Integration in Finite Terms with Special Functions: Polylogarithmic Integrals, Logarithmic Integrals and Error Functions.

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Dedicated to

My Mumma and Papa

Certificate

Certified that the work incorporated in the thesis entitled "*Integration in Finite Terms with Special Functions: Polylogarithmic Integrals, Logarithmic Integrals and Error Functions*", submitted by *Yashpreet Kaur* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Dr. Varadharaj Ravi Srinivasan
Thesis Supervisor

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Contents

Abstract

The thesis work concerns the problem of integration in finite terms with special functions. The main theorem extends the classical theorem of Liouville in the context of elementary functions to various classes of special functions: error functions, logarithmic integrals, dilogarithmic and trilogarithmic integrals. The results are important since they provide a necessary and sufficient condition for an element of the base field to have an antiderivative in a field extension generated by transcendental elementary functions and special functions. A special case of the theorem simplifies and generalizes Baddoura's theorem for integration in finite terms with dilogarithmic integrals. The main theorem can be naturally generalized to include polylogarithmic integrals and to this end, a conjecture is stated for integration in finite terms with transcendental elementary functions and polylogarithmic integrals.

Chapter 1

Introduction

In this thesis we prove various extensions of Liouville's Theorem on integration in finite terms that include special functions such as error functions, logarithmic integrals, dilogarithms and trilogarithms along with transcendental elementary functions. A special case of our result generalises Baddoura's theorem for integration in finite terms with dilogarithmic integrals. Precise statements of our results can be found in Theorem 4.3.3, Theorem 4.4.3, Theorem 5.2.9 and Conjecture 5.3.8. Our results can be naturally generalised to include polylogarithms and to this end, a conjecture for integration in finite terms with polylogarithmic integrals along with transcendental elementary functions is stated.

Throughout the thesis, a field always means a field of characteristic zero. For a field F equipped with a single derivation map $'$, the kernel of the map $'$ is a subfield of F, which we denote by C_F. We will be working with differential field extensions of the form $E = F(\theta_1, \ldots, \theta_n)$, $F_0 := F$, $F_i = F_{i-1}(\theta_i)$ such that one of the following holds:

(i) θ_i is algebraic over F_{i-1}.

(ii) $\theta_i' = u'\theta_i$ for some $u \in F_{i-1}$ (i.e. $\theta_i = e^u$ and is called exponential of u).

(iii) $\theta_i' = u'/u$ for some $u \in F_{i-1}$ (i.e. $\theta_i = \log(u)$ and is called logarithm of u).

(iv) $\theta_i' = u'/v$, where $v' = u'/u$ for some $u, v \in F_{i-1}$ (i.e. $\theta_i = \int u'/\log(u)$ and is called logarithmic integral of u, also denoted by $\ell i(u)$).

(v) $\theta_i' = u'v$, where $v' = (-u^2)'v$ for some $u, v \in F_{i-1}$ (i.e. $\theta_i = \int u'e^{-u^2}$ and is called error function of u, also denoted by $erf(u)$).

(vi) $\theta_i' = vu'/u$, where $v' = -(1-u)'/(1-u)$ for some $u, v \in F_{i-1}$ (i.e. $\theta_i = -\int \frac{u'}{u}\log(1-u)$ and is called dilogarithmic integral of u, also denoted by $\ell_2(u)$).

(vii) $\theta_i' = vu'/u$, where $v' = -(u'/u)\log(1-u)$ for some $u, v \in F_{i-1}$ (i.e. $\theta_i = \int \frac{u'}{u}\ell_2(u)$ and is called trilogarithmic integral of u, also denoted by $\ell_3(u)$).

A differential field extension $E = F(\theta_1, \ldots, \theta_n)$ of F, with $C_E = C_F$, is called a $\mathcal{DEL}-$*extension* (respectively an *elementary extension*) if each θ_i satisfies at least one of the cases i-vi (respectively i, ii or iii). Elements of an elementary extension field are called *elementary functions*.

History. The problem of integration in finite terms for elementary functions was considered by J. Liouville (1834-35) and by J.F. Ritt (1948). A. Ostrowski generalized it to a wider class of meromorphic functions in the regions of complex plane. His approach gave an algebraic aspect to the problem. However, M. Rosenlicht [13] was the first to give a purely algebraic solution to the problem. He showed that if E is an elementary field extension of F with $C_E = C_F$ and there is an element $u \in E$ such that $u' \in F$ then there are constants $r_1, \ldots, r_n$ and elements $w, g_1, \ldots, g_n \in F$ such that $u' = \sum_{i=1}^{n} r_i(g_i'/g_i) + w'$. That is, up to an element of F, u must be a constant linear combination of logarithms of elements of F. The problem of extending Liouville's Theorem to allow special functions was first studied by J. Moses [10, 11]. Later in [4, 5], G. Cherry proved an extension of the Theorem to include logarithmic integrals and error functions. In [16], p.968, M. Singer, B. Saunders and B. Caviness extended Liouville's Theorem to include a large class of functions which they called $\mathscr{E}\mathscr{L}-$elementary functions. In particular, if θ_i satisfies any one of the cases i–v then it is an $\mathscr{E}\mathscr{L}-$elementary function. They proved that if u lies in a field extension of F containing transcendental $\mathscr{E}\mathscr{L}-$elementary functions and $u' \in F$, then u' is a finite linear combination of derivatives of $\mathscr{E}\mathscr{L}-$elementary functions over C_F. However, cases vi and vii were not covered under the $\mathscr{E}\mathscr{L}-$class of functions. In [1], p.933, J. Baddoura extended Liouville's Theorem to include dilogarithmic integrals. He called the extensions that satisfy i, ii, iii or vi as dilogarithmic-elementary extensions. He proved that if E is a transcendental dilogarithmic-elementary extension of F having an algebraically closed field of constants C_F and if F is a liouvillian extension of C_F then any $u \in E$ with $u' \in F$ has the following form over F:

$$u = w + \sum_{i=1}^{m} r_i \log(g_i) + \sum_{j=1}^{n} c_j D(h_j),$$

where each $r_i, g_i, h_j, w \in F$, $\log(g_i)$ and $D(h_j)$ belong to some differential field extension of F and $D(h_j)' = \frac{-1}{2}\frac{h_j'}{h_j}\log(1-h_j) + \frac{1}{2}\frac{(1-h_j)'}{1-h_j}\log(h_j)$. Baddoura's proof involves producing equation of above form over F when one such equation for u is given over $F(\theta)$, where θ satisfies ii, iii or vi. The problematic terms here are $r_i \log(g_i)$, where both r_i and g_i are arbitrary elements of $F(\theta)$. Lengthy and involved calculations, along with a new dilogarithmic identity were needed to obtain the desired expression over F. In the spirit of Liouville's theorem as extended by Singer, Saunders and Caviness, and from an algorithmic view point, it is desirable to obtain an expression for u' in terms of elements of F. However, no such expression for u' was produced in [1].

Our Theorem 4.3.3 ([6], p.227) restricted to transcendental dilogarithmic-elementary extensions will yield the following expression for u' over F:

$$u' = \sum_{i\in I} r_i \frac{g_i'}{g_i} + \sum_{l\in L} s_l \frac{h_l'}{h_l} + w', \tag{1.1}$$

where for each $i \in I$, $l, t \in L$ there are constants c_i, d_{il}, b_{lt}, with $c_i \neq 0$ whenever $r_i' \neq 0$, such that

$$r_i' = c_i \frac{(1-g_i)'}{1-g_i} + \sum_{l\in L} d_{il} \frac{h_l'}{h_l} \quad \text{and} \quad s_l' = \sum_{i\in I} d_{il} \frac{g_i'}{g_i} + \sum_{t\in L} b_{lt} \frac{h_t'}{h_t}. \tag{1.2}$$

The converse also holds: if an element $v \in F$ admits an expression as in Equations 1.1 and 1.2 then an antiderivative of v can be found in some transcendental dilogarithmic-elementary extension of F. We say that $v \in F$ admits a $\mathcal{DEL}-$expression over F if there are finite indexing sets I, J, K and elements $r_i, g_i \in F$ for all $i \in I$, elements $u_j, \log(u_j) \in F$ and constants a_j for all $j \in J$, elements $v_k, e^{-v_k^2} \in F$ and constants b_k for all $k \in K$, and an element $w \in F$ such that

$$v = \sum_{i\in I} r_i \frac{g_i'}{g_i} + \sum_{j\in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + w',$$

where for each $i \in I$, there is an integer n_i such that $r_i' = \sum_{l=1}^{n_i} c_{il} h_{il}'/h_{il}$ for some constants c_{il} and elements $h_{il} \in F$. A $\mathcal{DEL}-$expression will be called a *special* $\mathcal{DEL}-$expression if for each $i \in I$, $r_i' = c_i(1-g_i)'/(1-g_i)$ for some constant c_i and a special $\mathcal{DEL}-$expression will be called $\mathcal{D}-$expression if each $a_j = b_k =$

0. A $\mathcal{DEL}-$extension $E = F(\theta_1, \ldots, \theta_n)$ is called a transcendental *dilogarithmic-elementary* extension if $C_E = C_F$ and for each i, θ_i is transcendental over F_{i-1} and satisfies either case ii or iii or vi. For a transcendental exponential θ over F, a $\mathcal{DEL}-$expression for $u' \in F$ over $F(\theta)$ does not in general reduce to a similar expression over F, however, when it is a special $\mathcal{DEL}-$expression, it does reduce. We utilize this fact and set up a special induction procedure to prove our main results. The problematic terms that appear in our proof are those $r_i(g_i'/g_i)$, where $0 \neq r_i' = \sum_{l=1}^{n_i} c_{il}\frac{h_{il}'}{h_{il}}$. However, we only need basic dilogarithmic identities, in particular, we do not require Baddoura's dilogarithmic identity, to handle these terms. Consequently, we obtain a simpler proof of Baddoura's Theorem which neither requires that F is a liouvillian extension of C_F nor that C_F is an algebraically closed field.

Many of our results concerning dilogarithms can be naturally extended to polylogarithms. In particular, the induction procedure used for the dilogarithmic set-up can also be extended. We shall call a differential field extension $E = F(\theta_1, \ldots, \theta_n)$ with $C_E = C_F$, a transcendental trilogarithmic-elementary extension if it satisfies either ii, iii, vi or vii. We prove that $u \in E, u' \in F$ and E is a transcendental trilogarithmic-elementary extension of F if and only if

$$u' = \sum_{i\in I} r_i g_i'/g_i + \sum_{j\in J} s_j h_j'/h_j + w' \tag{1.3}$$

over F, where I and J are some finite index sets and each w, g_i, h_j, r_i, s_j are elements in F such that

$$r_i' = t_i\frac{g_i'}{g_i} + \sum_{j\in J} r_{ij}\frac{h_j'}{h_j}, \quad s_j' = \sum_{i\in I} r_{ij}\frac{g_i'}{g_i} + \sum_{k\in J} s_{jk}\frac{h_k'}{h_k}$$

$$t_i' = -c_i\frac{(1-g_i)'}{1-g_i} + \sum_{j\in J} c_i c_{ij}\frac{h_j'}{h_j}, \quad r_{ij}' = c_i c_{ij}\frac{g_i'}{g_i} + \sum_{k\in J} e_{ijk}\frac{h_k'}{h_k} \quad \text{and}$$

$$s'_{jk} = \sum_{i \in I} e_{ijk} \frac{g'_i}{g_i} + \sum_{l \in J} f_{jkl} \frac{h'_l}{h_l},$$

where each c_i is a non-zero constant, each c_{ij}, e_{ijk}, f_{jkl} are some constants and each t_i, r_{ij} and s_{jk} are elements in an extension of F with $e_{ijk} = e_{ikj}$ and $s_{jk} = s_{kj}$ for every i, j and k.

Note that when u lies in trilogarithmic-elementary extension, the coefficients in the expression for u' in Equation 1.3 satisfies $\mathcal{DEL}-$expressions and further the coefficients in those $\mathcal{DEL}-$expressions are sum of logarithms. Whereas, when u lies in dilogarithmic-elementary extension, u' satisfies a $\mathcal{DEL}-$expression as in Equation 1.1 and its coefficients are sum of logarithms. One can restrict various theorems concerning trilogarithmic integrals to dilogarithmic integrals. In particular, Theorem 4.3.3 can be deduced from Theorem 5.2.9. However, since the results concerning trilogarithmic integrals are lengthy and complicated, the proofs of these theorems are written separately for the convenience of reader.

In a nutshell, we only use standard techniques from differential algebra and many calculations involved boils down to comparing terms of certain partial fraction expansions. Many people are interested in constructing algorithms for integration in finite terms (See [3] and [12] for integration with elementary integrals). Our results contain both necessary and sufficient conditions and therefore, these results will help in formatting algorithms for integration in finite terms with transcendental elementary functions and trilogarithmic integrals. We believe that the results concerning dilogarithmic and trilogarithmic integrals can be naturally generalised to polylogarithmic integrals. To this end, we conclude the thesis with a conjecture on integration in finite terms with polylogarithmic integrals.

The thesis is organized into four chapters.

Chapter 2: We reproduce several well-known results from differential algebra, many of which are due to Ostrowski, Kolchin and Rosenlicht.

Chapter 3: Several results concerning $\mathcal{DEL}-$extensions, $\mathcal{DEL}-$expressions and dilogarithmic identities are proved.

Chapter 4: The main results concerning $\mathcal{DEL}-$extensions along with a generalisation of Baddoura's theorem is proved. We conclude the chapter by providing non-trivial examples that explain our results.

Chapter 5: We extend our results concerning dilogarithmic extensions to trilogarithmic extensions and state a conjecture for integration in finite terms with polylogarithmic integrals along with transcendental elementary functions.

Chapter 2

Preliminaries

In this chapter we record several standard results and terminologies from differential algebra to make the thesis self-contained. In particular, we shall include a proof of Kolchin-Ostrowski Theorem due to Singer & Rubel [15] and Rosenlicht's proof of Liouville's Theorem [13].

2.1 Basic conventions

Definition 2.1.1. A field F equipped with a linear map $' : F \to F$ that satisfies the Leibnitz rule, that is, $(fg)' = fg' + f'g$ for all $f, g \in F$, is called *differential field* and the map $'$ is called a *derivation*.

For any element f in F and a non-zero element g in F, the derivation on fraction f/g is $(f/g)' = (f'g - fg')/g^2$ and for a natural number n, $(f^n)' = nf^{n-1}f'$. In particular, $1' = 0$. Elements $c \in F$ such that $c' = 0$ are called *constants.* The kernel of the map denoted by $C_F := \{c \in F : c' = 0\}$ forms a field and will be called *field of constants or constant field* of F.

Definition 2.1.2. A field extension $E \supseteq F$ is called a *differential field extension* of F if there exists a differential field structure on E which is compatible with the differential field structure of F.

In the next proposition, we shall show that every field extension of a differential field is also a differential field. For transcendental extensions, the technique of the proof follows from [9], p.2 and for algebraic extensions, one can also look into [9], p.9, Example 1.13.

Proposition 2.1.3. ([13], p.154) *Let F be a differential field and $E \supset F$ be any field extension of F. Then there exists a derivation on E that makes E differential field extension of F. If E is an algebraic extension of F then the derivation on E is unique.*

Proof. Let $'$ be the derivation on F. Assume $E = F(\theta)$ where θ is transcendental over F. Consider the ring of dual numbers over $F(\theta)$ i.e the ring $F(\theta)[\epsilon] = F(\theta) + F(\theta)\epsilon$ where $\epsilon^2 = 0$. Since ϵ is nilpotent, an element $x = a + b\epsilon$ of $F(\theta)[\epsilon]$ is a unit if and only if a is a unit of $F(\theta)$.

Define a map $a_D = (id, D) : F[\theta] \to F(\theta)[\epsilon]$ as $a_D(x) = x + x'\epsilon$ for every $x \in F$ and $a_D(\theta) = \theta + f(\theta)\epsilon$, where $f(\theta)$ is any element in $F(\theta)$. It is easy to check that $D : F[\theta] \to F(\theta)$, defined as $D(x) = x'$ for every $x \in F$ and $D(\theta) = f(\theta)$, satisfies a differential structure on $F[\theta]$ if and only if a_D is a ring homomorphism. Since θ is a unit in $F(\theta)$, $\theta + f(\theta)\epsilon$ is a unit in $F(\theta)[\epsilon]$. Thus we can extend a_D to the homomorphism $a_E : F(\theta) \to F(\theta)[\epsilon]$, which is of the form (id, E), where E is a derivation on $F(\theta)$ extending D as well as $'$.

Now assume θ is algebraic over F with minimal polynomial $P(X) = \sum_{i=0}^{n} p_i X^i$, where X is an indeterminate. As observed above, $D : F(X) \to F(X)$ mapping X to any rational function $Q(X) \in F(X)$ is a derivation on $F(X)$. Observe $D(P(X)) = \sum_{i=0}^{n} p_i' X^i + \sum_{i=1}^{n} i p_i X^{i-1} Q(X)$ and thus $\sum_{i=0}^{n} p_i' \theta^i + \sum_{i=1}^{n} i p_i \theta^{i-1} Q(\theta) = 0$. Clearly $\sum_{i=1}^{n} i p_i \theta^{i-1} \neq 0$ and therefore, we get a unique derivation on $F(\theta)$ given by

$$D(\theta) = Q(\theta) = -\frac{\sum_{i=0}^{n} p_i' \theta^i}{\sum_{i=1}^{n} i p_i \theta^{i-1}}.$$

Thus, any simple field extension of F is a differential field extension with derivation D. Using Zorn's lemma, the derivation D can be extended to any arbitrary field extension of F. □

In literature, differential field means a field with a family of derivations but throughout this thesis we fix a single derivation map $'$ on differential field F.

Definition 2.1.4. ([13], p.153) Let f, g be elements of a differential field F such that $g \neq 0$ and $f' = g'/g$ then in correspondence to the classical theory, f is called *logarithm* of g denoted by $\log g$ and g is called *exponential* of f denoted by e^f.

If g has a logarithm in the field F then it is unique up to an additive constant and if f has an exponential in F then it is unique up to a multiplicative constant. Therefore, for some constants c, d and elements g_1, g_2 in F, $\log(g_1 g_2) = \log g_1 + \log g_2 + c$ and $\log(-g_1) = \log(g_1) + d$.

2.2 Liouvillian extensions

Let $E \supset F$ be a differential field extension. An element $\theta \in E$ is called *primitive* over F if $\theta' \in F$. Note that a logarithm of some element in F is primitive over F.

Definition 2.2.1. ([8], p.408) A differential field extension $E = F(\theta_1, \ldots, \theta_n)$, $F_0 := F$, $F_i = F_{i-1}(\theta_i)$ is called *liouvillian extension* of F if for each i, either θ_i is algebraic over F_{i-1}, $\theta_i'/\theta_i \in F_{i-1}$ or $\theta_i' \in F_{i-1}$.

Now we shall describe some properties of liouvillian field extensions.

Proposition 2.2.2. ([6], pp.212-213) *Let $F \subsetneqq F(\theta)$ be differential fields and θ be algebraic over F. Then the following statements hold:*

(a) If $\theta' \in F$ then there is an element $x \in F$ such that $x' = \theta'$ and $C_{F(\theta)} \supsetneqq C_F$.

(b) If $\theta'/\theta \in F$ then there is an element $x \in F - 0$ and an integer n such that $x'/x = n\theta'/\theta$. Furthermore, if $C_{F(\theta)} = C_F$ then the minimal monic polynomial of θ over F is of the form $P(X) = X^n + cx$ for some $c \in C_F$.

(c) Every $c \in C_{F(\theta)}$ is algebraic over C_F.

Proof. Let $P(X) = \sum_{i=0}^{n} a_i X^i$, $a_n = 1$ and $n \geq 2$ be the minimal monic polynomial of θ over F. Differentiating $\sum_{i=0}^{n} a_i \theta^i = 0$, we obtain that θ is also a root of the polynomial

$$P'(X) = (n\theta' + a_{n-1}')X^{n-1} + \cdots + \left(ia_i\theta' + a_{i-1}'\right) X^{i-1} + \cdots + a_1\theta' + a_0' \in F[X].$$

If $\theta' \in F$ then by minimality of $P(X)$, $P'(X)$ must be the zero polynomial. In particular, $(-a_{n-1}/n)' = \theta'$ and $\theta + (a_{n-1}/n)$ is constant that is not in C_F. A similar calculation with the minimal monic polynomial over F of $c \in C_{F(\theta)}$ would give us that $a_i' = 0$ for all i and thus c is algebraic over C_F. If $\theta'/\theta = \alpha \in F$ then we shall rewrite

$$P'(X) = n\alpha X^n + (a_{n-1}' + (n-1)a_{n-1}\alpha)X^{n-1} + \cdots + a_0'$$

and observe that $P'(X) = n\alpha P(X)$. Then for each $i \in \{1, \ldots, n\}$, we have

$$a'_{i-1} = (n - (i-1))\alpha a_{i-1}.$$

In particular, $a'_0 = n\alpha a_0$ and since $a_0 \neq 0$, we have $n\theta'/\theta = a'_0/a_0$. Finally, if $C_{F(\theta)} = C_F$ then $a_i = 0$ for all $i \in \{1, \ldots, n-1\}$. Otherwise, $(\theta^{n-i}/a_i)' = 0$ and therefore $\theta^{n-i} + ca_i = 0$ for some non zero constant $c \in C_F$. This contradicts the assumption that $P(X)$ is of degree n. □

Remark 2.2.3. If $\overline{F}$ is an algebraic closure of the field F then it is clear from the part (c) of Proposition 2.2.2 that $C_{\overline{F}} = C_F$ if and only if C_F is algebraically closed field.

Proposition 2.2.4. ([6], p.213) *Let $F \subset F(\theta)$ be differential fields, θ be transcendental over F, $\theta' \in F$ and $v = \sum_{i=0}^{s} \beta_i\theta^i \in F[\theta]$ be a polynomial in θ over F. Suppose that there is a $w \in F(\theta) - F$ such that $w' = v$.*

(a) If $C_{F(\theta)} = C_F$ then $w = \sum_{i=0}^{t} \alpha_i\theta^i \in F[\theta]$, $\alpha_t \neq 0$ and $t \geq 1$.

(b) If $v = 0$, that is $C_{F(\theta)} \supsetneq C_F$, then there is a non zero constant $c \in C_F$ and $\alpha_0 \in F$ such that $(c\theta + \alpha_0)' = 0$.

(c) If $v \neq 0$, $C_{F(\theta)} = C_F$ and $s =$deg(v) then either deg$(w)= s$ or $s + 1$. In the former case $\alpha'_t = \beta_s$ and in the latter case $\alpha_t \in C_F$ and $(t\alpha_t\theta + \alpha_{t-1})' = \beta_s$.

(d) If $\alpha \in F$, $x' \neq \alpha$ for all $x \in F$ and $\theta' = \alpha$ then $C_{F(\theta)} = C_F$. In general, if $\alpha_1, \ldots, \alpha_n \in F$ are non zero elements then there is a differential field extension E of F such that $C_E = C_F$ and $E = F(\theta_1, \ldots, \theta_n)$, where $\theta'_i = \alpha_i$.

Proof. Let there be an element $w \in F(\theta)$ such that $w' = v$. Then there are relatively prime polynomials $P, Q \in F[\theta]$, where Q is monic, such that $w = P/Q$. Taking derivatives, we obtain

$$Q^2 v = P'Q - Q'P. \tag{2.1}$$

From the above equation, it is immediate that Q divides Q'. Since Q is monic and $\theta' \in F$ we have $\deg Q' < \deg Q$. This forces $Q' = 0$. If $C_{F(\theta)} = C_F$, then $Q = 1$ and thus $P = w \in F[\theta] - F$. Now suppose that $v = 0$. If $Q = 1$ then $\deg P \geq 1$ and $P' = 0$ and if $Q \neq 1$ then $\deg Q \geq 1$ and as observed earlier $Q' = 0$. Thus we have $\left(\sum_{i=0}^{t} \alpha_i \theta^i\right)' = 0$, $\alpha_t \neq 0$ and $t \geq 1$. Now we compare coefficients and obtain that $\alpha_t' = 0$ and $(t\alpha_t\theta + \alpha_{t-1})' = t\alpha_t\theta' + \alpha_{t-1}' = 0$. This proves (b).

From (a), we have

$$\alpha_t'\theta^t + (t\alpha_t\theta' + \alpha_{t-1}')\theta^{t-1} + \cdots + \alpha_1\theta' + \alpha_0' = \sum_{i=0}^{s} \beta_i\theta^i = v. \tag{2.2}$$

If $\deg(w) = s \geq 0$ and $C_{F(\theta)} = C_F$ then it is easy to see that $t = s$ or $t = s + 1$. If $t = s$ then $\alpha_t' = \beta_s$, where $\alpha_t \in F$ and if $t = s + 1$ then $\alpha_t \in C_F$ and $(t\alpha_t\theta + \alpha_{t-1})'$ $= t\alpha_t\theta' + \alpha_{t-1}' = \beta_s$.

Suppose that $\theta' = \alpha$ and $x' \neq \alpha$ for all $x \in F$. If $w \in F(\theta) - F$ and $w' = 0$ then from (b) there is a nonzero constant $c \in C_F$ and an element $\alpha_0 \in F$ such that $0 = (c\theta + \alpha_0)' = c\alpha + \alpha_0'$. Thus $(-\alpha_0/c)' = \alpha$ and this is a contradiction. Finally, let $F_0 = F$ and F_{n-1} be a differential field extension of F such that $C_{F_{n-1}} = C_F$ and $F_{n-1} = F(\theta_1, \ldots, \theta_{n-1})$, where $\theta_i' = \alpha_i$ for all $1 \leq i \leq n-1$. If there is no element $x \in F_{n-1}$ such that $x' = \alpha_n$ then let θ_n be a transcendental and define a derivation on $E := F_{n-1}(\theta)$ by defining $\theta_n' = \alpha_n$. Clearly, $C_E = C_{F_{n-1}} = C_F$. On the other hand if there is an element $x \in F_{n-1}$ such that $x' = \alpha_n$ then take θ_n to be x. □

We repeatedly use partial fraction expansions in our results. Thus in this spirit, it is useful to note the following proposition.

Proposition 2.2.5. ([6], p.214) *Let $F(\theta) \supset F$ be a transcendental liouvillian extension of F with $C_{F(\theta)} = C_F$. Let $v \in F(\theta)$, $\overline{F}$ be an algebraic closure of F and $v = \eta \prod_{j=1}^{s} (\theta - \alpha_j)^{m_j}$, where $\eta \in F$, $0 = \alpha_1, \ldots, \alpha_s$ are distinct elements in $\overline{F}$ and m_j are integers.*

(a) Suppose that $\theta' \in F$. Then each $\theta' - \alpha_j'$ is a non zero element of $\overline{F}$ and

$$\frac{v'}{v} = \frac{\eta'}{\eta} + \sum_{j=1}^{s} m_j \frac{\theta' - \alpha_j'}{\theta - \alpha_j} \tag{2.3}$$

is the partial fraction expansion of v'/v.

(b) Suppose that $\theta'/\theta \in F$. Then

(b.1)

$$\frac{v'}{v} = \mu + \sum_{j=2}^{s} m_j \frac{\mu_j}{\theta - \alpha_j}, \tag{2.4}$$

where $\mu = (\eta'/\eta) + \sum_{j=1}^{s} m_j(\theta'/\theta) \in F$ and $\mu_j = \alpha_j(\theta'/\theta) - \alpha_j' \in \overline{F} - \{0\}$, is the partial fraction expansion of v'/v.

(b.2) If v_0 is the constant term of the partial fraction expansion of v in $\overline{F}(\theta)$ then the constant term of v' is v_0'.

(c) If $v \in \overline{F}(\theta)$ has a pole of order $m \geq 1$ and $\theta' \in F$ then v' has a pole of order $m + 1$. Similarly, if $v \in \overline{F}(\theta)$ has a non-zero pole of order $m \geq 1$ and $\theta'/\theta \in F$ then v' has a pole of order $m + 1$.

Proof. If $\theta' \in F$ and $\theta' = \alpha_j'$ for some α_j then by Proposition 2.2.2, there is an element $x \in F$ such that $\theta' = x'$. Now $\theta - x \notin F$ is a constant of $F(\theta)$ and this contradicts our assumption that $C_{F(\theta)} = C_F$. Similarly, if $\theta'/\theta = x \in F$ and

$\alpha_j' = x\alpha_j$ then again from Proposition 2.2.2, there are an integer n and an element $y \in F$ such that $nx = n(\theta'/\theta) = y'/y$. Thus $\theta^n/y \in F(\theta) - F$ is a constant which again contradicts our assumption. A straightforward calculation shows that Equations 2.3 and 2.4 represents the partial fraction expansion of v'/v. Let

$$v = \sum_{i=1}^{l} \sum_{j=1}^{m_i} \frac{v_{ij}}{(\theta - \alpha_i)^j} + v_0 + v_1\theta + \cdots + v_n\theta^n,$$

where elements α_i, v_i and v_{ij} belong to $\overline{F}$ be the partial fraction expansion of v over $\overline{F}$. Note that

$$\left(\frac{v_{ij}}{(\theta - \alpha_i)^j}\right)' = \begin{cases} \dfrac{v_{ij}'}{(\theta - \alpha_i)^j} + \dfrac{-jv_{ij}(x - \alpha_i')}{(\theta - \alpha_i)^{j+1}} & \text{if } \ \theta' = x \in F \\ \dfrac{v_{ij}'}{(\theta - \alpha_i)^j} + \dfrac{-jv_{ij}(x\alpha_i - \alpha_i')}{(\theta - \alpha_i)^{j+1}} + \dfrac{-jv_{ij}x}{(\theta - \alpha_i)^j} & \text{if } \ \theta'/\theta = x \in F \end{cases} \tag{2.5}$$

and

$$(v_i\theta^i)' = \begin{cases} iv_ix\theta^{i-1} + v_i'\theta^i & \text{if } \ \theta' = x \in F \\ (v_i' + iv_ix)\theta^i & \text{if } \ \theta'/\theta = x \in F. \end{cases}$$

From this observation it follows that when $\theta'/\theta \in F$, the constant term of v' is v_0'. Suppose that v has a pole at α_i of order m_i. Then, $-m_iv_{im_i}(x - \alpha_i') \neq 0$ when $\theta' \in F$ and $-m_iv_{im_i}(x\alpha_i - \alpha_i') \neq 0$ when $\alpha_i \neq 0$ and $\theta'/\theta \in F$. Therefore, from Equation 2.5, we obtain that v' has a pole of order $m_i + 1$ at α_i. □

The following Proposition is due to M. Rosenlicht [13], p.155. Note that only partial fraction expansions are required to prove the result.

Proposition 2.2.6. ([13], p.155) *Let $F(\theta) \supset F$ be a liouvillian extension with θ transcendental over F and $C_{F(\theta)} = C_F$. Suppose that $v, u_1, \ldots, u_n \in F(\theta)$ and $w \in F$*

are elements such that

$$v' + \sum_{i=1}^{n} c_i(u_i'/u_i) = w,$$

where $c_1, \ldots, c_n$ are $\mathbb{Q}-$linearly independent constants then

(a) If $\theta' \in F$ then $u_i \in F$ for all i and $v = c\theta + \beta$ for some constant c and $\beta \in F$.

(b) If $\theta'/\theta \in F$ then $v \in F$ and for each i, $u_i = \eta_i\theta^{m_i}$ where $\eta_i \in F$ and m_i is an integer.

Proof. Let $\overline{F}$ be an algebraic closure of F and $u_i = \eta_i \prod_{j=1}^{t}(\theta - \alpha_j)^{m_{ij}}$, where for each i and j, $\eta_i \in F$, $0 = \alpha_1, \ldots, \alpha_t$ are distinct elements in $\overline{F}$ and m_{ij} are integers.

(a) From Proposition 2.2.5 part a,c, η_i'/η_i has poles of order 1 only and if v has poles then v' has poles of order greater than 1. For cancellation to take place we must have $\sum_{i=1}^{n} c_i m_{ij} = 0$ for each j. Since $c_i's$ are $\mathbb{Q}-$linearly independent constants, every $m_{ij} = 0$. Thus for each i, $u_i \in F$ and $v' \in F$. Using Proposition 2.2.4 we have $v = c\theta + \beta$ for some constant c and $\beta \in F$.

(b) Again from Proposition 2.2.5 part b,c, η_i'/η_i has non-zero poles of order 1 only and if v has non-zero poles then v' has poles of order greater than 1. Thus it follows that for all i and $j = 2, \ldots, t$, $m_{ij} = 0$, $u_i = \eta_i\theta^{m_{i1}}$ and $v \in F$. □

Proposition 2.2.7. ([14], p.338) *Let $E \supset F$ be an algebraic extension of F with $C_E = C_F$. Assume F is a liouvillian extension of C_F and suppose that there are $\mathbb{Q}-$linearly independent constants $c_1, \ldots, c_n$, elements $u_1, \ldots, u_n \in E^*$, $v \in E$ such that*

$$v' + \sum_{i=1}^{n} c_i(u_i'/u_i) \in F.$$

Then $v \in F$ and there is a non-zero integer m such that $u_i^m \in F$ for all i.

When F is liouvillian over its constant field C_F then M. Rosenlicht and M. Singer (See [14], p.338) proved this result for algebraic extensions, similar to previous proposition.

Proof. We use induction on tr $\deg F/C_F$ to prove the result. If tr $\deg F/C_F = 0$ then $C_F = F = E$ and result is trivial. Assume tr $\deg F/C_F > 0$ and suppose that the result is true for smaller degrees.

Case-I. Suppose that $v' + \sum_{i=1}^{n} c_i(u_i'/u_i) = 0$. Choose a liouvillian extension F_0 of C_F contained in F and an element θ such that θ is transcendental over F_0 and F is algebraic over $F_0(\theta)$.

If $\theta' \in F_0$ then from Proposition 2.2.6(a), we conclude that $u_1, \dots, u_n$ are algebraic over F_0 and there is a constant $c \in C_F$ such that $v + c\theta$ is algebraic over F_0. Thus $(v+c\theta)' + \sum_{i=1}^{n} c_i(u_i'/u_i) \in F_0$ and by induction hypothesis it follows that $v + c\theta \in F_0$, $v \in F$ and there is a non-zero integer m such that $u_i^m \in F_0 \subset F$ for all i.

If $\theta'/\theta \in F_0$ then again from Proposition 2.2.6(b), observe that v is algebraic over F_0 and there are integers $m_0, m_1, \dots, m_n$ with $m_0 \neq 0$ such that for each i, $u_i^{m_0}\theta^{m_i}$ is algebraic over F_0. Thus

$$m_0 v' + \sum_{i=1}^{n} c_i \frac{(u_i^{m_0}\theta^{m_i})'}{u_i^{m_0}\theta^{m_i}} = \sum_{i=1}^{n} c_i m_i \frac{\theta'}{\theta} \in F_0.$$

We again apply induction hypothesis to conclude that $v \in F_0 \subset F$ and that there exists a non-zero integer m such that $(u_i^{m_0}\theta^{m_i})^m \in F_0$. This implies $(u_i^{m_0})^m \in F_0(\theta) = F$. This prove the result in this particular case.

