On Using Entangled Systems of Two Particles to Beat the Standard Quantum Limit

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Certificate of Examination

This is to certify that the dissertation titled "On Using Entangled Systems of Two Particles to Beat the Standard Quantum Limit" submitted by Mr. Himanshu Patange (Reg. No. MS14066) for the partial ful- filment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Sandeep K. Goyal	Prof. Kavita Dorai	Prof. Arvind
		(Supervisor)

Dated: April 26, 2019

Declaration

The work in this dissertation has been carried out by me under the guidance of Dr. Arvind at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Himanshu Patange

(Candidate)

Dated: April 26, 2019

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Arvind

(Supervisor)

Acknowledgement

While the written thesis was produced on the keyboard during the last two months of my final year at IISER Mohali, it has been in formation on paper for the entirety of that year, if not from before. I am thankful to my supervisor, Professor Arvind, to have directed me to the exciting field of Quantum Metrology, and for the guidance and support he gave me. I am also thankful to the QCQI Group at IISER Mohali for providing a suitable environment and a paradigm for me to efficiently work in. I especially thank Jaskarn Nirankari and Rajendra Bhati for clearing my doubts to the best of their abilities. Without them, my thesis would have remained an unsolvable integral. My batchmate Joraver Singh also deserves a mention here for the same.

I would also like to thank all the people here at IISER Mohali that strive to perfection for five years with the unlearned mass of young junior college graduates they get saddled with at the start of each batch. I am of course talking about the faculty and staff here at IISER Mohali, especially the faculty whose courses I have studied over my five year stay at IISERM.

Finally, I would like to thank my family. My father who cultivated an interest of science in me from childhood and instilled the studying practices I have always found very useful, my mother for supporting my decisions in spite of her reservations, and my brother for providing an ideal to be admired and a competition to surpass.

Himanshu Patange

List of Figures

Notation

$\|\,\rangle$	A ket state
$\langle\,\|$	A bra state
$\|n\rangle$	Number State
$\boldsymbol{\Delta}$	Uncertainty or error
$\langle\,\rangle$	Expectation value
$\|\,\|$	Norm
$[\,,]$	Commutator
$\hat{\mathbf{H}}$	Hamiltonian
U_τ	Unitary transformation operator acting for time τ

Contents

Abstract

This thesis examines the genesis of and the advances in the field of Quantum Metrology. An overview of terminologies and definitions in this emerging field is given, with a focus on Shot Noise Limit and the Standard Quantum Limit for monitoring free mass position, and the efforts to overcome them are examined. Ideas from these two subfields are then taken together and developed in a unique way, namely, in using a free system of two coupled particles to beat the SQL in monitoring mass position. Four such entangled systems are developed to introduce new degrees of freedom to share the burden of uncertainty in position in order to help them beat the SQL.

Introduction

Metrology is the science of measurement. The pursuit of science always needs measurement to validate it, and so the field of measurement has always been one of the most important frontiers of science. It gains a life of its own in the fields of statistical analysis. However, the world was changed a hundred years ago by the emergence of Quantum Mechanics, and where before we could tell one electron from another (and still can, in truth), now we cannot even tell if the cat is dead or alive. Given the central role that measurement plays in quantum mechanics, it is no wonder that a new study of measurements was needed, with a quantum flavor.

Although the field proper itself was birthed a little later than the formulation of Quantum Mechanics as we know it today, the field today is full of an exciting buzz of activity, and finding applications in all multiple fields, from quantum computation to medical physics, to communication and information, to condensed and nanomatter technologies, and in lithography and biomedical physics. It has the potential to make revolutionary changes wherever there is a quantum interaction involved. Because at its core, Quantum Metrology is the science of beating classical limits on experiments by using quantum correlation.

Classical experimental designs often impose classical limitations on the outcomes of the results, limitations that can be overcome by using quantum tricks. The noise in the value of a parameter of an experiment scaling inversely with the square root of the number of independent measurements performed is a result of classical statistics, and classical physics experiments follow this result. However, we can break this limit by using quantum correlations, and bring down the scaling of noise to the ulitimate limit, derived from the fundamental Uncertainty Relations in Quantum Mechanics.

During the course of my final year, I have focussed on two subfields of Quantum Metrology. I have studied the efforts to beat the Shot Noise Limit in Interferometers and the attempts at defining and breaking the Standard Quantum Limit in monitoring the position of a free mass. Both the questions are crucial, among other things, to Gravitational Wave Detectors. However, they are always treated separately. In fact, while we have been successful in beating the Shot Noise Limit in the Interferometers, we are still struggling to understand the Standard Quantum Limit (SQL) in position measurement, let alone beat it. There have been and are many definitions of the SQL, some more popular than others, but they all deal with monitoring force on a single, free mass, sometime with one detector, sometimes with many. The regime of SQL shares with the Interferometer the identical detectors part, because in the Interferometer, photons in large numbers are the detectors, but there is a glaring difference between these approaches. There is always only one particle being monitored for its position in any schemes attempting to break the SQL, but in ther Interferometer, where these methods will be applied one day, there are two mirrors being monitored.

It is for this reason that for my MS thesis, I have explored ways to turn this multi-bodiedness of system to my advantage. This thesis thus serves to outline four ways, or four entanglements in two body systems, that can beat the SQL.

I shall begin this thesis with a brief introduction, an overview of the field and everything the reader must know to understand the final proceedings. Our daunting task will begin with the case of Shot Noise Limit in Chapter 1. After introducing the paradigm of Quantum Light and Interferometry, I shall derive the Shot Noise Limit and the fundamental Heisenberg Limit in Section 1. The next two sections will be devoted to examining the efforts in beating the Shot Noise Limit and achieve the Heisenberg Limit. The section will end chapter with comments on where this technology stands in terms of experiments.

The next chapter will be devoted to the SQL in monitoring mass position and shall follow a similar path as before. The first, introductory section will detail the weak measurement model as given by Von Neumann, followed by an account of the measurement protocol given by Aruthurs and Kelly to monitor conjugate observables simultaneously. Section 1 will then present the first derivation of the SQL as was done by Braginsky and Vorontsov, and then the next three sections will follow the efforts as were made to beat and/or to redifine the SQL. The chapter will close with a section introducing Quantum Nondemolition techniques and Backaction Evading measures.

In Chapter 3, we shall turn to my own explorations of how one might beat the SQL in repetitive position measurements by using a system of two coupled particles. In the first section of Chapter 3, I shall present my motivation for the idea and its origins. Here I shall justify my approach, and then in the next section, I shall present the four systems I have found to beat the SQL. I shall close the chapter, and this thesis, with concluding remarks and comments on where we might expect to go from here.

Chapter 1

Shot Noise Limit

A Michelson interferometer is a crucial component in many signal processing and analyzing techniques, but most notably, for the purpose of this thesis, it is one of the leading methods for Gravitational Wave Detection (from Wikipedia page on Michelson Interferometer)

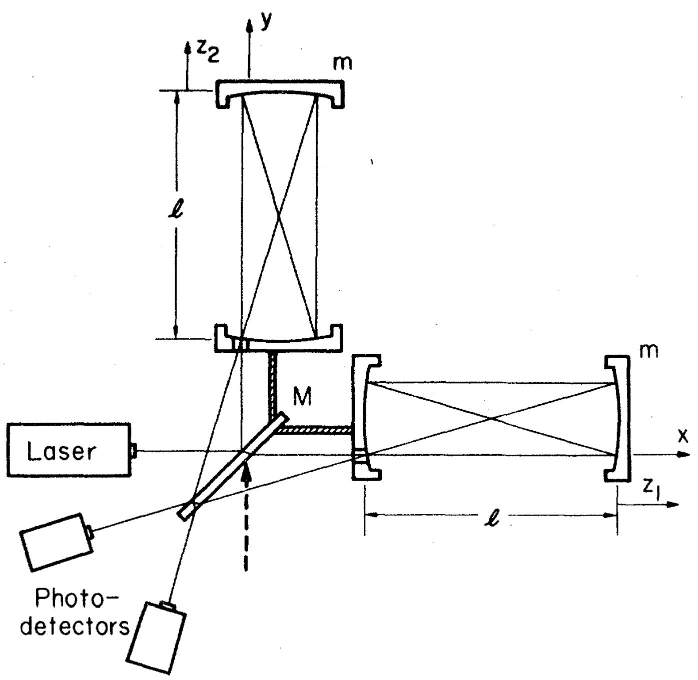

Figure 1.1: Schematic of a Michelson Interferometer working as a Gravitational Wave Detector (taken from [Caves 81])

The Interferometer, as the name indicates, uses interference of light to make inference about its properties. Its use in detecting gravitational waves comes from the fact that as gravitational waves passes over the interferometer, they act differently on the two perpendicular arms. One arm of the Interferometer is squeezed, and the other is elongated. The resulting difference in the lengths of the arms leads to a phase difference between the light beams traveling through these arms, phase difference which can be measured using the interference pattern of these two beams.

However, the detection of the induced phase difference is limited by the so called "Shot Noise Limit". In this chapter I shall derive the theory around the Shot Noise Limit, and present the methods used to beat this limit. But first, we will start with understanding the quantum treatment of light and the Interferometry.

1.0.1 Quantum Treatment of Light

Coherent light is represented as $|\alpha\rangle = \hat{\boldsymbol{D}}(\alpha)\,|0\rangle$ where $\hat{\boldsymbol{D}}(\alpha) = e^{\alpha\hat{\boldsymbol{a}}^{\dagger}-\alpha^{*}\hat{\boldsymbol{a}}}$ is the Displacement Operator, $|0\rangle$ is the vacuum state and α is a complex number. $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{a}}^{\dagger}$ are the annihilation and the creation operators of light, which are constructed from $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$ as

$$\hat{\boldsymbol{a}} = \sqrt{\frac{m\omega}{2\hbar}}\hat{\boldsymbol{x}} + i\frac{\hat{\boldsymbol{p}}}{\sqrt{2m\omega\hbar}} \tag{1.1}$$

giving $[\hat{\boldsymbol{a}}, \hat{\boldsymbol{a}}^{\dagger}] = 1$.

The Coherent state $|\alpha\rangle$ is an eigenstate of the operator $\hat{\boldsymbol{a}}$, obeying $\hat{\boldsymbol{a}}\,|\alpha\rangle = \alpha\,|\alpha\rangle$. $\hat{\mathbf{X}}_1$ and $\hat{\mathbf{X}}_2$ are the two quadratures of light, and are defined as

$$\hat{\boldsymbol{X}}_1 = \frac{\hat{\boldsymbol{a}} + \hat{\boldsymbol{a}}^{\dagger}}{2} \qquad \hat{\boldsymbol{X}}_2 = \frac{\hat{\boldsymbol{a}} - \hat{\boldsymbol{a}}^{\dagger}}{2i} \tag{1.2}$$

They obey the uncertainty relation

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4}, \tag{1.3}$$

and for the coherent light, take values $\Delta X_1 = \Delta X_2 = \frac{1}{2}$. The coherent state is the displaced ground state, and thus is a minimally uncertain state. However, as long as the equation 1.3 is obeyed, physics allows to us lower the uncertainty in one quadrature at the expense of the uncertainty in the other. This is called squeezing.

$\hat{\boldsymbol{S}}(\xi)$ is the Squeezing Operator, defined as

$$\hat{\boldsymbol{S}}(\xi) = e^{\frac{1}{2}(\xi^* \hat{\boldsymbol{a}}^2 - \xi \hat{\boldsymbol{a}}^{\dagger 2})} \tag{1.4}$$

where $\xi = re^i\theta$, r is the squeezing parameter, and $0 \leq \theta \leq 2\pi$ [Caves 81].

The Squeezing Operator satisfies relations

$$\hat{\boldsymbol{S}}^{-1}(\xi) = \hat{\boldsymbol{S}}^{\dagger}(\xi) = \hat{\boldsymbol{S}}(-\xi) \tag{1.5}$$

$$\hat{\boldsymbol{S}}^{\dagger}(\xi)\hat{\boldsymbol{a}}\hat{\boldsymbol{S}}(\xi) = \hat{\boldsymbol{a}}\cosh r - \hat{\boldsymbol{a}}^{\dagger} e^{i\theta}\sinh r \tag{1.6}$$

$$\hat{\boldsymbol{S}}^{\dagger}(\xi)\hat{\boldsymbol{a}}^{\dagger}\hat{\boldsymbol{S}}(\xi) = \hat{\boldsymbol{a}}^{\dagger}\cosh r - \hat{\boldsymbol{a}} e^{-i\theta}\sinh r \tag{1.7}$$

and acts on the vacuum state as

$$\hat{\boldsymbol{S}}(\xi)\left|0\right\rangle = \left|\xi\right\rangle . \tag{1.8}$$

For $\theta = 0$, the state $|\xi\rangle$ gives

$$(\Delta X_1)^2 = \frac{1}{4}e^{-2r} \leq \frac{1}{4} \tag{1.9}$$

$$(\Delta X_2)^2 = \frac{1}{4}e^{2r} \geq \frac{1}{4} \tag{1.10}$$

thereby preserving the Uncertainty Relation (4). For $\theta = 0$, squeezing is in the X_1 quadrature. For $\theta = \frac{\pi}{2}$, it will in the X_2 quadrature. For arbitrary θ, we can define $\hat{\boldsymbol{Y}}_1$ and $\hat{\boldsymbol{Y}}_2$ as

$$\begin{pmatrix} \hat{\boldsymbol{Y}}_1 \\ \hat{\boldsymbol{Y}}_2 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{X}}_1 \\ \hat{\boldsymbol{X}}_2 \end{pmatrix} \tag{1.11}$$

to get the squeezing as

$$(\Delta Y_1)^2 = \frac{1}{4}e^{-2r} \tag{1.12}$$

$$(\Delta Y_2)^2 = \frac{1}{4}e^{2r} \tag{1.13}$$

The effects of squeezing on a coherent state are compared with the normal coherent state in a complex amplitude plane in Figure 1.2.

