m},$$

$$\beta_{m,k,i} = S_{\sigma_1^m\rho_1^k,\rho_i} = \sigma_1^m\rho_1^k\rho_i\rho_1^{-k}\rho_1^{-1}\sigma_1^{-m},$$

where $m, k \in \mathbb{Z}$, $1 \leq i \leq n-1$. □

3.1. Observation: We note here that, $\alpha_{m,0,1} = 1$ for all $m \in \mathbb{Z}$. Also observe that for $i \geq 3$, $\alpha_{m,k,i} = \alpha_{0,0,i}$ and $\beta_{m,k,i} = \beta_{m,0,i}$, for any $m, k \in \mathbb{Z}$. Hence we can replace all those $\alpha_{m,k,i}$ and $\beta_{m,k,i}$ simply by α_i and $\beta_{m,i}$ respectively for $i \geq 3$. In this way we will get a set of generators for GVB_n' with no generator occurring more than once.

4. A set of defining relations for GVB'_n

To obtain defining relations for GVB'_n, following the Reidemeister-Schreier method, we apply re-writing process τ on $\lambda r_\mu \lambda^{-1}$, for all $\lambda \in \Lambda$, and r_μ the defining relators for GVB_n:

$$r_1 = \sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1},\ |i-j|>1;$$

$$r_2 = \rho_i\rho_j\rho_i^{-1}\rho_j^{-1},\ |i-j|>1;$$

$$r_3 = \sigma_i\rho_j\sigma_i^{-1}\rho_j^{-1},\ |i-j|>1;$$

$$r_4 = \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1};$$

$$r_5 = \rho_i\rho_{i+1}\rho_i\rho_{i+1}^{-1}\rho_i^{-1}\rho_{i+1}^{-1};$$

$$r_6 = \rho_i\sigma_{i+1}\sigma_i\rho_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1};$$

$$r_7 = \rho_{i+1}\sigma_i\sigma_{i+1}\rho_i^{-1}\sigma_{i+1}^{-1}\sigma_i^{-1}.$$

We have the following lemma.

LEMMA 4.3. *The generators of GVB'_n satisfy the following defining relations:*

$$\alpha_{m,k,1}\ \alpha_j\ \alpha_{m+1,k,1}^{-1}\ \alpha_j^{-1}\ = 1,\ j\geq 3; \tag{4.1}$$

$$\alpha_{m,k,2}\ \alpha_j\ \alpha_{m+1,k,2}^{-1}\ \alpha_j^{-1}\ = 1,\ j\geq 4; \tag{4.2}$$

$$\alpha_i\ \alpha_j\ \alpha_i^{-1}\ \alpha_j^{-1}\ = 1,\ i,j\geq 3,\ |i-j|>1; \tag{4.3}$$

$$\beta_{m,k,2}\ \beta_{m,j}\ \beta_{m,k+1,2}^{-1}\ \beta_{m,j}^{-1}\ = 1,\ j\geq 4; \tag{4.4}$$

$$\beta_{m,i}\ \beta_{m,j}\ \beta_{m,i}^{-1}\ \beta_{m,j}^{-1}\ = 1,\ i,j\geq 3,\ |i-j|>1; \tag{4.5}$$

$$\alpha_{m,k,1}\ \beta_{m+1,j}\ \alpha_{m,k+1,1}^{-1}\ \beta_{m,j}^{-1}\ = 1,\ j\geq 3; \tag{4.6}$$

$$\alpha_{m,k,2}\ \beta_{m+1,j}\ \alpha_{m,k+1,2}^{-1}\ \beta_{m,j}^{-1}\ = 1,\ j\geq 4; \tag{4.7}$$

$$\alpha_i\ \beta_{m+1,k,2}\ \alpha_i^{-1}\ \beta_{m,k,2}^{-1}\ = 1,\ i\geq 4; \tag{4.8}$$

$$\alpha_i\ \beta_{m+1,j}\ \alpha_i^{-1}\ \beta_{m,j}^{-1}\ = 1,\ i,j\geq 3,\ |i-j|>1; \tag{4.9}$$

$$\alpha_{m,k,1}\ \alpha_{m+1,k,2}\ \alpha_{m+2,k,1}\ \alpha_{m+2,k,2}^{-1}\ \alpha_{m+1,k,1}^{-1}\ \alpha_{m,k,2}^{-1}\ = 1; \tag{4.10}$$

$$\alpha_{m,k,2}\ \alpha_3\ \alpha_{m+2,k,2}\ \alpha_3^{-1}\ \alpha_{m+1,k,2}^{-1}\ \alpha_3^{-1}\ = 1; \tag{4.11}$$

$$\alpha_i \ \alpha_{i+1} \ \alpha_i \ \alpha_{i+1}^{-1} \ \alpha_i^{-1} \ \alpha_{i+1}^{-1} = 1, \ i \geq 3; \tag{4.12}$$

$$\beta_{m,k+1,2} \ \beta_{m,k+2,2}^{-1} \ \beta_{m,k,2}^{-1} \ = 1; \tag{4.13}$$

$$\beta_{m,k,2} \ \beta_{m,3} \ \beta_{m,k+2,2} \ \beta_{m,3}^{-1} \ \beta_{m,k+1,2}^{-1} \ \beta_{m,3}^{-1} \ = 1; \tag{4.14}$$

$$\beta_{m,i} \ \beta_{m,i+1} \ \beta_{m,i} \ \beta_{m,i+1}^{-1} \ \beta_{m,i}^{-1} \ \beta_{m,i+1}^{-1} \ = 1, \ i \geq 3; \tag{4.15}$$

$$\alpha_{m,k+1,2} \ \alpha_{m+1,k+1,1} \ \beta_{m+2,k,2}^{-1} \ \alpha_{m+1,k,1}^{-1} \ \alpha_{m,k,2}^{-1} \ = 1; \tag{4.16}$$

$$\beta_{m,k,2} \ \alpha_3 \ \alpha_{m+1,k+1,2} \ \beta_{m+2,3}^{-1} \ \alpha_{m+1,k,2}^{-1} \ \alpha_3^{-1} \ = 1; \tag{4.17}$$

$$\beta_{m,i} \ \alpha_{i+1} \ \alpha_i \ \beta_{m+2,i+1}^{-1} \ \alpha_i^{-1} \ \alpha_{i+1}^{-1} \ = 1, \ i \geq 3; \tag{4.18}$$

$$\beta_{m,k,2} \ \alpha_{m,k+1,1} \ \alpha_{m+1,k+1,2} \ \alpha_{m+1,k,2}^{-1} \ \alpha_{m,k,1}^{-1} \ = 1; \tag{4.19}$$

$$\beta_{m,3} \ \alpha_{m,k+1,2} \ \alpha_3 \ \beta_{m+2,k,2}^{-1} \ \alpha_3^{-1} \ \alpha_{m,k,2}^{-1} \ = 1; \tag{4.20}$$

$$\beta_{m,i+1} \ \alpha_i \ \alpha_{i+1} \ \beta_{m+2,i}^{-1} \ \alpha_{i+1}^{-1} \ \alpha_i^{-1} \ = 1, \ i \geq 3. \tag{4.21}$$

Proof. Choose any element $\lambda = \sigma_1^m \rho_1^k \in \Lambda$.

Rewriting $\lambda r_1 \lambda^{-1}$ we get:

$$\tau(\lambda r_1 \lambda^{-1}) = S_{\sigma_1^m \rho_1^k, \sigma_i} \ S_{\sigma_1^{m+1} \rho_1^k, \sigma_j} \ S^{-1}_{\sigma_1^{m+1} \rho_1^k, \sigma_i} \ S^{-1}_{\sigma_1^m \rho_1^k, \sigma_j}.$$

We have the following 3 possible cases:

Case 1: $i = 1, \ j \geq 3$; gives the relations: (4.1).

Case 2: $i = 2, \ j \geq 4$; gives the relations: (4.2).

Case 3: $i, j \geq 3, \ |i - j| > 1$; gives the relations: (4.3).

Rewriting $\lambda r_2 \lambda^{-1}$ we get:

$$\tau(\lambda r_2 \lambda^{-1}) = S_{\sigma_1^m \rho_1^k, \rho_i} \ S_{\sigma_1^m \rho_1^{k+1}, \rho_j} \ S^{-1}_{\sigma_1^m \rho_1^{k+1}, \rho_i} \ S^{-1}_{\sigma_1^m \rho_1^k, \rho_j}.$$

We have the following 3 possible cases:

Case 1: $i = 1, \ j \geq 3$; gives no nontrivial relation.

Case 2: $i = 2, \ j \geq 4$; gives the relations: (4.4).

Case 3: $i, j \geq 3,\ |i-j| > 1$; gives the relations: (4.5).

Rewriting $\lambda r_3 \lambda^{-1}$ we get:

$$\tau(\lambda r_3 \lambda^{-1}) = S_{\sigma_1^m \rho_1^k, \sigma_i}\ S_{\sigma_1^{m+1} \rho_1^k, \rho_j}\ S^{-1}_{\sigma_1^m \rho_1^{k+1}, \sigma_i}\ S^{-1}_{\sigma_1^m \rho_1^k, \rho_j}.$$

We have the following 5 possible cases:

Case 1: $i = 1,\ j \geq 3$; gives the relations: (4.6).

Case 2: $j = 1,\ i \geq 3$; gives no nontrivial relation.

Case 3: $i = 2,\ j \geq 4$; gives the relations: (4.7).

Case 4: $j = 2,\ i \geq 4$; gives the relations: (4.8).

Case 5: $i, j \geq 3,\ |i-j| > 1$; gives the relations: (4.9).

Rewriting $\lambda r_4 \lambda^{-1}$ we get:

$$\tau(\lambda r_4 \lambda^{-1}) = S_{\sigma_1^m \rho_1^k, \sigma_i}\ S_{\sigma_1^{m+1} \rho_1^k, \sigma_{i+1}}\ S_{\sigma_1^{m+2} \rho_1^k, \sigma_i}\ S^{-1}_{\sigma_1^{m+2} \rho_1^k, \sigma_{i+1}}\ S^{-1}_{\sigma_1^{m+1} \rho_1^k, \sigma_i}\ S^{-1}_{\sigma_1^m \rho_1^k, \sigma_{i+1}}.$$

We have the following 3 possible cases:

Case 1: $i = 1$; gives the relation: (4.10).

Case 2: $i = 2$; gives the relation: (4.11).

Case 3: $i \geq 3$; gives the relation: (4.12).

Rewriting $\lambda r_5 \lambda^{-1}$ we get:

$$\tau(\lambda r_5 \lambda^{-1}) = S_{\sigma_1^m \rho_1^k, \rho_i}\ S_{\sigma_1^m \rho_1^{k+1}, \rho_{i+1}}\ S_{\sigma_1^m \rho_1^{k+2}, \rho_i}\ S^{-1}_{\sigma_1^m \rho_1^{k+2}, \rho_{i+1}}\ S^{-1}_{\sigma_1^m \rho_1^{k+1}, \rho_i}\ S^{-1}_{\sigma_1^m \rho_1^k, \rho_{i+1}}.$$

We have the following 3 possible cases:

Case 1: $i = 1$; gives the relation: (4.13).

Case 2: $i = 2$; gives the relation: (4.14).

Case 3: $i \geq 3$; gives the relation: (4.15).

Rewriting $\lambda r_6 \lambda^{-1}$ we get:

$$\tau(\lambda r_6 \lambda^{-1}) = S_{\sigma_1^m \rho_1^k, \rho_i}\ S_{\sigma_1^m \rho_1^{k+1}, \sigma_{i+1}}\ S_{\sigma_1^{m+1} \rho_1^{k+1}, \sigma_i}\ S^{-1}_{\sigma_1^{m+2} \rho_1^k, \rho_{i+1}}\ S^{-1}_{\sigma_1^{m+1} \rho_1^k, \sigma_i}\ S^{-1}_{\sigma_1^m \rho_1^k, \sigma_{i+1}}.$$

We have the following 3 possible cases:

Case 1: $i = 1$; gives the relation: (4.16).

Case 2: $i=2$; gives the relation: (4.17).

Case 3: $i \geq 3$; gives the relation: (4.18).

Rewriting $\lambda r_7 \lambda^{-1}$ we get:

$$\tau(\lambda r_7\lambda^{-1}) = S_{\sigma_1^m\rho_1^k,\rho_{i+1}}\ S_{\sigma_1^m\rho_1^{k+1},\sigma_i}\ S_{\sigma_1^{m+1}\rho_1^{k+1},\sigma_{i+1}}\ S^{-1}_{\sigma_1^{m+2}\rho_1^k,\rho_i}\ S^{-1}_{\sigma_1^{m+1}\rho_1^k,\sigma_{i+1}}\ S^{-1}_{\sigma_1^m\rho_1^k,\sigma_i}.$$

We have the following 3 possible cases:

Case 1: $i=1$; gives the relation: (4.19).

Case 2: $i=2$; gives the relation: (4.20).

Case 3: $i \geq 3$; gives the relation: (4.21).

This completes the proof of the lemma. $\square$

Now, we will apply Tietze transformations on the above presentation for GVB'_n in order to obtain simpler presentations for GVB'_n, for different n's, with the aim to prove Theorem 4.1 .

5. Infinite generation of GVB'_3

From Lemma 4.2 and Lemma 4.3 we have the following presentation for GVB'_3:

Generators: $\{\ \alpha_{m,k,1},\ \alpha_{m,k,2},\ \beta_{m,k,2} \mid m,k \in \mathbb{Z}\ \}$.