Case-II. In general, let $v' + \sum_{i=1}^{n} c_i(u_i'/u_i) = w \in F$, where w is some element in F. Let L be smallest normal algebraic extension of F containing u_i, v. Let $N = [L : F]$

and consider the trace with respect to L then

$$\sum_{i=1}^{n} c_i \frac{\mathrm{Nr}(u_i)'}{\mathrm{Nr}(u_i)} + \mathrm{Tr}(v)' = Nw = N\sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + Nv'$$

and

$$\sum_{i=1}^{n} c_i \frac{(\mathrm{Nr}(u_i)u_i^{-N})'}{\mathrm{Nr}(u_i)u_i^{-N}} + (\mathrm{Tr}(v) - Nv)' = 0.$$

Now this reduces to case-I. Therefore, we have $\mathrm{Tr}(v) - Nv \in F$ and there exists a non-zero integer m such that $(\mathrm{Nr}(u_i)u_i^{-N})^m \in F$. Hence, $v \in F$ and $u_i^{mN} \in F$ for each i. □

Remark 2.2.8. The condition that F must be liouvillian over C_F is essential to the proof of Proposition 2.2.7(See [14], p.339). Consider the field of formal power series $\mathbb{C}((x))$ with the usual derivation $x' = 1$. Let $E = F((x^{1/2}))$ then $u'/u = v'$ where $u = exp(x^{1/2})$ and $v = x^{1/2}$. Here neither v nor any power of u lies in F.

2.3 Kolchin-Ostrowski Theorem

The Theorem provides a criterion for algebraic independence of exponentials and primitive elements of a differential field. It was first proved, using analytic techniques, by A. Ostrowski for a set of primitive elements over the field of meromorphic functions over complex numbers. Later, using the language of differential Galois theory, the theorem was reformulated and generalised by E. Kolchin ([7], p.1155) to include exponentials. Kolchin's proof of the Kolchin-Ostrowski Theorem uses his Galois theory of strongly normal extensions, whereas the one we provide here is due to L. Rubel and M. Singer (See [15], Appendix, p.366) and it uses only elementary techniques from differential algebra. In order to proceed with the proof of the theorem we need the following two lemmas. An alternate proof to the first lemma that uses the theory of differential ideals, can be found in [9], Chapter-1, pp.7-8.

Lemma 2.3.1. ([15], p.367) *Let $E \supset F$ be differential fields with $C_E = C_F$. Let $u \in E$ be an algebraic element over F.*

(a) If $u' \in F$ then $u \in F$.

(b) If $u'/u \in F$ then there is a non-zero integer m such that $u^m \in F$.

Proof. Let $P(X) = \sum_{i=0}^{n} p_i X^i$ be the minimal monic polynomial of u over F, where $n > 1$ and $p_n = 1$.

(a) When $u' \in F$, we differentiate $P(u)$ and observe that $\sum_{i=1}^{n}(p'_{i-1} + ip_i u')u^{i-1} = 0$. But the minimal polynomial of u over F is of degree n. Therefore $p'_{n-1} + nu' = 0$ which implies $u = (-1/n)p_{n-1} + c$ for some constant c and hence $u \in F$.

(b) When $u'/u = v \in F$ assume that for some $j \neq 0$, $p_j \neq 0$. Differentiating $P(u)$ we have $\sum_{i=0}^{n}(p'_i + ivp_i)u^i = 0$. Thus $P'(u)$ must be a multiple of $P(u)$ and hence $p'_j + jvp_j = nvp_j$. That is

$$\frac{p'_j}{p_j} = (n-j)\frac{u'}{u}.$$

Therefore, $(u^{n-j}/p_j)' = 0$. Since $C_E = C_F$ we have $u^{n-j}/p_j \in F$ and $u^{n-j} \in F$. This completes the proof. □

Lemma 2.3.2. ([15], p.367) *Let $E \supset F$ be differential fields with $C_E = C_F$. Let $v \in E$ be transcendental over F and $u \in E$ be algebraic over $F(v)$.*

(a) If $v' \in F$ and $u' \in F$ then there is a constant c such that $u + cv \in F$.

(b) If $v' \in F$ and $u'/u \in F$ then $u^m \in F$ for some non-zero integer m.

(c) If $v'/v \in F$ and $u'/u \in F$ then $u^m v^n \in F$, where m and n are some integers, both not zero.

Proof. (a) Since u is algebraic over $F(v)$ and $u' \in F \subset F(v)$, Lemma 2.3.1(a) implies $u \in F(v)$. From Proposition 2.2.6(a) we have $u = cv + w$ for some constant c and element $w \in F$.

(b) Since u is algebraic over $F(v)$ and $u'/u \in F \subset F(v)$, Lemma 2.3.1(b) implies that $u^m \in F(v)$ for some non-zero integer m. Now $(u^m)'/u^m \in F$ and $v' \in F$, therefore, from Proposition 2.2.6(a) we have $u^m \in F$.

(c) Again apply Lemma 2.3.1(b) and observe $u^m \in F(v)$ for some non-zero integer m. Since $v'/v \in F$ and $(u^m)'/u^m \in F$, from Proposition 2.2.6(b) we have $u^m = \eta v^n$, where $\eta \in F$ and n is any integer. Thus, $u^m v^{-n} \in F$. □

Kolchin-Ostrowski's Theorem. ([7], p.1155) *Let $E \supset F$ be differential fields with $C_E = C_F$. Let $v_1, \ldots, v_n$ be elements in E such that $v_j' \in F$ for each $j = 1, \ldots, n$ and $u_1, \ldots, u_m$ be non-zero elements in E such that $u_i'/u_i \in F$ for all $i = 1, \ldots, m$. If $u_1, \ldots, u_m, v_1, \ldots, v_n$ are algebraically dependent over F, then either there are constants $c_1, \ldots, c_n$, not all zero, such that $\sum_{j=1}^{n} c_j v_j \in F$ or there are integers $n_1, \ldots, n_m$, not all zero, such that $\prod_{i=1}^{m} u_i^{n_i} \in F$.*

Proof. We prove the result by using induction on $m + n$. When $m + n = 1$, the problem reduces to Lemma 2.3.1. Suppose $m + n > 1$.

Case-I. If $n \neq 0$ then $u_1, \ldots, u_m, v_2, \ldots, v_n$ are algebraically dependent over $F(v_1)$. Therefore, by induction either $\sum_{j=2}^{n} c_j v_j \in F(v_1)$ for some constants c_j, not all zero or $\prod_{i=1}^{m} u_i^{n_i} \in F(v_1)$, where n_i are some integers, not all zero. When v_1 is algebraic over F, we shall apply Lemma 2.3.1(a) to get $1.v_1 + 0.v_2 + \cdots + 0.v_n \in F$. When v_1 is transcendental over F, we apply 2.3.2 and obtain that either $v_1 + c\sum_{j=2}^{n} c_j v_j \in F$, where c is any constant or $\prod_{i=1}^{m} u_i^{n_i} \in F$.

Case-II. If $m \neq 0$ then similarly $u_2, \ldots, u_m, v_1, \ldots, v_n$ are algebraically dependent over $F(u_1)$. Therefore, by induction either $\sum_{j=1}^{n} c_j v_j \in F(u_1)$ for some constants c_j, not all zero or $\prod_{i=2}^{m} u_i^{n_i} \in F(u_1)$, where n_i are some integers, not all zero. When u_1 is algebraic over F, we apply Lemma 2.3.1(b) to get a non-zero integer n_1 such that $u_1^{n_1} \in F$. When u_1 is transcendental over F, we shall apply 2.3.2 and obtain $u_1^{n_1} \prod_{i=2}^{m} u_i^{\nu n_i} \in F$, where n_1, ν are some integers, not both zero. □

Corollary 2.3.3. ([6], p.215) *Let $E = F(\theta = \theta_1, \theta_2, \ldots, \theta_n)$ be a liouvillian extension of F with $C_E = C_F$.*

(a) If $y \in E$, $y' \in F$ and for each i, $\theta_i' \in F$ then $y = \sum_{i=1}^{n} c_i\theta_i + \eta$, where c_i are constants and $\eta \in F$.

(b) If $\theta_i' \in F$ for all $i \geq 2$ and θ is transcendental over F then there are elements $y_1, \ldots, y_t \in \{\theta_2, \ldots, \theta_n\}$ such that $\theta, y_1, \ldots, y_t$ are algebraically independent over F and $E = F(\theta, y_1, \ldots, y_t)$.

(c) Suppose that $n = 2$ and $E = F(\theta, \theta_2)$ is a transcendental liouvillian extension of F such that $\theta' \in F$ and $\theta_2' = v'/v$ for some $v \in F(\theta) - F$. If $y \in E$ and $y' \in F$ then $y \in F(\theta)$ and $y = c\theta + \eta$ for some constant c and $\eta \in F$.

Proof. (a) follows from Kolchin-Ostrowski Theorem. Let $\{y_1, \ldots, y_t\} \subset \{\theta_2, \ldots, \theta_n\}$ be such that $\theta, y_1, \ldots, y_t$ is a transcendence base of E over F. Suppose that there is a smallest integer i such that $\theta_i \notin F^* := F(\theta, y_1, \ldots, y_t)$. Then θ_i must be algebraic over F^* and since $\theta_i' \in F$ for all $i \geq 2$, the field F^* is a differential field. Now from Proposition 2.2.2, we obtain $C_{F^*(\theta_i)} \supsetneqq C_F$ and this contradicts our assumption that $C_E = C_F$. Finally, if $n = 2$, $\theta_2' = v'/v$ for some $v \in F(\theta) - F$ and $y \in E$ with $y' \in F$ then we shall use (a) to find some constant c and an element $f \in F(\theta)$ such that $y = c\theta_2 + f$. Taking derivatives, we obtain

$$y' = c(v'/v) + f'. \tag{2.6}$$

As in Proposition 2.2.5, we write $v = \eta \prod_{j=1}^{s}(\theta - \alpha_j)^{m_j}$, where $\eta \in F$, $0 = \alpha_1, \ldots, \alpha_s$ are distinct elements in $\overline{F}$ and m_j are integers. Since $v \in F(\theta) - F$, we must have a j such that $m_j \neq 0$. Now since $y' \in F \subset F(\theta)$ and $f' \in F(\theta)$, from Equation 2.3, we conclude that c must be zero for Equation 2.6 to hold. Thus $y - f \in C_F \subset F$ and that $y \in F(\theta)$. Now we apply (a) to obtain that $y = c\theta + \eta$ for some $c \in C_F$ and $\eta \in F$. □

2.4 Liouville's Theorem

Here we explain Rosenlicht's proof of Liouville's Theorem on integration in finite terms.

Definition 2.4.1. ([13], p.153) A differential field extension E over F is called *elementary extension* if there is a tower of differential fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = E$ such that for each $1 \leq i \leq n$, $F_i = F_{i-1}(\theta_i)$ and θ_i satisfies one of the following:

(i) θ_i is algebraic over F_{i-1}.

(ii) $\theta_i' = u'\theta_i$ for some $u \in F_{i-1}$ (i.e. $\theta_i = e^u$).

(iii) $\theta_i' = u'/u$ for some $u \in F_{i-1}$ (i.e. $\theta_i = \log(u)$).

Note that an elementary extension is always a liouvillian extension.

Remark 2.4.2. Note that if $u_1, u_2 \in F$ and $c \in C_F$ then for any $p/q \in \mathbb{Q}$, we have

$$c\frac{u_1'}{u_1} + c\frac{p}{q}\frac{u_2'}{u_2} = \frac{c}{q}\frac{(u_1^q u_2^p)'}{u_1^q u_2^p}.$$

In general, if $u_1, \ldots, u_m \in F$ and $a_1, \ldots, a_m \in C_F$ then $\sum_{i=1}^m a_i(u_i'/u_i) = \sum_{i=1}^p c_i(v_i'/v_i)$, where $c_1, \ldots, c_p$ is a $\mathbb{Q}-$basis for the vector space spanned by $a_1, \ldots, a_m$ over $\mathbb{Q}$ and $v_i = \prod_{j=1}^m u_j^{q_j}, q_j \in \mathbb{Z}$.

We recall that if E is an algebraic extension of F, $u \in E$ and $P(X) = X^m + \alpha_{n-1}X^{m-1} + \cdots + \alpha_0$ is the minimal monic polynomial of u over F then $\mathrm{tr}(u) := -\alpha_{n-1}$ and $\mathrm{nr}(u) = (-1)^m\alpha_0$. Let L be a finite Galois extension of F containing u with Galois group G and $n := [L : F]$. Define $\mathrm{Tr}(u) := \sum_{\sigma \in G}\sigma(u)$ and $\mathrm{Nr}(u) := \prod_{\sigma \in G}\sigma(u)$. It is easy to see that $\mathrm{Tr}(u)$ and $\mathrm{Nr}(u)$ belong to F and

$$\mathrm{Tr}(u) = \frac{n}{m}\mathrm{tr}(u) \quad \text{and} \quad \mathrm{Nr}(u) = \mathrm{nr}(u)^{\frac{n}{m}}.$$

Liouville's Theorem. ([13], pp.157-158) *Let $E \supset F$ be an elementary field extension of F with $C_E = C_F$. If there is an element $u \in E$ with $u' \in F$ then there are $\mathbb{Q}-$linearly independent constants $c_1, \ldots, c_n$, non-zero elements $g_1, \ldots, g_n \in F$ and an element $w \in F$ such that*

$$u' = \sum_{i=1}^{n} c_i \frac{g_i'}{g_i} + w'.$$

Proof. We prove the result by induction on length m of the tower

$$F = F_0 \subset F_1 \subset \cdots \subset F_m = E.$$

If $m = 0$ the result is trivial. Let $m > 0$, then by induction the result holds for the tower $F_1 \subset \cdots \subset F_m = E$, that is,

$$u' = \sum_{i=1}^{n} c_i \frac{g_i'}{g_i} + w',$$

where $c_i \in C_F$ and $g_i, w \in F_1$ for all i.

Case-I. When θ_1 is transcendental over F. This case further reduces to two sub cases.

Sub case-I. If θ_1 is logarithm over F, that is, $\theta_1' = x'/x$ for some $x \in F$ then we apply Proposition 2.2.6(a) and obtain $g_i \in F$ for all i and $w = c\theta_1 + w_0$ for some constant c and element $w_0 \in F$. Therefore,

$$u' = \sum_{i=1}^{n} c_i \frac{g_i'}{g_i} + c\frac{x'}{x} + w_0'.$$

If $c, c_1, \ldots, c_n$ are $\mathbb{Q}-$linearly dependent then as noted in Remark 2.4.2, the sum can be reduced further so that the constants are $\mathbb{Q}-$linearly independent.

Sub case-II. If θ_1 is exponential over F, that is, $\theta_1' = x'\theta_1$ for some $x \in F$ then we apply Proposition 2.2.6(b) and obtain $w \in F$ and $g_i = \eta_i \theta^{m_i}$ where $\eta_i \in F$ and $m_i \in \mathbb{Z}$ for all i. Therefore,

$$u' = \sum_{i=1}^{n} c_i \frac{\eta_i'}{\eta_i} + \sum_{i=1}^{n} c_i m_i x' + w'.$$

Case-II. When θ_1 is algebraic over F. Let L be a finite Galois extension of F that contains F_1 with Galois group G. Then for any $\sigma \in G$, we have

$$u' = \sum_{i=1}^{n} c_i \frac{(\sigma g_i)'}{\sigma g_i} + (\sigma w)'$$

and

$$[L:F]u' = \sum_{i=1}^{n} c_i \sum_{\sigma \in G} \frac{(\sigma g_i)'}{\sigma g_i} + \sum_{\sigma \in G} (\sigma w)' = \sum_{i=1}^{n} c_i \frac{\mathrm{Nr}(g_i)'}{\mathrm{Nr}(g_i)} + \mathrm{Tr}(w)'.$$

Since $\mathrm{Nr}(g_i), \mathrm{Tr}(w) \in F$ and constants $(1/[L:F])c_i$ are $\mathbb{Q}-$linearly independent, we obtained the desired result. □

Using this theorem, M. Rosenlicht (See [13], p.160) proved that error functions and logarithmic integrals are non-elementary functions over $F = C(z)$, where $C_F = C$ and z is an indeterminate with derivative $z' = 1$.

2.5 Error functions and logarithmic integrals

Let F be a differential field. For an element $u \in F$, *error function* ([10], p.18) is defined as

$$\int u' e^{-u^2}$$

and is denoted by $erf(u)$.

A *logarithmic integral* ([16], p.968) of an element $v \in F$ is defined as

$$\int \frac{v'}{\log v}$$

and is denoted by $li(v)$.

Suppose $F = \mathbb{C}(z, e^{-z^2})$, where $C_F = \mathbb{C}$ and $z' = 1$. If an antiderivative of e^{-z^2} lies in an elementary extension of F, then Liouville's Theorem implies that there are non-zero elements $u_1, \ldots, u_n$ in F, $\mathbb{Q}-$linearly independent constants $c_1, \ldots, c_n$ and an element $w \in F$ such that

$$e^{-z^2} = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + w'.$$

Note that if e^{-z^2} lies in some algebraic normal extension L of $\mathbb{C}(z)$ then for any σ in Galois group of L over $\mathbb{C}(z)$, we have $2[L : \mathbb{C}(z)]z = \sum_\sigma (\sigma e^{-z^2})'/(\sigma e^{-z^2}) =$

$\mathrm{Nr}(e^{-z^2})'/\mathrm{Nr}(e^{-z^2})$. That means $z = v'/v$ for some element $v \in \mathbb{C}(z)$, which is absurd. Therefore, e^{-z^2} is transcendental over $\mathbb{C}(z)$. Thus we shall apply Proposition 2.2.6 and obtain $w = w_1 e^{-z^2} + w_0$ for some $w_1, w_0 \in \mathbb{C}(z)$ and each u_i is a multiple of some power of e^{-z^2}. Comparing the coefficient on e^{-z^2}, we have $1 = w_1' - 2zw_1$, but there is no such element w_1 in $\mathbb{C}(z)$. Hence, $erf(z)$ is non-elementary function over $F = \mathbb{C}(z, e^{-z^2})$.

A similar calculation for $li(z)$ over $F = \mathbb{C}(z, \log z)$ will give rise to equation $1/z = w' + w$, which do not have a solution in $\mathbb{C}(z)$. Thus, $li(z)$ is also a non-elementary function over $F = \mathbb{C}(z, \log z)$.

Chapter 3

Dilogarithmic Integrals and $\mathcal{DEL}-$Expressions

3.1 Dilogarithmic integrals

A *dilogarithm or Spence's function*, named after William Spence, a Scottish mathematician in early nineteenth century, is the function defined by the power series

$$Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad \text{for} \quad |z| < 1.$$

The name and the definition of dilogarithm come from the analogy with the Taylor series of ordinary logarithm around 1,

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \qquad \text{for} \quad |z| < 1.$$

This similarly leads to the definition of *polylogarithm*

$$Li_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m} \qquad \text{for} \quad |z| < 1, \quad m \in \mathbb{N},$$

where $Li_1(z) = -\log(1-z)$. It is clear that for $m > 1$

$$\frac{\mathrm{d}}{\mathrm{d}z} Li_m(z) = \frac{z'}{z} Li_{m-1}(z).$$

Thus the analytic continuation of the dilogarithm function is

$$Li_2(z) = -\int_0^z \log(1-u)\frac{\mathrm{d}u}{u} \qquad \text{for} \quad z \in \mathbb{C} \setminus [1, \infty).$$

Keeping this analytic theory in mind, one can study dilogarithmic integrals from a purely algebraic stand point. The following algebraic definition of dilogarithmic integrals is due to Singer, Saunders and Caviness [16]:

Definition 3.1.1. ([16], p.968) Let $E \supset F$ be differential fields and $g \in F \setminus \{0, 1\}$ be any element. The integral

$$-\int \frac{g'}{g} \log(1-g)$$

in E is called *dilogarithmic integral* and is denoted by $\ell_2(g)$.

It is clear from the definition that if $\ell_2 \in E$ then $\log(1-g) \in E$ and $\ell_2(g)$ is primitive over $F(\log(1-g))$. We shall now explain some basic identities satisfied by dilogarithmic integrals.

Proposition 3.1.2. *Let $E \supset F$ be differential fields and $\ell_2(g) \in E$ for some $g \in F \setminus \{0, 1\}$. Then $\ell_2(1/g), \ell_2(1-g)$ lies in $E(\log g)$ and for a constant c,*

(i) $\ell_2\left(\frac{1}{g}\right) = -\ell_2(g) - \frac{1}{2}\log^2 g + c\log g.$

(ii) $\ell_2(1-g) = -\ell_2(g) - \log g \log(1-g)$.

Proof. From the definition of dilogarithmic integral,

$$\ell_2\left(\frac{1}{g}\right)' = -\frac{(1/g)'}{(1/g)}\log\left(1-1/g\right) = \frac{g'}{g}\log\left(\frac{g-1}{g}\right)$$

and

$$\ell_2(1-g)' = -\frac{(1-g)'}{1-g}\log g = \frac{g'}{g}\log(1-g) - (\log g \log(1-g))'.$$

Since $\log((g-1)/g) = \log(1-g) - \log g + c$ for some constant c. Thus integrating the above equations, we shall obtain the desired expressions for $\ell_2(1/g)$ and $\ell_2(1-g)$ and clearly $\ell_2(1/g), \ell_2(1-g)$ lies in $E(\log g)$. □

Let $F = C(z)$ be a differential field with $C_F = C$ and $z' = 1$. Let $E \supset F$ be differential field extension having constant field $C_E = C$. Assume there is an element $g \in F \setminus \{0, 1\}$ such that $\ell_2(g) \in E$ and $\ell_2(g)' \in F(\log(1-g))$. Suppose that E is an elementary extension over $F(\log(1-g))$. Then Liouville's Theorem implies that there exists $\mathbb{Q}-$linearly independent constants $c_1, \ldots, c_n$, elements $g_1, \ldots, g_n \in F(\log(1-g))^*$ and $w \in F(\log(1-g))$ such that

$$\ell_2(g)' = -\frac{g'}{g}\log(1-g) = \sum_{i=1}^{n} c_i \frac{g_i'}{g_i} + w'. \tag{3.1}$$

Clearly $\log(1-g)$ is transcendental over $F = C(z)$. Consider the partial fraction expansion of w and g_i for each i as done in Proposition 2.2.5 and note that w is a polynomial in $F[\log(1-g)]$ with $\deg(w) \leq 2$. Since $\sum_{i=1}^{n} c_i(g_i)'/g_i \in F[\log(1-g)]$, using Proposition 2.2.6 we obtain $g_i \in F$. Let $w = c\log^2(1-g) + w_1\log(1-g) + w_0$, where $c \in C$, $w_1, w_0 \in F$. Then comparing the coefficients of $\log(1-g)$, we obtain $w_1' = -g'/g - 2c(1-g')/(1-g)$. But there is no such element in F, therefore, we arrive at a contradiction. Thus E must be a non-elementary extension of $F(\log(1-g))$, and hence $\ell_2(g)$ is non-elementary function over $F(\log(1-g))$.

3.2 Special expressions and identities

First we recall the definitions of $\mathcal{DEL}-$extensions and $\mathcal{DEL}-$expressions from Chapter-1.

Definition 3.2.1. ([6], pp.210-211) A differential field $E \supset F$ is called a $\mathcal{DEL}-$extension of F if $C_E = C_F$ and there is a tower of differential fields F_i such that

$$F = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = E$$

and for each i, $F_i = F_{i-1}(\theta_i)$ and one of the following holds:

(i) θ_i is algebraic over F_{i-1}.

(ii) $\theta_i' = u'\theta_i$ for some $u \in F_{i-1}$ (i.e. $\theta_i = e^u$).

(iii) $\theta_i' = u'/u$ for some $u \in F_{i-1}$ (i.e. $\theta_i = \log(u)$).

(iv) $\theta_i' = u'/v$, where $v' = u'/u$ for some $u, v \in F_{i-1}$ (i.e. $\theta_i = \int u'/\log(u)$, also denoted by $\ell i(u)$).

(v) $\theta_i' = u'v$, where $v' = (-u^2)'v$ for some $u, v \in F_{i-1}$ (i.e. $\theta_i = \int u'e^{-u^2}$, also denoted by $erf(u)$).

(vi) $\theta_i' = vu'/u$, where $v' = (1-u)'/(1-u)$ (i.e. $\theta_i = \int \frac{u'}{u}\log(1-u)$, also denoted by $-\ell_2(u)$) for some $u, v \in F_{i-1}$.

Definition 3.2.2. ([6], p.211) We say that $v \in F$ admits a $\mathcal{DEL}-$expression over F if there are finite indexing sets I, J, K and elements $r_i, g_i \in F$ for all $i \in I$, elements $u_j, \log(u_j) \in F$ and constants a_j for all $j \in J$, elements $v_k, e^{-v_k^2} \in F$ and constants b_k for all $k \in K$, and an element $w \in F$ such that

$$v = \sum_{i \in I} r_i \frac{g_i'}{g_i} + \sum_{j \in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{k \in K} b_k v_k' e^{-v_k^2} + w', \tag{3.2}$$

where for each $i \in I$, there is an integer n_i such that $r_i' = \sum_{l=1}^{n_i} c_{il} h_{il}'/h_{il}$ for some constants c_{il} and elements $h_{il} \in F$.

Definition 3.2.3. ([6], pp.215-216) A $\mathcal{DEL}-$expression will be called

(a) a *special* $\mathcal{DEL}-$*expression* if for each $i \in I$, $r_i' = c_i(1-g_i)'/(1-g_i)$ for some constant c_i,

(b) a $\mathcal{L}-$*expression* if for all i, j, k, $r_i' = 0$, $a_j = 0$ and $b_k = 0$,

(c) a $\mathcal{D}-$*expression* if it is special and for all j, k, $a_j = b_k = 0$,

(d) a $\mathcal{DL}-$*expression* if $b_k = 0$ for all k.

An $\mathcal{L}-$expression $\sum_{i=1}^{p} c_i(v_i'/v_i)$ over F is said to be *reduced* if constants $c_1, \ldots, c_p$ are $\mathbb{Q}-$linearly independent. We observed in Remark 2.4.2 that if $v \in F$ admits a $\mathcal{L}-$expression over F then it also admits a reduced $\mathcal{L}-$expression over F. To this end, whenever we write $\sum_{i=1}^{m} a_i(u_i'/u_i)$, we shall assume that $a_1, \ldots, a_m$ are $\mathbb{Q}-$linearly independent. In particular,

(a) we assume that for each i, the $\mathcal{L}-$expression $\sum_{l=1}^{n_i} c_{il} h_{il}'/h_{il}$ that appear in the definition of $\mathcal{DEL}-$expression is reduced and

(b) if $I_1 = \{i \in I \mid r_i' = 0\}$ then $\sum_{i \in I_1} r_i(g_i'/g_i)$ is also reduced.

Definition 3.2.4. A differential field extension E of F will be called a *logarithmic extension* of F if $C_E = C_F$ and there are elements $h_1, \ldots, h_m \in F$ such that $E = F(\log(h_1), \ldots, \log(h_m))$.

From Proposition 3.1.2, we shall prove the following propositions.

Proposition 3.2.5. ([6], p.217) *Let $F(\theta) \supset F$ be a transcendental $\mathcal{DEL}-$extension and suppose that*

$$v := \sum_{i \in I} r_i(g_i'/g_i) + w' \tag{3.3}$$

is a $\mathcal{D}-$expression over a logarithmic extension E of $F(\theta)$. Then v admits a $\mathcal{D}-$expression:

$$v = \sum_{i \in I} \tilde{r}_i(\tilde{g}_i'/\tilde{g}_i) + \tilde{w}'$$

over some logarithmic extension $\tilde{E}$ of $F(\theta)$, containing E, such that for each i, $\tilde{r}_i$ is a constant or $\tilde{g}_i \in F(\theta)$ and $1 - \tilde{g}_i = \eta_i P_i/Q_i$, where P_i and Q_i are monic relatively prime polynomials over $F[\theta]$ and $\eta_i \in F$ having the following properties:

(a) θ is neither a factor of P_i nor a factor of Q_i and deg$(Q_i) \geq$ deg(P_i).

(b) If $\eta_i \neq 1$ then deg$(Q_i - \eta_i P_i) =$deg(Q_i) and if $\eta_i = 1$ then log$(\eta_i) \in C_F$.

(c) If ξ_i is the leading coefficient of $Q_i - \eta_i P_i$ then either deg$(P_i) =$deg(Q_i) or $\xi_i = 1$. Furthermore, either $\eta_i = 1$ or $\xi_i = 1$ or $\xi_i = 1 - \eta_i$ and in any event, $\log(\eta_i)(\xi_i'/\xi_i)$ is a $\mathcal{D}-$expression over $F(\log(\eta_i))$.

Proof. Let $E = F(\theta)(\log(y_1), \ldots, \log(y_n))$ for $y_1, \ldots, y_n \in F(\theta)$ and $\Lambda_p = F(\theta)(\log(y_1), \ldots, \log(y_{p-1}), \log(y_{p+1}), \ldots, \log(y_n))$. Observe that $r_i' - c_i(1 - g_i')/(1 -$

$g_i) = 0 \in \Lambda_p[\log(y_p)]$ and that $\Lambda_p(\log(y_p)) = E$. Apply Proposition 2.2.7 and obtain that $1 - g_i$, and therefore g_i, belongs to Λ_p. Thus $g_i \in F(\theta) = \cap_p \Lambda_p$ for each $i \in I$. Let $1 - g_i = \xi_i P_i/Q_i$, where P_i and Q_i are relatively prime monic polynomials over F and $\xi_i \in F$. Then θ can either divide P_i or Q_i but not both. Suppose that θ divides P_i. Then over the differential field $E(\log(g_i))$, using Proposition 3.1.2, we have $r_i(g_i'/g_i) = \tilde{r}_i \tilde{g}_i'/\tilde{g}_i + (\log(g_i)r_i)'$, where $\tilde{r}_i = -c_i \log(g_i)$ and $\tilde{g}_i = 1 - g_i$. Then

$$v = \sum_{j \in I, j \neq i} r_j(g_j'/g_j) + \tilde{r}_i(\tilde{g}_i'/\tilde{g}_i) + (\tilde{w}_i + w)',$$

where $\tilde{w}_i = \log(g_i)r_i$, is a $\mathcal{D}-$expression over $E(\log(g_i))$. Note that $Q_i - \xi_i P_i$ and Q_i are relatively prime polynomials such that θ neither divides $Q_i - \xi_i P_i$ nor Q_i. Since $1 - \tilde{g}_i = g_i = (Q_i - \xi_i P_i)/Q_i$, we shall factor the leading coefficient $\tilde{\eta}_i$ of $Q_i - \xi_i P_i$ and obtain for all such i, relatively prime monic polynomials $\tilde{P}_i$ and Q_i such that $\tilde{\eta}_i \tilde{P}_i = Q_i - \xi_i P_i$, $1 - \tilde{g}_i = g_i = \tilde{\eta}_i \tilde{P}_i/Q_i$ and that θ neither dividing $\tilde{P}_i$ nor Q_i.

Now we suppose that θ divides Q_i. We make use of Proposition 3.1.2 and write $r_i g_i'/g_i = -\hat{r}_i \hat{g}_i'/\hat{g}_i + ((1/2c_i)r_i^2)'$, where $\hat{r}_i' = c_i(1-\hat{g}_i)'/(1-\hat{g}_i)$. Since $1-\hat{g}_i = 1/(1-g_i) = (1/\xi_i)(Q_i/P_i)$, we have θ dividing the numerator polynomial Q_i. Therefore, we shall proceed as in the previous case and obtain $\tilde{r}_i$, $\tilde{g}_i$ and $\tilde{w}_i$. This proves the first part of (a). To prove the second part of (a), we simply apply Proposition 3.1.2, to those terms that have $\deg(P_j) \gneqq \deg(Q_j)$.

Since $\deg(Q_i) \geq \deg(P_i)$, if $\eta_i \neq 1$ then the leading coefficient of $\deg(Q_i - \eta_i P_i)$, which we shall call ξ_i is non zero and therefore $\deg(Q_i - \eta_i P_i)$ $=\deg(Q_i)$. Since $\log(\eta_i)' = \eta_i'/\eta_i$, if $\eta_i = 1$ then $\log(\eta_i)$ must be a constant. Note that if $\deg(Q_i) \gneqq \deg(P_i)$ then the polynomial $Q_i - \eta_i P_i$ must be monic, that is, $\xi_i = 1$. If neither $\eta_i = 1$ nor $\xi_i = 1$ then it is clear that $\deg(Q_i) = \deg(P_i)$ and therefore $\xi_i = 1 - \eta_i$. Thus we have the following observations on $\log(\eta_i)(\xi_i'/\xi_i)$:

$$\log(\eta_i)\frac{\xi_i'}{\xi_i} = \begin{cases} (\log(\eta_i)\log(\xi_i))' & \text{if } \eta_i = 1; \\ 0 & \text{if } \xi_i = 1; \\ \log(\eta_i)\frac{(1-\eta_i)'}{1-\eta_i} & \text{if } \xi_i = 1-\eta_i. \end{cases}$$

Thus, in any event, $\log(\eta_i)(\xi_i'/\xi_i)$ is a $\mathcal{D}-$expression over $F(\log(\eta_i))$. □

Proposition 3.2.6. ([6], p.217) *Let $F(\theta) \supset F$ be differential fields where θ is transcendental, $C_{F(\theta)} = C_F$ and either $\theta' \in F$ or $\theta'/\theta \in F$ and $v \in F(\theta)$. Suppose that*

$$v = \sum_{i\in I} r_i(g_i'/g_i) + w' \tag{3.4}$$

is a $\mathcal{D}-$expression over $F(\theta)$.

(a) If $\theta' \in F$ then for each i such that $r_i' \neq 0$, we have $g_i \in F$.

(b) For each i, $r_i = a_i\theta + \eta_i$ for some constant $a_i \in C_F$ and $\eta_i \in F$. Furthermore, each r_i belong to F when $\theta'/\theta \in F$.

