Understanding Decoherence: Quantum Harmonic Oscillator in Spin Environment

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Certificate of Examination

This is to certify that the dissertation titled "**Understanding Decoherence: Quantum Harmonic Oscillator in Spin Environment**" submitted by Rishabh (Reg. No. MS14028) for the partial fulfillment of BS-MS dual degree programme of the institute, has been examined by the thesis committee duly appointed by the institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Prof. Kavita Dorai   Dr. Sandeep K. Goyal   Prof. Arvind

(Supervisor)

Dated: May 11, 2019

Declaration

The work presented in this dissertation has been carried out by me under guidance of Prof. Arvind at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Rishabh

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Dated: May 11, 2019

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Arvind

(Supervisor)

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To Maa and Mumma.

Abstract

We believe Quantum mechanics to be the fundamental theory of nature yet the everyday world of our perception seems very different from what quantum mechanics suggests.The superposition principle is one of the key principles of quantum theory. The lack of certain macroscopic superposition appears to defy these notions.

This pseudo contradiction can be solved (at least partially) by realizing the importance of environmental interactions. The assumption of closed system does not hold good while applying quantum mechanics to macroscopic systems. Decoherence is one of the key fruits of this realization.

Decoherence explains why for macroscopic objects we observe only a subset of quantum mechanically allowed states and never their superposition, if the system to began with is in such a superposition, Such a superposition will decay or so to say decohere into mixture of microscopically observed states.

In this thesis, we will try to understand the process of decoherence in much more depth by focusing on a particular model. The model of interest throughout the thesis will be Quantum Harmonic Oscillator interacting with the spin environment.

First we will discuss the situation in which our environment will be made up of spin-$\frac{1}{2}$ and have no internal dynamics of its own (self-Hamiltonian is zero). After this, using Born-Markov master equation formalism we will study the case of environment made up of spin-1 particles with non-zero self-Hamiltonian.

Chapter 1

Introduction

1.1 Quantum - Classical Divide

Quantum Mechanics (QM) is the crowning jewel of physicists. QM has explained a large variety of phenomena with incredible success, these include lasers, solid state devices, superconductivity, quantum cryptography, MRI and many more. QM has been brought to test many times in a large number of experiments performed since its inception and no contradiction has ever been found. Most physicists believe QM to be the fundamental theory that should explain all the underpinnings of our world. But despite all its successes there are still some important questions regarding QM which remain unanswered, of utmost importance among them is the question of Quantum to Classical transition.

The classical world i.e. the world of our experience appears to be in contradiction with the laws of QM. One of the fundamental principle of QM is Principle of Superposition:

- If $|\Psi_1\rangle$ and $|\Psi_2\rangle$ describe two quantum mechanically allowed states of a system then their superposition will also describe another quantum mechanically allowed state of the system.

It is difficult to reconcile the superposition principle with our every day reality, for example, let $|\Psi_1\rangle$ and $|\Psi_2\rangle$ describe two states of a macroscopic system which is localized in space at two different positions, then their superposition $|\Psi_{12}\rangle = \alpha |\Psi_1\rangle + \beta |\Psi_2\rangle$[1]

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[1] α, β belong to complex numbers and $|\alpha|^2 + |\beta|^2 = 1$.

are no longer localized with respect to position. Such delocalized macroscopic states do not correspond to our perception, so our every day reality seem to be incompatible with QM.

To state it differently, in words of Zurek[2] - " the Hilbert Space is huge i.e. in the quantum space of possibilities not just states like $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are allowed but their superpositions are also allowed. Within this huge space, we need to understand the origin of a small classical corner, the sort of states we perceive." States such as $|\Psi_1\rangle$ and $|\Psi_2\rangle$ which corresponds to our classical reality are called pointer states. We need to explain why macroscopic objects are almost always found in a pointer states but never in their superposition and how these pointer states are selected?

1.2 Decoherence

The question of preferred states (pointer states) is answered by the process called decoherence[1–6]. So what is decoherence?

The whole idea of decoherence is based on a simple realization that quantum system exists submerged in an environment and in particular for macroscopic systems it is impossible to isolate them from the environment (Figure 1.1).

So, if you start with a system in state $|\Psi_S(0)\rangle$, which could be written as a superposition in some basis $\{|S_i\rangle\}$ and let it interact with the environment with the initial state $|E_0\rangle$.

$$|\Psi_{SE}(0)\rangle = |\Psi_S(0)\rangle \otimes |E_0\rangle = \left(\sum_i a_i |S_i\rangle \right) \otimes |E_0\rangle \tag{1.1}$$

Typically even though the system and environment are initially not entangled, they will eventually get entangled due to interaction between them. So after some time you will have one big entangled state describing system and environment.

$$|\Psi_{SE}(t)\rangle = \sum_i \left(a_i |S_i\rangle \otimes |E_0\rangle \right) \tag{1.2}$$

As we are interested in the system only, so we have to take the trace over the environment and this will give us the reduced density matrix (reduced density matrix) of

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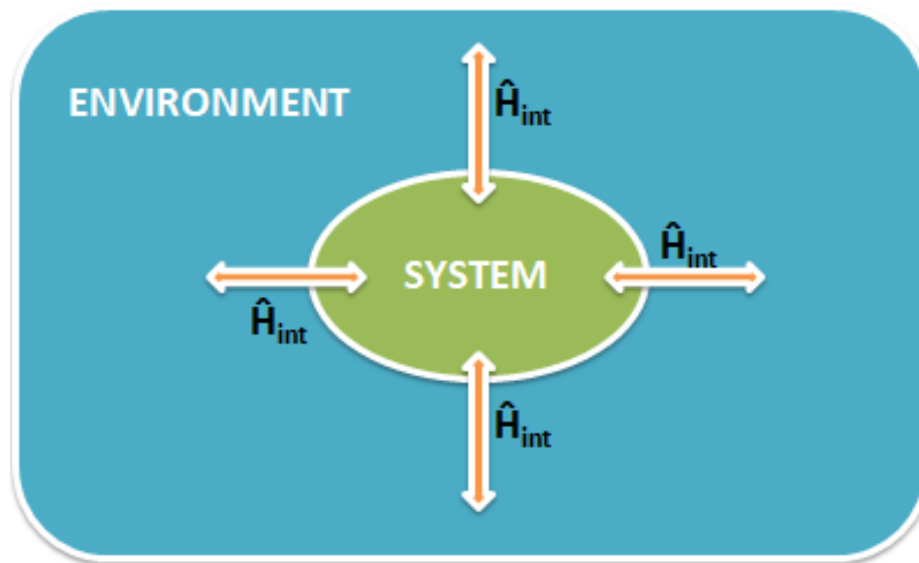[2]Taken from the talk given at Collge de France in March 2015.

FIGURE 1.1: Real world systems and not isolated, they interact with there environment.

the system.

$$\hat{\rho}_S(t) = Tr_E\left[\,|\Psi_{SE}(t)\rangle\,\langle\Psi_{SE}(t)|\,\right] = \sum_i a_i\,|S_i\rangle\,\langle S_i| \tag{1.3}$$

The important fact is that we can start with any initial state of the system and the environment, at the end the reduced density matrix of the system will always be diagonal in the same basis $\{|S_i\rangle\}$. So there is something very special about these basis states and if you start your system in one of these states, the state of the system will not get entangled with the environment.These states are the pointer states. Two important points to note about the pointer states from above discussion are:

- Pointer states are robust against environmental monitoring.

- Any initial superposition of pointer states is reduced to the classical mixture of pointer states after the system has interacted with environment.

The pointer states emerge purely from the interaction between the system and the environment and in particular the interaction Hamiltonian have a key deciding power as to which states emerge as pointer state. In simple situation when the interaction Hamiltonian dominates the dynamics, pointer states are simply selected because the observable which are diagonal in these pointer sates commute with the interaction Hamiltonian. In cases which are complicated one can go through a more sophisticated

process of going through the whole Hilbert space and see for each state how the purity of system changes after it interacts with the environment for suitable amount of time, pointer states are the states which remains least entangled with the environment and hence their purity changes the least. This process is called predictability sieve[3].

Example of Decoherence

Think of a spin-$\frac{1}{2}$ particle interacting with some environment. Consider that the self-Hamiltonian of both the bath and the environment are zero, so that the dynamics is completely governed by the interaction Hamiltonian ($\hat{H}_{int}$).

$$\hat{H}_{total} = \hat{H}_{int} = \hat{\sigma}_z \otimes \hat{E} \tag{1.4}$$

here $\hat{E}$ is the part of Hamiltonian acting on environment.