Defining Relations: For all $m,k \in \mathbb{Z}$,

$$\alpha_{m,k,1}\ \alpha_{m+1,k,2}\ \alpha_{m+2,k,1}\ \alpha^{-1}_{m+2,k,2}\ \alpha^{-1}_{m+1,k,1}\ \alpha^{-1}_{m,k,2}\ = 1; \tag{5.1}$$

$$\beta_{m,k+1,2}\ \beta^{-1}_{m,k+2,2}\ \beta^{-1}_{m,k,2}\ = 1; \tag{5.2}$$

$$\alpha_{m,k+1,2}\ \alpha_{m+1,k+1,1}\ \beta^{-1}_{m+2,k,2}\ \alpha^{-1}_{m+1,k,1}\ \alpha^{-1}_{m,k,2}\ = 1; \tag{5.3}$$

$$\beta_{m,k,2}\ \alpha_{m,k+1,1}\ \alpha_{m+1,k+1,2}\ \alpha^{-1}_{m+1,k,2}\ \alpha^{-1}_{m,k,1}\ = 1; \tag{5.4}$$

$$\alpha_{m,0,1} = 1. \tag{5.5}$$

We prove the following lemma:

Lemma 4.4. *GVB_3' is not finitely generated.*

Proof. From (5.4) we have:

$$\beta_{m,k,2} \;=\; \alpha_{m,k,1}\ \alpha_{m+1,k,2}\ \alpha^{-1}_{m+1,k+1,2}\ \alpha^{-1}_{m,k+1,1}.$$

Replacing these values of $\beta_{m,k,2}$ in the other relations and removing the generators $\beta_{m,k,2}$ from the set of generators we obtain an equivalent presentation for GVB_3' as follows:

Generators: $\{\ \alpha_{m,k,1},\ \alpha_{m,k,2} \mid m,k \in \mathbb{Z}\ \}$;

Defining relations: For all $m,k \in \mathbb{Z}$,

$$\alpha_{m,k,1}\ \alpha_{m+1,k,2}\ \alpha_{m+2,k,1}\ \alpha^{-1}_{m+2,k,2}\ \alpha^{-1}_{m+1,k,1}\ \alpha^{-1}_{m,k,2} \;=1; \tag{5.6}$$

$$\alpha_{m,k+1,1}\ \alpha_{m+1,k+1,2}\ \alpha^{-1}_{m+1,k+2,2}\ \alpha^{-1}_{m,k+2,1} \;= \tag{5.7}$$

$$\alpha_{m,k,1}\ \alpha_{m+1,k,2}\ \alpha^{-1}_{m+1,k+1,2}\ \alpha^{-1}_{m,k+1,1}\ \alpha_{m,k+2,1}\ \alpha_{m+1,k+2,2}\ \alpha^{-1}_{m+1,k+3,2}\ \alpha^{-1}_{m,k+3,1};$$

$$\alpha_{m,k+1,2}\,\alpha_{m+1,k+1,1}\,\alpha_{m+2,k+1,1}\,\alpha_{m+3,k+1,2}\,\alpha^{-1}_{m+3,k,2}\,\alpha^{-1}_{m+2,k,1}\,\alpha^{-1}_{m+1,k,1}\,\alpha^{-1}_{m,k,2} = 1; \tag{5.8}$$

$$\alpha_{m,0,1} = 1. \tag{5.9}$$

Now, consider the words $w_{m,k} \;=\; \alpha_{m,k,1}\ \alpha_{m+1,k,2}$ in GVB_3' for all $m,k \in \mathbb{Z}$. Now let's construct the quotient group GVB_3'/W, where $W = \overline{\langle w_{m,k} \mid m,k \in \mathbb{Z}\rangle}$, the normal subgroup generated by the words $w_{m,k}$. We obtain a presentation for GVB_3'/W by inserting the relations $\alpha_{m,k,1}\ \alpha_{m+1,k,2} \;=\; 1,\ m,k \in \mathbb{Z}$ in the last presentation for GVB_3'; and the obtained presentation for GVB_3'/W is as follows:

Generators: $\{\ \alpha_{m,k,1} \mid m,k \in \mathbb{Z}\ \}$;

Defining relations:

$$\alpha_{m+2,k,1} = \alpha^{-1}_{m-1,k,1}; \tag{5.10}$$

$$\alpha^{-1}_{m-1,k+1,1}\ \alpha_{m+1,k+1,1}\ \alpha^{-1}_{m+1,k,1}\ \alpha_{m-1,k,1} = 1; \tag{5.11}$$

$$\alpha_{m,0,1} = 1. \tag{5.12}$$

Now, we consider the words $v_{m,k} = \alpha^{-1}_{m+1,k,1}\ \alpha_{m-1,k,1}$ in GVB'_3/W for all $m, k \in \mathbb{Z}$. And we consider the quotient group $(GVB'_3/W)/V$, where $V = \overline{\langle v_{m,k} \mid m, k \in \mathbb{Z}\rangle}$, the normal subgroup generated by the words $v_{m,k}$. Similar to what we did earlier, we obtain a presentation for $(GVB'_3/W)/V$ by inserting the relations $\alpha_{m+1,k,1} = \alpha_{m-1,k,1},\ m, k \in \mathbb{Z}$ in the above presentation for GVB'_3/W; and the obtained presentation for $(GVB'_3/W)/V$ is as follows:

Generators: $\{\ \alpha_{0,k,1},\ \alpha_{1,k,1} \mid k \in \mathbb{Z}\ \}$;

Defining relations:

$$\alpha_{0,0,1} = \alpha_{1,0,1} = 1; \tag{5.13}$$

$$\alpha_{1,k,1} = \alpha^{-1}_{0,k,1} \tag{5.14}$$

Clearly, from (5.14) we can remove the generators $\alpha_{1,k,1}$ and a free presentation for the group $(GVB'_3/W)/V$ as follows:

$$(GVB'_3/W)/V = \langle\ \alpha_{0,k,1},\ k \in \mathbb{Z} - \{0\}\ \rangle$$

Hence, we have obtained the following quotient maps:

$$GVB'_3 \overset{\phi}{\to} GVB'_3/W \overset{\psi}{\to} (GVB'_3/W)/V \ \cong\ \langle\ \alpha_{0,k,1},\ k \in \mathbb{Z} - \{0\}\ \rangle \ \cong\ F^{\infty},$$

which gives us an onto homomorphism $\psi \circ \phi$ from the group GVB'_3 to the free group of infinite rank, F^{∞}. This proves that GVB'_3 is not finitely generated. □

6. Finite Generation of GVB'_4

From Lemma 4.2 and Lemma 4.3 we have the following presentation for GVB'_4:

Generators: $\{\ \alpha_{m,k,1},\ \alpha_{m,k,2},\ \alpha_3,\ \beta_{m,k,2},\ \beta_{m,3} \mid m, k \in \mathbb{Z}\ \}$.

Defining relations:

$$\alpha_{m,k,1}\ \alpha_3\ \alpha^{-1}_{m+1,k,1}\ \alpha^{-1}_3 = 1; \tag{6.1}$$

$$\alpha_{m,k,1}\ \beta_{m+1,3}\ \alpha^{-1}_{m,k+1,1}\ \beta^{-1}_{m,3} = 1; \tag{6.2}$$

$$\alpha_{m,k,1}\ \alpha_{m+1,k,2}\ \alpha_{m+2,k,1}\ \alpha^{-1}_{m+2,k,2}\ \alpha^{-1}_{m+1,k,1}\ \alpha^{-1}_{m,k,2}\ = 1; \tag{6.3}$$

$$\alpha_{m,k,2}\ \alpha_3\ \alpha_{m+2,k,2}\ \alpha_3^{-1}\ \alpha^{-1}_{m+1,k,2}\ \alpha_3^{-1}\ = 1; \tag{6.4}$$

$$\beta_{m,k+1,2}\ \beta^{-1}_{m,k+2,2}\ \beta^{-1}_{m,k,2}\ = 1; \tag{6.5}$$

$$\beta_{m,k,2}\ \beta_{m,3}\ \beta_{m,k+2,2}\ \beta^{-1}_{m,3}\ \beta^{-1}_{m,k+1,2}\ \beta^{-1}_{m,3}\ = 1; \tag{6.6}$$

$$\alpha_{m,k+1,2}\ \alpha_{m+1,k+1,1}\ \beta^{-1}_{m+2,k,2}\ \alpha^{-1}_{m+1,k,1}\ \alpha^{-1}_{m,k,2}\ = 1; \tag{6.7}$$

$$\beta_{m,k,2}\ \alpha_3\ \alpha_{m+1,k+1,2}\ \beta^{-1}_{m+2,3}\ \alpha^{-1}_{m+1,k,2}\ \alpha_3^{-1}\ = 1; \tag{6.8}$$

$$\beta_{m,k,2}\ \alpha_{m,k+1,1}\ \alpha_{m+1,k+1,2}\ \alpha^{-1}_{m+1,k,2}\ \alpha^{-1}_{m,k,1}\ = 1; \tag{6.9}$$

$$\beta_{m,3}\ \alpha_{m,k+1,2}\ \alpha_3\ \beta^{-1}_{m+2,k,2}\ \alpha_3^{-1}\ \alpha^{-1}_{m,k,2}\ = 1. \tag{6.10}$$

We prove the following lemma:

LEMMA 4.5. *GVB'_4 is finitely generated.*

PROOF. From (6.2), putting $k = 1$, we get:

$$\beta_{m+1,3} = \beta_{m,3}\ \alpha_{m,1,1}. \tag{6.11}$$

Using the above relations finitely many times we get:

$$\beta_{m,3} = \beta_{0,3}\ \alpha_{0,1,1} \dots \alpha_{m-1,1,1},\ \text{ if } m \geq 1; \tag{6.12}$$

$$\beta_{m,3} = \beta_{0,3}\ \alpha^{-1}_{-1,1,1}\ \dots\ \alpha^{-1}_{m,1,1},\ \text{ if } m \leq -1. \tag{6.13}$$

So, we can remove $\beta_{m,3}$, for all $m \neq 0$, from the set of generators by replacing these values in all the other relations.

After the above replacement (6.10) becomes:

$$\beta_{0,3}\ \alpha_{0,1,1} \dots \alpha_{m-1,1,1}\ \alpha_{m,k+1,2}\ \alpha_3\ \beta^{-1}_{m+2,k,2}\ \alpha_3^{-1}\ \alpha^{-1}_{m,k,2}\ = 1,\ \text{ for } m \geq 1; \tag{6.14}$$

$$\beta_{0,3}\ \alpha^{-1}_{-1,1,1}\ \dots\ \alpha^{-1}_{m,1,1}\ \alpha_{m,k+1,2}\ \alpha_3\ \beta^{-1}_{m+2,k,2}\ \alpha_3^{-1}\ \alpha^{-1}_{m,k,2}\ = 1,\ \text{ for } m \leq -1. \tag{6.15}$$

$$\beta_{0,3}\ \alpha_{0,k+1,2}\ \alpha_3\ \beta^{-1}_{2,k,2}\ \alpha_3^{-1}\ \alpha^{-1}_{0,k,2}\ = 1,\ \text{ for } m = 0. \tag{6.16}$$

From the above relations it is clear that we can express $\beta_{m,k,2}$ in terms of elements from $\{\ \alpha_{m,1,1},\ \alpha_{m,k,2},\ \alpha_3,\ \beta_{0,3}\ |\ m,k\in\mathbb{Z}\ \}$. We remove all $\beta_{m,k,2}$ from the generating set after replacing these values in all other relations.

Next we note that, after the above replacement in (6.5), iterating the transformed relation finitely many times we can express each $\alpha_{m,k,2}$ in terms of elements from the set

$$\{\ \alpha_{m,0,2},\ \alpha_{m,1,2},\ \alpha_{m,2,2},\ \alpha_{m,1,1},\ \alpha_3,\ \beta_{0,3}\ |\ m\in\mathbb{Z}\ \}.$$

But using (6.4) we can further simplify the expression for $\alpha_{m,k,2}$, and we write $\alpha_{m,k,2}$ in terms of elements from the set

$$\{\ \alpha_{0,0,2},\ \alpha_{1,0,2},\ \alpha_{0,1,2},\ \alpha_{1,1,2},\ \alpha_{0,2,2},\ \alpha_{1,2,2},\ \alpha_3,\ \beta_{0,3},\ \alpha_{m,1,1}\ |\ m\in\mathbb{Z}\ \}.$$

Then we replace these values of $\alpha_{m,k,2}$ in other relations and remove $\alpha_{m,k,2}$ from the generating set.

Finally we note that (6.1) is still unchanged after all the above replacements. Using this relation we have:

$$\alpha_{m,k,1}=\alpha_3^{-m}\ \alpha_{0,k,1}\ \alpha_3^{m}.$$

We replace these values in all the relations and remove all $\alpha_{m,k,1}$ with $m\neq 0$ from the generating set. Then using the relation (6.7) we express all $\alpha_{0,k,1}$ in terms of the elements $\alpha_{0,0,2},\ \alpha_{1,0,2},\ \alpha_{0,1,2},\ \alpha_{1,1,2},\ \alpha_{0,2,2},\ \alpha_{1,2,2},\ \alpha_3,\ \beta_{0,3},\ \alpha_{0,1,1}$ and remove all $\alpha_{0,k,1}$ except $\alpha_{0,0,1}$.

