Topological Complexity

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Certificate of Examination

This is to certify that the dissertation titled **"Topological Complexity"** submitted by **Mr. Ramandeep Singh Arora** (Reg. No. MS14030) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Pranab Sardar   Dr. Shane D'mello   Dr. Mahender Singh

(Supervisor)

Dated: April 23, 2019

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Mahender Singh at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

Ramandeep Singh Arora
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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Mahender Singh
(Supervisor)

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Notation

X	A topological space
X^n	The n-fold product $X \times \ldots \times X$
I	The unit interval $[0, 1]$
PX or X^I	The free path space of X
$\dim(X)$	The topological dimension of X
id_X	The identity map on X
id_U^X	The inclusion map from a subset U of X into X
π_i	The projection map onto the i^{th} factor
Δ_n	The n-fold diagonal map from X into X^n
$H_n(X)$	The n^{th} homology group of X with coefficients in $\mathbb{Z}$
$\widetilde{H}_n(X)$	The n^{th} reduced homology group of X with coefficients in $\mathbb{Z}$
$H^n(X)$	The n^{th} cohomology group of X with coefficients in $\mathbb{Z}$
$H^n(X; R)$	The n^{th} cohomology group of X with coefficients in ring R
$H^*(X; R)$	The cohomology ring of X with coefficients in ring R
$\widetilde{H}^*(X; R)$	The reduced cohomology ring of X with coefficients in ring R
$\mathrm{secat}(p)$	The Schwarz genus of fibration $p : E \to B$
$\mathrm{cat}(X)$	The Lusternik-Schnirelmann category of X
$\mathrm{TC}(X)$	The topological complexity of X
$\mathrm{TC}_n(X)$	The higher topological complexity of X

Abstract

The goal of this project is to study numerical homotopy invariants called the higher topological complexity $\mathrm{TC}_n(X)$ of a topological space X for $n \geq 2$. We begin by introducing the notion of Schwarz genus of a surjective fibration which provides us insights for understanding the numerical homotopy invariants - Lusternik-Schnirelmann (LS) category and higher topological complexity of spaces as both of them are the Schwarz genus of specific path space fibrations. We further explore the LS category of a space and study its bounds, since for any fibration $p : E \to B$ the Schwarz genus of p is bounded above by the LS category of the base space B. In particular, $\mathrm{TC}_n(X)$ is bounded above by the LS category of the base space of the corresponding path space fibration. We then implement the results associated with the Schwarz genus and LS category to study the higher topological complexity comprehensively.

Contents

Chapter 1

Prerequisite Knowledge

In this chapter, we study some basic concepts in topology required for a good understanding of the later chapters.

1.1 Path Spaces

In this section, we give a topology on the set of all paths of a topological space and study its fundamental properties. Some of the text in this section can be found in [3, Chapter 6].

Definition 1.1.1. Suppose X and Y are topological spaces. Let $C(X,Y)$ be the set of all continuous maps $f : X \to Y$. Give a topology on $C(X,Y)$ generated by sets of the form,

$$B(K,U) = \{f \in C(X,Y) \mid f(K) \subset U\},$$

where K is compact in X and U is open in Y. The topology obtained on $C(X,Y)$ is called **compact-open** topology.

Clearly the sets $B(K,U)$ forms a subbasis for the compact-open topology on $C(X,Y)$.

Definition 1.1.2. Let I be the unit interval $[0,1]$. Then $C(I,X)$ is called the **free path space** of X and is denoted by PX or X^I, i.e.,

$$PX = \{\alpha : I \to X \mid \alpha \text{ is continuous}\}.$$

Let x_0 be a fixed point in the space X. Define

$$P_{x_0}X = \{\alpha \in PX \mid \alpha(0) = x_0\},$$

and

$$\Omega_{x_0}X = \{\alpha \in PX \mid \alpha(0) = \alpha(1) = x_0\}.$$

The subspace $\Omega_{x_0}(X)$ is called the **loop space** of X at x_0.

Proposition 1.1.3. The evaluation map $\mathrm{ev} : PX \times I \to X$, given by $\mathrm{ev}(\alpha, t) = \alpha(t)$, is continuous.

Proof. Let U be an open subset of X. Then $\mathrm{ev}^{-1}(U) = \{(\alpha, t) \in PX \times I \mid \alpha(t) \in U\}$. Let $(\alpha, t) \in \mathrm{ev}^{-1}(U)$. Since α is continuous, there exists an open neighborhood of t, say J, such that $\alpha(J) \subset U$. Take K to be a compact set contained in J such that t is an interior point of K, then $\alpha(K) \subset U$. Hence $(\alpha, t) \in B(K, U) \times \mathrm{int}(K)$ is open in $PX \times I$. If $(\beta, t') \in B(K, U) \times \mathrm{int}(K)$, then $\beta(K) \subset U$. Since $t' \in \mathrm{int}(K)$, we have $\beta(t') \in U$, i.e., $(\beta, t') \in \mathrm{ev}^{-1}(U)$. Thus $B(K, U) \times \mathrm{int}(K)$ is an open subset of $\mathrm{ev}^{-1}(U)$ containing (α, t). Therefore the evaluation map ev is continuous. ■

Proposition 1.1.4. The map $H : Y \times I \to X$ is continuous if and only if the map $H' : Y \to PX$, given by $H'(y)(t) = H(y, t)$, is continuous.

Proof. Suppose the map $H' : Y \to PX$ is continuous. Consider the homotopy defined by the composition of the following maps

$$U \times I \xrightarrow{H' \times \mathrm{id}} PX \times I \xrightarrow{\mathrm{ev}} X.$$

Then

$$\mathrm{ev}((H' \times \mathrm{id})(y, t)) = \mathrm{ev}((H'(y), t)) = H'(y)(t) = H(y, t).$$

Thus H is continuous.

Conversely, if $H : Y \times I \to X$ is continuous. It is enough to check continuity of H' on subbasis of PX. Let $B(K, U)$ be a subbasis element of PX and $y \in H'^{-1}(B(K, U))$. Then $H'(y)(K) \subset U$ implying $\{y\} \times K \subset H^{-1}(U)$. Since H is continuous, for each $(y, t_i) \in \{y\} \times K$ there exists an open neighborhood $V_i \times W_i$ of (y, t_i) such that $H(V_i \times W_i) \subset U$. Using compactness of K, there exist finite number of W_i, say $i = 1, \ldots, n$, such that $\cup_{i=1}^{n} W_i \supset K$. Suppose $V = \cap_{i=1}^{n} V_i$ and $W = \cup_{i=1}^{n} W_i$, then $\{y\} \times K \subset V \times W \subset \cup_{i \in K}(V_i \times W_i)$ implying $H(V \times W) \subset U$. Since $K \subset W$, we have $H(V \times K) \subset U$, i.e. $H'(V) \subset B(K, U)$. Thus H' is continuous. ■

The restriction of the evaluation map $\mathrm{ev} : PX \times I \to X$ on $PX \times \{1\}$ gives a continuous map $\mathrm{ev}_1 : PX \to X$ given by $\mathrm{ev}_1(\alpha) = \alpha(1)$ for all $\alpha \in PX$. Similarly, restricting ev on $PX \times \{0\}$ gives a continuous map $\mathrm{ev}_0 : PX \to X$ given by $\mathrm{ev}_0(\alpha) = \alpha(0)$ for all $\alpha \in PX$. Thus we can define a continuous map

$$\widetilde{e} : PX \to X \times X$$

given by $\widetilde{e}(\alpha) = (\alpha(0), \alpha(1))$ for all $\alpha \in PX$. More generally, we can define a continuous map

$$\widetilde{e}_n : PX \to X \times \ldots \times X = X^n$$

given by $\widetilde{e}_n(\alpha) = (\alpha(0), \alpha(t_1), \ldots, \alpha(t_{n-2}), \alpha(1))$ where $t_i = i/(n-1)$ for $i \in \{0, \ldots, n-1\}$ and $n \geq 2$. Clearly $\widetilde{e}_2 = \widetilde{e}$.

Lemma 1.1.5. The free path space PX is homotopy equivalent to X. The map $\mathrm{ev}_j : PX \to X$, for $j = 0, 1$, is a homotopy equivalence with the map $i : X \to PX$, which maps $x \in X$ to the constant path c_x at x, as the homotopy inverse.

Proof. Let $i : X \to PX$ be the map which maps $x \in X$ to the constant path c_x at x. Then $\mathrm{ev}_j \circ i = \mathrm{id}_X$ and $i \circ \mathrm{ev}_j(\alpha) = c_{\alpha(j)}$ for $j = 0, 1$. Thus it is enough to show $i \circ \mathrm{ev}_j$ is homotopic to id_{PX}.

Let $F_j : PX \times I \to PX$ be defined by

$$F_j(\alpha, s)(t) = \alpha(t + s(j - t)), \text{ for } j = 0, 1.$$

Then $F_j(\alpha, 0) = \alpha$ and $F_j(\alpha, 1) = c_{\alpha(j)}$. Hence, ev_j and i are homotopy inverses. ■

Corollary 1.1.6. The space $P_{x_0}X$ is contractible.

Proof. Let $F_0 : P_{x_0}X \times I \to P_{x_0}X$ be defined by

$$F_0(\alpha, s)(t) = \alpha(t - st).$$

Then $F_0(\alpha, 0) = \alpha$ and $F_0(\alpha, 1) = c_{\alpha(0)} = c_{x_0}$. Thus $P_{x_0}X$ is contractible. ■

Definition 1.1.7. Let $f : X \to Y$ be a continuous map. Then the space $P_f = \{(\alpha, x) \in PY \times X \mid \alpha(0) = f(x)\}$ is called the **mapping path space** of f.

Lemma 1.1.8. Let $f : X \to Y$ be a continuous map. Then the space P_f is homotopy equivalent to X and the projection map $\pi_2 : P_f \to X$, given by $\pi_2(\alpha, x) = x$, is a homotopy equivalence.

Proof. Let $i : X \to P_f$ be the map which maps $x \in X$ to the tuple $(c_{f(x)}, x)$ where $c_{f(x)}$ is the constant path at $f(x)$. Then $\pi_2 \circ i = \mathrm{id}_X$ and $i \circ \pi_2(\alpha, x) = (c_{f(x)}, x)$. Thus it is enough to show $i \circ \pi_2$ is homotopic to id_{P_f}.

Let $F_0 : PY \times I \to PY$ be defined by

$$F_0(\alpha, s)(t) = \alpha(t - st).$$

Then $F_0(\alpha, 0) = \alpha$ and $F_0(\alpha, 1) = c_{\alpha(0)}$. Now, define $F : P_f \times I \to P_f$ by

$$F(\alpha, x, t) = (F_0(\alpha, t), x).$$

Then $F(\alpha, x, 0) = (F_0(\alpha, 0), x) = (\alpha, x)$ and $F(\alpha, x, 1) = (F_0(\alpha, 1), x) = (c_{\alpha(0)}, x) = (c_{f(x)}, x)$. Thus P_f is homotopy equivalent to X. ■

1.2 Fiber Bundles

In this section, we study a nice class of maps called fiber bundles. We refer to [7, Chapter 4] for details.

Definition 1.2.1. A **fiber bundle** structure (E, B, p, F) on space E, with fiber F, consists of a continuous map $p : E \to B$ such that for each point $b \in B$ there exists an open neighborhood U of b and a homeomorphism $\phi : p^{-1}(U) \to U \times F$ such that the following diagram commute

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\phi} & U \times F \\ & \searrow^{p} & \downarrow{\scriptstyle \pi_1} \\ & & U \end{array}$$

where π_1 is the projection onto the first factor.

The space B is called the **base space**, E the **total space**, and F the **fiber** of the bundle. The map p is called the **projection map** and the collection $\{(U_i, \phi_i)\}$ is called the **local trivialization** of the bundle.

A fiber bundle structure (E, B, p, F) is sometimes denoted as

$$F \longrightarrow E \xrightarrow{\ p\ } B,$$

a 'short exact sequence of spaces.'

Since $p = \pi_1 \circ \phi$ on $p^{-1}(U)$ and ϕ is a homeomorphism, it implies that ϕ maps $p^{-1}(b)$ homeomorphically onto $\{b\} \times F$ and the map p is surjective. The preimage $p^{-1}(b)$ is called the **fiber over p** and is homeomorphic to the fiber F. Thus the fiber bundle structure is determined by the projection map $p : E \to B$.

Example 1.2.2. The projection map $\pi_1 : B \times F \to B$ is a fiber bundle with fiber F.

Here E is not just locally a product $B \times F$ but also globally. Such a fiber bundle is called a **trivial bundle**.

Example 1.2.3. The Möbius band mapping onto its central circle is fiber bundle with fiber [0,1]. But it's not a trivial bundle.

Example 1.2.4. If the fiber F has discrete topology, then the map $p : E \to B$ is a covering map. Conversely, if $p : E \to B$ is a covering map such that $p^{-1}(b)$ has the same cardinality for all $b \in B$, then the map $p : E \to B$ is a fiber bundle with discrete fiber.

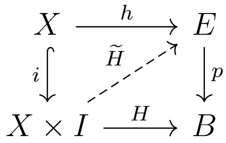
1.3 Fibrations

In this section, we study a class of maps called fibrations, which are very important from homotopy theory point of view. A fibration in a sense is a generalization of a fiber bundle, i.e., the fibers $p^{-1}(b)$ need not be homeomorphic as in the case of a fiber bundle, but they are homotopy equivalent. Most of the text in this section can be found in [3, Chapter 6].

If $H : X \times I \to Y$ is a homotopy, let $H_t : X \to Y$ denote the map $H_t(x) = H(x, t)$.

Definition 1.3.1. A continuous map $p : E \to B$ is said to have **homotopy lifting property** with respect to a topological space X if, given a homotopy $H : X \times I \to B$, and a map $h : X \to E$ lifting H_0, i.e. $p \circ h = H_0$, there exists a homotopy $\widetilde{H} : X \times I \to E$ lifting H, i.e. $p \circ \widetilde{H} = H$, and satisfying $\widetilde{H}_0 = h$.

Let $i : X \hookrightarrow X \times I$ be the inclusion map given by $i(x) = (x, 0)$. Then the diagram given below depicts the homotopy lifting property of the map $p : E \to B$ with respect to the space X.



The outer diagram commutes if and only if the hypotheses of homotopy lifting property are satisfied. A lifting $\widetilde{H}$ of H, satisfying $\widetilde{H}_0 = h$, corresponds to a dotted arrow making the whole diagram commute.

Definition 1.3.2. A continuous map $p : E \to B$ is said to be a **Hurewicz fibration** if it has homotopy lifting property with respect to any space X.

Definition 1.3.3. A continuous map $p : E \to B$ is said to be a **Serre fibration** if it has homotopy lifting property with respect to I^n for $n \geq 0$.

Unless mentioned the word ‘fibration’ will be used for Hurewicz fibration.

Remark 1.3.4. These are some trivial observations:

- A Hurewicz fibration is a Serre fibration.
- A homeomorphism is a fibration.
- Cartesian product and composition of two fibrations is a fibration.
- A covering map is a fibration since covering maps have homotopy lifting property for all spaces.

Example 1.3.5. The projection map $\pi_1 : B \times F \to B$ is a fibration.

Given a homotopy $H : X \times I \to B$ and a map $h : X \to B \times F$ lifting H_0, then the map $\widetilde{H} : X \times I \to B \times F$, given by $\widetilde{H}(x,t) = (H(x,t), \pi_2 \circ h(x))$, lifts H and $\widetilde{H}(x,0) = (H(x,0), \pi_2 \circ h(x)) = (H_0(x), \pi_2 \circ h(x)) = (\pi_1 \circ h(x), \pi_2 \circ h(x)) = h(x)$.

Example 1.3.6. Let $\pi_1 : \mathbb{R} \times I \to \mathbb{R}$ be the projection map onto the first factor. Suppose $H : \mathbb{R} \times I \to \mathbb{R}$ is a homotopy between the zero map, i.e., $H_0(x) = 0$ for all $x \in \mathbb{R}$, and the identity map on $\mathbb{R}$, i.e., $H_1(x) = x$ for all $x \in \mathbb{R}$. Let $h : \mathbb{R} \to \mathbb{R} \times I$ be the zero map, i.e., $h(x) = (0,0)$ for all $x \in \mathbb{R}$. Then h lifts H_0 with respect to the projection map π_1.

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{h} & \mathbb{R} \times I \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle \pi_1} \\ \mathbb{R} \times I & \xrightarrow{H} & \mathbb{R} \end{array}$$

The maps $\widetilde{H}(x,t) = (H(x,t), t)$ and $G(x,t) = (H(x,t), 0)$ both lifts H, and satisfies $\widetilde{H}(x,0) = G(x,0) = h(x)$. Thus the lift of a homotopy may not be unique.

Theorem 1.3.7. Let B be a paracompact space and $p : E \to B$ be a continuous map. Suppose there exists an open cover $\{U_\alpha\}_{\alpha \in I}$ of B such that $p : p^{-1}(U_\alpha) \to U_\alpha$ is a fibration for each $\alpha \in I$. Then $p : E \to B$ is a fibration.

Corollary 1.3.8. Let $p : E \to B$ be a fiber bundle and B a paracompact space. Then p is a fibration.

Proof. Let $p : E \to B$ be a fiber bundle, with fiber F. Then for each $b \in B$ there exists an open neighborhood U_b and a homeomorphism $\phi_b : p^{-1}(U_b) \to U_b \times F$ such that $\pi_1 \circ \phi_b = p$. Since both π_1 and ϕ_b are fibrations, the map $p = \pi_1 \circ \phi_b : p^{-1}(U_b) \to U_b$ is also a fibration for all $b \in B$. Thus $p : E \to B$ is a fibration. ■

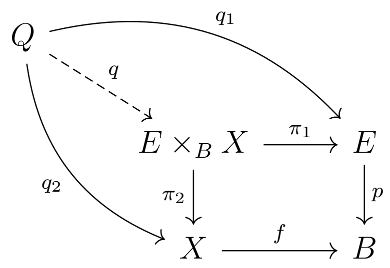
Theorem 1.3.9. Let $p : E \to B$ be a fibration. Then the fibers $p^{-1}(b)$ are homotopy equivalent over each path connected component of B.

Definition 1.3.10. Let $p : E \to B$ and $f : X \to B$ be continuous maps. Let $E \times_B X = \{(e,x) \in E \times X \mid p(e) = f(x)\}$ be the subspace of $E \times X$. Then the following diagram commutes

$$\begin{array}{ccc} E \times_B X & \xrightarrow{\pi_1} & E \\ {\scriptstyle \pi_2}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & B \end{array}$$

where π_1 and π_2 are the projection maps. The space $E \times_B X$ with the maps π_1 and π_2, denoted by $(E \times_B X, \pi_1, \pi_2)$, is called the **pullback** or the **fibered product** of p and f.

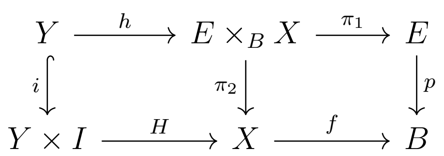
Suppose there exists a space Q with continuous maps $q_1 : Q \to E$ and $q_2 : Q \to X$ such that $p \circ q_1 = f \circ q_2$.



Then we can define a map $q : Q \to E \times_B X$ by $q = (q_1, q_2)$ which makes the above diagram commute, i.e., the pullback $(E \times_B X, \pi_1, \pi_2)$ is universal with respect to the above diagram. Thus the pullback $E \times_B X$ is unique up to homeomorphism.