Even at this stage because of the form of interaction Hamiltonian we can be pretty sure that the pointer basis is going to be eigenstates of $\hat{\sigma}_z$ ($\{|0\rangle, |1\rangle\}$) If we start with following system-environment initial state:

$$|\Psi(0)\rangle = \left(\alpha |0\rangle + \beta |1\rangle \right) \otimes |\Psi_E(0)\rangle \tag{1.5}$$

Then time evolved state is given by:

$$|\Psi(0)\rangle = \alpha |0\rangle |E_0\rangle + \beta |1\rangle |E_1\rangle \tag{1.6}$$

given sufficient time $\langle E_0|E_1\rangle$ will approach zero. Let's try to think of this in terms of reduced density matrix of the system. The reduced density matrix which corresponds to initial pure state will evolve to a reduced density matrix which is diagonal in $|0\rangle, |1\rangle$ basis:

$$\hat{\rho}_S(0) = \begin{pmatrix} |\alpha|^2 & \alpha^*\beta \\ \alpha\beta^* & |\beta|^2 \end{pmatrix} \xrightarrow{\hat{H}_{int}} \hat{\rho}_S(t) = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix} \tag{1.7}$$

So the central spin has decohered into pointer basis ($|0\rangle, |1\rangle$).

1.3   What's Ahead?

In this thesis, we will try to understand decoherence in much more depth. The model of interest in the rest of the chapters will be Quantum Harmonic Oscillator interacting with the spin environment. In Chapter 2, we will take our environment to be made up of spin-$\frac{1}{2}$ and we will assume the self-Hamiltonian of our environment to be zero. In Chapter 3 our environment will be made up of spin-1 particles and the self-Hamiltonian of environment will not be zero, this chapter will make use of Born-Markov master equation formalism.

Chapter 2

Quantum Harmonic Oscillator in Spin-$\frac{1}{2}$ Environment ($\hat{H}_E = 0$)

In this chapter we will study a toy model with Quantum Harmonic Oscillator (QHO) as a central system interacting with spin-$\frac{1}{2}$ environment having zero self-Hamiltonian.

2.1 Details of Model

Central System

The central system (or simply system), $\mathcal{S}$, is a QHO with mass m and angular frequency Ω. The Hamiltonian of the system ($\hat{H}_S$) is:

$$\hat{H}_S = \frac{\hat{p}^2}{2m} + \frac{m\Omega^2}{2}\hat{x}^2 \tag{2.1}$$

where $\hat{p}$ is the momentum operator and $\hat{x}$ is the position operator.

The Hilbert space of the system is denoted as $\mathcal{H}_S$ and the set of energy eigenstates spanning the $\mathcal{H}_S$ is given by $\{|\phi_k\rangle\}$.

$$\hat{H}_S |\phi_k\rangle = (k + \frac{1}{2})\Omega |\phi_k\rangle \,^1 \tag{2.2}$$

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[1] in this thesis $\hbar$ is always taken as 1 except in section 2.4.

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Environment

Environment consists of N spin-$\frac{1}{2}$ particles. The Hilbert space of environment is the tensor product of the Hilbert spaces of individual spins and hence is of 2^N-dimension and is denoted by $\mathcal{E}$.

The self Hamiltonian of the environment is assumed to be zero i.e.,

$$\hat{H}_E = 0 \tag{2.3}$$

The set $\{|\epsilon_n\rangle\}$ spans the Hilbert space $\mathcal{H}_S$, $\{|\epsilon_n\rangle\}$ here is the computational basis of environment i.e.,

$$|\epsilon_0\rangle = |0\rangle_1 |0\rangle_2 ... |0\rangle_{N-1} |0\rangle_N$$

$$|\epsilon_1\rangle = |0\rangle_1 |0\rangle_2 ... |0\rangle_{N-1} |1\rangle_N$$

$$\vdots$$

$$|\epsilon_{2^N-1}\rangle = |1\rangle_1 |1\rangle_2 ... |1\rangle_{N-1} |1\rangle_N$$

where $|0\rangle_i$ and $|1\rangle_i$ are the eigenstates of $\hat{\sigma}_z^{(i)}$: the pauli - Z operator of ith spin.

Hamiltonian

Interaction Term

The interaction Hamiltonian ($\hat{H}_{int}$) is given by:

$$\hat{H}_{int} = \hat{x} \otimes \sum_{i=1}^{N} \left(g_i \hat{\sigma}_z^{(i)} \bigotimes_{i' \neq i} \hat{I}_{i'} \right) \tag{2.4a}$$

$$\equiv \hat{x} \otimes \hat{E} \tag{2.4b}$$

here g_i is the coupling strength of the ith spin with QHO.

The computational basis of environment $\{|\epsilon_n\rangle\}$ are the eigenstates of operator $\hat{E}$ with

eigenvalue ϵ_n:

$$\hat{E}\,|\epsilon_n\rangle = \epsilon_n\,|\epsilon_n\rangle \tag{2.5a}$$

$$\text{where } \epsilon_n = \sum_{i=1}^{N} g_i n_i \tag{2.5b}$$

$$\text{with } n_i = \begin{cases} +1 & ,\ i^{th} \text{ spin is in state } |0\rangle_i \\ -1 & ,\ i^{th} \text{ spin is in state } |1\rangle_i \end{cases} \tag{2.5c}$$

Total Hamiltonian

$$\hat{H}_{SE} = \left(\hat{H}_S \otimes \hat{I}_E\right) + \left(\hat{I}_S \otimes \hat{H}_E\right) + \hat{H}_{int} \tag{2.6a}$$

$$= \hat{H}_S \otimes \hat{I}_E + \hat{x} \otimes \hat{E} \tag{2.6b}$$

$$\equiv \hat{H}_S + \hat{x} \otimes \hat{E} \tag{2.6c}$$

eq. (2.6b) follows from eq. (2.4b) and (2.3).

Initial State

We assume initial state of our model to be a pure product state $|\Psi_{SE}(0)\rangle$ with the state of the system being $|\Psi_S(0)\rangle$ and that of environment being $|\Psi_E(0)\rangle$.

$$|\Psi_{SE}(0)\rangle = |\Psi_S(0)\rangle \otimes |\Psi_E(0)\rangle \tag{2.7}$$

Later on we will take $|\Psi_S(0)\rangle$ to be the equal superposition of two symmetrically opposite (in phase space) coherent states[2][7].

$$|\Psi_S(0)\rangle = A(|\alpha\rangle + |-\alpha\rangle) \tag{2.8}$$

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[2]Coherent states are the eigenstates of annihilation operator, $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$, where $\hat{a}$ is the annihilation operator of original harmonic oscillator.

where $|\pm\alpha\rangle$ are the coherent states and A is the normalization constant.

$$|\pm\alpha\rangle = e^{-|\alpha|^2/2} \sum_{k=0}^{\infty} \frac{(\pm\alpha)^k}{k!} |\phi_k\rangle \tag{2.9a}$$

$$A = \left[2\left(1 + e^{-2|\alpha|^2/2}\right)\right]^{-1/2} \tag{2.9b}$$

2.2 Motivations

Decoherence of QHO coupled to a spin environment has not been studied much in the past.Only one such study of which I am aware of use an approximate method to find the dynamics of the system [8]. So the natural question to ask is can we find the exact evolution of QHO coupled to spin-$\frac{1}{2}$ environment.

In 2005 Cucchietti, Paz and Zurek published a paper[9] in which they studied the decoherence of spin-$\frac{1}{2}$ particle in spin-$\frac{1}{2}$ environment. We realized that if apply the same method to QHO in spin-$\frac{1}{2}$ environment we may be able to find exact solution provided we take self-Hamiltonian of the environment to be zero.

Based upon the past studies of Quantum Brownian Motion (QBM)[10, 11], it's pretty clear that for QHO coupled to it's environment via its position the pointer states will be localized in phase space. A very nice way to see decoherence of QHO using Wigner function was used in following paper [11]. They started with the equal superposition of two symmetrically opposite (in phase space) coherent states. The Wigner function of such a super position has an interference pattern in between two peaksFigure 3.1a, this interference signifies the coherence between the two coherent states. As the time passes system interacts with the environment and get decohered in to the mixture of two localized states in phase space , In Wigner function plot this is marked by the decay of interference pattern Figure 3.1b.