Hence we have shown that GVB_4' can be generated by the finite set of generators:

$$\{\alpha_{0,0,2},\ \alpha_{1,0,2},\ \alpha_{0,1,2},\ \alpha_{1,1,2},\ \alpha_{0,2,2},\ \alpha_{1,2,2},\ \alpha_3,\ \beta_{0,3},\ \alpha_{0,1,1}\}.$$

This completes the proof of the lemma. $\square$

7. Finite Generation of GVB_n', $n\geq 5$

We prove the following lemma:

LEMMA 4.6. *GVB_n' has a generating set with $3n-7$ generators, for all $n\geq 5$.*

PROOF. From (4.16) we get:

$$\beta_{m,k,2} \; = \alpha_{m-1,k,1}^{-1} \; \alpha_{m-2,k,2}^{-1} \; \alpha_{m-2,k+1,2} \; \alpha_{m-1,k+1,1}.$$

We replace these values of $\beta_{m,k,2}$ in all other defining relations and remove $\beta_{m,k,2}$ from the set of generators.

Note that the above substitution does not change the relation (4.7), which gives:

$$\alpha_{m,k+1,2} = \beta_{m,j}^{-1} \; \alpha_{m,k,2} \; \beta_{m+1,j}, \; j \geq 4.$$

Here we need $n \geq 5$. Choosing $j = 4$, we replace $\alpha_{m,k,2}$ by $\beta_{m,4}^{-k} \; \alpha_{m,0,2} \; \beta_{m+1,4}^{k}$ in all other relations, and remove all $\alpha_{m,k,2}$ with $k \neq 0$ from the generating set.

After this replacement (4.11) becomes:

$$\beta_{m,4}^{-k} \; \alpha_{m,0,2} \; \beta_{m+1,4}^{k} \; \alpha_3 \; \beta_{m+2,4}^{-k} \; \alpha_{m+2,0,2} \; \beta_{m+3,4}^{k} \; \alpha_3^{-1} \; \beta_{m+2,4}^{-k} \; \alpha_{m+1,0,2}^{-1} \; \beta_{m+1,4}^{k} \; \alpha_3^{-1} \; = 1.$$

Note that, using the above relations, we can express $\alpha_{m,0,2}$ in terms of $\alpha_{0,0,2}, \; \alpha_{1,0,2}, \; \alpha_3, \; \beta_{m,4}$. And, we remove all $\alpha_{m,0,2}$ except $\alpha_{0,0,2}$ and $\alpha_{1,0,2}$ from the generating set.

Now look at the relations (4.1) and (4.6). Note that, both the relations are untouched after all the above substitutions. The relation (4.6) gives us:

$$\alpha_{m,k+1,1} = \beta_{m,j}^{-1} \; \alpha_{m,k,1} \; \beta_{m+1,j}$$

Note here that, $\alpha_{m,0,1} = 1$. So for all $j \geq 3$, using the above relation finitely many times we deduce that,

$$\alpha_{m,k,1} = \beta_{m,j}^{-k} \; \alpha_{m,0,1} \; \beta_{m+1,j}^{k} = \beta_{m,j}^{-k} \; \beta_{m+1,j}^{k} \tag{7.1}$$

Now, if we put the values of $\alpha_{m,1,1}$ obtained from above relation in (4.1) we have:

$$\beta_{m,j}^{-1} \; \beta_{m+1,j} \; \alpha_l \; \beta_{m+2,j}^{-1} \; \beta_{m+1,j} \; \alpha_l^{-1} = 1, \; \text{for any } j, \; l \geq 3. \tag{7.2}$$

We remove all $\alpha_{m,k,1}$ from the set of generators by replacing the values of $\alpha_{m,k,1}$ as in (7.1) in all the relations.

And finally, for every $j \geq 3$, using (7.2) we can express $\beta_{m,j}$ in terms of $\beta_{0,j}, \; \beta_{1,j}, \; \alpha_j$. So for each $j \geq 3$, we can remove all $\beta_{m,j}$ with $m \neq 0, 1$ from the set of generators.

Hence, we can generate GVB_n', for all $n \geq 5$, with the finite generating set:

$$\{ \; \alpha_{0,0,2}, \; \alpha_{1,0,2}, \; \alpha_j, \; \beta_{0,j}, \; \beta_{1,j} \mid 3 \leq j \leq n-1 \; \}$$

which has $3n-7$ elements.

Hence, the proof of the lemma is complete. $\square$

8. Perfectness of GVB'_n

We prove the following lemma:

LEMMA 4.7. *GVB'_n is perfect for $n \geq 5$.*

PROOF. We abelianize the presentation for GVB'_n as in Lemma 4.3 by inserting the relations of the type $x^{-1}y^{-1}xy = 1$ for all x, y in the generating set, and obtain a presentation for $(GVB'_n)^{ab}$. We will now show that $(GVB'_n)^{ab} \cong \langle 1 \rangle$.

In the abelianized presentation we observe the following:

(1) From (4.1) we get $\alpha_{m+1,k,1} = \alpha_{m,k,1}$ for all $m, k \in \mathbb{Z}$. This implies that $\alpha_{m,k,1} = \alpha_{0,k,1}$ for all $m, k \in \mathbb{Z}$.

(2) From (4.2) we get $\alpha_{m+1,k,2} = \alpha_{m,k,2}$ for all $m, k \in \mathbb{Z}$. This implies that $\alpha_{m,k,2} = \alpha_{0,k,2}$ for all $m, k \in \mathbb{Z}$. (Note that here we need $n \geq 5$.)

(3) From (4.10) using the above 2 observations we get: $\alpha_{0,k,1} = \alpha_{0,k,2}$ for all $k \in \mathbb{Z}$.

(4) From (4.11) we get: $\alpha_3 = \alpha_{0,k,2}$ for all $k \in \mathbb{Z}$.

(5) From (4.12) we get $\alpha_i = \alpha_{i+1}$ for all $3 \leq i \leq n-2$.

(6) From all the above observations we have:

$$\alpha_3 = \alpha_{0,0,2} = \alpha_{0,0,1} = 1$$

$$\implies \alpha_{m,k,1} = \alpha_{m,k,2} = \alpha_i = 1, \text{ for all } m, k \in \mathbb{Z},\ 3 \leq i \leq n-1.$$

(7) From (4.19) and observation (6) we get: $\beta_{m,k,2} = 1$ for all $m, k \in \mathbb{Z}$.

(8) From (4.20) and observations (6) and (7) we get: $\beta_{m,3} = 1$ for all $m \in \mathbb{Z}$.

(9) From (4.21) we get: $\beta_{m,i+1} = \beta_{m+2,i}$ for all $m \in \mathbb{Z}$. Putting $i = 3$ we get $\beta_{m,4} = \beta_{m+2,3} = 1$ for all $m \in \mathbb{Z}$. Repeated use of this relation for increasing i's gives us: $\beta_{m,i} = 1$ for all $m \in \mathbb{Z},\ 3 \leq i \leq n-1$.

From the above observations we conclude that, in the presentation for $(GVB'_n)^{ab}$ all the generators are equal to 1. So, $(GVB'_n)^{ab} \cong \langle 1 \rangle$. Hence, GVB'_n is perfect for $n \geq 5$. $\square$

We prove the following lemma.

Lemma 4.8. *GVB_3' and GVB_4' are not perfect.*

Proof. The welded braid group WB_n is a homomorphic image of GVB_n. It is proved in Corollary 3.4 that WB_k is not adorable, for $k = 3, 4$. Hence, GVB_k is not adorable, for $k = 3, 4$. In particular, GVB_3' and GVB_4' are not perfect. □

Proof of Theorem 4.1

Combining Lemma 4.4, Lemma 4.5, Lemma 4.6, Lemma 4.7, and Lemma 4.8, we have the proof of Theorem 4.1.

CHAPTER 5

Flat Welded (and Virtual) Braid Groups

In this chapter, we investigate the commutator subgroups of the flat welded braid groups FWB_n and the flat virtual braid groups FVB_n.

1. Presentation for FWB_n and FVB_n

The group FVB_n is generated by a set of $2(n-1)$ generators:
$\{\sigma_i, \rho_i,\ i = 1, 2, \ldots, n-1\}$, satisfying the following set of defining relations:

$$\sigma_i^2 = 1;$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| > 1;$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1};$$

$$\rho_i^2 = 1;$$

$$\rho_i\rho_j = \rho_j\rho_i,\ |i-j| > 1;$$

$$\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1};$$

$$\sigma_i\rho_j = \rho_j\sigma_i,\ |i-j| > 1;$$

$$\rho_i\rho_{i+1}\sigma_i = \sigma_{i+1}\rho_i\rho_{i+1}.$$

The group FWB_n is generated by the same set of generators as FVB_n, and has a set of defining relations as the set of relations above along with the following extra relations:

$$\rho_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\rho_{i+1}.$$

2. Goal of the chapter

It is easy to see that the abelianizations of FWB_n and FVB_n are finite groups. So, the commutator subgroups FWB_n' and FVB_n' are finite index subgroups of the finitely presented groups FWB_n and FVB_n respectively; and hence FWB_n' and FVB_n' are also finitely presented.

We compute explicit finite presentations for FVB_n' and FWB_n' in Theorem 5.3.

We note the following proposition.

Proposition 5.1. *The flat virtual braid group FVB_n and the flat welded braid group FWB_n are adorable groups of degree 1 for $n \geq 5$; i.e. commutator subgroups of these groups are perfect for $n \geq 5$.*

The above proposition can be proved by showing that the abelianizations of the presentations for FVB_n' and FWB_n' which we get in Theorem 5.3 are identity. But we will give an alternative proof of Proposition 5.1.

We shall prove Theorem 5.3 in the rest of this section by deducing explicit finite presentations for FVB_n' and FWB_n' using Reidemeister-Schreier method and Tietze transformations. In Section 3, we compute sets of generators for FWB_n' and FVB_n'. In Section 4, we deduce sets of defining relations for FWB_n' and FVB_n' and prove Theorem 5.3. Proposition 5.1 is proved in Section 5.

3. Sets of generators for FWB_n' and FVB_n'

To simplify the writing, let $G = FWB_n$ or FVB_n.

Define the map ϕ:

$$1 \to G' \to G \xrightarrow{\phi} \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1$$

where, for $i = 1, \ldots, n-1$, $\phi(\sigma_i) = \overline{\sigma_1}$, $\phi(\rho_i) = \overline{\rho}_1$; here $\overline{\sigma}_1$ and $\overline{\rho}_1$ are the generators of the two copies of $\mathbb{Z}_2$. Note that, ϕ does have a section in the above short exact sequence and $\ker \phi = G'$.

Here, Image(ϕ) is isomorphic to the abelianization of G, denoted as G^{ab}. To prove this, we abelianize the presentations of FWB_n and FVB_n by inserting the relations $xy = yx$ in the presentations for all $x,\ y \in \{\ \sigma_i,\ \rho_i \mid 1 \leq i \leq n-1\ \}$. We find that in both the cases the resulting presentation is the following:

$$G^{ab} = <\sigma_1, \rho_1 \mid \sigma_1\rho_1 = \rho_1\sigma_1,\ \sigma_1^2 = 1,\ \rho_1^2 = 1\ > .$$

Clearly, G^{ab} is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. But as ϕ is onto, Image(ϕ) $= \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, Image($\phi$) is isomorphic to G^{ab}.

We have the following lemma.

LEMMA 5.2. *G' is generated by $\sigma_i\sigma_1 = a_i$, $\rho_i\rho_1 = b_i$, $\rho_1\sigma_i\rho_1\sigma_1 = c_i$, $\rho_1\rho_i = d_i$, $\sigma_1\sigma_i = e_i$, $\sigma_1\rho_i\rho_1\sigma_1 = f_i$, $\sigma_1\rho_1\sigma_i\rho_1 = g_i$, $\sigma_1\rho_1\rho_i\sigma_1 = h_i$ for $i = 1, 2, \ldots, n-1$.*

PROOF. Consider a Schreier set of coset representatives:

$$\Lambda = \{1, \sigma_1, \rho_1, \sigma_1\rho_1\}.$$

By the Reidemeister-Schreier method, the group G' is generated by the set:

$$\{S_{\lambda,a} = (\lambda a)(\overline{\lambda a})^{-1} \mid \lambda \in \Lambda,\ a \in \{\sigma_i, \rho_i \mid i = 1, 2, \ldots, n-1\}\}.$$

We compute the generators as follows:

(1) *For* $\lambda = 1$:

$$S_{1,\sigma_i} = \sigma_i\sigma_1 = a_i\ ,$$

$$S_{1,\rho_i} = \rho_i\rho_1 = b_i\ ;$$

(2) *For* $\lambda = \rho_1$:

$$S_{\rho_1,\sigma_i} = \rho_1\sigma_i\rho_1\sigma_1 = c_i\ ,$$

$$S_{\rho_1,\rho_i} = \rho_1\rho_i = d_i\ ;$$

(3) *For* $\lambda = \sigma_1$:

$$S_{\sigma_1,\sigma_i} = \sigma_1\sigma_i = e_i\ ,$$

$$S_{\sigma_1,\rho_i} = \sigma_1\rho_i\rho_1\sigma_1 = f_i\ ;$$

(4) *For* $\lambda = \sigma_1\rho_1$:

$$S_{\sigma_1\rho_1,\sigma_i} = \sigma_1\rho_1\sigma_i\rho_1 = g_i\ ,$$

$$S_{\sigma_1\rho_1,\rho_i} = \sigma_1\rho_1\rho_i\sigma_1 = h_i\ .$$

This proves the lemma. $\square$

4. Sets of defining relations for FWB'_n and FVB'_n

To obtain defining relations for G', following the Reidemeister-Schreier method, we apply re-writing process τ on $\lambda r_\mu \lambda^{-1}$, for all $\lambda \in \Lambda$, and r_μ the defining relators for G as follows.