Proof. We have $r_i' - c_i(1-g_i)'/(1-g_i) = 0$ for some $c_i \in C_F$. Suppose that $c_i \neq 0$ for some i. If $\theta' \in F$ then since $c_i \neq 0$, we apply Proposition 2.2.7 and obtain $1-g_i \in F$ and consequently, $g_i \in F$ for all i. On the other hand if $\theta'/\theta \in F$ then we have $1 - g_i = \xi_i\theta^{m_i}$ for some integer m_i and elements ξ_i and r_i in F. Thus for each i, $r_i' \in F$. From Proposition 2.2.7 it follows that $r_i = a_i\theta + \eta_i$ for some $a_i \in C_F$ and $\eta_i \in F$ and that each $r_i \in F$ (that is, $a_i = 0$) when $\theta'/\theta \in F$. □

Following lemma helps us to handle various terms that appear in $\mathcal{DEL}-$expressions.

Lemma 3.2.7. **([6], p.218)** *Let $F(\theta) \supset F$ be a transcendental $\mathcal{DEL}-$extension of F and $v \in F(\theta)$ be an element such that*

$$v = \sum_{j \in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{k \in K} b_k v_k' e^{-v_k^2} + \sum_{i=1}^{s} \delta_i \frac{w_i'}{w_i} + w',$$

where $u_j, \log(u_j), v_k, e^{-v_k^2}, w_i, w \in F(\theta)$, $\delta_i \in F$ and a_j, b_k are constants.

(i) Suppose that $\theta' \in F$. Then each u_j, v_k and $e^{-v_k^2}$ belong to F. Furthermore, if $v \in F[\theta]$ then $w \in F[\theta]$, $v - w' \in F$ and there is a subset $J_1 \subset J$ and elements $\xi_i \in F$ such that

$$v = \sum_{j \in J_1} a_j \frac{u_j'}{\log(u_j)} + \sum_{k \in K} b_k v_k' e^{-v_k^2} + \sum_{i=1}^{s} \delta_i \frac{\xi_i'}{\xi_i} + w'.$$

Finally, if $v \in F$ then $w = c\theta + w_0$ for some constant c and $w_0 \in F$.

(ii) Suppose that $\theta'/\theta \in F$. Then each $\log(u_j)$ and v_k belong to F and there are elements η_j and ζ_k in F and integers m_j and n_k such that $u_j = \eta_j \theta^{m_j}$ and $e^{-v_k^2} = \zeta_k \theta^{n_k}$. Furthermore, if $v \in F$, $\theta'/\theta = x'$ for some $x \in F$ and for each i, δ_i is a constant or the constant term of the partial fraction expansion of the corresponding w_i'/w_i is zero, then there are sets $J_1 = \{j \in J \mid m_j = 0\}$ and $K_1 = \{k \in K \mid n_k = 0\}$ and an element $\tilde{w} \in F$ such that

$$v = \sum_{j \in J_1} a_j \frac{u_j'}{\log(u_j)} + \sum_{k \in K_1} b_k v_k' e^{-v_k^2} + \sum_{i=1}^{s} \delta_i \frac{\xi_i'}{\xi_i} + \tilde{w}'.$$

Proof. Fix an algebraic closure $\overline{F}$ of F and write $w_i = \xi_i \prod_{l=1}^{n} (\theta - \alpha_l)^{m_{il}}$ with each $\alpha_l \in \overline{F}$, $\xi_i \in F$ and integers m_{il} as in Proposition 2.2.5. Consider the equations

$$\frac{\left(e^{-v_k^2}\right)'}{e^{-v_k^2}} = \left(-v_k^2\right)' \quad \text{and} \quad \log(u_j)' = \frac{u_j'}{u_j}.$$

(i) From Proposition 2.2.7, we have for each $k \in K$, $e^{-v_k^2} \in F$ and $-v_k^2 = c_k\theta + \eta_k$ for some constant c_k and $\eta_k \in F$. Since θ is transcendental, the latter equation holds only if $c_k = 0$ and that $v_k \in F$. Similarly, we have $u_j \in F$ and $\log(u_j) = c_j\theta + \zeta_j$, where c_j is a constant and $\zeta_j \in F$.

Now we further suppose that $v \in F[\theta]$. Write w_i as in Proposition 2.2.5 and observe that the partial fraction expansion of $\sum_{i=1}^{s} \delta_i(w_i'/w_i)$ contains only poles of order at most 1 and a constant term ζ, where

$$\zeta = \begin{cases} \sum_{i=1}^{s} \delta_i(\xi_i'/\xi_i) & \text{when} \quad \theta' \in F \\ \sum_{i=1}^{s} \delta_i(\xi_i'/\xi_i) + \left(\sum_{i=1}^{s}\sum_{l=1}^{n} m_{il}\delta_i\right)(\theta'/\theta) & \text{when} \quad \theta'/\theta \in F. \end{cases} \tag{3.5}$$

Let $J_1 = \{j \in J \mid \log(u_j) \in F\}$ and write

$$v = \sum_{j\in J_1} a_j \frac{u_j'}{\log(u_j)} + \sum_{j\in J-J_1} a_j \frac{u_j'}{c_j\theta + \zeta_j} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + \sum_{i=1}^{s} \delta_i \frac{w_i'}{w_i} + w'. \tag{3.6}$$

Since $v \in F[\theta]$, it is clear from the above equation that $v - w'$ has poles of order at most 1. Therefore, from Proposition 2.2.5, we obtain that w has no poles and thus $w \in F[\theta]$. Consequently, all the poles of Equation 3.6 must cancel out and we obtain

$$v = \sum_{j\in J_1} a_j \frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + \sum_{i=1}^{s} \delta_i \frac{\xi_i'}{\xi_i} + w'. \tag{3.7}$$

If $v \in F$ then $w' \in F$ and we have $w = c\theta + w_0$ for some constant c and $w_0 \in F$.

(ii) In this case we have each $-v_k^2$, and therefore v_k, belong to F and $e^{-v_k^2} = \eta_k\theta^{n_k}$ for elements $\eta_k \in F$ and integers n_k. We have $\log(u_j) \in F$ and each $u_j = \zeta_j\theta^{m_j}$ for some $\zeta_j \in F$ and integers m_j. Let $J_1 = \{j \in J \mid m_j = 0\}$ and $K_1 = \{k \in K \mid n_k = 0\}$ and rewrite

$$v = \sum_{j\in J_1} a_j \frac{u_j'}{\log(u_j)} + \sum_{j\in J-J_1} a_j \frac{\mu_j\zeta_j\theta^{m_j}}{\log(u_j)} + \sum_{k\in K_1} b_k v_k' e^{-v_k^2} + \sum_{k\in K-K_1} b_k v_k' \eta_k \theta^{n_k} + \sum_{i=1}^{s} \delta_i \frac{w_i'}{w_i} + w', \tag{3.8}$$

where $\mu_j = (\zeta_j'/\zeta_j) + m_j(\theta'/\theta) \in F$.

Let $v \in F$ and $\theta'/\theta = x'$ for some $x \in F$. We rearrange the terms of $\sum_{i=1}^{s} \delta_i(w_i'/w_i)$ and assume that $\delta_i \in C_F$ for $1 \le i \le p$ and the constant term of w_i'/w_i is zero for $p+1 \le i \le s$. By assumption, $\sum_{i=p+1}^{s} \delta_i(w_i'/w_i)$ is a sum of poles. Now use Equation 3.5, Proposition 2.2.5 and compare the constant terms of Equation 3.8 and obtain for some $w_0 \in F$ that

$$\begin{aligned} v &= \sum_{j\in J_1} a_j \frac{u_j'}{\log(u_j)} + \sum_{k\in K_1} b_k v_k' e^{-v_k^2} + \sum_{i=1}^{p} \delta_i \frac{\xi_i'}{\xi_i} + \left(\sum_{i=1}^{p}\sum_{l=1}^{n} m_{il}\delta_i\right)\frac{\theta'}{\theta} + w_0' \\ &= \sum_{j\in J_1} a_j \frac{u_j'}{\log(u_j)} + \sum_{k\in K_1} b_k v_k' e^{-v_k^2} + \sum_{i=1}^{p} \delta_i \frac{\xi_i'}{\xi_i} + \tilde{w}_0', \end{aligned} \tag{3.9}$$

where $\left(\sum_{i=1}^{p}\sum_{l=1}^{n} m_{il}\delta_i\right) x + w_0 =: \tilde{w}_0 \in F$. □

Chapter 4

An Extension of Liouville's Theorem

In this chapter, we provide necessary and sufficient condition for an element to have an antiderivative in a transcendental $\mathcal{DEL}-$extension. We also obtain a generalisation of Baddoura's theorem on integration in finite terms with dilogarithmic integrals. Several examples can also be found at the end of this chapter, that support our theorems.

4.1 Integration in $\mathcal{DEL}-$extensions

We recall that special $\mathcal{DEL}-$expression is a $\mathcal{DEL}-$expression

$$\sum_{i\in I} r_i \frac{g_i'}{g_i} + \sum_{j\in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + w',$$

where each $r_i' = c_i(1-g_i)'/(1-g_i)$ for some constant c_i. First we prove the sufficient condition that if we have a $\mathcal{DEL}-$expression, of a particular type, over a differential field then definitely its antiderivative lies in some $\mathcal{DEL}-$extension.

Theorem 4.1.1. *Let F be a differential field with constant field C_F. Let I, J, K and L be finite indexing sets such that $v \in F$ has $\mathcal{DEL}-$expression:*

$$v = \sum_{i\in I} r_i \frac{g_i'}{g_i} + \sum_{l\in L} s_l \frac{h_l'}{h_l} + \sum_{j\in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + w'$$

over F, where for each $i \in I$, $l, t \in L$ there are constants c_i, d_{il}, b_{lt}, with $c_i \neq 0$ whenever $r_i' \neq 0$, such that

$$r_i' = c_i \frac{(1-g_i)'}{1-g_i} + \sum_{l\in L} d_{il} \frac{h_l'}{h_l} \quad \text{and} \quad s_l' = \sum_{i\in I} d_{il} \frac{g_i'}{g_i} + \sum_{t\in L} b_{lt} \frac{h_t'}{h_t}.$$

Then there exists a $\mathcal{DEL}-$extension E of F that contains an antiderivative of v.

Proof. Clearly $r_i = c_i \log(1-g_i) + \sum_{l\in L} d_{il} \log h_l + e_i$ and $s_l = \sum_{i\in I} d_{il} \log g_i + \sum_{t\in L} b_{lt} \log h_t + d_l$, where e_i's and d_l's are some constants. Substitute these values in the expression for v and obtain

$$v = \sum_{i\in I} c_i \log(1-g_i) \frac{g_i'}{g_i} + \sum_{\substack{i\in I\\ l\in L}} d_{il} \left(\log g_i \log h_l\right)' + \sum_{l,t\in L} b_{lt} (\log h_l \log h_t)' + \sum_{j\in J} a_j \frac{u_j'}{\log(u_j)}$$
$$+ \sum_{k\in K} b_k v_k' e^{-v_k^2} + w'.$$

It is easy to observe that the integral of above equation is

$$\int v = -\sum_{i\in I} c_i \ell_2(g_i) + \sum_{\substack{i\in I\\ l\in L}} d_{il} \log g_i \log h_l + \sum_{l,t\in L} b_{lt} \log h_l \log h_t + \sum_{j\in J} a_j li(u_j)$$
$$+ \sum_{k\in K} b_k erf(v_k) + w.$$

Thus an antiderivative of v lies in field $E = F(\{\log h_t, \log g_i, \ell_2(g_i), li(u_j), erf(v_k)\})$ which is a $\mathcal{DEL}-$extension. $\square$

Recall that a differential field extension $E \supset F$ is called a logarithmic extension of F if $C_E = C_F$ and there are elements $h_1, \ldots, h_m \in F$ such that $E = F(\log(h_1), \ldots, \log(h_m))$. A $\mathcal{DEL}-$expression is called a special $\mathcal{DEL}-$expression if $r_i' = c_i(1-g_i)'/(1-g_i)$ for some constant c_i.

In the next proposition, we observe that any special $\mathcal{DEL}-$expression over a logarithmic extension of a field F can be reduced to a $\mathcal{DEL}-$expression over F. Also note that the non constant coefficients of logarithmic derivatives involved in the final $\mathcal{DEL}-$ expression are of the type mentioned in Theorem 4.1.1.

Proposition 4.1.2. *Let F be a differential field and $v \in F$ satisfies a special $\mathcal{DEL}-$expression over a logarithmic extension E of F. Then v satisfies a $\mathcal{DEL}-$expression:*

$$v = \sum_{i \in I} r_i \frac{g_i'}{g_i} + \sum_{l \in L} s_l \frac{h_l'}{h_l} + \sum_{j \in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{k \in K} b_k v_k' e^{-v_k^2} + w' \tag{4.1}$$

over F, where for each $i \in I$, $l, t \in L$ there are constants c_i, d_{il}, b_{lt}, with $c_i \neq 0$ whenever $r_i' \neq 0$, such that

$$r_i' = c_i \frac{(1-g_i)'}{1-g_i} + \sum_{l \in L} d_{il} \frac{h_l'}{h_l} \quad \textit{and} \quad s_l' = \sum_{i \in I} d_{il} \frac{g_i'}{g_i} + \sum_{t \in L} b_{lt} \frac{h_t'}{h_t}. \tag{4.2}$$

Proof. For a positive integer l, let $E = F(\log(h_1), \ldots, \log(h_l))$, where $h_1, \ldots, h_l \in F$. Suppose that there are finite indexing sets I, J, K such that

$$v = \sum_{i \in I} r_i \frac{g_i'}{g_i} + \sum_{j \in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{k \in K} b_k v_k' e^{-v_k^2} + w', \tag{4.3}$$

where for each $i \in I$, $r_i, g_i \in E$ and $r_i' = c_i(1-g_i)'/(1-g_i)$, $a_j \in C_F, u_j$ and $\log(u_j) \in E$ for each $j \in J$, $b_k \in C_F, v_k$ and $e^{-v_k^2} \in E$ for each $k \in K$ and $w \in E$. Without loss of generality, let $\{\log(1-g_1), \ldots, \log(1-g_n), \log(h_1), \ldots, \log(h_m)\}$ be transcendental base of E over F. Then, Corollary 2.3.3 implies that every r_i can be written as

$$r_i = c_i \log(1 - g_i) + e_i \qquad \text{for} \quad i = 1, \dots, n \tag{4.4}$$

$$\text{and} \quad r_i = \sum_{\nu=1}^{n} c_{i\nu} \log(1 - g_\nu) + \sum_{\mu=1}^{m} e_{i\mu} \log h_\mu + s_i, \tag{4.5}$$

where for each i, $c_i, e_i, c_{i\nu}, e_{i\mu}$ are some constants and $s_i \in F$. From Proposition 2.2.6 and Lemma 3.2.7, we obtain $g_i, u_j, e^{-v_k^2}, v_k \in F$ for all $i \in I, j \in J$ and $k \in K$. As noted in Proposition 2.2.5, if w has a pole of order 1 then w' has pole of order 2. Therefore w is a polynomial in $F[(\log(1-g_1), \dots, \log(1-g_n), \log(h_1), \dots, \log(h_m))]$. Let $J_1 \subseteq J$ be a finite index set such that $J_1 = \{j \in J | \log(u_j) \in F\}$. Since $\sum_{j \in J - J_1} a_j u_j' / \log(u_j)$ is the term containing poles of order 1 only, it must vanish. Using Proposition 2.2.4, we can write

$$\begin{aligned} w = &\sum_{p,q=1}^{n} a_{pq} \log(1-g_p) \log(1-g_q) + \sum_{\rho,\delta=1}^{m} b_{\rho\delta} \log h_\rho \log h_\delta + \sum_{l,t=1}^{n,m} d_{lt} \log(1-g_l) \log h_t \\ &+ \sum_{l=1}^{n} y_l \log(1-g_l) + \sum_{t=1}^{m} z_t \log h_t + w_0, \end{aligned}$$

where $a_{pq}, b_{\rho\delta}, d_{lt}$ are some constants and y_l, z_t, w_0 are some elements in F. Substituting derivative of w in Equation 4.3 and using Equations 4.4 and 4.5, we equate the coefficients of $\log(1-g_l)$ and $\log h_t$ to zero and obtain

$$-y_l' = c_l \frac{g_l'}{g_l} + \sum_{i>n} c_{il} \frac{g_i'}{g_i} + \sum_{p=1}^{n} (a_{pl} + a_{lp}) \frac{(1-g_p)'}{(1-g_p)} + \sum_{t=1}^{m} d_{lt} \frac{h_t'}{h_t} \tag{4.6}$$

and

$$-z_t' = \sum_{i>n} e_{it} \frac{g_i'}{g_i} + \sum_{\rho=1}^{m} (b_{\rho t} + b_{t\rho}) \frac{h_\rho'}{h_\rho} + \sum_{l=1}^{n} d_{lt} \frac{(1-g_l)'}{1-g_l}. \tag{4.7}$$

Thus, we have

$$v = \sum_{i>n} s_i \frac{g_i'}{g_i} + \sum_{l=1}^{n} y_l \frac{(1-g_l)'}{1-g_l} + \sum_{t=1}^{m} z_t \frac{h_t'}{h_t} + \sum_{j\in J_1} a_j \frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + w_0',$$

where s_i, y_l, z_t are elements as in Equations 4.5, 4.6 and 4.7 and therefore, we get a desired $\mathcal{DEL}-$ expression for v over F. □

It would be remarked in detail later that presence of algebraic elements along with exponentials and error functions can complicate the problem of integration in finite terms. However, for $\mathcal{DEL}-$extensions that do not contain exponentials and error functions, we have following theorem.

Theorem 4.1.3. **([6], pp.219-220)** *Let $E = F(\theta_1, \cdots, \theta_m) \supset F$ be a $\mathcal{DEL}-$extension of F and $u \in E$ be an element with $u' \in F$.*

(i) If each θ_i is neither algebraic over F_{i-1} nor an exponential of an element of F_{i-1} then u' admits a $\mathcal{DEL}-$expression over F.

(ii) If F be a liouvillian extension of C_F and each θ_i is neither an exponential of F_{i-1} nor an error function of F_{i-1} then u' admits a $\mathcal{DL}-$expression over F.

Proof. We prove the result using an induction on m. When $m = 1$, we apply Proposition 2.2.4 (c) and get $u = c\theta + \eta$ for some constant $c \in C_F$ and $\eta \in F$. Differentiating this equation we obtain the desired expression for u'. Let I, J, K be finite indexing sets such that

$$u' = \sum_{i\in I} r_i \frac{g_i'}{g_i} + \sum_{j\in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + w', \quad r_i' = \sum_{t=1}^{n_i} c_{it} h_{it}'/h_{it}, \qquad (4.8)$$

where elements $r_i, g_i, w, h_{it}, u_j, \log(u_j), v_k$ and $e^{-v_k^2}$ all belong to $F_1 := F(\theta_1)$, a_j and b_k are constants and $c_{i1}, \cdots, c_{in_i}$ are $\mathbb{Q}-$ linearly independent constants for each i, be a $\mathcal{DEL}-$expression for u' over $F_1 := F(\theta_1)$. Let $\overline{F}$ be an algebraic closure of F and $\beta_j \in \overline{F}$ and $\xi_i \in F$ be elements such that

$$g_i'/g_i = \xi_i'/\xi_i + \sum_{l=1}^{p} m_{il}\frac{\theta_1' - \beta_l'}{\theta_1 - \beta_l}, \tag{4.9}$$

where each m_{il} is an integer.

(i) We have $\theta_1' \in F$. Then from Proposition 2.2.7 each h_{it} belongs to F and from Lemma 3.2.7 each u_j, $e^{-v_k^2}$ and v_k belong to F. If all r_i and $\log(u_j)$ belong to F then we shall apply Lemma 3.2.7 and obtain

$$u' = \sum_{i\in I} r_i\frac{\xi_i'}{\xi_i} + \sum_{j\in J} a_j\frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + c_1\theta_1' + w_0'. \tag{4.10}$$

From the definition of θ_1, it is clear that the above expression is a $\mathcal{DEL}-$expression of u' over F.

Now suppose that there is an $r \in \{r_i, \log(u_j) \mid i \in I, j \in J\}$ and $r \notin F$. Since $F(\theta_1) = F(r)$, we shall find constants c_i and element $\eta_i \in F$ such that $r_i = c_i r + \eta_i \in F[\theta_1]$. We shall take $\theta_1 = r$ in Equation 4.9 and rewrite Equation 4.8 over $\overline{F}(r)$ as

$$\begin{aligned} u' = {} & \sum_{i\in I} c_i\frac{\xi_i'}{\xi_i} r + \Big(\sum_{i\in I}\sum_{l=1}^{p} m_{il}c_i\frac{r' - \beta_l'}{r - \beta_l}\Big) r + \sum_{i\in I}\eta_i\frac{\xi_i'}{\xi_i} + \sum_{i\in I}\sum_{l=1}^{p} m_{il}\eta_i\frac{r' - \beta_l'}{r - \beta_l} \\ & + \sum_{j\in J} a_j\frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + w'. \end{aligned} \tag{4.11}$$

Thus,

$$u' - \sum_{i\in I} c_i\frac{\xi_i'}{\xi_i} r - \sum_{i\in I}\eta_i\frac{\xi_i'}{\xi_i} = \sum_{l=1}^{p}\delta_l\frac{r' - \beta_l'}{r - \beta_l} + \sum_{j\in J} a_j\frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + w', \tag{4.12}$$

where $\delta_l = \sum_{i\in I} m_{il}(c_i\beta_l + \eta_i)$ and w is replaced with $w + \sum_{i\in I}\sum_{l=1}^{p} m_{il}c_i(r - \beta_l)$.

Note that $u' - \sum_{i\in I} c_i \frac{\xi_i'}{\xi_i} r - \sum_{i\in I} \eta_i \frac{\xi_i'}{\xi_i} \in F[r]$, apply Lemma 3.2.7 to the fields $\overline{F}(r) \supset \overline{F}$ and obtain the following expression for u':

$$u' = \sum_{i\in I} c_i \frac{\xi_i'}{\xi_i} r + \sum_{i\in I} \eta_i \frac{\xi_i'}{\xi_i} + \sum_{j\in J_1} a_j \frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + w', \tag{4.13}$$

where for $j \in J_1$, $\log(u_j) \in \overline{F}$. But we know that $\log(u_j) \in F(r)$ for all j and thus $\log(u_j) \in F$ for all $j \in J_1$. We already know that each $v_k, e^{-v_k^2}$ belongs to F. Since $u' - \sum_{i\in I} c_i \frac{\xi_i'}{\xi_i} r - \sum_{i\in I} \eta_i \frac{\xi_i'}{\xi_i} \in F[r]$, which is a polynomial of degree one and that $w \in \overline{F}(r)$, we obtain $w' \in F[r] \subset F(r)$. Now from Proposition 2.2.4, we shall replace w with an element of $F(r)$ that satisfies Equation 4.13 and we also conclude $w = dr^2 + w_1 r + w_0$, where $w_1, w_0 \in F$ and $d \in C_F$. Comparing the coefficients of r in Equation 4.13, we obtain $w_1' = -2dr' - \sum_{i\in I} c_i \xi_i'/\xi_i$ and comparing constant terms, we obtain

$$u' = \sum_{i\in I} \eta_i \frac{\xi_i'}{\xi_i} + w_1 r' + \sum_{j\in J_1} a_j \frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + w_0'. \tag{4.14}$$

Since, for some j, $r = r_j$ or $r = \log(u_j)$, we have $r' = \sum_{l=1}^{q} e_l (h_l'/h_l)$ for some constants e_l and elements $h_l \in F$. Therefore, rewriting Equation 4.14, we obtain the following $\mathcal{DEL}-$expression for u':

$$u' = \sum_{i\in I} \eta_i \frac{\xi_i'}{\xi_i} + w_1 \sum_{l=1}^{q} e_l \frac{h_l'}{h_l} + \sum_{j\in J_1} a_j \frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + w', \tag{4.15}$$

where

$$\eta_i' = \sum_{t=1}^{n_i} c_{it} \frac{h_{it}'}{h_{it}} - c_i \sum_{l=1}^{q} e_l \frac{h_l'}{h_l},\ w_1' = -2d \sum_{l=1}^{q} e_l \frac{h_l'}{h_l} - \sum_{i\in I} c_i \frac{\xi_i'}{\xi_i}. \tag{4.16}$$

(ii) Let θ_1 be algebraic over F and $b_k = 0$ for all $k \in K$. From Proposition 2.2.7, each $r_i, \log(u_j)$ belong to F and each u_j'/u_j and h_{it}'/h_{it} belong to F. For every element

$v \in \{u_j, h_{it} \mid j \in J, i \in I, 1 \leq t \leq n_i\}$, we can choose the smallest integer $n_v \geq 0$ such that $v^{n_v} \in F$ (See Proposition 2.2.2 (b)). Let N be a finite algebraic Galois extension of F containing $F(\theta_1)$ with Galois group G. Then either $0 = \mathrm{tr}(v) = \mathrm{Tr}(v)$ or $v \in F$ and $\mathrm{Tr}(v) = [N : F]v$. Moreover for $\sigma \in G$ we have $\sigma(y') = \sigma(y)'$, and thus $\mathrm{Tr}(y') = \mathrm{Tr}(y)'$ for all $y \in N$ and $\frac{\mathrm{Nr}(u)'}{\mathrm{Nr}(u)} = \sum_{\sigma \in G} \frac{\sigma(u)'}{\sigma(u)}$. Let J_1 be the subset of J such $\mathrm{Tr}(u_j) \neq 0$ for all $j \in J_1$. Then $u_j \in F$ and $\mathrm{Tr}(u_j) = [N : F]u_j$ for all $j \in J_1$. For each $\sigma \in G$ we have

$$u' = \sum_{i \in I} r_i \frac{\sigma(g_i)'}{\sigma(g_i)} + \sum_{j \in J} d_j \frac{\sigma(u_j)'}{\log(u_j)} + (\sigma w)'.$$

Therefore, we sum over all $\sigma \in G$ and obtain

$$[N : F]u' = \sum_{i \in I} r_i \sum_{\sigma} \frac{\sigma(g_i)'}{\sigma(g_i)} + \sum_{j \in J} d_j \sum_{\sigma} \frac{\sigma(u_j)'}{\log(u_j)} + \sum_{\sigma} (\sigma w)' \text{ and thus}$$

$$u' = \frac{1}{[N : F]} \sum_{i \in I} r_i \frac{\mathrm{Nr}(g_i)'}{\mathrm{Nr}(g_i)} + \sum_{j \in J_1} d_j \frac{u_j'}{\log(u_j)} + \frac{1}{[N : F]} \mathrm{Tr}(w)'. \tag{4.17}$$

Using Proposition 2.2.7 (c), we choose an integer $n \geq 0$ such that $h_{it}^n \in F$ for all i, t and observe that

$$r_i' = \sum_{t=1}^{n_i} \frac{c_{it}}{n} \frac{(h_{it}^n)'}{h_{it}^n}.$$

Thus Equation 4.17 provides a $\mathcal{DL}-$expression for u' over F. □

Remark 4.1.4. (Problem of Algebraic Elements) When dealing with expressions involving algebraic elements, the only method known, to obtain a similar expression over the base field is the standard method of taking "Trace" as done in Theorem 4.1.3(ii). This method fails when we deal with error functions: If u, e^{-u^2} belongs to an algebraic extension of F with the same field of constants as F then from equation

$$(-u^2)' = \frac{(e^{-u^2})'}{e^{-u^2}},$$

we only obtain $u^2 \in F$ and that $(e^{u^2})^2 \in F$ (see [16] , p.977, Theorem 4.1). Moreover, such error functions do exists (see [16], p.968, Example 1.1). Thus $\mathrm{Tr}(u'e^{-u^2})$ can't be simplified further to obtain a similar expression over F. Similarly, for some $v \in F$ if $v = r\frac{g'}{g}$, where $r' = c\frac{(1-g)'}{1-g}$, is a $\mathcal{D}-$expression of v over an algebraic extension of F then r must be in F. However, we only know that some power of $1-g$ belongs to F. Now taking Trace, we obtain

$$v = \frac{r}{m}\frac{Nr(g)'}{Nr(g)} \quad \text{and} \quad \left(\frac{r}{m}\right)' = \frac{c}{m^2}\frac{Nr(1-g)'}{Nr(1-g)}. \tag{4.18}$$

Since $Nr(1-g)$ need not equal a constant multiple of $1-Nr(g)$, the above equation does not provide a $\mathcal{D}-$expression over F. Nonetheless, Equation 4.18 does provide a $\mathcal{DL}-$expression over F and indeed, this argument was used in the proof of Theorem 4.1.3 (ii).

4.2 Induction step

Let $E = F(\theta_1, \ldots, \theta_n)$ be a $\mathcal{DEL}-$extension of F. The proof of our main theorem uses an induction on n and the crucial argument (the induction step) is the following: First we show that if

$$u' = \sum_{i\in I} r_i\frac{g_i'}{g_i} + \sum_{j\in J} a_j\frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + w',$$
$$r_i' = c_i(1-g_i)'/(1-g_i)$$

is a special $\mathcal{DEL}-$expression over a logarithmic extension $F_{l-1}(\log(f_1), \ldots, \log(f_p))$ of $F_{l-1} = F_{l-2}(\theta_{l-1})$, where $f_1, \ldots, f_p \in F_{l-1}$ then one can find elements $h_1, \ldots, h_q \in F_{l-2}$ such that u' admits a special $\mathcal{DEL}-$expression over $F_{l-1}(\log(h_1), \ldots, \log(h_q))$. Next, we show that there is an element $h \in F_{l-2}$ such that u' admits a special

$\mathcal{DEL}-$expression over $F_{l-2}(\log(h), \log(h_1), \ldots, \log(h_q))$ which then completes the induction argument.

Remark 4.2.1. The induction process used is different from the usual induction. The reason we did not follow the usual induction is that a $\mathcal{DEL}-$expression over a field $F(\theta)$, where θ is a transcendental exponential, may not reduce to a $\mathcal{DEL}-$expression over F. For example, let $v = rg'/g \in F$, where $r, g \in F(\theta)$ and $r' = h'/h$ for some $h \in F(\theta)$. Clearly v holds a $\mathcal{DEL}-$expression over $F(\theta)$. Assume $\theta'/\theta = x'$ for some $x \in F$. Then using Proposition 2.2.6, we can write $v = r\eta'/\eta + rnx'$ and $r' = \xi'/\xi + mx'$, where η, ξ are the constant coefficients in the partial fraction of g and h, respectively, n, m are integers and $r \in F$. From the expression of v itself, it is clear that expression need not necessarily be a $\mathcal{DEL}-$expression over F.

The following lemma contains our crucial induction argument.

Lemma 4.2.2. ([6], p.223) *Let $F(\theta) \supset F$ be a transcendental $\mathcal{DEL}-$extension of F. If $v \in F$ admits a special $\mathcal{DEL}-$expression over the differential field $F(\theta)(\log(y_1), \ldots, \log(y_n))$, each $y_i \in F(\theta)$, having the same field of constants as F then there is a differential field $M = F(\log(h_1), \ldots, \log(h_m), \theta)$, where $h_i \in F$ and having the same field of constants as F such that v admits a special $\mathcal{DEL}-$expression over M. Moreover, if θ is exponential over F then v admits a special $\mathcal{DEL}-$expression over $F(\log(h_1), \ldots, \log(h_m))$.*

Proof. Let $v = \sum_{i\in I_1} r_i \frac{g_i'}{g_i} + \sum_{j\in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + w'$ be a special $\mathcal{DEL}-$expression over some logarithmic extension $E = F(\theta)(\log(y_1), \ldots, \log(y_n))$

of $F(\theta)$. For convenience, we shall rewrite

$$v = \sum_{i\in I} r_i \frac{g_i'}{g_i} + \sum_{j\in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{l\in I_1-I} r_l \frac{g_l'}{g_l} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + w', \tag{4.19}$$

where $I = \{i \in I_1 \mid r_i \text{ is not a constant}\}$. We can apply Proposition 3.2.5 to the $\mathcal{D}-$expression $\sum_{i\in I_1} r_i \frac{g_i'}{g_i} + w'$, enlarge E to a logarithmic extension of $F(\theta)$ to include $\log(g_i)$ and assume that $1 - g_i = \eta_i P_i/Q_i$, where $P_i, Q_i \in F[\theta]$ are relatively prime monic polynomials, θ neither divides P_i nor Q_i, $\deg(Q_i) \geq \deg(P_i)$ and $\eta_i \in F$. Now since $r_i' \in F(\theta)$, there are constants c_{ip} such that $r_i - \sum_{p=1}^{n} c_{ip}\log(y_p) \in F(\theta)$ and in particular, $r_i \in \Lambda_p[\log(y_p)]$ for each i, where $\Lambda_p = F(\theta)(\log(y_1), \ldots, \log(y_{p-1}), \log(y_{p+1}), \ldots, \log(y_n))$. Observe that $v - \sum_{i\in I} r_i(g_i'/g_i) \in \Lambda_p[\log(y_p)]$ and that

$$v - \sum_{i\in I} r_i(g_i'/g_i) = \sum_{j\in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{l\in I_1-I} r_l \frac{g_l'}{g_l} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + w'.$$

Let $\gamma := v - \sum_{i\in I} r_i(g_i'/g_i) - w'$ and apply Lemma 3.2.7 to get that $\gamma \in \Lambda_p$ for each p. Thus $\gamma \in F(\theta)$. Now a repeated application of Lemma 3.2.7 to the field extension E of $F(\theta)$, with $v = \gamma$ and $w = 0$, would tell us that we could assume $u_j, \log(u_j), v_k, e^{-v_k^2}$ and g_l belong to $F(\theta)$. We enlarge E to include $\log(g_l)$ and replace w with $w + \sum_{l\in I_1-I} r_l \log(g_l)$ and write

$$v = \sum_{i\in I} r_i \frac{g_i'}{g_i} + S + w', \tag{4.20}$$

where r_i, w is in some logarithmic extension E of $F(\theta)$, $g_i \in F(\theta)$ and $S = \sum_{j\in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2}$.

Fix $\overline{F}$ an algebraic closure of F. It is easy to see that there is a subset $A = \{0 = \alpha_1, \ldots, \alpha_t\}$ of $\overline{F}$ such that the following holds:

(i) $P_i = \prod_{j=1}^{t}(\theta - \alpha_j)^{l_{ij}}$, $Q_i = \prod_{j=1}^{t}(\theta - \alpha_j)^{m_{ij}}$ and $Q_i - \eta_i P_i = \xi_i \prod_{j=1}^{t}(\theta - \alpha_j)^{n_{ij}}$ for some $\xi_i \in F$, where l_{ij}, m_{ij} and n_{ij} are non negative integers.

(ii) $w \in M_1(\{\log(\theta - \alpha) \mid \alpha \in A\})$, where $M_1 = \overline{F}(\{\log(\eta_i), \log(\xi_i), \theta \mid i \in I\})$.