Inspired by these we initialized our system in the equal superposition of two symmetrically opposite (in phase space) coherent states and we plotted the Wigner function at different times to see whether the interference pattern decays or not.
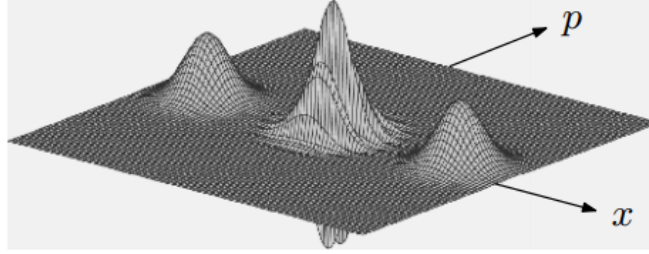
(A) Wigner function for initial state.
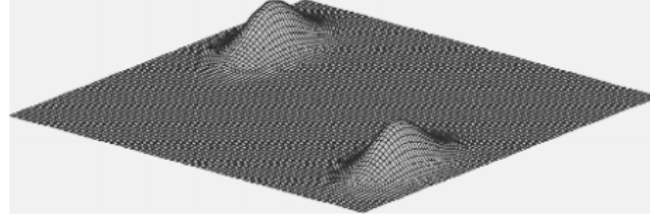
(B) Wigner function after decoherence.

FIGURE 2.1: Wigner function plot of QHO interacting with bath of QHOs.

2.3   Calculating the $\hat{\rho}_S(t)$

The initial state of environment could be expanded in $\{|\epsilon_n\rangle\}$ basis:

$$|\Psi_E(0)\rangle = \sum_{n=1}^{2^N-1} c_n |\epsilon_n\rangle \tag{2.10}$$

Using above equation combined initial state system - environment could be written as:

$$|\Psi_{SE}(0)\rangle = \sum_{n=1}^{2^N-1} c_n |\Psi_S(0)\rangle |\epsilon_n\rangle \tag{2.11}$$

Time Evolution

In this section we will calculate the system - environment state at any time:

$$|\Psi^{SE}(t)\rangle = \hat{U}^{SE}(t) |\Psi^{SE}(0)\rangle \tag{2.12a}$$

$$= \sum_{n=1}^{2^N-1} c_n e^{-i\hat{H}_{SE}t} |\Psi_S(0)\rangle |\epsilon_n\rangle \tag{2.12b}$$

To simplify it further consider the action of $\hat{H}_{SE}$ on $|\Psi_S(0)\rangle\,|\epsilon_n\rangle$:

$$\hat{H}^{SE}\left(\,|\Psi^S(0)\rangle\,|\epsilon_n\rangle\,\right) = \left(\hat{H}_S + \epsilon_n\hat{x}\right)|\Psi_S(0)\rangle\,|\epsilon_n\rangle \tag{2.13a}$$

$$\implies \left(\hat{H}^{SE}\right)^k\left(\,|\Psi^S(0)\rangle\,|\epsilon_n\rangle\,\right) = \left(\hat{H}_S + \epsilon_n\hat{x}\right)^k|\Psi_S(0)\rangle\,|\epsilon_n\rangle \tag{2.13b}$$

$$\implies e^{-i\hat{H}_{SE}t/}\left(\,|\Psi^S(0)\rangle\,|\epsilon_n\rangle\,\right) = \left(e^{-i(\hat{H}_S + \epsilon_n\hat{x})t}\,|\Psi_S(0)\rangle\,\right)|\epsilon_n\rangle \tag{2.13c}$$

let's define the following terms:

$$\hat{H}_S^n \equiv \hat{H}_S + \epsilon_n\hat{x} \tag{2.14a}$$

$$\hat{U}_S^n(t) \equiv e^{-i\hat{H}_S^n t} \tag{2.14b}$$

In terms of $\hat{U}_S^n(t)$ the $|\Psi_{SE}(t)\rangle$ could be written as:

$$|\Psi_{SE}(t)\rangle = \sum_{n=1}^{2^N-1}\left(\hat{U}_S^n(t)\,|\Psi^S(0)\rangle\,\right)|\epsilon_n\rangle \tag{2.15}$$

Reduced Density Matrix of system - Part A

From $|\Psi_{SE}(t)\rangle$ it is straight forward to calculate the combined system-environment density matrix:

$$\hat{\rho}_{SE}(t) = |\Psi_{SE}(t)\rangle\,\langle\Psi_{SE}(t)| \tag{2.16a}$$

$$= \sum_{n_1,n_2} c_n c_{n'}^*\left(\hat{U}_S^n(t)\,|\Psi^S(0)\rangle\,\langle\Psi^S(0)|\,\hat{U}_S^{n'\dagger}(t)\right)\otimes|\epsilon_n\rangle\,\langle\epsilon_{n'}| \tag{2.16b}$$

Now to calculate the reduced density matrix of system, we have to take the trace over environment:

$$\hat{\rho}_S(t) = Tr_E\left[\hat{\rho}_{SE}(t)\right] \tag{2.17a}$$

$$= \sum_n^{2^N-1}|c_n|^2\hat{U}_S^n(t)\,|\Psi^S(0)\rangle\,\langle\Psi^S(0)|\,\hat{U}_S^{n\dagger}(t) \tag{2.17b}$$

$$= \sum_n^{2^N-1}|c_n|^2\hat{U}_S^n(t)\hat{\rho}_s(0)\hat{U}_S^{n\dagger}(t) \tag{2.17c}$$

So, to calculate $\hat{\rho}_S(t)$ we need to find how the initial state of the system evolves under the unitary $\hat{U}_S^n(t)$.

Evolution of a coherent state under $\hat{U}_S^n(t)$

$\hat{U}_S^n(t)$ is the time evolution operator generated by $\hat{H}_S^n$.

$$\hat{H}_S^n = \hat{H}_S + \epsilon_n \hat{x} \tag{2.18a}$$

$$= \frac{\hat{p}^2}{2m} + \frac{m\Omega^2}{2}\hat{x}^2 + \epsilon_n \hat{x} \tag{2.18b}$$

$$= \frac{\hat{p}^2}{2m} + \frac{m\Omega^2}{2}\left(\hat{x} + \frac{\epsilon_n}{m\Omega^2}\hat{I}_S\right)^2 - \left(\frac{\epsilon_n^2}{2m\Omega^2}\hat{I}_S\right) \tag{2.18c}$$

Now let's define few terms:

$$x_n \equiv \frac{\epsilon_n}{m\Omega^2} \tag{2.19a}$$

$$\lambda_n \equiv \frac{\epsilon_n^2}{2m\Omega^2} \tag{2.19b}$$

$$\hat{y}_n \equiv \hat{x} + x_n \hat{I}_S \tag{2.19c}$$

$$\implies \hat{H}_S^n = \frac{\hat{p}^2}{2m} + \frac{m\Omega^2}{2}\hat{y}_n^2 - \lambda_n \hat{I}_S \tag{2.20}$$

So, $\hat{H}_S^n$ is the Hamiltonian representing an harmonic oscillator with the potential shifted by x_n and energy spectrum shifted by λ_n. Let $\{|\chi\rangle_k\}$ be the energy eigenstates of this shifted harmonic oscillator.

The coherent state of the original harmonic oscillator $|\alpha\rangle$ is also the coherent state of shifted harmonic oscillator.

$$\hat{a}_n |\alpha\rangle = (\alpha + x_n \sqrt{\frac{m\Omega}{2}}) |\alpha\rangle \equiv (\alpha + \gamma_n) |\alpha\rangle \tag{2.21}$$

where $\hat{a}_n$ is the annihilation operator for shifted harmonic oscillator.