The defining relators for FVB_n are:

$$r_1 = \sigma_i\sigma_j\sigma_i\sigma_j,\ |i-j|>1;$$

$$r_2 = \sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1};$$

$$r_3 = \sigma_i^2;$$

$$r_4 = \rho_i^2;$$

$$r_5 = \rho_i\rho_j\rho_i\rho_j,\ |i-j|>1;$$

$$r_6 = \rho_i\rho_{i+1}\rho_i\rho_{i+1}\rho_i\rho_{i+1};$$

$$r_7 = \sigma_i\rho_j\sigma_i\rho_j,\ |i-j|>1;$$

$$r_8 = \rho_i\rho_{i+1}\sigma_i\rho_{i+1}\rho_i\sigma_{i+1}.$$

The extra defining relators for FWB_n are as follows:

$$r_9 = \rho_i\sigma_{i+1}\sigma_i\rho_{i+1}\sigma_i\sigma_{i+1}.$$

We prove the following theorem.

THEOREM 5.3. *FVB'_n has the following finite presentation:*
Set of generators:

$$c_1, c_2, f_2, a_i, b_i,\ i = 2,\ldots,n-1$$

Set of defining relations:

$$a_2^3 = b_2^3 = c_2^3 = f_2^3 = 1;$$

$$a_i^2 = b_i^2 = (b_ic_1)^2 = 1,\ i = 3,\ldots,n-1;$$

$$b_2^{-1}f_2a_2^{-1} = 1;$$

$$b_2c_1f_2^{-1}c_2^{-1} = 1;$$

$$(a_2a_i)^2 = (b_2b_i)^2 = (c_2a_i)^2 = (f_2b_ic_1)^2 = 1, i \geq 4;$$

$$(a_2a_3)^3 = (b_2b_3)^3 = (c_2a_3)^3 = (f_2b_3c_1)^3 = 1;$$

$$a_2b_ic_1 = b_ic_2, i \geq 4;$$

$$a_i f_2 = b_2 a_i, i \geq 4;$$

$$b_2 b_3 a_2 b_3 c_1 f_2^{-1} a_3 = 1;$$

$$b_2^{-1} b_3 c_2 b_3 c_1 f_2 a_3 = 1;$$

$$(a_i a_j)^2 = (b_i b_j)^2 = 1,\ i,j \geq 3,\ |i-j| > 1;$$

$$(a_i a_{i+1})^3 = (b_i b_{i+1})^3 = 1,\ i \geq 3;$$

$$b_j^{-1} a_i^{-1} b_j a_i = c_1,\ i,j \geq 3,\ |i-j| > 1;$$

$$b_i b_{i+1} a_i = a_{i+1} b_i b_{i+1},\ i \geq 3.$$

The group FWB_n' is generated by the same set of generators as above and has a set of defining relations as the set of relations above along with the following relations:

$$a_2^{-1} c_2 c_1^{-1} b_2^{-1} = 1;$$

$$a_2 c_2^{-1} c_1 f_2^{-1} = 1;$$

$$a_2 a_3 b_2 a_3 c_2^{-1} b_3 = 1;$$

$$a_2^{-1} a_3 f_2 a_3 c_2 b_3 c_1 = 1;$$

$$a_2 b_i c_1 = b_i c_2,\ i \geq 4.$$

Proof. We apply re-writing process to the conjugates (by elements of Λ) of each of the defining relators of the above presentations of FWB_n and FVB_n in order to get the defining relations for the commutator subgroups, i.e. FWB_n' and FVB_n'.

Consider the first defining relator: $r_1 = \sigma_i \sigma_j \sigma_i \sigma_j,\ |i-j| > 1$.
We conjugate this relator by each element $\lambda \in \Lambda = \{1, \sigma_1, \rho_1, \sigma_1\rho_1\}$ and rewrite them as follows:

(1) For $\lambda = 1$: $\tau(r_1) = \tau(\sigma_i \sigma_j \sigma_i \sigma_j)$
$= S_{1,\sigma_i} S_{\sigma_1,\sigma_j} S_{1,\sigma_i} S_{\sigma_1,\sigma_j} = (a_i e_j)^2,\ |i-j| > 1;$

(2) For $\lambda = \sigma_1$: $\tau(\sigma_1 r_1 \sigma_1) = \tau(\sigma_1 \sigma_i \sigma_j \sigma_i \sigma_j \sigma_1)$
$= S_{1,\sigma_1} S_{\sigma_1,\sigma_i} S_{1,\sigma_j} S_{\sigma_1,\sigma_i} S_{1,\sigma_j} S_{\sigma_1,\sigma_1} = (e_i a_j)^2,\ |i-j| > 1;$

(3) For $\lambda = \rho_1$: $\tau(\rho_1 r_1 \rho_1) = \tau(\rho_1\sigma_i\sigma_j\sigma_i\sigma_j\rho_1)$
$= S_{1,\rho_1} S_{\rho_1,\sigma_i} S_{\sigma_1\rho_1,\sigma_j} S_{\rho_1,\sigma_i} S_{\sigma_1\rho_1,\sigma_j} S_{\rho_1,\rho_1} = (c_i g_j)^2,\ |i-j| > 1;$

(4) For $\lambda = \sigma_1\rho_1$: $\tau(\sigma_1\rho_1 r_1 \rho_1\sigma_1) = \tau(\sigma_1\rho_1\sigma_i\sigma_j\sigma_i\sigma_j\rho_1\sigma_1)$
$= S_{1,\sigma_1} S_{\sigma_1,\rho_1} S_{\sigma_1\rho_1,\sigma_i} S_{\rho_1,\sigma_j} S_{\sigma_1\rho_1,\sigma_i} S_{\rho_1,\sigma_j} S_{\sigma_1\rho_1,\rho_1} S_{\sigma_1,\sigma_1} = (g_i c_j)^2,\ |i-j| > 1.$

In this way, we get some of the defining relations for FVB'_n and FWB'_n , namely $(a_i e_j)^2 = 1,\ |i-j| > 1$ and $(c_i g_j)^2 = 1,\ |i-j| > 1$.

In a similar manner we re-write the conjugates of other defining relators i.e. $r_2, r_3, \ldots r_8$, and deduce the remaining defining relations for FVB'_n:

$$(a_i e_{i+1})^3 = 1,\ (e_i a_{i+1})^3 = 1,\ a_i e_i = 1;$$
$$(b_i d_{i+1})^3 = 1,\ (d_i b_{i+1})^3 = 1,\ b_i d_i = 1;$$
$$(f_i h_{i+1})^3 = 1,\ (h_i f_{i+1})^3 = 1,\ f_i h_i = 1;$$
$$(c_i g_{i+1})^3 = 1,\ (g_i c_{i+1})^3 = 1,\ c_i g_i = 1;$$
$$(b_i d_j)^2 = 1,\ (f_i h_j)^2 = 1,\ |i-j| > 1;$$
$$a_i f_j g_i d_j = 1,\ e_i b_j c_i h_j = 1,\ |i-j| > 1;$$
$$b_i d_{i+1} a_i f_{i+1} h_i e_{i+1} = 1;$$
$$f_i h_{i+1} e_i b_{i+1} d_i a_{i+1} = 1;$$
$$d_i b_{i+1} c_i h_{i+1} f_i g_{i+1} = 1;$$
$$h_i f_{i+1} g_i d_{i+1} b_i c_{i+1} = 1.$$

For FWB'_n we have these extra relations (rewriting conjugates of r_9):

$$b_i c_{i+1} g_i d_{i+1} a_i e_{i+1} = 1;$$
$$f_i g_{i+1} c_i h_{i+1} e_i a_{i+1} = 1;$$
$$d_i a_{i+1} e_i b_{i+1} c_i g_{i+1} = 1;$$
$$h_i e_{i+1} a_i f_{i+1} g_i c_{i+1} = 1.$$

Now we apply several Tietze transformations on the presentations obtained above in order to complete the proof of Theorem 5.3.

Replacing e_i, d_i, g_i, h_i by $a_i^{-1}, b_i^{-1}, c_i^{-1}, f_i^{-1}$ respectively, we get the defining relations for FVB'_n in terms of the generators a_i, b_i, c_i, f_i as follows:

$$(a_i a_j^{-1})^2 = 1, \ |i-j| > 1; \tag{4.1}$$

$$(b_i b_j^{-1})^2 = 1, \ |i-j| > 1; \tag{4.2}$$

$$(c_i c_j^{-1})^2 = 1, \ |i-j| > 1; \tag{4.3}$$

$$(f_i f_j^{-1})^2 = 1, \ |i-j| > 1; \tag{4.4}$$

$$(a_i a_{i+1}^{-1})^3 = 1; \tag{4.5}$$

$$(b_i b_{i+1}^{-1})^3 = 1; \tag{4.6}$$

$$(c_i c_{i+1}^{-1})^3 = 1; \tag{4.7}$$

$$(f_i f_{i+1}^{-1})^3 = 1; \tag{4.8}$$

$$a_i f_j = b_j c_i, \ |i-j| > 1; \tag{4.9}$$

$$a_1 = b_1 = f_1 = 1; \tag{4.10}$$

$$b_i b_{i+1}^{-1} a_i f_{i+1} f_i^{-1} a_{i+1}^{-1} = 1; \tag{4.11}$$

$$b_i^{-1} b_{i+1} c_i f_{i+1}^{-1} f_i c_{i+1}^{-1} = 1. \tag{4.12}$$

For FWB'_n we have the extra defining relations:

$$a_i a_{i+1}^{-1} b_i c_{i+1} c_i^{-1} b_{i+1}^{-1} = 1; \tag{4.13}$$

$$a_i^{-1} a_{i+1} f_i c_{i+1}^{-1} c_i f_{i+1}^{-1} = 1. \tag{4.14}$$

Observe that, putting $i = 1$ in the relations (4.1) - (4.8), we get the following:

$$a_j^2 = b_j^2 = c_j^2 = f_j^2 = 1, \ j = 3, \ldots, n-1,$$

$$a_2^3 = b_2^3 = c_2^3 = f_2^3 = 1.$$

From the relation (4.9), if $|i-j| > 1$, we have $a_i f_j = b_j c_i$. Putting $j = 1$, we get $c_i = a_i$ for $i = 3, \ldots, n-1$. And putting $i = 1$, we get $f_j = b_j c_1$ for $j = 3, \ldots, n-1$. We replace c_i by a_i and f_i by $b_i c_1$ for $i = 3, \ldots, n-1$.

Putting $i = 1$ in the relations (4.11) and (4.12), we have:

$$b_2^{-1} f_2 a_2^{-1} = 1;$$

$$b_2 c_1 f_2^{-1} c_2^{-1} = 1.$$

Putting $i = 1$ in the relations (4.13) and (4.14), we have:

$$a_2^{-1} c_2 c_1^{-1} b_2^{-1} = 1;$$

$$a_2 c_2^{-1} c_1 f_2^{-1} = 1.$$

Similarly, considering the cases $i = 2, j \geq 4$, and, $i \geq 4, j = 2$, in the above relations we get the following set of relations for FVB_n':

$$(a_2 a_i)^2 = (b_2 b_i)^2 = (c_2 a_i)^2 = (f_2 b_i c_1)^2 = 1, \ i \geq 4;$$

$$(a_2 a_3)^3 = (b_2 b_3)^3 = (c_2 a_3)^3 = (f_2 b_3 c_1)^3 = 1;$$

$$a_2 b_i c_1 = b_i c_2, \ i \geq 4;$$

$$a_i f_2 = b_2 a_i, \ i \geq 4;$$

$$b_2 b_3 a_2 b_3 c_1 f_2^{-1} a_3 = 1;$$

$$b_2^{-1} b_3 c_2 b_3 c_1 f_2 a_3 = 1.$$

And, the extra relations for FWB_n':

$$a_2 a_3 b_2 a_3 c_2^{-1} b_3 = 1;$$

$$a_2^{-1} a_3 f_2 a_3 c_2 b_3 c_1 = 1.$$

Lastly, we consider the case $i, j \geq 3$. And we get the following defining relations for FVB_n':

$$(a_i a_j)^2 = (b_i b_j)^2 = 1, \ i, j \geq 3, \ |i - j| > 1;$$

$$(a_i a_{i+1})^3 = (b_i b_{i+1})^3 = 1, \ i \geq 3;$$

$$b_j^{-1} a_i^{-1} b_j a_i = c_1, \ i, j \geq 3, \ |i - j| > 1;$$

$$b_i b_{i+1} a_i = a_{i+1} b_i b_{i+1}, \ i \geq 3.$$

And, the extra relations for FWB_n':

$$a_i a_{i+1} b_i = b_{i+1} a_i a_{i+1}, \ i \geq 3;$$

$$c_1 a_i a_{i+1} b_i c_1 = b_{i+1} a_i a_{i+1}, \ i \geq 3.$$

This completes the proof of the theorem. $\square$

In particular, for $n = 3$, we have the following nice presentations:

$$FWB_3' = \langle a, b, c, x \mid a^3 = b^3 = c^3 = 1,\ abc = 1,\ axc = bax = xcb \rangle;$$

$$FVB_3' = \langle a, b, x, y \mid a^3 = b^3 = (ab)^3 = (xy)^3 = 1,\ y^{-1} = axb \rangle.$$

5. Proof of Proposition 5.1

In Lemma 4.7, we have proved that GVB_n' is perfect, for $n \geq 5$. Hence, $doa(GVB_n) \leq 1$, for $n \geq 5$. As FVB_n and FWB_n both are homomorphic images of GVB_n, for $n \geq 5$ we have: $doa(FVB_n) \leq doa(GVB_n) \leq 1$ and $doa(FWB_n) \leq doa(GVB_n) \leq 1$. Clearly, FVB_n and FWB_n are not perfect, i.e. not adorable of degree 0 (as their abelianizations are non-trivial groups); hence they are adorable of degree 1, for $n \geq 5$. This completes the proof of Proposition 5.1.