Proposition 1.3.11. Let $p : E \to B$ be a fibration and $f : X \to B$ be a continuous map. Then $\pi_2 : E \times_B X \to X$ is a fibration.

Proof. Let $H : Y \times I \to X$ be a homotopy and h be a map lifting H_0 with respect to π_2, i.e., $\pi_2 \circ h = H_0$. Then we have a commutative diagram represented below



Since p is a fibration, there exists a homotopy $G : Y \times I \to E$ such that $G_0 = \pi_1 \circ h$ and $p \circ G = f \circ H$. Define $\widetilde{H} : Y \times I \to E \times_B X$ by

$$\widetilde{H}(y, t) = (G(y, t), H(y, t)).$$

Clearly $\widetilde{H}$ lifts H with respect to π_2 and $\widetilde{H}(y, 0) = (G(y, 0), H(y, 0)) = (\pi_1 \circ h(y), \pi_2 \circ h(y)) = h(y)$. Thus $\pi_2 : E \times_B X \to X$ is a fibration. ∎

Corollary 1.3.12. Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two fibrations with the base space B. Then the map $p : E_1 \times_B E_2 \to B$, given by $p(e_1, e_2) = p_1(e_1) = p_2(e_2)$, is a fibration and is called the **product** of the fibrations p_1 and p_2.

Proof. Since p_1 and p_2 are fibrations, it follows from the preceding theorem that π_1 and π_2 are fibrations. Thus $p = p_1 \circ \pi_1 = p_2 \circ \pi_2$ is a fibration. ∎

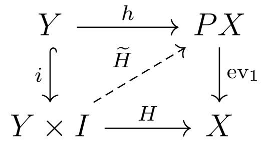
1.3.1 Path Space Fibrations

In this section, we show that the evaluation maps $\mathrm{ev}_0, \mathrm{ev}_1 : PX \to X$, and $\widetilde{e}_n : PX \to X^n$ defined in the Section 1.1 are fibrations. Moreover, the restriction of the map ev_1

to the subspace $P_{x_0}X$ is also a fibration. Some of the text in this section can be found in [3, Chapter 6].

Theorem 1.3.13. The map $\mathrm{ev}_1 : PX \to X$, given by $\mathrm{ev}_1(\alpha) = \alpha(1)$ for all $\alpha \in PX$, is a fibration.

Proof. Let $H : Y \times I \to X$ be a homotopy and $h : Y \to PX$ be a map lifting H_0, i.e., $\mathrm{ev}_1(h(y)) = H(y, 0)$.



For each $y \in Y$, $h(y)$ is a path in X ending at $H(y, 0)$, i.e., $h(y)(1) = H(y, 0)$. Define a map $\widetilde{H} : Y \times I \to PX$ given by

$$\widetilde{H}(y,s)(t) = \begin{cases} h(y)((1+s)t) & \text{if } 0 \leq t \leq 1/(1+s), \\ H(y,(1+s)t-1) & \text{if } 1/(1+s) \leq t \leq 1. \end{cases}$$

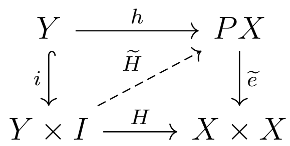
By pasting lemma and Theorem 1.1.4, the map $\widetilde{H}$ is continuous. Also, $\widetilde{H}(y, 0) = h(y)$ and $\mathrm{ev}_1(\widetilde{H}(y, s)) = \widetilde{H}(y, s)(1) = H(y, s)$. Thus ev_1 is a fibration. ■

Theorem 1.3.14. The map $\mathrm{ev}_1 : P_{x_0}X \to X$, given by $\mathrm{ev}_1(\alpha) = \alpha(1)$ for all $\alpha \in P_{x_0}X$, is a fibration.

Proof. This theorem has the same proof as the previous theorem. The fact that $h(y)$ is a path starting at x_0 implies $\widetilde{H}(y, s)$ is also a path starting at x_0. ■

Theorem 1.3.15. The map $\widetilde{e} : PX \to X \times X$, given by $\widetilde{e}(\alpha) = (\alpha(0), \alpha(1))$ for all $\alpha \in PX$, is a fibration.

Proof. Let $H : Y \times I \to X \times X$ be a homotopy and $h : Y \to PX$ be a map lifting H_0, i.e., $\widetilde{e}(h(y)) = H(y, 0)$.



For each $y \in Y$, $h(y)$ is a path in X beginning at $\pi_1 \circ H(y, 0)$ and ending at $\pi_2 \circ H(y, 0)$, i.e., $h(y)(0) = \pi_1 \circ H(y, 0)$ and $h(y)(1) = \pi_2 \circ H(y, 0)$. Define a map $\widetilde{H} : Y \times I \to PX$ given by

$$\widetilde{H}(y,s)(t) = \begin{cases} \pi_1 \circ H(y, s-(1+s)t) & \text{if } 0 \leq t \leq s/(1+s), \\ h(y)[\{(1+s)t - s\}/(1-s)] & \text{if } s/(1+s) \leq t \leq 1/(1+s), \\ \pi_2 \circ H(y,(1+s)t-1) & \text{if } 1/(1+s) \leq t \leq 1. \end{cases}$$

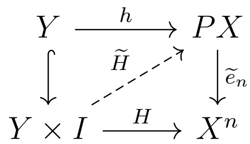
By pasting lemma and Theorem 1.1.4, the map $\widetilde{H}$ is continuous. Also, $\widetilde{H}(y,0) = h(y)$ and $\widetilde{e}(\widetilde{H}(y,s)) = (\widetilde{H}(y,s)(0), \widetilde{H}(y,s)(1)) = (\pi_1 \circ H(y,s), \pi_2 \circ H(y,s)) = H(y,s)$. Thus $\widetilde{e}$ is a fibration. ■

Theorem 1.3.16. The map $\widetilde{e}_n : PX \to X \times \ldots \times X = X^n$, given by

$$\widetilde{e}_n(\alpha) = (\alpha(0), \alpha(t_1), \ldots, \alpha(t_{n-2}), \alpha(1))$$

where $t_i = i/(n-1)$ for $i \in \{0, \ldots, n-1\}$ and $n \geq 2$, is a fibration.

Proof. Let $H : Y \times I \to X^n$ be a homotopy and $h : Y \to PX$ be a map lifting H_0, i.e., $\widetilde{e}_n(h(y)) = H(y,0)$.



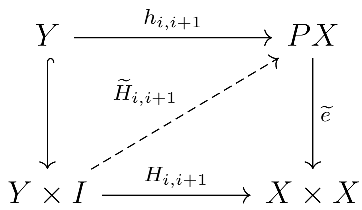
Let $\pi_i : X^n \to X$ be the projection map onto the i^{th} factor. For each $i \in \{1, \ldots, n-1\}$ define maps $h_{i,i+1} : Y \to PX$ and $H_{i,i+1} : Y \times I \to X \times X$ by

$$\begin{aligned} h_{i,i+1}(y)(t) &= h(y)((1-t)t_{i-1} + tt_i) \\ &= h(y)(t_{i-1} + t/(n-1)), \end{aligned}$$

and

$$H_{i,i+1}(y,t) = (\pi_i H(y,t), \pi_{i+1} H(y,t))$$

respectively. Then $h_{i,i+1}(y)(0) = h(y)(t_{i-1}) = \pi_i H(y,0)$ and $h_{i,i+1}(y)(1) = h(y)(t_i) = \pi_{i+1} H(y,0)$, i.e., $\widetilde{e}(h_{i,i+1}(y)) = (\pi_i H(y,0), \pi_{i+1} H(y,0)) = H_{i,i+1}(y,0)$. Thus for each $i \in \{1, \ldots, n-1\}$ we have a commutative diagram



Since $\widetilde{e}$ is a fibration, there exists $\widetilde{H}_{i,i+1}$, for each $i \in \{1, \ldots, n-1\}$, making the above diagram commute.

Define a map $\widetilde{H} : Y \times I \to PX$ given by

$$\widetilde{H}(y,s)(t) = \widetilde{H}_{i,i+1}(y,s)((n-1)(t - t_{i-1})) \text{ if } t \in [t_{i-1}, t_i]$$

for $i \in \{1, \ldots, n-1\}$. By pasting lemma and Theorem 1.1.4, $\widetilde{H}$ is continuous. Also,

$\widetilde{H}$ satisfies

$$\begin{aligned}\widetilde{H}(y,0)(t) &= \widetilde{H}_{i,i+1}(y,0)((n-1)(t-t_{i-1})) \qquad \text{for some } i \in \{1,\dots,n-1\} \\ &= h_{i,i+1}(y)((n-1)(t-t_{i-1})) \\ &= h(y)(t_{i-1} + (n-1)(t-t_{i-1})/(n-1)) \\ &= h(y)(t),\end{aligned}$$

and

$$\widetilde{H}(y,s)(t_i) = \widetilde{H}_{i,i+1}(y,s)((n-1)(t_i-t_{i-1})) = \widetilde{H}_{i,i+1}(y,s)(1) = \pi_{i+1}H(y,s),$$

i.e., $\widetilde{H}(y,0) = h(y)$ and $\widetilde{e}_n\widetilde{H} = H$. Thus $\widetilde{e}_n$ is a fibration. ■

Corollary 1.3.17. The maps $\widetilde{e}_{n,i} : PX \to X$, given by $\widetilde{e}_{n,i}(\alpha) = \alpha(t_i)$ where $t_i = i/(n-1)$ for $i \in \{0,\dots,n-1\}$ and $n \geq 2$, is a fibration. In particular, the map $\mathrm{ev}_0 : PX \to X$, given by $\mathrm{ev}_0(\alpha) = \alpha(0)$ for all $\alpha \in PX$, is a fibration.

Proof. The maps $\widetilde{e}_{n,i-1} = \pi_i \circ \widetilde{e}_n$ and $\mathrm{ev}_0 = \widetilde{e}_{n,0}$ where π_i denotes the projection map onto the i^{th} factor. Since the projection maps are fibrations and composition of two fibrations is a fibration, it follows that $\widetilde{e}_{n,i}$ and ev_0 are fibrations. ■

Recall that the mapping path space of a continuous map $f : X \to Y$ was defined to be the space $P_f = \{(\alpha, x) \in PY \times X \mid \alpha(0) = f(x)\}$. By Lemma 1.1.8, P_f is homotopy equivalent to X and the projection map π_2 onto the second factor X is a homotopy equivalence. Moreover, the mapping path space of a continuous map $f : X \to Y$ can also be seen as the pullback of the fibration $\mathrm{ev}_0 : PY \to Y$ and $f : X \to Y$.

$$\begin{array}{ccc} P_f & \xrightarrow{\pi_1} & PY \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} \\ X & \xrightarrow{f} & Y \end{array}$$

Thus, by Proposition 1.3.11, we have that the projection map π_2 is a fibration.

Theorem 1.3.18. Let $f : X \to Y$ be a continuous map and P_f be the mapping path space of f. Then $p : P_f \to Y$, given by $p(\alpha, x) = \alpha(1)$, is a fibration and is called the **mapping fibration** of f.

Proof. Let Q be the pullback of $\widetilde{e} : PY \to Y \times Y$ and $f \times \mathrm{id} : X \times Y \to Y \times Y$. Consider the diagram

$$\begin{array}{ccccc} P_f & \xrightarrow{i} & Q & \xrightarrow{\pi_1} & PY \\ & & {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \widetilde{e}} \\ Y & \xleftarrow[j]{} & X \times Y & \xrightarrow{f\times\mathrm{id}} & Y \times Y \end{array}$$

where $j : Y \times X \to X$ is the projection map onto the second factor Y and $i : P_f \to Q$ is given by $i(\alpha, x) = (\alpha, x, \alpha(1))$. Clearly i is a homeomorphism with i^{-1} given by $i^{-1}(\alpha, x, y) = (\alpha, x)$. By Proposition 1.3.11 and Theorem 1.3.15, π_2 is a fibration. Thus $j \circ \pi_2 \circ i : P_f \to Y$, given by $j \circ \pi_2 \circ i(\alpha, x) = j \circ \pi_2(\alpha, x, \alpha(1)) = j(x, \alpha(1)) = \alpha(1)$, is also a fibration ■

Corollary 1.3.19. Any continuous map $f : X \to Y$ can be factored in the form

$$X \xrightarrow{i} P_f \xrightarrow{p} Y,$$

where i is a homotopy equivalence and p is a fibration. The map p is said to be a **fibrational substitute** for the map f.

Proof. Let $i : X \to P_f$ be the map which maps $x \in X$ to the tuple $(c_{f(x)}, x)$ where $c_{f(x)}$ is the constant path at $f(x)$. Let $p : P_f \to Y$ be the map given by $p(\alpha, x) = \alpha(1)$. Then, by Lemma 1.1.8, i is a homotopy equivalence, and by Theorem above, p is a fibration. Moreover, $p \circ i(x) = p(c_{f(x)}, x) = c_{f(x)}(1) = f(x)$. ■

Example 1.3.20. The n-fold diagonal map $\Delta_n : X \to X^n$ can be factored in the form

$$X \xrightarrow{i} PX \xrightarrow{\widetilde{e}_n} X^n,$$

where the map i takes $x \in X$ to the constant path c_x at x. The map i is a homotopy equivalence by Lemma 1.1.5 and the map $\widetilde{e}_n$ is a fibration by Theorem 1.3.16. Thus the map $\widetilde{e}_n$ is a fibrational substitute for the n-fold diagonal map Δ_n for $n \geq 2$.

1.3.2 Sum of Fibrations

In this section, we define the sum of n-copies of a fibration p. It will play a key role in obtaining a lower bound on Schwarz genus of the fibration p. Most of the text in this section can be found in [3, Chapter 6] and [13, Chapter II].

Definition 1.3.21. The **mapping cylinder** M_p of a continuous map $p : E \to B$ is the quotient space $(E \times I) \sqcup B$ with respect to the equivalence relation $\sim$ generated by $(e, 0) \sim b$ if $p(e) = b$.

Let $a : (E \times I) \sqcup B \to M_p$ be the quotient map. The spaces E and B are naturally embedded in M_p by the mappings $i : E \hookrightarrow M_p$ and $j : B \hookrightarrow M_p$ given by $i(e) = a(e, 1)$ and $j(b) = a(b)$ respectively. The space M_p can also be continuously mapped onto the space B via the map $f : M_p \to B$, defined by $f(a(e, t)) = p(e)$ and $f(a(b)) = b$.

Lemma 1.3.22. Let $p : E \to B$ be a continuous map. Then the space M_p is homotopy equivalent to B and the map $f : M_p \to B$, defined by $f(a(e, t)) = p(e)$ and

$f(a(b)) = b$, is a homotopy equivalence where $a : (E \times I) \sqcup B \to M_p$ is the quotient map.

Proof. Let $j : B \hookrightarrow M_p$ be the embedding given by $j(b) = a(b)$. Then $f \circ j = \mathrm{id}_B$ and $j(f(a(e,t))) = a(p(e))$ and $j(f(a(b))) = a(b)$. Let $F : M_p \times I \to M_p$ be defined by

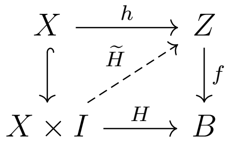
$$F(a(e,t), s) = a(e, t - st),$$
$$F(a(b), s) = a(b).$$

Then $F(a(e,t), 0) = a(e,t)$ and $F(a(e,t), 1) = a(e, 0) = a(p(e))$. Thus M_p is homotopy equivalent to B. ■

For the rest of this section, we shall assume that $p : E \to B$ is a fibration and Z denotes the mapping cylinder M_p of the fibration p.

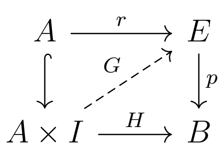
Theorem 1.3.23. Let $p : E \to B$ be a fibration and Z be the mapping cylinder of p. Then the map $f : Z \to B$, defined by $f(a(e,t)) = p(e)$ and $f(a(b)) = b$, is a fibration where $a : (E \times I) \sqcup B \to Z$ is the quotient map..

Proof. Let $H : X \times I \to B$ be a homotopy and $h : X \to Z$ be a map lifting H_0.



Let $A = \{x \in X \mid h(x) \notin j(B)\}$ be a subset of X. Since the set $Z \setminus j(B)$ is homeomorphic to $E \times (0,1]$, the map h when restricted to A will be of the form $h(x) = a(r(x), s(x))$, for all $x \in A$ where $r : A \to E$ and $s : A \to (0,1]$ are continuous maps.

Since h is the lift of H_0, we have $H(x,0) = f(h(x)) = f(a(r(x), s(x))) = p(r(x))$ for all $x \in A$. Consider the diagram below.



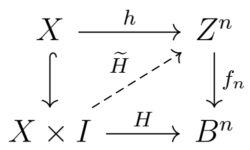
Since the outer-diagram commutes, we have a map $G : A \times I \to E$ such that $G(x, 0) = r(x)$ and $p \circ G = H$ for all $x \in A$.

Define $\widetilde{H} : X \times I \to Z$ by

$$\widetilde{H}(x,t) = \begin{cases} a(G(x,t), s(x)) & \text{if } x \in A, \\ j(H(x,t)) & \text{if } x \in X \setminus A. \end{cases}$$

The map $\widetilde{H}$ is continuous. Also, $\widetilde{H}(x,0) = h(x)$ and $f \circ \widetilde{H} = H$ and for all $x \in X$. Thus f is a fibration. ■

Let $f_n : Z^n \to B^n$ be the product of n-copies of the map f. Let $H : X \times I \to B^n$ be a homotopy and $h : X \to Z^n$ a map lifting H_0.



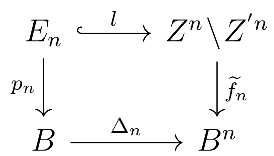
It is clear from the definition of f_n that the map $\pi_i \circ h$ lifts $\pi_i \circ H_0$ with respect to f where π_i is the projection map onto the i^{th} factor. Hence, we can define a lift $\widetilde{H} = (\widetilde{H}_1, \ldots, \widetilde{H}_n)$ of H where $\widetilde{H}_i : X \times I \to Z$ is the lift of $\pi_i \circ H$. Thus the map f_n is also a fibration.

Let $Z^{'} = Z \backslash i(E)$. Observe that if the map $h : X \to Z^n$ is such that $h(X) \notin Z'^n$, i.e., $h(x) = (a(e_1, t_1), \ldots, a(e_n, t_n))$ and $\max\{t_1, \ldots, t_n\} < 1$ for all $x \in X$, then the image of the lift $\widetilde{H}$, defined using Theorem 1.3.23 and argument in the preceding paragraph, also doesn't lie in Z'^n. Thus the restriction of f_n on $Z^n \backslash Z'^n$, say $\widetilde{f_n}$, is also a fibration.

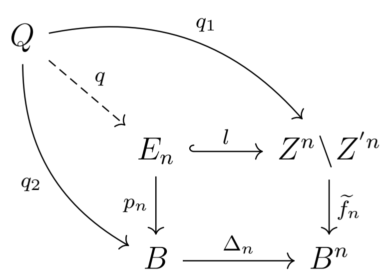
Let $\Delta_n : B \to B^n$ be the n-fold diagonal map. Define $E_n = \{(z_1, \ldots, z_n) \in Z^n \backslash\, Z'^n \mid f(z_1) = \ldots = f(z_n)\}$ and $l : E_n \hookrightarrow Z^n \backslash Z'^n$ be the inclusion map. Then $p_n : E_n \to B$, given by

$$p_n(z_1, \ldots, z_n) = f(z_1) = \ldots = f(z_n),$$

makes the diagram given below commute.