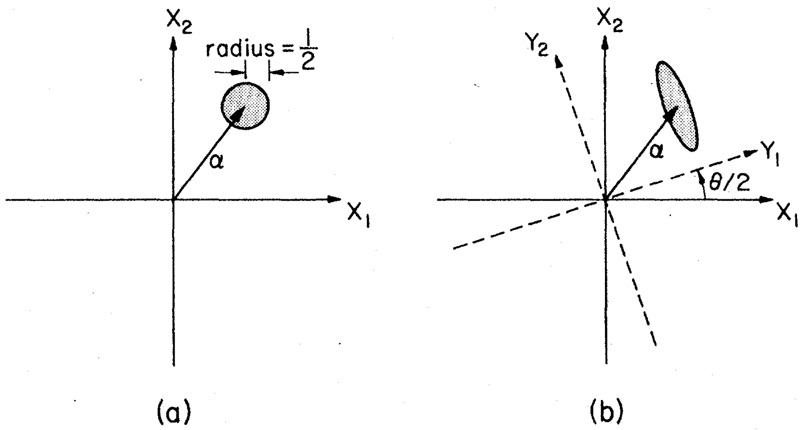

Figure 1.2: (a)Error circle for the coherent state$|\alpha\rangle$. (b) Error ellipse for the squeezed coherent state $|\alpha, re^{i\theta}\rangle$. (Figure taken from [Caves 81])

1.0.2 Interferometry with Coherent Light

An idealized Interferometer has been shown in Figure 1.3. The Interferometer has two input ports, 0 and 1. We can associate creation and annihilation operators $\hat{\boldsymbol{a}}_0$, $\hat{\boldsymbol{a}}_0^\dagger$ and $\hat{\boldsymbol{a}}_1$, $\hat{\boldsymbol{a}}_1^\dagger$ respectively for these ports for photons associated with them. These make for two modes of light in the Interferometer. These modes, and by extensions, the operators corresponding to them, in the Heisenberg Picture, transform into modes 2

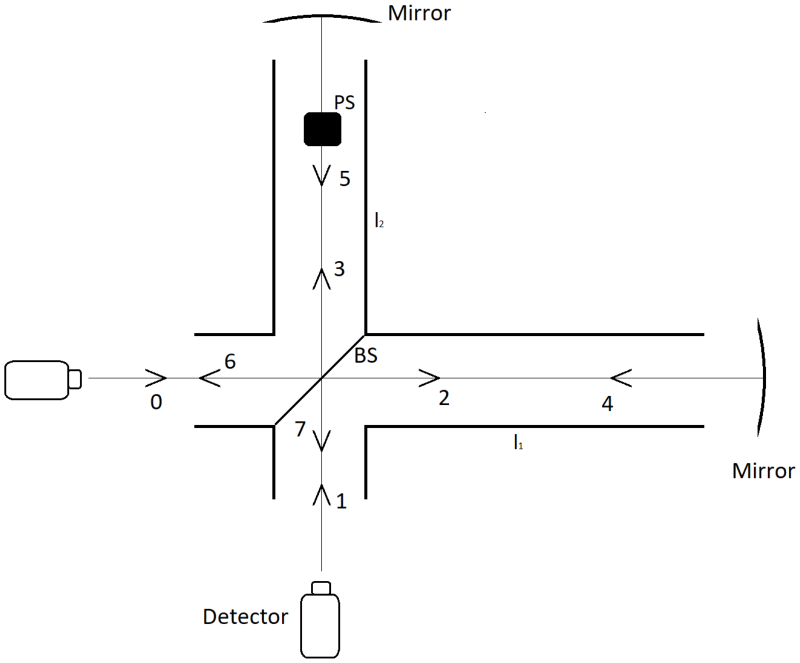

Figure 1.3: Idealized Michelson Interferometer with arm lengths l_1 and l_2 and effective phase difference ϕ with indicated respective modes of light.

and 3 as follows:

$$\hat{\boldsymbol{a}}_0 \rightarrow \frac{\hat{\boldsymbol{a}}_2 - i\hat{\boldsymbol{a}}_3}{\sqrt{2}} \tag{1.14}$$

$$\hat{\boldsymbol{a}}_1 \rightarrow \frac{-i\hat{\boldsymbol{a}}_2 + \hat{\boldsymbol{a}}_3}{\sqrt{2}}. \tag{1.15}$$

Upon reflecton from the mirrors, we get transformations

$$\hat{\boldsymbol{a}}_2 \rightarrow \hat{\boldsymbol{a}}_4 \tag{1.16}$$

$$\hat{\boldsymbol{a}}_3 \rightarrow e^{-i\phi}\hat{\boldsymbol{a}}_5, \tag{1.17}$$

where the phase shift from reflection in both modes has been ignored as it will be the same for both, and an effective phase shift of $\phi = \omega(l_1 - l_2)/c$ has been assumed in the light travelling through the vertical arm due to path length differences between the two arms. These operators, upon the next action of the Beam Splitter (BS2), transform for the last time as

$$\hat{\boldsymbol{a}}_4 \rightarrow \frac{\hat{\boldsymbol{a}}_6 - i\hat{\boldsymbol{a}}_7}{\sqrt{2}} \tag{1.18}$$

$$\hat{\boldsymbol{a}}_5 \rightarrow \frac{-i\hat{\boldsymbol{a}}_6 + \hat{\boldsymbol{a}}_7}{\sqrt{2}}. \tag{1.19}$$

Thus, in the Heisenberg Picture, the output modes are related to the input modes by the relation

$$\hat{\boldsymbol{a}}_6 = \frac{(1 - e^{i\phi})\hat{\boldsymbol{a}}_0 + i(1 + e^{i\phi})\hat{\boldsymbol{a}}_1}{2} \tag{1.20}$$

$$\hat{\boldsymbol{a}}_7 = \frac{i(1 + e^{i\phi})\hat{\boldsymbol{a}}_0 - (1 - e^{i\phi})\hat{\boldsymbol{a}}_1}{2} \tag{1.21}$$

If we are using only the port 0 of the interferometer, injecting coherent light into it, the state of the input ports is $|\Psi_{0,1}\rangle = |\alpha\rangle_0 |0\rangle_1$, signifying the fact that when port 1 isn't being used, it is in a state of vacuum, which is the ground state $|0\rangle$. Working in the Schrödinger picture, from the above transformations in equations 1.14 to 1.21, we can derive the action of the Interferometer $\hat{\mathbf{\Gamma}} = (\mathbf{B\hat{S}2})(\mathbf{\hat{P}S})(\mathbf{B\hat{S}1})$, where $\mathbf{B\hat{S}1}$ and $\mathbf{B\hat{S}2}$ are the first and the second actions of the Beam Splitter respectively, and $\mathbf{\hat{P}S}$ is the action of an assumed phase shifter.

$$\mathbf{B\hat{S}1} = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \mathbf{B\hat{S}2} \tag{1.22}$$

$$\mathbf{\hat{P}S} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \tag{1.23}$$

Thus,

$$\begin{aligned} |\Psi_{6,7}\rangle &= \hat{\mathbf{\Gamma}} |\alpha\rangle_0 |0\rangle_1 \\ &= (\mathbf{B\hat{S}2})(\mathbf{\hat{P}S})(\mathbf{B\hat{S}1}) \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \alpha - \alpha e^{-i\phi} \\ i\alpha + i\alpha e^{-i\phi} \end{pmatrix} \\ \Rightarrow |\Psi_{6,7}\rangle &= \left| \frac{\alpha - \alpha e^{-i\phi}}{2} \right\rangle_6 \left| \frac{i\alpha + i\alpha e^{-i\phi}}{2} \right\rangle_7 \end{aligned} \tag{1.24}$$

Ports 6 and 7 are output ports, light from which is incident onto detectors to get an interference pattern, which is denoted by $\left\langle \hat{\mathbf{N}} \right\rangle = \left\langle \hat{\mathbf{N}}_7 \right\rangle - \left\langle \hat{\mathbf{N}}_6 \right\rangle$, where

$$\begin{aligned} \hat{\mathbf{N}}_i &= \hat{\boldsymbol{a}}_i^\dagger \hat{\boldsymbol{a}}_i. &(1.25)\\ \text{Thus, } \left\langle \hat{\mathbf{N}}_6 \right\rangle &= \langle \Psi_{6,7} | \hat{\mathbf{N}}_6 | \Psi_{6,7} \rangle = \|\alpha\|^2 \sin^2 \frac{\phi}{2} &(1.26)\\ \text{and } \left\langle \hat{\mathbf{N}}_7 \right\rangle &= \langle \Psi_{6,7} | \hat{\mathbf{N}}_7 | \Psi_{6,7} \rangle = \|\alpha\|^2 \cos^2 \frac{\phi}{2}. &(1.27)\\ \Rightarrow \left\langle \hat{\mathbf{N}} \right\rangle &= \|\alpha\|^2 \cos\phi. &(1.28) \end{aligned}$$

Therefore, from the interference pattern, we can deduce the phase difference that is induced by the arms of the Interferometer. In Gravitational Wave Detectors, the lengths of the arms are equal, i. e., $l_1 = l_2$, which means that ϕ is zero. However, a passing gravitational wave introduces changes in arm lengths, and then a change in interference pattern can be observed according to equation 1.28. However, there is an error associated with this measurment, and that is where we shall go next.

1.1 Shot Noise

The previous section focused on counting the photon number as a way to measure the phase shift the two light beams undergo with respect to each other. However, with each measurement, there exists an associated uncertainty, and this is even more true in a Quantum framework.

Given that our intended final measurement is the difference in lengths of the arms, we write the phase difference ϕ between the two light beams in the respective arms as

$$\phi = \frac{z\omega}{c}, \tag{1.29}$$

where $z = l_1 - l_2$ is the difference between the lengths of the two arms, ω is the frequency of light used, and c is the speed of light. As the average trips the light would make inside the interferometer will remain the same for all time, we are taking it to be one. The error $\hat{\mathbf{z}}$ depends on the error in ϕ via the radiation pressure error ($\Delta\hat{\mathbf{z}}_{rp}$) - which arises due to the bombardment of photons on the mirrors and momentum they exchange - and on photon counting error($\Delta\hat{\mathbf{z}}_{pc}$) - which arises from the power fluctuations in the source of light (which is where the name Shot Noise comes from). We assume that the these two errors are independent of each other, and write

$$\Delta^2\hat{\mathbf{z}} = \Delta^2\hat{\mathbf{z}}_{rp} + \Delta^2\hat{\mathbf{z}}_{pc}. \tag{1.30}$$

These errors are derived below, starting with the radiation pressure error.

1.1.1 Radiation Pressure Error

Intuitively, an error associated with reflection will depend on the number of photons striking it, and thereby will be related to input power fluctuations. However, given that the Beam Splitter splits the input power equally into the two arms, it should distribute the error in pressure equally, thereby cancelling it out for an interference pattern, which is essentially a subtracting operation. However, Caves [Caves 81] showed that the Radiation Pressure error was not due to power fluctuations, but rather due to interference of the vacuum injected from port 1.

Radiation pressure $\Delta\hat{\mathbf{z}}_{rp}$ is then related to the momentum exchanged as

$$\Delta\hat{\mathbf{z}}_{rp} = \frac{\Delta\hat{\mathbf{p}}\tau}{2m}, \tag{1.31}$$

where τ is the time interval under consideration. The momentum exchanged depends only on the number of photons entering the respective arms.

$$\begin{aligned} \langle\hat{\mathbf{p}}\rangle &= \frac{2\hbar\omega}{c}\left\langle \hat{\boldsymbol{a}}_3^\dagger\hat{\boldsymbol{a}}_3 - \hat{\boldsymbol{a}}_2^\dagger\hat{\boldsymbol{a}}_2 \right\rangle \\ &= \frac{2\hbar\omega}{c}\left\langle \hat{\boldsymbol{a}}_1^\dagger\hat{\boldsymbol{a}}_0 - \hat{\boldsymbol{a}}_0^\dagger\hat{\boldsymbol{a}}_1 \right\rangle \quad \text{(from equations 1.16-1.17)} \qquad (1.32) \\ &= 0 \qquad (1.33) \end{aligned}$$

which confirms our intuition of the power being distribution being symmetric and cancelling each other out. However, $\langle\Delta^2\hat{\mathbf{p}}\rangle = \langle\hat{\mathbf{p}}^2\rangle - \langle\hat{\mathbf{p}}\rangle^2$ (from now on, the uncertainty expectation values such as $\langle\Delta^2\hat{\mathbf{p}}\rangle$ shall be written as $\Delta^2 p$), and for $\langle\hat{\mathbf{p}}^2\rangle$, we find

$$\begin{aligned}\left\langle\hat{\mathbf{p}}^2\right\rangle &= \frac{4\hbar^2\omega^2}{c^2}\left\langle\hat{\boldsymbol{a}}_1^\dagger\hat{\boldsymbol{a}}_0\hat{\boldsymbol{a}}_1^\dagger\hat{\boldsymbol{a}}_0 - \hat{\boldsymbol{a}}_1^\dagger\hat{\boldsymbol{a}}_0\hat{\boldsymbol{a}}_0^\dagger\hat{\boldsymbol{a}}_1 - \hat{\boldsymbol{a}}_0^\dagger\hat{\boldsymbol{a}}_1\hat{\boldsymbol{a}}_1^\dagger\hat{\boldsymbol{a}}_0 + \hat{\boldsymbol{a}}_0^\dagger\hat{\boldsymbol{a}}_1\hat{\boldsymbol{a}}_0^\dagger\hat{\boldsymbol{a}}_1\right\rangle \\ &= \frac{4\hbar^2\omega^2}{c^2}|\alpha|^2 \qquad (1.34)\\ \Rightarrow \Delta^2 p &= \frac{4\hbar^2\omega^2}{c^2}|\alpha|^2 \qquad (1.35)\\ \Rightarrow \Delta\hat{\mathbf{z}}_{rp} &= \frac{\hbar\omega\tau}{mc}|\alpha|. \qquad (1.36)\end{aligned}$$

1.1.2 Photon Counting Error

Unlike the radiation pressure error, where the expectation value of pressure difference on the two mirrors itself was zero, the value of $\left\langle\hat{\mathbf{N}}\right\rangle$ is not zero, and is also dependent on ϕ (equation 1.28). Thus, the error will depend on the phase accumulated. Therefore, the entire evolution of light in the Interferometer needs to be considered to derive the error in photon counting. From equation 1.28, we have

$$\left\langle\hat{\mathbf{N}}\right\rangle = |\alpha|^2\cos\phi\,.$$

and from equation 1.24, we can find out $\left\langle\hat{\mathbf{N}}^2\right\rangle$

$$\begin{aligned}\left\langle N^2\right\rangle &= |\alpha|^2(1+|\alpha|^2\cos^2\phi) \qquad (1.37)\\ \Delta^2 N &= \Delta^2 N_7 + \Delta^2 N_6 \\ &= |\alpha|^2. \qquad (1.38)\end{aligned}$$

As the detected number of photons at a time depends on the phase accumulated, we have $\Delta N = \Delta\phi\left[\frac{\partial N}{\partial\phi}\right]$. But also, since $\phi = \frac{2z\omega}{c}$, we have $\Delta\phi = \Delta z\frac{2\omega}{c}$. Thus, if we only consider contribution from photon counting error, we have

$$\begin{aligned}\Delta z_{pc} &= \frac{c}{2\omega}\Delta N\left[\frac{\partial N}{\partial\phi}\right]^{-1} \\ &= \frac{c}{2\omega}\frac{1}{|\alpha|}\frac{1}{\sin\phi} \qquad (1.39)\end{aligned}$$

which is the inverse relationship between the error and number of photons than the one we found coming from radiation pressure error.

Hence we see that if we increase the power of the source, it will decrease the error coming from $\Delta z_{pc},$, but it shall increase the contribution from $\Delta(z)_{rp}$. The overall error Δz at $\cos\phi = 0$ is

$$\Delta^2 z = \Delta^2 z_{pc} + \Delta^2 z_{rp} \tag{1.40}$$

$$= \frac{c^2}{4\omega^2}\frac{1}{|\alpha|^2} + \frac{\hbar^2\omega^2\tau^2}{m^2c^2}|\alpha|^2. \tag{1.41}$$

Minimizing this error with respect the laser power $|\alpha|^2$ gives optimal power P_0 for an optimal error.

$$|\alpha|^2_{min} = P_0 = \frac{mc^2}{2\hbar\omega^2\tau}[\text{Caves 81}] \tag{1.42}$$

$$\Rightarrow \Delta z_{min} = \sqrt{\frac{\hbar\tau}{m}}, \tag{1.43}$$

which is incidently equal to the SQL in position measurement of a free mass.