CHAPTER 6

Twin Groups

In this chapter, we investigate about the commutator subgroups of the twin groups.

1. Presentation for TW_n

Let $n \geq 2$. The *twin group on n arcs*, denoted by TW_n, is generated by a set of $(n-1)$ generators: $\{\tau_i \mid i = 1, 2, \ldots, n-1\}$, satisfying the following set of defining relations:

$$\tau_i^2 = 1, \quad \text{for all } i, \tag{1.1}$$

$$\tau_i \tau_j = \tau_j \tau_i, \quad \text{if } |i-j| > 1. \tag{1.2}$$

2. Goal of the chapter

Note that TW_2 is the cyclic group of order two, and hence the commutator subgroup TW_2' is trivial. It is easy to see that TW_n' is a finite index subgroup of the finitely presented group TW_n, hence it is clear that TW_n' is finitely presented. We obtain an explicit finite presentation for TW_n', for $n \geq 3$. Since for $m \geq 1$, TW_{m+2} is isomorphic to Grothendieck's m-dimensional cartographical group $\mathcal{C}_m$, we obtain a finite presentation for the group $\mathcal{C}_m'$. We prove the following theorem.

THEOREM 6.1. *For $m \geq 1$, TW_{m+2}' has the following presentation:*

Generators: $\quad \beta_p(j), \qquad 0 \leq p < j \leq m.$

Defining relations: $\quad$ *For all* $l \geq 3,\ 1 \leq k \leq j,\ j+2 \leq t \leq m,$

$$\beta_{j-k}(j)\ \beta_{t-(j+l)}(t) = \beta_{t-(j+l)}(t)\ \beta_{j-k}(j),$$

$$\beta_{t-k}(t) = \beta_{j-k}(j)^{-1}\ \beta_{t-(j+1)}(t)\ \beta_{j-k}(j).$$

We obtain the rank of TW'_n in terms of n, the number of 'arcs' of the twin group TW_n. We prove the following.

THEOREM 6.2. *For $m \geq 1$, the group TW'_{m+2} has rank $2m-1$.*

The following is a consequence of the above two theorems.

COROLLARY 6.3. *For $m \geq 1$, the quotient group TW'_{m+2}/TW''_{m+2}, is isomorphic to the free abelian group of rank $2m-1$, i.e. the group $\bigoplus_{i=1}^{2m-1} \mathbb{Z}$. In particular, TW'_{m+2} is not perfect for any $m \geq 1$.*

We characterize freeness of TW'_n in the following corollary.

COROLLARY 6.4. *TW'_{m+2} is a free group if and only if $m \leq 3$. The group TW'_3 is infinite cyclic. The groups TW'_4 and TW'_5 are free groups of rank 3 and 5 respectively.*

As applications to the above results, we derive some geometric properties of the ambient group TW_n. We prove the following characterization for word-hyperbolicity of TW_n.

COROLLARY 6.5. *The group TW_{m+2} is word-hyperbolic if and only if $m \leq 3$.*

We characterize the twin groups that does not contain any surface group, in the following corollary.

COROLLARY 6.6. *The group TW_{m+2} does not contain a surface group if and only if $m \leq 3$.*

We also prove the following.

COROLLARY 6.7. *The automorphism group of TW_{m+2} is finitely presented for $m \leq 3$.*

We have proved Theorem 6.1 by using the Reidemeister-Schreier method and the Tietze transformations. In Section 3, we compute a set of generators for TW'_n, $n \geq 3$. In Section 4, a set of defining relations for TW'_n involving these generators is obtained. We then apply Tietze transformations to the obtained presentation to prove Theorem 6.1 in Section 5. In Section 6, we prove Theorem 6.2, and the corollaries.

3. A set of generators for TW_n'

For $n \geq 3$, define the following map:

$$\phi : TW_n \longrightarrow \underbrace{\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{\text{(n − 1) copies}} = \bigoplus_{i=1}^{n-1} \mathbb{Z}_2$$

where, for $i = 1, \ldots, n-1$, ϕ maps τ_i to the generator of the i th copy of $\mathbb{Z}_2$ in the product $\bigoplus_{i=1}^{n-1} \mathbb{Z}_2$.

Here, Image(ϕ) is isomorphic to the abelianization of TW_n, denoted as TW_n^{ab}. To prove this, we abelianize the above presentation for TW_n by inserting the relations $\tau_i\tau_j = \tau_j\tau_i$ (for all i, j) in the presentation. The resulting presentation is the following:

$$\langle \tau_1, \ldots, \tau_{n-1} \mid \tau_i\tau_j = \tau_j\tau_i,\ \tau_i^2 = 1,\ i, j \in \{1, 2, \ldots n-1\}\rangle.$$

Clearly, the above is a presentation for $\bigoplus_{i=1}^{n-1} \mathbb{Z}_2$. Thus, TW_n^{ab} is isomorphic to $\bigoplus_{i=1}^{n-1} \mathbb{Z}_2$. But as ϕ is onto, Image(ϕ) $= \bigoplus_{i=1}^{n-1} \mathbb{Z}_2$, i.e. Image($\phi$) is isomorphic to TW_n^{ab}. Hence, we get the following short exact sequence:

$$1 \rightarrow TW_n' \hookrightarrow TW_n \xrightarrow{\phi} \bigoplus_{i=1}^{n-1} \mathbb{Z}_2 \rightarrow 1.$$

We have the following lemma.

Lemma 6.8. *For $n \geq 3$, TW_n' is generated by the conjugates of $\tau_j\tau_{j+1}\tau_j\tau_{j+1}$ and $\tau_{j+1}\tau_j\tau_{j+1}\tau_j$ by the elements $\tau_{i_1}\tau_{i_2}\ldots\tau_{i_s}$ for all $j \in \{1, 2, \ldots, n-2\}$ and $1 \leq i_1 < i_2 < \cdots < i_s < j$.*

Proof. Consider a Schreier set of coset representatives for TW_n' in TW_n:

$$\Lambda = \{\tau_1^{\epsilon_1}\tau_2^{\epsilon_2}\ldots\tau_{n-1}^{\epsilon_{n-1}} \mid \epsilon_i \in \{0, 1\},\ i = 1, 2, \ldots, n-1\}.$$

By the Reidemeister-Schreier method, TW_n' is generated by the set

$$\{S_{\lambda,a} = (\lambda a)(\overline{\lambda a})^{-1} \mid \lambda \in \Lambda,\ a \in \{\ \tau_i \mid i = 1, 2, \ldots, n-1\}\ \}.$$

Hence, TW_n' is generated by the elements:

$$\{ \ S_{\tau_{i_1}\tau_{i_2}\dots\tau_{i_k}, \ \tau_j} \mid 1 \le i_1 < i_2 < \dots < i_k \le n-1 \text{ and } 1 \le j \le n-1 \ \}.$$

We calculate these elements below.

Case 1: $i_k \le j$ **:** In this case,
$$\begin{aligned} S_{\tau_{i_1}\tau_{i_2}\dots\tau_{i_k},\tau_j} &= \tau_{i_1}\tau_{i_2}\dots\tau_{i_k}\tau_j\overline{(\tau_{i_1}\tau_{i_2}\dots\tau_{i_k}\tau_j)}^{-1} \\ &= \tau_{i_1}\tau_{i_2}\dots\tau_{i_k}\tau_j \ (\tau_{i_1}\tau_{i_2}\dots\tau_{i_k}\tau_j)^{-1} \\ &= 1. \end{aligned}$$

Hence we don't get any nontrivial generator from this case.

Case 2: $i_k > j$ **:** We divide this case into following 3 subcases.

Subcase 2A: $i_k > j$ *and* $(j+1) \in \{i_1, i_2, \dots, i_k\}$ *but* $j \notin \{i_1, i_2, \dots, i_k\}$*:*

Suppose $j+1 = i_{s+1}$. Then we have:

$$\begin{aligned} S_{\tau_{i_1}\tau_{i_2}\dots\tau_{i_k},\tau_j} &= \tau_{i_1}\tau_{i_2}\dots\tau_{i_s}\tau_{j+1}\tau_{i_{s+2}}\dots\tau_{i_k}\tau_j \ \overline{(\tau_{i_1}\tau_{i_2}\dots\tau_{i_s}\tau_{j+1}\tau_{i_{s+2}}\dots\tau_{i_k}\tau_j)}^{-1} \\ &= \tau_{i_1}\tau_{i_2}\dots\tau_{i_s}\tau_{j+1}\tau_{i_{s+2}}\dots\tau_{i_k}\tau_j \ (\tau_{i_1}\tau_{i_2}\dots\tau_{i_s}\tau_j\tau_{j+1}\tau_{i_{s+2}}\dots\tau_{i_k})^{-1} \\ &= \tau_{i_1}\tau_{i_2}\dots\tau_{i_s}\tau_{j+1}\tau_j\tau_{i_{s+2}}\dots\tau_{i_k} \ (\tau_{i_1}\tau_{i_2}\dots\tau_{i_s}\tau_j\tau_{j+1}\tau_{i_{s+2}}\dots\tau_{i_k})^{-1} \\ &= \tau_{i_1}\tau_{i_2}\dots\tau_{i_s}\tau_{j+1}\tau_j\tau_{i_{s+2}}\dots\tau_{i_k}\tau_{i_k}\dots\tau_{i_{s+2}}\tau_{j+1}\tau_j\tau_{i_s}\dots\tau_{i_2}\tau_{i_1} \\ &= \tau_{i_1}\tau_{i_2}\dots\tau_{i_s}\tau_{j+1}\tau_j\tau_{j+1}\tau_j\tau_{i_s}\dots\tau_{i_2}\tau_{i_1}. \end{aligned}$$

(Here we assume $i_1 < (j+1) < i_k$. The cases $(j+1) = i_1, i_k$ are similar and give same form of elements.)

So, we get some of the generators for TW_n' as follows:
$\{\tau_{i_1}\tau_{i_2}\dots\tau_{i_s}(\tau_{j+1}\tau_j\tau_{j+1}\tau_j)\tau_{i_s}\dots\tau_{i_2}\tau_{i_1} \mid j \in \{1, 2, \dots n-2\}$ and $i_1 < i_2 < \dots < i_s < j$ where $i_1, i_2, \dots, i_s, j$ are consecutive integers $\}$.

Subcase 2B: $i_k > j$ *and* $j, (j+1) \in \{i_1, i_2, \dots, i_k\}$*:*

Suppose $j = i_s, \ j+1 = i_{s+1}$. Then we have:

$$\begin{aligned} S_{\tau_{i_1}\tau_{i_2}\dots\tau_{i_k},\tau_j} &= \tau_{i_1}\tau_{i_2}\dots\tau_{i_{s-1}}\tau_j\tau_{j+1}\tau_{i_{s+2}}\dots\tau_{i_k}\tau_j \ \overline{(\tau_{i_1}\tau_{i_2}\dots\tau_{i_{s-1}}\tau_j\tau_{j+1}\tau_{i_{s+2}}\dots\tau_{i_k}\tau_j)}^{-1} \\ &= \tau_{i_1}\tau_{i_2}\dots\tau_{i_{s-1}}\tau_j\tau_{j+1}\tau_{i_{s+2}}\dots\tau_{i_k}\tau_j \ (\tau_{i_1}\tau_{i_2}\dots\tau_{i_{s-1}}\tau_{j+1}\tau_{i_{s+2}}\dots\tau_{i_k})^{-1} \\ &= \tau_{i_1}\tau_{i_2}\dots\tau_{i_{s-1}}\tau_j\tau_{j+1}\tau_j\tau_{i_{s+2}}\dots\tau_{i_k} \ (\tau_{i_1}\tau_{i_2}\dots\tau_{i_{s-1}}\tau_{j+1}\tau_{i_{s+2}}\dots\tau_{i_k})^{-1} \end{aligned}$$

$= \tau_{i_1}\tau_{i_2}\dots\tau_{i_{s-1}}\tau_j\tau_{j+1}\tau_j\tau_{i_{s+2}}\dots\tau_{i_k}\tau_{i_k}\dots\tau_{i_{s+2}}\tau_{j+1}\tau_{i_{s-1}}\dots\tau_{i_2}\tau_{i_1}$
$= \tau_{i_1}\tau_{i_2}\dots\tau_{i_{s-1}}\tau_j\tau_{j+1}\tau_j\tau_{j+1}\tau_{i_{s-1}}\dots\tau_{i_2}\tau_{i_1}.$

(Here we assume $i_1 < j < (j+1) < i_k$. The cases $j = i_1$ and $(j+1) = i_k$ are similar and give same form of elements.)

So, we get some of the generators for TW'_n as follows:
$\{\tau_{i_1}\tau_{i_2}\dots\tau_{i_s}(\tau_j\tau_{j+1}\tau_j\tau_{j+1})\tau_{i_s}\dots\tau_{i_2}\tau_{i_1} \mid j \in \{1, 2, \dots n-2\}$ and $i_1 < i_2 < \cdots < i_s < j$ where $i_1, i_2, \dots, i_s, j$ are consecutive integers $\}$.