Moreover, suppose there exists a space Q with continuous maps $q_1 : Q \to Z^n \backslash Z'^n$ and $q_2 : Q \to B$ such that $\widetilde{f_n} \circ q_1 = \Delta_n \circ q_2$.



Since $\widetilde{f}_n \circ q_1 = \Delta_n \circ q_2$, we have $q_1(Q) \subset E_n$. Hence, we can define a map $q : Q \to E_n$ by $q(x) = q_1(x)$ for all $x \in Q$. Clearly $q_1 = l \circ q$. Moreover, we have

$$\Delta_n q_2 = \widetilde{f}_n q_1 = \widetilde{f}_n l q = \Delta_n p_n q$$

implying $q_2 = p_n \circ q$. Thus (E_n, l, p_n) is universal with respect to the above diagram, i.e., (E_n, l, p_n) is the pullback of the maps $\widetilde{f}_n$ and Δ_n. Since the map $\widetilde{f}_n$ is a fibration, the map p_n is a fibration by Proposition 1.3.11 and is called the **sum** of n-copies of fibration p.

1.4 Dimension Theory

In this section, we introduce the concept of topological dimension of a space. The relevant references are [2, Appendix A], [8], and [9, Chapter 8].

Definition 1.4.1. Let $\mathcal{A}$ be a collection of subsets of X. The **order** of $\mathcal{A}$ is the smallest number n, if it exists, such that each point of X lies in at most n elements of $\mathcal{A}$.

Definition 1.4.2. The **topological dimension** (or **Lebesgue covering dimension**) of X, denoted $\dim(X)$, is the minimum number n such that for every open cover $\mathcal{A}$ of X, there is an open refinement of $\mathcal{A}$ which has order at most $n+1$. If no such minimal n exists, the space X is said to be of infinite topological dimension.

Remark 1.4.3. The topological dimension agrees with the "usual" definition of dimension for good spaces like manifolds, finite simplicial complexes and finite CW-complexes.

Lemma 1.4.4. Let $\mathcal{U} = \{U_i\}_{i \in J}$ be an open cover of X of order n with a partition of unity subordinate to the cover. Then there exists an open cover $\mathcal{W} = \{W_k\}_{k=1}^{n}$ of X such that W_k $(k = 1, \ldots, n)$ is a disjoint union of open sets each of which contained in some U_i.

Proof. Let $\{\rho_i\}_{i \in J}$ be a partition of unity subordinate to the open cover $\mathcal{U}$. For each $x \in X$, define

$$S(x) = \{i \in J \mid \rho_i(x) > 0\}.$$

Since the order of the cover $\mathcal{U}$ is n, no $x \in X$ can belong to more than n elements of the $\mathcal{U}$. Thus $S(x)$ is a finite set for each $x \in X$ and can have a maximum cardinality n.

For each finite subset $S \subset J$, define

$$W(S) = \{x \in X \mid \rho_i(x) > \rho_j(x) \text{ for all } i \in S \text{ and } j \notin S\}.$$

If J is finite, then define $W(J) = \{x \in X \mid \rho_i(x) > 0 \text{ for all } i \in J\}$. The space $W(S)$ can be written as the intersection of

$$A_1 = \{x \in X \mid \rho_i(x) > 0 \text{ for all } i \in S \text{ and } j \in S_1\},$$

and

$$A_2 = \{x \in X \mid \rho_i(x) > \rho_j(x) \text{ for all } i \in S \text{ and } j \in S_2\},$$

where $S_1 = \{j \notin S : \rho_j(x) = 0\}$ and $S_2 = \{j \notin S : \rho_j(x) \neq 0\}$. Moreover,

$$\begin{aligned} A_1 &= \{x \in X \mid \rho_i(x) > 0 \text{ for all } i \in S \text{ and } j \in S_1\} \\ &= \{x \in X \mid \rho_i(x) > 0 \text{ for all } i \in S\} \\ &= \bigcap_{i \in S} \rho_i^{-1}(0, 1], \end{aligned}$$

and

$$\begin{aligned} A_2 &= \{x \in X \mid \rho_i(x) > \rho_j(x) \text{ for all } i \in S \text{ and } j \in S_2\} \\ &= \bigcap_{\substack{i \in S \\ j \in S_2}} (\rho_i - \rho_j)^{-1}(0, 1]. \end{aligned}$$

Since $\{\rho_i\}$ is a partition of unity, we have S_2 is finite. Therefore W(S) is open.

Let S and S' be two finite subsets of J such that $W(S) \cap W(S') \neq \emptyset$. Suppose $S \not\subset S'$, then there exists $i_0 \in S \backslash S'$. Let $x \in W(S) \cap W(S')$. Then $\rho_{i_0}(x) > \rho_j(x)$ for all $j \notin S$ and $\rho_i(x) > \rho_{i_0}(x)$ for all $i \in S'$. Thus $\rho_i(x) > \rho_j(x)$ for all $i \in S'$ and $j \notin S$. This implies $S' \subset S$. Therefore if S and S' are two distinct finite subsets of I with same cardinality then $W(S)$ and $W(S')$ are disjoint.

If $x \in W(S)$, then $\rho_i(x) > 0$ for all $i \in S$, i.e. $x \in$ support(ρ_i) for all $i \in S$, thus $x \in \cap_{i \in S}$ support(ρ_i). Therefore we have

$$W(S) \subset \cap_{i \in S} \text{ support}(\rho_i) \subset \cap_{i \in S} U_i.$$

Moreover, $x \in W(S(x))$ since $\rho_i(x) > 0$ and $\rho_j(x) = 0$ for all $i \in S(x)$ and $j \notin S(x)$. Therefore $\{W(S(x))\}_{x \in X}$ forms an open refinement of the cover $\mathcal{U}$.

Define W_k (for $k = 1, 2, \ldots, n$) to be the union of $W(S(x))$ such that $S(x)$ has k elements. Thus $\mathcal{W} = \{W_k\}_{k=1}^n$ forms an open cover of X such that W_k $(k = 1, \ldots, n)$ is a disjoint union of open sets each of which contained in some U_i. ■

Theorem 1.4.5. Let X be a paracompact Hausdorff space of dimension n and $\mathcal{U} = \{U_i\}_{i \in J}$ be any open cover of X. Then there exists an open cover $\mathcal{W} = \{W_k\}_{k=1}^{n+1}$ of X such that W_k $(k = 1, \ldots, n+1)$ is a disjoint union of open sets each of which contained in some U_i.

Proof. Since the dimension of X is n, there exists an open refinement, say $\mathcal{B}$, of $\mathcal{U}$ which has order at most $n+1$. Since the space X is paracompact Hausdorff, X admits an partition of unity, say $\{\rho_\beta\}$, subordinate to the cover $\mathcal{B}$. Thus, by preceding lemma, there exists an open cover $\mathcal{W} = \{W_k\}_{k=1}^{n+1}$ of X such that W_k $(k = 1, \ldots, n+1)$ is a disjoint union of open sets each of which contained in some U_i. ■

Chapter 2

Schwarz Genus

In this chapter, we study the Schwarz genus of a fibration which leads to an equivalent approach to defining the numerical homotopy invariants - LS category and higher topological complexity of spaces. The relevant reference for this chapter is [13, Chapter III and Chapter VI].

2.1 Schwarz Genus of a Fibration

Definition 2.1.1. Let $p : E \to B$ be a surjective fibration. The **Schwarz genus** or **sectional category** of p, denoted $\mathrm{secat}(p)$, is defined to be the smallest number n such that there exists an open cover $\{U_i\}_{i=0}^n$ of the base space B such that over each open set U_i there exists a continuous section $s_i : U_i \to E$ of p, i.e., $p \circ s_i = \mathrm{id}_{U_i}^B$ where $\mathrm{id}_{U_i}^B$ is the inclusion map from U_i into B. If no such n exists, then $\mathrm{secat}(p) = \infty$.

It is clear from the definition that the genus of a surjective fibration equals 0 if and only if it admits a continuous global section. In what follows we shall assume that all fibrations under consideration are surjective.

Proposition 2.1.2. Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be fibrations with the common base space B and $f : E_1 \to E_2$ be a continuous map satisfying $p_2 \circ f = p_1$. Then $\mathrm{secat}(p_2) \leq \mathrm{secat}(p_1)$.

Proof. If $\mathrm{secat}(p_1) = n$ and $\{U_i\}_{i=0}^n$ is an open cover of B with a continuous section $s_i : U_i \to E_1$ of p_1 over each U_i, then $\{U_i\}_{i=0}^n$ is an open cover of B with a continuous section $f \circ s_i$ of p_2 over each U_i. Thus $\mathrm{secat}(p_2) \leq n$. ■

2.2 An Upper Bound for Schwarz Genus

Definition 2.2.1. A subset U of a topological space X is said to be **categorical** if it is contractible in X, i.e., the inclusion map $\mathrm{id}_U^X : U \hookrightarrow X$ is null-homotopic. A cover of X is said be categorical if it consists of categorical sets.

Proposition 2.2.2. Let X be a path-connected space. Let $\mathrm{ev}_1 : P_{x_0}X \to X$ be the fibration given by $\mathrm{ev}_1(\alpha) = \alpha(1)$ for all $\alpha \in P_{x_0}X$. Then a subset U of X is categorical if and only if there exists a continuous section $s : U \to P_{x_0}X$ of ev_1, i.e., $\mathrm{ev}_1 \circ s = \mathrm{id}_U^X$.

Proof. Let $s : U \to P_{x_0}X$ be a continuous map such that $\mathrm{ev}_1 \circ s = \mathrm{id}_U^X$. Consider the homotopy $H : U \times I \to X$ defined as the composition of the following maps

$$U \times I \xrightarrow{s\times\mathrm{id}} P_{x_0}X \times I \xrightarrow{\mathrm{ev}} X,$$

where ev is the evaluation. Then

$$H(x,0) = \mathrm{ev}((s \times \mathrm{id})(x,0)) = \mathrm{ev}(s(x),0) = s(x)(0) = x_0,$$

and

$$H(x,1) = \mathrm{ev}((s \times \mathrm{id})(x,1)) = \mathrm{ev}(s(x),1) = s(x)(1) = \mathrm{ev}_1(s(x)) = x.$$

Thus the inclusion map id_U^X is null-homotopic, i.e., U is categorical.

Conversely, suppose that U is categorical then the inclusion map $\mathrm{id}_U^X : U \hookrightarrow X$ is null-homotopic, i.e., there exists a homotopy $H : U \times I \to X$ between a constant map and the inclusion map. Since X is path connected, we can assume that the constant map is based at x_0. Since $H : U \times I \to X$ is continuous, by Proposition 1.1.4, it induces a continuous map $s : U \to PX$ given by $s(x)(t) = H(x,t)$ for $x \in U$ and $t \in I$. But,

$$s(x)(0) = H(x,0) = x_0$$

implies $s(U) \subset P_{x_0}X$, i.e., $s : U \to P_{x_0}X$ is continuous. Moreover,

$$\mathrm{ev}_1(s(x)) = s(x)(1) = H(x,1) = x$$

implies $\mathrm{ev}_1 \circ s = \mathrm{id}_U^X$. ∎

It is straightforward to see that the fibration $\mathrm{ev}_1 : P_{x_0}X \to X$ is surjective if and only if the space X is path-connected. Thus the Schwarz genus of the fibration $\mathrm{ev}_1 : P_{x_0}X \to X$, given by $\mathrm{ev}_1(\alpha) = \alpha(1)$, for a path-connected space X can equivalently be defined as the smallest number n such that there exists an open categorical cover $\{U_i\}_{i=0}^n$ of X. The number n is called the **Lusternik-Schnirelmann category** or **LS category** of the topological space X and is denoted by $\mathrm{cat}(X)$. We shall discuss LS category in detail in Chapter 3.

Corollary 2.2.3. Let X be a path-connected space. Let $\mathrm{ev}_1 : P_{x_0}X \to X$ be the fibration given by $\mathrm{ev}_1(\alpha) = \alpha(1)$ for all $\alpha \in P_{x_0}X$. Then $\mathrm{secat}(\mathrm{ev}_1) = \mathrm{cat}(X)$.

Theorem 2.2.4. Let $p : E \to B$ be a fibration. Then

$$\operatorname{secat}(p) \le \operatorname{cat}(B).$$

Proof. Let U be an open categorical subset of B. Then there exists a homotopy $H : U \times I \to B$ such that $H(b, 0) = b_0$ and $H(b, 1) = b$ for all $b \in U$. Let $e_0 \in E$ be a point in $p^{-1}(b_0)$. Consider the diagram

$$\begin{array}{ccc} U & \xrightarrow{\ h\ } & E \\ {\scriptstyle i}\big\downarrow & \overset{\widetilde{H}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ U \times I & \xrightarrow{\ H\ } & B \end{array}$$

where $h : U \to E$ is the constant map given by $h(b) = e_0$ and $i : U \hookrightarrow U \times I$ is the inclusion map given by $i(b) = (b, 0)$ for all $b \in U$. Since the outer diagram commutes, there exists $\widetilde{H} : U \times I \to E$ making the diagram commute. Thus $p \circ \widetilde{H} = H$ implies that $p(\widetilde{H}(b, 1)) = H(b, 1) = b$ for all $b \in U$. Hence, $\widetilde{H}$ when restricted to $U \times \{1\}$ is a section of p on U.

If $\operatorname{cat}(B) = n$ and $\{U_j\}_{j=0}^{n}$ is an open categorical cover of B, then $\{U_j\}_{j=0}^{n}$ is an open cover of B with a continuous section of p induced by $\widetilde{H}_j$ over each U_j. Thus $\operatorname{secat}(p) \le n$. ■

Thus if B is a space such that each point $b \in B$ is contained in an open categorical subset U_b of B (which is a much weaker condition than assuming that the space B is locally contractible, i.e., for every $b \in B$ and every open subset V of B containing b there exists an open subset U of B containing b such that $U \subset V$ and U is contractible), then B has an open cover $\{U_b\}_{b \in B}$ such that for any fibration $p : E \to B$ there exist continuous sections $s_b : U_b \to E$ of p for all $b \in B$.

Proposition 2.2.5. Let B be a locally contractible compact space and $p : E \to B$ be a fibration. Then $\operatorname{secat}(p)$ is finite.

Proof. Since B is locally contractible, B has an open cover $\{U_b\}_{b \in B}$ such that there are sections $s_b : U_b \to E$ of p. By compactness of B, it is possible to select a finite subcover of $\{U_b\}_{b \in B}$ for B. Thus $\operatorname{secat}(p)$ is finite. ■

For example, if the base space B is a finite CW complex or a compact manifold and $p : E \to B$ is a fibration, then $\operatorname{secat}(p)$ is finite.

Theorem 2.2.6. Let $p : E \to B$ be a fibration. If E is contractible, then

$$\operatorname{secat}(p) = \operatorname{cat}(B).$$

Proof. Suppose $p : E \to B$ is a fibration and U is an open subset of B with a section $s : U \to E$ of p. Let $H : E \times I \to E$ be a homotopy such that $H(e, 0) = e$ and $H(e, 1) = e_0$ for all $e \in E$. Then the homotopy $G : U \times I \to B$ defined as composition of the following maps

$$U \times I \xrightarrow{s \times \mathrm{id}} E \times I \xrightarrow{H} E \xrightarrow{p} B$$

satisfies

$$G(b, 0) = p(H((s \times \mathrm{id})(b, 0))) = p(H(s(b), 0)) = p(s(b)) = b,$$

and

$$G(b, 1) = p(H((s \times \mathrm{id})(b, 1))) = p(H(s(b), 1)) = p(e_0),$$

for all $b \in U$. Thus the inclusion map id_U^B is null-homotopic, i.e., U is categorical.

If $\mathrm{secat}(p) = n$ and $\{U_i\}_{i=0}^n$ is an open cover of B with a continuous section $s_i : U_i \to E$ of p over each U_i, then $\{U_i\}_{i=0}^n$ is an open categorical cover of B. Thus $\mathrm{cat}(B) \leq n$. Thus, by Theorem 2.2.4, we get $\mathrm{secat}(p) = \mathrm{cat}(B)$. ■

Suppose X is a path-connected space. Then Corollary 1.1.6, which states that the space $P_{x_0}X$ is contractible, and Theorem 2.2.6 provides an alternate proof of Corollary 2.2.3 which states that $\mathrm{secat}(\mathrm{ev}_1 : P_{x_0}X \to X) = \mathrm{cat}(X)$.

2.3 A Lower Bound for Schwarz Genus

Let $p : E \to B$ be a fibration and Z be the mapping cylinder of $p : E \to B$, i.e., Z is the quotient space of $(E \times I) \sqcup B$ with respect to the equivalence relation $\sim$ generated by $(e, 0) \sim b$ if $p(e) = b$. Let $a : (E \times I) \sqcup B \to Z$ be the quotient map. The spaces E and B are naturally embedded in Z by the mappings $i : E \hookrightarrow Z$ and $j : B \hookrightarrow Z$ given by $i(e) = a(e, 1)$ and $j(b) = a(b)$ respectively. The space Z can also be continuously mapped into the space B by the map $f : Z \to B$, defined by $f(a(e, t)) = p(e)$ and $f(a(b)) = b$.

We denote by Z' the space $Z \backslash i(E)$, by $f_n : Z^n \to B^n$ the product of n-copies of the map f, by $\widetilde{f_n} : Z^n \backslash Z'^n \to B^n$ the restriction of f_n on $Z^n \backslash Z'^n$, by E_n the space $\{(z_1, \ldots, z_n) \in Z^n \setminus Z'^n \mid f(z_1) = \ldots = f(z_n)\}$, by $p_n : E_n \to B$ be the map given by $p_n(z_1, \ldots, z_n) = f(z_1) = \ldots = f(z_n)$, by $l : E_n \hookrightarrow Z^n \backslash Z'^n$ the inclusion map, and by $\Delta_n : B \to B^n$ the n-fold diagonal map.

We showed at the end of the Section 1.3.2 that (E_n, l, p_n) is the pullback of $\widetilde{f_n}$ and Δ_n. Since $\widetilde{f_n}$ is a fibration, the map $p_n : E_n \to B$ is a fibration called the **sum** of n-copies of fibration p.

Thus we have a commutative diagram

$$\begin{array}{ccccc} E_n & \xhookrightarrow{l} & Z^n \setminus Z'^n & \xhookrightarrow{\mu} & Z^n \\ {\scriptstyle p_n}\big\downarrow & & {\scriptstyle \widetilde{f}_n}\big\downarrow & \swarrow {\scriptstyle f_n} & \\ B & \xrightarrow{\Delta_n} & B^n & & \end{array}$$

where μ is the inclusion map.

Proposition 2.3.1. Let B be a paracompact Hausdorff space and $p : E \to B$ a fibration. Then $p_n : E_n \to B$ has a global section if and only if there is an open cover $\{U_i\}_{i=1}^n$ of B such that over each U_i there exists a section of p.