For a gravitational wave detector with mirros of mass $m \approx 10^5 g$, interaction time $\tau \approx 2\times10^{-3}$ sec, $\omega \approx 4\times10^{15}$ rad sec^{-1}, this minimum optimal power comes out to be approximately $8\times10^3 w$[Caves 81], which is a power far higher than laser power available for GWDs. At available powers, phase resolution of these interferometers isn't limited by the SQL, but rather by the photon counting error, popularly called as the Shot Noise limit, which shows a scaling of

$$\Delta z_{SNL} \propto \frac{1}{\sqrt{N}} = \frac{1}{|\alpha|}. \tag{1.44}$$

The Shot Noise Limit, thus, comes from the limitations of the experimental design. The true limit, called the Heisenberg Limit, is derived below.

1.1.3 The Heisenberg Limit

The so called Heisenberg Limit on the minimum error in photon counting is calculated from the energy-time uncertainty relation as follows. Let $\hat{\mathbf{B}}$ be a time independent operator. Then the Ehrenfest Theorem gives

$$\frac{d}{dt}\left\langle \hat{\mathbf{B}} \right\rangle = -\frac{i}{\hbar}\left\langle [\hat{\mathbf{H}}, \hat{\mathbf{B}}] \right\rangle \tag{1.45}$$

At the same time, the uncertainty relation says

$$\Delta H \Delta B \geq \frac{1}{2}\left|\left\langle [\hat{\mathbf{H}}, \hat{\mathbf{B}}] \right\rangle\right|.$$

Putting the value of $[\hat{\mathbf{H}}, \hat{\mathbf{B}}]$ from equation 1.45, and writing $\Delta\hat{\mathbf{B}} = \Delta t \frac{d}{dt}\left\langle \hat{\mathbf{B}} \right\rangle$ we get,

$$\begin{aligned} \left\langle \Delta\hat{\mathbf{H}} \right\rangle \Delta t \frac{d}{dt}\left\langle \hat{\mathbf{B}} \right\rangle &\geq \frac{\hbar}{2}\frac{d}{dt}\left\langle \hat{\mathbf{B}} \right\rangle \\ \Rightarrow \Delta\hat{\mathbf{H}}\Delta t &\geq \frac{\hbar}{2} \end{aligned} \tag{1.46}$$

We know $\hat{\mathbf{H}} = \hbar\omega(\hat{\mathbf{N}} + \frac{1}{2})$ for light waves, and Δt can be written as $\Delta t = \frac{\Delta\phi}{\omega}$, which gives us

$$(\hbar\omega\Delta N)(\omega\Delta\phi) \geq \frac{\hbar}{2} \tag{1.47}$$

$$\Rightarrow \Delta N \Delta\phi \geq \frac{1}{2}. \tag{1.48}$$

Given that the largest value ΔN can have is N, we find that the minimum possible sensitivity to phase is

$$\Delta\phi \propto \frac{1}{N} \tag{1.49}$$

or, for a coherent state $|\alpha\rangle$,

$$\Delta\phi \propto \frac{1}{|\alpha|^2}. \tag{1.50}$$

This is called as the Heisenberg Limit. As we can see from equation 1.44, this limit is an order of magnitude smaller than the Shot Noise. Deriving as it does from the Uncertainty Relation and not from any limitations of the apparatus of the experiment or the type of light used, this is thought to be a fundamental limit[Giovannetti 04] that cannot be surpassed, unlike the Shot Noise Limit. In the next section, we shall examine the scheme given by C. M. Caves to surpass the Shot Noise Limit.

1.2 Squeezing the Vacuum to Beat the SNL

As we have seen, the experimental regime sets the Shot Noise Limit on the sensitivity to phase change in an Interferometer. In 1982, C. M. Caves effectively opened the field of Quantum Metrology by proposing a method to use squeezed vacuum states to beat the Shot Noise Limit.

It was Caves who proved that the radiation pressure error was arising due to the vacuum state injected from the unused port interfering with the coherent state input from port 0. To reduce the overall error, Caves proposed the input state be

$$|\Psi\rangle = |\alpha\rangle_0 \, |\xi\rangle_1$$

in place of $|\alpha\rangle_0 \, |0\rangle_1$, where $|\xi\rangle$ denotes a squeezed state as defined in 1.0.1. For this input, we find that both the photon counting error and radiation pressure error get modified.

1.2.1 Photon Counting Error

For an input of $|\psi\rangle = |\alpha\rangle_1 \, |\xi\rangle_0$, where $\xi = re^{i\theta}$ we get

$$\begin{aligned} \left\langle \hat{\mathbf{N}} \right\rangle &= \left\langle \hat{\mathbf{N}}_7 - \hat{\mathbf{N}}_6 \right\rangle = (|\alpha|^2 - \sinh^2 r)\cos\phi, \\ \Delta^2 N &= \Delta^2 N_7 + \Delta^2 N_6 \\ &= |\alpha|^2 \cos^2\phi + 2\cos^2\phi \cosh^2 r \sinh^2 r + sin^2\phi(|\alpha|^2 e^{-2r} + \sinh^2 r) \end{aligned}$$

then, going as before,

$$\begin{aligned} \Delta\hat{\mathbf{z}}_{pc} &= \frac{c}{2\omega}\Delta N \left[\frac{\partial N}{\partial\phi}\right]^{-1} \\ &= \left[\frac{\cot^4\phi}{|\alpha|^2} + \frac{2\cot^2\phi\cosh^2 r\sinh^2 r}{|\alpha|^4} + \frac{e^{-2r}}{|alpha|^2} + \frac{\sinh^2 r}{|\alpha|^4}\right]^{1/2} \end{aligned} \tag{1.51}$$

Equation 1.51 has terms coming from the individual powers of the light sources, as well as terms coming from the interference between them. Working at the point

$\cos\phi = 0$ in the interference pattern, and under the assumption that the power of the source of the squeezed vacuum state is very low as compared to the power of the coherent source, i. e., $\sinh^2 r \ll |\alpha|^2$, we get

$$(\Delta\hat{\mathbf{z}})_{pc} \simeq \frac{c}{2\omega}\frac{1}{\alpha}\frac{1}{e^r}. \tag{1.52}$$

Thus, the photon counting error is reduced when squeezing is introduced with $r \geq 0$. We shall return to this quantity after examining the change in radiation pressure error.

1.2.2 Radiation Pressure Error

Again, after considering the number of photons entering the different arms, we get the radiation pressure difference as

$$\begin{aligned} \langle\hat{\mathbf{p}}\rangle &= \langle\xi,\alpha|\hat{\boldsymbol{a}}_3^\dagger\hat{\boldsymbol{a}}_3 - \hat{\boldsymbol{a}}_2^\dagger\hat{\boldsymbol{a}}_2|\xi,\alpha\rangle_{0,1} \\ &= 0, \end{aligned}$$

and the radiation pressure error as

$$(\Delta\hat{\mathbf{p}})^2 = \frac{4\hbar^2\omega^2}{c^2}(|\alpha|^2 e^{2r} + \sinh^2 r) \tag{1.53}$$

$$\Rightarrow \Delta\hat{\mathbf{z}}_{rp} = \frac{\hbar\omega\tau}{mc}\left[|\alpha|^2 e^{2r} + \sinh^2 r\right]^{\frac{1}{2}} \tag{1.54}$$

We again see inverse relationship of the radiation pressure error and the photon counting error via the power of the light sources. In this case, increasing the squeezing (r) increases the radiation pressure while decreasing the photon counting error. However, we find that the required power of the coherent source for the minimum total error also changes with a change in squeezing of the vacuum source.

$$\begin{aligned}
\Delta\hat{\mathbf{z}} &= \left[(\Delta z)^2_{pc} + (\Delta z)^2_{rp}\right]^{\frac{1}{2}} && (1.55)\\
&= \left[\frac{c^2}{4\omega^2}\frac{1}{|\alpha|^2 e^{2r}} + \frac{\hbar^2\omega^2\tau^2}{m^2c^2}|\alpha|^2 e^{2r}\right]^{\frac{1}{2}}. && (1.56)\\
\Rightarrow |\alpha|^2_{opt} &\simeq \frac{mc^2}{2\hbar\omega^2\tau} \times e^{-2r} && (1.57)\\
&\simeq |\alpha|^2_0 \times e^{-2r} && (1.58)
\end{aligned}$$

Thus, we can lower the optimum power by increasing the squeezing parameter. The minimum error again is

$$\Delta z_{min} = \sqrt{\frac{\hbar\tau}{m}}, \qquad (1.59)$$

which is equal to the SQL in determining position.

Thus, we have reduced the required power to get the minimum error. The Shot Noise Limit scaled as $\frac{1}{|\alpha|}$, but, from the equation (of optimum power), we can see that the new limit scales as $\sqrt{\hbar\tau/m} \propto e^r/|\alpha|_{opt}$. However, because we have neglected the $\sinh^2 r$ term in the photon counting error, there is a limit on how much we can reduce the optimum power by increasing r. Caves gives $r_{max} = \frac{1}{3}\ln|\alpha|_0$[Caves 81] to be the maximum value of squeezing before the term coming from the power of the squeezed source begins dominating the $e^{-r}/|\alpha|$ term in optimum sensitivity. At this level of squeezing, we find

$$\begin{aligned}
(|\alpha|^2_{opt})_{min} &\simeq (|\alpha|^2_0)^{\frac{4}{3}}. && (1.60)\\
\text{Thus} & && (1.61)\\
N_{min} &\propto \left[\frac{\hbar\omega}{\tau}\right]^{-\frac{2}{3}} = (\Delta z_{min})^{-\frac{4}{3}} &&\\
\text{or,} & && (1.62)\\
\Delta z_{min} &\propto \frac{1}{(N_{min})^{3/4}} && (1.63)
\end{aligned}$$

Therefore, the minimum error scales with the $N^{-3/4}$, which, while an improvement on the Shot Noise scaling of $N^{-1/2}$, hasn't yet reached the Heisenberg Limit of N^{-1}

scaling. In the next section, we shall examine another approach to increase the phase sensitivity of the interferometers, this time reaching the Heisenberg Limit.

1.3 NOON States

Coherent states of light are easy to produce. In fact, due to the statistics they follow, they are called the most classical of the quantum states. However, light comes in many different colors, and the most fundamental of them is the number state.

1.3.1 Number States

We have, of course, already been using a number state. $|0\rangle$ is the vacuum state, or in other words, the state that has $n = 0$ photons in it. Defined as it is in a similar vein to the Simple Harmonic Oscillator in a metallic cavity which is then extended to infinity, there are creation and annihilation operators corresponding to the states of light as defined in section 1.1. They act on the number state $|n\rangle$ as

$$\hat{\boldsymbol{a}}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle\,; \qquad \hat{\boldsymbol{a}}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,. \tag{1.64}$$

Thus, the excited state $|n\rangle$ is related to the ground state $|0\rangle$ as

$$|n\rangle = \frac{(\hat{\boldsymbol{a}}^\dagger)^n}{\sqrt{n!}}\,|0\rangle \tag{1.65}$$

More imprtant for us, however, while studying the evolution of light in the interferometer, is the time evolution of light, or, in the Heisenberg picture, the time evolution of the creation and the annihilation operator. Given that the two operators follow the commutator relation $\left[\hat{\boldsymbol{a}}, \hat{\boldsymbol{a}}^\dagger\right] = 1$, we can see how these operators evolve with time in the Heisenberg picture by the Heisenberg Equation of Motion

$$\frac{d\hat{\boldsymbol{a}}(t)}{dt} = \frac{i}{\hbar}\left[\hat{\mathbf{H}}, \hat{\boldsymbol{a}}\right], \tag{1.66}$$

where $\hat{\mathbf{H}} = \hbar\omega\left(\hat{\boldsymbol{a}}^\dagger\hat{\boldsymbol{a}} + \frac{1}{2}\right)$ is the Hamiltonian by which the photon/number state evolves. If $\hat{\boldsymbol{a}}_0$ is the annihilation operator at $t = 0$, then we find

$$\hat{\boldsymbol{a}}(t) = \hat{\boldsymbol{a}}_0 e^{-i\omega t} \quad \hat{\boldsymbol{a}} \dagger (t) = \hat{\boldsymbol{a}}_0^\dagger e^{i\omega t}.$$

Writing $\omega t = \phi$, we get

$$\hat{\boldsymbol{a}}(t) = \hat{\boldsymbol{a}}_0 e^{-i\phi}; \qquad \hat{\boldsymbol{a}}^\dagger(t) = \hat{\boldsymbol{a}}_0^\dagger e^{i\phi} \tag{1.67}$$

This type of evolution leads to some interesting reasults. The major one being the difference between the phase accumulation by coherent light and the number states. The expression of Coherent Light, $|\alpha, t = 0\rangle = e^{\alpha \hat{\boldsymbol{a}}_0^\dagger - \alpha^* \hat{\boldsymbol{a}}_0} |0\rangle$ leads to time evolution as

$$\begin{aligned} |\alpha, t\rangle &= e^{\alpha \hat{\boldsymbol{a}}_0^\dagger e^{i\phi} - \alpha^* \hat{\boldsymbol{a}}_0 e^{-i\phi}} |0\rangle \\ &= e^{(\alpha e^{i\phi}) \hat{\boldsymbol{a}}_0^\dagger - (\alpha e^{i\phi})^* \hat{\boldsymbol{a}}_0} |0\rangle \\ &= \left| e^{i\phi} \alpha \right\rangle . \end{aligned} \tag{1.68}$$

But the expression is very different for the number state $|n\rangle$

$$\begin{aligned} |n, t\rangle &= \frac{(\hat{\boldsymbol{a}}^\dagger e^{i\phi})^n}{\sqrt{n!}} |0\rangle \\ &= e^{i\,n\,\phi} \frac{(\hat{\boldsymbol{a}}^\dagger)^n}{\sqrt{n!}} |0\rangle \\ &= e^{i\,n\,\phi} |n, t = 0\rangle \end{aligned} \tag{1.69}$$

Thus, we see the phase accumulation in the number state depends on the number of photons in the number state. More the number of photons, more will be the accumulated phase. This result is drastically different than the result for coherent light.

It is now easy to see why the nonclassical light states which we call as number states might be tried in interferometers. Couple this with the goal of increasing phase sensitivity for a number of photons used in an Interferometer, and we get the NOON states.

1.3.2 NOON State Interferometry

NOON states have interested scientists as being a type of Schrödinger Cat States since the 1980s. But the interest in them was regenerated when they were rediscovered by J. P. dowling in 2000 in connection with its properties with regards to lithography[P. Dowling 09].

NOON states are maximally entangled states. For modes a and b, they are represented as

$$|NOON\rangle = \frac{|N\rangle_a |0\rangle_b + |0\rangle_a |N\rangle_b}{\sqrt{2}}. \tag{1.70}$$

From the representation, it is clear why these states would be called Schrödinger Cat States. The photon NOON State is a superposition of photon states such that detecting the photons in one mode will make the number of photons in the other mode collapse to zero. The cat detected as alive won't be dead, and vice versa.