Let $M = F(\{\log(\eta_i), \log(\xi_i), \theta \mid i \in I\})$ and choose a differential subfield M^* of M_1 such that $\overline{F} \subset M^*$, θ is transcendental over M^* and $M^*(\theta) = M_1$ (See Corollary 2.3.3 (b)). Let $a_{ij} = l_{ij} - m_{ij}$ and $b_{ij} = n_{ij} - m_{ij}$ and observe that $\sum_{j=1}^{t} a_{ij} = \deg(P_i) - \deg(Q_i)$ and $\sum_{j=1}^{t} b_{ij} = \deg(Q_i - \eta_i P_i) - \deg(Q_i)$. We have

$$r_i = c_i \log(\eta_i) + \sum_{j=2}^{t} c_i a_{ij} \log(\theta - \alpha_j) + e_i \quad \text{for some } e_i \in C_F \tag{4.21}$$

$$g_i'/g_i = \xi_i'/\xi_i + \sum_{j=1}^{t} b_{ij} \frac{\theta' - \alpha_j'}{\theta - \alpha_j} \quad \text{and therefore} \tag{4.22}$$

$$v = \sum_{i \in I} r_i \frac{g_i'}{g_i} + S + w'$$

$$= \sum_{j=2}^{t} u_j \log(\theta - \alpha_j) + \sum_{i \in I} c_i \log(\eta_i) \frac{g_i'}{g_i} + S + w', \tag{4.23}$$

where $u_j := \sum_{i \in I} c_i a_{ij} (g_i'/g_i)$ is a $\mathcal{L}-$expression over $F(\theta)$ and $w \in M_1(\{\log(\theta - \alpha) \mid \alpha \in A\})$.

Consider the differential fields $M_1 = M^*(\theta) \supset M^*$. It is easy to see that $\log(\theta - \alpha_2), \ldots, \log(\theta - \alpha_t)$ are algebraically independent over M^* (See [1], pp.931-933, Propositions 3 and 4). Also observe $w' \in M_1[\{\log(\theta - \alpha) | \alpha \in A\}]$, then using Proposition 2.2.4 we can write $w = \sum_{j,l=2}^{t} c_{jl} \log(\theta - \alpha_j) \log(\theta - \alpha_l) + \sum_{j=2}^{t} f_j \log(\theta - \alpha_j) + w_0$, where c_{kl} are constants and $f_j, w_0 \in M_1$. Then comparing the constant terms we obtain

$$v = \sum_{j=2}^{t} f_j \frac{\theta' - \alpha_j'}{\theta - \alpha_j} + \sum_{i \in I} c_i \log(\eta_i) \frac{g_i'}{g_i} + S + w_0', \quad f_j' = -u_j - \sum_{l=2}^{t} c_{jl} \frac{\theta' - \alpha_l'}{\theta - \alpha_l}. \tag{4.24}$$

Substituting Equation 4.22 in Equation 4.24 we obtain the following expression for v over $M^*(\theta)$:

$$v = \sum_{j=1}^{t} \left(f_j + \sum_{i \in I} b_{ij} c_i \log(\eta_i) \right) \frac{\theta' - \alpha_j'}{\theta - \alpha_j} + \sum_{i \in I} c_i \log(\eta_i)(\xi_i'/\xi_i) + S + w_0', \tag{4.25}$$

where $f_1 = 0$ and for all $j > 1$,

$$f_j' = -\sum_{i \in I} c_i a_{ij} \frac{\xi_i'}{\xi_i} - \sum_{l=1}^{t} \sum_{i \in I} c_i a_{ij} b_{il} \frac{\theta' - \alpha_l'}{\theta - \alpha_l} - \sum_{l=2}^{t} c_{jl} \frac{\theta' - \alpha_l'}{\theta - \alpha_l}. \tag{4.26}$$

From Proposition 3.2.5, we have $\sum_{i \in I} c_i \log(\eta_i)(\xi_i'/\xi_i)$ is a $\mathcal{D}-$expression over M and that $S + \sum_{i \in I} c_i \log(\eta_i)(\xi_i'/\xi_i)$ is a special $\mathcal{DEL}-$expression over M. We only need to the handle the first and last term appearing on the above expression of v.

First we suppose that $\theta' \in F$. Since $\left(\sum_{i \in I} b_{ij} c_i \log(\eta_i)\right)' = \sum_{i \in I} b_{ij} c_i (\eta_i'/\eta_i)$, from Equation 4.26 and Proposition 2.2.7, we have $f_j + \sum_{i \in I} b_{ij} c_i \log(\eta_i) = d_j \theta + h_j$ for some $d_j \in C_F$ and $h_j \in M^*$. Similarly, $\log(\eta_i) - \lambda_i \theta \in M^*$ for some $\lambda_i \in C_F$. Thus

$$\begin{aligned} v &= \sum_{j=1}^{t} (d_j \theta + h_j) \frac{\theta' - \alpha_j'}{\theta - \alpha_j} + \sum_{i \in I} c_i \log(\eta_i) \frac{\xi_i'}{\xi_i} + S + w_0' \\ &= \sum_{j=1}^{t} (d_j \alpha_j + h_j) \frac{\theta' - \alpha_j'}{\theta - \alpha_j} + \sum_{i \in I} c_i \log(\eta_i) \frac{\xi_i'}{\xi_i} + S + \tilde{w}', \end{aligned} \tag{4.27}$$

where $\tilde{w} = w_0 + \sum_{j=1}^{t} d_j (\theta - \alpha_j) \in M^*(\theta)$ and $d_j \alpha_j + h_j \in F$. Since $v - \sum_{i=1}^{p} c_i \log(\eta_i)(\xi_i'/\xi_i) \in M^*[\theta]$, we shall apply Lemma 3.2.7 to the fields $M^*(\theta) \supset M^*$ and obtain

$$v = \sum_{i \in I} c_i \log(\eta_i) \frac{\xi_i'}{\xi_i} + \sum_{j \in J_1} a_j \frac{u_j'}{\log(u_j)} + \sum_{k \in K} v_k' e^{-v_k^2} + \tilde{w}', \tag{4.28}$$

where $v - \tilde{w}'$ is a special $\mathcal{DEL}$−expression over $M = F(\{\log(\eta_i), \log(\xi_i), \theta \mid i \in I\})$. This implies that $\tilde{w}' \in M$. Since $\tilde{w} \in M^*[\theta] \subset M_1$, which is an algebraic extension of M, we shall apply Proposition 2.2.2 and replace $\tilde{w}$ with an element $w \in M$ such that $w' = \tilde{w}'$. This settles the case when $\theta' \in F$.

Now we suppose that $\theta'/\theta = x' \in F$ for some $x \in F$ and consider Equations 4.25 and 4.26. We have for each i and j both f_j and $\log(\eta_i)$ belong to M^*. Using the fact that

$$\frac{\theta' - \alpha_j'}{\theta - \alpha_j} = x' + \frac{x'\alpha_j - \alpha_j'}{\theta - \alpha_j},$$

we rewrite Equation 4.25 as

$$v = \sum_{j=1}^{t}\left(f_j + \sum_{i\in I} b_{ij}c_i \log(\eta_i)\right)\left(x' + \frac{x'\alpha_j - \alpha_j'}{\theta - \alpha_j}\right) + \sum_{i\in I} c_i \log(\eta_i)(\xi_i'/\xi_i) + S + w_0'. \tag{4.29}$$

As in Equation 3.8, we find sets $J_1 = \{j \in J \mid m_j = 0\}$ and $K_1 = \{k \in K \mid n_k = 0\}$ and we rewrite

$$\begin{aligned} v = & \sum_{j=1}^{t}(f_j + \sum_{i\in I} b_{ij}c_i \log(\eta_i))x' + \sum_{j=1}^{t}\left(f_j + \sum_{i\in I} b_{ij}c_i \log(\eta_i)\right)\left(\frac{x'\alpha_j - \alpha_j'}{\theta_j - \alpha_j}\right) \\ & + \sum_{i\in I} c_i \log(\eta_i)(\xi_i'/\xi_i) + \sum_{j\in J_1} a_j \frac{u_j'}{\log(u_j)} + \sum_{j\in J-J_1} a_j \frac{\mu_j \zeta_j \theta^{m_j}}{\log(u_j)} + \sum_{k\in K_1} b_k v_k' e^{-v_k^2} \\ & + \sum_{k\in K-K_1} b_k v_k' \eta_k \theta^{n_k} + w_0', \end{aligned}$$

where $\mu_j = (\zeta_j'/\zeta_j) + m_j(\theta'/\theta) \in F$. Now comparing the constant coefficients, we obtain

$$\begin{aligned} v = & \sum_{j=1}^{t} f_j x' + \left(\sum_{i\in I}\sum_{j=1}^{t} b_{ij}c_i \log(\eta_i)\right)x' + \sum_{i\in I} c_i \log(\eta_i)\frac{\xi_i'}{\xi_i} + \sum_{j\in J_1} a_j \frac{u_j'}{\log(u_j)} \\ & + \sum_{k\in K_1} b_k v_k' e^{-v_k^2} + w_{00}'. \end{aligned} \tag{4.30}$$

Consider $\sum_{j=1}^{t} b_{ij}c_i \log(\eta_i)$. If $\eta_i \neq 1$ then $\deg(Q_i - \eta_i P_i) = \deg(Q_i)$ and therefore $\sum_{j=1}^{t} b_{ij} = 0$ and thus $\sum_{j=1}^{t} b_{ij}c_i \log(\eta_i) = 0$. If $\eta_i = 1$ then $\log(\eta_i) \in C_F$ and we have $\sum_{j=1}^{t} b_{ij}c_i \log(\eta_i) \in C_F$. Thus, in any event, $d := \sum_{i\in I}\sum_{j=1}^{t} b_{ij}c_i \log(\eta_i) \in C_F$. Therefore

$$v = fx' + \sum_{i\in I} c_i \log(\eta_i)\frac{\xi_i'}{\xi_i} + \sum_{j\in J_1} a_j \frac{u_j'}{\log(u_j)} + \sum_{k\in K_1} b_k v_k' e^{-v_k^2} + (w_{00} + dx)', \qquad (4.31)$$

where $f = \sum_{j=1}^{t} f_j = \sum_{j=2}^{t} f_j$. As observed earlier, $\sum_{i\in I} c_i \log(\eta_i)(\xi_i'/\xi_i)$ is a dilogarithmic expression over M. Now we will show that f is either a constant or $fx' = h'$ for some $h \in F$, and $w_{00} \in F(\{\log(\eta_i), \log(\xi_i) \mid i \in I\})$. Comparing the constant term of the equation 4.26, we obtain

$$f' = -\sum_{i\in I}\sum_{j=2}^{t} a_{ij}\frac{\xi_i'}{\xi_i} - cx', \quad \text{for some constant } c \in C_F.$$

If $\sum_{j=2}^{t} a_{ij} \neq 0$ then since $a_{i1} = 0$ for all $i \in I$, we have $\deg(Q_i) \gneqq \deg(P_i)$ and as observed in Proposition 3.2.5, $\xi_i = 1$ for all such i. Thus $\sum_{i\in I}\sum_{j=2}^{t} a_{ij}\frac{\xi_i'}{\xi_i} = 0$ and therefore $f' = -cx'$. Now it follows that either $f' = 0$ or $(-f^2/(2c))' = fx'$. Let $\tilde{w} := w_{00} + dx - f^2/(2c)$ and observe from Equation 4.31 that $\tilde{w}' \in F(\{\log(\eta_i), \log(\xi_i) | i \in I\})$. Therefore, from Proposition 2.2.6 we obtain $\tilde{w} \in \overline{F}(\{\log(\eta_i), \log(\xi_i) | i \in I\})$. Since $\overline{F}(\{\log(\eta_i), \log(\xi_i)\})$ is an algebraic extension of $F(\{\log(\eta_i), \log(\xi_i)\})$, we shall apply Proposition 2.2.2 and find an element $w \in F(\{\log(\eta_i), \log(\xi_i) | i \in I\})$ such that $w' = \tilde{w}'$. □

4.3 Extension theorems

Recall that a differential field extension $E \supset F$ is called a logarithmic extension of F if $C_E = C_F$ and there are elements $h_1, \ldots, h_m \in F$ such that $E =$

$F(\log(h_1), \dots, \log(h_m))$.

Theorem 4.3.1. **([6], p.226)** *Let $F(\theta) \supset F$ be transcendental $\mathcal{DEL}-$extension of F such that $F(\theta) \neq F(\log(h))$ for any $h \in F$. If $v \in F$ admits a special $\mathcal{DEL}-$expression over $F(\theta)$ then v admits a special $\mathcal{DEL}-$expression over F.*

Proof. As in Equation 4.19, let

$$v = \sum_{i \in I} r_i \frac{g_i'}{g_i} + \sum_{j \in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{l \in I_1 - I} r_l \frac{g_l'}{g_l} + \sum_{k \in K} b_k v_k' e^{-v_k^2} + w', \tag{4.32}$$

$$r_i' = c_i(1 - g_i)'/(1 - g_i) \text{ where each } c_i \neq 0, \tag{4.33}$$

be a special $\mathcal{DEL}-$expression over $F(\theta)$. Suppose that $\theta'/\theta = x'$ for some $x \in F$. We use Proposition 2.2.7 to Equation 4.33 and if necessary, Proposition 3.1.2 and obtain $r_i \in F$ and $1 - g_i = \eta_i/\theta^{m_i}$ for some integer $m_i \geq 0$ and $\eta_i \in F$. Now

$$\frac{g_i'}{g_i} = \frac{(\theta^{m_i} - \eta_i)'}{\theta^{m_i} - \eta_i} - \frac{(\theta^{m_i})'}{\theta^{m_i}} = \frac{m_i x' \eta_i - \eta_i'}{\theta^{m_i} - \eta_i}$$

and therefore g_i'/g_i has no constant term when $m_i \gneqq 0$. We now apply Lemma 3.2.7 and obtain a special $\mathcal{DEL}-$expression for v over F. On the other hand, if $\theta' \in F$ then $(r_i/c_i)' = (1 - g_i)'/(1 - g_i)$ and therefore $g_i \in F$. From our hypothesis $r_i \in F$. Now we have $v - \sum_{i \in I} r_i(g_i'/g_i) \in F$ and we can apply Lemma 3.2.7 to conclude that v admits a special $\mathcal{DEL}-$expression over F. □

We recall from Chapter-1 that a differential field extension $E = F(\theta_1, \dots, \theta_n)$ is called a transcendental dilogarithmic-elementary extension of F if for each i, θ_i is transcendental over F_{i-1} and satisfies either case ii or iii or vi in the definition of $\mathcal{DEL}-$extensions.

The following theorems provide an extension of Liouville's Theorem.

Theorem 4.3.2. ([6], p.227) *Let $E = F(\theta_1, \dots, \theta_n)$ be a transcendental $\mathcal{DEL}-$extension of F with $C_E = C_F$. Suppose that there is an element $u \in E$ with $u' \in F$ then u' admits a special $\mathcal{DEL}-$expression over some logarithmic extension of F. Furthermore, if E is a transcendental dilogarithmic-extension of F then u' admits a $\mathcal{D}-$expression over some logarithmic extension of F.*

Proof. We shall use an induction on n to prove the theorem. The case when $n = 1$, we have $u \in F(\theta)$ and that $u = c\theta + w$ for some $c \in C_F$ and $w \in F$ and therefore $u' = c\theta' + w'$. Now the definition of θ' proves that, in fact, u' admits a special $\mathcal{DEL}-$expression over F itself. Note that $u' \in F \subset F(\theta)$ and suppose that u' admits a special $\mathcal{DEL}-$expression (respectively a $\mathcal{D}-$expression) over some logarithmic extension of $F(\theta)$ having the same field of constants as F. Then we shall apply Lemma 4.2.2 and obtain that u' admits a special $\mathcal{DEL}-$expression (respectively a $\mathcal{D}-$expression) over $M = F(\theta, \log(h_1), \dots, \log(h_m))$, where $h_1, \dots, h_m \in F$ and $C_M = C_F$. Let $F^* = F(\log(h_1), \dots, \log(h_m))$ and suppose that $F^*(\theta) = F^*(\log(h))$ for some $h \in F^*$. If there is an integer p such that $h \in \Lambda_p(\log(h_p)) - \Lambda_p$, where Λ_p is the field generated by F and all $\log(h_i)$ except $\log(h_p)$, then we shall apply Theorem 2.3.3 to the fields $M = \Lambda_p(\log(h_p))(\log(h)) \supset F^* = \Lambda_p(\log(h_p)) \supset \Lambda_p$ and obtain that $\theta \in F^*$. This implies M is a logarithmic extension of F. If no such p exists then $h \in F$ and we have $M = F(\log(h), \log(h_1), \dots, \log(h_m))$, which is again a logarithmic extension of F. On the other hand if for any $h \in F^*$, $\log(h) \notin F^*(\theta) - F^*$ then we shall apply Theorem 4.3.1 and show that u' admits a special $\mathcal{DEL}-$expression (respectively a $\mathcal{D}-$expression) over the logarithmic extension F^*. This completes the induction argument. □

Theorem 4.3.3. *Let $E = F(\theta_1, \dots, \theta_n)$ be a transcendental $\mathcal{DEL}-$extension of F with $C_E = C_F$. Suppose that there is an element $u \in E$ with $u' \in F$ then there are* *finite indexing sets I, J, K and L such that u' satisfies $\mathcal{DEL}-$expression:*

$$u' = \sum_{i\in I} r_i \frac{g_i'}{g_i} + \sum_{l\in L} s_l \frac{h_l'}{h_l} + \sum_{j\in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + w'$$

over F, where for each $i \in I$, $l,t \in L$ there are constants c_i, d_{il}, b_{lt}, with $c_i \neq 0$ whenever $r_i' \neq 0$, such that

$$r_i' = c_i \frac{(1-g_i)'}{1-g_i} + \sum_{l\in L} d_{il} \frac{h_l'}{h_l} \quad \textit{and} \quad s_l' = \sum_{i\in I} d_{il} \frac{g_i'}{g_i} + \sum_{t\in L} b_{lt} \frac{h_t'}{h_t}.$$

Proof. From Theorem 4.3.2, we know that u' admits a special $\mathcal{DEL}-$expression over a logarithmic extension of F. We shall now apply Theorem 4.1.2 and obtain a $\mathcal{DEL}-$expression for u' over F. □

4.4 Generalisation of Baddoura's theorem

Using techniques from Proposition 2, p.923 of [1] and our Theorem 4.3.2, we shall generalise and provide a proof of Baddoura's Theorem. We recall that a $\mathcal{DEL}-$extension $E = F(\theta_1, \ldots, \theta_n)$ is called a transcendental dilogarithmic-elementary extension if $C_E = C_F$ and for each i, θ_i is transcendental over F_{i-1} and satisfies either case ii or iii or vi. Before we proceed to the proof of Baddoura's Theorem, we recall the definitions of $\ell_2(g)$ and $D(g)$ from [1], p.912. Let E be a differential field extension of F and $g \in F \setminus \{0,1\}$ be an element of F. If $y \in E$ is an element such that

$$y' = -\log(1-g)\frac{g'}{g}$$

then we shall pick one such element y and denote it by $\ell_2(g)$. Note that any other element in E whose derivative equal $\ell_2(g)$ differs from $\ell_2(g)$ by some constant of

E. The element $\ell_2(g) + (1/2)\log(g)\log(1-g)$ will be denoted by $D(g)$, which is sometimes called the *Bloch-Wigner-Spence function* of g and its derivative is:

$$D(g)' = -\frac{1}{2}\frac{g'}{g}\log(1-g) + \frac{1}{2}\frac{(1-g)'}{1-g}\log(g).$$

Baddoura uses Bloch Wigner Spence function in place of dilogarithmic integrals, if the dilogarithmic integrals appear in the integral, and proved that the Bloch Wigner Spence functions appear in linear way while the logarithms can appear in a possible non-linear way. The extension of Liouville's theorem to include dilogarithmic integrals, as done by Baddoura (Theorem, p.933, [1]), is stated as

Theorem 4.4.1. ([1], p.933) *Let F be a differential field of characteristic zero and the field of constants C_F be algebraically closed. Assume F is liouvillian extension of C_F. Let there be a transcendental dilogarithmic-elementary extension E of F and an element $v \in F$ such that $\int v \in E$ then*

$$\int v = \sum_{j=1}^{m} c_j D(g_j) + \sum_{i=1}^{n} f_i \log(h_i) + w,$$

where $c_j's$ are constants and each $g_j, f_i, h_i, w \in F$.

To prove this theorem, Baddoura stated and proved two identities of dilogarithmic integrals ([1], pp.922-923, Lemma 2 and Proposition 2), the former gives a relation between $D(1/g)$ and $D(g)$, while the latter provides an expansion of Spence function of a rational function. We shall state the latter identity here:

Proposition 4.4.2. ([1], p.923) *Let F be a differential field of characteristic zero, and θ be transcendental over F with $C_{F(\theta)} = C_F$. Let $g \in F(\theta)$ and E be splitting field of g and $1-g$. Let α and β be zero or pole of g and $1-g$, respectively, then*

$$D(g) = D(\eta) + \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} ord_\alpha(g) ord_\beta(1-g) D\left(\frac{\theta-\beta}{\theta-\alpha}\right)$$

modulo the vector space generated by constant multiples of logarithms over $E(\theta)$, *where* $\eta \in F$ *is the constant in partial fraction expansion of* g *and* $ord_\alpha(g)$ *denotes the multiplicity of* $\theta - \alpha$ *in* g *which is positive if* α *is zero and negative if* α *is pole of* g.

To prove the above proposition, Baddoura considered partial fraction expansions of g and $1-g$ and compared their poles in E. The proof involves lengthy calculations. On the other hand, we prove Theorem 4.4.1 without the hypothesis that C_F is algebraically closed and that F is liouvillian over C_F and our proof is relatively simpler. Furthermore, we do not require dilogarithmic identity from Proposition 4.4.2.

Theorem 4.4.3. ([6], p.228) *Let* $E = F(\theta_1, \dots, \theta_n)$ *be a transcendental dilogarithmic-elementary extension of* F. *Suppose that there is an element* $u \in E$ *with* $u' \in F$. *Then*

$$u = \sum_{j=1}^{m} c_j D(g_j) + \sum_{i=1}^{n} f_i \log(h_i) + w, \tag{4.34}$$

where each $f_i, h_i, g_j, w \in F$, c_j *are constants and* $\log(h_i)$ *and* $D(g_j)$ *belong to some dilogarithmic-elementary extension of* F.

Proof. From Theorem 4.3.2 we have the $\mathcal{D}-$expression $u' = \sum_{i \in I_1} r_i \frac{g_i'}{g_i} + w'$ over a logarithmic extension $E = F(\log(h_1), \dots, \log(h_m))$ of F. As done in Equation 4.19, we shall rewrite this expression as:

$$u' = \sum_{i \in I} r_i \frac{g_i'}{g_i} + \sum_{l=1}^{q} d_l \frac{w_l'}{w_l} + w',$$

$$r_i' = -c_i (1-g_i)'/(1-g_i), \tag{4.35}$$

where each c_i is a non zero constant. From Remark 2.4.2, we shall assume that $\{d_1, \ldots, d_q\}$ are $\mathbb{Q}-$linearly independent constants of F.

Claim: Each g_i and w_l belong to F and E can be chosen to be the differential field $F(\{\log(1-g_i) \mid i \in I\})$.

As denoted earlier, let Λ_p be the field generated by F and all $\log(h_i)$ except $\log(h_p)$. We know from Proposition 3.2.6 that each $g_i \in \Lambda_p$ and each r_i is a polynomial in $\log(h_p)$ of degree one. Thus $u' - \sum_{i\in I} r_i \frac{g_i'}{g_i} \in \Lambda_p[\log(h_p)]$. Now, from Proposition 2.2.5 and from the fact that $d_1, \ldots, d_q$ are $\mathbb{Q}-$linearly independent, it follows that $w_l \in \Lambda_p$. Since $p \in I$ is arbitrary, we have $g_i \in F$ and that $w_l \in F$. Note that each $r_i \in E$ and consider the differential subfield $F^* := F(\{\log(1-g_i) \mid i \in I\})$ of E. Then $r_i \in F^*$ and observe that $v := u' - \sum_{i\in I} r_i \frac{g_i'}{g_i} - \sum_{l=1}^{q} d_l \frac{w_l'}{w_l} \in F^*$. Since $w' = v$, we apply Theorem 2.3.3 and write $w = \sum_{j=1}^{m} a_j \log(h_j) + \tilde{w}$, for some constants $a_j \in C_F$ and $\tilde{w} \in F^*$. Thus

$$u' = \sum_{i\in I} r_i \frac{g_i'}{g_i} + \sum_{l=1}^{q} d_l \frac{w_l'}{w_l} + \sum_{j=1}^{m} a_j \frac{h_j'}{h_j} + \tilde{w}'$$

and this proves the claim.

Thus g_i and w_l belong to F and we shall assume $E = F(\{\log(1-g_i) \mid i \in I\})$. "Taking integrals" we see that there is some dilogarithmic elementary extension E^* of F containing E such that

$$u = \sum_{i\in I} c_i \ell_2(g_i) + \sum_{l=1}^{q} d_l \log(w_l) + w + c, \tag{4.36}$$

where $c \in C_F$ and each $\ell_2(g_i)$ and $\log(w_l)$ belong to E^*. We shall first show that $w \in F[\log(1-g_1), \ldots, \log(1-g_m)]$ is a polynomial of total degree at most 2 and then show how to combine terms of w with $\ell_2(g_i)$ of Equation 4.36 to obtain Equation

4.34[1]. Without loss of generality, assume that $I = \{1, 2, \ldots, m\}$ and $\{\log(1 - g_1), \ldots, \log(1 - g_n)\}$ is a transcendence base of E over F for some n, $1 \leq n \leq m$. Then $E = F(\log(1 - g_1), \ldots, \log(1 - g_n))$ and since $r_i' \in F$, we have constants c_i and e_i such that

$$r_i = -c_i \log(1 - g_i) + e_i, \text{ for } 1 \leq i \leq m \tag{4.37}$$

and we shall also rewrite r_i for $n + 1 \leq i \leq m$ as

$$r_i = \sum_{s=1}^{n} c_{is} \log(1 - g_s) - z_i, \tag{4.38}$$

where c_{is} are constants and elements $z_i \in F$. In particular, r_i is a polynomial in $\log(1 - g_p)$ over Λ_p of degree at most 1 and $w \in \Lambda_p[\log(1 - g_p)]$ for any p, $1 \leq p \leq n$. Now from Proposition 2.2.4, we have w is a polynomial in $\log(1 - g_p)$ over Λ_p of degree at most 2 whose leading coefficient is a constant. Since p is arbitrary, we have $w \in F[\log(1 - g_1), \ldots, \log(1 - g_n)]$ is a polynomial of total degree at most 2. Write

$$w = \sum_{s=1}^{n} a_s \log(1 - g_s)^2 + \sum_{\substack{s,t=1 \\ s \lneqq t}}^{n} x_{st} \log(1 - g_s) \log(1 - g_t) + \sum_{s=1}^{n} x_s \log(1 - g_s) + w_0, \tag{4.39}$$

where $a_s \in C_F$ and x_{st}, x_s, w_0 belong to F. Then

$$\begin{aligned} w' = &\sum_{s=1}^{n} \left(2a_s \frac{(1 - g_s)'}{1 - g_s} + x_s' + \sum_{t=1, s \lneqq t}^{n} \left(x_{st}' \log(1 - g_t) + x_{st} \frac{(1 - g_t)'}{1 - g_t} \right) \right) \log(1 - g_s) \\ &+ \sum_{s,t=1, s \lneqq t}^{n} x_{st} \frac{(1 - g_s)'}{1 - g_s} \log(1 - g_t) + \sum_{s=1}^{n} x_s \frac{(1 - g_s)'}{1 - g_s} + w_0'. \end{aligned} \tag{4.40}$$

[1]The technique used to combine terms is taken from Proposition 1, in particular pp.920-921, of [1].

Substituting Equations 4.38 and 4.40 in Equation 4.35 and comparing the coefficients of $\log(1-g_s)$, we obtain

$$-c_s\frac{g_s'}{g_s}+\sum_{i=n+1}^{m}c_{is}\frac{g_i'}{g_i}+2a_s\frac{(1-g_s)'}{1-g_s}+x_s'+\sum_{\substack{t=1\\ s\lneqq t}}^{n}\left(x_{st}'\log(1-g_t)+x_{st}\frac{(1-g_t)'}{1-g_t}\right)$$
$$+\sum_{\substack{t=1\\ t\lneqq s}}^{n}x_{ts}\frac{(1-g_t)'}{(1-g_t)}=0. \tag{4.41}$$

Therefore $\sum_{t=1,s\lneqq t}^{n}x_{st}'\log(1-g_t)\in F$ and since $\{\log(1-g_i)\mid i=1,2,\cdots,n\}$ is algebraically independent over F, we must have $x_{st}'=0$ for all $s\lneqq t$. Now it follows that there is a constant $a\in C_F$ such that

$$a_s\log(1-g_s)+\sum_{\substack{t=1\\ s\lneqq t}}^{n}\frac{x_{st}}{2}\log(1-g_t)+\sum_{\substack{t=1\\ t\lneqq s}}^{n}\frac{x_{ts}}{2}\log(1-g_t)=$$
$$\frac{c_s}{2}\log(g_s)-\sum_{i=n+1}^{m}\frac{c_{is}}{2}\log(g_i)-\frac{x_s}{2}+a.$$

Now we multiply the above equation by $\log(1-g_s)$ and sum over all of s to obtain

$$\sum_{s=1}^{n}a_s\log(1-g_s)^2+\sum_{\substack{t=1,s=1\\ s\lneqq t}}^{n}x_{st}\log(1-g_t)=$$
$$\sum_{s=1}^{n}\left(\frac{c_s}{2}\log(g_s)-\sum_{i=n+1}^{m}\frac{c_{is}}{2}\log(g_i)\right)(\log(1-g_s))-\sum_{s=1}^{n}\left(\frac{x_s}{2}-a\right)\log(1-g_s). \tag{4.42}$$

We have

$$u=\sum_{i=1}^{n}c_i\ell_2(g_i)+\sum_{s=1}^{n}\left(a_s\log(1-g_s)+\sum_{\substack{t=1\\ s\lneqq t}}^{n}x_{st}\log(1-g_t)\right)\log(1-g_s)$$
$$+\sum_{i=n+1}^{m}c_i\ell_2(g_i)+\sum_{s=1}^{n}x_s\log(1-g_s)+\sum_{l=1}^{q}d_l\log(w_l)+w_0+c.$$

Replace $w_0 + c$ by w_0 and we shall use Equation 4.42 to rewrite the above equation and get

$$u = \sum_{i=1}^{n} \Big(c_i \ell_2(g_i) + \frac{c_i}{2} \log(g_i) \log(1-g_i) \Big) + \sum_{i=n+1}^{m} \left(\sum_{s=1}^{n} \frac{-c_{is}}{2} \log(1-g_s) \right) \log(g_i)$$
$$+ \sum_{i=n+1}^{m} c_i \ell_2(g_i) + \sum_{s=1}^{n} \frac{x_s}{2} \log(1-g_s) + a \sum_{s=1}^{n} \log(1-g_s) + \sum_{l=1}^{q} d_l \log(w_l) + w_0.$$

Now from Equations 4.38 and 4.37, we have

$$\sum_{s=1}^{n} \frac{-c_{is}}{2} \log(1-g_s) = -\frac{r_i}{2} - \frac{z_i}{2} = \frac{c_i}{2} \log(1-g_i) - \frac{z_i + e_i}{2}$$

for each i, $n+1 \leq i \leq m$. Therefore

$$u = \sum_{i=1}^{n} \Big(c_i \ell_2(g_i) + \frac{c_i}{2} \log(g_i) \log(1-g_i) \Big) + \sum_{i=n+1}^{m} \Big(c_i \ell_2(g_i) + \frac{c_i}{2} \log(g_i) \log(1-g_i) \Big)$$
$$+ \sum_{s=1}^{n} \frac{x_s}{2} \log(1-g_s) - \sum_{i=n+1}^{m} \frac{z_i + e_i}{2} \log(g_i) + a \sum_{s=1}^{n} \log(1-g_s) + \sum_{l=1}^{q} d_l \log(w_l) + w_0$$

and thus we shall rewrite

$$u = \sum_{j=1}^{m} c_j D(g_j) + \sum_{i=1}^{n} f_i \log(h_i) + w$$

for suitable f_i, h_i and w in F. □

4.5 Examples

Example 4.5.1. Let $F = \mathbb{C}(z, \log(1+z))$ be the ordinary differential field with derivation $' := d/dz$ and $E = F_2 \supset F_1 \supset F_0 = F$ be the dilogarithmic-elementary extension of F, where

(a) $F_1 = F(\log z)$,

(b) $F_2 = F_1(\ell_2(1-z))$.

Assume $u \in E - F_1$ such that $u' \in F$. Then for some $w \in F_1$ and constant c, we have $u = c\ell_2(1-z) + w$. That is,

$$u' = -c\frac{(1-z)'}{1-z}\log z + w'. \tag{4.43}$$

Now w' is a polynomial in $F[\log z]$, therefore using Proposition 2.2.4 we get that w is a polynomial in $F[\log z]$ with $\deg(w) \leq 2$. Let $w = c_1 \log^2 z + w_1 \log z + w_0$, for some constant c_1 and elements $w_1, w_0 \in F$. Then comparing the coefficients of $\log z$ in the expression of u', we get

$$w_1' = c\frac{(1-z)'}{1-z} - 2c_1\frac{z'}{z}.$$

It is obvious that $c = c_1 = 0$ and $u \in F_1$, which is a contradiction. Therefore there is no element u in $E - F_1$ whose derivative u' lies in F. □

Example 4.5.2. Let $F = \mathbb{C}(z, e^z)$ be the ordinary differential field with derivation $' := d/dz$ and $E = F_2 \supset F_1 \supset F_0 = F$ be the dilogarithmic-elementary extension of F, where

(a) $F_1 = F(\log(1-e^z))$,

(b) $F_2 = F_1(\ell_2(e^z))$.

Assume $u \in E$ such that $u' \in F$. Then for some $w \in F_1$ and constant c, we have $u = c\ell_2(e^z) + w$. That is,

$$u' = -c\frac{(e^z)'}{e^z}\log(1-e^z) + w' = -c\log(1-e^z) + w', \tag{4.44}$$

which is a special $\mathcal{DEL}-$expression over F_1. Then using Proposition 2.2.4, we can write $w = c_1 \log^2(1-e^z) + w_1 \log(1-e^z) + w_0$, for some constant c_1 and elements $w_1, w_0 \in F$. Substituting w' in Equation 4.44 and comparing the coefficients of $\log(1-e^z)$, obtain

$$w_1' = c - 2c_1 \frac{(1-e^z)'}{1-e^z}.$$

Clearly $c_1 = 0$ and u' satisfies $\mathcal{DEL}-$expression

$$u' = w_1 \frac{(1-e^z)'}{1-e^z} + w_0',$$

where $w_1' = c = c\frac{(e^z)'}{e^z}$.