Proof. Let $s : B \to E_n$ be a section of p_n. Define $s_i : B \to Z$, for $1 \leq i \leq n$, by composing s with the inclusion map $\mu \circ l$ into Z^n followed by the projection map onto the i^{th} factor. Consider the sets $U_i = \{b \in B \mid s_i(b) \notin j(B)\}$, then the sets U_i are open in B as they are the inverse images of $s_i^{-1}(Z \setminus j(B))$. Suppose there exist $b \in B$ such that $b \notin U_i$ for all i, i.e., $s_i(b) \in j(B)$ for all i. Then $s(b) = (s_1(b), \ldots, s_n(b)) \in j(B)^n \subset Z'^n$ which is a contradiction. Thus the set $\{U_i\}_{i=1}^n$ forms an open cover of B. The map s_i when restricted to U_i gives a continuous map $s_i : U_i \to Z \setminus j(B)$ and this map will be of the form $s_i(b) = a(\rho_i(b), h_i(b))$ where $\rho_i : U_i \to E$ and $h_i : U_i \to (0, 1]$ are continuous maps. Since $p_n \circ s(b) = b$ for all $b \in B$, it implies $p \circ \rho_i(b) = b$ for all $b \in U_i$.

Conversely, suppose $\{U_i\}_{i=1}^n$ is an open cover of B such that for each $i \in \{1, \ldots, n\}$ there exist a map $\rho_i : U_i \to E$ with $p \circ \rho_i(b) = b$ for all $b \in U_i$. By Theorem A.2, there exist continuous maps $h_i : U_i \to \mathbb{R}$, for $i \in \{1, \ldots, n\}$, satisfying (a) $0 \leq h_i \leq 1$; (b) support$(h_i) \subset U_i$; (c) at each point $b \in B$ there exist $i \in \{1, \ldots, n\}$ such that $h_i(b) = 1$. For each $i \in \{1, \ldots, n\}$ define a map $s_i : B \to Z$ by

$$s_i(b) = \begin{cases} a(\rho_i(b), h_i(b)) & \text{if } b \in \text{support}(h_i), \\ a(b) & \text{if } b \in h_i^{-1}(0). \end{cases}$$

If $b \in \text{support}(h_i) \cap h_i^{-1}(0)$, then $a(\rho_i(b), h_i(b)) = a(\rho_i(b), 0) = a(p(\rho_i(b))) = a(b)$. Thus, by pasting lemma, s_i is continuous. Clearly $f \circ s_i(b) = b$ for all $b \in B$. Let $s : B \to Z^n$ be the map defined by $s(b) = (s_1(b), \ldots, s_n(b))$ for all $b \in B$. Since for each $b \in B$ there exist $i \in \{1, \ldots, n\}$ such that $h_i(b) = 1$, we have $s(b) \in Z^n \setminus Z'^n$. Since $f \circ s_i(b) = b$ for all i, we have $s(b) \in E_n$. Thus $s : B \to E_n$ is a section of p_n. ■

Corollary 2.3.2. Let B be a paracompact Hausdorff space and $p : E \to B$ a fibration. Then $p_n : E_n \to B$ has a global section if and only if $\text{secat}(p) < n$.

Proof. It follows from the fact that $\text{secat}(p) < n$ if and only if there is an open cover $\{U_i\}_{i=1}^n$ of B such that there exists a section of p over each U_i. ■

In what follows we shall assume that (co)homology for a topological space being used in this text is the singular homology. Of course for good spaces like CW complexes it agrees with the cellular homology, refer to Hatcher's book [7]. Thus we can use them interchangeably whenever necessary.

Let R be a commutative ring with identity. The cohomology $H^*(X;R)$ is a graded ring with multiplication defined by the cup product homomorphism

$$\cup : H^*(X;R) \otimes_R H^*(X;R) \to H^*(X;R)$$

given by $\alpha_1 \otimes \alpha_2 \mapsto \alpha_1 \cup \alpha_2$. Let $\cup_n$ denote the n-fold cup product homomorphism

$$\cup_n : H^*(X;R) \otimes_R \ldots \otimes_R H^*(X;R) \to H^*(X;R)$$

given by $\alpha_1 \otimes \ldots \otimes \alpha_n \mapsto \alpha_1 \cup \ldots \cup \alpha_n$ for $n \geq 2$. The n-fold cup product homomorphism $\cup_n$ can be factored as $\cup_n = \times_n \Delta_n^*$ where $\times_n$ denotes the n-fold cross product homomorphism and Δ_n denotes the n-fold diagonal map of the space X (refer to Appendix for more details).

Suppose $p_n : E_n \to B$ has a global section s. Then $p_n^* : H^*(B;R) \to H^*(E_n;R)$ is injective since $s^* \circ p_n^*$ is the identity map.

Proposition 2.3.3. Let B be a paracompact Hausdorff space and $p : E \to B$ a fibration. Suppose $\mathrm{secat}(p) < n$ and there are elements $\xi_i \in H^*(B, ; R)$, for $i \in \{1, \ldots, n\}$, such that $\xi_1 \cup \ldots \cup \xi_n \neq 0$. Then $\lambda^* f_n^*(\xi_1 \times \ldots \times \xi_n) \neq 0$ where $\xi_1 \times \ldots \times \xi_n \in H^*(B^n; R)$ and $\lambda = \mu \circ l : E_n \hookrightarrow Z^n$ is the inclusion map.

Proof. From the commutative diagram we have $f_n \lambda = \Delta_n p_n$ which implies $\lambda^* f_n^* = p_n^* \Delta_n^*$. Thus

$$\lambda^* f_n^*(\xi_1 \times \ldots \times \xi_n) = p_n^* \Delta_n^*(\xi_1 \times \ldots \times \xi_n) = p_n^*(\xi_1 \cup \ldots \cup \xi_n).$$

Since $\mathrm{secat}(p) < n$, we have a global section of p_n implying p_n^* is injective. Thus $p_n^*(\xi_1 \cup \ldots \cup \xi_n) \neq 0$. ■

Definition 2.3.4. A **pair** of topological spaces (X, A) is a space X together with a subspace A of X.

Definition 2.3.5. The **product of the pairs** (X, A) and (Y, B), denoted $(X, A) \times (Y, B)$, is defined as $(X \times Y, (X \times B) \cup (A \times Y))$.

It is easy to check that the product of pair of spaces is associative. Moreover,

$$(X, A)^n = (X^n, (A \times X \times \ldots \times X) \cup (X \times A \times X \times \ldots \times X) \cup \ldots \cup (X \times \ldots \times X \times A))$$

where $(X, A)^n$ denotes the n-fold product of the pair (X, A).

Lemma 2.3.6. Let $p : E \to B$ be a fibration. Let $\xi_i \in H^*(B;R)$ such that $p^*(\xi_i) = 0$ for $i \in \{1, \ldots, n\}$. Then $\mu^* f_n^*(\xi_1 \times \ldots \times \xi_n) = 0$ where $\xi_1 \times \ldots \times \xi_n \in H^*(B^n;R)$.

Proof. Let $\sigma : Z \hookrightarrow (Z, i(E))$ be the inclusion map. Then, from the exact cohomology sequence of the pair $(Z, i(E))$ and $f \circ i = p$, we have a commutative diagram

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & H^k(Z, i(E); R) & \xrightarrow{\sigma^*} & H^k(Z;R) & \longrightarrow & H^k(i(E);R) & \longrightarrow & \ldots \\
 & & & & {\scriptstyle f^*}\uparrow & & \downarrow\wr & & \\
 & & & & H^k(B;R) & \xrightarrow{p^*} & H^*(E;R) & &
\end{array}$$

Since, by Lemma 1.3.22, $f : Z \to B$ is a homotopy equivalence, f^* is an isomorphism. Thus, by exactness of the sequence and $p^*(\xi_i) = 0$, there exist elements $\beta_i \in H^*(Z, i(E); R)$, for $i \in \{1, \ldots, n\}$, such that $\sigma^*(\beta_i) = f^*(\xi_i)$.

Observe first that Z'^n, by definition, is the subset of Z^n with no points of the type $(a(e_1, 1), \ldots, a(e_n, 1))$. Thus

$$Z^n \backslash Z'^n = (i(E) \times Z \times \ldots \times Z) \cup (Z \times i(E) \times Z \times \ldots \times Z) \cup \ldots \cup (Z \times \ldots \times Z \times i(E)).$$

Therefore $(Z, i(E))^n = (Z^n, Z^n \backslash Z'^n)$.

Consider the diagrams

$$\begin{array}{ccccc}
B^n & \xleftarrow{f_n} & Z^n & \xhookrightarrow{\sigma_n} & (Z^n, Z^n \backslash Z'^n) \\
\downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_i} \\
B & \xleftarrow{f} & Z & \xhookrightarrow{\sigma} & (Z, i(E))
\end{array}$$

and

$$\begin{array}{ccccc}
\otimes^n H^*(B;R) & \xrightarrow{\otimes^n f^*} & \otimes^n H^*(Z;R) & \xleftarrow{\otimes^n \sigma^*} & \otimes^n H^*(Z, i(E); R) \\
\downarrow{\scriptstyle \times_n} & & \downarrow{\scriptstyle \times_n} & & \downarrow{\scriptstyle \times_n} \\
H^*(B^n;R) & \xrightarrow{f_n^*} & H^*(Z^n;R) & \xleftarrow{\sigma_n^*} & H^*(Z^n, Z^n \backslash Z'^n; R)
\end{array}$$

where σ_n is the inclusion map, π_i is the projection map onto the i^{th} factor, $\times_n$ is the n-fold cross product homomorphism, and $\otimes^n f^*$ and $\otimes^n \sigma^*$ are n-tensor product of the maps f^* and σ^*. Clearly the first diagram commutes. Moreover,

$$\begin{aligned}
\times_n \otimes^n f^*(\xi_1 \otimes \ldots \otimes \xi_n) &= \times_n(f^*(\xi_1) \otimes \ldots \otimes f^*(\xi_n)) \\
&= f^*(\xi_1) \times \ldots \times f^*(\xi_n) = \pi_1^* f^*(\xi_1) \cup \ldots \cup \pi_n^* f^*(\xi_n),
\end{aligned}$$

and

$$\begin{aligned}
f_n^* \times_n (\xi_1 \otimes \ldots \otimes \xi_n) &= f_n^*(\xi_1 \times \ldots \times \xi_n) \\
&= f_n^*(\pi_1^*(\xi_1)) \cup \ldots \cup f_n^*(\pi_n^*(\xi_n)) = \pi_1^* f^*(\xi_1) \cup \ldots \cup \pi_n^* f^*(\xi_n).
\end{aligned}$$

Similarly, $\times_n \otimes^n \sigma^* = \sigma_n^* \times_n$. Thus the second diagram also commutes and we get $f_n^*(\xi_1 \times \ldots \times \xi_n) = \sigma_n^*(\beta_1 \times \ldots \times \beta_n)$ since $\sigma^*(\beta_i) = f^*(\xi_i)$.

From the exact sequence of the pair $(Z^n, Z^n \backslash Z'^n)$

$$\ldots \longrightarrow H^k(Z^n, Z^n \backslash Z'^n; R) \xrightarrow{\sigma_n^*} H^k(Z^n; R) \xrightarrow{\mu^*} H^k(Z^n \backslash Z'^n; R) \longrightarrow \ldots,$$

we have $\mu^* \sigma_n^* = 0$. Thus $\mu^* f_n^*(\xi_1 \times \ldots \times \xi_n) = \mu^* \sigma_n^*(\beta_1 \times \ldots \times \beta_n) = 0$. ■

Theorem 2.3.7. Let B be a paracompact Hausdorff space and $p : E \to B$ a fibration. Suppose there are $\xi_i \in H^*(B; R)$, for $i \in \{1, \ldots, n\}$, such that $p^*(\xi_i) = 0$ and $\xi_1 \cup \ldots \cup \xi_n \neq 0$. Then $\mathrm{secat}(p) \geq n$.

Proof. By preceding lemma, we have $\mu^* f_n^*(\xi_1 \times \ldots \times \xi_n) = 0$. Then

$$\lambda^* f_n^*(\xi_1 \times \ldots \times \xi_n) = l^* \mu^* f_n^*(\xi_1 \times \ldots \times \xi_n) = l^*(0) = 0.$$

If $\mathrm{secat}(p) < n$, by Proposition 2.3.3, then $\lambda^* f_n^*(\xi_1 \times \ldots \times \xi_n) \neq 0$ which leads to a contradiction. ■

Definition 2.3.8. The **length** of a fibration $p : E \to B$ is the greatest number n for which there exist n elements; say $\xi_1, \ldots, \xi_n$, in $H^*(B; R)$ satisfying $p^*(\xi_i) = 0$ and $\xi_1 \cup \ldots \cup \xi_n \neq 0$. It is denoted by $\mathrm{long}(p)$.

It is equivalent to saying that $\mathrm{long}(p)$ is the smallest number n such that all $(n+1)$-fold cup product of elements in the kernel of $p^* : H^*(B; R) \to H^*(E; R)$ vanish in the cohomology ring $H^*(B; R)$. Moreover, the preceding Definition and Theorem 2.3.7 yields the following lower bound for the Schwarz genus of a fibration.

Theorem 2.3.9. Let $p : E \to B$ be a fibration. Suppose B is a paracompact Hausdorff space. Then

$$\mathrm{long}(p) \leq \mathrm{secat}(p).$$

Chapter 3

Lusternik-Schnirelmann Category

In this chapter, we study the numerical homotopy invariant - LS category of a space which will later help us in providing bounds on the higher topological complexity of a space. The relevant reference is [2, Chapter 1 and Chapter 8].

3.1 LS Category

Definition 3.1.1. The **Lusternik-Schnirelmann category** or **LS category** of a topological space X is the smallest number n such that there exists an open categorical cover $\{U_i\}_{i=0}^{n}$ of X. The LS category is denoted by $\mathrm{cat}(X)$. If no such n exists, we say $\mathrm{cat}(X) = \infty$.

If the space X is path-connected, then a member U_i of the cover may consist of several components since each component may be contracted separately and then moved along a path to a fixed point in X. Moreover, it is clear from the definition that the LS category of a space X equals 0 if and only if X is contractible.

The following result states that LS category is a homotopy invariant. This reduces the effort in the computation of $\mathrm{cat}(X)$ as the space X can now be simplified.

Theorem 3.1.2. The LS category of a topological space X is a homotopy invariant.

Proof. Suppose X dominates Y, i.e., there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to the identity map of Y, say id_Y.

Let U be an open categorical subset of X, i.e., the inclusion map $i : U \hookrightarrow X$ is null-homotopic. Then $V = g^{-1}(U)$ is open in Y. Since i is null-homotopic, the map $f \circ i = f|_U$ is null-homotopic. Thus $f \circ g|_V$ is null-homotopic as $g(V) \subset U$. Since $f \circ g$ is homotopic to id_Y, it follows that $f \circ g|_V$ is homotopic to the inclusion map $V \hookrightarrow Y$. Thus the inclusion map $V \hookrightarrow Y$ is null-homotopic, i.e., V is an open categorical subset of Y.

Explicitly, we can define $K : V \times I \to Y$ by

$$K(v,t) = \begin{cases} G(v, 2t) & \text{if } 0 \le t \le 1/2, \\ f(H(g(v), 2t-1)) & \text{if } 1/2 \le t \le 1. \end{cases}$$

where $G : Y \times I \to Y$ is a homotopy between id_Y and $f \circ g$ and $H : U \times I \to X$ is a homotopy between the inclusion map $i : U \hookrightarrow X$ and the constant map c_{x_0}. By pasting lemma, K is continuous with $K(v, 0) = v$ and $K(v, 1) = f(x_0)$. Thus the inclusion map $V \hookrightarrow Y$ is null-homotopic, i.e., V is an open categorical subset of Y.

If $\mathrm{cat}(X) = n$ and $\{U_i\}_{i=0}^n$ is an open categorical cover of X, then $\{V_i\}_{i=0}^n$ forms an open categorical cover of Y. Thus $\mathrm{cat}(Y) \le n$. Similarly, if Y dominates X, then we get $\mathrm{cat}(X) \le \mathrm{cat}(Y)$. ∎

Proposition 3.1.3. Let X be a topological space. Suppose $X = A \cup B$ where A and B are open in X. Then $\mathrm{cat}(X) \le \mathrm{cat}(A) + \mathrm{cat}(B) + 1$.

Proof. Let $\mathrm{cat}(A) = n$ and $\mathrm{cat}(B) = m$. Let $\mathcal{U} = \{U_i\}_{i=0}^n$ and $\mathcal{V} = \{V_i\}_{i=0}^m$ be open categorical covers of A and B respectively. Since A and B are open in X, $\mathcal{U} \cup \mathcal{V}$ forms an open cover of X. Since U_i and V_i are categorical in A and B respectively, U_i and V_i are also categorical in X. Thus $\mathcal{U} \cup \mathcal{V}$ forms an open categorical cover of X of cardinality $n + m + 2$ and therefore $\mathrm{cat}(X) \le n + m + 1$. ∎

Definition 3.1.4. Let X be a topological space. A finite sequence $V_0, V_1, \dots, V_n = X$ of open subspaces with $V_0 = \emptyset$ is said to be a **categorical sequence** of length n if each of the differences $V_i \setminus V_{i-1}$ for $i = 1, \dots, n$ is contained in an open categorical set U_i of X.

Suppose $V_0, V_1, \dots, V_n = X$ be a categorical sequence of length n. Define $W_i = \cup_{j=0}^i V_j$. Then $W_{i+1} \setminus W_i \subset V_{i+1} \setminus V_i$, i.e., we have a increasing categorical sequence $W_0 \subset W_1 \subset \dots \subset W_n = X$ of length n. Thus the above definition is equivalent to taking the sequence of open subspaces to be increasing as the other way implication is obvious.

Lemma 3.1.5. A space X has a categorical sequence of length $n + 1$ if and only if $\mathrm{cat}(X) \le n$.

Proof. Suppose X has a categorical sequence, say $V_0, V_1, \dots, V_{n+1} = X$, of length $n + 1$. Then there exist open categorical sets U_i, for $1 \le i \le n + 1$, of X such that $V_i \setminus V_{i-1} \subset U_i$. Since the sets $V_i \setminus V_{i-1}$ forms a cover of X, it follows $\{U_i\}_{i=1}^{n+1}$ is an open categorical cover of X. Thus $\mathrm{cat}(X) \le n$.