So α could be written in the $\{|\chi\rangle_k\}$ basis as:

$$|\alpha\rangle = e^{-|\alpha + \gamma_n|^2/2} \sum_{k=0}^{\infty} \frac{(\alpha + \gamma_n)^k}{k!} |\chi_k\rangle \tag{2.22}$$

The time of the coherent state could be easily solved [7]:

$$\hat{U}_S^n(t)\ket{\alpha} = e^{-it(\frac{\Omega}{2} - \lambda_n)} e^{-|\alpha'_n(t) + \gamma_n|^2/2} \sum_{k=0}^{\infty} \frac{(\alpha'_n(t) + \gamma_n)^k}{k!} \ket{\chi_k} \tag{2.23}$$

where $\alpha'_n(t) = (\alpha + \gamma_n)e^{-i\Omega t} - \gamma_n$.

$$\hat{a}\left(\hat{U}_S^n(t)\ket{\alpha}\right) = \alpha'_n(t)\left(\hat{U}_S^n(t)\ket{\alpha}\right) \tag{2.24}$$

So $\hat{U}_S^n(t)\ket{\alpha}$ is the eigenstate of annihilation operator of original harmonic oscillator with eigenvalue $\alpha'_n(t)$ so in standard notation:

$$\hat{U}_S^n(t)\ket{\alpha} = \ket{\alpha'_n(t)} \tag{2.25}$$

Evolution of a initial state of system under $\hat{U}_S^n(t)$

Remember the initial state of the system was:

$$\ket{\Psi_S(0)} = A\left[\ket{\alpha} + \ket{-\alpha}\right]$$

Time evolved state:

$$\hat{U}_S^n(t)\ket{\Psi_S(0)} = A\left[\ket{\alpha'_n(t)} + \ket{-\alpha'_n(t)}\right] \tag{2.26a}$$

$$\equiv \ket{\Psi_S^n(t)} \tag{2.26b}$$

Reduced Density Matrix of system - Part B

The reduced density matrix at any time t (using eq. 2.17c):

$$\hat{\rho}_S(t) = \sum_n^{2^N - 1} |c_n|^2 \ket{\Psi_S^n(t)}\bra{\Psi_S^n(t)} \tag{2.27}$$

Now this could very well be written as:

$$\hat{\rho}_S(t) = \int_\epsilon J(\epsilon) \ket{\Psi_S^\epsilon(t)}\bra{\Psi_S^\epsilon(t)} \tag{2.28}$$

where $J(\epsilon)$ is given by:

$$J(\epsilon) = \sum_{n}^{2^N-1} |c_n|^2 \delta(\epsilon - \epsilon_n) \tag{2.29}$$

In [9] authors have shown that in the limit of large N, $J(\epsilon)^3$ will approach Gaussian behaviour:

$$J(\epsilon) \approx \frac{1}{\sqrt{2\pi s_N^2}} \exp -(\epsilon^2 - \bar{\epsilon}_N) 2s_N^2 \tag{2.30}$$

where $\bar{\epsilon}_N$ and s_N are the average value and standard deviation of $J(\epsilon)$. for fixed environment size and fixed initial environmental state s_N and $\bar{\epsilon}_N$ depend on the distribution of g_i, the coupling strength of the ith spin in the environment.

If the environment is initially in the product state:

$$|\Psi_E(0)\rangle = \bigotimes_{i=1}^{N} \left(a_i \, |0\rangle_i + b_i \, |1\rangle_i \right) \tag{2.31}$$

Then:

$$\bar{\epsilon}_N = \sum_{i}^{N} \left(|a_i|_i^2 - |b_i|^2 \right) g_i \tag{2.32a}$$

$$s_N^2 = \sum_{i^N} 4|a_i|_i^2 |b_i|^2 g_i^2 \tag{2.32b}$$

2.4   Wigner Function Plots

Now since we have calculated the reduced density matrix of our system at any arbitrary time t, we are ready to plot the Wigner function $W_S(x, p, t)$[5] of our system for different scenarios.

$$W_S(x, p, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipy} \langle x - \frac{y}{2}| \, \hat{\rho}_S(t) \, |x + \frac{y}{2}\rangle \, dy \tag{2.33}$$

For simplicity we have made following assumptions:

- m = 1

- $\Omega = 1$

---

[3]$J(\epsilon)$ describes the distribution of the environment energies.

- $\hbar = \frac{1}{2}$.[4]

- All the spins be in same state $\frac{1}{\sqrt{2}}[|0\rangle + |1\rangle]$.

- All coupling constant are taken to be equal to same constant g.

For above assumptions $\bar{\epsilon}_N = 0$ and $s_N^2 = g^2 N^2$.

On next four pages you will find the Wigner function plots pertaining to four different situations, in first three situation the initial state will be the superposition of two coherent states separated in x by 6 units ($\alpha = 3$) and in the last one the initial state is the superposition of two coherent states separated in p by 6 units ($\alpha = 3i$). These cases also differ from each other in the s_N value, larger s_N value implies stronger system-environment coupling. Details of all the four cases are below:

- **Case 1:** $s_N = 0.01$, $\alpha = 3$ (Figure 2.2).

- **Case 2:** $s_N = 0.5$, $\alpha = 3$ (Figure 2.3).

- **Case 3:** $s_N = 10$, $\alpha = 3$ (Figure 2.4).

- **Case 4:** $s_N = 0.5$, $\alpha = 3i$ (Figure 2.5).

The last figure of the chapter (Figure 2.6) shows the behaviour of the $x = 0$ and $p = 0$ point (peak of interference pattern) in Wigner function plot for different situations.

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[4]only in this section we have assumed $\hbar$ to be $1/2$ in rest of the thesis it is taken to be 1.

(A) t = 0

(B) t = 1

(C) t = 3

(D) t = 4

(E) t = 5

(F) t = 6
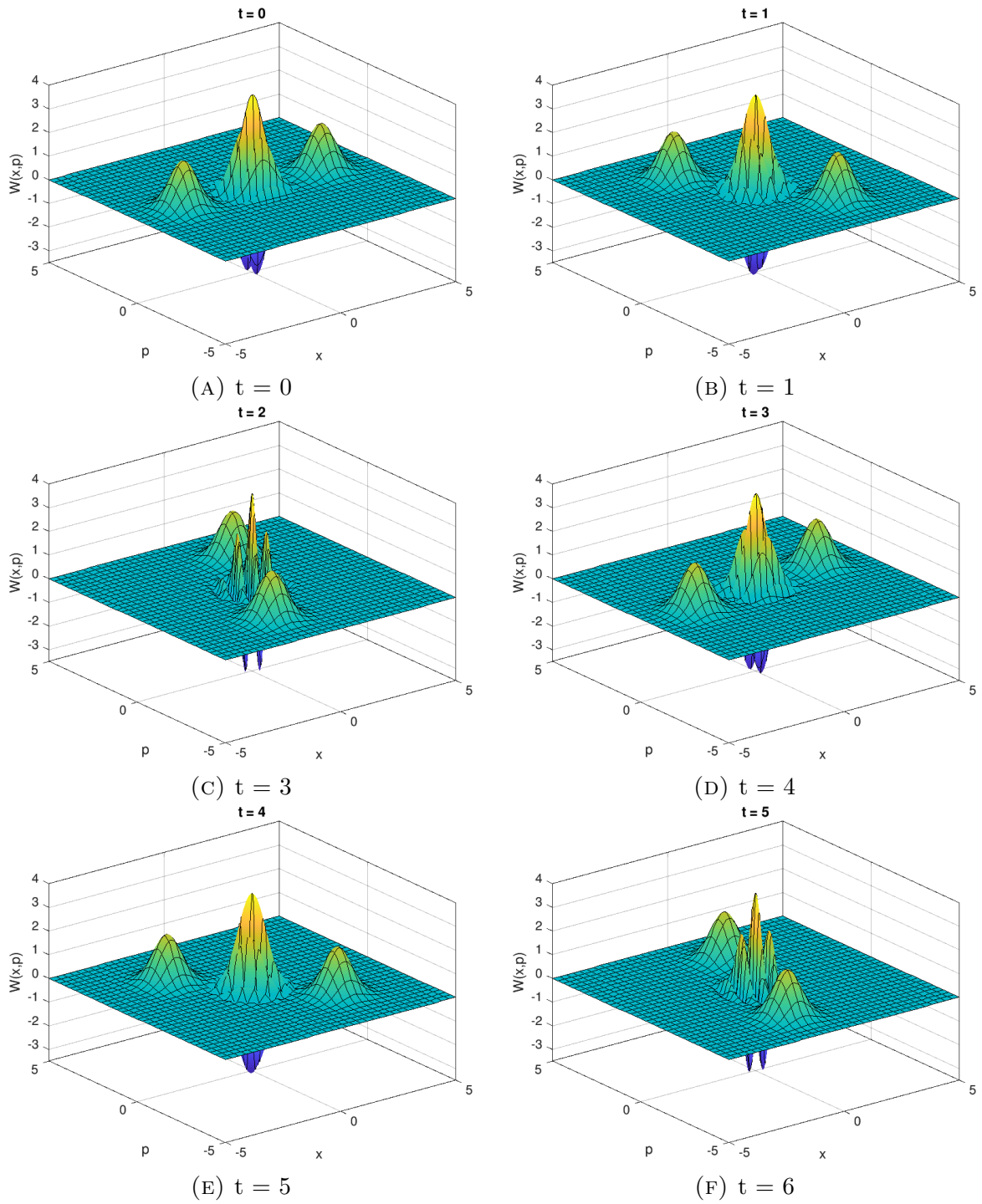
FIGURE 2.2: **Case 1:** $s_N = 0.01$ $\alpha = 3$

(A) t = 0

(B) t = 1

(C) t = 2

(D) t = 3.