□

Example 4.5.3. ([6], pp.231-233) Let $\log(z)$, $\log(z-1)$, $\log(z+1)$ and $\log(z^2+z-1)$ be designated solutions of the differential equations $y' = 1/z$, $y' = 1/(z-1)$, $y' = 1/(z+1)$ and $y' = (2z+1)/(z^2+z-1)$ respectively. Denote $\log(z)+\log(z-1)+\log(z^2+z-1)$ and $\log(z)+\log(z+1)$ by $\log(z(z-1)(z^2+z-1))$ and $\log(z(z+1))$ respectively. Let $F = \mathbb{C}\left(z, \log(z+1), \log(z(z-1)(z^2+z-1))\right)$ be the ordinary differential field with the derivation $' := d/dz$ and $E = F_3 \supset F_2 \supset F_1 \supset F_0 = F$ be the dilogarithmic-elementary extension of F, where

(a) $F_1 = F(\log z)$,

(b) $F_2 = F_1\left(\ell_2(1-z)\right)$, $\quad \ell_2(1-z)' = -\frac{(1-z)'}{1-z}\log z \quad$ and

(c) $F_3 = F_2\left(\ell_2\left(1-z(z+1)\right)\right)$, $\quad \ell_2\left(1-z(z+1)\right)' = -\frac{(1-z(z+1))'}{1-z(z+1)}\log(z(z+1))$.

Note that

$$\log(z)+\log(z-1)+\log(z^2+z-1) = \log(z)+\log(1-z)+\log(1-z-z^2)+c, \quad (4.45)$$

for some constant $c \in C_F$ and consider the element

$$-\log(z+1)\frac{(1-z(z+1))'}{1-z(z+1)} + \log\left(z(z-1)(z^2+z-1)\right)\frac{z'}{z} + v_0' =: v \in F, \quad (4.46)$$

where $v_0 \in F$ is arbitrary.

Note that for $g_1 := 1 - z - z^2$, $g_2 := z \in F$,

$$v = r_1 \frac{g_1'}{g_1} + r_2 \frac{g_2'}{g_2} + v_0',$$

$$\text{where} \quad r_1' = -\frac{(1-g_1)'}{1-g_1} + \frac{g_2'}{g_2} = -\frac{(1+z)'}{1+z} \quad \text{and}$$

$$r_2' = \frac{(1-g_2)'}{1-g_2} + \frac{g_1'}{g_1} + \frac{g_2'}{g_2} = \frac{(1-z)'}{1-z} + \frac{z'}{z} + \frac{(1-z-z^2)'}{1-z-z^2}$$

Over the field F_1, we rewrite v as

$$\begin{aligned} v = &-\frac{(1-z(z+1))'}{1-z(z+1)}\log(z(z+1)) - \frac{(1-z)'}{1-z}\log z \\ &+ \left(\frac{(1-z(z+1))'}{1-z(z+1)} + \frac{(1-z)'}{1-z}\right)\log z + \log\left(z(z-1)(z^2+z-1)\right)\frac{z'}{z} + v_0'. \end{aligned} \quad (4.47)$$

Let $w = -(1/2)\log^2(z) + \log\left(z(z-1)(z^2+z-1)\right)\log(z) + v_0$ and observe that

$$v = -\frac{(1-z(z+1))'}{1-z(z+1)}\log(z(z+1)) - \frac{(1-z)'}{1-z}\log z + w'. \quad (4.48)$$

Thus we have $u := \ell_2(1 - z(z+1)) + \ell_2(1-z) + w \in E$, and from Equations 4.46 and 4.48, we have $u' = v \in F$. From Equations 4.46, 4.48 we see that v admits a $\mathcal{DEL}-$expression over F and a special $\mathcal{DEL}-$expression over the extension field $F_1 = F(\log(z))$ respectively.

Representation of u' in terms of Bloch Wigner Spence function: Observe that

$$u = D(1 - z(z+1)) - \frac{1}{2}\log(z(z+1))\log(1-z(z+1)) + D(1-z)$$
$$- \frac{1}{2}\log z \log(1-z) - \frac{1}{2}\log^2(z) + \log\big(z(z-1)(z^2+z-1)\big)\log(z) + v_0.$$
$$= D(1-z(z+1)) + D(1-z) - \frac{1}{2}\log(z)\Big(\log(1-z(z+1)) + \log(1-z) + \log(z)\Big)$$
$$+ \log(z)\log\big(z(z-1)(z^2+z-1)\big) - \frac{1}{2}\log(z+1)\log(1-z(z+1)) + v_0.$$

Now we shall substitute Equation 4.45 in the above equation to obtain

$$u = D(1-z(z+1)) + D(1-z) + \frac{1}{2}\log(z)\log\big(z(z-1)(z^2+z-1)\big) + \frac{1}{2}c\log(z) \tag{4.49}$$
$$- \frac{1}{2}\log(z+1)\log(1-z(z+1)) + v_0.$$

Since the elements $\log\left(z(z-1)(z^2+z-1)\right)$ and $\log(z+1)$ belong to F, the above equation provides the Bloch-Wigner-Spence function representation of u over F.

Now we will show that u' does not admit a special $\mathcal{DEL}-$expression over F of the form:

$$v = u' = \sum_{i\in I} r_i \frac{g_i'}{g_i} + w_0', \quad r_i' = c_i\frac{(1-g_i)'}{1-g_i}, \; r_i, g_i, w_0 \in F \text{ and } c_i \in \mathbb{C}. \tag{4.50}$$

Suppose that u' does admit such an expression over F. Then since $r_i' - c_i\dfrac{(1-g_i)'}{1-g_i} = 0$, which belong to the rings $\mathbb{C}\left(z, \log(z+1)\right)\left[\log(z(z-1)(z^2+z-1))\right]$ as well as $\mathbb{C}\big(z, \log(z(z-1)(z^2+z-1))\big)[\log(z+1)]$, we apply Proposition 2.2.7 and obtain that $1 - g_i \in \mathbb{C}(z)$. On the other hand, by Theorem 2.3.3, there are constants $c_{i1}, c_{i2} \in \mathbb{C}$ and an element $d_i \in \mathbb{C}(z)$ such that

$$r_i = c_{i1}\log(z+1) + c_{i2}\log(z(z-1)(z^2+z-1)) + d_i. \tag{4.51}$$

Taking derivatives, we obtain

$$\frac{(1-g_i)'}{1-g_i} = m_{i1}\frac{1}{z+1} + m_{i2}\left(\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-\omega_1}+\frac{1}{z-\omega_2}\right)+d_i', \tag{4.52}$$

where $\omega_1 = (-1+\sqrt{5})/2$, $\omega_2 = (-1-\sqrt{5})/2$ and $m_{ij} = c_{ij}/c_i$ for $j = 1, 2$. For any $x \in \mathbb{C}(z)$ we write

$$x = \frac{c\prod_{i=1}^{n} z-\alpha_i}{\prod_{j=1}^{m} z-\beta_j},$$

where $c, \alpha_i, \beta_j \in \mathbb{C}$ and observe that x' has no poles of order 1 and that x'/x is either zero (that is $x \in \mathbb{C}$) or a sum of poles of order 1. Thus from Equation 4.52, we obtain $d_i \in \mathbb{C}$, m_{i1} and m_{i2} are integers and that $1 - g_i = a_i(z+1)^{m_{i1}}\left(z(z-1)(z^2+z-1)\right)^{m_{i2}}$ for some constant $a_i \in \mathbb{C}$. We shall use Proposition 3.1.2 and assume that $m_{i2} \geq 0$. From Equations 4.46, 4.50, 4.51 and 4.52, we obtain

$$\begin{aligned} w' =& \log(z+1)\left(\frac{(1-z(z+1))'}{1-z(z+1)} + \sum_{i\in I} c_{i1}\frac{g_i'}{g_i}\right) \\ &+ \log\left(z(z-1)(z^2+z-1)\right)\left(-\frac{z'}{z}+\sum_{i\in I} c_{i2}\frac{g_i'}{g_i}\right) + \sum_{i\in I} d_i\frac{g_i'}{g_i}, \end{aligned} \tag{4.53}$$

where $w = v_0 - w_0$. It follows that w must be a polynomial in $\log(z+1)$ and $\log(z(z-1)(z^2+z+1))$ over $\mathbb{C}(z)$ of total degree 2. Write

$$\begin{aligned} w =& e_1\log^2(z(z-1)(z^2+z-1)) + e_2\log^2(z+1) \\ &+ e_3\log\left(z(z-1)(z^2+z-1)\right)\log(z+1) + \alpha_1\log\left(z(z-1)(z^2+z-1)\right) \\ &+ \alpha_2\log(z+1)+\beta \end{aligned}$$

and substitute in Equation 4.53 and obtain e_1, e_2 and e_3 are constants. Moreover, from algebraic independence of logarithms, we also obtain that the coefficients of $\log(z+1)$ and $\log(z(z-1)(z^2+z-1))$ must be zero. Thus

$$-\frac{1}{z}+\sum_{i\in I} c_{i2}\frac{g_i'}{g_i} = 2e_1\left(\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-\omega_1}+\frac{1}{z-\omega_2}\right)+e_3\frac{1}{z+1}+\alpha_1',$$

where e_1 and e_3 are constants, $\alpha_1 \in \mathbb{C}(z)$. Note that α_1' has no poles of order 1. Any pole of $\sum_{i\in I} c_{i2}(g_i'/g_i)$ is a pole of g_i'/g_i for some i such that $c_{i2} \neq 0$. If for some i, $c_{i2} \gneqq 0$ (that is, when $m_{i2} \neq 0$) then g_i'/g_i has no poles at $0, 1, \omega_1$ and ω_2. Thus, $\sum_{i\in I} c_{i2}(g_i'/g_i)$ has no poles at $0, 1, \omega_1$ and ω_2. Now by comparing poles of the above equation, one arrives at a contradiction. □

Example 4.5.4. Let $\log(z)$, $\log(1-z)$, $\log(1+z)$ and $\log(1-z^2)$ be designated solutions of the differential equations $y' = 1/z$, $y' = 1/(1-z)$, $y' = 1/(1+z)$ and $y' = -2z/1-z^2$ respectively. Denote $\log(1-z) + \log(1+z)$ by $\log(1-z^2)$. Let $F = \mathbb{C}\left(z, \log(z), \log(1-z^2)\right)$ be the ordinary differential field with the derivation $' := d/dz$ and $E_1 = F(\ell_2(z^2), \ell_2(1+z))$ be a dilogarithmic-elementary extension of F.

Note that the element $u = \ell_2(z^2) + 2\ell_2(1+z) + u_0$, where u_0 is arbitrary element in F, has derivative:

$$u' = -2\frac{z'}{z}\log(1-z^2) - 2\frac{(1+z)'}{1+z}\log z + u_0', \tag{4.54}$$

which is $\mathcal{D}-$expression over F itself.

Over a logarithmic extension $F(\log(1+z)) \supset F$, we can rewrite u' as:

$$u' = -2\frac{z'}{z}(\log(1-z) + \log(1+z)) - 2\frac{(1+z)'}{1+z}\log z + u_0'. \tag{4.55}$$

Note that $\log(1-z) \in F(\log(1+z))$ because $\log(1-z) = -\log(1+z) + \log(1-z^2)$. For $w := -2\log(z)\log(1+z) + u_0$ we have

$$u' = -2\frac{z'}{z}\log(1-z) + w' = 2\ell_2(z)' + w' \tag{4.56}$$

in a dilogarithmic-elementary extension $E_2 = F(\log(1+z), \ell_2(z))$.

We claim that E_1 and E_2 are two distinct dilogarithmic-elementary extensions.

Assume $\ell_2(z^2) \in E_2$. Since $\ell_2(z^2)$ is primitive over $F(\log(1+z))$, by Kolchin-Ostrowski Theorem, $\ell_2(z^2) = c\ell_2(z) + v$ for some constant $c \neq 0$ and element v in $F(\log(1+z))$. Therefore,

$$\begin{aligned} -2\frac{z'}{z}\log(1-z^2) &= -c\frac{z'}{z}\log(1-z) + v' \\ &= -c\frac{z'}{z}(\log(1-z^2) - \log(1+z)) + v'. \end{aligned} \tag{4.57}$$

Since $\log(1+z)$ is transcendental over F, we apply Proposition 2.2.4 and obtain $v = c_1\log^2(1+z) + w_1\log(1+z) + v_0$, for some constant c_1 and $w_1, v_0 \in F$. We compare the coefficients of $\log(1+z)$ in above equation and get

$$-2\frac{z'}{z}\log(1-z^2) = -c\frac{z'}{z}\log(1-z^2) - c\frac{(1+z)'}{1+z}\log(z) + v_0'. \tag{4.58}$$

Now we know that $\log(z)$ is transcendental over $\mathbb{C}\left(z, \log(1-z^2)\right)$, apply Proposition 2.2.4 again and observe $c = 0$, which is a contradiction. □

Example 4.5.5. Let $\log(z)$, $\log(1-z)$, $\log(1+z)$ and $\log(1+z-z^2)$ be designated solutions of the differential equations $y' = 1/z$, $y' = 1/(1-z)$, $y' = 1/(1+z)$ and $y' = (1-2z)/1+z-z^2$ respectively. Let $F = \mathbb{C}\left(z, e^z, \log(1+z), \log(1+z-z^2)\right)$ be the ordinary differential field with the derivation $' := d/dz$ and $E = F(\ell_2(1+e^z), \ell_2(z^2/(1+z)), \ell_2(z(z-1)))$ be a dilogarithmic-elementary extension of F.

Now the element $u = \ell_2(1+e^z), \ell_2(z^2/(1+z)) - 2\ell_2(z(z-1)) + 2\ell_2(-z)$ in E has derivative:

$$\begin{aligned} u' = &-\frac{(1+e^z)'}{1+e^z}z - \left(2\frac{z'}{z} - \frac{(1+z)'}{1+z}\right)\log\left(\frac{1+z-z^2}{1+z}\right) \\ &+ 2\left(\frac{z'}{z} + \frac{(z-1)'}{z-1}\right)\log(1+z-z^2) - 2\frac{z'}{z}\log(1+z) \end{aligned} \tag{4.59}$$

which is $\mathcal{D}-$expression over F itself. Further simplification of above expression gives

$$\begin{aligned} u' &= -\frac{(1+e^z)'}{1+e^z}z + \frac{((1+z)(1-z)^2)'}{(1+z)(1-z)^2}\log(1+z-z^2) \\ &= -\frac{(1+e^z)'}{1+e^z}z + \frac{((1+z)(1-z)^2)'}{(1+z)(1-z)^2}\left(\log z(1+z-z^2) - \log z\right) \\ &\quad - \frac{z'}{z}\left(\log(1+z)(1-z)^2 - 2\log(1-z)\right) + \frac{(-z)'}{(-z)}\log(1+z). \end{aligned} \tag{4.60}$$

Let $g_1 := 1+e^z$, $g_2 := (1+z)(1-z)^2$, $g_3 := z$ and $g_4 := -z \in F$ then we have $r_1 = -\log(1-g_1) = z$, $r_2 = \log(1-g_2) - \log g_3 = \log z(1+z-z^2) - \log z = \log(1+z-z^2) \in F$, $r_3 = -2\log(1-g_3) + \log g_2 = -2\log(1-z) + \log((1+z)(1-z)^2) = \log(1+z) \in F$ and $r_4 = \log(1-g_4) = \log(1+z) \in F$.

Thus the expression of u' in Equation 4.60 is also a $\mathcal{DEL}-$expression of u' over F, that satisfies our hypothesis in Theorem 4.1.1. □

If in the Theorem 4.1.1, the constant coefficients of $\frac{h_l'}{h_l}$ in r_i' and $\frac{g_i'}{g_i}$ in s_l', that is, d_{il}'s are not same then v need not satisfy a special $\mathcal{DEL}-$expression over any logarithmic extension of F and thus antiderivative of v need not to be in a $\mathcal{DEL}-$extension.

Example 4.5.6. Let $\log(z)$, $\log(1-z)$, $\log(1+z)$ and $\log(1-z-z^2)$ be designated solutions of the differential equations $y' = 1/z$, $y' = 1/(1-z)$, $y' = 1/(1+z)$ and $y' = (-1-2z)/1+z-z^2$ respectively. Let $F = \mathbb{C}\left(z, \log z, \log(1-z), \log(1-z-z^2)\right)$ be the ordinary differential field with the derivation $' := d/dz$. Let $v \in F$ satisfies a $\mathcal{DEL}-$expression:

$$v = \frac{z'}{z}\left(\log(1-z) + \log(1-z-z^2)\right) + 2\frac{(1-z-z^2)'}{1-z-z^2}\log z + w' \tag{4.61}$$

over F. Replacing $\log(z)\log(1-z-z^2) + w$ with w, we obtain

$$v = \frac{z'}{z}\log(1-z) + \frac{(1-z-z^2)'}{1-z-z^2}\log z + w'. \tag{4.62}$$

Claim: v cannot be written as a special $\mathcal{DEL}-$expression over any logarithmic extension of F.

Let $L = F(\log(z-\alpha_1),\ldots,\log(z-\alpha_l))$, α_j's $\in \mathbb{C}$, be a logarithmic extension of F. Since $F = \mathbb{C}(z, \log z, \log(1-z), \log(1-z-z^2))$, we can assume $L = \mathbb{C}(z, \log(z-\alpha_1),\ldots,\log(z-\alpha_n))$ where $\log(z-\alpha_j)$'s are algebraically independent over $\mathbb{C}$. Suppose

$$v = \sum_{j=1}^{n} r_j \frac{(z-\alpha_j)'}{z-\alpha_j} + v_0' = \sum_{j=1}^{n} c_j \log(1-z-\alpha_j)\frac{(z-\alpha_j)'}{z-\alpha_j} + v_0',$$

where $c_j \in \mathbb{C}$ and $v_0 \in L$, is a special $\mathcal{DEL}-$expression of v over L. It is evident that $w - v_0$ is a quadratic polynomial in $\mathbb{C}(z)[\log(z-\alpha_1),\ldots,\log(z-\alpha_n)]$.

Let $w-v_0 = \sum_{j=1}^{n} d_j \log^2(z-\alpha_j) + \sum_{j=1}^{n} w_j \log(z-\alpha_j) + w_0$, where d_j's are constants and w_j's are in $\mathbb{C}(z)$. Thus

$$\begin{aligned}\frac{z'}{z}\log(1-z) + \frac{(1-z-z^2)'}{1-z-z^2}\log z = {} & \sum_{j=1}^{n} c_j \log(1-z-\alpha_j)\frac{(z-\alpha_j)'}{z-\alpha_j} \\ & - \left(\sum_{j=1}^{n} d_j \log^2(z-\alpha_j) + \sum_{j=1}^{n} w_j \log(z-\alpha_j) + w_0\right)'.\end{aligned}$$

Let $\alpha_1 = 0$ and $\alpha_2 = 1$, compare the coefficient of $\log(z-\alpha_1)$ i.e $\log z$ and obtain

$$\frac{(1-z-z^2)}{1-z-z^2} = c_2\frac{(z-1)'}{z-1} - 2d_1\frac{z'}{z} - w_1'$$

which is absurd because no such element w_1 lies in $\mathbb{C}(z)$. This proves our claim. Therefore, v does not hold antiderivative in any $\mathcal{DEL}-$extension. □

Chapter 5

Integration with Polylogarithmic Integrals

We study integration in finite terms with polylogarithmic integrals along with transcendental elementary functions in this chapter. In the first two sections of this chapter, we define polylogarithmic integral of order 3, namely trilogarithmic integrals and extend the Liouville's Theorem to include trilogarithmic integrals. Though the proofs concerning trilogarithmic integrals are quite lengthy and calculations are exhaustive, the techniques used are similar to that of dilogarithmic integrals. In the third section we note that one can inductively extend these results to polylogarithmic integral of order m. We expect proofs concerning polylogarithmic integrals to be quite complicated and therefore we shall only state the conjecture for integration in finite terms with polylogarithmic integrals.

5.1 Trilogarithmic integrals

In Chapter-3, we defined polylogarithm as

$$Li_m(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}z} Li_m(z) = \frac{z'}{z} Li_{m-1}(z), \quad \text{for any positive integer } m.$$

In particular, for $m = 3$

$$\frac{\mathrm{d}}{\mathrm{d}z} Li_3(z) = \frac{z'}{z} Li_2(z).$$

Therefore from an algebraic point of view, we shall define trilogarithmic integrals as below:

Definition 5.1.1. Let $E \supset F$ be differential fields and $g \in F - \{0, 1\}$ be any element. Then the integral

$$\int \frac{g'}{g} \ell_2(g)$$

in E is called *trilogarithmic integral* of g and is denoted by $\ell_3(g)$.

It is clear from the definition that for $g \in F$, $\ell_3(g)$ is primitive over the field $F(\log(1-g), \ell_2(g))$. Now we look into some identities of trilogarithmic integrals whose proofs involve only basic algebra. One of the fact which we will use widely is noted as a remark below. The approach to the remark is similar to the one used by Baddoura in [1], pp.924-925.

Remark 5.1.2. Let $f \in F(\theta)$ be any non-zero rational element and assume θ is transcendental over F. Let $\overline{F}$ be an algebraic closure of F containing all the zeroes and poles of f and $1-f$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_t\}$ be the set of all zeroes and poles of f and $1-f$ in $\overline{F}$. Write

$$f = \eta\frac{P}{Q} = \eta\prod_{j=1}^{t}(\theta-\alpha_j)^{a_j} \quad \text{and} \quad 1-f = \frac{Q-\eta P}{Q} = \xi\frac{R}{Q} = \xi\prod_{j=1}^{t}(\theta-\alpha_j)^{b_j},$$

where $\eta, \xi \in F$, P, Q and R are co-prime monic polynomials in $\overline{F}[\theta]$ and a_j, b_j are some integers. Without loss of generality, assume $P(\alpha_j) = 0$ for $j = 1, \ldots, m$, $Q(\alpha_j) = 0$ for $j = m+1, \ldots, n$ and $R(\alpha_j) = 0$ for $j = n+1, \ldots, t$. Since P, Q and R are co-prime polynomials, it is clear that $b_1 = \cdots = b_m = 0$, $a_{n+1} = \cdots = a_t = 0$ and $a_j = b_j$ for $j = m+1, \ldots, n$.

Consider the expression

$$T = \sum_{\substack{j,k=1\\k\neq j}}^{t}(a_k b_j - a_j b_k)\frac{\alpha_j' - \alpha_k'}{\alpha_j - \alpha_k}v_k,$$

where v_k is any element in some extension of $\overline{F}(\theta)$ and divide T into three parts $T = T_1 + T_2 + T_3$ where

$$\begin{aligned}
T_1 &= \sum_{k=1}^{m} a_k \left(\sum_{j=m+1}^{t} b_j \frac{\alpha_j' - \alpha_k'}{\alpha_j - \alpha_k}\right) v_k \\
T_2 &= -\sum_{k=n+1}^{t} b_k \left(\sum_{j=1}^{n} a_j \frac{\alpha_j' - \alpha_k'}{\alpha_j - \alpha_k}\right) v_k \\
T_3 &= \sum_{k=m+1}^{n} \left(a_k \sum_{j=m+1}^{t} b_j \frac{\alpha_j' - \alpha_k'}{\alpha_j - \alpha_k} - b_k \sum_{j=1}^{n} a_j \frac{\alpha_j' - \alpha_k'}{\alpha_j - \alpha_k}\right) v_k.
\end{aligned}$$

Since $\eta P + \xi R = Q$, if $P(\alpha_k) = 0$ for some $k = 1, \ldots, m$ then $\xi R(\alpha_k) = Q(\alpha_k)$ and its logarithmic derivative is $\frac{R(\alpha_k)'}{R(\alpha_k)} - \frac{Q(\alpha_k)'}{Q(\alpha_k)} = -\frac{\xi'}{\xi}$. Thus,

$$\sum_{j=m+1}^{t} b_j \frac{\alpha_j' - \alpha_k'}{\alpha_j - \alpha_k} = \frac{R(\alpha_k)'}{R(\alpha_k)} - \frac{Q(\alpha_k)'}{Q(\alpha_k)} = -\frac{\xi'}{\xi}.$$

Similarly for some $k = n+1, \ldots, t$,

$$\sum_{j=1}^{n} a_j \frac{\alpha_j' - \alpha_k'}{\alpha_j - \alpha_k} = \frac{P(\alpha_k)'}{P(\alpha_k)} - \frac{Q(\alpha_k)'}{Q(\alpha_k)} = -\frac{\eta'}{\eta}.$$

Note that for $k = m+1, \ldots, n$, $a_k = b_k$ and

$$a_k \sum_{j=m+1}^{n} b_j \frac{\alpha_j' - \alpha_k'}{\alpha_j - \alpha_k} - b_k \sum_{j=m+1}^{n} a_j \frac{\alpha_j' - \alpha_k'}{\alpha_j - \alpha_k} = 0.$$

Also, $Q(\alpha_k) = 0$ for any $k = m+1, \ldots, n$ and $\eta P(\alpha_k) = -\xi R(\alpha_k)$. Therefore,

$$\begin{aligned}\sum_{j=m+1}^{t} b_j \frac{\alpha_j' - \alpha_k'}{\alpha_j - \alpha_k} - \sum_{j=1}^{n} a_j \frac{\alpha_j' - \alpha_k'}{\alpha_j - \alpha_k} &= \sum_{j=n+1}^{t} b_j \frac{\alpha_j' - \alpha_k'}{\alpha_j - \alpha_k} - \sum_{j=1}^{m} a_j \frac{\alpha_j' - \alpha_k'}{\alpha_j - \alpha_k} \\ &= -\frac{P(\alpha_k)'}{P(\alpha_k)} + \frac{R(\alpha_k)'}{R(\alpha_k)} \\ &= \frac{\eta'}{\eta} - \frac{\xi'}{\xi}.\end{aligned}$$

Combining the above three parts, T becomes

$$\begin{aligned}T &= \sum_{k=n+1}^{t} b_k v_k \frac{\eta'}{\eta} - \sum_{k=1}^{m} a_k v_k \frac{\xi'}{\xi} + \sum_{k=m+1}^{n} v_k \left(b_k \frac{\eta'}{\eta} - a_k \frac{\xi'}{\xi} \right) \\ &= \sum_{k=1}^{t} \left(b_k \frac{\eta'}{\eta} - a_k \frac{\xi'}{\xi} \right) v_k.\end{aligned}$$

□

Since derivative of trilogarithmic integral involves dilogarithmic integrals, it is useful to have some more identities of dilogarithmic integrals. Baddoura [1] described an identity for dilogarithmic integrals in terms of Bloch-Wigner-Spence function. We shall state and prove a similar identity which involves only dilogarithmic integrals and its proof is similar to one mentioned in [1], p.923, Proposition 2 (which we have also noted here as Proposition 4.4.2).

Proposition 5.1.3. *Let $F(\theta) \supset F$ be a transcendental field extension with $C_{F(\theta)} = C_F$. Let $f(\theta)$ be a rational element in $F(\theta)$ and $\{\alpha_j; j = 1, \ldots, t\}$ be the set of all zeroes and poles of $f(\theta)$ and $1 - f(\theta)$ in an algebraic closure of F. Also, for some integers a_j, b_j, let $f(\theta) = \eta \prod_{j=1}^{t} (\theta - \alpha_j)^{a_j}$, $1 - f(\theta) = \xi \prod_{j=1}^{t} (\theta - \alpha_j)^{b_j}$, then for some constant c,*

$$\begin{aligned} \ell_2(f(\theta)) = \ell_2(\eta) - \sum_{\substack{j,k=1 \\ k \neq j}}^{t} a_j b_k \ell_2 \left(\frac{\theta - \alpha_j}{\theta - \alpha_k} \right) - \frac{1}{2} \sum_{j,k=1}^{t} a_j b_k \log^2(\theta - \alpha_k) \\ - \sum_{k=1}^{t} a_k \log(\theta - \alpha_k) \log \xi - \sum_{\substack{j,k=1 \\ k \neq j}}^{t} a_j b_k \log \left(\frac{\theta - \alpha_j}{\theta - \alpha_k} \right) \log(\alpha_j - \alpha_k) + c. \end{aligned}$$

Proof. From the definition of dilogarithm, we have

$$\ell_2'(f(\theta)) = -\frac{f'}{f} \log(1 - f) \tag{5.1}$$

Replacing $f, 1 - f$ with their partial fraction expansion and rearranging the terms, obtain

$$\begin{aligned} \ell_2(f) = -\frac{\eta'}{\eta} \log \xi + \sum_{j=1}^{t} a_j \log(\theta - \alpha_j) \frac{\xi'}{\xi} - \sum_{k=1}^{t} b_k \log(\theta - \alpha_k) \frac{\eta'}{\eta} \\ - \sum_{j,k=1}^{t} a_j b_k \frac{\theta' - \alpha_j'}{\theta - \alpha_j} \log(\theta - \alpha_k) - \left(\sum_{j=1}^{t} a_j \log(\theta - \alpha_j) \log \xi \right)'. \end{aligned} \tag{5.2}$$

From the definition of dilogarithm, observe

$$\begin{aligned} \sum_{j,k=1,k \neq j}^{t} a_j b_k \ell_2' \left(\frac{\theta - \alpha_j}{\theta - \alpha_k} \right) = \sum_{j,k=1,k \neq j}^{t} a_j b_k \left(\log(\theta - \alpha_j) - \log(\theta - \alpha_k) \right) \frac{\alpha_j' - \alpha_k'}{\alpha_j - \alpha_k} \\ + \sum_{j,k=1,k \neq j}^{t} a_j b_k \left(\frac{\theta' - \alpha_j'}{\theta - \alpha_j} - \frac{\theta' - \alpha_k'}{\theta - \alpha_k} \right) \log(\theta - \alpha_k) \\ - \left(\sum_{j,k=1,k \neq j}^{t} a_j b_k \log \left(\frac{\theta - \alpha_j}{\theta - \alpha_k} \right) \log(\alpha_j - \alpha_k) \right)'. \end{aligned} \tag{5.3}$$

Using remark 5.1.2, the Equation 5.3 can be written as

$$\sum_{k=1}^{t}\left(b_k\frac{\eta'}{\eta}-a_k\frac{\xi'}{\xi}\right)\log(\theta-\alpha_k)+\sum_{j,k=1}^{t}a_jb_k\frac{\theta'-\alpha_j'}{\theta-\alpha_j}\log(\theta-\alpha_k)=\sum_{\substack{j,k=1\\k\neq j}}^{t}a_jb_k\ell_2'\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right)$$
$$+\sum_{j,k=1}^{t}a_jb_k\frac{\theta'-\alpha_k'}{\theta-\alpha_k}\log(\theta-\alpha_k)+\left(\sum_{\substack{j,k=1\\k\neq j}}^{t}a_jb_k\log\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right)\log(\alpha_j-\alpha_k)\right)'.$$

From Equation 5.2, we see that LHS of above equation equals $-\ell_2(f)-\frac{\eta'}{\eta}\log\xi-\left(\sum_{j=1}^{t}a_j\log(\theta-\alpha_j)\log\xi\right)'$. Therefore,

$$\ell_2'(f(\theta))=-\frac{\eta'}{\eta}\log\xi-\sum_{\substack{j,k=1\\k\neq j}}^{t}a_jb_k\ell_2'\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right)-\sum_{j,k=1}^{t}a_jb_k\frac{\theta-\alpha_k'}{\theta-\alpha_k}\log(\theta-\alpha_k)$$
$$-\left(\sum_{k=1}^{t}a_k\log(\theta-\alpha_k)\log\xi\right)'-\left(\sum_{\substack{j,k=1\\k\neq j}}^{t}a_jb_k\log\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right)\log(\alpha_j-\alpha_k)\right)'.$$

When $\deg(P)<\deg(Q)$ or $\deg(P)=\deg(Q)$ with $\eta=1$, in both cases the term $\frac{\eta'}{\eta}\log\xi=0$, otherwise $\xi=1-\eta$ and $-\frac{\eta'}{\eta}\log\xi=\ell_2'(\eta)$. Thus integrating the above equation we shall obtain desired result. □

We note one more basic identity for dilogarithmic integrals in the following remark.

Remark 5.1.4. Let $F(\theta)\supset F$ be a transcendental differential field extension and $C_{F(\theta)}=C_F$. Let α,β be algebraic over F. Then, for some constants $c_{\alpha\beta}$ and c in F,

$$\ell_2\left(\frac{\theta-\alpha}{\theta-\beta}\right)=-\ell_2\left(\frac{\theta-\beta}{\theta-\alpha}\right)+\log(\theta-\alpha)\log(\theta-\beta)$$
$$-\frac{1}{2}\left(\log^2(\theta-\alpha)+\log^2(\theta-\beta)\right)+c_{\alpha\beta}\log\left(\frac{\theta-\alpha}{\theta-\beta}\right)+c.$$

Proof. From the definition of dilogarithm, it is clear that

$$\ell_2'\left(\frac{\theta-\alpha}{\theta-\beta}\right)=-\left(\frac{\theta'-\alpha'}{\theta-\alpha}-\frac{\theta'-\beta'}{\theta-\beta}\right)\log\left(\frac{\alpha-\beta}{\theta-\beta}\right)$$

and

$$\ell_2'\left(\frac{\theta-\beta}{\theta-\alpha}\right) = -\left(\frac{\theta'-\beta'}{\theta-\beta} - \frac{\theta'-\alpha'}{\theta-\alpha}\right)\log\left(\frac{\beta-\alpha}{\theta-\alpha}\right)$$
$$= -\left(\frac{\theta'-\beta'}{\theta-\beta} - \frac{\theta'-\alpha'}{\theta-\alpha}\right)\left(\log\left(\frac{\alpha-\beta}{\theta-\alpha}\right) + c_{\alpha\beta}\right).$$

Adding the above two equation gives

$$\ell_2'\left(\frac{\theta-\alpha}{\theta-\beta}\right) + \ell_2'\left(\frac{\theta-\beta}{\theta-\alpha}\right) = (\log(\theta-\alpha)\log(\theta-\beta))' - \frac{\theta'-\alpha'}{\theta-\alpha}\log(\theta-\alpha)$$
$$- \frac{\theta'-\beta'}{\theta-\beta}\log(\theta-\beta) + c_{\alpha\beta}\left(\frac{\theta'-\alpha'}{\theta-\alpha} - \frac{\theta'-\beta'}{\theta-\beta}\right).$$

Rearranging the above terms and integrating gives us the desired relation. □

We are dealing with dilogarithmic integrals of the form $\ell_2(\frac{\theta-\alpha}{\theta-\beta})$. The question of algebraic independence of such dilogarithmic integrals is natural. From above remark it is clear that $\ell_2(\frac{\theta-\alpha}{\theta-\beta})$ and $\ell_2(\frac{\theta-\beta}{\theta-\alpha})$ are algebraically dependent over a field containing $\log(\theta-\alpha)$ and $\log(\theta-\beta)$. Now we will show that the set $\{\ell_2\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right); k > j, \alpha_j \neq \alpha_k \text{ for } j \neq k\}$ is algebraically independent over a logarithmic extension of $F(\theta)$.