Conversely, suppose $\text{cat}(X) \leq n$. Then there exists an open categorical cover $\{U_i\}_{i=1}^{n+1}$ of X. Define

$$V_0 = \emptyset \text{ and } V_k = \bigcup_{i=1}^{k} U_i \text{ for } k = 1, \dots, n+1.$$

Observe $V_{n+1} = X$ and $V_k \setminus V_{k-1} \subset U_k$ which is open and categorical. Thus $V_0 \subset V_1 \subset \ldots \subset V_n = X$ is a categorical sequence of length $n+1$. ■

Theorem 3.1.6. Let E be a path-connected space and $p : E \to B$ a covering map. Then

$$\text{cat}(E) \leq \text{cat}(B).$$

Proof. Let U be an open categorical subset of B. Then there exists a homotopy $H : U \times I \to B$ such that $H(b, 0) = b$ and $H(b, 1) = b_0$ for all $b \in U$. Define

$$G : p^{-1}(U) \times I \to B$$

by $G = H(p \times \text{id})$. Then $G(e, 0) = p(e)$ and $G(e, 1) = b_0$ for all $e \in p^{-1}(U)$. Observe that we have the following commutative diagram

$$\begin{array}{ccc} p^{-1}(U) & \hookrightarrow & E \\ {\scriptstyle i}\downarrow & \overset{\widetilde{G}}{\nearrow} & \downarrow{\scriptstyle p} \\ p^{-1}(U) \times I & \xrightarrow{G} & B \end{array}$$

where $p^{-1}(U) \hookrightarrow E$ is the inclusion map and $i : p^{-1}(U) \hookrightarrow p^{-1}(U) \times I$ is given by $i(e) = (e, 0)$ for all $e \in p^{-1}(U)$. Since covering maps are fibrations, there exists a homotopy $\widetilde{G} : p^{-1}(U) \times I \to E$ making the above diagram commute. Thus $\widetilde{G}(e, 0) = e$ and $p\widetilde{G} = G$ for all $e \in p^{-1}(U)$. Since $p\widetilde{G}(e, 1) = G(e, 1) = b_0$ for all $e \in p^{-1}(U)$, it follows that $\widetilde{G}_1(p^{-1}(U)) \subset p^{-1}(b_0)$. Thus $\widetilde{G}_1$ can be factored as

$$p^{-1}(U) \to p^{-1}(b_0) \hookrightarrow E$$

where $p^{-1}(b_0) \hookrightarrow E$ is the inclusion map. Since p is covering map, $p^{-1}(b_0)$ is discrete subset of E. Thus it follows that the inclusion map $p^{-1}(b_0) \hookrightarrow E$ is null-homotopic as E is path-connected. If $L : p^{-1}(b_0) \times I \to E$ is the homotopy between the inclusion map $p^{-1}(b_0) \hookrightarrow E$ and a constant map, then

$$\widetilde{L} : p^{-1}(U) \times I \to E$$

given by $\widetilde{L}(e, t) = L(\widetilde{G}_1(e), t)$ is a homotopy between $\widetilde{G}_1$ and a constant map. Thus $\widetilde{G}_1$ is null-homotopic. Since $\widetilde{G}_1$ is homotopic to $\widetilde{G}_0$ which is the inclusion map $p^{-1}(U) \hookrightarrow$

E, it follows that $p^{-1}(U) \hookrightarrow E$ is null-homotopic.

If $\text{cat}(B) = n$ and $\{U_j\}_{j=0}^n$ is an open categorical cover of B, then the set $\{p^{-1}(U_j)\}_{j=0}^n$ forms an open categorical cover of E. Thus $\text{cat}(E) \leq n$. ■

3.2 An Upper Bound for LS Category

Theorem 3.2.1. Let X be a path-connected locally contractible paracompact Hausdorff space. Then

$$\text{cat}(X) \leq \dim(X).$$

Proof. Suppose $\dim(X) = n$. Since X is locally contractible there exists an open categorical cover $\{U_i\}_{i \in J}$ of X. Since $\dim(X) = n$, by Theorem 1.4.5, we have an open cover $\mathcal{W} = \{W_k\}_{k=1}^{n+1}$ of X such that W_k $(k = 1, \ldots, n+1)$ is a disjoint union of open sets each of which contained in some U_i. Since U_i are categorical and each open set making W_k lies inside some U_i, we have open sets making W_k are categorical. Since X is path-connected and open sets making W_k are disjoint and categorical, we have W_k is categorical for $k = 1, \ldots, n+1$. Thus $\mathcal{W}$ forms an open categorical cover of X of cardinality $n+1$ and therefore $\text{cat}(X) \leq n$. ■

3.3 A Lower Bound for LS Category

Definition 3.3.1. Let X be a topological space and R be a commutative ring with unity. The **cup-length** of X with coefficients in R is the greatest number n for which there exist n elements; say $\xi_1, \ldots, \xi_n$; of degree ≥ 1 in the cohomology ring $H^*(X; R)$ such that $\xi_1 \cup \ldots \cup \xi_n \neq 0$. It is denoted by $\text{cup}_R(X)$.

It is equivalent to saying that $\text{cup}_R(X)$ is the least integer n such that all $(n+1)$-fold cup product of elements of degree ≥ 1 vanish in the cohomology ring $H^*(X; R)$.

Theorem 3.3.2. Let X be a topological space and R be a commutative ring with unity. Then

$$\text{cup}_R(X) \leq \text{cat}(X).$$

Proof. Suppose $\text{cat}(X) = n$. Then there exists an open categorical cover $\{U_i\}_{i=0}^n$ of X. Let $\{x_i\}_{i=0}^n$ be arbitrary elements of cohomology ring $H^*(X; R)$ of degree ≥ 1. Let $j_i : U_i \hookrightarrow X$ and $k_i : X \hookrightarrow (X, U_i)$ be the inclusion maps. These inclusions induce a long exact sequence of cohomology groups

$$\ldots \longrightarrow H^m(X, U_i; R) \xrightarrow{k_i^*} H^m(X; R) \xrightarrow{j_i^*} H^m(U_i; R) \longrightarrow \ldots \ .$$

Since $j_i : U_i \hookrightarrow X$ is null-homotopic, it follows $j_i^* = 0$ for all $m \geq 1$. Since the sequence is exact and degree of each x_i is ≥ 1, for each $x_i \in H^*(X; R)$ there exist a element $y_i \in H^*(X, U_i; R)$ such that $k_i^*(y_i) = x_i$. Observe that $y_0 \cup \ldots \cup y_n$ belongs to $H^*(X, \cup_{i=0}^n U_i; R) = H^*(X, X; R) = 0$. Thus $y_0 \cup \ldots \cup y_n = 0$.

Consider the diagram

$$
\begin{array}{ccc}
X & \xhookrightarrow{\quad k \quad} & (X, \cup_{i=0}^n U_i) \\
\big\downarrow{\scriptstyle \Delta_{n+1}} & & \big\downarrow{\scriptstyle \Delta_{n+1}} \\
X \times \ldots \times X & \xhookrightarrow{k_0 \times \ldots \times k_n} & (X, U_0) \times \ldots \times (X, U_n) \\
\big\downarrow{\scriptstyle \pi_i} & & \big\downarrow{\scriptstyle \pi_i} \\
X & \xhookrightarrow{\quad k_{i-1} \quad} & (X, U_{i-1})
\end{array}
$$

where k is the inclusion map, π_i is the projection maps onto the i^{th} factor and Δ_{n+1} is the $(n+1)$-fold diagonal map. Since the diagram is commutative, it follows $k^*\Delta_{n+1}^* = \Delta_{n+1}^*(k_0 \times \ldots \times k_n)^*$ and $(k_0 \times \ldots \times k_n)^* \pi_i^* = \pi_i^* k_{i-1}^*$. Thus

$$
\begin{aligned}
k^*(y_0 \cup \ldots \cup y_n) &= k^*\Delta_{n+1}^*(y_0 \times \ldots \times y_n) \\
&= \Delta_{n+1}^*(k_0 \times \ldots \times k_n)^*(y_0 \times \ldots \times y_n) \\
&= \Delta_{n+1}^*(k_0 \times \ldots \times k_n)^*(\pi_1^*(y_0) \cup \ldots \cup \pi_{n+1}^*(y_n)) \\
&= \Delta_{n+1}^*(\pi_1^* k_0^*(y_0) \cup \ldots \cup \pi_{n+1}^* k_n^*(y_n)) \\
&= k_0^*(y_0) \cup \ldots \cup k_n^*(y_n) \\
&= x_0 \cup \ldots \cup x_n.
\end{aligned}
$$

Since $y_0 \cup \ldots \cup y_n = 0$, it follows $x_0 \cup \ldots \cup x_n = 0$. Thus $\text{cup}_R(X) \le n$. ∎

3.4 Product Inequality

In this section, we estimate $\text{cat}(X \times Y)$ in terms of $\text{cat}(X)$ and $\text{cat}(Y)$. A proof of the following theorem using categorical sequences introduced in the section 3.1 is given in [2, Chapter 1]. Here we give a simpler proof which is based on the proof of the product inequality for the topological complexity in [5].

Let X and Y be topological spaces with $\text{cat}(X) = n$ and $\text{cat}\ (Y) = m$. Suppose $\{U_i\}_{i=0}^n$ and $\{V_j\}_{j=0}^m$ be open categorical covers of X and Y respectively. Then $\{U_i \times V_j\}$ forms an open categorical cover of $X \times Y$. Thus $\text{cat}(X \times Y) \le (n+1)(m+1) - 1 = \text{cat}(X)\text{cat}(Y) + \text{cat}(X) + \text{cat}(Y)$. But this is not a good upper bound for computation of LS category for the product of spaces. In the following Theorem, we provide an improved upper bound for $\text{cat}(X \times Y)$ in terms of $\text{cat}(X)$ and $\text{cat}(Y)$.

Theorem 3.4.1. Let X and Y be path-connected paracompact Hausdorff spaces.

Then

$$\mathrm{cat}(X \times Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y).$$

Proof. Let $\mathrm{cat}(X) = n$ and $\mathrm{cat}(Y) = m$. Then there exist open categorical covers $\{U_i\}_{i=0}^n$ and $\{V_j\}_{j=0}^m$ of X and Y respectively. Since X and Y are paracompact Hausdorff spaces, there exist partition of unity $\{f_i\}_{i=0}^n$ and $\{g_j\}_{j=0}^m$ subordinate to the covers $\{U_i\}_{i=0}^n$ and $\{V_j\}_{j=0}^m$ respectively.

For each pair of nonempty subsets $S \subset \{0, 1, \ldots, n\}$ and $T \subset \{0, 1, \ldots, m\}$ define $W(S, T) \subset X \times Y$ by $(x, y) \in W(S, T)$ if

$$f_i(x).g_j(y) > f_{i'}(x).g_{j'}(y)$$

for any $(i, j) \in S \times T$ and for any $(i', j') \notin S \times T$. If $S = \{0, 1, \ldots, n\}$ and $T = \{0, 1, \ldots, m\}$, then define $W(S, T) \subset X \times Y$ by

$$W(S, T) = \{(x, y) \in X \times Y \mid f_i(x).g_j(y) > 0 \text{ for any } (i, j) \in S \times T\}.$$

Define a map $f_i.g_j : X \times Y \to \mathbb{R}$ by

$$f_i.g_j(x, y) = f_i(x).g_j(y).$$

The map $f_i.g_j$ is the composition of the following maps

$$X \times Y \xrightarrow{f_i \times g_j} \mathbb{R} \times \mathbb{R} \xrightarrow{\ h\ } \mathbb{R}$$

where h is the multiplication of real numbers. Thus $f_i.g_j$ is continuous and $(f_i.g_j - f_{i'}.g_{j'})^{-1}(0, 1]$ is open.

Since

$$W(S, T) = \bigcap_{\substack{(i,j) \in S \times T \\ (i',j') \notin S \times T}} (f_i.g_j - f_{i'}.g_{j'})^{-1}(0, 1]$$

and the intersection is over finitely many elements, it follows that $W(S, T)$ is open.

Let $S' \subset \{0, 1, \ldots, n\}$ and $T' \subset \{0, 1, \ldots, m\}$ such that $W(S, T) \cap W(S', T') \neq \emptyset$. If $S \times T \not\subset S' \times T'$, then there exist $(i_0, j_0) \in (S \times T) \setminus (S' \times T')$. Let $(x, y) \in W(S, T) \cap W(S', T')$. Then $f_{i_0}(x).g_{j_0}(y) > f_i(x).g_j(y)$ for any $(i, j) \notin S \times T$ and $f_k(x).g_l(y) > f_{i_0}(x).g_{j_0}(y)$ for any $(k, l) \in S' \times T'$. Thus $f_k(x).g_l(y) > f_i(x).g_j(y)$ for any $(k, l) \in S' \times T'$ and for any $(i, j) \notin S \times T$. This implies $S' \times T' \subset S \times T$. Therefore $W(S, T)$ and $W(S', T')$ are disjoint if neither $S \times T \subset S' \times T'$ nor $S' \times T' \subset S \times T$.

Suppose $(x, y) \in X \times Y$. Let

$$S = \{i \in \{0, \ldots, n\} \mid f_i(x) = \max\{f_0(x), \ldots, f_n(x)\}\},$$

and

$$T = \{j \in \{0, \ldots, m\} \mid g_j(y) = \max\{g_0(y), \ldots, g_m(y)\}\}.$$

Then $(x, y) \in W(S, T)$. Thus the sets $W(S, T)$ forms an open cover of $X \times Y$.

Let $(i, j) \in S \times T$. Since $f_i(x).g_j(y) > 0$ for all $(x, y) \in W(S, T)$, it follows that $W(S, T) \subset U_i \times V_j$. Thus $W(S, T)$ is categorical since $U_i \times V_j$ is categorical.

Let W_k $(k = 2, \ldots, n+m+2)$ denote the union of $W(S, T)$ such that $|S|+|T| = k$. The set $\{W_k\}_{k=2}^{n+m+2}$ forms an open cover of $X \times Y$. Suppose $|S|+|T| = |S'|+|T'|$ and $S \times T \neq S' \times T'$. Then $W(S, T)$ and $W(S', T')$ are disjoint since neither $S \times T \subset S' \times T'$ nor $S' \times T' \subset S \times T$. Thus W_k is a disjoint union of open categorical sets. Since X and Y are path-connected, $X \times Y$ is also path-connected, and it follows W_k is categorical. Hence $\{W_k\}_{k=2}^{n+m+2}$ forms an open categorical cover of $X \times Y$. Thus $\mathrm{cat}(X \times Y) \leq n + m$. ■

3.5 Category Weight

Definition 3.5.1. The **category of a map** $f : A \to X$, denoted $\mathrm{cat}(f)$, is the smallest integer n such that there exists an open cover $\{U_i\}_{i=0}^n$ of A such that $f|_{U_i}$ is null-homotopic for each i.

Thus the category of a space X, denoted $\mathrm{cat}(X)$, can equivalently be defined as the $\mathrm{cat}(\mathrm{id}_X)$ where id_X is the identity map of X.

Proposition 3.5.2. Let $f : A \to X$ be a continuous map. Then

$$\mathrm{cat}(f) \leq \min\{\mathrm{cat}(A), \mathrm{cat}(X)\}.$$

Proof. Let $\mathrm{cat}(A) = n$ and $\{U_i\}_{i=0}^n$ be an open categorical cover of A. Since U_i is a categorical set of A, the inclusion map $\mathrm{id}_{U_i}^A : U_i \hookrightarrow A$ is null-homotopic. Then the map $f \circ \mathrm{id}_{U_i}^A = f|_{U_i}$ is null-homotopic. Since $\{U_i\}_{i=0}^n$ is an open cover of A, it follows that $\mathrm{cat}(f) \leq n$.

Let $\mathrm{cat}(V) = m$ and $\{V_j\}_{j=0}^m$ be an open categorical cover of X. Let $W_j = f^{-1}(V_j)$. Then $\widetilde{f_j} : W_j \to V_j$, given by $\widetilde{f_j}(a) = f(a)$, is continuous. Since V_j is a categorical set of X, i.e., the inclusion map $\mathrm{id}_{V_j}^X : V_j \hookrightarrow X$ is null-homotopic, it follows that $\mathrm{id}_{V_j}^X \circ \widetilde{f_j} = f|_{W_j}$ is null-homotopic. Since $\{W_j\}_{j=0}^m$ forms an open cover of A, it follows that $\mathrm{cat}(f) \leq m$. ■

Definition 3.5.3. The **category weight** of a non-zero cohomology class $u \in H^*(X; R)$, denoted $\mathrm{wgt}(u)$, is defined by

$$\mathrm{wgt}(u) \geq k \text{ if and only if } f^*(u) = 0 \text{ for any } f : A \to X \text{ with } \mathrm{cat}(f) < k.$$

Proposition 3.5.4. Category weight satisfies the following properties:

(a) $\mathrm{wgt}(u) \le \mathrm{cat}(X)$ for all $u \in H^*(X; R)$.

(b) $f : A \to X$ such that $f^*(u) \neq 0$ then $\mathrm{wgt}(f^*(u)) \ge \mathrm{wgt}(u)$.

(c) $\mathrm{wgt}(u \cup v) \ge \mathrm{wgt}(u) + \mathrm{wgt}(v)$.

Proof. For (a) and (b), we refer to [2, Chapter 8, Proposition 8.22].

(c) Let $\mathrm{wgt}(u) = n$ and $\mathrm{wgt}(v) = m$. Let $f : A \to X$ be any continuous map such that $\mathrm{cat}(f) < n+m$. Since $\mathrm{cat}(f) < n+m$, there exist open sets $U_1, \ldots, U_n, V_1, \ldots, V_m$ such that $f|_{U_i}$ and $f|_{V_j}$ is null-homotopic for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Let

$$U = U_1 \cup \ldots \cup U_n \quad \text{and} \quad V = V_1 \cup \ldots \cup V_m.$$

Since $f|_{U_i}$ and $f|_{V_j}$ are null-homotopic, it follows that $\mathrm{cat}(f|_U) < n$ and $\mathrm{cat}(f|_V) < m$. Since $\mathrm{wgt}(u) = n$ and $\mathrm{wgt}(v) = m$, it follows that $f|_U^*(u) = 0$ and $f|_V^*(v) = 0$.

Since the following diagram commutes

$$\begin{array}{ccccc} & \overset{f|_U}{\nearrow} & X & & \\ & & \uparrow f & & \\ U & \hookrightarrow & A & \overset{k_1}{\hookrightarrow} & (A, U), \end{array}$$

we have the following commutative diagram

$$\begin{array}{ccccc} & & H^*(X) & & \\ & \overset{f|_U^*}{\swarrow} & \downarrow f^* & & \\ H^*(U) & \longleftarrow & H^*(A) & \overset{k_1^*}{\longleftarrow} & H^*(A, U) \end{array}$$

Since $f|_U^*(u) = 0$ and the lower sequence of the preceding commutative diagram is exact, it follows that there exists $\bar{u} \in H^*(A, U)$ such that it maps to $f^*(u)$ by the homomorphism $k_1^* : H^*(A, U) \to H^*(A)$.

$$\begin{array}{ccccc} H^*(U) & \longleftarrow & H^*(A) & \overset{k_1^*}{\longleftarrow} & H^*(A, U) \\ 0 & \longleftarrow\!\shortmid & f^*(u) & \longleftarrow\!\shortmid & \bar{u} \end{array}$$

Similarly, there exists $\bar{v} \in H^*(A, V)$ such that it maps to $f^*(v)$ by the homomorphism $k_2^* : H^*(A, V) \to H^*(A)$.

$$\begin{array}{ccccc} H^*(V) & \longleftarrow & H^*(A) & \overset{k_2^*}{\longleftarrow} & H^*(A, V) \\ 0 & \longleftarrow\!\shortmid & f^*(v) & \longleftarrow\!\shortmid & \bar{v} \end{array}$$

Now, $\bar{u} \cup \bar{v} \in H^*(A, U \cup V) = H^*(A, A) = 0$, thus $\bar{u} \cup \bar{v} = 0$.