In the Michelson Interferometer, to exploit the properties of the NOON states, the treatment has to be different than the one for coherent light. NOON states need to be introduced into the arms 2 and 3 *after* the first beam splitter action, and the evolved states need to be measured *before* the second beam splitter action. That is, the evolution will be

$$\frac{|N\rangle_2 |0\rangle_3 + |0\rangle_2 |N\rangle_3}{\sqrt{2}} \to \frac{|N\rangle_4 |0\rangle_5 + e^{-iN\phi} |0\rangle_4 |N\rangle_5}{\sqrt{2}}$$

This state then needs to be measured for the operator

$$\begin{aligned} \hat{\mathbf{A}} &= |N\rangle_a |0\rangle_b \langle 0|_a \langle N|_b + |0\rangle_a |N\rangle_b \langle N|_a \langle 0|_b \\ &= |N, 0\rangle \langle 0, N| + |0, N\rangle \langle N, 0| . \end{aligned} \tag{1.71}$$

The expectation value of $\hat{\mathbf{A}}$ is then found to be

$$\left\langle \hat{\mathbf{A}} \right\rangle = N \cos N\phi, \tag{1.72}$$

which is now giving us a signal that changes faster by a factor of N than the signal in the coherent light input case. Calculating the uncertainty of $\hat{\mathbf{A}}$ gives us

$$\Delta^2 A = N \sin^2 N\phi. \tag{1.73}$$

Thus, evaluating $\Delta\phi = \Delta A \left[\frac{\partial A}{\partial \phi}\right]^{-1}$, we find

$$\Delta\phi = \frac{1}{N}, \tag{1.74}$$

which is the Heisenberg Limit.

However, while the NOON state interferometry is a quadratic improvement on classical interferometry , in reality, by 2004, only NOON states uptil $N = 4$ photons had been kept going long enough to demonstrate the Heisenberg Limit sensitivity[P. Dowling 09]. A similar picture exists in spectroscopy, where NOON states comprising of atom spin states with $N = 6$ been successfully used at the Heisnberg Limit in 2011[Pezzè 18].

The difficulty with NOON states is, as with squeezed states of light, not only protecting the entanglement until after the measurement process is complete, but also the generation of these states. After arranging single photon inputs at each ports, itself a difficult task, an input of $|\Psi\rangle = |1\rangle_0 |1\rangle_1$ to the Interferometer will, due to the Hong-Ou-Mandel effect, introduce NOON states into the Interferomter with $N = 2$. But for $N \geq 2$, the 50-50 Beam Splitter produces states of the form[P. Dowling 09]

$$\left|\frac{N}{2}\right\rangle_0 \left|\frac{N}{2}\right\rangle_1 \xrightarrow{BS_I} C_N |N\rangle_2 |0\rangle_3 + C_{N-2} |N-2\rangle_2 |2\rangle_3 + \cdots + C_{N-2} |2\rangle_2 |N-2\rangle_3 + C_N |0\rangle_2 |N\rangle_3 , \tag{1.75}$$

where the states with photons $\leq N$ cannot achieve Heisenberg Limit. NOON states then are needed to be produced by unwanted states by post-selection, or by simply not processing the signals further when the detectors click in both arms of the Interferometer. But these types of approaches reduce the odds of NOON states appearing, thereby increasing the number of photons used. And all this only for NOON states with a low N.

1.4 Conclusion

While squeezing and producing NOON states might seem difficult, the SNL itself allows for the LIGO interferometers to reach a sensitivity of a $10^{-18}m$ ($\Delta\phi = 10^{-12}$). This sensitivity is enough to detect gravitational waves, expected themselves to bring about a relative shift in the position of the mirrors on the scale of $10^{-18}m$. In fact, detection of the Gravitational Waves announced in 2016 did not use any quantum tricks on the light used, instead they eliminated as many sources of classical noise as they could and reached the SNL.

However, this does not negate the importance of the techniques developed above. Squeezing and NOON states are both used in spectroscopy[Pezzè 18], and NOON states allow us to beat the Rayleigh Diffraction Limit in Lithography to allow for a sensitivity of $\frac{\lambda}{N}$[N. Boto 99]. And there exist methods other than NOON states and squeezed vacuum to allow for precision higher than the respective classical limits in different fields[Barnett 03],[P. Dowling 09], the study of which were beyond the scope of my thesis.

We shall then leave our discussion of Interferometers here, for the time being, and turn to a more somber field, one that seems only to be progressing on paper. Nonetheless, the question of surpassing the SQL in measurement of position is just as important to the realm of Interferometers as it is to anything else. In the next chapter, I shall give an overview of what upheavals that area of Quantum Metrology has underwent.

Chapter 2

SQL in Monitoring Position of a Mass

Introduction

To monitor a force acting on a mass, one needs to know the position of the mass at at least two points in space. But such a measurement is limited by what is called the Standard Quantum Limit (SQL) in monitoring position of a free mass.

The SQL has multiple formulations, some of them voided, some of them beaten, some of them contested. The multitude of papers on the topic is in sharp contrast with the experimental advances in realizing any of these measurements, however. Indeed, while the discussions of breaking the Shot Noise Limit and the Standard Quantum Limit began around similar times and around the same topic of detecting Gravitational Waves, it is only in the area of Shot Noise that we have significant experiments to compliment our theoretical achievements.

Not only is the position of a free mass continuously changing through feedback from the particle's momentum, the schemes proposed to measure the position of particles through quantum interactions have no physical realizations as of yet, and this is the simple reason for the situation described in the above paragraph. So for the time being, we shall ignore the question of whether the interactions and situations described in this thesis can have a physical realization. Indeed, even the question of what is measurement itself will be raised in the course of the review this thesis will present. And that is where we must start. The next two subsections will develop the paradigm of weak and approximate measurements, which will set the stage for the derivation of the SQL.

2.0.1 Von Neumann Model of Approximate Measurements

Unlike spin or the number of photons, position is a continuous variable. It is only for discrete variables that we have found state measurement with infinite precision to be possible["Ozawa" 15], whereas for continuous variables, the most one can do is to determine their values within an interval with a probability. One of the way to measure quantum variables was outlined by Von Neumann, presented here by following the treatment given by Ozawa["Ozawa" 15] and Wheeler[Wheeler 12] in the context of continuous variables, more particularly, the measurement of position.

Von Neumann defined two kinds of measurements. The Measurement of the First kind and the Measurement of the Second Kind. All measurements involve a system which is to be measured, and a detector which will perform this measurement. The Measurement of the Second Kind is the one where the quantum state of the system is destroyed through its measurement by a detector. The macroscopic measurements of photon in a photon counting detector are a measurement of the second kind. The measurement of the first kind, however, involves quantum interaction between a detector system and the target system in such a way that the target system retains its quantum state, and a measurement of the second kind of the detector gives us information about the target system, acquired by the detector during the the interaction. As the current technology stands, a measurement of the first kind always has to be preceeded by a measurement of the second kind. However, recently a photon has been measured without being destroyed[Nogues 99]), so this might change one day.

Von Neumann model of approximate measurement is a measurement of the first kind, and involves a linear coupling between two properties of the probe and the target system respectively, the justification being that the system and the detector interact by some properties associated with them. For a target system ψ with variables $\{\hat{\mathbf{x}}, \hat{\mathbf{p}}\}$ and a probe ϕ described with $\{\hat{\mathbf{Q}}, \hat{\mathbf{P}}\}$, a standard coupling used to detect the position of the target system is

$$H_{int} = k\hat{\mathbf{x}}\hat{\mathbf{P}} \tag{2.1}$$

where k is the strength of the coupling. The coupling is supposed to be brief enough that the free evolution of two particles is ignored. In other words, the impulse approximation is used and the enviornment is ignored. The standard procedure is to solve this Hamiltonian by using Schrödinger Equation, but because it will be relevant in this article later, I shall present a solution using the Heisenberg Equation of Motion as well.

Schrödinger Equation

If the composite system of the target-detector is $\Psi(x, Q, t)$, then at just before the interaction is turned 'on', the state is $\Psi(x, Q, 0) = \psi(x)\phi(Q)$ (where, the two quantum states are assumed to be saperable because they are two different particles). Then, working in the Schrödinger Picture, the Schrödinger Equation says

$$\frac{\partial \Psi}{\partial t} = -iH\Psi, \tag{2.2}$$

where $\hbar$ is set to 1, here and henceforth in this article, unless it is taken for the sake of complying with the standard equation. The subscript of H_{int} shall also be supressed. Putting in the values, we get,

$$\frac{\partial \Psi}{\partial t} = -k\hat{\mathbf{x}}\frac{\partial}{\partial Q}\psi(x)\phi(Q)$$

Taking Fourier Transform with respect to Q,

$$\int_{-\infty}^{\infty} \frac{\partial \Psi}{\partial t} e^{-iPQ} dQ = -kx \int_{-\infty}^{\infty} \Psi(x, Q) e^{-iPQ} dQ$$

$$\Rightarrow \frac{\partial \hat{\mathbf{\Psi}}}{\partial t} = -ikxP\hat{\mathbf{\Psi}}$$

$$\text{where } \hat{\mathbf{\Psi}} = \int_{-\infty}^{\infty} \psi e^{-iPQ} dQ$$

$$\Rightarrow \hat{\mathbf{\Psi}}(P, x, t) = A_0 e^{-ikxpt}.$$

Thus, given that $\Psi(x, Q, 0) = \psi(x)\phi(Q)$,

$$\Psi(x, Q, t) = \int_{-\infty}^{\infty} A_0 e^{-ikxPt + iPQ} dP = F(x, Q - xt/\tau) \tag{2.3}$$

where k is taken to be equal to the reciprocal of the interaction time τ, here and henceforth wherever needed. The final state at the end of the interaction then is

$$\Psi(x, Q, \tau) = \psi(x)\phi(Q - x) \tag{2.4}$$

If we prepare the probe ϕ in such a way that at the start of the interaction, the expectation value of its position is 0, i.e., $\langle \hat{\mathbf{Q}} \rangle_0 = \left\langle \phi(Q, t = 0) \middle| \hat{\mathbf{Q}} \middle| \phi(Q, t = 0) \right\rangle = 0$, then, after the interaction is complete and the position of ϕ is measured destructively, we find $\langle \hat{\mathbf{Q}} \rangle = x$, from which we can infer that the position of ψ was x at the time of the measurement.

Heisenberg Equation of Motion

Heisenberg Equation of Motion says that, in the Heisenberg Picture, for a time independent operator $\hat{\mathbf{A}}$ on a system whose evolution followes the Hamiltonian H,

$$\frac{d\hat{\mathbf{A}}(t)}{dt} = i\left[\hat{\mathbf{H}}, \hat{\mathbf{A}}\right]. \tag{2.5}$$

For our system $\Psi(x, Q) = \psi(x)\phi(Q)$ and $H_{int} = k\hat{\mathbf{x}}\hat{\mathbf{P}}$, The Heisenberg Equations of Motion are

$$\begin{aligned} \frac{d\hat{\mathbf{x}}(t)}{dt} &= i\left[\hat{\mathbf{H}}, \hat{\mathbf{x}}\right] = 0 \\ \Rightarrow \hat{\mathbf{x}}(t) &= \hat{\mathbf{x}}(0) \\ \frac{d\hat{\mathbf{Q}}(t)}{dt} &= i\left[\hat{\mathbf{H}}, \hat{\mathbf{Q}}\right] = k\hat{\mathbf{x}} \\ \Rightarrow \hat{\mathbf{Q}}(t) &= \hat{\mathbf{Q}}(0) + k\hat{\mathbf{x}}(0)t \end{aligned} \tag{2.6}$$

From this, we can directly get the expectation value of $\hat{\mathbf{Q}}(\tau)$ to be $x(0)$ with the assumptions that we had made while solving in the Schrödinger Picture, that is, $\hat{\mathbf{Q}}(0) = 0$ and $\tau = 1/k$. However, if we want to go back from here to the Schrödinger Picture, we must realize that the operator Q in the Schrödinger Picture is the same at time t and 0. To reconcile this with equation 2.3, we can look at this as a shift of origin. The state ϕ has moved from the point Q to $Q - kxt$. Thus, we have

$$\Psi(x, Q, \tau) = \psi(x)\phi(Q - x) \quad \text{and} \quad \langle \hat{\mathbf{Q}} \rangle = x \tag{2.7}$$

From the above equations, one can easily see that the momentum of the target system isn't conserved [Kosugi 10] in all frames of references. For some reason, this is an acceptable consequence to the scientific community at large, given that I only found a reference to this fact in one paper, though I read many talking on this measurement. The largest flaw, it can be argued, is that this type of measurement doesn't have a physical realization as of yet. This, of course, does not invalidate the approach. And as far as blind steps go, this one goes a long way in following quantum mechanics and increasing our knowledge of where one might find the right way.

As noted above, this measurement method has been very useful, especially in the development of the concept of weak measurement protocols which can leave the target system undisturbed while still giving us some information about it. Another area is that this type of protocol allows us to measure commutating variables simultaneously, the description of which is given below.

2.0.2 Simultaneous Measurement of Commutating Observabes

Based on the ideal measurement given by Von Neumann, Arthurs and Kelly[Arthurs 65] devised a method to measure two conujugate non-commuting variables $\hat{\mathbf{q}}$ and $\hat{\mathbf{p}}$ of a system simultaneously. Following Wheeler[Wheeler 12], we define two near-variants of $\hat{\mathbf{q}}$ and $\hat{\mathbf{p}}$

$$\hat{\mathbf{Q}} = \hat{\mathbf{q}} + \hat{\mathbf{A}} \quad (2.8)$$

$$\hat{\mathbf{P}} = \hat{\mathbf{p}} + \hat{\mathbf{B}}, \quad (2.9)$$

where $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ model the noise terms coming due to the imprecise measurements of $\hat{\mathbf{q}}$ and $\hat{\mathbf{p}}$ we are making by measuring $\hat{\mathbf{Q}}$ and $\hat{\mathbf{P}}$. On these operators, impose the condition that they can be simultaneously measured:

$$\begin{aligned} \left[\hat{\mathbf{Q}}, \hat{\mathbf{P}}\right] &= [\hat{\mathbf{q}}, \hat{\mathbf{p}}] + \left[\hat{\mathbf{q}}, \hat{\mathbf{B}}\right] + \left[\hat{\mathbf{A}}, \hat{\mathbf{p}}\right] + \left[\hat{\mathbf{A}}, \hat{\mathbf{B}}\right] = 0 \\ \Rightarrow \quad \left[\hat{\mathbf{A}}, \hat{\mathbf{B}}\right] &= -iI - \left[\hat{\mathbf{q}}, \hat{\mathbf{B}}\right] - \left[\hat{\mathbf{A}}, \hat{\mathbf{p}}\right] \end{aligned} \quad (2.10)$$

when $\hat{\mathbf{q}}$ and $\hat{\mathbf{p}}$ are position and momentum operator respectively. Compute ΔQ and ΔP, and we have

$$\Delta^2 Q = \Delta^2 q + \langle \hat{\mathbf{A}}^2 \rangle \quad \text{and} \quad \Delta^2 P = \Delta^2 p + \langle \hat{\mathbf{B}}^2 \rangle \tag{2.11}$$