Recall that a differential field extension E of F is called a logarithmic extension of F if $C_E = C_F$ and there are elements $h_1, \dots, h_m \in F$ such that $E = F(\log(h_1), \dots, \log(h_m))$.

Lemma 5.1.5. *Let $F(\theta) \supset F$ be a transcendental differential field extension of F and $C_{F(\theta)} = C_F$. Let $\alpha_1, \dots, \alpha_t$ be distinct elements in F. Then the set $\{\ell_2\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right); k > j\}$ is algebraically independent over the logarithmic extension $E = F(\theta)(\{\log(\alpha_j-\alpha_k), \log(\theta-\alpha_j); j,k = 1,\dots,t\})$ of $F(\theta)$.*

Proof. Suppose the set $\{\ell_2\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right); k > j\}$ is algebraically dependent over E. Since for each $k > j$, the derivative $\ell_2'\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right)$ lies in E, we shall apply Corollary 2.3.3 and obtain

$$\ell_2\left(\frac{\theta-\alpha_1}{\theta-\alpha_2}\right) = \sum_{\substack{j,k=1 \\ k>j, k\neq 2}}^{t} c_{jk}\ell_2\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right) + v,$$

where each c_{jk} is a constant and $v \in E$. Taking derivative of the above equation, we get

$$\begin{aligned} -\left(\frac{\theta'-\alpha_1'}{\theta-\alpha_1} - \frac{\theta'-\alpha_2'}{\theta-\alpha_2}\right)\log\left(\frac{\alpha_1-\alpha_2}{\theta-\alpha_2}\right) &= \\ -\sum_{\substack{j,k=1 \\ k>j, k\neq 2}}^{t} c_{jk}\left(\frac{\theta'-\alpha_j'}{\theta-\alpha_j} - \frac{\theta'-\alpha_k'}{\theta-\alpha_k}\right)\log\left(\frac{\alpha_j-\alpha_k}{\theta-\alpha_k}\right) + v'. \end{aligned} \tag{5.4}$$

Since $\log(\theta-\alpha_2)$ is transcendental over the field $F_2 = F(\theta)(\{\log(\alpha_j-\alpha_k); j,k = 1,\ldots,t\})(\{\log(\theta-\alpha_j); j \neq 2\})$, we can apply Proposition 2.2.4 and therefore, for some constant c_1 and elements $v_1, v_0 \in F_2$, we shall write $v = c_1\log^2(\theta-\alpha_2) + v_1\log(\theta-\alpha_2) + v_0$. Comparing the coefficient of $\log(\theta-\alpha_2)$ in the above equation, we will obtain

$$\frac{\theta'-\alpha_1'}{\theta-\alpha_1} - \frac{\theta'-\alpha_2'}{\theta-\alpha_2} = 2c_1\frac{\theta'-\alpha_2'}{\theta-\alpha_2} + v_1'.$$

It is obvious that $c_1 = -1/2$ and for a constant c, $v_1 = \log(\theta-\alpha_1) + c$. Thus the Equation 5.4 becomes

$$\begin{aligned} -\left(\frac{\theta'-\alpha_1'}{\theta-\alpha_1} - \frac{\theta'-\alpha_2'}{\theta-\alpha_2}\right)\log(\alpha_1-\alpha_2) &= \\ -\sum_{\substack{j,k=1 \\ k>j, k\neq 2}}^{t} c_{jk}\left(\frac{\theta'-\alpha_j'}{\theta-\alpha_j} - \frac{\theta'-\alpha_k'}{\theta-\alpha_k}\right)\log\left(\frac{\alpha_j-\alpha_k}{\theta-\alpha_k}\right) &+ \frac{\theta'-\alpha_2'}{\theta-\alpha_2}(\log(\theta-\alpha_1)+c) + v_0'. \end{aligned}$$

Now since $\log(\theta-\alpha_1)$ is transcendental over $F_1 = F(\theta)(\{\log(\alpha_j-\alpha_k); j,k = 1,\ldots,t\})(\{\log(\theta-\alpha_j); j \neq 1,2\})$, we reapply Proposition 2.2.4 and repeat the same process. For some constant c_2 and elements $w_1, w_0 \in F_1$, we write $v_0 = c_2 \log^2(\theta - \alpha_1) + w_1 \log(\theta - \alpha_1) + w_0$. Comparing the coefficient of $\log(\theta - \alpha_1)$ in the above equation, we shall obtain

$$\frac{\theta' - \alpha_2'}{\theta - \alpha_2} + 2c_2 \frac{\theta' - \alpha_1'}{\theta - \alpha_1} + w_1' = 0.$$

It is clear that there exists no such w_1 in F_1. Thus we arrive at a contradiction and therefore, the set $\{\ell_2\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right); k > j\}$ is algebraically independent over the logarithmic extension E of F. □

The trilogarithmic integrals $\ell_3(1-g)$ and $\ell_3(g)$ have no known identity. However, $\ell_3(1/g)$ and $\ell_3(g)$ do satisfy a relation which we note in the next remark.

Remark 5.1.6. Let $E \supset F$ be differential fields and $v, r, g, w \in E$ such that

$$v = r\frac{g'}{g} + w',$$

where $r' = -c\log(1-g)g'/g$, that is, $r = c\ell_2(g)$. Since $\ell_2(g) = -\ell_2(1/g) - (1/2)\log^2 g$,

$$v = -c\ell_2(1/g)\frac{g'}{g} - \frac{c}{2}\log^2 g \frac{g'}{g} + w' = c\ell_2(1/g)\frac{(1/g)'}{(1/g)} + \tilde{w}',$$

where $\tilde{w} = -(c/6)\log^3 g + w$. In other words, $\ell_3(1/g) = \ell_3(g)$+some element in the field containing $\log g$.

5.2 Liouville's theorem for $\mathcal{T}-$extensions

We shall consider trilogarithmic integrals in our field of definition and extend Liouville's Theorem to such extensions.

Definition 5.2.1. A differential field $E \supset F$ is called a $\mathcal{T}-$*extension* of F if $C_E = C_F$ and there is a tower of differential fields F_i such that

$$F = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = E$$

and for each i, $F_i = F_{i-1}(\theta_i)$ and one of the following holds:

(i) θ_i is algebraic over F_{i-1}.

(ii) $\theta_i' = u'\theta_i$ for some $u \in F_{i-1}$ (i.e. $\theta_i = e^u$).

(iii) $\theta_i' = u'/u$ for some $u \in F_{i-1}$ (i.e. $\theta_i = \log(u)$).

(iv) $\theta_i' = vu'/u$, where $v' = (1-u)'/(1-u)$ for some $u, v \in F_{i-1}$ (i.e. $\theta_i = \int \frac{u'}{u}\log(1-u)$, also denoted by $-\ell_2(u)$).

(v) $\theta_i' = vu'/u$, where $v' = -u'\log(1-u)/u$ for some $u, v \in F_{i-1}$ (i.e. $\theta_i = \int \frac{u'}{u}\ell_2(u)$, also denoted by $\ell_3(u)$).

Definition 5.2.2. We say that $v \in F$ admits a $\mathcal{T}-$*expression* over F if there are finite indexing sets I, J and elements $r_i, g_i \in F$ for all $i \in I$, elements $s_j, h_j \in F$ for all $j \in J$ and an element $w \in F$ such that

$$v = \sum_{i \in I} r_i \frac{g_i'}{g_i} + \sum_{j \in J} s_j \frac{h_j'}{h_j} + w', \tag{5.5}$$

where for each $i \in I$, there is a constant c_i and $\log(1-g_i) \in F$ such that $r_i' = -c_i g_i' \log(1-g_i)/g_i$, and for each $j \in J$ there is a constant d_j such that $s_j' = d_j(1-h_j)'/(1-h_j)$.

If in expression 5.5, r_i''s satisfy some $\mathcal{DEL}-$expressions and s_j's are constant linear combinations of logarithms over F then the expression: $\sum_{i\in I} r_i \frac{g_i'}{g_i} + \sum_{j \in J} s_j \frac{h_j'}{h_j} + w'$ will be called *general* $\mathcal{T}-$*expression*.

Definition 5.2.3. We call a differential field extension E of F to be a *dilogarithmic extension* of F if their field of constants coincides and there are elements $y_1, \ldots, y_n, z_1, \ldots, z_m \in F$ such that $E = F(\log(y_1), \ldots, \log(y_n), \ell_2(z_1), \ldots, \ell_2(z_m))$.

Note that for a $\mathcal{T}-$expression: $\sum_{i\in I} r_i \frac{g_i'}{g_i} + \sum_{j\in J} s_j \frac{h_j'}{h_j} + w'$ over a dilogarithmic extension $M \supset F$, where $r_i' = -c_i g_i' \log(1-g_i)/g_i$ and $s_j' = d_j(1-h_j)'/(1-h_j)$, we shall combine the term $\sum_{j\in J} s_j \frac{h_j'}{h_j} = (\sum_{j\in J} d_j \ell_2(h_j))'$ with w' and consider the $\mathcal{T}-$expression over M as $\sum_{i\in I} r_i \frac{g_i'}{g_i} + w'$ only. Similar to Proposition 4.1.2 for $\mathcal{DEL}-$expressions, we first prove that a $\mathcal{T}-$expression over a dilogarithmic extension of field F can be written as a general $\mathcal{T}-$expression: $\sum_{i\in I} r_i g_i'/g_i + w'$, where r_i' satisfies some $\mathcal{DEL}-$expressions.

Proposition 5.2.4. *Let F be a differential field and $v \in F$ satisfies a $\mathcal{T}-$expression over a dilogarithmic extension E of F. Then v satisfies a general $\mathcal{T}-$expression:*

$$v = \sum_{i\in I} r_i g_i'/g_i + \sum_{j\in J} s_j h_j'/h_j + w' \tag{5.6}$$

over F, where I and J are some finite index sets and each w, g_i, h_j, r_i, s_j are elements in F such that

$$r_i' = t_i \frac{g_i'}{g_i} + \sum_{j\in J} r_{ij} \frac{h_j'}{h_j}, \quad s_j' = \sum_{i\in I} r_{ij} \frac{g_i'}{g_i} + \sum_{k\in J} s_{jk} \frac{h_k'}{h_k}, \tag{5.7}$$

$$t_i' = -c_i \frac{(1-g_i)'}{1-g_i} + \sum_{j\in J} c_i c_{ij} \frac{h_j'}{h_j}, \quad r_{ij}' = c_i c_{ij} \frac{g_i'}{g_i} + \sum_{k\in J} e_{ijk} \frac{h_k'}{h_k} \quad \text{and}$$

$$s_{jk}' = \sum_{i\in I} e_{ijk} \frac{g_i'}{g_i} + \sum_{l\in J} f_{jkl} \frac{h_l'}{h_l}, \tag{5.8}$$

where each c_i is a non-zero constant whenever $r_i' \neq 0$, each c_{ij}, e_{ijk}, f_{jkl} are some constants and each t_i, r_{ij} and s_{jk} are in some extension of F with $e_{ijk} = e_{ikj}$ and $s_{jk} = s_{kj}$ for every i, j and k.

Proof. Suppose $E = F(\log y_1, \dots, \log y_n, \ell_2(z_1), \dots, \ell_2(z_m))$ is a dilogarithmic extension over F, where $y_1, \dots, y_n, z_1, \dots, z_m \in F$ forms a transcendental base of E over F. We can also assume that each $\log(1 - z_j)$ lies in $F(\log y_1, \dots, \log y_n)$. Let $v = \sum_{i \in I} r_i \frac{g_i'}{g_i} + w'$ be $\mathcal{T}$−expression over E. Since each $\ell_2(z_j)$ is primitive over $E_1 = F(\log y_1, \dots, \log y_n)$, we shall apply Corollary 2.3.3 and write

$$r_i = \sum_{j=1}^{m} c_{ij} \ell_2(z_j) + s_i$$

for some constants c_{ij} and elements s_i in E_1. Considering Proposition 2.2.4, we assume $w = \sum_{j,k=1}^{m} d_{jk} \ell_2(z_j) \ell_2(z_k) + \sum_{j=1}^{m} w_j \ell_2(z_j) + w_0$, where each d_{jk} is a constant and each w_j, w_0 lies in E_1. Since $\{\ell_2(z_j)\}$ are transcendental over E_1, we shall compare the coefficients of $\ell_2(z_j)$ in the expression of v and obtain

$$v = \sum_{i \in I} s_i \frac{g_i'}{g_i} - \sum_{j=1}^{m} w_j \log(1 - z_j) \frac{z_j'}{z_j} + w_0', \tag{5.9}$$

$$\text{where} \quad s_i = c_i \ell_2(g_i) - \sum_{j=1}^{m} c_{ij} \ell_2(z_j) \tag{5.10}$$

$$\text{and} \quad w_j' = -\sum_{i \in I} c_{ij} \frac{g_i'}{g_i} + \sum_{k=1}^{m} (d_{jk} + d_{kj}) \log(1 - z_k) \frac{z_k'}{z_k}. \tag{5.11}$$

Since dilogarithmic integrals are non-elementary functions and $\ell_2(z_1), \dots, \ell_2(z_m)$ are algebraically independent over E_1, $d_{jk} + d_{kj}$ must be 0. Thus, $w_j' = -\sum_{i \in I} c_{ij} \frac{g_i'}{g_i}$. Now each $s_i \in E_1$ and s_i' is a polynomial in $F[\log y_1, \dots, \log y_n]$. We shall apply Proposition 2.2.4 and write

$$s_i = \sum_{k,l=1}^{n} e_{ikl} \log y_k \log y_l + \sum_{k=1}^{n} s_{ik} \log y_k + t_i, \tag{5.12}$$

for some constants c_{ikl} and elements $s_{ik}, t_i \in F$. Also using Corollary 2.3.3, write

$$w_j = \sum_{k=1}^{n} e_{jk} \log y_k + w_{j0}, \tag{5.13}$$

$$\log(1 - g_i) = \sum_{k=1}^{n} d_{ik} \log y_k + \alpha_i \quad \text{and} \tag{5.14}$$

$$\log(1 - z_j) = \sum_{k=1}^{n} f_{jk} \log y_k + \beta_j, \tag{5.15}$$

where each e_{jk}, d_{ik} and f_{jk} is constant and each $w_{j0}, \alpha_i, \beta_j$ is some element in F. From Equation 5.9, it is clear that $w_0' \in F[\log y_1, \ldots, \log y_n]$ is a polynomial of degree 2. Therefore, a repeated application of Proposition 2.2.4 gives

$$w_0 = \sum_{j,k,l=1}^{n} c_{jkl} \log y_j \log y_k \log y_l + \sum_{k,l=1}^{n} v_{kl} \log y_k \log y_l + \sum_{k=1}^{n} v_k \log y_k + v_0,$$

where $c_{jkl} \in C_F$ and $v_{jk}, v_k, v_0 \in F$. Take derivative of s_i in Equation 5.10 and substitute values of $\log(1 - g_i)$ and $\log(1 - z_j)$ in it. Also take the derivative of s_i in Equation 5.12. Comparing these two expressions for s_i', we shall obtain

$$s_{ik}' = -c_i d_{ik} \frac{g_i'}{g_i} + \sum_{j=1}^{m} c_{ij} f_{jk} \frac{z_j'}{z_j} - \sum_{l=1}^{n} (e_{ikl} + e_{ilk}) \frac{y_l'}{y_l} \quad \text{and} \tag{5.16}$$

$$t_i' = -c_i \alpha_i \frac{g_i'}{g_i} + \sum_{j=1}^{m} c_{ij} \beta_j \frac{z_j'}{z_j} - \sum_{k=1}^{n} s_{ik} \frac{y_k'}{y_k}. \tag{5.17}$$

Now substituting the values of $s_i, w_j, \log(1 - z_j)$ and w_0 in Equation 5.9 and comparing the coefficients of $\log^2 y_k$ and $\log y_k$, observe that

$$(v_{kl} + v_{lk})' = -\sum_{i \in I} (e_{ikl} + e_{ilk}) \frac{g_i'}{g_i} + \sum_{j=1}^{m} (e_{jk} f_{jl} + e_{jl} f_{jk}) \frac{z_j'}{z_j} - \sum_{j=1}^{n} e_j \frac{y_j'}{y_j} \tag{5.18}$$

$$\text{and} \quad -v_k' = \sum_{i\in I} s_{ik}\frac{g_i'}{g_i} - \sum_{j=1}^{m}(w_{j0}f_{jk} + e_{jk}\beta_j)\frac{z_j'}{z_j} + \sum_{l=1}^{n}(v_{kl} + v_{lk})\frac{y_l'}{y_l}, \tag{5.19}$$

where $e_j = \sum_{\sigma\in S_3} c_{\sigma(j)\sigma(k)\sigma(l)}$. Comparing the constant terms of Equation 5.9, we obtain

$$v = \sum_{i\in I} t_i\frac{g_i'}{g_i} - \sum_{j=1}^{m} w_{j0}\beta_j\frac{z_j'}{z_j} + \sum_{k=1}^{n} v_k\frac{y_k'}{y_k} + v_0'.$$

The above expression is the desired general $\mathcal{T}-$expression for v, where the derivatives of the coefficients, that is, t_i', $(w_{j0}\beta_j)'$ and v_k' satisfies the $\mathcal{DEL}-$expressions given by Equations 5.17, 5.13, 5.15 and 5.19. Further observe that coefficients in these $\mathcal{DEL}-$expressions satisfies $\mathcal{L}-$expressions given by Equations 5.14, 5.15, 5.16 and 5.18. □

Remark 5.2.5. Note that in Proposition 4.1.2, when we reduce a $\mathcal{D}-$expression over a logarithmic extension of F to a general $\mathcal{D}-$expression over F, we obtained "two sets of equations", namely, 4.1, 4.2. In a similar way, in Proposition 5.2.4 when we reduce a $\mathcal{T}-$expression over a dilogarithmic extension of F to a general $\mathcal{T}-$expression over F, we obtained "three sets of equations", namely, 5.6, 5.7 and 5.8. Thus inductively we can conjecture that for any polylogarithmic extension of order m, a $\mathcal{P}-$expression will reduce to a general $\mathcal{P}-$expression consisting of "m-sets of equations". We will explain this in detail in the next section.

Proposition 5.2.6. *Let F be a differential field and $v \in F$ satisfies a general $\mathcal{T}-$expression: $\sum_{i\in I} r_i g_i'/g_i + w$ over F, where the elements $r_i \in F$ are chosen as in Proposition 5.2.4. Then there exists a dilogarithmic extension K of F such that v satisfies a $\mathcal{T}-$expression over K and there exists a $\mathcal{T}-$extension E of F that contains an antiderivative of v.*

Proof. As in Proposition 5.2.4, consider a general $\mathcal{T}-$expression:

$$\sum_{i\in I} r_i g_i'/g_i + \sum_{j\in J} s_j h_j'/h_j + w'$$

over F, where I and J are some finite index sets and each w, g_i, h_j, r_i, s_j are elements in F such that

$$r_i' = t_i \frac{g_i'}{g_i} + \sum_{j\in J} r_{ij}\frac{h_j'}{h_j}, \quad s_j' = \sum_{i\in I} r_{ij}\frac{g_i'}{g_i} + \sum_{k\in J} s_{jk}\frac{h_k'}{h_k},$$

$$t_i' = -c_i\frac{(1-g_i)'}{1-g_i} + \sum_{j\in J} c_i c_{ij}\frac{h_j'}{h_j}, \quad r_{ij}' = c_i c_{ij}\frac{g_i'}{g_i} + \sum_{k\in J} e_{ijk}\frac{h_k'}{h_k} \text{ and}$$

$$s_{jk}' = \sum_{i\in I} e_{ijk}\frac{g_i'}{g_i} + \sum_{l\in J} f_{jkl}\frac{h_l'}{h_l},$$

where each c_i is a non-zero constant whenever $r_i' \neq 0$, each c_{ij}, e_{ijk}, f_{jkl} are some constants, $e_{ijk} = e_{ikj}$ and $s_{jk} = s_{kj}$ for every i, j and k.

Over the extension $F(\{\log g_i, \log h_j\})$, we can replace $w - \sum_{i\in I} r_i \log g_i - \sum_{j\in J} s_j \log h_j$ with w and write v as

$$v = -\sum_{i\in I} r_i' \log g_i - \sum_{j\in J} s_j' \log h_j + w'.$$

Substituting values of r_i' and s_j', we obtain

$$v = -\sum_{i\in I}\left(t_i\frac{g_i'}{g_i} + \sum_{j\in J} r_{ij}\frac{h_j'}{h_j}\right)\log g_i - \sum_{j\in J}\left(\sum_{i\in I} r_{ij}\frac{g_i'}{g_i} + \sum_{k\in J} s_{jk}\frac{h_k'}{h_k}\right)\log h_j + w'.$$

We shall replace w with $w - 1/2\sum_{i\in I} t_i \log^2 g_i - \sum_{i\in I}\sum_{j\in J} r_{ij}\log h_j \log g_i - 1/2\sum_{j,k\in J} s_{jk}\log h_j \log h_k$, we get

$$v = \frac{1}{2}\sum_{i\in I} t_i' \log^2 g_i + \sum_{i\in I, j\in J} r_{ij}' \log h_j \log g_i + \frac{1}{2}\sum_{j,k\in J} s_{jk}' \log h_j \log h_k + w'.$$

Now substitute t_i', r_{ij}' and s_{jk}' in the above expression for v, thus

$$
\begin{aligned}
v =& \frac{1}{2}\sum_{i\in I}\left(-c_i\frac{(1-g_i)'}{1-g_i}+\sum_{j\in J}c_ic_{ij}\frac{h_j'}{h_j}\right)\log^2 g_i\\
&+\sum_{i\in I, j\in J}\left(c_ic_{ij}\frac{g_i'}{g_i}+\sum_{k\in J}e_{ijk}\frac{h_k'}{h_k}\right)\log h_j\log g_i\\
&+\frac{1}{2}\sum_{j,k\in J}\left(\sum_{i\in I}e_{ijk}\frac{g_i'}{g_i}+\sum_{l\in J}f_{jkl}\frac{h_l'}{h_l}\right)\log h_j\log h_k+w',
\end{aligned}
$$

where f_{jkl} are some constants. Note that for some constants c_{jkl}, we can assume $f_{jkl}=\sum_{\sigma\in S_3}c_{\sigma(j)\sigma(k)\sigma(l)}$ and thus the term $\sum_{j,k,l\in J}f_{jkl}\frac{h_l'}{h_l}\log h_j\log h_k$ equals $(\sum_{j,k,l\in J}c_{jkl}\log h_j\log h_k\log h_l)'$. Also note that the last term in second sum and first term in the third sum combines to give $(\sum_{i\in I,j,k\in J}e_{ijk}\log g_i\log h_j\log h_k)'$. Similarly last term in first sum and first term in the second sum combines to give $(\frac{1}{2}\sum_{i\in I}\sum_{j\in J}c_ic_{ij}\log h_j\log^2 g_i)'$. We shall replace w with $\frac{1}{2}\sum_{i\in I,j\in J}c_ic_{ij}\log h_j\log^2 g_i$ $+\sum_{i\in I,j,k\in J}e_{ijk}\log g_i\log h_j\log h_k+\frac{1}{2}\sum_{j,k,l\in J}c_{jkl}\log h_j\log h_k\log h_l+w$ and observe

$$v=-\frac{1}{2}\sum_{i\in I}c_i\frac{(1-g_i)'}{1-g_i}\log^2 g_i+w'.$$

Again replace w with $w-\frac{1}{2}\sum_{i\in I}c_i\log(1-g_i)\log^2 g_i$ and obtain

$$v=\sum_{i\in I}c_i\frac{g_i'}{g_i}\log(1-g_i)\log g_i+w'=-\sum_{i\in I}c_i(\ell_2(g_i))'\log g_i+w'.$$

Now consider the differential field extension $F(\{\log g_i,\log h_j,\ell_2(g_i)\})$, which is obviously a dilogarithmic extension of F and replace w with the element $\sum_{i\in I}c_i\ell_2(g_i)$ $\log g_i+w$. Therefore,

$$v=\sum_{i\in I}c_i\ell_2(g_i)\frac{g_i'}{g_i}+w'.$$

Thus v satisfies a special $\mathcal{T}-$expression over the dilogarithmic extension

$F(\{\log g_i, \log h_j, \ell_2(g_i)\})$ of F and if we consider the trilogarithmic extension $E = F(\{\log g_i, \log h_j, \ell_2(g_i)\})(\{\ell_3(g_i)\})$ then $\int v = \sum_{i\in I} c_i \ell_3(g_i) + w$ lies in E. □

From Propositions 5.2.4 and 5.2.6, we obtain necessary and sufficient condition that v satisfies a $\mathcal{T}-$expression over some dilogarithmic extension of F if and only if v satisfies a general $\mathcal{T}-$expression over F whose coefficients satisfies the relation given in Proposition 5.2.4. Proposition 5.2.6 also gives a sufficient condition for existence of an element $v \in F$ whose antiderivative lies in some trilogarithmic extension of F. The necessary condition for this result is discussed in Theorem 5.2.10.

The following lemma, whose proof is lengthy and involved, plays a role similar to that of Lemma 4.2.2 and is required to prove the Theorem 5.2.10.

Lemma 5.2.7. *Let $F(\theta) \supset F$ be a transcendental $\mathcal{T}-$extension of F. Suppose there is an element $v \in F$ such that v admits a $\mathcal{T}-$expression over the differential field $E = F(\theta)(\log(y_1), \dots, \log(y_n), \ell_2(z_1), \dots, \ell_2(z_m))$ where each $y_i, z_i \in F(\theta)$ and $C_E = C_F$. Then there is a differential field $M = F(\log(p_1), \dots, \log(p_l), \ell_2(q_1), \dots, \ell_2(q_t), \theta)$, where each $p_i, q_i \in F$, having the same field of constants as F such that v admits a $\mathcal{T}-$expression over M. Moreover, if θ is exponential over F then v admits a $\mathcal{T}-$expression over $F(\log(p_1), \dots, \log(p_l), \ell_2(q_1), \dots, \ell_2(q_t))$.*

Proof. Let I, J be finite indexing sets and there be constants $c_i(\neq 0), d_j$ in F for all $i \in I$ and $j \in J$ such that

$$v = \sum_{i\in I} r_i \frac{g_i'}{g_i} + \sum_{j\in J} s_j \frac{h_j'}{h_j} + w', \tag{5.20}$$

$$r_i' = -c_i \log(1-g_i)\frac{g_i'}{g_i}, \;\; s_j' = d_j \frac{(1-h_j)'}{(1-h_j)}. \tag{5.21}$$

where $r_i, g_i, \log(1-g_i), s_j, h_j, w \in E$ for all $i \in I$ and $j \in J$. That is, v admits $\mathcal{T}-$expression over E. Since $s_j \frac{h_j'}{h_j} = -d_j \ell_2'(h_j)$ or $s_j \frac{h_j'}{h_j} = (s_j \log(h_j))'$ (when $d_j = 0$), if necessary we enlarge E and assume $\ell_2(h_j) \in E$. Replace w' with $w' + \sum_{j\in J} s_j \frac{h_j'}{h_j}$ and write

$$v = \sum_{i\in I} r_i \frac{g_i'}{g_i} + w'. \tag{5.22}$$

Observe that $\log'(1-g_i) - \frac{(1-g_i)'}{1-g_i} = 0 \in F(\theta)$, we use Proposition 2.2.6 repeatedly and obtain $1-g_i$ and thus g_i lies in $F(\theta)$ for each i. Assume $g_i = \eta_i \frac{P_i}{Q_i}$, where $\eta_i \in F$ and $P_i, Q_i \in F[\theta]$ are relatively prime monic polynomials. If for some i, $\deg(P_i) >\deg(Q_i)$ then use Remark 5.1.4 so that for each i, $\deg(P_i) \leq\deg(Q_i)$. Let $\overline{F}$ be an algebraic closure of F and $A = \{0 = \alpha_1, \ldots, \alpha_t\}$ be a subset of $\overline{F}$ such that $P_i = \prod_{j=1}^t (\theta - \alpha_j)^{l_{ij}}, Q_i = \prod_{j=1}^t (\theta-\alpha_j)^{m_{ij}}$ and $Q_i - \eta_i P_i = \xi_i \prod_{j=1}^t (\theta - \alpha_j)^{n_{ij}}$ for some $\xi_i \in F$, where l_{ij}, m_{ij}, and n_{ij} are non negative integers.

Let $M_1 = \overline{F}(\{\log(\eta_i), \log(\xi_i), \ell_2(\eta_i)|i \in I\}, \{\log(\alpha_j - \alpha_k)|j, k = 1, \ldots, t, j \neq k\}\theta)$ be a differential extension such that $w \in M_1\left(\{\log(\theta-\alpha), \ell_2\left(\frac{\theta-\alpha}{\theta-\beta}\right)|\alpha, \beta \in A\}\right)$. Consider $M = F(\{\log(\alpha_j - \alpha_k)|j, k = 1, \ldots, t, j \neq k\}, \{\log(\eta_i), \log(\xi_i), \ell_2(\eta_i)|i \in I\}, \theta)$ and let M^* be a differential subfield of M_1 such that $\overline{F} \subset M^*, \theta$ is transcendental over M^* and $M^*(\theta) = M_1$. This setup is similar to the one in Lemma 4.2.2. Let $a_{ij} = l_{ij} - m_{ij}$ and $b_{ij} = n_{ij} - m_{ij}$ and note that $\sum_{j=1}^t a_{ij} = \deg(P_i) - \deg(Q_i)$ and $\sum_{j=1}^t b_{ij} = \deg(Q_i - \eta_i P_i) - \deg(Q_i)$. Using Proposition 5.1.3 we have

$$\begin{aligned} r_i =& c_i e_i + c_i\Bigg(\ell_2(\eta_i) - \sum_{\substack{j,k=1\\k\neq j}}^t a_{ij}b_{ik}\ell_2\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right) - \frac{1}{2}\sum_{j,k=1}^t a_{ij}b_{ik}\log^2(\theta-\alpha_k) \\ &- \sum_{k=1}^t a_{ik}\log(\theta-\alpha_k)\log\xi_i - \sum_{\substack{j,k=1\\k\neq j}}^t a_{ij}b_{ik}\log\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right)\log(\alpha_j-\alpha_k)\Bigg), \end{aligned} \tag{5.23}$$

where e_i are some constants. It is clear from Remark 5.1.4 that for some non-negative integers j, k, $\ell_2\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right)$ and $\ell_2\left(\frac{\theta-\alpha_k}{\theta-\alpha_j}\right)$ are algebraically dependent over $M_1(\{\log(\theta-\alpha)|\alpha\in A\})$. In order to make these factors algebraically independent we use Remark 5.1.4 and observe

$$\begin{aligned} r_i = c_i\Bigg(&\ell_2(\eta_i) - \sum_{\substack{j,k=1\\k>j}}^{t}(a_{ij}b_{ik}-a_{ik}b_{ij})\ell_2\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right) - \sum_{\substack{j,k=1\\k>j}}^{t} a_{ik}b_{ij}\log(\theta-\alpha_j)\log(\theta-\alpha_k) \\ &- \sum_{j=1}^{t} a_{ij}\log(\theta-\alpha_j)\log\xi_i - \sum_{\substack{j,k=1\\k\neq j}}^{t} a_{ij}b_{ik}\log\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right)\log(\alpha_j-\alpha_k) \\ &- \frac{1}{2}\sum_{\substack{j,k=1\\k>j}}^{t}(a_{ij}b_{ik}-a_{ik}b_{ij})\log^2(\theta-\alpha_k) - \frac{1}{2}\sum_{j=1}^{t} a_{ij}b_{ij}\log^2(\theta-\alpha_j) + e_i\Bigg). \end{aligned} \tag{5.24}$$

Using Remark 5.1.2, $-\sum_{j=1}^{t} a_{ij}\log(\theta-\alpha_j)\log\xi_i - \sum_{\substack{j,k=1\\k\neq j}}^{t} a_{ij}b_{ik}\log\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right)\log(\alpha_j-\alpha_k)$ can be replaced with $-\sum_{k=1}^{t} b_{ik}\log\eta_i\log(\theta-\alpha_k)$ $+\sum_{\substack{j,k=1\\k\neq j}}^{t} f_{jk}\log\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right)$. Therefore,

$$\begin{aligned} v = \sum_{i\in I} c_i\Bigg(&\ell_2(\eta_i) - \sum_{\substack{j,k=1\\k>j}}^{t}(a_{ij}b_{ik}-a_{ik}b_{ij})\ell_2\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right) + \sum_{\substack{j,k=1\\k\neq j}}^{t} f_{jk}\log\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right) \\ &- \sum_{\substack{j,k=1\\k>j}}^{t} a_{ik}b_{ij}\log(\theta-\alpha_j)\log(\theta-\alpha_k) - \sum_{k=1}^{t} b_{ik}\log\eta_i\log(\theta-\alpha_k) \\ &- \frac{1}{2}\sum_{\substack{j,k=1\\k>j}}^{t}(a_{ij}b_{ik}-a_{ik}b_{ij})\log^2(\theta-\alpha_k) - \frac{1}{2}\sum_{j=1}^{t} a_{ij}b_{ij}\log^2(\theta-\alpha_j)\Bigg)\frac{g_i'}{g_i} + w'. \end{aligned} \tag{5.25}$$

Note that here we have replaced $\sum_{i\in I} c_i e_i g_i'/g_i + w'$ with w'. Now since $w \in M_1(\{\log(\theta-\alpha_j), \ell_2\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right)|k > j, j = 1,\ldots,t\})$ and $w' \in M_1(\{\log(\theta-\alpha)|\alpha \in$

$A\})\left[\{\ell_2\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right)|k>j, j=1,\ldots,t\}\right]$, we apply Proposition 2.2.4 and assume

$$w=\sum_{\substack{j,k,l,m=1\\k>j,m>l}}^{t} c_{jklm}\ell_2\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right)\ell_2\left(\frac{\theta-\alpha_l}{\theta-\alpha_m}\right)+\sum_{\substack{j,k=1\\k>j}}^{t} w_{jk}\ell_2\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right)+w_0,$$

where c_{jklm} are some constants and each w_{jk}, w_0 are some elements in $M_1(\log(\theta-\alpha)|\alpha\in A)$.