Consider the diagrams

$$
\begin{array}{ccc}
A & \overset{k}{\hookrightarrow} & (A, U \cup V) \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\
A \times A & \overset{k_1 \times k_2}{\hookrightarrow} & (A, U) \times (A, V) \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} \\
A & \overset{k_1}{\hookrightarrow} & (A, U)
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \overset{k}{\hookrightarrow} & (A, U \cup V) \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\
A \times A & \overset{k_1 \times k_2}{\hookrightarrow} & (A, U) \times (A, V) \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} \\
A & \overset{k_2}{\hookrightarrow} & (A, V)
\end{array}
$$

where k is the inclusion map, π_i is the projection map onto the i^{th} factor, Δ is the diagonal map and $(A, U) \times (A, V) = (A \times A, (A \times V) \cup (U \times A))$. Since the diagrams are commutative, it follows that $k^*\Delta^* = \Delta^*(k_1 \times k_2)^*$ and $(k_1 \times k_2)^*\pi_i^* = \pi_i^* k_i^*$. Thus

$$
\begin{aligned}
k^*(\bar{u} \cup \bar{v}) = k^*\Delta^*(\bar{u} \times \bar{v}) &= \Delta^*(k_1 \times k_2)^*(\bar{u} \times \bar{v}) \\
&= \Delta^*(k_1 \times k_2)^*(\pi_1^*(\bar{u}) \cup \pi_2^*(\bar{v})) \\
&= \Delta^*(\pi_1^* k_1^*(\bar{u}) \cup \pi_2^* k_2^*(\bar{v})) \\
&= k_1^*(\bar{u}) \cup k_2^*(\bar{v}) \\
&= f^*(u) \cup f^*(v) \\
&= f^*(u \cup v).
\end{aligned}
$$

Since $\bar{u} \cup \bar{v} = 0$, it follows that $f^*(u \cup v) = 0$. Thus $\text{wgt}(u \cup v) \geq n + m$. ■

3.6 Some Examples

Example 3.6.1.

- If X is contractible, then X forms an open categorical cover of itself. Thus $\text{cat}(X) = 0$. This also implies that $\text{cup}_R(X) = 0$ for all R, where R is a commutative ring with identity. Hence $H^p(X; R) = 0$ for all $p \geq 1$.

- The sphere S^n can be covered by two open hemispherical sets extended slightly to overlap. Each of these open sets is contractible and hence are open categorical subsets of S^n. Thus we have $\text{cat}(S^n) \leq 1$. Since S^n is not contractible, it follows $\text{cat}(S^n) > 0$. Thus $\text{cat}(S^n) = 1$.

Theorem 3.6.2. Let $X = S^m \times \ldots \times S^m$ be the Cartesian product of n copies of m-dimensional sphere S^m. Then $\text{cat}(X) = n$.

Proof. Since S^m is a path-connected smooth manifold, by Theorem 3.4.1 and preceding example, we get

$$\text{cat}(X) \leq n \ \text{cat}(S^m) = n.$$

The cohomology groups of S^n are given by

$$H^p(S^m;\Bbbk) = \begin{cases} \Bbbk & \text{for } p = 0, m; \\ 0 & \text{otherwise.} \end{cases}$$

Let a be the fundamental class of S^m, i.e., a is the generator of $H^m(S^m;\Bbbk)$ as a $\Bbbk$-module. Then the cohomology ring of S^m is given by

$$H^*(S^m;\Bbbk) = \Bbbk[a]/(a^2).$$

Let $\pi_i : X \to S^m$ be the projection map onto the i^{th} factor of X. Suppose $a_i \in H^m(S^m;\Bbbk)$ is the fundamental class of the i^{th} factor and let $u_i = \pi_i^*(a_i)$ for each $i = 1, \ldots, n$. Since $H_j(S^m)$ is finitely generated for all j, by the Künneth Theorem [10, Chapter 7, Theorem 61.6], the n-fold cross product homomorphism

$$H^*(S^m;\Bbbk) \otimes_{\Bbbk} \ldots \otimes_{\Bbbk} H^*(S^m;\Bbbk) \xrightarrow{\times_n} H^*(X;\Bbbk)$$

is an isomorphism of $\Bbbk$-algebras. Thus

$$a_1 \otimes \ldots \otimes a_n \mapsto \pi_1^*(a_1) \cup \ldots \cup \pi_n^*(a_n) = u_1 \cup \ldots \cup u_n \neq 0.$$

Under the n-fold cross product isomorphism, each u_i is an element of degree m in the graded ring $H^*(X;\Bbbk)$ and $u_1 \cup \ldots \cup u_n \neq 0 \in H^*(X;\Bbbk)$. Thus $n \leq \mathrm{cup}_{\Bbbk}(X)$. Thus, by Theorem 3.3.2, we obtain $\mathrm{cat}(X) = n$. ■

Theorem 3.6.3. Let Σ_g be the connected compact orientable surface (without boundary) of genus g. Then

$$\mathrm{cat}(\Sigma_g) = \begin{cases} 1 & \text{if } g = 0, \\ 2 & \text{if } g \geq 1. \end{cases}$$

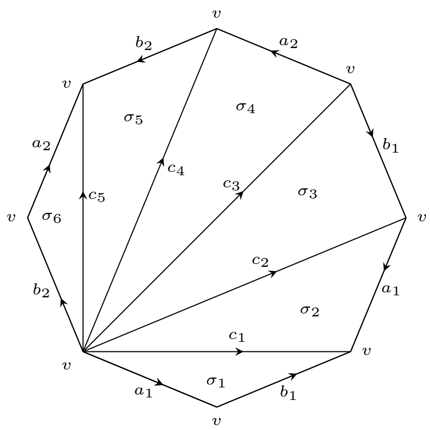
Proof. If $g = 0$, then Σ_g is the 2-dimensional sphere S^2. It follows from Example 3.6.1 that $\mathrm{cat}(S^2) = 1$. If $g = 1$, then Σ_g is the 2-dimensional torus T^2. Since T^2 is homeomorphic to $S^1 \times S^1$, by preceding Theorem, we obtain $\mathrm{cat}(T^2) = 2$.

Let us consider the case $g \geq 2$. Since Σ_g is a path-connected smooth manifold, by Theorem 3.2.1, we have

$$\mathrm{cat}(\Sigma_g) \leq \dim(\Sigma_g) = 2 \qquad \text{for all } g.$$

The cohomology groups of Σ_g is given by

$$H^n(\Sigma_g;\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } n = 0, 2, \\ \mathbb{Q}^{2g} & \text{if } n = 1, \\ 0 & \text{if } n \geq 3. \end{cases}$$



where $H^1(\Sigma_g;\mathbb{Q})$ is generated by the equivalence classes of the cocycles ϕ_i and ψ_i, $1 \leq i \leq g$, which takes value 1 on elementary 1-chain a_i and b_i respectively; and $H^2(\Sigma_g;\mathbb{Q})$ is generated by the equivalence class of the cocycle φ which takes 1 on 2-chain $\Sigma(-1)^{\xi_i}\sigma_i$ where $\xi_i = 0$ if orientation of σ_i is anticlockwise and $\xi_i = 1$ if orientation of σ_i is clockwise.

Consider the case of $g = 2$ with the Δ-complex structure drawn above with the 2-cycles σ_1, σ_4 and σ_5 having anticlockwise orientation and the 2-cycles σ_2, σ_3 and σ_6 having clockwise orientation starting from the common vertex v of all σ_i. We have 1-cocycles ϕ_1, ϕ_2, ψ_1 and ψ_2. Since $\delta\phi_i = \delta\psi_i = 0$, where δ is coboundary operator, the cocycle ϕ_1 takes value 1 on a_1, c_1; the cocycle ψ_1 takes value 1 on b_1, c_1, c_2; the cocycle ϕ_2 takes value 1 on a_2, c_4, c_5; the cocycle ψ_2 takes value 1 on b_2, c_5; and 0 otherwise.

Computing the cup products of 1-cocycles we get

$$(\phi_1 \cup \psi_1)(\sigma_i) = \begin{cases} 1, & \text{if i} = 1, \\ 0, & \text{otherwise.} \end{cases} \qquad (\phi_2 \cup \psi_2)(\sigma_i) = \begin{cases} 1, & \text{if i} = 5, \\ 0, & \text{otherwise.} \end{cases}$$

and $\phi_i \cup \phi_j = \psi_i \cup \psi_j = 0$ for all $i, j \in \{1, 2\}$ and $\phi_i \cup \psi_j = 0$ if $i \neq j$.

Let c denote the 2-chain $\Sigma(-1)^{\xi_i}\sigma_i$ where $\xi_i = 0$ if orientation of σ_i is anticlockwise and $\xi_i = 1$ if orientation of σ_i is clockwise. Thus $(\phi_1 \cup \psi_1)(c) = 1$ and $(\phi_2 \cup \psi_2)(c) = 1$, i.e., $\phi_1 \cup \psi_1$ and $\phi_2 \cup \psi_2$ takes value 1 on the 2-chain c. This implies that $\phi_1 \cup \psi_1 = \phi_2 \cup \psi_2 = \varphi$.

Similarly, for all $g \geq 2$ we can find non-zero 1-cocycles ϕ_1, ϕ_2, ψ_1 and ψ_2 and a non-zero 2-cocycle φ such that $\phi_1 \cup \psi_1 = \phi_2 \cup \psi_2 = \varphi$ and $\phi_i \cup \phi_j = \psi_i \cup \psi_j = 0$ for all $i, j \in \{1, 2\}$ and $\phi_i \cup \psi_j = 0$ if $i \neq j$.

Let u_i, v_i and Λ be the equivalence class of the cocycles ϕ_i, ψ_i and φ respectively. Then $u_i \cup v_i = \Lambda \neq 0$. Thus $2 \leq \text{cup}_{\mathbb{Q}}(\Sigma_g)$. Thus, by Theorem 3.3.2, $\text{cat}(\Sigma_g) = 2$ for $g \geq 2$. ■

Chapter 4

Higher Topological Complexity

In this chapter, we study the numerical homotopy invariants - higher topological complexity $\mathrm{TC}_n(X)$ of a space X, for $n \geq 2$, which measures the discontinuity of motion planning in X. The topological complexity $\mathrm{TC}(X) := \mathrm{TC}_2(X)$ was introduced by M. Farber in [5] to study the motion planning problem for the configuration spaces of mechanical systems. Later Yuli B. Rudyak introduced higher analogs $\mathrm{TC}_n(X)$ of topological complexity in [12]. In the sections 4.1 - 4.3 we reproduce the proofs in [5] for $\mathrm{TC}(X)$ to show similar results for $\mathrm{TC}_n(X)$, and the sections 4.4 - 4.7 are the expositions of the corresponding sections in [5]. Then in the section 4.8 we classify spaces with $\mathrm{TC}_n(X) = n - 1$ for some $n \geq 2$. This is a recent result of M. Grant et al. (2013) in [6].

4.1 Topological Complexity of Motion Planning

Let $\widetilde{e}_n : PX \to X \times \ldots \times X = X^n$ be the fibration which evaluates a path $\alpha \in PX$ as follows

$$\widetilde{e}_n(\alpha) = (\alpha(0), \alpha(t_1), \ldots, \alpha(t_{n-2}), \alpha(1))$$

where $t_i = i/(n-1)$ for $i \in \{0, \ldots, n-1\}$ and $n \geq 2$. In view of Example 1.3.20, the map $\widetilde{e}_n$ is a fibrational substitute for the n-fold diagonal map $\Delta_n : X \to X^n$ for $n \geq 2$.

It is straightforward to see that the fibration $\widetilde{e}_n$ is surjective if and only if the space X is path-connected. Since in this chapter we would like to study $\mathrm{secat}(\widetilde{e}_n)$ for which we require the fibration $\widetilde{e}_n$ to be surjective, we shall assume that the space X under consideration is path-connected.

Definition 4.1.1. The space of all possible configurations of a mechanical system is called the **configuration space** of the system.

Let X be the configuration space of a mechanical system. In most of the practical applications the configuration space X is equipped with a topological space structure. We shall assume that the configuration space X is path-connected.

Definition 4.1.2. Let X be a path-connected space and $n \geq 2$. A **motion planning** is a set map $s : X^n \to PX$ such that $\widetilde{e}_n \circ s$ is the identity map of X^n.

A motion planning s associates to each point $(x_1, \dots, x_n) \in X^n$ a path $s(x_1, \dots, x_n)$ which starts at x_1 and ends at x_n such that it goes from x_1 to x_n via $n-2$ equally timed intermediate points in X, i.e., it describes the movement of the mechanical system from the initial configuration x_1 to the final configuration x_n such that there are additional $n-2$ equally timed intermediate configurations. Even if the points $(x_1, \dots, x_n)$ and $(x'_1, \dots, x'_n)$ are arbitrarily close the associated paths may not be. Thus the continuity of the motion planning is essential.

Theorem 4.1.3. A continuous motion planning $s : X^n \to PX$ exists for $n \geq 2$ if and only if the space X is contractible.

Proof. Suppose $s : X^n \to PX$ is a continuous motion planning. Fix a point $x_0 \in X$. Consider the homotopy $H : X \times I \to X$ defined as the composition of the following maps

$$X \times I \longrightarrow (X \times \{x_0\}^{n-1}) \times I \xrightarrow{s\times \mathrm{id}} PX \times I \xrightarrow{\mathrm{ev}} X,$$

where $\{x_0\}^{n-1}$ denotes the Cartesian product of $n-1$ copies of $\{x_0\}$, the first map is the natural homeomorphism and the map ev is the evaluation map. Then $H(x,t) = s(x, x_0, \dots, x_0)(t)$. Thus $H(x,0) = s(x, x_0, \dots, x_0)(0) = x$ and $H(x,1) = s(x, x_0, \dots, x_0)(1) = x_0$, i.e., H is a homotopy between the identity map of X and the constant map at x_0. Thus X is contractible.

Conversely, suppose X is contractible, i.e., there exists a homotopy $H : X \times I \to X$ such that $H(x,0) = x$ and $H(x,1) = x_0$ for all $x \in X$. Let $s_{i,i+1} : X^n \to PX$ be defined as

$$s_{i,i+1}(x_1, \dots, x_n)(t) = \begin{cases} H(x_i, 2t) & 0 \leq t \leq 1/2, \\ H(x_{i+1}, 2-2t) & 1/2 \leq t \leq 1. \end{cases}$$

for $i \in \{1, \dots, n-1\}$. By pasting lemma and Theorem 1.1.4, $s_{i,i+1}$ is continuous. Also, $s_{i,i+1}$ satisfies $s_{i,i+1}(x_1, \dots, x_n)(0) = H(x_i, 0) = x_i$ and $s_{i,i+1}(x_1, \dots, x_n)(1) = H(x_{i+1}, 0) = x_{i+1}$.

Let $t_i = i/(n-1)$ for $i \in \{0, \dots, n-1\}$. Define $s : X^n \to PX$ as

$$s(x_1, \dots, x_n)(t) = s_{i,i+1}(x_1, \dots, x_n)((n-1)(t - t_{i-1})) \text{ if } t \in [t_{i-1}, t_i]$$

for $i \in \{1, \dots, n-1\}$. By pasting lemma and Theorem 1.1.4, s is continuous. Also, s

satisfies

$$s(x_1, \ldots, x_n)(t_i) = s_{i,i+1}(x_1, \ldots, x_n)(1) = x_{i+1},$$

i.e.,

$$\widetilde{e}_n(s(x_1, \ldots, x_n)) = (s(x_1, \ldots, x_n)(t_0), \ldots, s(x_1, \ldots, x_n)(t_{n-1})) = (x_1, \ldots, x_n).$$

Thus s is a continuous motion planning. ■

Definition 4.1.4. Let X be a path-connected space and $n \geq 2$. The **higher topological complexity** of motion planning in X, denoted $\mathrm{TC}_n(X)$, of a space X is defined to be the Schwarz genus of the fibration $\widetilde{e}_n$, i.e., $\mathrm{TC}_n(X) = \mathrm{secat}(\widetilde{e}_n)$.

In other words, the higher topological complexity $\mathrm{TC}_n(X)$ is the smallest number m such that there exists an open cover $\{U_i\}_{i=0}^m$ of X^n such that over each open set U_i there exists a continuous motion planning $s_i : U_i \to PX$, i.e., $\widetilde{e}_n \circ s_i$ is the inclusion map from U_i into X^n. If no such m exists, then $\mathrm{TC}_n(X) = \infty$. Thus, intuitively, $\mathrm{TC}_n(X)$ measures the discontinuity of motion planning in X in which not only the initial and the final points are inputted but also an additional $n - 2$ intermediate points.

Definition 4.1.5. Let X be a path-connected space. The **topological complexity** of motion planning in X, denoted $\mathrm{TC}(X)$, of a space X is defined as $\mathrm{TC}(X) = \mathrm{TC}_2(X)$.

The Definition 4.1.4 and Theorem 4.1.3 yields the following result.

Theorem 4.1.6. Let X be a path-connected space. The higher topological complexity $\mathrm{TC}_n(X) = 0$ for $n \geq 2$ if and only if X is contractible.

Example 4.1.7. Let X be a convex subset of the Euclidean space $\mathbb{R}^n$. Since X is contractible, we have $\mathrm{TC}_n(X) = 0$. We can also directly define a continuous motion planning in X by $s : X^n \to PX$ defined as

$$s(x_1, \ldots, x_n)(t) = x_i + (n-1)(t - t_{i-1})(x_{i+1} - x_i) \text{ if } t \in [t_{i-1}, t_i]$$

where $t_i = i/(n-1)$ for $i \in \{0, \ldots, n-1\}$.

The following result states that TC_n is a homotopy invariant. This reduces the effort in computation of $\mathrm{TC}_n(X)$ as the space X can now be simplified.

Theorem 4.1.8. Let X and Y be path-connected topological spaces. Suppose X and Y are homotopy equivalent. Then $\mathrm{TC}_n(X) = \mathrm{TC}_n(Y)$ for $n \geq 2$.

Proof. Suppose there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to the identity map of Y, say id_Y.

Let U be an open subset of X^n with a continuous section $s : U \to PX$ of $\widetilde{e}_n$. Let $g^n : Y^n \to X^n$ denote the n-fold Cartesian product of the map g. Then $V = (g^n)^{-1}(U) \subset Y^n$ is open. Let $t_i = i/(n-1)$ for $i \in \{0, \dots, n-1\}$. For each $i \in \{1, \dots, n-1\}$ define a homotopy $G_{i,i+1} : V \times I \to Y$ as composition of the following maps

$$V \times I \xrightarrow{(g^n)\times \mathrm{id}} U \times I \xrightarrow{s\times \mathrm{id}} PX \times I \xrightarrow{\mathrm{id}\times\phi_i} PX \times [t_{i-1}, t_i] \xrightarrow{\mathrm{ev}} X \xrightarrow{f} Y$$

where ϕ_i is the linear homeomorphism which maps 0 to t_{i-1} and 1 to t_i, and ev is the evaluation map. Then

$$G_{i,i+1}(y_1, \dots, y_n, 0) = f(s(g(y_1), \dots, g(y_n))(t_{i-1})) = f(g(y_i))$$

and

$$G_{i,i+1}(y_1, \dots, y_n, 1) = f(s(g(y_1), \dots, g(y_n))(t_i)) = f(g(y_{i+1})).$$

Let $H : Y \times I \to Y$ be a homotopy between id_Y and $f \circ g$. For each $i \in \{1, \dots, n-1\}$ define a map $r_{i,i+1} : V \to PY$ by

$$r_{i,i+1}(y_1, \dots, y_n)(t) = \begin{cases} H(y_i, 3t) & 0 \le t \le 1/3, \\ G_{i,i+1}(y_1, \dots, y_n, 3t-1) & 1/3 \le t \le 2/3, \\ H(y_{i+1}, 3-3t) & 2/3 \le t \le 1. \end{cases}$$

By pasting lemma and Theorem 1.1.4, the map $r_{i,i+1}$ is continuous Also $r_{i,i+1}$ satisfies $r_{i,i+1}(y_1, \dots, y_n, 0) = H(y_i, 0) = y_i$ and $r_{i,i+1}(y_1, \dots, y_n, 1) = H(y_{i+1}, 0) = y_{i+1}$. Define a map $r : V \to PY$ as

$$r(y_1, \dots, y_n)(t) = r_{i,i+1}(y_1, \dots, y_n)((n-1)(t - t_{i-1})) \text{ if } t \in [t_{i-1}, t_i]$$

for $i \in \{1, \dots, n-1\}$. By pasting lemma and Theorem 1.1.4, r is continuous. Also, r satisfies

$$r(y_1, \dots, y_n)(t_i) = r_{i,i+1}(y_1, \dots, y_n)(1) = y_{i+1},$$

i.e., $\widetilde{e}_n(r(y_1, \dots, y_n)) = (r(y_1, \dots, y_n)(t_0), \dots, r(y_1, \dots, y_n)(t_{n-1})) = (y_1, \dots, y_n)$. Thus r is a section of $\widetilde{e}_n$.