Hence,

$$\begin{aligned} \Delta^2 \hat{\mathbf{Q}} \Delta^2 \hat{\mathbf{P}} &= \Delta^2 \hat{\mathbf{q}} \Delta^2 \hat{\mathbf{p}} + 2 \frac{\Delta^2 \hat{\mathbf{q}} \langle \hat{\mathbf{B}}^2 \rangle + \Delta^2 \hat{\mathbf{p}} \langle \hat{\mathbf{A}}^2 \rangle}{2} + \langle \hat{\mathbf{A}}^2 \rangle \langle \hat{\mathbf{B}}^2 \rangle \\ &\geq \Delta^2 \hat{\mathbf{q}} \Delta^2 \hat{\mathbf{p}} + 2 \Delta \hat{\mathbf{q}} \Delta \hat{\mathbf{p}} \sqrt{\langle \hat{\mathbf{A}}^2 \rangle \langle \hat{\mathbf{B}}^2 \rangle} + \langle \hat{\mathbf{A}}^2 \rangle \langle \hat{\mathbf{B}}^2 \rangle \\ \Delta \hat{\mathbf{Q}} \Delta \hat{\mathbf{P}} &\geq \frac{1}{2} + \sqrt{\langle \hat{\mathbf{A}}^2 \rangle \langle \hat{\mathbf{B}}^2 \rangle}, \end{aligned} \tag{2.12}$$

where the AM-GM inequality has been used, and the Heisenberg Uncertainty Relation, $\Delta \hat{\mathbf{q}} \Delta \hat{\mathbf{p}} \geq \frac{1}{2}$, has been used. Assuming that the expectation value of the noise terms is zero, and assuming

$$\langle \hat{\mathbf{q}} \hat{\mathbf{A}} \rangle = \langle \hat{\mathbf{A}} \hat{\mathbf{q}} \rangle = \langle \hat{\mathbf{p}} \hat{\mathbf{B}} \rangle = \langle \hat{\mathbf{B}} \hat{\mathbf{p}} \rangle = 0 \tag{2.13}$$

we get

$$\begin{aligned} \Delta^2 \hat{\mathbf{A}} \Delta^2 \hat{\mathbf{B}} = \langle \hat{\mathbf{A}}^2 \rangle \langle \hat{\mathbf{B}}^2 \rangle &\geq \frac{1}{2} \left| \left\langle \left[\hat{\mathbf{A}}, \hat{\mathbf{B}} \right] \right\rangle \right|^2 \\ &= \frac{1}{2} \left| - \langle iI \rangle - \left\langle \left[\hat{\mathbf{q}}, \hat{\mathbf{B}} \right] \right\rangle - \left\langle \left[\hat{\mathbf{A}}, \hat{\mathbf{p}} \right] \right\rangle \right| \\ &= \frac{1}{4}. \end{aligned} \tag{2.14}$$

Putting this in equation 2.12, we get

$$\begin{aligned} \Delta \hat{\mathbf{Q}} \Delta \hat{\mathbf{P}} &\geq \Delta \hat{\mathbf{q}} \Delta \hat{\mathbf{p}} + \frac{1}{2} \\ &\geq 1 \end{aligned} \tag{2.15}$$

Thus, we find that such a simultaneous measurement will follow an uncertainty relation twice as strong as the Heisenberg Relation. Arthurs and Kelly demonstrate such a measurement for position and momentum of a system. For a system ψ in with variables $\{q, p\}$, Arthurs and Kelly give an interaction

$$H = k(xP_x - pP_y), \tag{2.16}$$

where x, P_x and y, P_y are position and momenta operators of the two detectors $M(x)$ and $N(y)$ respectively[Arthurs 65].

Though I have followed the solution of the above Hamiltonian as given by Wheeler [Wheeler 12] (Arthurs and Kelly do not make it easy to follow their solution, as none is offered), who solves it by the method of Schrödinger Equation, I shall here present a solution by the Heisenberg Equation of Motion as it is easier to follow and doesn't require *Mathematica* to be solved.

Proceeding with respect to the equation 2.5, we write the time evolution of the position and momenta operators of the entire enseble:

$$\hat{\mathbf{q}}'(t) = k\hat{\mathbf{P}}_y \qquad \hat{\mathbf{P}}'_y(t) = 0 \tag{2.17}$$

$$\hat{\mathbf{x}}'(t) = k\hat{\mathbf{q}} \qquad \hat{\mathbf{P}}'_x(t) = 0 \tag{2.18}$$

$$\hat{\mathbf{y}}'(t) = k\hat{\mathbf{p}} \qquad \hat{\mathbf{p}}'(t) = -k\hat{\mathbf{P}}_x \tag{2.19}$$

Solving these simple differential equations is much easier than solving the cumbersom, second order differential equation that appears in the Schrödinger Equation. The solutions we get, after rearranging the equations so that the original operators at time $t = 0$ appear on the left side and the operators at time t on the right, are:

$$\hat{\mathbf{q}}(0) = \hat{\mathbf{q}}(t) - k\hat{\mathbf{P}}_y(t)t \tag{2.20}$$

$$\hat{\mathbf{x}}(0) = \hat{\mathbf{x}}(t) - k\hat{\mathbf{q}}(t) + \frac{1}{2}k^2 P_y(t)t^2 \tag{2.21}$$

$$\hat{\mathbf{y}}(0) = \hat{\mathbf{y}}(t) - k\hat{\mathbf{p}}(t) + \frac{3}{2}k^2 P_x(t)t^2 \tag{2.22}$$

giving a state at the end of interaction time $\tau = 1/k$ as

$$\Psi(q, x, y, t) = \psi(q - P_y)M(x - q + \frac{1}{2}P_y)N(y - p + \frac{3}{2}P_x) \tag{2.23}$$

After this, one can follow Wheeler [Wheeler 12] to get an agreement with his answers upto a multiplicative phase factor. Or, we can continue with the Heisenberg Equations of Motion. The new operators at $t = 1/k$ when the coupling is turned off are

$$\begin{aligned}
\hat{\mathbf{q}}(\tau) &= \hat{\mathbf{q}}(0) + \hat{\mathbf{P}}_y(0) && (2.24)\\
\hat{\mathbf{p}}(\tau) &= \hat{\mathbf{p}}(0) + \hat{\mathbf{P}}_x(0) && (2.25)\\
\hat{\mathbf{x}}(\tau) &= \hat{\mathbf{x}}(0) + \hat{\mathbf{q}}(0) + \frac{1}{2}P_y(0) && (2.26)\\
\hat{\mathbf{y}}(\tau) &= \hat{\mathbf{y}}(0) + \hat{\mathbf{p}}(0) - \frac{1}{2}P_x(0) && (2.27)
\end{aligned}$$

To impose the conditions of noise described in 2.10 and 2.13, Arthurs and Kelly choose gaussian detectors M and N whose initial conditions are

$$\begin{aligned}
M(x) &= \left(\frac{2}{\pi b}\right)^{\frac{1}{4}} e^{-\frac{x^2}{b}}\\
N(y) &= \left(\frac{2b}{\pi}\right)^{\frac{1}{4}} e^{-by^2}.
\end{aligned} \qquad (2.28)$$

This immediately makes $\langle\hat{\mathbf{x}}(0)\rangle = \langle\hat{\mathbf{y}}(0)\rangle = \langle\hat{\mathbf{P}}_x(0)\rangle = \langle\hat{\mathbf{P}}_y(0)\rangle = 0$ and we see that the expectation value of $\hat{\mathbf{x}}$ is the expectation value of position of the system just before the measurement, and the expectation value of $\hat{\mathbf{y}}$ is the expectation value of the momentum of the system just before the measurement. Moreover, the new state of the system will reflect the values of $\langle\hat{\mathbf{x}}(t)\rangle$ and $\langle\hat{\mathbf{y}}(t)\rangle$ respectively in its position and momentum. That is, the state of the system after the measurement is determined by the outcome of the measurement. We can calculate the uncertainties in the projective measurements of x and y by using equations 2.24 to 2.27:

$$\begin{aligned}
\Delta^2\hat{\mathbf{x}}(t) &= \Delta^2\hat{\mathbf{x}}(0) + \Delta^2\hat{\mathbf{q}}(0) + \frac{1}{4}\Delta^2\hat{\mathbf{P}}_{\mathbf{y}}(0) && (2.29)\\
\Delta^2\hat{\mathbf{y}}(t) &= \Delta^2\hat{\mathbf{y}}(0) + \Delta^2\hat{\mathbf{p}}(0) + \frac{1}{4}\Delta^2\hat{\mathbf{P}}_{\mathbf{x}}(0). && (2.30)\\
& && (2.31)
\end{aligned}$$

From equations 2.28, we can see that

$$\Delta^2\hat{\mathbf{x}}(0) = \frac{b}{4} \qquad \Delta^2\hat{\mathbf{P}}_{\mathbf{x}}(0) = \frac{1}{b}$$
$$\Delta^2\hat{\mathbf{y}}(0) = \frac{1}{4b} \qquad \Delta^2\hat{\mathbf{P}}_{\mathbf{y}}(0) = b$$

Thus, the measurement uncertainties become become

$$\Delta^2\hat{\mathbf{x}}(t) = \Delta^2\hat{\mathbf{q}}(0) + \frac{1}{2}b \tag{2.32}$$
$$\Delta^2\hat{\mathbf{y}}(t) = \Delta^2\hat{\mathbf{p}}(0) + \frac{1}{2b} \quad \text{(compare with equation 2.11)} \tag{2.33}$$

Therefore, while the state of the system after the measurement reflects the readout values of the detectors, the errors in these redouts depend on the uncertainties in the system before the measurement. Minimizing the product $\Delta\hat{\mathbf{x}}(t)\Delta\hat{\mathbf{y}}(t)$ with respect to b and using the uncertainty relation of q and p, we get

$$\Delta\hat{\mathbf{x}}(t)\Delta\hat{\mathbf{y}}(t) \geq 1 \tag{2.34}$$

In conclusion, Arthurs and Kelly have given a protocol for measuring conjugate observables simultaneously, with the uncertainty relation between these measurement being twice as large as the uncertainty relation between momentum and position of a particle.

2.1 The Standard Quantum Limit

After the rather lengthy introduction to Quantum Measurements, we are now equipped to define the Standard Quantum Limit for repeated measurements of free mass position. Bragginsky and Vorontsov first defined the SQL in 1974 when deriving the SQL for determining a force F acting on a free mass. A force F acting on a mass m for time τ will produce a displacement of $\delta x = F\tau^2/2m$ for the mass[Kosugi 10]. Thus, if we measure the position of the mass at $t = 0$ and $t = \tau$ and the displacement differs from what it would be under a free evolution, we can attribute the change to the act of the force and thereby measure the force. However, if the change in the readout from the value expected from free evolution is lesser than the position uncertainty of the mass, we cannot know if the change we are seeing is due to the quantum uncertainty in the position or due to a force acting on the mass. Worse, as momentum is a conjugate observable of position *and* appears in its free evolution, it worsens the position uncertainty of the particle as it evolves through time.

Over the years, the formulation of the SQL has changed, however, the need for it has remained the same, to monitor the force acting on a free particle. The standard procedure to determine the SQL followed by Braginsky involved studying the free evolution of the position uncertainty. From the free Hamiltonian

$$H = \frac{p^2}{2m}$$

we get the Heisenberg Equation of motion for position as

$$\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}(0) + \frac{t}{m}\hat{\mathbf{p}}(0). \tag{2.35}$$

This gives the expression of the uncertainty at time t as

$$\Delta^2 x(t) = \Delta^2 x(0) + \frac{t^2}{m^2}\Delta^2 p(0) + \frac{t}{m}\left\langle \Delta x(0)\Delta p(0) + \Delta p(0)\Delta x(0)\right\rangle . \tag{2.36}$$

The standard assumption was that the last term in equation2.36, called the correlation term in statistics, was always non-negative and thus could be dropped, giving

$$\begin{aligned}
\Delta^2 x(t) = \Delta^2 x(0) + \frac{t^2}{m^2}\Delta^2 p(0) \quad &\geq \quad 2\frac{t}{m}\Delta x(0)\Delta p(0) && (2.37)\\
&\geq \quad \frac{t}{m}|[\hat{\mathbf{x}}(0), \hat{\mathbf{p}}(0)]| && (2.38)\\
&= \quad \frac{\hbar t}{m} && (2.39)
\end{aligned}$$

Thus, the SQL is written as

$$\Delta x(\tau) = \sqrt{\frac{\hbar \tau}{m}}. \tag{2.40}$$

However, there were some problems raised concerning this picture, as we shall see in the next section.

2.2 Yuen's Contractive States

Yuen [P. Yuen 83] showed that there might exist some states for which the correlation term in equation [**?**] might be negative. A set of states he names as 'Twisted Coherent States' $|\mu\nu\alpha\omega\rangle$ are eigenstates of the operator $\mu\hat{\boldsymbol{a}}+\nu\hat{\boldsymbol{a}}^\dagger$, with arbitrary ω of unit sec^{-1} and $|\mu|^2 - |\nu|^2 = 1$. The twisted coherent states are written in the position space as

$$\langle x|\mu\nu\alpha\omega\rangle = \left(\frac{m\omega}{\pi\hbar|\mu-\nu|^2}\right)^{\frac{1}{4}} exp\left\{-\frac{m\omega}{2\hbar}\frac{1+i\xi}{|\mu-\nu|^2}\left[x-\left(\frac{2\hbar}{m\omega}\right)^{\frac{1}{2}}\alpha_1\right]^2 + i\left(\frac{2m\omega}{\hbar}\right)^{\frac{1}{2}}\alpha_2\left[x-\left(\frac{2\hbar}{m\omega}\right)^{\frac{1}{2}}\alpha_1\right]\right\} \tag{2.41}$$

$$(\mu\hat{\boldsymbol{a}} + \nu\hat{\boldsymbol{a}}^\dagger)\,|\mu\nu\alpha\omega\rangle = (\mu\alpha + \nu\alpha^*) \tag{2.42}$$

where $\xi = \text{Im}(\mu^*\nu)$, $\alpha = \alpha_1 + i\alpha_2$, and α_1, α_2 are real. Here, the operator $\hat{\boldsymbol{a}}$ is adopted from the annihilation operator of harmonic oscillators, and defined for a mass m with position x and momentum p as [P. Yuen 83]

$$\hat{\boldsymbol{a}} = x\left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}} + ip(2\hbar m\omega)^{\frac{1}{2}},[\hat{\boldsymbol{a}}, \hat{\boldsymbol{a}}\dagger] = 1 \tag{2.43}$$

Then the free Hamiltonian can be written as

$$H = \frac{p^2}{2m} = \frac{1}{2}\hbar\omega\left(\hat{\boldsymbol{a}}^\dagger\hat{\boldsymbol{a}} - \frac{1}{2}\hat{\boldsymbol{a}}^2 - \frac{1}{2}\hat{\boldsymbol{a}}^{\dagger 2} + \frac{1}{2}\right). \tag{2.44}$$

Under these conditions, the state $|\mu\nu\alpha\omega\rangle$ shows following properties

$$\langle\hat{\mathbf{x}}\rangle = (\frac{2\hbar}{m\omega})^{\frac{1}{2}}\alpha_1, \qquad \langle\hat{\mathbf{p}}\rangle = (2\hbar m\omega)^{\frac{1}{2}}\alpha_2 \tag{2.45}$$

$$\langle\Delta^2 x\rangle = \frac{2\hbar\zeta}{m\omega}, \qquad \langle\Delta^2 p\rangle = 2\hbar m\omega\eta \tag{2.46}$$

$$\zeta = \frac{|\mu-\nu|^2}{4}, \qquad \eta = \frac{|\mu+\nu|^2}{4}, \qquad \zeta\eta = \frac{1+4\xi^2}{16} \tag{2.47}$$

$$\langle\Delta\hat{\mathbf{x}}\Delta\hat{\mathbf{p}} - \Delta\hat{\mathbf{p}}\Delta\hat{\mathbf{x}}\rangle = 2\xi\hbar \tag{2.48}$$

$$\frac{\langle\hat{\mathbf{p}}^2\rangle}{2m} = \hbar\omega(\alpha_2^2 + \eta), \tag{2.49}$$

$$\langle\Delta^2\hat{\mathbf{x}}(t)\rangle = \frac{2\hbar}{m}\left(\frac{\zeta}{\omega} - \xi t + \eta\omega t^2\right). \tag{2.50}$$

Thus, the position uncertainty of the TCS has a quadratic dependence on time, with a minima at

$$t_m = \frac{\xi}{2\eta\omega}, \tag{2.51}$$

$$\text{giving } \Delta^2 x(t_m) = \frac{2\hbar t_m}{m}\frac{1}{8\xi} \tag{2.52}$$

for $\xi \geq 0$.