Substitute w' in equation 5.25 and compare the coefficients of $\ell_2\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right)$, for each j,k, we get

$$-\sum_{i\in I} c_i(a_{ij}b_{ik}-a_{ik}b_{ij})\frac{g_i'}{g_i}+\sum_{\substack{l,m=1\\m>l}}^{t} c_{jklm}\ell_2'\left(\frac{\theta-\alpha_l}{\theta-\alpha_m}\right)+w_{jk}'=0.$$

It is clear that $c_{jklm}=0$ for each j,k,l,m and $w_{jk}=\sum_{i\in I} c_i(a_{ij}b_{ik}-a_{ik}b_{ij})\log g_i+e_{jk}$, where e_{jk} are some constants in F. Thus,

$$\begin{aligned}
v=&\sum_{i\in I} c_i\Bigg(\ell_2(\eta_i)-\sum_{\substack{j,k=1\\k>j}}^{t} a_{ik}b_{ij}\log(\theta-\alpha_j)\log(\theta-\alpha_k)-\sum_{k=1}^{t} b_{ik}\log\eta_i\log(\theta-\alpha_k)\\
&+\sum_{\substack{j,k=1\\k\neq j}}^{t} f_{jk}\log\left(\frac{\theta-\alpha_j}{\theta-\alpha_k}\right)-\frac{1}{2}\sum_{\substack{j,k=1\\k>j}}^{t}(a_{ij}b_{ik}-a_{ik}b_{ij})\log^2(\theta-\alpha_k)\\
&-\frac{1}{2}\sum_{j=1}^{t} a_{ij}b_{ij}\log^2(\theta-\alpha_j)\Bigg)\frac{g_i'}{g_i}\\
&+\sum_{\substack{j,k=1\\k>j}}^{t}\left(\sum_{i\in I} c_i(a_{ij}b_{ik}-a_{ik}b_{ij})\log g_i+e_{jk}\right)\left(\frac{\theta'-\alpha_k'}{\theta-\alpha_k}-\frac{\theta-\alpha_j'}{\theta-\alpha_j}\right)\log\left(\frac{\alpha_j-\alpha_k}{\theta-\alpha_k}\right)\\
&+w_0'.
\end{aligned}\tag{5.26}$$

We divide the rest of proof in two parts.

Case I: When θ is an antiderivative, that is, $\theta' \in F$.

As observed in Lemma 4.2.2, it is easy to see that $\log(\theta-\alpha_1),\ldots,\log(\theta-\alpha_t)$ are algebraically independent over M^*. Apply Proposition 2.2.4 to w_0 and write

$$w_0 = \sum_{j,k,l=1}^{t} c_{jkl}\log(\theta-\alpha_j)\log(\theta-\alpha_k)\log(\theta-\alpha_l) + \sum_{j,k=1}^{t} v_{jk}\log(\theta-\alpha_j)\log(\theta-\alpha_k)$$
$$+\sum_{j=1}^{t} v_j\log(\theta-\alpha_j) + w_{00},$$

for some constants c_{jkl}, and elements $v_{jk}, v_j, w_{00} \in M_1$. Expand $\log g_i$ as $\log\eta_i + \sum_{l=1}^{t} a_{il}\log(\theta-\alpha_l)$. Substitute $w_0, \log g_i$ in Equation 5.26 and compare the constant coefficients, we have

$$v = \sum_{\substack{j,k=1\\k>j}}^{t}\left(\sum_{i\in I} c_i(a_{ij}b_{ik}-a_{ik}b_{ij})\log\eta_i + e_{jk}\right)\left(\frac{\theta'-\alpha_k'}{\theta-\alpha_k} - \frac{\theta'-\alpha_j'}{\theta-\alpha_j}\right)\log(\alpha_j-\alpha_k)$$
$$+\sum_{i\in I} c_i\ell_2(\eta_i)\frac{g_i'}{g_i} + \sum_{k=1}^{t} v_k\frac{\theta'-\alpha_k'}{\theta-\alpha_k} + w_{00}'. \tag{5.27}$$

Again with the use of Remark 5.1.2, we have

$$\sum_{\substack{j,k=1\\k>j}}^{t}\left(\sum_{i\in I} c_i(a_{ij}b_{ik}-a_{ik}b_{ij})\log\eta_i\right)\left(\frac{\theta'-\alpha_k'}{\theta-\alpha_k} - \frac{\theta'-\alpha_j'}{\theta-\alpha_j}\right)\log(\alpha_j-\alpha_k) =$$
$$\sum_{k=1}^{t}(a_{ik}\log\xi_i - b_{ik}\log\eta_i + d_{ik})\log\eta_i\frac{\theta'-\alpha_k'}{\theta-\alpha_k}$$

for some constants d_{ik}. Therefore,

$$v = \sum_{i\in I} c_i\ell_2(\eta_i)\frac{g_i'}{g_i} + \sum_{\substack{j,k=1\\k>j}}^{t} e_{jk}\left(\frac{\theta'-\alpha_k'}{\theta-\alpha_k} - \frac{\theta'-\alpha_j'}{\theta-\alpha_j}\right)\log(\alpha_j-\alpha_k)$$
$$+\sum_{k=1}^{t}(a_{ik}\log\xi_i - b_{ik}\log\eta_i + d_{ik})\log\eta_i\frac{\theta'-\alpha_k'}{\theta-\alpha_k} + \sum_{k=1}^{t} v_k\frac{\theta'-\alpha_k'}{\theta-\alpha_k} + w_{00}'. \tag{5.28}$$

To know more about the element v_k, we compare the coefficients of $\log(\theta-\alpha_j)\log(\theta-\alpha_k)$ and $\log(\theta-\alpha_k)$ for each j,k in Equation 5.26 and observe that

$$(v_{jk}+v_{kj})' = a_{ik}b_{ij}\frac{\eta_i'}{\eta_i} \qquad \text{for} \quad k>j,$$

$$v_{kk}' = \frac{1}{2}\left(\sum_{1\leq j<k}(a_{ij}b_{ik}-a_{ik}b_{ij})+a_{ik}b_{ik}\right)\frac{\eta_i'}{\eta_i}$$

and

$$\begin{aligned} v_k' = \sum_{i\in I} c_i\Bigg(& b_{ik}\log\eta_i\frac{\eta_i'}{\eta_i}+\sum_{j>k}^{t} a_{ij}b_{ik}\log\eta_i\frac{\theta'-\alpha_j'}{\theta-\alpha_j}+\sum_{1\leq j<k}(f_{jk}-f_{kj})\frac{g_i'}{g_i} \\ & -a_{ik}\sum_{j=1}^{t}(a_{ij}\log\xi_i-b_{ij}\log\eta_i+d_{ij})\frac{\theta'-\alpha_j'}{\theta-\alpha_j}\Bigg)-\sum_{1\leq j<k}e_{jk}\left(\frac{\theta'-\alpha_j'}{\theta-\alpha_j}-\frac{\theta'-\alpha_k'}{\theta-\alpha_k}\right). \end{aligned}$$

Note that for each i, $\log\eta_i, \log\xi_i, \ell_2(\eta_i)\in M^*(\theta)$ where θ is transcendental over M^*. Also, for some i, if $\ell_2(\eta_i)\in M^*$ then it is obvious that $\log\xi_i\in M^*$. Therefore, there are only three sub cases possible:

(a) $\log\eta_i, \log\xi_i, \ell_2(\eta_i)\in M^*$.

(b) $\log\eta_i, \log\xi_i\in M^*$ and $\ell_2(\eta_i)\in M^*(\theta)$.

(c) $\log\xi_i, \ell_2(\eta_i)\in M^*$ and $\log\eta_i\in M*(\theta)$.

We divide the index I into three subsets I_1, I_2 and I_3 consisting of those $i's$ for which (a),(b) and (c) holds, respectively.

Then clearly

$$v_k' = \sum_{i\in I_1\cup I_2} c_i\left(b_{ik}\log\eta_i+\sum_{1\leq j<k}(f_{jk}-f_{kj})\right)\frac{\eta_i'}{\eta_i}+\sum_{i\in I_3}c_i\left(b_{ik}\log\eta_i\frac{\eta_i'}{\eta_i}+A_{ik}\theta'\right)+\sum_{j=1}^{t}B_{jk}\alpha_j'$$

which implies

$$v_k = \frac{1}{2}\sum_{i\in I} c_i b_{ik}\log^2\eta_i + \sum_{i\in I} A_i\log\eta_i + \sum_{j=1}^{t} B_{jk}\alpha_j + e_k,$$

for some constants A_i, B_{jk}, e_k. Suppose for each i, j, k, $\log(\alpha_j - \alpha_k) - c_{jk}, \log\eta_i - d_i\theta, \ell_2(\eta_i) - f_i \in M^*$ where c_{jk}, d_i, f_i are constants. Then Equation 5.28 reduces to

$$\begin{aligned} v =& \sum_{i\in I} c_i\ell_2(\eta_i)\frac{\eta_i'}{\eta_i} + \sum_{i\in I_2} c_i f_i \sum_{k=1}^{t} a_{ik}(\theta' - \alpha_k') + \sum_{\substack{j,k=1\\ j<k}}^{t} e_{jk}c_{jk}(\alpha_j' - \alpha_k') \\ &- \sum_{i\in I_3} c_i d_i \sum_{k=1}^{t} b_{ik}(\theta' - \alpha_k') + \sum_{i\in I_3} c_i \sum_{k=1}^{t}(a_{ik}\log\xi_i - \frac{1}{2}b_{ik}\log\eta_i)\log\eta_i\frac{\theta' - \alpha_k'}{\theta - \alpha_k} \\ &+ \text{terms containing poles} + w_{00}'. \end{aligned} \tag{5.29}$$

Let $L = \sum_{i\in I_3} c_i \sum_{k=1}^{t}(a_{ik}\log\xi_i - \frac{1}{2}b_{ik}\log\eta_i)\log\eta_i\frac{\theta'-\alpha_k'}{\theta-\alpha_k}$ and $\log\eta_i = d_i\theta + \beta_i$, where $\beta_i \in M^*$. Then the constant term in L is $\sum_{i\in I_3} c_i \sum_{k=1}^{t}(a_{ik}\log\xi_i d_i - \frac{1}{2}b_{ik}(d_i^2\theta + 2d_i\beta_i + d_i^2\alpha_k))(\theta' - \alpha_k')$. Since we assumed $\deg(P_i) \le \deg(Q_i)$, if $\deg(P_i) < \deg(Q_i)$ then $\log\xi_i$ is a constant and $\sum_{k=1}^{t} b_{ik} = 0$. So the constant term in L becomes $\sum_{i\in I_3} c_i\frac{1}{2}\sum_{k=1}^{t} b_{ik}((d_i^2\theta + 2d_i\beta_i)\alpha_k' - d_i^2\theta\alpha_k')$ which can be further written as $-\sum_{i\in I_3} c_i \sum_{k=1}^{t} b_{ik}(d_i^2\theta' + 2d_i\beta_i')\alpha_k + \rho' = -\sum_{i\in I_3} c_i\sum_{k=1}^{t} b_{ik}d_i\frac{\eta_i'}{\eta_i}\alpha_k + \rho'$ for some $\rho \in M^*$. From Remark 5.1.2, we have $-\sum_{k=1}^{t} b_{ik}\frac{\eta_i'}{\eta_i}\alpha_k = \sum_{k=1}^{t} b_{ik}\sum_{j=1,j\neq k}^{t} a_{ij}\frac{\alpha_j'-\alpha_k'}{\alpha_j-\alpha_k}\alpha_k = \sum_{k=1}^{t} b_{ik}\sum_{j=1,j\neq k}^{t} a_{ij}\frac{\alpha_j'-\alpha_k'}{\alpha_j-\alpha_k}\alpha_j - \sum_{k=1}^{t} b_{ik}\sum_{j=1,j\neq k}^{t} a_{ij}(\alpha_j' - \alpha_k') = -\sum_{j=1}^{t} a_{ij}\frac{\xi_i'}{\xi_i}\alpha_j - \sum_{k=1}^{t} b_{ik}\sum_{j=1,j\neq k}^{t} a_{ij}(\alpha_j' - \alpha_k') = -\sum_{k=1}^{t} b_{ik}\sum_{j=1,j\neq k}^{t} a_{ij}(\alpha_j' - \alpha_k')$. Thus the constant term in L is a derivative in M^*. Similarly, if $\deg(P_i) = \deg(Q_i)$ and $\xi_i = 1 - \eta_i$, we proceed in the same manner and observe that v equals sum of derivatives of trilogarithmic integrals over M and an element over M_1. Since M_1 is algebraic over M, using the Proposition 2.2.2 we can find a suitable element $\tilde{w} \in M$ such that v satisfies a $\mathcal{T}-$expression over M.

Case II: When θ is exponential over F, that is, for an element $x \in F, \theta' = x'\theta$. Then $\log(\theta - \alpha_1) = x \in F$. Using Remark 5.1.2, v can be rewritten as

$$
\begin{aligned}
v = \sum_{i \in I} c_i \Bigg(\Bigg(\ell_2(\eta_i) - \sum_{k>1}^{t} a_{ik} b_{i1} x \log(\theta - \alpha_k) - \sum_{\substack{j,k>1 \\ k>j}}^{t} a_{ik} b_{ij} \log(\theta - \alpha_j) \log(\theta - \alpha_k) \\
&- b_{i1} x \log \eta_i - \sum_{k>1}^{t} b_{ik} \log \eta_i \log(\theta - \alpha_k) + \sum_{k>1}^{t} f_{1k}(x - \log(\theta - \alpha_k)) \\
&+ \sum_{k>1}^{t} f_{k1}(\log(\theta - \alpha_k) - x) - \frac{1}{2} \sum_{k>1}^{t} (a_{i1} b_{ik} - a_{ik} b_{i1}) \log^2(\theta - \alpha_k) - \frac{1}{2} a_{i1} b_{i1} x^2 \\
&- \frac{1}{2} \sum_{\substack{j,k>1 \\ k>j}}^{t} (a_{ij} b_{ik} - a_{ik} b_{ij}) \log^2(\theta - \alpha_k) - \frac{1}{2} \sum_{j>1}^{t} a_{ij} b_{ij} \log^2(\theta - \alpha_j) \Bigg) \frac{g_i'}{g_i} \\
&+ \sum_{j=1}^{t} (a_{ij} \log \xi_i - b_{ij} \log \eta_i + d_{ij}) \log(g_i) \frac{\theta - \alpha_j'}{\theta - \alpha_j} \\
&- \sum_{\substack{j,k=1 \\ k>j}}^{t} (a_{ij} b_{ik} - a_{ik} b_{ij}) \log(g_i) \log(\theta - \alpha_k) \left(\frac{\theta' - \alpha_k'}{\theta - \alpha_k} - \frac{\theta - \alpha_j'}{\theta - \alpha_j} \right) \Bigg) \\
&+ \sum_{\substack{j,k=1 \\ k>j}}^{t} e_{jk} \left(\frac{\theta' - \alpha_k'}{\theta - \alpha_k} - \frac{\theta - \alpha_j'}{\theta - \alpha_j} \right) \log \left(\frac{\alpha_j - \alpha_k}{\theta - \alpha_k} \right) + w_0'. \qquad (5.30)
\end{aligned}
$$

Again apply Proposition 2.2.4 to w_0. Thus for some constants c_{jkl} and elements $v_{jk}, v_j, w_{00} \in M_1$, we shall write

$$
\begin{aligned}
w_0 = \sum_{j,k,l>1}^{t} c_{jkl} \log(\theta - \alpha_j) \log(\theta - \alpha_k) \log(\theta - \alpha_l) + \sum_{j,k>1}^{t} v_{jk} \log(\theta - \alpha_j) \log(\theta - \alpha_k) \\
&+ \sum_{j>1}^{t} v_j \log(\theta - \alpha_j) + w_{00}.
\end{aligned}
$$

Expand $\log g_i$ as $\log \eta_i + a_{i1} x + \sum_{l>1}^{t} a_{il} \log(\theta - \alpha_l)$. Equate the coefficient of the product $\log(\theta - \alpha_j) \log(\theta - \alpha_k)$, for each j, k such that $k > j$, in Equation 5.30 to zero and after removal of the pole part in it(which has to be zero), we get

$$(v_{jk} + v_{kj})' = \sum_{i\in I} c_i a_{ik} b_{ij} \left(\frac{\eta_i'}{\eta_i} + a_{i1}x'\right) + h_{jk}x', \tag{5.31}$$

where $h_{jk} = -\sum_{i\in I} c_i\big((a_{i1}b_{ik} - a_{ik}b_{i1})a_{ij} + (a_{i1}b_{ij} - a_{ij}b_{i1})a_{ik}\big)$ is constant. Similarly we equate coefficient of $\log^2(\theta - \alpha_k)$, for each k, in Equation 5.30 to zero and after removal of the pole part in it, we get

$$\begin{aligned} v_{kk}' =& \frac{1}{2}\sum_{i\in I} c_i\Big((a_{i1}b_{ik} - a_{ik}b_{i1}) + \sum_{1<j<k}(a_{ij}b_{ik} - a_{ik}b_{ij}) + a_{ik}b_{ik}\Big)\left(\frac{\eta_i'}{\eta_i} + a_{i1}x'\right) \\ &- \sum_{i\in I} c_i(a_{i1}b_{ik} - a_{ik}b_{i1})a_{ik}x'. \end{aligned} \tag{5.32}$$

Therefore, it is obvious that $v_{jk} + v_{kj}$ and v_{kk} are some elements in M^*. Compare the coefficient of $\log(\theta - \alpha_k)$ in Equation 5.30 and obtain

$$\begin{aligned} &\sum_{i\in I} c_i\Bigg((-a_{ik}b_{i1}x - b_{ik}\log\eta_i - f_{1k} + f_{k1})\frac{g_i'}{g_i} + \sum_{j=1}^{t}(a_{ij}\log\xi_i - b_{ij}\log\eta_i + d_{ij})a_{ik}\frac{\theta' - \alpha_j'}{\theta - \alpha_j} \\ &- \sum_{1\le j<k}(a_{ij}b_{ik} - a_{ik}b_{ij})(\log\eta_i + a_{i1}x)\left(\frac{\theta' - \alpha_k'}{\theta - \alpha_k} - \frac{\theta - \alpha_j'}{\theta - \alpha_j}\right)\Bigg) \\ &- \sum_{1\le j<k} e_{jk}\left(\frac{\theta' - \alpha_k'}{\theta - \alpha_k} - \frac{\theta - \alpha_j'}{\theta - \alpha_j}\right) + \sum_{j>1}^{t}(v_{jk} + v_{kj})\frac{\theta' - \alpha_j'}{\theta - \alpha_j} + v_k' = 0. \end{aligned} \tag{5.33}$$

Divide the indices $j > 1$ into three parts, replace the fraction g_i'/g_i with its partial fraction expansion and compare the coefficients of $(\theta' - \alpha_j')/(\theta - \alpha_j)$ as follows:

Sub-case I. When $j < k$:

$$\begin{aligned} &\sum_{i\in I} c_i\Big((-a_{ik}b_{i1}x - b_{ik}\log\eta_i - f_{1k} + f_{k1})a_{ij} + (a_{ij}b_{ik} - a_{ik}b_{ij})(\log\eta_i + a_{i1}x) + e_{jk} \\ &+ (a_{ij}\log\xi_i - b_{ij}\log\eta_i + d_{ij})a_{ik}\Big) - e_{jk} + v_{jk} + v_{kj} = 0. \end{aligned} \tag{5.34}$$

Sub-case II. When $j = k$:

$$\sum_{i\in I} c_i\Big((-a_{ik}b_{i1}x - b_{ik}\log\eta_i - f_{1k} + f_{k1})a_{ik} - ((a_{i1}b_{ik} - a_{ik}b_{i1})(\log(\eta_i) + a_{i1}x) + e_{1k})$$
$$- \sum_{l<k}((a_{il}b_{ik} - a_{ik}b_{il})(\log\eta_i + a_{i1}x) + e_{lk}) + (a_{ik}\log\xi_i - b_{ik}\log\eta_i + d_{ik})a_{ik}\Big)$$
$$+ \sum_{j=1}^{k} e_{jk} + 2v_{kk} = 0. \tag{5.35}$$

Sub-case III. When $j > k$:

$$\sum_{i\in I} c_i((-a_{ik}b_{i1}x - b_{ik}\log\eta_i - f_{1k} + f_{k1})a_{ij} + (a_{ij}\log\xi_i - b_{ij}\log\eta_i + d_{ij})a_{ik})$$
$$+ v_{jk} + v_{kj} = 0. \tag{5.36}$$

Adding the above three equations we get

$$\sum_{i\in I} c_i\Big(\sum_{j>1}^{t}(-a_{ik}b_{i1}x - b_{ik}\log\eta_i - f_{1k} + f_{k1})a_{ij} - (a_{i1}b_{ik} - a_{ik}b_{i1})(\log\eta_i + a_{i1}x)$$
$$- e_{1k} + \sum_{j>1}^{t}(a_{ij}\log\xi_i - b_{ij}\log\eta_i + d_{ij})a_{ik}\Big) + \sum_{j>1}^{t}(v_{jk} + v_{kj}) = 0. \tag{5.37}$$

The constant term in the Equation 5.33 is

$$\sum_{i\in I} c_i\Big((-a_{ik}b_{i1}x - b_{ik}\log\eta_i - f_{1k} + f_{k1})\Big(\frac{\eta_i'}{\eta_i} + a_{i1}x'\Big) + ((a_{i1}b_{ik} - a_{ik}b_{i1})(\log\eta_i$$
$$+ a_{i1}x) + e_{1k})x' + (a_{i1}\log\xi_i - b_{i1}\log\eta_i + d_{i1})a_{ik}x'\Big) + v_k' \tag{5.38}$$

which equals 0. Substitute the term $\sum_{i\in I} c_i(a_{i1}b_{ik} - a_{ik}b_{i1})(\log\eta_i + a_{i1}x) + e_{1k}$ from Equation 5.37 and observe

$$\sum_{i\in I} c_i\Bigg((-a_{ik}b_{i1}x - b_{ik}\log\eta_i - f_{1k} + f_{k1})\Big(\frac{\eta_i'}{\eta_i} + \sum_{j=1}^{t} a_{ij}x'\Big)$$
$$+ \sum_{j=1}^{t}(a_{ij}\log\xi_i - b_{ij}\log\eta_i + d_{ij})a_{ik}x'\Bigg) + \sum_{j>1}^{t}(v_{jk} + v_{kj})x' + v_k' = 0. \tag{5.39}$$

Since $v_k \in M^*(\theta)$ and $v'_k \in M^*$, it is clear from Proposition 2.2.6 that $v_k \in M^*$ for each $k > 1$. Now we move back to the expression of v in Equation 5.30 and look at the constant terms (i.e terms without $\log(\theta - \alpha_j)$). Thus,

$$v = \sum_{i \in I} c_i \Bigg(\Big(\ell_2(\eta_i) - b_{i1} x \log \eta_i + \sum_{k>1} (f_{1k} - f_{k1}) x - \frac{1}{2} a_{i1} b_{i1} x^2 \Big) \frac{g'_i}{g_i} + \sum_{j=1}^{t} (a_{ij} \log \xi_i - b_{ik} \log \eta_i + d_{ij})(\log \eta_i + a_{i1} x) \frac{\theta' - \alpha'_j}{\theta - \alpha_j} \Bigg) + \sum_{\substack{j,k=1 \\ k>j}}^{t} e_{jk} \left(\frac{\theta' - \alpha'_k}{\theta - \alpha_k} - \frac{\theta' - \alpha'_j}{\theta - \alpha_j} \right) \log(\alpha_j - \alpha_k) + \sum_{j>1}^{t} v_j \frac{\theta' - \alpha'_j}{\theta - \alpha_j} + w'_{00}. \tag{5.40}$$

Compare the coefficients of $(\theta' - \alpha'_j)/(\theta - \alpha_j)$ in above equation, we have

$$\sum_{i \in I} c_i \Bigg(a_{ij} \Big(\ell_2(\eta_i) - b_{i1} x \log \eta_i + \sum_{k>1}^{t} (f_{1k} - f_{k1}) x - \frac{1}{2} a_{i1} b_{i1} x^2 \Big) + (\log \eta_i + a_{i1} x)(a_{ij} \log \xi_i - b_{ij} \log \eta_i + d_{ij}) \Bigg) + e_{1j} \log(-\alpha_j) + \sum_{1<k<j} e_{kj} \log(\alpha_k - \alpha_j) - \sum_{k>j}^{t} e_{jk} \log(\alpha_j - \alpha_k) + v_j = 0 \tag{5.41}$$

Take the summation of above equation for $1 < j < t$, observe

$$\sum_{i \in I} c_i \sum_{j>1}^{t} \Bigg(\Big(\ell_2(\eta_i) - b_{i1} x \log \eta_i + \sum_{k>1}^{t} (f_{1k} - f_{k1}) x - \frac{1}{2} a_{i1} b_{i1} x^2 \Big) a_{ij} + (\log \eta_i + a_{i1} x)(a_{ij} \log \xi_i - b_{ij} \log \eta_i + d_{ij}) \Bigg) + \sum_{j>1}^{t} e_{1j} \log(-\alpha_j) + \sum_{j>1}^{t} v_j = 0 \tag{5.42}$$

The constant term in the expression 5.40,which is

$$v = \sum_{i \in I} c_i \Bigg(\Big(\ell_2(\eta_i) - b_{i1} x \log \eta_i + \sum_{k>1}^{t} (f_{1k} - f_{k1}) x - \frac{1}{2} a_{i1} b_{i1} x^2 \Big) \Big(\frac{\eta'_i}{\eta_i} + a_{i1} x' \Big) + (\log \eta_i + a_{i1} x)(a_{i1} \log \xi_i - b_{i1} \log \eta_i + d_{i1}) x' \Bigg) - \sum_{k>1}^{t} e_{1k} \log(-\alpha_k) x' + w'_{00} \tag{5.43}$$

can be rewritten (using Equation 5.42) as

$$v = \sum_{i\in I} c_i \Bigg(\Big(\ell_2(\eta_i) - b_{i1} x \log \eta_i + \sum_{k>1}^{t} (f_{1k} - f_{k1}) x - \frac{1}{2} a_{i1} b_{i1} x^2 \Big) \Big(\frac{\eta_i'}{\eta_i} + \sum_{j=1}^{t} a_{ij} x' \Big) \\ + \sum_{j=1}^{t} (\log \eta_i + a_{i1} x)(a_{ij} \log \xi_i - b_{ij} \log \eta_i + d_{ij}) x' \Bigg) + \sum_{k>1}^{t} v_k x' + w_{00}'. \tag{5.44}$$

Replace w_{00} with $w_{00} - \sum_{k>1}^{t} v_k x$ to get

$$v = \sum_{i\in I} c_i \Bigg(\Big(\ell_2(\eta_i) - b_{i1} x \log \eta_i + \sum_{k>1}^{t} (f_{1k} - f_{k1}) x - \frac{1}{2} a_{i1} b_{i1} x^2 \Big) \Big(\frac{\eta_i'}{\eta_i} + \sum_{j=1}^{t} a_{ij} x' \Big) \\ + \sum_{j=1}^{t} (\log \eta_i + a_{i1} x)(a_{ij} \log \xi_i - b_{ij} \log \eta_i + d_{ij}) x' \Bigg) - \sum_{k>1}^{t} v_k' x + w_{00}'. \tag{5.45}$$

Substitute the value of v_k' from Equation 5.39 and obtain

$$v = \sum_{i\in I} c_i \Bigg(\Big(\ell_2(\eta_i) - b_{i1} x \log \eta_i + \sum_{k>1}^{t} (f_{1k} - f_{k1}) x - \frac{1}{2} a_{i1} b_{i1} x^2 \Big) \Big(\frac{\eta_i'}{\eta_i} + \sum_{j=1}^{t} a_{ij} x' \Big) \\ + \sum_{j=1}^{t} (\log \eta_i + a_{i1} x)(a_{ij} \log \xi_i - b_{ij} \log \eta_i + d_{ij}) x' \\ + \sum_{k>1}^{t} (-a_{ik} b_{i1} x - b_{ik} \log \eta_i - f_{1k} + f_{k1}) \Big(\frac{\eta_i'}{\eta_i} + \sum_{j=1}^{t} a_{ij} x' \Big) x \\ + \sum_{k>1}^{t} \sum_{j=1}^{t} (a_{ij} \log \xi_i - b_{ij} \log \eta_i + d_{ij}) a_{ik} x' x \Bigg) + \sum_{j,k>1}^{t} (v_{jk} + v_{kj}) x x' + w_{00}'. \tag{5.46}$$

From Equations 5.31 and 5.32, we have

$$v = \sum_{i\in I} c_i\Big(\ell_2(\eta_i) - \sum_{k=1}^{t} b_{ik}x\log\eta_i - \frac{1}{2}a_{i1}b_{i1}x^2 - \sum_{k>1}^{t} a_{ik}b_{i1}x^2\Big)\Big(\frac{\eta_i'}{\eta_i} + \sum_{j=1}^{t} a_{ij}x'\Big)$$
$$+ 2\sum_{\substack{j,k>1\\k>j}}^{t}\sum_{i\in I} c_i a_{ik}b_{ij}(\log\eta_i + a_{i1}x)xx'$$
$$+ \sum_{k>1}^{t}\sum_{i\in I} c_i\Big((a_{i1}b_{ik} - a_{ik}b_{i1}) + \sum_{j>1}^{k}(a_{ij}b_{ik} - a_{ik}b_{ij}) + a_{ik}b_{ik}\Big)\Big(\log\eta_i + a_{i1}x\Big)xx'$$
$$+ \sum_{\substack{j,k>1\\k>j}}^{t} h_{jk}x^2x' - 2\sum_{i\in I} c_i(a_{i1}b_{ik} - a_{ik}b_{i1})x^2x' + w_{00}'. \tag{5.47}$$

Since we assumed $\deg(P_i) \le \deg(Q_i)$, then either $\sum_{j=1}^{t} a_{ij} = 0$, $\sum_{j=1}^{t} b_{ij} = 0$ or both. If $\sum_{j=1}^{t} a_{ij} \neq 0$ then $\xi_i = 1$ and note that in the expansion of $\log(1-g_i)$, there will be no constant term, that is, $\log\xi_i = 0$. So lets divide the indexing set I into three sets I_1, I_2 and I_3, where for each $i \in I_1$, $\sum_{j=1}^{t} a_{ij} = \sum_{j=1}^{t} b_{ij} = 0$, for each $i \in I_2$ we have $\sum_{j=1}^{t} a_{ij} = 0$ and $\sum_{j=1}^{t} b_{ij} \neq 0$ (in this case $\eta_i = 1$) and for $i \in I_3$, $\sum_{j=1}^{t} a_{ij} \neq 0$ and $\sum_{j=1}^{t} b_{ij} = 0$. Therefore,

$$v = \sum_{i\in I_1} c_i\Big(\Big(\ell_2(\eta_i) + \frac{1}{2}a_{i1}b_{i1}x^2\Big)\Big(\frac{\eta_i'}{\eta_i}\Big) + a_{i1}b_{i1}\log\eta_i xx'\Big)$$
$$+ \sum_{i\in I_2} c_i\Big(\Big(\ell_2(\eta_i) + \frac{1}{2}a_{i1}b_{i1}x^2\Big)\Big(\frac{\eta_i'}{\eta_i}\Big) + a_{i1}b_{i1}\log\eta_i xx'\Big)$$
$$+ \sum_{i\in I_3} c_i\Big(\Big(\ell_2(\eta_i) - \frac{1}{2}a_{i1}b_{i1}x^2 - \sum_{k>1}^{t} a_{ik}b_{i1}x^2\Big)\Big(\frac{\eta_i'}{\eta_i} + \sum_{j=1}^{t} a_{ij}x'\Big)$$
$$+ (-a_{i1}b_{i1} - 2\sum_{k>1}^{t} a_{ik}b_{i1})xx'(\log\eta_i + \sum_{j=1}^{t} a_{ij}x)\Big) + \bar{w}'. \tag{5.48}$$

For $i \in I_3$, $\ell_2(\eta_i)$ is just a constant because $\ell_2(\eta_i)' = -\frac{\eta_i'}{\eta_i}\log\xi_i = 0$. Also as $\bar{w}' \in M$ and M^* is algebraic over M, therefore, for a suitable $\tilde{w}' \in M$ we obtain

$$v = \sum_{i \in I_1 \cup I_2} c_i \ell_2(\eta_i) \frac{\eta_i'}{\eta_i} + \tilde{w}', \tag{5.49}$$

which is $\mathcal{T}-$expression over $F(\{\log(\eta_i), \log \xi_i, \ell_2(\eta_i) | i \in I\})$. □

The difficulty in the setup lies in exponential case. If our field of definition involve only antiderivatives (i.e logarithms, dilogarithmic integrals and trilogarithmic integrals) then the proof of Lemma 5.2.7 does not require dilogarithmic identity in Proposition 5.1.3 and can be simplified as follows:

Remark 5.2.8. *Alternate proof of Lemma 5.2.7 when θ is an antiderivative.*

Since $v \in F$ admits a $\mathcal{T}-$expression over dilogarithmic extension E of $F(\theta)$, consider $v = \sum_{i \in I} r_i \frac{g_i'}{g_i}$, where each r_i, g_i lies in some dilogarithmic extension of $F(\theta)$ and $r_i' = -c_i \log(1 - g_i) \frac{g_i'}{g_i}$, as done in proof of Lemma 5.2.7. If necessary, enlarge E and assume $\log g_i \in E$. Since w belongs to dilogarithmic extension of $F(\theta)$ that contains each r_i and $\log g_i$, replace w with $w - \sum_{i \in I} r_i \log g_i$ and observe

$$v = \sum_{i \in I} c_i \frac{g_i'}{g_i} \log(1 - g_i) \log g_i + w'.$$

By our assumption $\log g_i, \log(1 - g_i) \in F_1 = F(\theta)(\log y_1, \ldots, \log y_n)$ and therefore, $w' \in \log g_i, \log(1 - g_i) \in F_1$. We use Proposition 2.2.4 and write $w = \sum_{j=1}^{m} e_j \ell_2(z_j) + w_0$, for some element $w_0 \in F_1$ and constants e_j. Thus

$$v = \sum_{i \in I} c_i \frac{g_i'}{g_i} \log(1 - g_i) \log g_i - \sum_{j=1}^{m} e_j \frac{z_j'}{z_j} \log(1 - z_j) + w_0'.$$

As observed earlier, $\log'(1 - g_i) - \frac{(1-g_i)'}{1-g_i} = 0, \log(1 - z_j - \frac{(1-z_j)'}{1-z_j}) \in F(\theta)$, we use Proposition 2.2.6 repeatedly and obtain $1 - g_i, 1 - z_j$ and thus g_i, z_j lies in $F(\theta)$ for each i, j.