If $\mathrm{TC}_n(X) = k$ and $\{U_j\}_{j=0}^k$ is an open cover of X^n with a continuous motion planning $s_j : U_j \to PX$ over each U_j, then $\{V_j\}_{j=0}^k$ forms an open cover of Y^n with a continuous motion planning $r_j : V_j \to PY$ over each V_j. Thus $\mathrm{TC}_n(Y) \le k$. Similarly, if $g \circ f$ is homotopic to the identity map of X, then we get $\mathrm{TC}_n(X) \le \mathrm{TC}_n(Y)$. ■

4.2 An Upper Bound for $\mathrm{TC}_n(X)$

Proposition 4.2.1. Let X be a path-connected space. Then

$$\mathrm{cat}(X^{n-1}) \leq \mathrm{TC}_n(X) \leq \mathrm{cat}(X^n).$$

for $n \geq 2$.

Proof. Suppose $s : U \to PX$ is a section of $\widetilde{e}_n$ where U is a open subset of X^n. Let $x_0 \in X$ be a fixed point. Define

$$V = \{\bar{x} \in X^{n-1} \mid (x_0, \bar{x}) \in U\} \quad \text{where } \bar{x} = (x_1, \ldots, x_{n-1}).$$

Then V is open in X^{n-1} since it is the inverse image of $U \cap (\{x_0\} \times X^{n-1})$ under the natural homeomorphism between X^{n-1} and $\{x_0\} \times X^{n-1}$.

Since s is a section of $\widetilde{e}_n$, s satisfies the following property that $s(x_0, \bar{x})(t_i) = x_i$ for $i = 0, \ldots, n-1$ where $\bar{x} = (x_1, \ldots, x_{n-1})$ and $t_i = i/(n-1)$.

For each $i \in \{1, \ldots, n-1\}$ define a homotopy $s_i : V \times I \to X$ as the composition of the following maps

$$(\{x_0\} \times V) \times I \xrightarrow{s \times \mathrm{id}} PX \times I \xrightarrow{\mathrm{id} \times f_i} PX \times [0, t_i] \xrightarrow{\mathrm{ev}} X$$

where $f_i : I \to [0, t_i]$ is the linear homeomorphism which takes 0 to 0 and 1 to $t_i = i/(n-1)$; and ev is the evaluation map. Then $s_i((x_0, \bar{x}), 0) = s(x_0, \bar{x})(0) = x_0$ and $s_i((x_0, \bar{x}), 1) = s(x_0, \bar{x})(t_i) = x_i$ where $\bar{x} = (x_1, \ldots, x_{n-1})$.

Define a homotopy $H : V \times I \to X^{n-1}$ as

$$H(\bar{x}, t) = (s_1((x_0, \bar{x}), t), \ldots, s_{n-1}((x_0, \bar{x}), t))$$

where $\bar{x} = (x_1, \ldots, x_{n-1})$. Then

$$H(\bar{x}, 0) = (s_1((x_0, \bar{x}), 0), \ldots, s_{n-1}((x_0, \bar{x}), 0)) = (x_0, \ldots, x_0)$$

and

$$H(\bar{x}, 1) = (s_1((x_0, \bar{x}), 1), \ldots, s_{n-1}((x_0, \bar{x}), 1)) = (x_1, \ldots, x_{n-1}) = \bar{x}.$$

Thus the inclusion map $V \hookrightarrow X^{n-1}$ is null-homotopic, i.e., V is categorical.

If $\mathrm{TC}_n(X) = k$ and $\{U_j\}_{j=0}^k$ is an open cover of X^n with a continuous motion planning $s_j : U_j \to PX$ over each U_j, then $\{V_j\}_{j=0}^k$ forms an open categorical cover of X^{n-1} since $\{x_0\} \times V_j = U_j \cap (\{x_0\} \times X^{n-1})$. Thus $\mathrm{cat}(X^{n-1}) \leq k$. Since $\mathrm{TC}_n(X)$ is the Schwarz genus of $\widetilde{e}_n$, by Theorem 2.2.4, it follows that $\mathrm{TC}_n(X) \leq \mathrm{cat}(X^n)$. ∎

Proposition 4.2.2. Let X be path-connected paracompact Hausdorff space. Then

$$\mathrm{TC}_n(X) \leq n \ \mathrm{cat}(X)$$

for $n \geq 2$.

Proof. It follows from Theorem 3.4.1 and Proposition 4.2.1. ■

Theorem 4.2.3. Let X be a path-connected locally contractible paracompact Hausdorff space. Then

$$\mathrm{TC}_n(X) \leq n \ \dim(X)$$

for $n \geq 2$.

Proof. It follows from Theorem 3.2.1 and Proposition 4.2.2 . ■

If X is a path-connected smooth manifold, then X locally contractible paracompact Hausdorff space. Thus the bounds $\mathrm{TC}_n(X) \leq n \ \mathrm{cat}(X)$ and $\mathrm{TC}_n(X) \leq n \ \dim(X)$ are valid for all path-connected smooth manifolds.

4.3 A Lower Bound for $\mathrm{TC}_n(X)$

Let X be a path-connected space and R be a commutative ring with unity. Then, by Theorem 3.3.2 and Proposition 4.2.1, we get a lower bound on $\mathrm{TC}_n(X)$, i.e.,

$$\mathrm{cup}_R(X^{n-1}) \leq \mathrm{TC}_n(X).$$

But this is not a good lower bound for the computation of $\mathrm{TC}_n(X)$. In this section, we provide an improved lower bound for $\mathrm{TC}_n(X)$.

Let $\Bbbk$ be a field. Let $\cup_n$ denote the n-fold cup product homomorphism

$$\cup_n : H^*(X;\Bbbk) \otimes_{\Bbbk} \ldots \otimes_{\Bbbk} H^*(X;\Bbbk) \to H^*(X;\Bbbk) \tag{4.1}$$

given by $\alpha_1 \otimes \ldots \otimes \alpha_n \mapsto \alpha_1 \cup \ldots \cup \alpha_n$ for $n \geq 2$. The n-fold cross product homomorphism becomes a k-algebra homomorphism if we define multiplication in $H^*(X;\Bbbk) \otimes_{\Bbbk} \ldots \otimes_{\Bbbk} H^*(X;\Bbbk)$ by

$$(\alpha_1 \otimes \ldots \otimes \alpha_n).(\beta_1 \otimes \ldots \otimes \beta_n) = (-1)^{\xi}((\alpha_1 \cup \beta_1) \otimes \ldots \otimes (\alpha_n \cup \beta_n)),$$
$$\xi = |\beta_1|(|\alpha_2| + \ldots + |\alpha_n|) + |\beta_2|(|\alpha_3| + \ldots + |\alpha_n|) + \ldots + |\beta_{n-1}||\alpha_n|,$$

where $|\alpha_i|$ and $|\beta_i|$ denotes the degree of the cohomology class α_i and β_i in $H^*(X;\Bbbk)$ respectively. The n-fold cup product homomorphism $\cup_n$ can be factored as $\cup_n = \times_n \Delta_n^*$

where $\times_n$ denotes the n-fold cross product homomorphism and Δ_n denotes the n-fold diagonal map of the space X (refer to Appendix for more details).

Definition 4.3.1. The kernel of the n-fold cup product homomorphism (4.1), for $n \geq 2$, is called the **ideal of *n*-fold zero divisors** of $H^*(X;\Bbbk)$. It is denoted by $\ker(\cup_n(X))$. The length of the longest non-trivial product in the $\ker(\cup_n(X))$ is called the ***n*-fold zero divisors cup-length** of $H^*(X;\Bbbk)$. It is denoted by $\mathrm{nil}(\ker(\cup_n(X)))$.

Theorem 4.3.2. Let X be a path-connected topological space and $\Bbbk$ be any field. Let $H_i(X)$ denote the i^{th} homology group with coefficients in $\mathbb{Z}$. Suppose that the space X^n is paracompact Hausdorff and $H_i(X)$ is finitely generated for all i. Then

$$\mathrm{nil}(\ker(\cup_n(X))) \leq \mathrm{TC}_n(X)$$

for $n \geq 2$.

Proof. Let $i : X \to PX$ be the map which maps $x \in X$ to the constant path c_x at x. By Lemma 1.1.5, i is a homotopy equivalence, thus i^* is an isomorphism. Since $H_i(X)$ finitely generated for all i, by the Künneth Theorem [10, Chapter 7, Theorem 61.6], the n-fold cross product homomorphism is an $\Bbbk$-algebra isomorphism. Furthermore, $\widetilde{e}_n \circ i = \Delta_n$. Now consider the diagram

$$\begin{array}{ccccc} H^*(X;\Bbbk)\otimes_{\Bbbk}\ldots\otimes_{\Bbbk} H^*(X;\Bbbk) & \xrightarrow{\times_n} & H^*(X^n;\Bbbk) & \xrightarrow{\Delta_n^*} & H^*(X;\Bbbk) \\ & & \downarrow{\scriptstyle \widetilde{e}_n^*} & \nearrow{\scriptstyle i^*} & \\ & & H^*(PX;\Bbbk) & & \end{array}$$

Then $\ker(\cup_n(X)) = \ker(\Delta_n^*) = \ker(\widetilde{e}_n^*)$. Thus $\mathrm{nil}(\ker(\cup_n(X))) = \mathrm{long}(\widetilde{e}_n)$. By Theorem 2.3.9, we have $\mathrm{nil}(\ker(\cup_n(X))) \leq \mathrm{secat}(\widetilde{e}_n) = \mathrm{TC}_n(X)$. ■

4.4 Product Inequality

Theorem 4.4.1. Let X and Y be path-connected topological spaces. Suppose that the spaces $X \times X$ and $Y \times Y$ are paracompact Hausdorff. Then

$$\mathrm{TC}(X \times Y) \leq \mathrm{TC}(X) + \mathrm{TC}(Y).$$

Proof. Let $\mathrm{TC}(X) = n$ and $\mathrm{TC}(Y) = m$. Then there exists an open cover $\{U_i\}_{i=0}^n$ of $X \times X$ with a continuous motion planning $r_i : U_i \to PX$ for each $i \in \{1, \ldots, n\}$. Similarly, there exists an open cover $\{V_j\}_{j=0}^m$ of $Y \times Y$ with a continuous motion planning $s_j : V_j \to PY$ for each $j \in \{1, \ldots, m\}$. Since $X \times X$ and $Y \times Y$ are paracompact

Hausdorff spaces, there exist partition of unity $\{f_i\}_{i=0}^n$ and $\{g_j\}_{j=0}^m$ subordinate to the covers $\{U_i\}_{i=0}^n$ and $\{V_j\}_{j=0}^m$ respectively.

For each pair of nonempty subsets $S \subset \{0, 1, \ldots, n\}$ and $T \subset \{0, 1, \ldots, m\}$ define $W(S,T) \subset (X \times Y) \times (X \times Y)$ by $(A, B, C, D) \in W(S,T)$ if

$$f_i(A,C).g_j(B,D) > f_{i'}(A,C).g_{j'}(B,D)$$

for any $(i, j) \in S \times T$ and for any $(i', j') \notin S \times T$. If $S = \{0, 1, \ldots, n\}$ and $T = \{0, 1, \ldots, m\}$, then define $W(S,T) \subset (X \times Y) \times (X \times Y)$ by

$$W(S,T) = \{(A,B,C,D) \mid f_i(A,C).g_j(B,D) > 0 \text{ for any } (i,j) \in S \times T\}.$$

Define a map $f_i.g_j : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$f_i.g_j(A,B,C,D) = f_i(A,C).g_j(B,D).$$

The map $f_i.g_j$ is the composition of the following maps

$$(X \times Y) \times (X \times Y) \xrightarrow{L} (X \times X) \times (Y \times Y) \xrightarrow{f_i \times g_j} \mathbb{R} \times \mathbb{R} \xrightarrow{h} \mathbb{R}$$

where L is the natural homeomorphism given by $L(A, B, C, D) = (A, C, B, D)$ and h is the multiplication of real numbers. Thus $f_i.g_j$ is continuous and $(f_i.g_j - f_{i'}.g_{j'})^{-1}(0, 1]$ is open.

Since

$$W(S,T) = \bigcap_{\substack{(i,j) \in S \times T \\ (i',j') \notin S \times T}} (f_i.g_j - f_{i'}.g_{j'})^{-1}(0,1]$$

and the intersection is over finitely many elements, it follows that $W(S,T)$ is open.

Let $S' \subset \{0, 1, \ldots, n\}$ and $T' \subset \{0, 1, \ldots, m\}$ such that $W(S,T) \cap W(S',T') \neq \emptyset$. If $S \times T \not\subset S' \times T'$, then there exists $(i_0, j_0) \in (S \times T) \setminus (S' \times T')$. Let $(A, B, C, D) \in W(S,T) \cap W(S',T')$. Then $f_{i_0}(A,C).g_{j_0}(B,D) > f_i(A,C).g_j(B,D)$ for any $(i, j) \notin S \times T$ and $f_k(A,C).g_l(B,D) > f_{i_0}(A,C).g_{j_0}(B,D)$ for any $(k, l) \in S' \times T'$. Thus $f_k(A,C).g_l(B,D) > f_i(A,C).g_j(B,D)$ for any $(k, l) \in S' \times T'$ and for any $(i, j) \notin S \times T$. This implies $S' \times T' \subset S \times T$. Therefore $W(S,T)$ and $W(S',T')$ are disjoint if neither $S \times T \subset S' \times T'$ nor $S' \times T' \subset S \times T$.

Suppose $(A, B, C, D) \in (X \times Y) \times (X \times Y)$. Let

$$S = \{i \in \{0, \ldots, n\} \mid f_i(A,C) = \max\{f_0(A,C), \ldots, f_n(A,C)\}\},$$

and

$$T = \{j \in \{0, \ldots, m\} \mid g_j(B,D) = \max\{g_0(B,D), \ldots, g_m(B,D)\}\}.$$

Then $(A, B, C, D) \in W(S, T)$. Thus the sets $W(S, T)$ forms an open cover of $(X \times Y) \times (X \times Y)$.

Let $(i, j) \in S \times T$. Since $f_i(A, C).g_j(B, D) > 0$ for all $(A, B, C, D) \in W(S, T)$, it follows that $W(S, T) \subset L^{-1}(U_i \times V_j)$. Thus there exist a continuous motion planning over $W(S, T)$ given by the map $\sigma_{i,j} : W(S, T) \to P(X \times Y)$ defined as $\sigma_{i,j}(A, B, C, D) = (r_i(A, C), s_j(B, D))$ for all $(A, B, C, D) \in W(S, T)$.

Let W_k $(k = 2, \ldots, n+m+2)$ denote the union of $W(S, T)$ such that $|S|+|T| = k$. The set $\{W_k\}_{k=2}^{n+m+2}$ forms an open cover of $(X \times Y) \times (X \times Y)$. Suppose $|S| + |T| = |S'| + |T'|$ and $S \times T \neq S' \times T'$. Then $W(S, T)$ and $W(S', T')$ are disjoint since neither $S \times T \subset S' \times T'$ nor $S' \times T' \subset S \times T$. Thus W_k is disjoint union of open sets each having a continuous motion planning. Hence there exist a continuous motion planning over W_k. Thus $\mathrm{TC}(X \times Y) \leq n + m$. ■

4.5 Topological Complexity of Spheres

In this section, we compute the topological complexity of a n-dimensional sphere S^n which will later help us in calculating the topological complexity of motion planning for a robot arm in Section 4.7. The text in this section can be found in [5].

Theorem 4.5.1. The topological complexity of a n-dimensional sphere S^n is given by

$$\mathrm{TC}(S^n) = \begin{cases} 1 & \text{for } n \text{ odd,} \\ 2 & \text{for } n \text{ even.} \end{cases}$$

Proof. Let $u \in H^n(S^n; \Bbbk)$ be the fundamental class and $1 \in H^0(S^n; \Bbbk)$ be the identity element of $H^*(S^n; \Bbbk)$. The elements $u \otimes u$ and $1 \otimes u - u \otimes 1$ belongs to the kernel of the cup product homomorphism (4.1), since their images are $u^2 \in H^{2n}(S^n; \Bbbk) = 0$ and $1.u - u.1 = 0$ respectively. Then

$$\begin{aligned}(1 \otimes u - u \otimes 1)^2 =& (1 \otimes u - u \otimes 1).(1 \otimes u - u \otimes 1) \\ =& (1 \otimes u)^2 - (1 \otimes u).(u \otimes 1) - (u \otimes 1).(1 \otimes u) + (u \otimes 1)^2 \\ =& (-1)^{n^2+1}(u \otimes u) - (u \otimes u).\end{aligned}$$

If n is odd, then $(1 \otimes u - u \otimes 1)^2 = 0$ and if n is even then $(1 \otimes u - u \otimes 1)^2 = -2(u \otimes u)$. Thus if $\Bbbk$ is not a field of characteristic 2, by Theorem 4.3.2, we get

$$\mathrm{TC}(S^n) \geq \mathrm{nil}(\ker(\cup_2(S^n))) \geq \begin{cases} 1 & \text{for } n \text{ odd,} \\ 2 & \text{for } n \text{ even.} \end{cases}$$

Since S^n is a path-connected smooth manifold, it follows from Theorem 4.2.1 that

$$\mathrm{TC}(S^n) \leq 2\ \mathrm{cat}(S^n) = 2.$$

This shows that $\mathrm{TC}(S^n) = 2$ if n is even. If n is odd, we are going to construct a motion planning on a cover of $S^n \times S^n$ containing two open sets implying $\mathrm{TC}(S^n) \leq 1$.

Let $U_1 = \{(x, y) \in S^n \times S^n \mid x \neq -y\}$ and $U_2 = \{(x, y) \in S^n \times S^n \mid x \neq y\}$ be open sets covering $S^n \times S^n$. Define $s_1 : U_1 \to PS^n$ by

$$s_1(x, y)(t) = \frac{(1-t)x + ty}{||(1-t)x + ty||_2}$$

where $||z||_2 = \sqrt{z_1^2 + \ldots + z_{n+1}^2}$ for $z = (z_1, \ldots, z_{n+1})$. It can be seen that $||(1-t)x + ty||_2 = \sqrt{1 - 2t(1-t)(1 - \Sigma x_i y_i)}$ and for $t \in (0, 1)$ we have

$$||(1-t)x + ty||_2 = 0 \iff \Sigma x_i y_i = 1 - \frac{1}{2t(1-t)} \iff \Sigma x_i y_i = -1 \iff x = -y.$$

Thus, by and Theorem 1.1.4, s_1 is continuous with $s_1(x, y)(0) = x$ and $s_1(x, y)(1) = y$. Hence s_1 is a continuous motion planning on U_1.