(E) t = 4

(F) t = 5
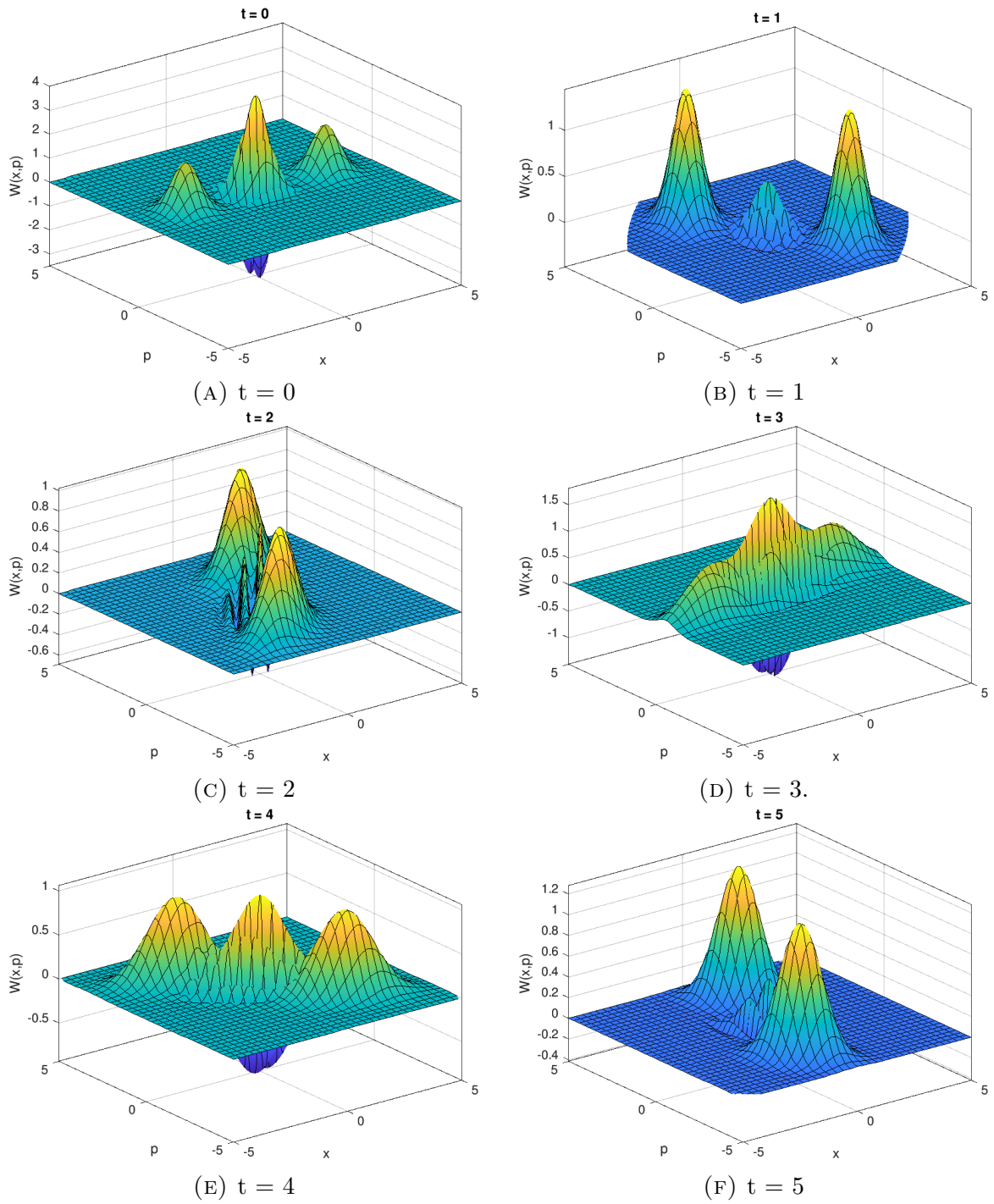
FIGURE 2.3: **Case 2:** $s_N = 0.5 \ \alpha = 3$

(A) t = 0

(B) t = 1

(C) t = 2

(D) t = 3

(E) t = 5

(F) t = 6
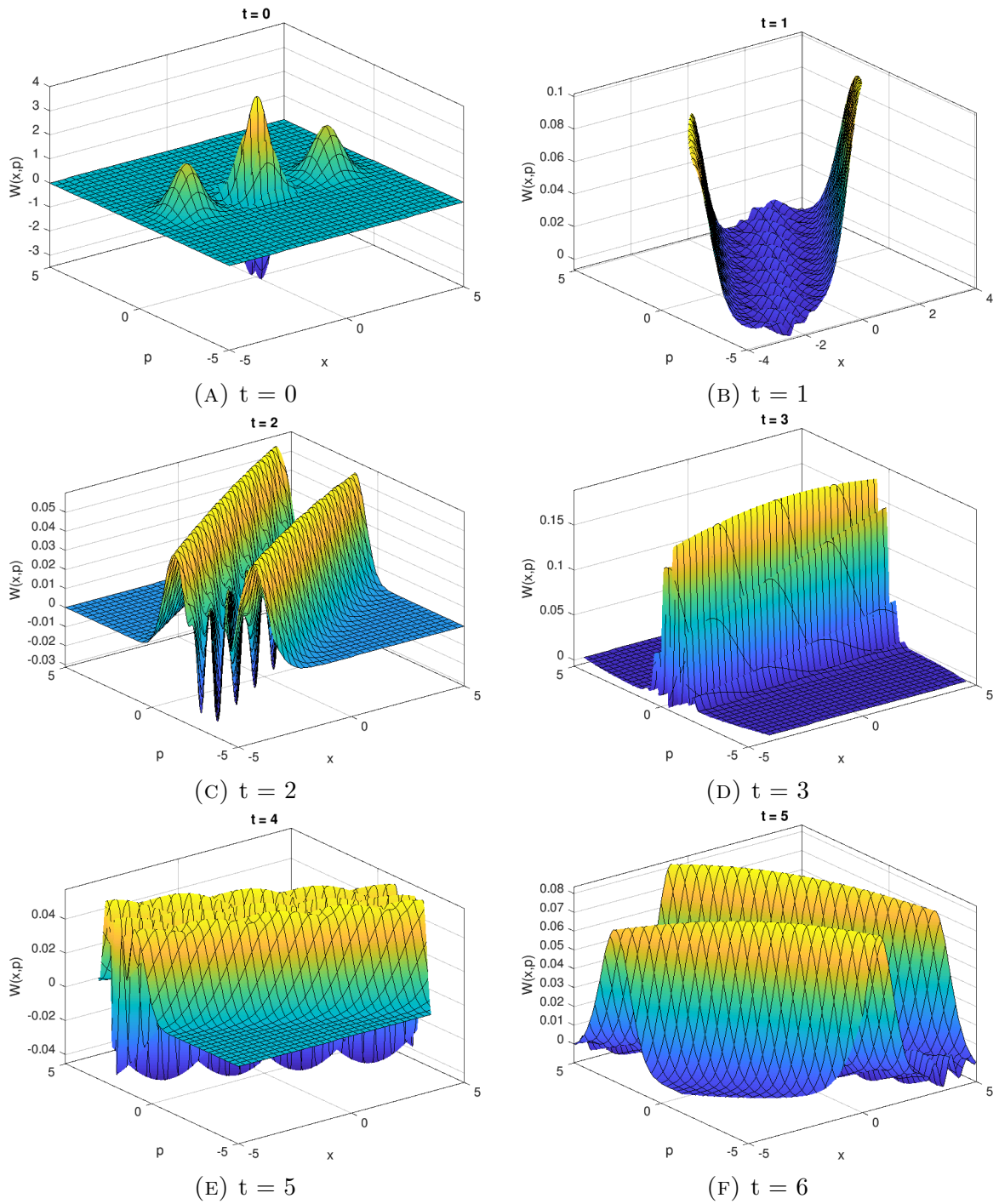
FIGURE 2.4: **Case 3:** $s_N = 0.10$ $\alpha = 3$