Let $\overline{F}$ be an algebraic closure of F that contains all the zeroes and poles of $g_i, 1-g_i, z_j, 1-z_j$. Let $A = \alpha_1, \ldots, \alpha_t$ be the set of all zeroes and poles of $g_i, 1-g_i, z_j, 1-z_j$ and assume $g_i = \eta_i \prod_{k=1}^{t}(\theta-\alpha_k)^{a_{ik}}$, $1-g_i = \xi \prod_{k=1}^{t}(\theta-\alpha_k)^{b_{ik}}$, $z_j = \gamma_j \prod_{k=1}^{t}(\theta-\alpha_k)^{p_{jk}}$ and $1-z_j = \rho_j \prod_{k=1}^{t}(\theta-\alpha_k)^{q_{jk}}$, where each $\eta_i, \xi_i, \gamma_j, \rho_j \in F$ and $a_{ik}, b_{ik}, p_{jk}, q_{jk}$ are some integers. Then instead of F_1 we can consider the field $\overline{F}(\theta)(\{\log\eta_i, \log\xi_i, \log\gamma_j, \log\rho_j\})(\{\log(\theta-a_k)\})$. Using partial fraction expansion, we have

$$v = \sum_{i\in I} c_i\Big(\frac{\eta_i'}{\eta_i} + \sum_{k=1}^{t} a_{ik}\frac{\theta'-\alpha_k'}{\theta-\alpha_k}\Big)(\log\xi_i + \sum_{k=1}^{t} b_{ik}\log(\theta-\alpha_k))(\log\eta_i + \sum_{k=1}^{t} a_{ik}\log(\theta-\alpha_k)) - \sum_{j=1}^{m} e_j\left(\frac{\gamma_j'}{\gamma_j} + \sum_{k=1}^{t} p_{jk}\frac{\theta'-\alpha_k'}{\theta-\alpha_k}\right)(\log\rho_j + \sum_{k=1}^{t} q_{jk}\log(\theta-\alpha_k)) + w_0'.$$

Now the set $\{\log(\theta-\alpha_k)\}$ is algebraically independent over the differential field $\overline{F}(\theta)(\{\log\eta_i, \log\xi_i, \log\gamma_j, \log\rho_j\})$. We enlarge our field under consideration to include $\ell_2(\eta_i)$ for each i, and name it M_1. Consider the differential field $M = F(\theta)(\{\log\eta_i, \log\xi_i, \log\gamma_j, \log\rho_j\})(\{\ell_2(\eta_i)\})$ and M^* be a subfield of M_1 which is algebraic over M and $M^*(\theta) = M$ (See Corollary 2.3.3 (b)). Using Proposition 2.2.4, we also write $w_0 = \sum_{j,k,l=1}^{t} c_{jkl}\log(\theta-\alpha_j)\log(\theta-a_k)\log(\theta-\alpha_l) + \sum_{k,l=1}^{t} w_{kl}\log(\theta-\alpha_k)\log(\theta-\alpha_l) + \sum_{k=1}^{t} w_k\log(\theta-\alpha_k) + v_0$, where $w_{jk}, w_k, v_0 \in M_1$. Comparing the coefficients of $\log(\theta-\alpha_k)\log(\theta-\alpha_l)$ and $\log(\theta-\alpha_k)$ in the expression of v, we obtain each $c_{jkl} = 0$, $(w_{kl}+w_{lk})' = -\sum_{i\in I} c_i(a_{ik}b_{il} + a_i b_{ik})\frac{\eta_i'}{\eta_i}$ and

$$w_k' = -\sum_{i\in I} c_i\frac{\eta_i'}{\eta_i}(a_{ik}\log\xi_i + b_{ik}\log\eta_i) - \sum_{i\in I} c_i a_{ik}\sum_{l=1}^{t}(a_{il}\log\xi_i - b_{il}\log\eta_i)\frac{\theta'-\alpha_l'}{\theta-\alpha_l}$$
$$= \sum_{i\in I} c_i a_{ik}\ell_2'(\eta_i) - \frac{1}{2}\sum_{i\in I} c_i b_{ik}(\log^2\eta_i)' - \sum_{i\in I} c_i a_{ik}\sum_{l=1}^{t}(a_{il}\log\xi_i - b_{il}\log\eta_i)\frac{\theta'-\alpha_l'}{\theta-\alpha_l}.$$

Since $\log \eta_i, \log \xi_i$ and θ are primitive over M^*, assume $\log \eta_i = d_i\theta + D_i$ and $\log \xi_i = f_i\theta + F_i$, where d_i, f_i are some constants. Therefore, with the help of Proposition 2.2.6, we obtain

$$w_k' = \sum_{i\in I} c_i a_{ik}\ell_2'(\eta_i) - \frac{1}{2}\sum_{i\in I} c_i b_{ik}(\log^2\eta_i)' - \sum_{i\in I} c_i a_{ik}\sum_{l=1}^{t}(a_{il}d_i - b_{il}f_i)(\theta' - \alpha_l').$$

Thus the constant term in the expression of v remains

$$v = \sum_{i\in I} c_i\left(\frac{\eta_i'}{\eta_i} + \sum_{k=1}^{t} a_{ik}\frac{\theta'-\alpha_k'}{\theta-\alpha_k}\right)\log\xi_i\log\eta_i - \sum_{j=1}^{m} e_j\left(\frac{\gamma_j'}{\gamma_j} + \sum_{k=1}^{t} p_{jk}\frac{\theta'-\alpha_k'}{\theta-\alpha_k}\right)\log\rho_j$$
$$+ \sum_{k=1}^{t} w_k\frac{\theta'-\alpha_k'}{\theta-a_k} + v_0'.$$

Substituting the value of w_k, we get

$$v = \sum_{i\in I} c_i\frac{\eta_i'}{\eta_i}\log\xi_i\log\eta_i + \sum_{\substack{i\in I\\ k=1}}^{t} c_i\left(a_{ik}\ell_2(\eta_i) - \frac{1}{2}b_{ik}\log^2\eta_i + a_{ik}\log\xi_i\log\eta_i\right)\frac{\theta'-\alpha_k'}{\theta-\alpha_k}$$
$$+ v_{00}',$$

where v_{00} is some element in M_1. Note that the coefficient of $\frac{\theta'-\alpha_k'}{\theta-\alpha_k}$ is same as we obtained in Proof of Lemma 5.2.7, Case-I after the Equation 5.40. So we divide the index set I into three parts and proceed in the same manner, and observe that the term $\sum_{k=1}^{t}\sum_{i\in I} c_i\left(a_{ik}\ell_2(\eta_i) - \frac{1}{2}b_{ik}\log^2\eta_i + a_{ik}\log\xi_i\log\eta_i\right)\frac{\theta'-\alpha_k'}{\theta-\alpha_k}$ sums up as a derivative of some element in M_1. We adjoin this term with v_{00}' and get

$$v = \sum_{i\in I} c_i\frac{\eta_i'}{\eta_i}\log\xi_i\log\eta_i + v_{00}' = -\sum_{i\in I} c_i(\ell_2(\eta_i))'\log\eta_i + v_{00}'.$$

Replace v_{00} with $v_{00} + \sum_{i\in I} c_i\ell_2(\eta_i)\log\eta_i$ and observe

$$v = \sum_{i\in I} c_i\ell_2(\eta_i)\frac{\eta_i'}{\eta_i} + v_{00}'.$$

Note that $\sum_{i\in I} c_i\ell_2(\eta_i)\frac{\eta_i'}{\eta_i}$ is sum of derivatives of trilogarithmic integrals over M and v_{00} is an element in $M^*(\theta)$. Since M_1 is algebraic over M, using the Proposition 2.2.2 we can find a suitable element $\tilde{w} \in M$ such that v satisfies a $\mathcal{T}-$expression over M.

□

We recall that a differential field extension E of F to be a dilogarithmic extension of F if their field of constants coincides and there are elements $y_1, \ldots, y_n, z_1, \ldots, z_m \in F$ such that $E = F(\log(y_1), \ldots, \log(y_n), \ell_2(z_1), \ldots, \ell_2(z_m))$.

The following theorems provide an extension of Liouville's Theorem.

Theorem 5.2.9. *Let $E = F(\theta_1, \ldots, \theta_n)$ be a transcendental $\mathcal{T}-$extension of F. Suppose that there is an element $u \in E$ with $u' \in F$ then u' admits a $\mathcal{T}-$ expression over some dilogarithmic extension of F.*

Proof. We prove the theorem using induction on n. For $n = 1$, we have $u \in F(\theta)$ with $u' \in F$ then from Proposition 2.2.4, $u = c\theta + w$ for some constant c and $w \in F$. Therefore, $u' = c\theta' + w'$ and from the definition of θ' it is clear that u' admits a $\mathcal{T}-$expression over F. Now suppose that u' admits a $\mathcal{T}-$expression over some dilogarithmic extension of $F(\theta)$. Then we shall apply the Lemma 5.2.7 and obtain that u' admits a $\mathcal{T}-$expression over $M = F(\theta, \log(y_1), \ldots, \log(y_n), \ell_2(z_1), \ldots, \ell_2(z_m))$, where $y_1, \ldots, y_n, z_1, \ldots z_m \in F$ and constant field of M coincides with that of F. If $\theta'/\theta \in F$ then it is evident from the case-II of Lemma 5.2.7 that u' admits a $\mathcal{T}-$expression over $F(\log(y_1), \ldots, \log(y_n), \ell_2(z_1), \ldots, \ell_2(z_m))$, which is a dilogarithmic extension of F. If θ is logarithm or dilogarithm over F, then M is indeed a dilogarithmic extension of F. So the only case left is when θ is a trilogarithmic integral over F. Let

$$u' = \sum_{i\in I} r_i \frac{g_i'}{g_i} + \sum_{j\in J} s_j \frac{h_j'}{h_j} + w',$$
$$r_i' = -c_i \log(1-g_i)\frac{g_i'}{g_i} \quad \text{and} \quad s_j' = d_j \frac{(1-h_j)'}{(1-h_j)}, \tag{5.50}$$

where I, J are finite indexing sets, each $c_i \neq 0, d_j$ is constant. From Proposition 2.2.6 it is clear that each $g_i, h_j \in F(\log(y_1), \ldots, \log(y_n), \ell_2(z_1), \ldots, \ell_2(z_m))$. Since trilogarithmic integrals cannot be written as sum of dilogarithm and logarithms over constants, we have $r_i, s_j \in F(\log(y_1), \ldots, \log(y_n), \ell_2(z_1), \ldots, \ell_2(z_m))$. Using Proposition 2.2.4 we can write $w = c\theta + w_0$ where $w_0 \in F(\log(y_1), \ldots, \log(y_n), \ell_2(z_1), \ldots, \ell_2(z_m))$ and thus $w' = c\theta' + w_0'$. So the definition of θ' proves that u' admits a $\mathcal{T}-$expression over the dilogarithmic extension of F. This completes the argument. □

Theorem 5.2.10. *Let $E = F(\theta_1, \ldots, \theta_n)$ be a transcendental $\mathcal{T}-$extension of F. Suppose that there is an element $u \in E$ with $u' \in F$ then*

$$u' = \sum_{i\in I} r_i g_i'/g_i + \sum_{j\in J} s_j h_j'/h_j + w'$$

over F, where I and J are some finite index sets and each w, g_i, h_j, r_i, s_j are elements in F such that

$$r_i' = t_i \frac{g_i'}{g_i} + \sum_{j\in J} r_{ij}\frac{h_j'}{h_j}, \quad s_j' = \sum_{i\in I} r_{ij}\frac{g_i'}{g_i} + \sum_{k\in J} s_{jk}\frac{h_k'}{h_k},$$
$$t_i' = -c_i \frac{(1-g_i)'}{1-g_i} + \sum_{j\in J} c_i c_{ij}\frac{h_j'}{h_j}, \quad r_{ij}' = c_i c_{ij}\frac{g_i'}{g_i} + \sum_{k\in J} e_{ijk}\frac{h_k'}{h_k} \quad \textit{and}$$
$$s_{jk}' = \sum_{i\in I} e_{ijk}\frac{g_i'}{g_i} + \sum_{l\in J} f_{jkl}\frac{h_l'}{h_l},$$

where each c_i is a non-zero constant whenever $r_i' \neq 0$, each c_{ij}, e_{ijk}, f_{jkl} are some constants, $e_{ijk} = e_{ikj}$ and $s_{jk} = s_{kj}$ for every i, j and k.

Proof. From Theorem 5.2.9 we know that u' admits $\mathcal{T}-$expression over some dilogarithmic extension $M = F(\log y_1, \ldots, \log y_n, \ell_2(z_1), \ldots, \ell_2(z_m))$ of F. Now we apply Proposition 5.2.4 and obtain the desired result. □

5.3 Integration with polylogarithmic integrals

In the view of Theorems 4.3.3 and 5.2.10, we shall inductively state a conjecture for integration in finite terms involving polylogarithmic integrals along with transcendental elementary functions. We shall include polylogarithmic integrals in our field of definition and provide an extension of Liouville's Theorem.

The following definition is due to J. Baddoura (See [2], p.232)

Definition 5.3.1. Let $E \supset F$ be differential fields and $g \in F\backslash\{0,1\}$ be any element. Then for an integer $m > 0$, the integral

$$\int \frac{g'}{g}\ell_{m-1}(g)$$

in E is called *polylogarithmic integral* of order m and is denoted by $\ell_m(g)$.

Note that for $m = 2, 3$ we called the polylogarithmic integral a dilogarithmic integral and trilogarithmic integral, respectively, and $\ell_1(g) = -\log(1-g)$.

We shall now provide an identity for polylogarithms.

Proposition 5.3.2. *Let F be a differential field and $g \in F$ be any non-zero, non-identity element. Then for every integer $m > 0$, there is a polynomial P of degree m in $C_F[X]$, where X is an indeterminate, such that*

$$\ell_m\left(\frac{1}{g}\right) + (-1)^m \ell_m(g) = P(\log g).$$

Proof. We prove this identity by induction. For $m = 1$,

$$\begin{aligned}\ell_1\left(\frac{1}{g}\right) = -\log\left(1-\frac{1}{g}\right) &= -\log(g-1) + \log g \\ &= -\log(1-g) + \log g + c, \text{ for some constant c} \\ &= \ell_1(g) + P_1(\log g),\end{aligned}$$

where $P_1 = X + c$ is a polynomial of order 1 in $C_F(X)$. For $m = 2$, the result is clearly true from Proposition 3.1.2. Assume the result is true for any integer $k > 0$. From the definition of polylogarithmic integral

$$\ell'_{k+1}\left(\frac{1}{g}\right) = \ell_k\left(\frac{1}{g}\right)\frac{(1/g)'}{(1/g)}.$$

By induction, for some polynomial P_k of degree k in $C_F(X)$, we have

$$\begin{aligned}\ell'_{k+1}\left(\frac{1}{g}\right) &= \left(-(-1)^k\ell_k(g) + P_k(\log g)\right)\frac{-g'}{g} \\ &= (-1)^k\ell'_{k+1}(g) + P_k(\log g)\frac{g'}{g}.\end{aligned}$$

Note that since $P_k(\log g)$ is a polynomial in $\log g$ over C_F, the term $P_k(\log g)\frac{g'}{g}$ is a derivative of some polynomial P of degree k over C_F. Therefore, we obtain

$$\ell_{k+1}\left(\frac{1}{g}\right) + (-1)^{k+1}\ell_{k+1}(g) = P(\log g).$$

Thus, by induction the result is true for any integer $m > 0$. □

Definition 5.3.3. A differential field $E \supset F$ is called a $\mathcal{P}-$*extension* of order m if $C_E = C_F$ and there is a tower of differential fields F_i such that

$$F = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = E$$

and for each i, $F_i = F_{i-1}(\theta_i)$ and one of the following holds:

(i) θ_i is algebraic over F_{i-1}.

(ii) $\theta_i' = u'\theta_i$ for some $u \in F_{i-1}$ (i.e. $\theta_i = e^u$).

(iii) $\theta_i' = u'/u$ for some $u \in F_{i-1}$ (i.e. $\theta_i = \log(u)$).

(iv) $\theta_i' = vu'/u$, where $v = \ell_{j-1}(u)$ for some $u, v \in F_{i-1}$ and $j \leq m$ (i.e. $\theta_i = \int \frac{u'}{u}\ell_{j-1}(u)$, also denoted by $\ell_j(u)$).

Definition 5.3.4. We say that $v \in F$ admits a *general* $\mathcal{P}-$*expression* of order m over F if there are finite index sets $I_1, \ldots, I_m$ and elements $r_{i_j}, g_{i_j} \in F$ for all $i_j \in I_j$ and an element $w \in F$ such that

$$v = \sum_{j=1}^{m} \sum_{i_j \in I_j} r_{i_j} \frac{g'_{i_j}}{g_{i_j}} + w', \tag{5.51}$$

where for each $i_j \in I_j$, r_{i_j} is sum of polylogarithmic integrals of order $\leq j-1$.

Definition 5.3.5. A general $\mathcal{P}-$expression will be called a $\mathcal{P}$-*expression* if for each $i_j \in I_j$, $r_{i_j} = c_{i_j}\ell_{j-1}(g_{i_j})$ where c_{i_j}'s are constants .

Definition 5.3.6. A differential field extension E of F will be called a *polylogarithmic extension* of F of order m if $C_E = C_F$ and the base of E over F consists of polylogarithmic integrals of order less than or equal to m over F.

Now in this context of polylogarithmic integrals, we shall inductively extend the Lemmas 4.2.2 and 5.2.7 and conjecture the following lemma.

Lemma 5.3.7. *Let $F(\theta) \supset F$ be a transcendental $\mathcal{P}-$extension of order m. Suppose there is an element $v \in F$ such that v admits a $\mathcal{P}-$expression of order m over a polylogarithmic extension of $F(\theta)$ of order $m-1$. Then there is polylogarithmic extension M of F of order $m-1$ such that v admits a $\mathcal{P}-$expression of order m over $M(\theta)$.*

Using this lemma we can obtain a general $\mathcal{P}-$expression for v over F. Thus the main extension theorem for polylogarithmic integrals can be conjectured as follows:

Conjecture 5.3.8. *Let $E = F(\theta_1, \ldots, \theta_n)$ be a transcendental $\mathcal{P}-$extension of order m. Suppose there is an element $u \in E$ with $u' \in F$ then u' admits a general $\mathcal{P}-$expression of order m over the field F.*

5.3.1 A Note on Polylogarithmic Integrals of Order 4

Let $E \supset M \supset F$ be differential fields such that E is a $\mathcal{P}-$extension of order 4 over F and M is a polylogarithmic extension of order 3 over F. Then the results of Propositions 4.1.2 and 5.2.4 can be extended to polylogarithmic integrals of order 4. That is, if $v \in F$ satisfies a $\mathcal{P}-$expression of order 4 over M then one can check that v satisfies a general $\mathcal{P}-$expression over F:

$$v = \sum_{i\in I} r_i \frac{g_i'}{g_i} + \sum_{j\in J} s_j \frac{h_j'}{h_j} + w',$$

where

$$r_i' = t_i \frac{g_i'}{g_i} + \sum_{j\in J} r_{ij} \frac{h_j'}{h_j}, \quad s_j' = \sum_{i\in I} r_{ij} \frac{g_i'}{g_i} + \sum_{k\in J} s_{jk} \frac{h_k'}{h_k},$$

$$t_i' = p_i \frac{g_i'}{g_i} + \sum_{j\in J} t_{ij} \frac{h_j'}{h_j}, \quad r_{ij}' = t_{ij} \frac{g_i'}{g_i} + \sum_{k\in J} r_{ijk} \frac{h_k'}{h_k}, \quad s_{jk}' = \sum_{i\in I} r_{ijk} \frac{g_i'}{g_i} + \sum_{l\in J} s_{jkl} \frac{h_l'}{h_l},$$

$$p_i' = -c_i \frac{(1-g_i)'}{1-g_i} + \sum_{j\in J} c_i c_{ij} \frac{h_j'}{h_j}, \; t_{ij}' = c_i c_{ij} \frac{g_i'}{g_i} + \sum_{k\in J} c_i e_{ijk} \frac{h_k'}{h_k},$$

$$r_{ijk}' = \sum_{i\in I} c_i e_{ijk} \frac{g_i'}{g_i} + \sum_{l\in J} f_{ijkl} \frac{h_l'}{h_l} \;\text{ and }\; s_{jkl}' = \sum_{i\in I} f_{ijkl} \frac{g_i'}{g_i} + \sum_{m\in J} e_{jklm} \frac{h_m'}{h_m},$$

where each c_i is a non-zero constant whenever $r_i' \neq 0$, each $c_{ij}, e_{ijk}, f_{ijkl}, e_{jklm}$ are constants, each r_i, s_j, g_i, h_j and w are some elements in F, $t_i, r_{ij}, s_{jk}, p_i, t_{ij}, r_{ijk}, s_{jkl}$

are elements in M with $e_{ijk} = e_{ikj}$, $s_{jk} = s_{kj}$, $r_{ijk} = r_{ikj}$, $s_{jkl} = s_{\sigma(j)\sigma(k)\sigma(l)}$, $f_{ijkl} = f_{i\sigma(j)\sigma(k)\sigma(l)}$ and $e_{jklm} = e_{\rho(j)\rho(k)\rho(l)\rho(m)}$ for each i, j, k, l, m and any permutation σ in S_3 and ρ in S_4.

Conversely (Similar to Propositions 4.1.1 and 5.2.6), in the next proposition we shall show that if we have such an expression for an element v in F then there exists a $\mathcal{P}-$extension of F that contains an antiderivative of v. The major work here is to show that if there is an element u in a transcendental $\mathcal{P}-$extension E of order 4 over F such that u' lies in F, then u' satisfies a $\mathcal{P}-$expression over a polylogarithmic extension M of order 3. Therefore, in order to prove the conjecture for any order m, this part would be crucial.

Proposition 5.3.9. *Let $v \in F$ satisfies a general $\mathcal{P}-$expression over F:*

$$v = \sum_{i \in I} r_i \frac{g_i'}{g_i} + \sum_{j \in J} s_j \frac{h_j'}{h_j} + w',$$

where

$$r_i' = t_i \frac{g_i'}{g_i} + \sum_{j \in J} r_{ij} \frac{h_j'}{h_j}, \quad s_j' = \sum_{i \in I} r_{ij} \frac{g_i'}{g_i} + \sum_{k \in J} s_{jk} \frac{h_k'}{h_k},$$

$$t_i' = p_i \frac{g_i'}{g_i} + \sum_{j \in J} t_{ij} \frac{h_j'}{h_j}, \quad r_{ij}' = t_{ij} \frac{g_i'}{g_i} + \sum_{k \in J} r_{ijk} \frac{h_k'}{h_k}, \quad s_{jk}' = \sum_{i \in I} r_{ijk} \frac{g_i'}{g_i} + \sum_{l \in J} s_{jkl} \frac{h_l'}{h_l},$$

$$p_i' = -c_i \frac{(1-g_i)'}{1-g_i} + \sum_{j \in J} c_i c_{ij} \frac{h_j'}{h_j}, \ t_{ij}' = c_i c_{ij} \frac{g_i'}{g_i} + \sum_{k \in J} c_i e_{ijk} \frac{h_k'}{h_k},$$

$$r_{ijk}' = \sum_{i \in I} c_i e_{ijk} \frac{g_i'}{g_i} + \sum_{l \in J} f_{ijkl} \frac{h_l'}{h_l} \ \ \textit{and} \ \ s_{jkl}' = \sum_{i \in I} f_{ijkl} \frac{g_i'}{g_i} + \sum_{m \in J} e_{jklm} \frac{h_m'}{h_m},$$

where each c_i is a non-zero constant whenever $r_i' \neq 0$, each $c_{ij}, e_{ijk}, f_{ijkl}$ are constants, each r_i, s_j, g_i, h_j and w are some elements in F, $t_i, r_{ij}, s_{jk}, p_i, t_{ij}, r_{ijk}, s_{jkl}$ are elements in M with $e_{ijk} = e_{ikj}$, $s_{jk} = s_{kj}$, $r_{ijk} = r_{ikj}$, $s_{jkl} = s_{\sigma(j)\sigma(k)\sigma(l)}$, $f_{ijkl} = f_{i\sigma(j)\sigma(k)\sigma(l)}$ and $e_{jklm} = e_{\rho(j)\rho(k)\rho(l)\rho(m)}$ for each i, j, k, l, m and any permutation σ in

S_3 and ρ in S_4. Then there exists a $\mathcal{P}-$extension E of F of order 4 that contains $\int v$.

Proof. Replace w with an element $w - \sum_{i\in I} r_i \log g_i - \sum_{j\in J} s_j \log h_j$ in a logarithmic extension of F and obtain

$$v = -\sum_{i\in I} r_i' \log g_i - \sum_{j\in J} s_j' \log h_j + w'.$$

Substitute the given values of r_i' and s_j', we shall get

$$v = -\sum_{i\in I}\left(t_i\frac{g_i'}{g_i} + \sum_{j\in J} r_{ij}\frac{h_j'}{h_j}\right)\log g_i - \sum_{j\in J}\left(\sum_{i\in I} r_{ij}\frac{g_i'}{g_i} + \sum_{k\in J} s_{jk}\frac{h_k'}{h_k}\right)\log h_j + w'.$$

We shall combine the second and third term and replace w with $w+1/2\sum_{i\in I} t_i \log^2 g_i$ $+\sum_{i\in I, j\in J} r_{ij}\log h_j \log g_i + 1/2\sum_{j,k\in J} s_{jk}\log h_j \log h_k$ to obtain

$$v = \frac{1}{2}\sum_{i\in I} t_i' \log^2 g_i + \sum_{i\in I,j\in J} r_{ij}' \log h_j \log g_i + \frac{1}{2}\sum_{j,k\in J} s_{jk}' \log h_j \log h_k + w'.$$

Substituting the values of t_i', r_{ij}', s_{jk}', we have

$$\begin{aligned} v =& \frac{1}{2}\sum_{i\in I}\left(p_i\frac{g_i'}{g_i} + \sum_{j\in J} t_{ij}\frac{h_j'}{h_j}\right)\log^2 g_i + \sum_{i\in I,j\in J}\left(t_{ij}\frac{g_i'}{g_i} + \sum_{k\in J} r_{ijk}\frac{h_k'}{h_k}\right)\log h_j \log g_i \\ &+ \frac{1}{2}\sum_{j,k\in J}\left(\sum_{i\in I} r_{ijk}\frac{g_i'}{g_i} + \sum_{l\in J} s_{jkl}\frac{h_l'}{h_l}\right)\log h_j \log h_k + w'. \end{aligned}$$

Now we shall combine second and third term, fourth and fifth term,respectively and replace w with $w - 1/6\sum_{i\in I} c_i p_i \log^3 g_i - 1/2\sum_{i\in I,j\in J} c_i t_{ij}\log h_j \log^2 g_i -$ $1/2\sum_{i\in I,j,k\in J} r_{ijk}\log g_i \log h_j \log h_k - 1/6\sum_{j,k,l\in J} s_{jkl}\log h_j \log h_k \log h_l$. Here we are using the fact that $r_{ijk} = r_{ikj}$ and $s_{jkl} = s_{\sigma(j)\sigma(k)\sigma(l)}$ for any σ in S_3. Then v becomes

$$\begin{aligned} v =& -\frac{1}{6}\sum_{i\in I} p_i' \log^3 g_i - \frac{1}{2}\sum_{i\in I,j\in J} t_{ij}' \log h_j \log^2 g_i - \frac{1}{2}\sum_{i\in I,j,k\in J} r_{ijk}' \log g_i \log h_j \log h_k \\ &-\frac{1}{6}\sum_{j,k,l\in J} s_{jkl}' \log h_j \log h_k \log h_l + w'. \end{aligned}$$

We shall again substitute p'_i, t'_{ij}, r'_{ijk} and s'_{jkl} and obtain

$$
\begin{aligned}
v = & -\frac{1}{6}\sum_{i\in I}\left(-c_i\frac{(1-g_i)'}{1-g_i}+\sum_{j\in J}c_ic_{ij}\frac{h'_j}{h_j}\right)\log^3 g_i \\
& -\frac{1}{2}\sum_{i\in I,j\in J}\left(c_ic_{ij}\frac{g'_i}{g_i}+\sum_{k\in J}c_ie_{ijk}\frac{h'_k}{h_k}\right)\log h_j\log^2 g_i \\
& -\frac{1}{2}\sum_{i\in I,j,k\in J}\left(\sum_{i\in I}c_ie_{ijk}\frac{g'_i}{g_i}+\sum_{l\in J}f_{ijkl}\frac{h'_l}{h_l}\right)\log g_i\log h_j\log h_k \\
& -\frac{1}{6}\sum_{j,k,l\in J}\left(\sum_{i\in I}f_{ijkl}\frac{g'_i}{g_i}+\sum_{m\in J}\frac{h'_m}{h_m}\right)\log h_j\log h_k\log h_l + w'.
\end{aligned}
$$

Combine the terms second and third, fourth and fifth, sixth and seventh, respectively and replace w with $w+\frac{1}{6}\sum_{i\in I,j\in J}c_ic_{ij}\log h_j\log^3 g_i+\frac{1}{4}\sum_{i\in I,j,k\in J}c_ie_{ijk}\log h_j\log h_k$ $\log^2 g_i+\frac{1}{6}\sum_{i\in I,j,k,l\in J}f_{ijkl}\log g_i\log h_j\log h_k\log h_l+\frac{1}{24}\sum_{j,k,l,m\in J}\log h_j\log h_k\log h_l\log h_m$. Thus, v reduces to

$$v=\frac{1}{6}\sum_{i\in I}c_i\log^3 g_i\frac{(1-g_i)'}{1-g_i}+w'.$$

Observe that in a polylogarithmic extension M of order 3 containing $\ell_2(g_i),\ell_3(g_i)$, we can replace w with $w-1/6\sum_{i\in I}c_i\log^3 g_i\log(1-g_i)-1/2\sum_{i\in I}c_i\log^2 g_i\ell_2(g_i)+\sum_{i\in I}\log g_i\ell_3(g_i)$ and thus obtain

$$v=\sum_{i\in I}c_i\ell_3(g_i)\frac{g'_i}{g_i}+w' \quad \text{and}$$

$$\int v=\sum_{i\in I}c_i\ell_4(g_i)+w.$$

Hence, antiderivative of v lies in a $\mathcal{P}-$extension $E\supset M\supset F$. $\square$

We shall conclude the thesis with the following example.

Example 5.3.10. Consider a transcendental $\mathcal{P}-$extension $E = F(\log(1-g_i), \ell_2(g_i), \ell_3(g_i), \ell_4(g_i) \mid i = 1, 2)$ of order 4 over F. Through this example, we shall characterize all elements u of E whose derivative lies in F. Assume there is an element $u \in E$ such that $u' \in F$ then there exists some constants c_1, c_2 such that

$$u' = \sum_{i=1,2} c_i \ell_3(g_i) \frac{g_i'}{g_i} + w', \tag{5.52}$$

where w is some element in $F(\log(1 - g_i), \ell_2(g_i), \ell_3(g_i)| \ i = 1, 2)$. Since w' lies in the polynomial ring $F(\log(1 - g_i), \ell_2(g_i))[\ell_3(g_i)]$, we shall use Proposition 2.2.4 and write

$$w = \sum_{i,j=1,2} a_{ij} \ell_3(g_i)\ell_3(g_j) + \sum_{i=1,2} w_i \ell_3(g_i) + w_0,$$

where a_{ij}'s are constants and $w_i, w_0 \in F(\log(1 - g_i), \ell_2(g_i)| \ i = 1, 2)$ Since $\ell_3(g_1), \ell_3(g_2)$ are transcendental over $F(\log(1-g_i), \ell_2(g_i)| \ i = 1, 2)$, we shall compare the coefficients in Equation 5.52 and obtain

$$u' = \sum_{i=1,2} w_i \ell_2(g_i) \frac{g_i'}{g_i} + w_0' \quad \text{and} \quad w_i' = -c_i \frac{g_i'}{g_i} \tag{5.53}$$

for each i. Now the Equation 5.53 is similar to Equation 5.52 and w_0' is an element in the polynomial ring $F(\log(1 - g_i))[\ell_2(g_i)]$, we shall repeat the same process and write

$$w_0 = \sum_{i,j=1,2} b_{ij} \ell_2(g_i)\ell_2(g_j) + \sum_{i=1,2} w_{i0} \ell_2(g_i) + w_{00}$$

for some constants b_{ij} and elements w_{i0}, w_{00} in $F(\log(1 - g_i) \ |i = 1, 2)$. Comparing the coefficients in Equation 5.53, we obtain

$$u' = -\sum_{i=1,2} w_{i0} \log(1 - g_i) \frac{g_i'}{g_i} + w_{00}' \quad \text{and} \quad w_{i0}' = -w_i \frac{g_i'}{g_i}. \tag{5.54}$$

Again this equation is similar to Equation 5.52 and w_{00}' lies in the polynomial ring $F[\log(1 - g_i)]$. Therefore, we shall write $w_{00} = \sum_{i,j=1,2} c_{ij} \log(1 - g_i) \log(1 - g_j) +$

$\sum_{i=1,2} w_{i00} \log(1 - g_i) + v$ for some elements $v, w_{i00} \in F$ and constants c_{ij}. Thus, comparing the coefficients we shall obtain

$$u' = \sum_{i=1,2} w_{i00} \frac{g_i'}{g_i} + v', \tag{5.55}$$

where

$$w_{i00}' = w_{i0} \frac{g_i'}{g_i}, \quad w_{i0}' = -w_i \frac{g_i'}{g_i} \text{ and } w_i' = -c_i \frac{g_i'}{g_i}.$$

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List of Notations

F	A differential field
E	A differential field extension of F
$\overline{F}$	Algebraic closure of F
C_F	Field of constants of F
$[E : F]$	Index of E over F
tr deg E/F	Transcendence degree of E over F
$\mathbb{C}$	The field of complex numbers
$\mathbb{Q}$	The field of rational numbers
u'	Derivative of u
$\int v$	Integral of v
e^u	Exponential of u
$\log(u)$	Logarithm of u
$\log^m(u)$	m fold product of $\log(u)$ i.e. $(\log u)^m$
u'/u	Logarithmic derivative of g
$\ell i(u)$	Logarithmic integral of u
$erf(u)$	Error function of u
$\ell_2(u)$	Dilogarithmic integral of u
$\ell_3(u)$	Trilogarithmic integral of u

$\ell_m(u)$	Polylogarithmic integral of u of order m
$D(g)$	Bloch-Wigner Spence function of g
$\deg(P(X))$	Degree of polynomial $P(X)$
η	Constant term in the partial fraction expansion of g
ξ	Constant term in the partial fraction expansion of $1-g$