Thus, for a $\xi \geq 0$, the TCS show a reduction in its position uncertainty for some time. Yuen calls Contractive States, and he argues that for large ξ, their uncertainty at time t can be made arbitrarily small thereby violating the SQL. One just has to make the second measurement before the time $t = 2t_m$.

However, to bring the free mass into a contractive state at $t = 0$, Yuen suggests a set of Gordon-Louisell measurements, of a class whose validity in under question. Gordon Loisell measurements are represented by operators of the form $\left|\psi^M\right\rangle\left\langle\psi^S\right|$ and will bring the system to the state $\left|\psi^M\right\rangle$ after the measurement with probability $\left|\left\langle\psi^S\middle|\psi^S\right\rangle\right|^2$. However, due to being a measurement of a non self-adjoint operator, not everyone accepts their validity.

2.3 Caves's redifination of the SQL

Regardless of whether or not Gordon-Louisell measurements exist, Yuen had found a serious flaw in the derivation of the SQL. Following that, Caves gave a new formulation of the SQL that holds for specific models of measurement. He gave the following definition of the SQL[Caves 85]:

> Let a free mass m undergo unitary evolution during the time τ between two measurements of its position x, made with identical measuring apparatuses; the result of the second measurement cannot be predicted with uncertainty smaller than $(\hbar\tau/m)^{1/2}$.

Caves defines the variance of the measurement (Δ) as the error in the result, and takes it to be the sum of the resolution (σ) of the apparatus/detector and the uncertainty

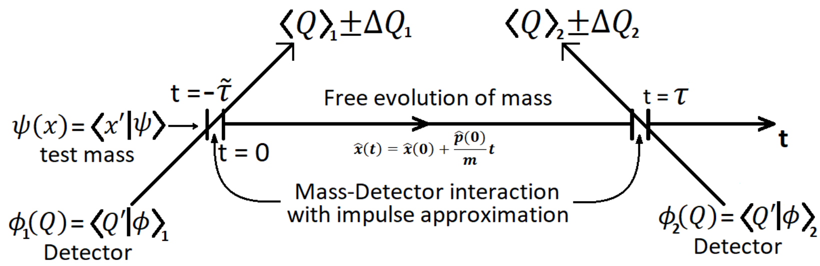

Figure 2.1: A repetitive measurement scheme with mass $\psi(x)$ and detector $\phi(Q)$

of the system (Δx).

$$\text{variance of the measurement } = \Delta^2 = \sigma^2 + \Delta^2 x \tag{2.53}$$

He assumes that after a measurement of the system by a detector of resolution σ, the state of the system is reduced to one with the uncertainty $\Delta x \leq \sigma$. Thus, for repeated measurement, the first measurement ensures that

$$\Delta x(0) \leq \sigma \tag{2.54}$$

Then, the error Δ_2^2 of the next measurement of the freely evolving mass by the same detector $\phi(Q)$ will be

$$\Delta_2^2 = \sigma^2 + \Delta^2 \hat{\mathbf{x}}(\tau) \geq \Delta^2 \hat{\mathbf{x}}(0) + \Delta^2 \hat{\mathbf{x}}(\tau) \tag{2.55}$$
$$\geq 2\Delta\hat{\mathbf{x}}(0)\Delta\hat{\mathbf{x}}(\tau) \tag{2.56}$$
$$\geq \frac{\hbar\tau}{m} \tag{2.57}$$

Ozawa["Ozawa" 15] points out flaws in this construction. His objection is mainly to the assumption Caves makes about the uncertainty of the free mass after the measurement falling below the resolution of the detector before the measurement.

In the regime of Quantum Measurement, it isn't yet entirely clear what the results of measurement are. It is a point of debate whether the prior state of the system caused the measurement readout to be what it is and the subsequent state of the system might be different because of the effect of the measurement process, or whether the readout tells the state of the system *after* the measurement because the measurement changes the state of the system which might be in related with the readout. In areas of computing errors, this might get even more confusing.

Ozawa thus defines two error terms related to measurement of mass position. First, precision ϵ is the error of the measurement if the mass was in a position eigenstate just before the measurement. Second, resolution σ is the deviation of the mass position from the readout just after the measurement. For a position measurement of ψ where the readout result is a, he gives following expressions for ϵ and σ

$$\epsilon^2(x) = \int da\,(a-x)^2 G(a,x) \qquad \epsilon^2(\psi) = \int dx\,\epsilon^2(x)|\psi(x)|^2 \tag{2.58}$$

$$\sigma^2(a) = \int dx\,(a-x)^2|\psi_a(x)|^2 \qquad \sigma^2(\psi) = \int da\,\sigma^2(a)P(a|\psi), \tag{2.59}$$

where $\psi_a(x)$ is the state of the system just after the measuremen, and $P(a|\psi)$ is the probability of getting the readout value a:

$$P(a|\psi) = \int dx\,G(a,x)|\psi(x)|^2, \tag{2.60}$$

and $G(a,x)$ is the normalized conditional probability density of the readout a given that the free mass m is in the position x at the time of measurement. $G(a,x) = \delta(x-a)$ for the case of noiseless/discrete case, giving $P(a|\psi) = |\psi(a)|^2$. With the assumption that the mean value of the readout is equal to the mean position of the mass just before the measurement $\left(\int aP(a|\psi)da = \int x|\psi(x)|^2dx\right)$, we get the expressions

$$\Delta^2 = \epsilon^2(\psi) + \Delta^2 x(\psi) \tag{2.61}$$

$$\sigma^2(\psi) = \int da\,P(a|\psi)\Delta^2 x(\psi_a) + \int da\,P(a|\psi)(a - \langle\psi_a|\hat{\mathbf{x}}|\psi_a\rangle)^2 \tag{2.62}$$

where Δ is the variance of the measurement. The proofs of these expressions are as follows. From the equation 2.58:

$$\begin{aligned}\epsilon^2(\psi) &= \int dx\,\epsilon^2(x)|\psi(x)|^2 \\ &= \int dx\,da\,(a-x)^2G(a,x)|\psi(x)|^2 \\ &= \int dx\,da\,a^2G(a,x)|\psi(x)|^2 + \int dx\,da\,x^2G(a,x)|\psi(x)|^2 - 2\int dx\,da\,axG(a,x)|\psi(x)|^2\end{aligned} \tag{2.63}$$

We have, from definitions and assumptions, including normalization of states, relations saying

$$\int da\, G(a,x) = 1 \qquad \int dx\, |\psi(x)|^2 = 1$$
$$\int da\, a^2 G(a,x) = \langle a^2\rangle \qquad \int dx\, x^2|\psi(x)|^2 = \langle x^2\rangle$$
$$\int aP(a|\psi)\, da = \int x|\psi(x)|^2\, dx$$
$$\Rightarrow \int aG(a,x)|\psi(x)|^2\, da\, dx = \int x|\psi(x)|^2\, dx, \quad 2\int dx\, da\, a\, xG(a,x)|\psi(x)|^2 = \int x^2|\psi(x)|^2\, dx.$$

Hence, 2.63 reduces to

$$\begin{aligned} \epsilon^2(\psi) &= \langle a^2\rangle - \langle x^2\rangle \\ &= \langle a^2\rangle - \langle a\rangle^2 - \langle x^2\rangle + \langle x\rangle^2 \\ &= \Delta^2 - \Delta^2 x(\psi) \\ \Rightarrow \Delta^2 &= \epsilon^2(\psi) + \Delta^2 x(\psi). \end{aligned} \tag{2.64}$$

Following a similar procedure for the resolution, we can write

$$\begin{aligned} \sigma^2(\psi) &= \int da\, P(a|\psi) \int dx\, (a-x)^2|\psi_a(x)|^2 \\ &= \int da\, dx\, a^2 P(a|\psi)|\psi_a(x)|^2 + \int da\, dx\, x^2 P(a|\psi)|\psi_a(x)|^2 - 2\int da\, dx\, aP(a|\psi)x|\psi_a(x)|^2 \\ &= \int da\, dx\, a^2 P(a|\psi)|\psi_a(x)|^2 + \int da\, dx\, P(a|\psi)\, \langle x^2(\psi_a)\rangle - \int da\, dx\, P(a|\psi)\, \langle x(\psi_a)\rangle^2 \\ &\qquad + \int da\, dx\, P(a|\psi)\, \langle x(\psi_a)\rangle^2 - 2\int da\, dx\, aP(a|\psi)x|\psi_a(x)|^2 \\ &= \int da\, P(a|\psi)\Delta^2 x(\psi_a) + \int da\, P(a|\psi)(a - \langle\psi_a|\hat{\mathbf{x}}|\psi_a\rangle)^2 \end{aligned}$$

With these definitions Ozawa shows that in the second measurement at the time τ is

$$\Delta_2^2(\tau,\psi,a) = \epsilon^2(U_\tau\psi_a) + \Delta^2 x(U_\tau\psi_a), \tag{2.65}$$

where $U_\tau\psi_a$ is the state of the mass after evolving freely for time τ after the first measurement. Caves argues that since we do not know what the readout of the first measurmene will be, one must average the error in the second measurement over all the possible values of the readout to predict the SQL. That is, take the average uncertainty in the second measurement:

$$\Delta_2^2(\tau,\psi) = \int da\, P(a|\psi)\epsilon^2(U_\tau\psi_a) + \int da\, P(a|\psi)\Delta^2 x(U_\tau\psi_a) \tag{2.66}$$

Ozawa shows that if $\sigma^2(\psi) \leq [\epsilon^2(U_\tau\psi_a)]$, where the brackets [] signifies an average over all possible readout values a, then the SQL given by Caves holds:

$$\begin{aligned}
[\epsilon^2(U_\tau\psi_a)] &\geq \sigma^2(\psi) && (2.67)\\
&= \int da\, P(a|\psi)\Delta^2 x(\psi_a) + \int da\, P(a|\psi)(a - \langle\psi_a|\hat{\mathbf{x}}|\psi_a\rangle)^2 \\
&\geq \int da\, P(a|\psi)\Delta^2 x(\psi_a) = [\Delta^2 x(\psi_a)] && (2.68)\\
\Rightarrow \Delta_2^2 &= [\epsilon^2(U_\tau\psi_a)] + [\Delta^2 x(U_\tau\psi_a)] && (2.69)\\
&\geq [\Delta^2 x(\psi_a)] + [\Delta^2 x(U_\tau\psi_a)] && (2.70)\\
&\geq 2\Delta x(\psi_a)\Delta x(U_\tau\psi_a) && (2.71)\\
&\geq [\hat{\mathbf{x}}(0), \hat{\mathbf{x}}(\tau)] && (2.72)\\
&= \frac{\hbar\tau}{m} && (2.73)
\end{aligned}$$

Caves demonstrates the validity of this SQL for Von Neumann's approximation measurement model. Here, I present his measurement scheme in the framework given by Ozawa. Keeping with the Von Neumann model of approximate measurements, consider the system ψ in the Hilbert space $\{x,p\}$ and the detector ϕ in $\{Q,P\}$ to be entangled at $t=-\tilde{\tau}$ until $t=0$ under the impulse approximation by the Hamiltonian

$$H = k\hat{\mathbf{x}}\hat{\mathbf{P}}, \tag{2.74}$$

where $k=\frac{1}{\tilde{\tau}}$. Assume that $\left\langle\phi\middle|\hat{\mathbf{Q}}\middle|\phi\right\rangle = \left\langle\phi\middle|\hat{\mathbf{P}}\middle|\phi\right\rangle = 0$. As seen before, first measurement then results in the transformation,

$$\Psi(x,Q,t=-\tilde{\tau}) = \psi(x)\phi(Q) \Rightarrow \Psi(x,Q,t=0) = \psi(x)\phi(Q-x) \tag{2.75}$$

If $\overline{Q}$ is the readout of the measurement of the detector, then, by the Hamiltonian above, we have

$$\begin{aligned}
\overline{Q} &= \int dQ\, dx\, \hat{\mathbf{Q}} |\Psi(x, Q-x)|^2 && (2.76)\\
&= \int dQ\, dx\, \hat{\mathbf{Q}} |\psi(x)|^2 |\phi(Q-x)|^2 && (2.77)\\
&= \int dQ\, dx\, (Q-x) |\psi(x)|^2 |\phi(Q-x)|^2 && (2.78)\\
&= \int dQ\, dx\, Q |\psi(x)|^2 |\phi(Q-x)|^2 - \int dQ\, dx\, Q |\psi(x)|^2 |\phi(Q-x)|^2 && (2.79)\\
&= \int dQ\, dx\, Q |\psi(x)|^2 |\phi(Q-x)|^2 && (2.80)\\
&= \langle x \rangle_0 && (2.81)
\end{aligned}$$

The same relation can be obtained by Heisenberg Equation of Motion. The variance of measurement (Δ_1^2) can also be calculated by the Heisenberg Equation of Motion.

$$\begin{aligned}
\hat{\mathbf{Q}}(0) &= \hat{\mathbf{Q}}(-\tilde{\tau}) + \hat{\mathbf{x}}(-\tilde{\tau}) && (2.82)\\
\Rightarrow \Delta^2 \hat{\mathbf{Q}}(0) &= \Delta^2 Q + \Delta^2 \hat{\mathbf{x}}(-\tilde{\tau}) = \Delta_1^2, && (2.83)
\end{aligned}$$

Where we have taken the initial variance in Q as $\Delta^2 Q$. Thus, we find

$$\epsilon^2(\psi) = \Delta^2 Q \quad (2.84)$$

The same result can be arrived at by following the procedure in the proof of the precision and recognizing that $G(a, x) = \left|\Psi(\overline{Q} - x)\right|^2$, $P(\overline{Q}|\psi) = \left|\phi(\overline{Q} - x)\right|^2$, and that the posterior state $\psi_{\overline{Q}}(x)$ is $\psi(x)$ itself. We get the resolution to be

$$\begin{aligned}
\sigma^2(\psi) &= \int d\overline{Q}\, P(\overline{Q}|\psi) \Delta^2 x(\psi) + \int d\overline{Q}\, P(\overline{Q}|\psi)(\overline{Q} - \left\langle \psi \middle| \hat{(x)} \middle| \psi \right\rangle)^2 && (2.85)\\
&= \int d\overline{Q}\, x^2\, P(\overline{Q}|\psi) + \int d\overline{Q}\, \overline{Q}^2\, P(\overline{Q}|\psi) - \int d\overline{Q}\, x^2\, P(\overline{Q}|\psi) && (2.86)\\
&= \Delta^2 Q && (2.87)
\end{aligned}$$

Thus, the condition 2.67 given by Ozawa is met as $\epsilon = \sigma = \Delta Q$ for the model of Von Neumann measurement. And, thus, the SQL holds for Von Neumann's measurement model. However, Ozawa holds that it is not at all obvious that this condition will be met by every measurement protocol["Ozawa" 15]. He gives his own Hamiltonian to beat the SQL.