(A) t = 0

(B) t = 1

(C) t = 2

(D) t = 3

(E) t = 4

(F) t = 5

FIGURE 2.5: **Case 4:** $s_N = 0.5 \ \alpha = 3i$

(A) $s_N = 0.01$, $\alpha = 3$

(B) $s_N = 0.5$, $\alpha = 3$

(C) $s_N = 1$, $\alpha = 3$

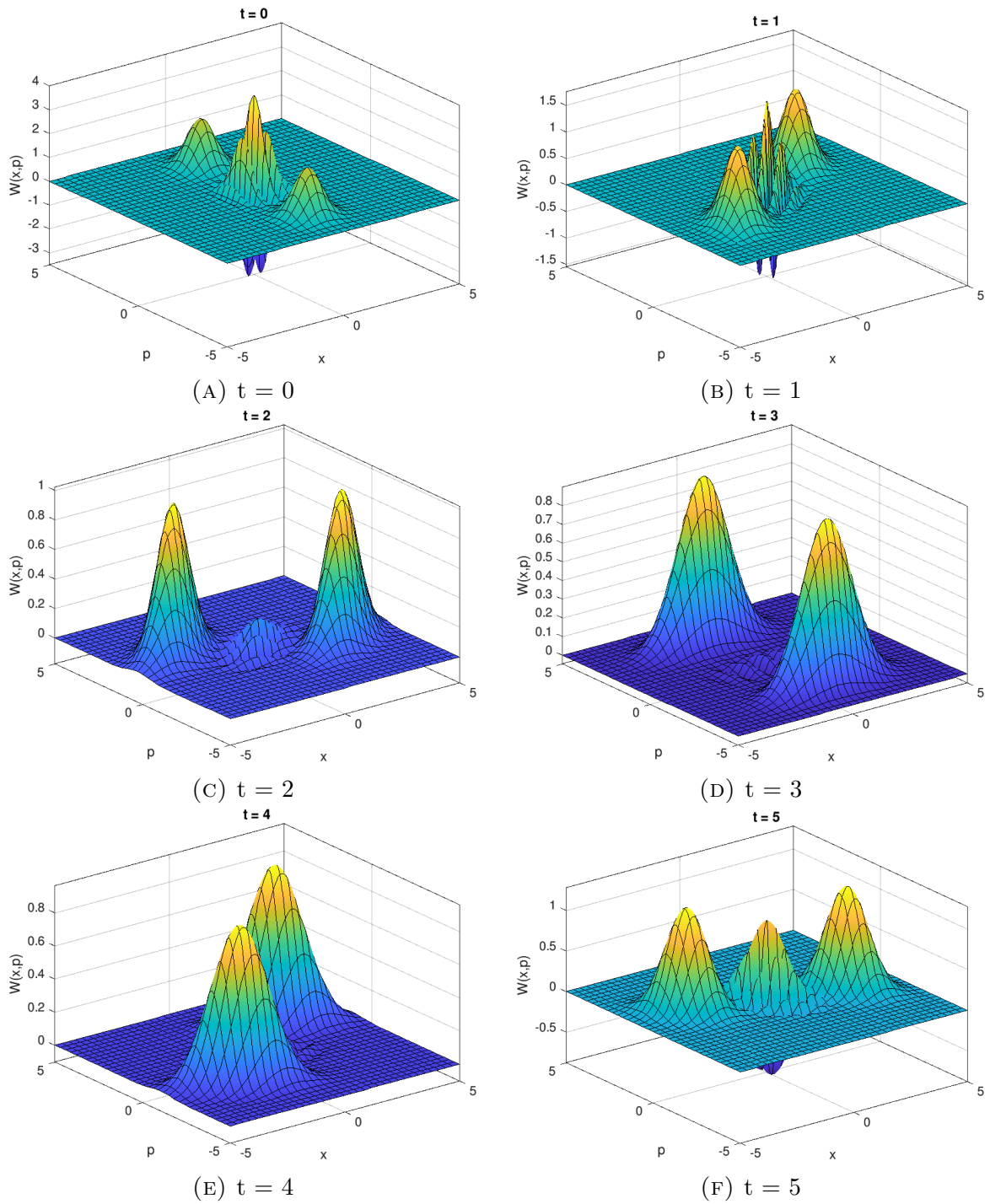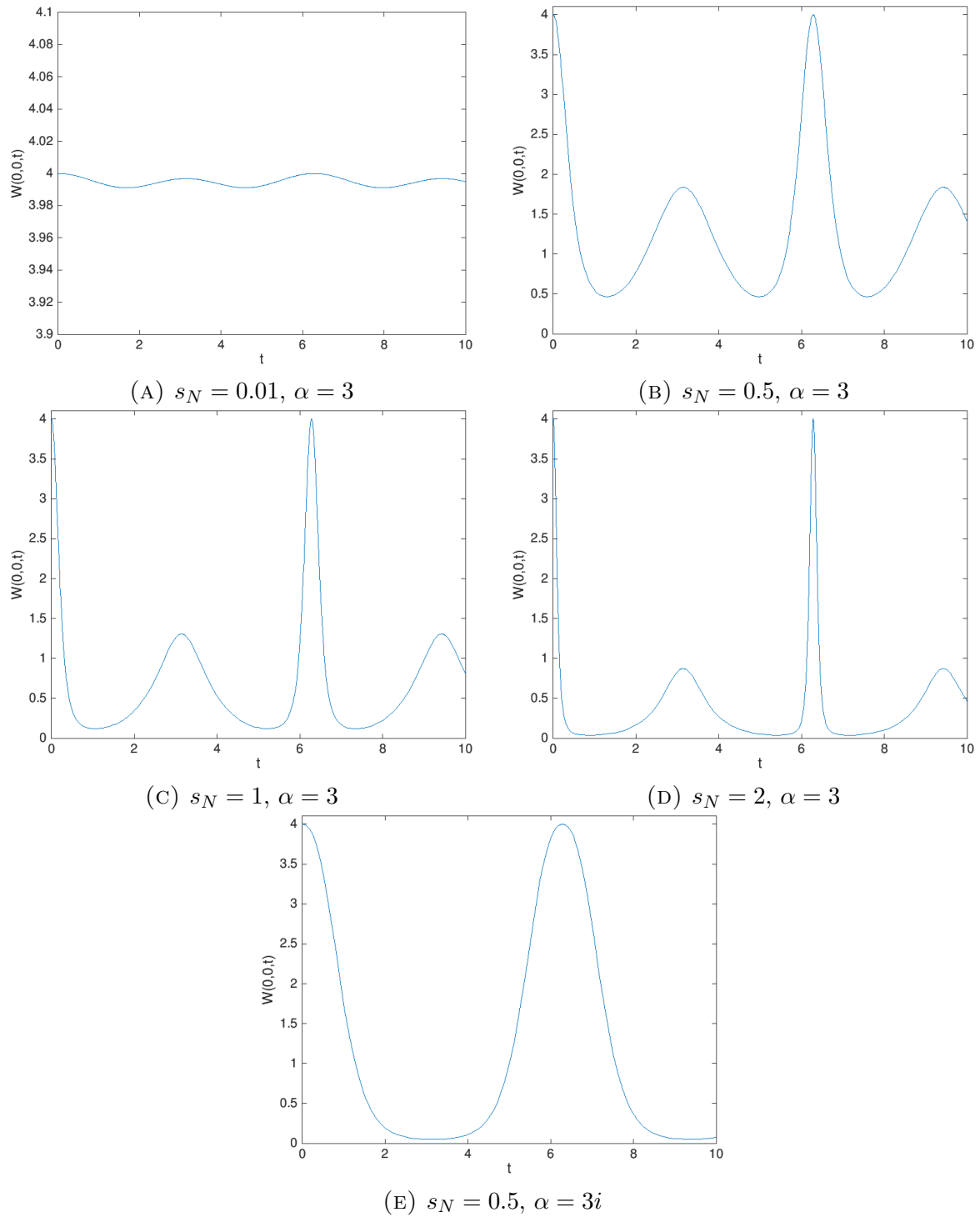(D) $s_N = 2$, $\alpha = 3$

(E) $s_N = 0.5$, $\alpha = 3i$

FIGURE 2.6: Wigner function at $x = 0$ and $p = 0$ from $t = 0$ to $t = 10$.