Subcase 2C: $i_k > j$ *and* $(j+1) \notin \{i_1, i_2, \dots, i_k\}$*:*

There is $i_s \in \{i_1, i_2, \dots, i_k\}$ such that $i_s \le j < i_{s+1}$.
As $(j+1) \notin \{i_1, i_2, \dots, i_k\}$, $|i_{s+1} - j| > 1$. So we have:

$$S_{\tau_{i_1}\tau_{i_2}\dots\tau_{i_k},\tau_j} = \tau_{i_1}\tau_{i_2}\dots\tau_{i_s}\tau_{i_{s+1}}\dots\tau_{i_k}\tau_j\,\overline{(\tau_{i_1}\tau_{i_2}\dots\tau_{i_s}\tau_{i_{s+1}}\dots\tau_{i_k}\tau_j)}^{-1}$$
$$= \tau_{i_1}\tau_{i_2}\dots\tau_{i_s}\tau_j\tau_{i_{s+1}}\dots\tau_{i_k}\,\overline{(\tau_{i_1}\tau_{i_2}\dots\tau_{i_s}\tau_j\tau_{i_{s+1}}\dots\tau_{i_k})}^{-1} = 1.$$

So, this case does not give any nontrivial generator for TW'_n. $\square$

3.1. Notation: Let us introduce some notations as follows:

For $1 \le i_1 < i_2 < \cdots < i_s < j \le n-2$ let us denote

$$\alpha(i_1, i_2, \dots, i_s\ ;\ j) := \tau_{i_1}\tau_{i_2}\dots\tau_{i_s}(\tau_j\tau_{j+1}\tau_j\tau_{j+1})\tau_{i_s}\dots\tau_{i_2}\tau_{i_1},$$

$$\beta(i_1, i_2, \dots, i_s\ ;\ j) := \tau_{i_1}\tau_{i_2}\dots\tau_{i_s}(\tau_{j+1}\tau_j\tau_{j+1}\tau_j)\tau_{i_s}\dots\tau_{i_2}\tau_{i_1},$$

$$\alpha(j) := \tau_j\tau_{j+1}\tau_j\tau_{j+1}, \qquad \beta(j) := \tau_{j+1}\tau_j\tau_{j+1}\tau_j.$$

4. A set of defining relations for TW'_n

To obtain defining relations for TW'_n, following the Reidemeister-Schreier algorithm, we rewrite $\lambda r_\mu \lambda^{-1}$ for all $\lambda \in \Lambda$ and r_μ, the defining relators for TW_n:

$$r_1 = \tau_j^2; \tag{4.1}$$

$$r_2 = \tau_t\tau_j\tau_t\tau_j, \ |t-j| > 1. \tag{4.2}$$

We have the following lemma.

LEMMA 6.9. *The generators* $\alpha(j)$, $\beta(j)$, $\alpha(i_1, i_2, \ldots, i_s \ ; \ j)$, $\beta(i_1, i_2, \ldots, i_s \ ; \ j)$ *satisfy the following defining relations in* TW'_n*:*

$$\alpha(j)\ \beta(j) = 1, \quad \textit{for all} \ \ j \in \{1, 2, \ldots, n-2\},$$

$$\alpha(i_1, \ldots, i_s \ ; \ j)\ \beta(i_1, \ldots, i_s \ ; \ j) = 1, \quad \textit{when} \ \ 1 \le i_1 < i_2 < \cdots < i_s < j \le n-2.$$

PROOF. We apply the re-writing process η on all the conjugates (by the elements $\tau_{i_1}\tau_{i_2}\ldots\tau_{i_k}$ of Λ) of the defining relators for TW_n.

Note: Here we will use the notation η to denote the re-writing process, as τ is being used in the notation for elements $\tau_i \in TW_n$.

For all $j \in \{1, 2, \ldots, n-1\}$, we have the relation $\tau_j^2 = 1$ in TW_n. We apply the re-writing process η on the conjugates of the relator as follows.

For any element $\tau_{i_1}\tau_{i_2}\ldots\tau_{i_k} \in \Lambda$ we have,

$\eta\ (\tau_{i_1}\tau_{i_2}\ldots\tau_{i_k}(\tau_j\tau_j)\ \tau_{i_k}\ldots\tau_{i_2}\tau_{i_1})$

$= S_{1,\tau_{i_1}} S_{\overline{\tau_{i_1}},\tau_{i_2}} \ldots S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}},\tau_j}\ S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_j},\tau_j}\ S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}},\tau_{i_k}} \ldots S_{\overline{\tau_{i_1}\tau_{i_2}},\tau_{i_2}} S_{\overline{\tau_{i_1}},\tau_{i_1}}$

$= S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}},\tau_j}\ S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_j},\tau_j}.$

For $i_k \le j$ the above expression vanishes.

If we have $i_k > j$ and $(j+1) \notin \{i_1, i_2, \ldots, i_k\}$ the above expression vanishes.

In case $i_k > j$ and $j, (j+1) \in \{i_1, i_2, \dots, i_k\}$, assuming $j = i_s$, the above expression equals

$$\alpha(i_1, i_2, \dots, i_{s-1} \ ; \ j) \ \beta(i_1, i_2, \dots, i_s - 1 \ ; \ j).$$

And, if $s = 1$, then we have:

$$\alpha(j) \ \beta(j).$$

For $i_k > j$ and $(j+1) \in \{i_1, i_2, \dots, i_k\}$ but $j \notin \{i_1, i_2, \dots, i_k\}$, assuming $j + 1 = i_{s+1}$, the above expression equals

$$\beta(i_1, i_2, \dots, i_s \ ; \ j) \ \alpha(i_1, i_2, \dots, i_s \ ; \ j).$$

Hence, corresponding to the relation $\tau_j^2 = 1$ in TW_n we have the following defining relations for TW_n':

$$\alpha(j) \ \beta(j) = 1, \ \text{for all} \ 1 \leq j \leq n-2, \tag{4.3}$$

$$\alpha(i_1, i_2, \dots, i_s \ ; \ j) \ \beta(i_1, i_2, \dots, i_s \ ; \ j) = 1, \tag{4.4}$$

for all $i_1, i_2, \dots, i_s, j$ such that $1 \leq i_1 < i_2 < \cdots < i_s < j \leq n-2$. $\square$

Now, we will find the defining relations in TW_n' corresponding to the defining relations $\tau_t \tau_j \tau_t \tau_j = 1$, $|t - j| > 1$, in TW_n.

We have the following lemma.

Lemma 6.10. *The generators* $\alpha(j)$, $\beta(j)$, $\alpha(i_1, i_2, \dots, i_s \ ; \ j)$, $\beta(i_1, i_2, \dots, i_s \ ; \ j)$ *satisfy the following defining relations in* TW_n'*:*

For all $i_1, i_2, \dots, i_r, j, t$ *where* $1 \leq i_1 < i_2 < \cdots < i_r < t \leq n-2$, $j \leq t-2$, *we have*

$$\alpha(i_1, \dots, j, \ \widehat{j+1}, \dots, i_r; t) \ \beta(i_1, \dots, \widehat{j}, \ \widehat{j+1}, \dots, i_r; t) = 1,$$

$$\beta(i_1, \dots, j, \ \widehat{j+1}, \dots, i_r; t) \ \alpha(i_1, \dots, \widehat{j}, \ \widehat{j+1}, \dots, i_r; t) = 1,$$

$$\beta(i_1, \dots, i_s, \ \widehat{j}, \ j+1, \dots, i_r; t) \ \beta(i_1, \dots, i_s; j)$$
$$.\alpha(i_1, \dots, i_s, \ j, \ j+1, \dots, i_r; t) \ \alpha(i_1, \dots, i_s; j) = 1,$$

$$\alpha(i_1, \dots, i_s, \ \widehat{j}, \ j+1, \dots, i_r; t) \ \beta(i_1, \dots, i_s; j)$$
$$.\beta(i_1, \dots, i_s, \ j, \ j+1, \dots, i_r; t) \ \alpha(i_1, \dots, i_s; j) = 1.$$

Proof. For $|t-j|>1$, in TW_n we have the relation: $\tau_t\tau_j\tau_t\tau_j=1$. We rewrite this relation below.

$\eta\ (\tau_{i_1}\tau_{i_2}\ldots\tau_{i_k}(\tau_t\tau_j\tau_t\tau_j)\ \tau_{i_k}\ldots\tau_{i_2}\tau_{i_1})$
$= S_{1,\tau_{i_1}} S_{\overline{\tau_{i_1}},\tau_{i_2}} \ldots S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}},\tau_t} S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t},\tau_j} S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j},\tau_t} S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j\tau_t},\tau_j}$
$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j\tau_t\tau_j},\tau_{i_k}} \ldots S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j\tau_t\tau_j\tau_{i_k}\cdots\tau_{i_2}},\tau_{i_1}}$
$= S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}},\tau_t} S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t},\tau_j} S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j},\tau_t} S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j\tau_t},\tau_j}.$

We need to calculate the above expression in all possible cases in order to get all the remaining defining relations for TW_n'.

Without loss of generality, we may assume that $j<t$.

We can only have the following 3 cases:

Case 1: $i_k \le j < t$;
Case 2: $j < i_k \le t$;
Case 3: $j < t < i_k$.

Case 1: $i_k \le j < t$**.** In this case we have:

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}},\tau_t}=1,$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t},\tau_j}=1,$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j},\tau_t}=1,$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j\tau_t},\tau_j}=1.$$

Hence, this case gives no nontrivial defining relation for TW_n'.

Case 2: $j < i_k \le t$**.** We further divide this case into 3 subcases.
Subcase 2A. $(j+1)\in\{i_1,i_2,\ldots,i_k\}$ but $j\notin\{i_1,i_2,\ldots,i_k\}$:

Assume, $(j+1)=i_{s+1}$. Then we have:

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}},\tau_t}=1,$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t},\tau_j}=\beta(i_1,\ldots,i_s;j),$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j},\tau_t} = 1,$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j\tau_t},\tau_j} = \alpha(i_1, \ldots, i_s; j).$$

Hence, we get the relations:

$$\beta(i_1, \ldots, i_s; j)\ \alpha(i_1, \ldots, i_s; j) = 1.$$

Subcase 2B. $j, (j+1) \in \{i_1, i_2, \ldots, i_k\}$:

Assume, $(j+1) = i_{s+1},\ j = i_s$. Then we have:

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}},\tau_t} = 1,$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t},\tau_j} = \alpha(i_1, \ldots, i_{s-1}; j),$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j},\tau_t} = 1,$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j\tau_t},\tau_j} = \beta(i_1, \ldots, i_{s-1}; j).$$

So, we get the relations:

$$\alpha(i_1, \ldots, i_{s-1}; j)\ \beta(i_1, \ldots, i_{s-1}; j) = 1.$$

Subcase 2C. $(j+1) \notin \{i_1, i_2, \ldots, i_k\}$: In this case we have:

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}},\tau_t} = 1,$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t},\tau_j} = 1,$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j},\tau_t} = 1,$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j\tau_t},\tau_j} = 1.$$

So, we do not get any nontrivial relation from this subcase.

Case 3: $j < t < i_k$**.** We need to divide this case into 9 subcases.

Subcase 3A. $(j+1) \in \{i_1, i_2, \ldots, i_k\}$, $j \notin \{i_1, i_2, \ldots, i_k\}$, $(t+1) \in \{i_1, i_2, \ldots, i_k\}$, $t \notin \{i_1, i_2, \ldots, i_k\}$:

Assume, $(j+1) = i_{s+1},\ (t+1) = i_{r+1}$. In this case we have:

$$S_{\overline{\tau_{i_1}\tau_{i_2}\ldots\tau_{i_k}},\tau_t} = \beta(i_1, \ldots, \widehat{j}, \ldots, i_r; t),$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\ldots\tau_{i_k}\tau_t},\tau_j} = \beta(i_1, \ldots, i_s; j),$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\ldots\tau_{i_k}\tau_t\tau_j},\tau_t} = \alpha(i_1, \ldots, j, \ldots, i_r; t),$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\ldots\tau_{i_k}\tau_t\tau_j\tau_t},\tau_j} = \alpha(i_1, \ldots, i_s; j).$$

($\widehat{j}$ denotes absence of j)

Hence, we get the relations:

$$\beta(i_1, \ldots, i_s,\ \widehat{j},\ j+1, \ldots, i_r; t)\ \beta(i_1, \ldots, i_s; j)$$

$$.\alpha(i_1, \ldots, i_s,\ j,\ j+1, \ldots, i_r; t)\ \alpha(i_1, \ldots, i_s; j) = 1.$$

Subcase 3B. $(j+1) \in \{i_1, i_2, \ldots, i_k\}$, $j \notin \{i_1, i_2, \ldots, i_k\}$, and $t, (t+1) \in \{i_1, i_2, \ldots, i_k\}$:

Assume, $(j+1) = i_{s+1},\ (t+1) = i_{r+1},\ t = i_r$. In this case we have:

$$S_{\overline{\tau_{i_1}\tau_{i_2}\ldots\tau_{i_k}},\tau_t} = \alpha(i_1, \ldots, \widehat{j}, \ldots, i_{r-1}; t),$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\ldots\tau_{i_k}\tau_t},\tau_j} = \beta(i_1, \ldots, i_s; j),$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\ldots\tau_{i_k}\tau_t\tau_j},\tau_t} = \beta(i_1, \ldots, j, \ldots, i_{r-1}; t),$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\ldots\tau_{i_k}\tau_t\tau_j\tau_t},\tau_j} = \alpha(i_1, \ldots, i_s; j).$$