2.4 The Hamiltonian by Ozawa for Beating the SQL

In his paper Realization of Measurement and the Standard Quantum Limit["Ozawa" 15], Ozawa proves that every Gordon Louisell measurement is realizable by the standards given by Von Neumann. He also gives a Hamiltonian that realizes a Gordon-Louisell Measurement of the type envisoned by Yuen that leaves the free mass in contractive state. For a particle ψ in Hilbert Space $\{x, p\}$ and the detector ϕ in space $\{Q, P\}$, he proposes an entanglement given by the Hamiltonian

$$H = \frac{k\pi}{3\sqrt{3}} \left(2 \left(\hat{\mathbf{x}}\hat{\mathbf{P}} - \hat{\mathbf{p}}\hat{\mathbf{Q}} \right) + \left(\hat{\mathbf{x}}\hat{\mathbf{p}} - \hat{\mathbf{Q}}\hat{\mathbf{P}} \right) \right). \tag{2.88}$$

Solving this evolution the help of Heisenberg Equation of Motion, we find

$$\hat{\mathbf{Q}}(t) = \hat{\mathbf{Q}}(0) \left(\cos\frac{k\pi t}{3} - \frac{1}{\sqrt{3}} \sin\frac{k\pi t}{3} \right) + \frac{2}{\sqrt{3}} \hat{\mathbf{x}}(0) \sin\frac{k\pi t}{3} \tag{2.89}$$

$$\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}(0) \left(\cos\frac{k\pi t}{3} + \frac{1}{\sqrt{3}} \sin\frac{k\pi t}{3} \right) - \frac{2}{\sqrt{3}} \hat{\mathbf{q}}(0) \sin\frac{k\pi t}{3} \tag{2.90}$$

From these equations, we can derive the operators at time $t = 0$ in terms of opeartors at time t, and find the state of the composite system $\Psi(x, Q, t)$ to be

$$\Psi(x, Q, t) = \psi \left(\frac{2}{\sqrt{3}} \left(x \sin\frac{(1-kt)\pi}{3} + Q \sin\frac{k\pi t}{3} \right) \left(\phi \left(\frac{2}{\sqrt{3}} \left(Q \sin\frac{(1+kt)\pi}{3} - x \sin\frac{k\pi t}{3} \right) \right) \right.$$

$$\Rightarrow \Psi(x, Q, t = \tau = 1/k) = \psi(Q)\phi(Q - x)$$

Therefore, we can see that for this measurement, we have $G(\overline{Q}, x) = \delta(\overline{Q} - x)$ and $P(\overline{Q}|\psi) = \int dx \left|\Psi(x, \overline{Q}, t)\right|^2 = \left|\psi(\overline{Q})\right|^2$, which means $\epsilon(\psi) = 0$. Ozawa argues that if the probe is initially prepared in a contractive state $|\mu\nu a\omega\rangle$ such that before the coupling is turned on, $\langle Q \rangle = a = 0$ and $\langle P \rangle = 0$, then the posterior state of the free mass after the measurement is given by

$$\psi(x|\overline{Q}) = \left[\frac{1}{P(\overline{Q}|\psi)}\right]^{1/2} \Psi(x, \overline{Q}) \tag{2.91}$$

$$= \left[\frac{\psi(\overline{Q})}{\psi(\overline{Q})}\right] \phi(\overline{Q} - x) \tag{2.92}$$

$$= C \left\langle x \middle| \mu\nu\overline{Q}\omega \right\rangle \tag{2.93}$$

which is now a contractive state. Then, by the analysis given by Yuen, and the fact that the precision of the measurement is zero and cannot be greater than the average uncertainty of the free mass after the measurement which runs counter to the condition 2.67 given by Ozawa for the SQL to hold, the SQL is beaten.

In his paper defending the SQL after Yuen's Contractive State paper was published, Caves concedes that he does not prove the SQL, but rather, demonstrates it for a certain type of measurements, namely the Von Neumann model for approximate measurement. He asserts that to beat the SQL, one must look at other types of interactions, interactions that follow Hamiltonians which don't utilize linear relationships observed in Von Neumann's model. However, as Caves points out, in such models, it is often difficult to ascertain just what is being measured [Caves 85]. Ozawa's Hamiltonian, however, uses the type of linear relationship witnessed in Von Neumann measurement model, and is used to beat the SQL by realizing a Gordon-Louisell measurement (this interaction doesn't conserve momentum of the particle-detection system either, however[Kosugi 10]). Thus, it would seem there is at least one protocol for measurement that can beat the SQL.

We are near the end of the literature review necessary for this thesis. But before going on to discuss the idea of using two entangled particles to beat the SQL, we must first introduce ourselves with a closely related field of QND measurements.

2.5 Quantum Nondemolition Measurements and Back-action evading measures

Apart from the Hamiltonian given by Ozawa, there exist other methods to beat the SQL (including one given by Kosugi [Kosugi 10], which incidently also conserves the momentum of the system during the measurement process). However, most of these methods employ more than one detectors to interact with the system during the measurement, which almost always means that one of the detectors performs the actual measurement, while the subsequent detector interactions prepare the state for the subsequent measurement["Ozawa" 15],[Ni 86]. While these setups may be necessary and optimal for some experiments, they do not, in theory, invalidate the SQL, which insists on similar measurements and free evolution of the system.

Another field that is closely related to the topic of the SQL and precision measurements is Quantum Nondemolition methods and Backaction Evading measurements. Seeing as we have done that the SQL depends on the commutator $[\hat{\mathbf{x}}(0), \hat{\mathbf{x}}(t)]$, and that this commutator isn't zero due to the presence of $\hat{\mathbf{p}}(0)$ in the expression for $\hat{\mathbf{x}}(t)$, it is important to note that position is not only observable whose conjugate interfers in its subsequent measurement. The commutator of the type $\left[\hat{\mathbf{A}}(0), \hat{\mathbf{A}}(t)\right]$ belongs to the field of Quantum Nondemolition Measurements (QND Measurements). The observable $\hat{\mathbf{A}}$ for which the commutator

$$[\hat{\mathbf{x}}(0), \hat{\mathbf{x}}(t)] = 0 \quad \forall\ t \tag{2.94}$$

is called a QND observable. And the observables for which this commutator is zero at certain times are called Stroboscopic QND observables[Bahrami 14], measurements of which need to be done very quickly at the prescribed times in order to evade backaction from the conjugate observables.

Given that continuous QND variables exist, it is obvious that not all conjugate variables interfer with the measurement of their cojugate. For instance, position of a particle does not give any back action to its momentum measurement, and hence, the commutator $[\hat{\mathbf{p}}(0), \hat{\mathbf{p}}(t)] = 0$, which makes the momentum of a free particle a continuous QND variable.

To better demonstrate stroboscopic QND observables and back action evding measures, let us take the example of a a Simple Harmonic Oscillator We can write the evolution of position and momentum for an SHO with frequency ω and mass m as

$$\begin{aligned}\hat{\mathbf{x}}(t) &= \hat{\mathbf{x}}(0)\cos\omega t + \frac{\hat{\mathbf{p}}(0)}{m\omega}\sin\omega t &(2.95)\\ \hat{\mathbf{p}}(t) &= -m\omega\hat{\mathbf{x}}(0)\sin\omega t + \hat{\mathbf{p}}(0)\cos\omega t &(2.96)\end{aligned}$$

Here, we find that

$$\begin{aligned}[\hat{\mathbf{x}}(0), \hat{\mathbf{x}}(t)] &= i\frac{\sin\omega t}{m\omega}\\ &= 0 \text{ when } \sin\omega t = 0.\\ \Delta^2 x(t) &= \Delta^2 x(0)\cos^2\omega t + \frac{\Delta^2 p(0)}{m^2\omega^2}\sin^2\omega t \qquad (2.97)\\ &+ \frac{\sin\omega t\cos\omega t}{m\omega}\langle\Delta x(0)\Delta p(0) + \Delta p(0)\Delta x(0)\rangle\\ &= \Delta^2 x(0) \text{ when } \sin\omega t = 0.\end{aligned}$$

Thus, the operators $\hat{\mathbf{x}}(0)$ and $\hat{\mathbf{x}}(t)$ commute at the times when $\sin\omega t = 0$. They share all their eigenstates, and therefore, their uncertainties are equal. Thus, there is no back action from the momentum of the particle affecting the position measurement at these times. This makes the position of a SHO a stroboscopic observable. And when we examine the quadratures of the Oscillator, we find that

$$\begin{aligned}X_1(t) &= \frac{\hat{\boldsymbol{a}}^\dagger(t) + \hat{\boldsymbol{a}}(t)}{\sqrt{2}} = \frac{\hat{\boldsymbol{a}}^\dagger(0)e^{i\omega t} + \hat{\boldsymbol{a}}(0)e^{-i\omega t}}{\sqrt{2}}\\ X_2(t) &= \frac{\hat{\boldsymbol{a}}^\dagger(t) - \hat{\boldsymbol{a}}(t)}{\sqrt{2}} = \frac{\hat{\boldsymbol{a}}^\dagger(0)e^{i\omega t} - \hat{\boldsymbol{a}}(0)e^{-i\omega t}}{\sqrt{2}},\end{aligned}$$

where $\hat{\boldsymbol{a}}$ is the annihilation operator of the Oscillator and ω is the frequency. Thus, we find that the two conjugates X_1 and X_2 do not appear in the time evolution of each other, thereby allowing us measure one quadrature as precisely as we want at the expense of increasing the uncertainty in the other quadrature but evading any back action thereof. Thus, these measurements can be classified as backaction evading measurements.

After a long and arduous background review, we are finally in a position to get on with explorations of using two mass systems to beat the SQL, or rather, using two mass systems to engineer QND and backaction evading measures. For this, we proceed to the next chapter.

Chapter 3

Entanglement to beat the SQL

Introduction

The study of Quantum Metrology is both theoretic and experiment intensive. There is often assymetric development across the board of theoretic knowledge and experimental ability, especially in Quantum Mechanics. For the purpose of beating the SQL by using entangled systems of two particles, I have made a few assumptions, the motivations and justifications for which I shall provide in the next section. Does this approach then indeed beat the SQL or is it only classified as backaction evading and/or stroboscopic QND measurements, I shall leave for the reader (and my thesis comitee) to decide.

3.1 Assumptions and their Justifications

We have examined two subfields of Quantum Metrology. The field of the Interferometers and the Shot Noise Limit, and the field of the Standard Quantum Limit in monitoring mass position. In both the cases, we are concerned with measurement. In both the cases there are detectors and the systems to be measured; in the context of the Interferometer, the photons are the detectors, as they are interacting with the Interferometer and getting entangled with the position of its mirrors, and the mirror or the arms of the Interferometer, or the Interferometer itself, is the system to be measured.

From this comparison, one thing discrepency jumps out. While the debate over the SQL was concerned with only one system, indeed, even when there might be more than one detector, there was only one system to be measured, there are two masses, namely the mirrors, whose position is being measured in the Interferometers. As Quantum Metrology is the science of using quantum correlations to reduce errors in measurements, it is natural to ask what correlations or entanglements between these two particles can we engineer to reduce measurement error.

In the following chapter, I give four such entanglements that are seen to bring the error of the second measurement below the derived SQL for two unentangled particles. My first assumption is that the measurements of these particles by the detectors follows the Impulse Approximation, which here means that in excluding the free evolution of the particles, the measurement protocol will also exclude the entanglement between the particles. I feel that this is an appropriate assumption given that we can control the timespan of the measurement process (in theory), and can also set the strength of the coupling between our two particles.

My second assumption is about what signifies a free system. As we have seen, the SQL is defined for one particle. A Hydrogen atom travelling freely in space is a free particle, but the sole proton and the electron that form it, when taken saperately, aren't free particles. One can argue that the position of the center of mass of these two particles is evolving freely, that is, its momentum is conserved. Thus the coupled system of the proton and electron, which is they Hydrogen atom, is a free system. However, in the entanglements I have taken, the momentum of the center of mass isn't always conserved. But the same is true for Von Neumann's approximate measurement model, Arthurs and Kelly's general protocol to measure position and momentum simultaneously, and Ozawa's Hamiltonian given to beat the SQL. Hence, I have tailored the intuitive definition of a free system to say "the system is free if the environment does not appear in the Hamiltonian of the system", rather than basing it on the momentum of the center of mass.

However, the question still remains of whether these types of system beat the SQL. If we take two particles ψ_1 and ψ_2 with variables $\{x_1, p_1\}$ and $\{x_2, p_2\}$ respectively, and set up a detector ϕ with variables $\{x, p\}$ to measure the position difference $x_2 - x_1$

between them (taking after what the Interference pattern of a coherent state input in the Interferometer tells us) by taking the interaction Hamiltonian between them to be

$$H = k\left(\hat{\mathbf{x}}_2\hat{\mathbf{p}} - \hat{\mathbf{x}}_1\hat{\mathbf{p}}\right), \tag{3.1}$$

we find

$$\hat{\mathbf{x}}(\tau = 1/k) = \hat{\mathbf{x}}(0) + (\hat{\mathbf{x}}_2(0) - \hat{\mathbf{x}}_1(0)) \tag{3.2}$$

$$\hat{\mathbf{x}}_1(\tau) = \hat{\mathbf{x}}_1(0) \tag{3.3}$$

$$\hat{\mathbf{x}}_2(\tau) = \hat{\mathbf{x}}_2(0) \tag{3.4}$$

$$\Delta^2 x(\tau) = \Delta_1^2 = \Delta^2 x(0) + \Delta^2 x_1(0) + \Delta^2 x_2(0) \tag{3.5}$$

$$\Rightarrow \quad \epsilon^2(\psi_1, \psi_2) = \Delta^2 x = \sigma(\psi_1, \psi_2) \geq [\Delta^2 x_1(0)] + [\Delta^2 x_2(0)] \tag{3.6}$$

$$\Rightarrow \quad \Delta_2^2 = [\epsilon(U_\tau \psi_1 \psi_2)] + [\Delta^2 x_1(\psi_1)] + [\Delta^2 x_1(\psi_2)] \tag{3.7}$$

$$\begin{aligned} &\geq [\Delta^2 x_1(0)] + [\Delta^2 x_2(0)] + [\Delta^2 x_1(\tau)] + [\Delta^2 x_1(\tau)] \\ &\geq \frac{2\hbar\tau}{m} \end{aligned} \tag{3.8}$$

Thus, for two particles, the SQL to be beaten is $2\hbar\tau/m$, twice the limit for one mass. The same limit is arrived at even if we use two different detectors to measure the positions of the two particles. However, the SQL as formulated by Caves insists on free evolution of the mass between measurements, and indeed, methods where this free evolution is not allowed are judged to not violate the SQL. And so, in our formalism of two entangled particles, where we have seen that the two particles taken individually do not undergo free evolution, the question of whether my methods beat the SQL as given by Caves, I leave for the reader to decide. These methods however, certainly beat the SQL for two particles as derived in 3.8. Now I shall detail these methods.