2.5 Results and Summary

- For Case 1: ($s_N = 0.01$ and $\alpha = 3$) the coupling is so weak that there is almost no effect of environment is seen, the interference pattern does not decay.(Figure 2.2)

- For Case 2: ($s_N = 0.5$ and $\alpha = 3$) interference pattern decays at first but then reappears after some time.(Figure 2.3)

- For Case 3: ($s_N = 10$ and $\alpha = 3$) interference pattern decays at first but then reappears after some time and the localization seems to be strong only in one direction.(Figure 2.4)

- For Case 4: ($s_N = 0.5$ and $\alpha = 3i$) interference pattern decays but the decay is slower than cases with initial separation in x. Interference reappears after some time.(Figure 2.5)

- All cases show the cyclic decay and birth of interference, with a period of $t = 2\pi$, since interference pattern does not decay permanently our model does not show decoherence.

- The cyclic behaviour comes because $\hat{\rho}_S(t)$ itself is cyclic.

Chapter 3

Quantum Harmonic Oscillator in Spin-1 Environment

In this chapter we will focus on QHO in spin-1 environment using the Born-Markov (B-M) Master equation formalism. This chapter will closely follow the approach used by Schlosshauer, Hines, and Milburn[8, 12, 13] to study QHO in spin-$\frac{1}{2}$ environment.

3.1 Details of Model

As in the last chapter the central system is still the same, but instead of spin-$\frac{1}{2}$ environment we have spin-1 environment and this time the self Hamiltonian of the environment is not taken to be zero. All the symbols in this chapter will have the same meaning as in previous chapter unless otherwise stated with just one caveat that the symbols pertaining to environment have to be understood in the context of spin-1 rather than spin-$\frac{1}{2}$.

Hamiltonian

Self-Hamiltonian of Environment

The self-Hamiltonian of environment is given by:

$$\hat{H}_E = -\sum_{i=1}^{N} \left(\frac{\omega_i}{2} \hat{\sigma}_x^{(i)} \bigotimes_{i' \neq i} \hat{I}_{i'} \right) \tag{3.1a}$$

$$\equiv \sum_{i=1}^{N} \hat{H}_E^{(i)} \tag{3.1b}$$

where $\hat{\sigma}_x^{(i)}$ is a spin-1 Pauli matrix (in $\hat{\sigma}_z^{(i)}$ basis):

$$\hat{\sigma}_x^{(i)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.2}$$

Interaction Hamiltonian

The interaction Hamiltonian is of same form as in the preceding chapter, just $\hat{\sigma}_z^{(i)}$ here refers to spin-1 Z-Pauli matrix.

$$\hat{H}_{int} = \hat{x} \otimes \sum_{i=1}^{N} \left(g_i \hat{\sigma}_z^{(i)} \bigotimes_{i' \neq i} \hat{I}_{i'} \right) \tag{3.3a}$$

$$\equiv \hat{x} \otimes \hat{E} \tag{3.3b}$$

Total Hamiltonian

$$\hat{H}_{SE} = \left(\hat{H}_S \otimes \hat{I}_E \right) + \left(\hat{I}_S \otimes \hat{H}_E \right) + \hat{H}_{int} \tag{3.4a}$$

$$\equiv \hat{H}_S + \hat{I}_E + \hat{x} \otimes \hat{E} \tag{3.4b}$$

3.2   Motivation

In [8] authors have studied the B-M Master Equation for an QHO in spin-$\frac{1}{2}$ environment and they found striking difference from the previously studied QBM (QHO in QHO bath)[10, 11].

In the case of harmonic oscillator environment the noise kernel which is responsible for decoherence has a temperature dependence of form $coth(1/T)$ and hence decoherence is also temperature dependent (increases with T), whereas the dissipation kernel responsible for dissipation is temperature independent.[1]

Schlosshauer and his coauthors have shown that if you replace the harmonic oscillator environment with spin-$\frac{1}{2}$ the noise kernel will become temperature independent and dissipation kernel will decrease with temperature as $tanh(1/T)$, so in this case decoherence will not be affected by temperature change but dissipation will decrease with increase in temperature.

So we wanted to see how will the temperature dependence of these kernels will change if instead of spin-$\frac{1}{2}$ or QHO the environment is composed of spin-1 particles.

3.3   Born-Markov Master Equation for QHO in Spin Environment

In this section we will state few important results of B-M master equation formalism[5, 14] relevant to our work. B-M formalism relies on following two assumption:

- Born Approximation: $\hat{\rho}_{SE}(t) \approx \hat{\rho}_S(t) \otimes \hat{\rho}_E(0)$

- Markov Approximation: "Memory effect" of environment can be neglected.

Using these two assumption one can derive the B-M master equation for the evolution of our system, if the central system is harmonic and interaction Hamiltonian is of the form:

$$\hat{H}_{int} = \hat{x} \otimes \hat{E} \tag{3.5}$$

---

[1]for defination of noise and dissipation kernal see next section.

and the self-correlation function of environment $C(t')$ [2] is of following form:

$$C(t') = \nu(t') - i\eta(t') \tag{3.6}$$

then B-M equation is of following form:

$$\frac{d}{dt}\hat{\rho}_S(t) = -i\left[\hat{H}_S + \frac{1}{2}m\bar{\Omega}^2\hat{x}^2, \hat{\rho}_S(t)\right] - i\gamma\left[\hat{x}, \{\hat{p}, \hat{\rho}_S(t)\}\right] - D\left[x, [\hat{x}, \hat{\rho}_S(t)]\right] - f\left[x, [\hat{p}, \hat{\rho}_S(t)]\right]\hat{H}_{int} = \hat{x}\otimes\hat{E} \tag{3.7}$$

here,

$$\bar{\Omega}^2 \equiv -\frac{2}{m}\int_0^\infty dt'\eta(t')cos(\Omega t') \tag{3.8a}$$

$$\gamma \equiv \frac{1}{m\Omega}\int_0^\infty dt'\eta(t')sin(\Omega t') \tag{3.8b}$$

$$D \equiv \int_0^\infty dt'\nu(t')cos(\Omega t') \tag{3.8c}$$

$$f \equiv -\frac{1}{m\Omega}\int_0^\infty dt'\nu(t')sin(\Omega t') \tag{3.8d}$$

3.4   Calculating Self-Correlation function for Spin-1 Environment

The self-correlation function of the environment $(C(t'))$ is given by:

$$C(t') = \langle\hat{E}^I(t')\hat{E}\rangle_{\hat{\rho}_E} \tag{3.9}$$

where $\hat{E}^I(t')$ is operator $\hat{E}$ in interaction picture and hence is given by:

$$\hat{E}^I(t') = e^{i\hat{H}_E t'}\hat{E}e^{-i\hat{H}_E t'} \tag{3.10}$$

As the the spins in environment do not interact with each other therefore there operators will commute with each other, we can write:

$$e^{i\hat{H}_E t'} = \prod_{i=1}^N e^{i\hat{H}_E^{(i)} t'} \tag{3.11}$$

---

[2]$C(t')$ is defined as $\langle\hat{E}^I(t')\hat{E}\rangle_{\hat{\rho}_E}$ where $\hat{E}^I(t')$ is operator $\hat{E}$ in interaction picture

Using eq. (3.11) we can simplify the expression of $\hat{E}^I(t')$:

$$\hat{E}^I(t') = e^{i\hat{H}_E t'}\left(\sum_{i=1}^{N} g_i \hat{\sigma}_z^{(i)}\right)e^{-i\hat{H}_E t'} \tag{3.12a}$$

$$= \sum_{i=1}^{N} e^{i\hat{H}_E t'} g_i \hat{\sigma}_z^{(i)} e^{-i\hat{H}_E t'} \tag{3.12b}$$

$$= \sum_{i=1}^{N} g_i e^{i\hat{H}_E^{(i)} t'} \hat{\sigma}_z^{(i)} e^{-i\hat{H}_E^{(i)} t'} \tag{3.12c}$$

$$= \sum_{i=1}^{N} g_i \hat{\sigma}_z^{(i)}(t') \tag{3.12d}$$

where $\hat{\sigma}_z^{(i)}(t')$ is the $\hat{\sigma}_z^{(i)}$ in the interaction picture.