So, we get the relations:

$$\alpha(i_1, \ldots, i_s,\ \widehat{j},\ j+1, \ldots, i_{r-1}; t)\ \beta(i_1, \ldots, i_s; j)$$

$$.\beta(i_1, \ldots, i_s,\ j,\ j+1, \ldots, i_{r-1}; t)\ \alpha(i_1, \ldots, i_s; j) = 1.$$

Subcase 3C. $(j{+}1) \in \{i_1, i_2, \ldots, i_k\}, j \notin \{i_1, i_2, \ldots, i_k\},\ (t{+}1) \notin \{i_1, i_2, \ldots, i_k\}$:

Assume, $(j+1) = i_{s+1}$. In this case we have:

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}},\tau_t} = 1,$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t},\tau_j} = \beta(i_1, \ldots, i_s; j),$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j},\tau_t} = 1,$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j\tau_t},\tau_j} = \alpha(i_1, \ldots, i_s; j).$$

Hence, we get the relations:

$$\beta(i_1, \ldots, i_s; j)\ \alpha(i_1, \ldots, i_s; j) = 1.$$

Subcase 3D. $j, (j{+}1) \in \{i_1, i_2, \ldots, i_k\}, (t{+}1) \in \{i_1, i_2, \ldots, i_k\}, t \notin \{i_1, i_2, \ldots, i_k\}$:

Assume, $(j+1) = i_{s+1},\ j = i_s,\ (t+1) = i_{r+1}$. In this case we have:

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}},\tau_t} = \beta(i_1, \ldots, j, \ldots, i_r; t),$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t},\tau_j} = \alpha(i_1, \ldots, i_{s-1}; j),$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j},\tau_t} = \alpha(i_1, \ldots, \widehat{j}, \ldots, i_r; t),$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j\tau_t},\tau_j} = \beta(i_1, \ldots, i_{s-1}; j).$$

So, we get the relations:

$$\begin{aligned}\beta(i_1, \ldots, i_{s-1},\ j,\ j+1, \ldots, i_r; t)\ \alpha(i_1, \ldots, i_{s-1}; j)\\ .\alpha(i_1, \ldots, i_{s-1},\ \widehat{j},\ j+1, \ldots, i_r; t)\ \beta(i_1, \ldots, i_{s-1}; j) = 1.\end{aligned}$$

Subcase 3E. $j, (j+1) \in \{i_1, i_2, \ldots, i_k\}$, and $t, (t+1) \in \{i_1, i_2, \ldots, i_k\}$:

Assume, $(j+1) = i_{s+1},\ j = i_s,\ (t+1) = i_{r+1},\ t = i_r$. In this case we have:

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}},\tau_t} = \alpha(i_1, \ldots, j, \ldots, i_{r-1}; t),$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t},\tau_j} = \alpha(i_1, \ldots, i_{s-1}; j),$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j},\tau_t} = \beta(i_1, \ldots, \widehat{j}, \ldots, i_{r-1}; t),$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j\tau_t},\tau_j} = \beta(i_1, \ldots, i_{s-1}; j).$$

Hence, we get the relations:

$$\alpha(i_1,\ldots,i_{s-1},\ j,\ j+1,\ldots,i_{r-1};t)\ \alpha(i_1,\ldots,i_{s-1};j)$$
$$.\beta(i_1,\ldots,i_{s-1},\ \widehat{j},\ j+1,\ldots,\ i_{r-1};t)\ \beta(i_1,\ldots,i_{s-1};j)=1.$$

Subcase 3F. $j,(j+1)\in\{i_1,i_2,\ldots,i_k\}$, and $(t+1)\notin\{i_1,i_2,\ldots,i_k\}$:

Assume, $(j+1)=i_{s+1},\ j=i_s$. In this case we have:

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}},\tau_t}=1,$$
$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t},\tau_j}=\alpha(i_1,\ldots,i_{s-1};j),$$
$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j},\tau_t}=1,$$
$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j\tau_t},\tau_j}=\beta(i_1,\ldots,i_{s-1};j).$$

So, we get the relations:

$$\alpha(i_1,\ldots,i_{s-1};j)\ \beta(i_1,\ldots,i_{s-1};j)=1.$$

Subcase 3G. $(j+1)\notin\{i_1,i_2,\ldots,i_k\}$, and $(t+1)\in\{i_1,i_2,\ldots,i_k\}$, $t\notin\{i_1,i_2,\ldots,i_k\}$:

Assume, $(t+1)=i_{r+1}$.

In this case we have:

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}},\tau_t}=\begin{cases}\beta(i_1,\ldots,j,\ldots,i_r;t) & \text{if } j\in\{i_1,i_2,\ldots,i_k\},\\ \beta(i_1,\ldots,\widehat{j},\ldots,i_r;t) & \text{if } j\notin\{i_1,i_2,\ldots,i_k\},\end{cases}$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t},\tau_j}=1,$$
$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j},\tau_t}=\begin{cases}\alpha(i_1,\ldots,\widehat{j},\ldots,i_r;t) & \text{if } j\in\{i_1,i_2,\ldots,i_k\},\\ \alpha(i_1,\ldots,j,\ldots,i_r;t) & \text{if } j\notin\{i_1,i_2,\ldots,i_k\},\end{cases}$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j\tau_t},\tau_j}=1.$$

So, we get the relations:

$$\beta(i_1,\ldots,j,\ldots,i_r;t)\ \alpha(i_1,\ldots,\widehat{j},\ldots,i_r;t)=1,$$
$$\beta(i_1,\ldots,\widehat{j},\ldots,i_r;t)\ \alpha(i_1,\ldots,j,\ldots,i_r;t)=1.$$

Subcase 3H. $(j+1) \notin \{i_1, i_2, \ldots, i_k\}$, and $t, (t+1) \in \{i_1, i_2, \ldots, i_k\}$:

Assume, $(t+1) = i_{r+1},\ t = i_r$. In this case we have:

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}},\tau_t} = \begin{cases} \alpha(i_1, \ldots, j, \ldots, i_{r-1}; t) & \text{if } j \in \{i_1, i_2, \ldots, i_k\}, \\ \alpha(i_1, \ldots, \widehat{j}, \ldots, i_{r-1}; t) & \text{if } j \notin \{i_1, i_2, \ldots, i_k\}, \end{cases}$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t},\tau_j} = 1,$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j},\tau_t} = \begin{cases} \beta(i_1, \ldots, \widehat{j}, \ldots, i_{r-1}; t) & \text{if } j \in \{i_1, i_2, \ldots, i_k\}, \\ \beta(i_1, \ldots, j, \ldots, i_{r-1}; t) & \text{if } j \notin \{i_1, i_2, \ldots, i_k\}, \end{cases}$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j\tau_t},\tau_j} = 1.$$

Hence, we get the relations:

$$\alpha(i_1, \ldots, j, \ldots, i_{r-1}; t)\ \beta(i_1, \ldots, \widehat{j}, \ldots, i_{r-1}; t) = 1,$$

$$\alpha(i_1, \ldots, \widehat{j}, \ldots, i_{r-1}; t)\ \beta(i_1, \ldots, j, \ldots, i_{r-1}; t) = 1.$$

Subcase 3I. $(j+1) \notin \{i_1, i_2, \ldots, i_k\}$, and $(t+1) \notin \{i_1, i_2, \ldots, i_k\}$:

In this case we have:

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}},\tau_t} = 1,$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t},\tau_j} = 1,$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j},\tau_t} = 1,$$

$$S_{\overline{\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}\tau_t\tau_j\tau_t},\tau_j} = 1.$$

So, we do not get any nontrivial relation from this subcase.

Collecting the relations obtained in all the above cases we have the lemma. $\square$

5. Proof of Theorem 6.1

In this section, we will simplify the presentation for TW_n' that we deduced in the previous section. We will apply Tietze transformations on the current presentation for TW_n' in order to deduce another presentation for TW_n' with less number of generators and relations than the last one. We begin with the following lemma.

Lemma 6.11. *For $n \geq 3$, TW_n' has the following presentation:*

Generators: $\beta(j),\ \beta(i_1, i_2, \ldots, i_s\ ;\ j)$, *for* $1 \leq i_1 < i_2 < \cdots < i_s < j \leq n-2$, *Defining relations:*

$$\beta(i_1,\ldots,i_s,\ j,\ \widehat{j+1},\ldots,i_r;t)=\beta(i_1,\ldots,i_s,\ \widehat{j},\ \widehat{j+1},\ldots,i_r;t),$$

$$\beta(i_1,\ldots,i_s,\ j,\ j+1,\ldots,i_r;t)=$$
$$\beta(i_1,\ldots,i_s;j)^{-1}\ \beta(i_1,\ldots,i_s,\ \widehat{j},\ j+1,\ldots,i_r;t)\ \beta(i_1,\ldots,i_s;j),$$

where $1\le i_1<i_2<\cdots<i_s<j<\cdots<i_r<t\le n-2,\ j\le t-2.$

Proof. From Lemma 6.9, we have $\alpha(j)=\beta(j)^{-1}$, $\alpha(i_1,i_2,\ldots,i_s\ ;\ j)=\beta(i_1,i_2,\ldots,i_s\ ;\ j)^{-1}$. Hence, we replace $\alpha(j)$ by $\beta(j)^{-1}$ and $\alpha(i_1,i_2,\ldots,i_s\ ;\ j)$ by $\beta(i_1,i_2,\ldots,i_s\ ;\ j)^{-1}$ in all other defining relations for TW_n', and remove all $\alpha(j)$, $\alpha(i_1,i_2,\ldots,i_s\ ;\ j)$ from the set of generators. This completes the proof of Lemma 6.11. $\square$

5.1. Observation. Consider the defining relations:

$$\beta(i_1,\ldots,i_s,\ j,\ \widehat{j+1},\ldots,i_r;t)=\beta(i_1,\ldots,i_s,\ \widehat{j},\ \widehat{j+1},\ldots,i_r;t).$$

Note that here we have $j\le t-2$. Let us look at the following example.

Consider the generator $\beta(3,4,6,7,9,10,11;12)$ in TW_{15}'. From the above set of relations, as '5' does not appear in $\beta(3,4,6,7,9,10,11;12)$, we can conclude that

$$\beta(3,4,6,7,9,10,11;12)=\beta(3,6,7,9,10,11;12).$$

As '4' is missing in $\beta(3,6,7,9,10,11;12)$, we get

$$\beta(3,6,7,9,10,11;12)=\beta(6,7,9,10;11).$$

We can go further. Using the same relations we get

$$\beta(6,7,9,10;11)=\beta(6,9,10;11)=\beta(9,10;11).$$

From the above observation it is clear that using the above defining relations finitely many times, any generator $\beta(i_1,i_2,\ldots,i_s;j)$ can be shown to be equal to a generator of the form $\beta(j-p,j-p+1,\ldots,j-1\ ;\ j)$ for some $p<j$, or be equal to $\beta(j)$. Let's call these the *normal forms* of the generators.

5.2. Notation: We will follow the notations for the normal forms as below:

$$\text{For } 1 \leq p < j, \quad \beta_p(j) := \beta(j-p, \ldots, j-1; j), \quad \text{and} \quad \beta_0(j) := \beta(j).$$

As every generator is equal to its normal form, we replace all the generators with their normal forms in all the defining relations and remove all the generators except the normal forms from the generating set. For clarity of exposition, we define the following.

DEFINITION 6.12. For a generator $\beta(i_1, i_2, \ldots, i_s; j)$ we define the *highest missing entry in* $\beta(i_1, i_2, \ldots, i_s; j)$ *to be the integer* k, *where* $i_1 - 1 \leq k \leq j-1$, *if* $k \notin \{i_1, i_2, \ldots, i_s, j\}$ *but for any* m *with* $k < m \leq j$, $m \in \{i_1, i_2, \ldots, i_s, j\}$.

Now we are ready to prove Theorem 6.1.

5.3. Proof of Theorem 6.1. From Lemma 6.11, we have the following defining relations in the presentation for TW_n', $n \geq 3$:

$$\beta(i_1, \ldots, i_s,\ j,\ j+1, \ldots, i_r; t) =$$
$$\beta(i_1, \ldots, i_s; j)^{-1}\ \beta(i_1, \ldots, i_s,\ \widehat{j},\ j+1, \ldots, i_r; t)\ \beta(i_1, \ldots, i_s; j),$$

where $1 \leq i_1 < i_2 < \cdots < i_s < j < \cdots < i_r < t \leq n-2$, $j \leq t-2$.

We replace the generators appearing in these relations by their normal forms $\beta_p(j)$'s. Our goal is to find the modified relations after the substitution.

Consider the left hand side of the above relations. We have $\beta(i_1, \ldots, i_s,\ j,\ j+1, \ldots, i_r; t)$. Note that the highest missing entry in $\beta(i_1, \ldots, i_s,\ j,\ j+1, \ldots, i_r; t)$ cannot be j or $j+1$, as both are present as entries. So we can have 2 possibilities. We examine the 2 cases separately below.

Case 1: The highest missing entry in $\beta(i_1, \ldots, i_s,\ j,\ j+1, \ldots, i_r; t)$ is greater than $j+1$.

Suppose the highest missing entry in $\beta(i_1, \ldots, i_s,\ j,\ j+1, \ldots, i_r; t)$ is $j+(l-1)$ for some $l \geq 3$. Also suppose the highest missing entry in $\beta(i_1, \ldots, i_s; j)$ is $m-1$ for some $1 \leq m \leq j$.