3.2 Entangled Systems to beat the SQL

The first entanglement I consider is modeled after Von Neumann's approximation model. I assume two particles ψ_1 and ψ_2 with variables $\{x_1, p_1\}$ and $\{x_2, p_2\}$ respectively to be entangled according to the Hamiltonian

$$H = \frac{\hat{\mathbf{p}}_1^2}{2m} + \frac{\hat{\mathbf{p}}_2^2}{2m} + k\hat{\mathbf{x}}_2\hat{\mathbf{p}}_1, \tag{3.9}$$

where I have assumed that the two particles have equal masses for simplicity. As promised, the Hamiltonian only has terms concerning to the two particles taken, and the environment isn't present, so by my assumption, this is taken as a free and isolated system. The positions of these two particles then evolve as following

$$\hat{\mathbf{x}}_1(t) = \hat{\mathbf{x}}_1(0) + \frac{t}{m}\hat{\mathbf{p}}_1(0) + k\,t\,\hat{\mathbf{x}}_2(0) + \frac{k}{m}\left(\frac{1}{2}t^2\hat{\mathbf{p}}_2(0) - \frac{1}{6}k\,t^3\hat{\mathbf{p}}_1(0)\right) \tag{3.10}$$

$$\hat{\mathbf{x}}_2 = \hat{\mathbf{x}}_1(0) + \frac{t}{m}\hat{\mathbf{p}}_2(0) - \frac{k\,t^2}{2\,m}\hat{\mathbf{p}}_1(0) \tag{3.11}$$

$$\Rightarrow |[\hat{\mathbf{x}}_1(0), \hat{\mathbf{x}}_1(t)]| = \frac{\hbar t}{m} - \frac{\hbar k^2 t^3}{6\,m} \tag{3.12}$$

$$\Rightarrow |[\hat{\mathbf{x}}_2(0), \hat{\mathbf{x}}_2(t)]| = \frac{\hbar t}{m} \tag{3.13}$$

From equation 2.72, we know that the commutator of the position operators at time 0 and t decides the SQL. From equations 3.12 and 3.13, we see that the commutator we are concerned with for the first particle goes to zero at time $t = \sqrt{6}/k$, whereas the same commutator for the second particle does not go to zero at all after $t = 0$. But there is a bigger problem here. If we write the position uncertainty terms of the first mass (we don't need to do this for the second mass as we already know that it is not going to beat the SQL), we find

$$\Delta^2 x_1(t) = \Delta^2 X_1 + \frac{t^2}{m^2}\Delta^2 P_1 + k^2 t^2 \Delta^2 X_2 + \frac{k^2}{m^2}\left(\frac{1}{4}t^4\Delta^2 P_2 + \frac{1}{36}k^2 t^6 \Delta^2 P_1\right)$$
$$+ (\Delta X_1 \Delta P_1 + \Delta P_1 \Delta X_1)\left(\frac{t}{m} - \frac{k^2 t^3}{6m}\right) + \frac{k^2 t^3}{6m}(\Delta X_2 \Delta P_2 + \Delta P_2 \Delta X_2), \tag{3.14}$$

where $X = \hat{\mathbf{x}}(0)$ and $P = \hat{\mathbf{p}}(0)$ for the respective particles. Comparing this with the uncertainty of the free particle

$$\Delta^2 x_{free}(t) = \Delta^2 X + \frac{t^2}{m^2}\Delta^2 P + \frac{t}{m}\left(\Delta X \Delta P + \Delta P \Delta X\right)$$

$$\begin{aligned}\Delta^2 x_1(t) - \Delta^2 x_{free}(t) &= k^2 t^2 \Delta^2 X_2 + \frac{k^2 t^4}{4m^2}\Delta^2 P_2 + \frac{k^2 t^3}{2m}(\Delta X_2 \Delta P_2 + \Delta P_2 \Delta X_2) \\ &+ \frac{k^4 t^6}{36m^2}\Delta^2 P_1 - \frac{k^2 t^3}{6m}(\Delta X_1 \Delta P_1 + \Delta P_1 \Delta X_1), \quad (3.15)\end{aligned}$$

we find that the entanglement has added randomness to the picture. However, since it is clear that the second particle is not going to break the SQL as a free mass, we can take liberties with it. Remembering Yuen's contractive states, if we prepare the second particle in a contractive state, then we can reduce the contribution of the second particle in the uncertainty of the first particle as much as we want, and the equation 3.15 reduces to

$$\Delta^2 x_1(t) - \Delta^2 x_{free}(t) \geq \frac{k^2 t^2}{2m}\frac{1}{16\omega\eta} + \frac{k^4 t^6}{36m^2}\Delta^2 P_1 - \frac{k^2 t^3}{6m}(\Delta X_1 \Delta P_1 + \Delta P_1 \Delta X_1), \quad (3.16)$$

This equation then can go below zero, but only for specific values of $\Delta^2 p_1(0)$ and that of the correlation term of the first particle. As $\hat{\mathbf{p}}$ is a continuous QND variable, it is easy to establish its value and uncertainty for a free mass without disturbing the system too much, but the correlation term might be harder to keep track of. Thus, this entanglement will not break the SQL as a general case, but only if the correlation term is known. Also, in an interferometer, if such a scheme were to be realized, a total of four particles must be used, two pairs of entanglements as detailed above. In all, this has not been a very satisfactory result.

The next entanglement I consider is similar to the previous one. The Hamiltonian is given by

$$H = \frac{\hat{\mathbf{p}}_1^2}{2m} + \frac{\hat{\mathbf{p}}_2^2}{2m} + k(\hat{\mathbf{x}}_1 \hat{\mathbf{p}}_2 - \hat{\mathbf{x}}_2 \hat{\mathbf{p}}_1) \quad (3.17)$$

The two particles evolve as

$$\hat{\mathbf{x}}_1(t) = \hat{\mathbf{x}}_1(0)\cos k\,t - \hat{\mathbf{v}}_2(0)\sin k\,t + \frac{t}{m}(\hat{\mathbf{p}}_1(0)\cos k\,t - \hat{\mathbf{p}}_2(0)\sin k\,t) \tag{3.18}$$

$$\hat{\mathbf{x}}_2(t) = \hat{\mathbf{x}}_1(0)\sin k\,t + \hat{\mathbf{v}}_2(0)\cos k\,t + \frac{t}{m}(\hat{\mathbf{p}}_1(0)\sin k\,t + \hat{\mathbf{p}}_2(0)\cos k\,t) \tag{3.19}$$

$$\Rightarrow |[\hat{\mathbf{x}}_1(0), \hat{\mathbf{x}}_1(t)]| = |[\hat{\mathbf{x}}_2(0), \hat{\mathbf{x}}_2(t)]| = \frac{t}{m}\cos kt = 0 \quad \text{when} \;\; \cos kt = 0 \tag{3.20}$$

So we see that the time evolution commutator for position of both particles goes to zero at the same time. But again, the uncertainties of the particles compare with the uncertainty of the free particle with a twist.

$$\Delta^2 x_1(t) - \Delta^2 x_{free}(t) = [(\Delta^2 X_2 - \Delta^2 X_1) + \frac{t^2}{m^2}(\Delta^2 P_2 - \Delta^2 P_1) + \frac{t}{m}((\Delta X_1 \Delta P_1 + \Delta P_1 \Delta X_1) - (\Delta X_2 \Delta P_2 + \Delta P_2 \Delta X_2))]\sin^2 kt \tag{3.21}$$

$$\Delta^2 x_2(t) - \Delta^2 x_{free}(t) = -(\Delta^2 x_1(t) - \Delta^2 x_{free}(t)) \tag{3.22}$$

We see that only one of the particles can have reduced uncertainty than the unentangled case at a time. Again, only one of the particles can beat the SQL at a time, and like the picture before, the Interferometer again would need four particles. However, this time we have a lot of control over the reduction, in that the reduction in the uncertainty of the target particle can be reduced to arbitrary levels by increasing the initial uncertainty in the accompanying particle.

The third Hamiltonian I have considered is a quantumm analog of two particles on a spring.

$$H = \frac{\hat{\mathbf{p}}_1^2}{2m} + \frac{\hat{\mathbf{p}}_2^2}{2m} + k(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2)^2 \tag{3.23}$$

$$= \frac{\hat{\mathbf{p}}_+^2}{2m} + \frac{\hat{\mathbf{p}}_-^2}{2m} + kx_-^2, \tag{3.24}$$

where $p_+ = \frac{p_1+p_2}{\sqrt{2}}$ and $p_- = \frac{p_1-p_2}{\sqrt{2}}$, and a similar relationship for the position variables. The coordinate frame transform helps us ignore the motion of the center of mass and focus on the motion of the particles with respect to the center of mass, which makes the calculations easier (It also tells us that the momentum of the center of mass is conserved in this entanglement).

The system then evolves as

$$\hat{\mathbf{x}}_+(t) = \hat{\mathbf{p}}_+(0) + \frac{t}{m}\hat{\mathbf{p}}_+(0) \tag{3.25}$$

$$\hat{\mathbf{x}}_-(t) = \hat{\mathbf{x}}_(0)\cos\omega\, t + \frac{1}{m}\hat{\mathbf{p}}_-(0)\frac{\sin\omega\, t}{\omega}, \qquad \text{where } \omega = \sqrt{\frac{k}{m}} \tag{3.26}$$

$$\hat{\mathbf{x}}_1(t) = \frac{1}{2}\hat{\mathbf{x}}_1(0)(1+\cos\omega\, t) + \hat{\mathbf{x}}_2(0)(1-\cos\omega\, t) + \frac{\hat{\mathbf{p}}_1(0)}{m}\left(t + \frac{\sin\omega\, t}{\omega}\right) + \frac{\hat{\mathbf{p}}_2(0)}{m}\left(t - \frac{\sin\omega\, t}{\omega}\right) \tag{3.27}$$

$$\hat{\mathbf{x}}_2(t) = \frac{1}{2}\hat{\mathbf{x}}_1(0)(1-\cos\omega\, t) + \hat{\mathbf{x}}_2(0)(1+\cos\omega\, t) + \frac{\hat{\mathbf{p}}_1(0)}{m}\left(t - \frac{\sin\omega\, t}{\omega}\right) + \frac{\hat{\mathbf{p}}_2(0)}{m}\left(t + \frac{\sin\omega\, t}{\omega}\right) \tag{3.28}$$

$$\Rightarrow |[\hat{\mathbf{x}}_1(0), \hat{\mathbf{x}}_1(t)]| = |[\hat{\mathbf{x}}_2(0), \hat{\mathbf{x}}_2(t)]| = \frac{1}{2m}\left(t + \frac{\sin\omega\, t}{\omega}\right) \geq \frac{1}{2m}\left(t - \frac{1}{\omega}\right). \tag{3.29}$$

Thus, we see that both the particles, while either one not being stroboscopic QND variables, beat the SQL at times when $\sin\omega\, t \leq 0$. And the problems with that arose in the previous pictures do not arise here.

$$\begin{aligned}\Delta^2 x_1(t) - \Delta^2 x_{free}(t) =& \frac{1}{4}(\cos^2\omega t + 2\cos\omega t - 3)\Delta^2 X_1 + \frac{1}{4}(\cos^2\omega t - 2\cos\omega t - 3)\Delta^2 X_2\\ &+ \frac{1}{m^2}\left(\frac{\sin\omega\, t}{\omega} + 2t\right)\frac{\sin\omega\, t}{\omega}\Delta_1^P\\ &+ \frac{1}{m^2}\left(\frac{\sin\omega\, t}{\omega} - 2t\right)\frac{\sin\omega\, t}{\omega}\Delta_2^P\\ &+ \frac{1}{2m}\left(\frac{\sin\omega t\cos\omega t}{\omega} + \frac{\sin\omega t}{\omega} + t\cos\omega t - t\right)(\Delta X_1\Delta P_1 + \Delta P_1\Delta X_1)\\ &+ \frac{1}{2m}\left(\frac{\sin\omega t\cos\omega t}{\omega} - \frac{\sin\omega t}{\omega} - t\cos\omega t - t\right)(\Delta X_2\Delta P_2 + \Delta P_2\Delta X_2)\end{aligned} \tag{3.30}$$

If we examine the coefficients of $\Delta^2 X_1$, $\Delta^2 P_1$, $\Delta^2 X_2$, $\Delta^2 P_2$ and of the two correlation terms in 3.30, we find that we can determine when the uncertainty of our entangled particle will be lesser than that of the free particle. The coefficient of $\Delta^2 X_1$ and $\Delta^2 X_2$ are negative at all times. The coefficient of $\Delta^2 P_1$ is negative when $\sin\omega t$ is negative, and that of $\Delta^2 P_2$ is negative when $\sin\omega t$ is positive. As for the correlation terms, both of their coefficients are negative when $\sin \omega t = 0$. The result is similar for the second particle due to the symmetry in the entanglement, except that the cases of the correlation terms are interchanged. Thus, it is easy to identify times when the entangled uncertainty of the particles go below the free uncertainty by knowing what state we have prepared them in. The lowest value for the SQL determining commutator is $\hbar t/2m$ for when $\omega \to \infty$, which is an improvement over the SQL by a factor of $\sqrt{2}$.

Of the three entanglements we have discussed, this one might be the most realizable in practice, as coherent states of simple harmonic oscillators have been demonstrated using trapped ions.[Kienzler 15]. Moreover, this entanglement, as noted before, also conserves the momentum of the system through it's evolution between measurements.

The last entanglement that I have considered, characterized by the Hamiltonian

$$H = \frac{\hat{\mathbf{p}}_1^2}{2m} + \frac{\hat{\mathbf{p}}_2^2}{2m} + k(\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2)^2 \tag{3.31}$$

$$H = \frac{\hat{\mathbf{p}}_+^2}{2m} + \frac{\hat{\mathbf{p}}_-^2}{2m} + kx_+^2, \tag{3.32}$$

follows the same treatment and exhibits the same results as the previous one in terms of position uncertainty, although I couldn't find a classical analog for it. Maybe that is why it doesn't conserve momentum of the system either.

3.3 Concludig Remarks and The Future

Thus, I have derived four types of entanglements that can help us beat the SQL in an experimental setting. The problem of how to bring the mass into a contractive state to enjoy their contractiveness has been sidestepped by allowing for particles to be prepared in those states at the beginning itself, which should offset any concerns about using twice as many particles as before. For the last two Hamiltonians, even this trick is not necessary, although it can certainly be applied if necessary.

The work in this thesis should be seen as a first step to beating the SQL, especially in the regime of the Interferometers, and wherever there are more than one particles being used. When I first started studying this field, reviewing the literature on Interferometry and Shot Noise Limit, it struck me that the reason why we were able to counter the Photon Counting Error and the Radiation Pressure Error was the fact that we could see how they appeared via the quantum treatment of the evolution of light. But we haven't been able to do so for the SQL error in the Interferometer, because we do not have a completely quantum treatment of the photon's interaction with the mirrors. That is where our next step should be. To realize position measurement by a measurement of the first kind. Hopefully, once we understand more about light particle interaction, we may have a basis for where to apply the entanglements outlined in the previous section.

And the entanglements themselves need to be examined. This thesis was only the first step. The next natural step is to examine real world entanglements and how we can engineer them in the contexts of interferometers and wherever else they might be needed. For these, particle-particle interactions need to be investigated to see what types of entanglements they might show and how we can exploit them.

Another step one can take, and probably the easiest of all of these, is to apply such entanglements in atom interferometry. In Atom Interferometry, the roles of light and matter are reversed from their roles in the traditional Michelson Interferometer. Atom waves are injected into the arms of the interferometers and light pulses are used as Beam Splitters and mirrors. We know more about light entanglement than we do about matter entanglement, and this should make the construction of SQL beating entanglements and their implementation easier. After that, it might even be possible to bring these techniques back to the traditional interferometers and beat the SQL once and for all for all practical purposes.

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