$$C(t') = \sum_{i,k} g_i g_k \langle \hat{\sigma}_z^{(i)}(t')\hat{\sigma}_z^{(k)}\rangle_{\hat{\rho}_E} \tag{3.13}$$

because spins do not interact with each other and are not correlated, for $i \neq k$:

$$\langle \hat{\sigma}_z^{(i)}(t')\hat{\sigma}_z^{(k)}\rangle_{\hat{\rho}_E} = \langle \hat{\sigma}_z^{(i)}(t')\rangle_{\hat{\rho}_E}\langle \hat{\sigma}_z^{(k)}\rangle_{\hat{\rho}_E} \tag{3.14}$$

$$\implies C(t') = \sum_{i}\left(g_i\langle \hat{\sigma}_z^{(i)}(t')\rangle_{\hat{\rho}_E}\sum_{k\neq i}g_k\langle \hat{\sigma}_z^{(k)}\rangle_{\hat{\rho}_E}\right) + \sum_{i}g_i^2\langle \hat{\sigma}_z^{(i)}(t')\hat{\sigma}_z^{(i)}\rangle_{\hat{\rho}_E} \tag{3.15}$$

we can always assume the following:

$$\sum_{i} g_i\langle \hat{\sigma}_z^{(i)}\rangle_{\hat{\rho}_E} = 0 \tag{3.16}$$

this implies $\sum_{k\neq i} g_k\langle \hat{\sigma}_z^{(k)}\rangle_{\hat{\rho}_E}$ will also tend to 0. This simplifies the expression of $C(t')$ to a great extent,

$$C(t') = \sum_{i} g_i^2\langle \hat{\sigma}_z^{(i)}(t')\hat{\sigma}_z^{(i)}\rangle_{\hat{\rho}_E} \tag{3.17a}$$

$$= \sum_{i} g_i^2 \frac{1}{z_i} Tr_{E_i}[e^{-\hat{H}_E^{(i)}/k_B T}\hat{\sigma}_z^{(i)}(t')\hat{\sigma}_z^{(i)}] \tag{3.17b}$$

z_i is the partition function of ith spin.

In $\hat{\sigma}_x^{(i)}$ basis[3] :

$$\hat{H}_E^{(i)} = \frac{\omega_i}{2}\left|-1\right\rangle\left\langle-1\right| - \frac{\omega_i}{2}\left|1\right\rangle\left\langle1\right| \tag{3.18a}$$

$$\hat{\sigma}_z^{(i)} = \frac{1}{\sqrt{2}}\Bigl(\left|-1\right\rangle\left\langle0\right| + \left|0\right\rangle\left\langle-1\right| + \left|0\right\rangle\left\langle1\right| + \left|1\right\rangle\left\langle0\right|\Bigr) \tag{3.18b}$$

$$\hat{\sigma}_z^{(i)}(t') = \frac{e^{i\omega_i t'/2}}{\sqrt{2}}\Bigl(\left|-1\right\rangle\left\langle0\right| + \left|0\right\rangle\left\langle1\right|\Bigr) + \frac{e^{-i\omega_i t'/2}}{\sqrt{2}}\Bigl(\left|1\right\rangle\left\langle0\right| + \left|0\right\rangle\left\langle-1\right|\Bigr) \tag{3.18c}$$

Using the above expression we calculated z_i,

$$z_i = Tr_{E_i}\left[e^{-\hat{H}_E^{(i)}/k_B T}\right] \tag{3.19a}$$

$$= 1 + e^{\omega_i/2k_B T} + e^{-\omega_i/2k_B T} \tag{3.19b}$$

Now we can calculate $C(t')$:

$$C(t') = \sum_i g_i^2\left(\frac{2 + e^{\omega_i/2k_B T} + e^{-\omega_i/2k_B T}}{2(1 + e^{\omega_i/2k_B T} + e^{-\omega_i/2k_B T})}cos(\frac{\omega_i t'}{2}) - i\frac{e^{\omega_i/2k_B T} - e^{-\omega_i/2k_B T}}{2(1 + e^{\omega_i/2k_B T}e^{-\omega_i/2k_B T})}sin(\frac{\omega_i t'}{2})\right) \tag{3.20}$$

Let's introduce $J(\omega)$

$$J(\omega) = \sum_i g_i^2\delta(\omega - \omega_i) \tag{3.21}$$

$C(t')$ in terms of $J(\omega)$:

$$C(t') = \int_0^\infty d\omega J(\omega)\left(\frac{2 + e^{\omega/2k_B T} + e^{-\omega/2k_B T}}{2(1 + e^{\omega/2k_B T} + e^{-\omega/2k_B T})}cos(\frac{\omega t'}{2}) - i\frac{e^{\omega/2k_B T} - e^{-\omega/2k_B T}}{2(1 + e^{\omega/2k_B T}e^{-\omega/2k_B T})}sin(\frac{\omega t'}{2})\right) \tag{3.22a}$$

$$\equiv \nu(t') - i\eta(t') \tag{3.22b}$$

---

[3]to make things less messy we will omit the index i and x from our ket and bras

$\nu(t')$ and $\eta(t')$ are called the noise kernel and dissipation kernel respectively,

$$\nu(t') = \int_0^\infty d\omega J(\omega) \frac{2 + e^{\omega/2k_BT} + e^{-\omega/2k_BT}}{2(1 + e^{\omega/2k_BT} + e^{-\omega/2k_BT})} cos(\frac{\omega t'}{2}) \tag{3.23a}$$

$$\eta(t') = \int_0^\infty d\omega J(\omega) \frac{e^{\omega/2k_BT} - e^{-\omega/2k_BT}}{2(1 + e^{\omega/2k_BT}e^{-\omega/2k_BT})} sin(\frac{\omega t'}{2}) \tag{3.23b}$$

compare these with spin-$\frac{1}{2}$ case [8]:

$$\nu(t') = \int_0^\infty d\omega J(\omega) cos(\frac{\omega t'}{2}) \tag{3.24a}$$

$$\eta(t') = \int_0^\infty d\omega J(\omega) tanh(\frac{\omega}{2k_BT}) sin(\frac{\omega t'}{2}) \tag{3.24b}$$

and Harmonic oscillator case[5]:

$$\nu(t') = \int_0^\infty d\omega J(\omega) coth(\frac{\omega}{2k_BT}) cos(\frac{\omega t'}{2}) \tag{3.25a}$$

$$\eta(t') = \int_0^\infty d\omega J(\omega) sin(\frac{\omega t'}{2}) \tag{3.25b}$$

3.5   Results and Summary

We have plotted the temperature dependent part of the of $\nu(t')$ and $\eta(t')$ for harmonic oscillator environment, spin-$\frac{1}{2}$ environment and spin-1 environment with respect to temperature (measured in the units of $\frac{\omega}{2k_B}$), we have kept the ω constant.
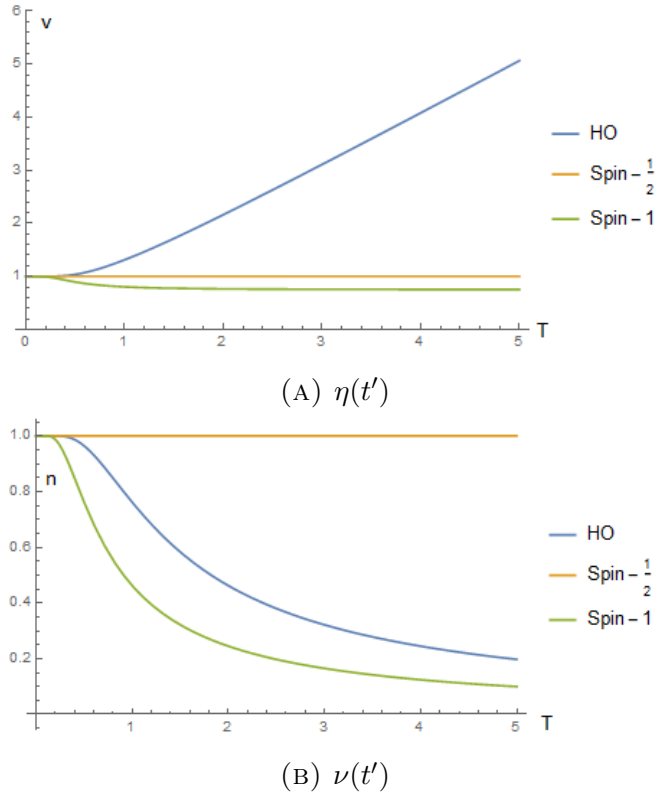
(A) $\eta(t')$

(B) $\nu(t')$

FIGURE 3.1: Temperature dependence of $\nu(t')$ and $\eta(t')$

The temperature dependence of $\nu(t')$ and $\eta(t')$ is almost of same form for spin-1 and spin-$\frac{1}{2}$ environment.

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