Then, the relations are equivalent to the following relations:
for all $l \geq 3,\ 1 \leq m \leq j,\ j \leq t-2$,

$$\beta(j+l,\ldots,t-1;t) = \beta(m,\ldots,j-1;j)^{-1}\beta(j+l,\ldots,t-1;t)\beta(m,\ldots,j-1;j).$$

So, after the substitution by normal forms the relations become:

$$\beta_{t-(j+l)}(t) = \beta_{j-m}(j)^{-1}\ \beta_{t-(j+l)}(t)\ \beta_{j-m}(j), \quad \text{for all } l \geq 3,\ 1 \leq m \leq j,\ j \leq t-2.$$

Equivalently,

$$\beta_{j-m}(j)\ \beta_{t-(j+l)}(t) = \beta_{t-(j+l)}(t)\ \beta_{j-m}(j), \quad \text{for all } l \geq 3,\ 1 \leq m \leq j,\ j \leq t-2.$$

Case 2: The highest missing entry in $\beta(i_1,\ldots,i_s,\ j,\ j+1,\ldots,i_r;\ t)$ is less than j.

Suppose the highest missing entry in $\beta(i_1,\ldots,i_s,\ j,\ j+1,\ldots,i_r;t)$ is $m-1$ for some $1 \leq m \leq j$. Then clearly the highest missing entry in $\beta(i_1,\ldots,i_s;j)$ is also $m-1$.

So, after the substitution by normal forms the relations become:

$$\beta_{t-m}(t) = \beta_{j-m}(j)^{-1}\ \beta_{t-(j+1)}(t)\ \beta_{j-m}(j), \quad \text{for all } 1 \leq m \leq j,\ j \leq t-2.$$

This completes the proof of Theorem 6.1.

6. Proof of Theorem 6.2 and corollaries

We shall further reduce the number of generators in the presentation by removing all $\beta_p(j)$ with $p > 1$ by using the defining relations:

$$\beta_{t-m}(t) = \beta_{j-m}(j)^{-1}\ \beta_{t-(j+1)}(t)\ \beta_{j-m}(j),$$

for $m,\ j,\ t \in \{1,\ldots,n-2\}$ with $1 \leq m \leq j \leq t-2$.

Note that, if we consider the cases where $j = m$ in the above set of relations, we obtain the following set of relations:

$$\beta_{t-m}(t) = \beta_0(m)^{-1}\ \beta_{t-(m+1)}(t)\ \beta_0(m),$$

for all $m,\ t \in \{1,\ldots,n-2\}$ with $1 \leq m \leq t-2$.

So, if $t-m \geq 2$, we can express $\beta_{t-m}(t)$ as the conjugate of $\beta_{t-(m+1)}(t)$ by $\beta_0(m)$. We do this iteratively to express $\beta_{t-m}(t)$ as the conjugate of $\beta_1(t)$

by the element $\beta_0(t-2)\ldots\beta_0(m)$ and thus remove all $\beta_p(j)$ with $p \geq 2$ from the set of generators after replacing them with the above values in all the remaining relations.

We have the following lemma.

LEMMA 6.13. *For $n \geq 3$, TW'_n has a finite presentation with $(2n-5)$ generators.*

PROOF. After performing the above substitution we are left with $\beta_p(j)$ with $p \leq 1$ and $1 \leq j \leq n-2$. Hence, corresponding to every $2 \leq j \leq n-2$ we have 2 generators $\beta_0(j)$ and $\beta_1(j)$. For $j = 1$, we have only 1 generator, namely $\beta_0(1)$. So, we have total $2 \times (n-3) + 1 = 2n-5$ generators in the final presentation for TW'_n for $n \geq 3$.

Note that the presentation given in Theorem 6.1 has finitely many defining relations. As finitely many $\beta_p(j)$ are being replaced and each $\beta_p(j)$ appears finitely many times in all the defining relations, after the above substitution we will have finitely many defining relations in the final presentation.
This completes the proof of the lemma. $\square$

6.1. Proof of Theorem 6.2. We consider the abelianization of TW'_n for $n \geq 3$, $(TW'_n)^{ab} = TW'_n/TW''_n$. In order to find a presentation for $(TW'_n)^{ab}$ we insert all possible commuting relations $\beta_p(j)\ \beta_q(i) = \beta_q(i)\ \beta_p(j)$, for all $i, j \in \{1, \ldots, n-2\}$, $0 \leq p < j$, $0 \leq q < i$, in the presentation for TW'_n. This gives the following presentation for $(TW'_n)^{ab}$:

Generators: $\beta_p(j), \quad 0 \leq p < j \leq n-2.$

Defining relations: $\beta_p(j)\ \beta_q(i) = \beta_q(i)\ \beta_p(j),\ \forall i, j \in \{1, \ldots, n-2\},$

$$\beta_{t-m}(t) = \beta_{t-(j+1)}(t),\ 1 \leq m \leq j,\ j+2 \leq t \leq n-2.$$

Iterating the last set of relations, we deduce that $\beta_p(j) = \beta_1(j)$ for all $p \geq 2$ and for all $j \geq 3$. Hence, we remove all $\beta_p(j)$ with $p \geq 2$ from the set of generators by replacing them with $\beta_1(j)$. After this replacement we get the following presentation for $(TW_n')^{ab}$:

Generators: $\beta_0(1),\ \beta_0(j),\ \beta_1(j),\ 2 \leq j \leq n-2.$

Defining relations: $\beta_p(j)\,\beta_q(i) = \beta_q(i)\,\beta_p(j),\ \forall i, j \in \{1, \ldots, n{-}2\},\ p, q \in \{0, 1\}.$

Clearly, this is the presentation for direct sum of $(2n-5)$ copies of $\mathbb{Z}$, i.e. $\mathbb{Z}^{2n-5}$. So, $(TW_n')^{ab}$ is isomorphic to $\mathbb{Z}^{2n-5}$. Hence, rank of $(TW_n')^{ab}$ is $(2n-5)$.

As, $(TW_n')^{ab}$ is the homomorphic image of TW_n' under the quotient homomorphism $TW_n' \longrightarrow (TW_n')^{ab}$, rank of $(TW_n')^{ab}$ is less than or equal to the rank of TW_n'. Thus, rank(TW_n') $\geq$ rank($(TW_n')^{ab}$) $= 2n-5$. From Lemma 6.13 we get rank(TW_n') $\leq 2n-5$. So, we conclude that rank(TW_n') $= 2n-5$.

6.2. Proof of Corollary 6.3: In the proof of Theorem 6.2 we observed that for $n \geq 3$, TW_n'/TW_n'' is isomorphic to direct sum of $(2n-5)$ copies of $\mathbb{Z}$. So, we conclude that $TW_n' \neq TW_n''$, and hence TW_n' is not perfect for any $n \geq 3$. This proves the corollary.

For $n \leq 5$, TW_n' are well known groups. We have the following proposition.

PROPOSITION 6.14. *We have the following:*
(i) TW_2' *is the identity group* $\{1\}$.
(ii) TW_3' *is the infinite cyclic group* $\mathbb{Z}$.
(iii) TW_4' *and* TW_5' *are free groups of rank* 3 *and* 5, *respectively.*

PROOF. Note that, $TW_2 = \langle\ \tau_1 \mid \tau_1^2 = 1\ \rangle = \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ is an abelian group. Hence, TW_2' is the identity group.

From Theorem 6.1 it follows that $TW_3' = \langle\ \beta_0(1)\ \rangle$ which is isomorphic to the infinite cyclic group $\mathbb{Z}$.

From Theorem 6.1 it follows that $TW_4' = \langle\ \beta_0(1),\ \beta_0(2),\ \beta_1(2)\ \rangle$ which is the free group of rank 3.

From Theorem 6.1 it follows that:

$$TW_5' = \langle\, \beta_0(1),\ \beta_0(2),\ \beta_1(2),\ \beta_0(3),\ \beta_1(3),\ \beta_2(3) \mid \beta_2(3) = \beta_0(1)^{-1}\, \beta_1(3)\, \beta_0(1) \,\rangle$$
$$= \langle\, \beta_0(1),\ \beta_0(2),\ \beta_1(2),\ \beta_0(3),\ \beta_1(3) \,\rangle.$$

Hence, TW_5' is free of rank 5. This completes the proof of Proposition 6.14. □

From [**PV16**] we have a necessary and sufficient condition for the commutator subgroup of a right-angled Coxeter group to be free. Since TW_n is a right-angled Coxeter group, we check the condition for TW_n. We note the following definitions.

Definition 6.15. A graph Γ is called *chordal* if for every cycle in Γ with at least 4 vertices there is an edge (called chord) in Γ joining 2 non-adjacent vertices of the cycle.

Definition 6.16. The *Coxeter graph* Γ_{TW_n} corresponding to TW_n is defined as follows. Corresponding to each generator τ_i of TW_n we have a vertex v_i in Γ_{TW_n}. Corresponding to each commuting defining relation $\tau_i\tau_j = \tau_j\tau_i$, $|i-j| > 1$, we have an edge in Γ_{TW_n} joining v_i and v_j.

We have the following proposition.

Proposition 6.17. *For $n \geq 6$, TW_n' is not a free group.*

Proof. As proved in [**PV16**], for a right-angled Coxeter group G, the commutator subgroup G' is free if and only if the Coxeter graph of G, Γ_G, is chordal.

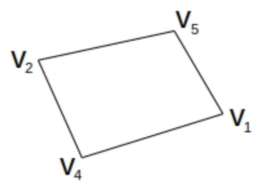

Figure 1. Cycle with 4 vertices but no chord in Γ_{TW_n}, $n \geq 6$

Consider the Coxeter graph Γ_{TW_n} corresponding to TW_n. Note that for $n \geq 6$, Γ_{TW_n} contains the cycle $v_1v_4v_2v_5v_1$ joining the vertices v_1, v_4, v_2, v_5 (as in the figure above). Clearly this cycle does not have any chord; as τ_1, τ_2 do not commute and τ_4, τ_5 do not commute. This shows that for $n \geq 6$ Γ_{TW_n} is not chordal.

Consequently, TW_n' is not free for $n \geq 6$, proving Proposition 6.17. □

6.3. Proof of Corollary 6.4. Corollary 6.4 follows from Proposition 6.14. and Proposition 6.17.

6.4. Proof of Corollary 6.5. It is clear from the presentation in Theorem 6.1 that for $m \geq 4$, TW'_{m+2} contains free abelian subgroups of rank ≥ 2. By [**Mou88**, Theorem B], this shows that TW_{m+2} is not word-hyperbolic for $m \geq 4$. Whereas from Corollary 6.4 we observe that TW_{m+2} is virtually free for $m \leq 3$. Hence TW_{m+2} is word-hyperbolic for $m \leq 3$.

6.5. Proof of Corollary 6.6. Gordon, Long and Reid proved in [**GLR04**] that a Coxeter group G is virtually free if and only if G does not contain a surface group. Since TW_{m+2} is finitely generated, by Corollary 6.5, it can not be virtually free for $m \geq 4$.

6.6. Proof of Corollary 6.7. According to [**KL04**, Theorem B], and also [**Krs92**, Theorem 1], any finite extension of a free group of finite rank has a finitely presented automorphism group. Noting that TW_n/TW'_n is a finite group and using Corollary 6.4 it is clear that the automorphism group of TW_{m+2} is finitely presented for $m \leq 3$.

7. Remarks

Presentation for TW'_6. As follows from the above, TW'_6 is the first non-free group in the family of TW'_n, $n \geq 3$. Here, we note down a presentation for TW'_6 with minimal number of generators:

Generators: $\beta_0(1),\ \beta_0(2),\ \beta_1(2),\ \beta_0(3),\ \beta_1(3),\ \beta_0(4),\ \beta_1(4).$

Defining relations:

$$\beta_0(1)\ \beta_0(4) \ = \ \beta_0(4)\ \beta_0(1),$$
$$\beta_1(2)^{-1}\ \beta_1(4)\ \beta_1(2) \ = \ \beta_0(1)^{-1}\ \beta_0(2)^{-1}\ \beta_1(4)\ \beta_0(2)\ \beta_0(1).$$

Note that, the rank of TW'_n mentioned in Theorem 6.2 is attained in the above presentation for TW'_6.

CHAPTER 7

Future research directions

Here we note down some research directions (suggested by the referees) related to the generalized braid groups.

Problem 1: Investigate whether the groups WB_n, GVB_n, FVB_n, FWB_n and TW_n are linear groups, i.e. whether they admit faithful linear representations.

Problem 2: If possible, express WB_n' as a nontrivial amalgamated product. Do the same for the groups GVB_n', FVB_n', FWB_n' and TW_n'.

Problem 3: For each of the generalized braid groups considered in this thesis, construct explicit CW-complex whose fundamental group is the group in consideration; and prove finite generation or finite presentability of the group using geometric techniques.

Problem 4: Find the commutator widths of the generalized braid groups.

Problem 5: Does there exist a torsion free group G which is non-adorable and none of the terms in the derived series of G is free?

Problem 6: Is TW_n non-adorable?

Let G be a group and ϕ be an endomorphism of G. Then, two elements $x, y \in G$ are said to be Reidemeister equivalent if there exists another element $z \in G$ such that $x = z\ y\ \phi(z)^{-1}$; and the corresponding equivalence classes are called the *Reidemeister classes* or *twisted conjugacy classes*. The *Reidemeister number* of ϕ, denoted by $R(\phi)$, is the number of Reidemeister classes in G corresponding to ϕ. A group G has the R_∞ property if for every automorphism ϕ of G, $R(\phi)$ is infinite.

Problem 7: Do the generalized braid groups have the R_∞ property?

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