Since n is odd there exist a continuous non-vanishing tangent vector field v on S^n, for example $(x_1, \ldots, x_{n+1}) \mapsto (-x_2, x_1, -x_4, x_3, \ldots, -x_{n+1}, x_n)$. We can assume that $||v(x)||_2 = 1$ for all $x \in S^n$, since v is non-vanishing. Thus $v : S^n \to S^n$ is a continuous map such that $v(x) \perp x$. Define $\widetilde{v} : S^n \times I \to S^n$ by $\widetilde{v}(x, t) = -\cos(\pi t)x + \sin(\pi t)v(x)$. Define $s_2 : U_2 \to PS^n$ by

$$s_2(x, y)(t) = \begin{cases} s_1(x, -y)(2t) & \text{if } 0 \leq t \leq 1/2, \\ \widetilde{v}(y, 2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

By pasting lemma and Theorem 1.1.4, s_2 is continuous with $s_2(x, y)(0) = x$ and $s_2(x, y)(1) = y$. Hence s_2 is a continuous motion planning on U_2. ■

4.6 Topological Complexity of Compact Orientable Surfaces

Theorem 4.6.1. The topological complexity of a connected compact orientable surface (without boundary) Σ_g of genus g is given by

$$\mathrm{TC}(\Sigma_g) = \begin{cases} 2 & \text{if } g \leq 1, \\ 4 & \text{if } g \geq 2. \end{cases}$$

Proof. If $g = 0$, then Σ_g is the 2-dimensional sphere S^2. It follows from Theorem 4.5.1 that $\mathrm{TC}(S^2) = 2$. If $g = 1$, then Σ_g is the 2-dimensional torus T^2. Since T^2 is homeomorphic to $S^1 \times S^1$, by Theorem 4.1.8 and Theorem 4.4.1, we obtain $\mathrm{TC}(T^2) = \mathrm{TC}(S^1 \times S^1) \leq 2\ \mathrm{TC}(S^1) = 2$. Moreover, by Proposition 4.2.1 and Theorem 3.6.2, we obtain $2 = \mathrm{cat}(T^2) \leq \mathrm{TC}(T^2)$. Thus $\mathrm{TC}(T^2) = 2$.

Let us consider the case $g \geq 2$. Since Σ_g is a path-connected smooth manifold, by Theorem 4.2.3, we have

$$\mathrm{TC}(\Sigma_g) \leq 2\dim(\Sigma_g) = 4 \qquad \text{for all } g.$$

For all $g \geq 2$ we can find non-zero cohomology classes u_1, u_2, v_1 and $v_2 \in H^1(\Sigma_g; \mathbb{Q})$ and a non-zero cohomology class $\Lambda \in H^2(\Sigma_g; \mathbb{Q})$ such that $u_1 \cup v_1 = u_2 \cup v_2 = \Lambda$ and $u_i \cup u_j = v_i \cup v_j = 0$ for all $i, j \in \{1, 2\}$ and $u_i \cup v_j = 0$ if $i \neq j$ (refer to Theorem 3.6.3 for more details).

The elements $1 \otimes u_i - u_i \otimes 1$ and $1 \otimes v_i - v_i \otimes 1$ belongs to the kernel of the cup product homomorphism (4.1), since their images are $1.u_i - u_i.1 = 0$ and $1.v_i - v_i.1 = 0$ respectively.

Consider the product

$$\begin{aligned}\prod_{i=1}^{2}(1 \otimes u_i - u_i \otimes 1)(1 \otimes v_i - v_i \otimes 1) &= \prod_{i=1}^{2}(1 \otimes \Lambda + v_i \otimes u_i - u_i \otimes v_i + \Lambda \otimes 1)\\ &= 2(\Lambda \otimes \Lambda) \neq 0.\end{aligned}$$

Then, by Theorem 4.3.2, $\mathrm{TC}(\Sigma_g) \geq \mathrm{nil}(\ker(\cup_2(\Sigma_g))) \geq 4$ for $g \geq 2$. Thus $\mathrm{TC}(\Sigma_g) = 4$ for $g \geq 2$. ■

4.7 Motion Planning for a Robot Arm

Consider a robot arm consisting of n bars $l_1, l_2, \ldots, l_n$ such that:

(a) the initial point of l_1 is fixed,

(b) l_i and l_{i+1} are connected by a flexible joint for $i \in \{1, 2, \ldots, n-1\}$, and

(c) no obstacles are present, i.e., the bars don't interact with each other.

If the bars are allowed to move in a plane satisfying (a), (b) and (c), then a configuration of the robot arm is determined by n angles $\alpha_1, \alpha_2, \ldots, \alpha_n$, where α_i is the angle that the bar l_i makes with the x-axis. Thus the configuration space of the robot arm with n bars in the planar case is the n-dimensional torus $T^n = S^1 \times \ldots \times S^1$.

Similarly, if the bars are allowed to move in the 3-dimensional space $\mathbb{R}^3$ satisfying (a), (b) and (c), then the configuration space of the robot arm with n bars in the 3-space $\mathbb{R}^3$ is the Cartesian product of n copies of 2-dimensional spheres S^2.

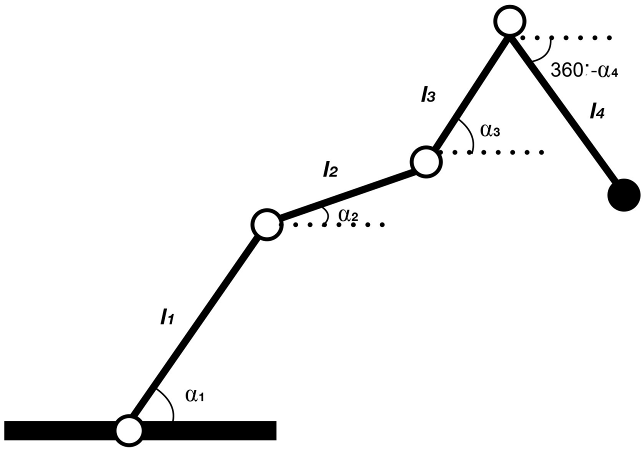


Figure 4.1: A planar robot arm with 4 bars

Theorem 4.7.1. Let $X = S^m \times \ldots \times S^m$ be the Cartesian product of n copies of m-dimensional sphere S^m. Then

$$\mathrm{TC}(X) = \begin{cases} n & \text{if } m \text{ is odd,} \\ 2n & \text{if } m \text{ is even.} \end{cases}$$

Proof. Since S^m is a path-connected smooth manifold, by Theorem 4.4.1 and Theorem 4.5.1, we get

$$\mathrm{TC}(X) \leq n\ \mathrm{TC}(S^m) = \begin{cases} n & \text{if } m \text{ is odd,} \\ 2n & \text{if } m \text{ is even.} \end{cases}$$

Let $\pi_i : X \to S^m$ be the projection map onto the i^{th} factor of X. Suppose $a_i \in H^m(S^m; \mathbb{Q})$ is the fundamental class of the i^{th} factor and let $u_i = \pi_i^*(a_i)$ for each i. Since $a_i^2 = 0$, we have $u_i^2 = 0$. Since $H_j(S^m)$ is finitely generated for all j, by the Künneth Theorem [10, Chapter 7, Theorem 61.6], the n-fold cross product homomorphism

$$H^*(S^m; \mathbb{Q}) \otimes_{\mathbb{Q}} \ldots \otimes_{\mathbb{Q}} H^*(S^m; \mathbb{Q}) \xrightarrow{\times_n} H^*(X; \mathbb{Q})$$

is an isomorphism of algebras. Thus

$$a_1 \otimes \ldots \otimes a_n \mapsto \pi_1^*(a_1) \cup \ldots \cup \pi_n^*(a_n) = u_1 \cup \ldots \cup u_n \neq 0$$

The elements $u_i \otimes u_i$ and $1 \otimes u_i - u_i \otimes 1$ belongs to the kernel of the cup product homomorphism (4.1), since their images are $u_i^2 = 0$ and $1.u_i - u_i.1 = 0$ respectively.

Moreover,

$$\begin{aligned}(1 \otimes u_i - u_i \otimes 1)^2 =&(1 \otimes u_i - u_i \otimes 1).(1 \otimes u_i - u_i \otimes 1)\\ =&(1 \otimes u_i)^2 - (1 \otimes u_i).(u_i \otimes 1) - (u_i \otimes 1).(1 \otimes u_i) + (u_i \otimes 1)^2\\ =&(-1)^{m^2+1}(u_i \otimes u_i) - (u_i \otimes u_i).\end{aligned}$$

Thus $(1 \otimes u_i - u_i \otimes 1)^2 = 0$ if m is odd and $(1 \otimes u_i - u_i \otimes 1)^2 = -2(u_i \otimes u_i)$ if m is even. The product

$$\prod_{i=1}^{n}(1 \otimes u_i - u_i \otimes 1) \neq 0 \in H^*(X;\mathbb{Q}) \otimes_{\mathbb{Q}} H^*(X;\mathbb{Q}),$$

since there is a non-zero term $1 \otimes u_1 \dots u_n$ left in the product obtained by multiplying the first term of $(1 \otimes u_i - u_i \otimes 1)$ for each i. Thus $\mathrm{TC}(X) \geq n$, implying $\mathrm{TC}(X) = n$ if m is odd.

If m is even, then the product

$$\prod_{i=1}^{n}(1 \otimes u_i - u_i \otimes 1)^2 = \prod_{i=1}^{n}(-2)(u_i \otimes u_i) = (-2)^n(u_1 \dots u_n) \otimes (u_1 \dots u_n) \neq 0.$$

Thus $\mathrm{TC}(X) \geq 2n$, implying $\mathrm{TC}(X) = 2n$ if m is even. ■

Corollary 4.7.2. Consider a robot arm consisting of n bars $l_1, l_2, \dots, l_n$ such that the initial point of l_1 is fixed, and l_i and l_{i+1} are connected by a flexible joint for $i \in \{1, 2, \dots, n-1\}$. Then the topological complexity of motion planning of a n bar robot arm in a plane equals n. The topological complexity of motion planning of n-bar robot arm in 3-space $\mathbb{R}^3$ equals $2n$.

4.8 Spaces with $\mathrm{TC}_n(X) = n - 1$

Theorem 4.1.6 classifies spaces with $\mathrm{TC}_n(X) = 0$, i.e., $\mathrm{TC}_n(X) = 0$ if and only if X is contractible. Suppose X is not contractible, then we can readily show that $\mathrm{TC}_n(X) \geq n - 1$ using Proposition 4.2.1 and Theorem 3.3.2. In this section, we classify spaces with $\mathrm{TC}_n(X) = n - 1$ for some $n \geq 2$. The relevant reference is [6].

Let $H_i(X)$ denote the i^{th} homology group with coefficients in $\mathbb{Z}$ and $\widetilde{H}_i(X)$ denote the i^{th} reduced homology group with coefficients in $\mathbb{Z}$. Let $\dim_{\Bbbk}(\widetilde{H}^*(X;\Bbbk))$ denote the dimension of $\widetilde{H}^*(X;\Bbbk)$ as a graded $\Bbbk$-vector space.

Definition 4.8.1. A CW complex is said to be of **finite type** if it has finitely many cells of each dimension.

Proposition 4.8.2. Suppose X is a CW complex of finite type. Then $H_i(X)$ is finitely generated abelian group for all i.

Proof. It is a well known result that if X is a CW complex with k i-cells, then $H_i(X)$ is generated by at most k elements (refer to the consequence (ii) of [7, Chapter 2, Theorem 2.35]). Since X has finitely many cells of each dimension, it follows that $H_i(X)$ is finitely generated abelian group for all i. ■

Definition 4.8.3. A space X is said to be an **integral homology sphere** if X has integral homology groups isomorphic to that of an n-sphere S^n for some $n \geq 1$, i.e.,

$$H_i(X) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n, \\ 0 & \text{else.} \end{cases}$$

Definition 4.8.4. A space X is said to be **acyclic** if the reduced integral homology groups of X vanishes in all dimensions, i.e., $\widetilde{H}_i(X) = 0$ for all $i \geq 0$.

Proposition 4.8.5. Let X be a path-connected CW complex of finite type. Suppose $\dim_{\Bbbk}(\widetilde{H}^*(X;\Bbbk)) \leq 1$ for any field $\Bbbk$. Then either X is acyclic or X is an integral homology sphere.

Proof. If X is acyclic, then $\widetilde{H}_i(X) = 0$ for all $i \geq 0$. For all $i \geq 1$, by the Universal Coefficient Theorem C.7, we have a split exact sequence

$$0 \longrightarrow \operatorname{Ext}(H_{i-1}(X), \Bbbk) \longrightarrow H^i(X;\Bbbk) \longrightarrow \operatorname{Hom}(H_i(X), \Bbbk) \longrightarrow 0$$

for any field $\Bbbk$. Moreover, $\operatorname{Ext}(H_0(X), \Bbbk) \cong \operatorname{Ext}(\mathbb{Z}, \Bbbk) = 0$ and $\operatorname{Ext}(H_i(X), \Bbbk) = \operatorname{Ext}(0, \Bbbk) = 0$ for all $i \geq 1$. Thus $\operatorname{Ext}(H_i(X), \Bbbk) = 0$ for all $i \geq 0$. Therefore, by exactness of the sequence, we get $H^i(X;\Bbbk) \cong \operatorname{Hom}(H_i(X), \Bbbk) = \operatorname{Hom}(0, \Bbbk) = 0$ for all $i \geq 1$. Thus $\widetilde{H}^*(X;\Bbbk) = 0$ for any field $\Bbbk$.

Now suppose that X is not acyclic. Let $H_r(X)$ be the first non-trivial integral homology group of X, $r \geq 1$. If $r = 1$, then $\operatorname{Ext}(H_{r-1}(X), \Bbbk) = \operatorname{Ext}(H_0(X), \Bbbk) \cong \operatorname{Ext}(\mathbb{Z}, \Bbbk) = 0$. If $r > 1$, then $\operatorname{Ext}(H_{r-1}(X), \Bbbk) = \operatorname{Ext}(0, \Bbbk) = 0$. Thus we have $\operatorname{Ext}(H_{r-1}(X), \Bbbk) = 0$ for any field $\Bbbk$. Since X is of finite type, by Proposition 4.8.2, either

$$H_r(X) \cong \mathbb{Z}^n \quad \text{or} \quad H_r(X) \cong \mathbb{Z}^n \oplus \mathbb{Z}/p^k \oplus \mathbb{Z}/p_1^{k_1} \oplus \mathbb{Z}/p_2^{k_2} \oplus \ldots \oplus \mathbb{Z}/p_l^{k_l}$$

for some $n \geq 0$, primes $p \leq p_1 \leq p_2 \leq \ldots \leq p_l$ and natural numbers $k, k_1, k_2, \ldots, k_l$.

Let us suppose that the torsion part of $H_r(X)$ is non-trivial. Then there is atleast one summand, say $\mathbb{Z}_{p^k}$, in the decomposition of $H_r(X)$. Since $\operatorname{Ext}(H_{r-1}(X), \mathbb{Z}_p) = 0$, we have

$$\begin{aligned} H^r(X;\mathbb{Z}_p) &\cong \mathrm{Hom}(H_r(X),\mathbb{Z}_p) \\ &\cong \mathrm{Hom}(\mathbb{Z}_{p^k},\mathbb{Z}_p)\oplus S \\ &\cong \ker(\mathbb{Z}_p \xrightarrow{p^k} \mathbb{Z}_p)\oplus S \qquad \text{by Proposition C.6} \\ &\cong \mathbb{Z}_p \oplus S \end{aligned}$$

where S is a $\mathbb{Z}_p$ vector space. Moreover,

$$\begin{aligned} H^{r+1}(X;\mathbb{Z}_p) &\supseteq \mathrm{Ext}(H_r(X),\mathbb{Z}_p) \\ &\cong \mathrm{Ext}(\mathbb{Z}_{p^k},\mathbb{Z}_p)\oplus T \\ &\cong (\mathbb{Z}_p)/(p^k\mathbb{Z}_p)\oplus T \qquad \text{by Proposition C.5} \\ &\cong \mathbb{Z}_p \oplus T \end{aligned}$$

where T is a $\mathbb{Z}_p$ vector space. Then it follows that $\dim_{\mathbb{Z}_p}(\widetilde{H}^*(X;\mathbb{Z}_p)) \geq 2$ which leads to a contradiction. Thus $H_r(X)$ is torsion-free.

Let us suppose that $H_r(X) \cong \mathbb{Z}^n$ for some $n \geq 2$. Since $\mathrm{Ext}(H_{r-1}(X),\mathbb{Q}) = 0$, we have

$$H^r(X;\mathbb{Q}) \cong \mathrm{Hom}(H_r(X),\mathbb{Q}) \cong \mathrm{Hom}(\mathbb{Z}^n,\mathbb{Q}) \cong \mathbb{Q}^n.$$

Then it follows that $\dim_{\mathbb{Q}}(\widetilde{H}^*(X;\mathbb{Q})) \geq 2$ which leads to a contradiction. Thus $H_r(X) \cong \mathbb{Z}$.

Since $H_r(X) \cong \mathbb{Z}$, we have $\mathrm{Ext}(H_r(X),\Bbbk) \cong \mathrm{Ext}(\mathbb{Z},\Bbbk) = 0$ for all field $\Bbbk$. Thus the same argument as before shows that $H_{r+1}(X)$ is torsion-free. Moreover, if $H_{r+1}(X) \cong \mathbb{Z}^n$ for some $n \geq 1$, then using $H^{r+1}(X;\mathbb{Q}) \cong \mathrm{Hom}(H_{r+1}(X),\mathbb{Q})$, we get $H^{r+1}(X;\mathbb{Q}) \cong Q^n$ for some $n \geq 1$. This implies $\widetilde{H}^*(X;\mathbb{Q})$ has dimension atleast 2 over $\mathbb{Q}$ since $H^r(X;\mathbb{Q}) \cong \mathbb{Q}$. Thus $H_{r+1}(X)$ is trivial.

Thus we get $\mathrm{Ext}(H_{r+1}(X),\Bbbk) \cong \mathrm{Ext}(0,\Bbbk) = 0$ for all field $\Bbbk$. Then, by inductive argument, we get that $H_i(X) = 0$ for all $i > r$. Thus X is an integral homology r-sphere. ■

Proposition 4.8.6. Let X be path-connected CW complex. Suppose the fundamental group of X has a non-trivial element of finite order. Then $\mathrm{cat}(X^n) \geq 2n$ for each $n \geq 1$.

Proof. Since the fundamental group of X has a non-trivial element of finite order, the fundamental group of X has an element of prime order. Thus $\mathbb{Z}_p$ is a subgroup of the fundamental group of X for some prime p. Since CW complexes are locally path-connected and semi-locally path-connected, by [7, Chapter 1, Proposition 1.36], there exists a path-connected covering space Y of X whose fundamental group is isomorphic to $\mathbb{Z}_p$. Thus $H_1(Y) \cong \mathbb{Z}_p$. Moreover, by [7, Exercise 1, p. 529], Y is also a CW complex.

Consider the long exact sequence of cohomology groups of Y associated to the short exact sequence of coefficients $0 \to \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_{p^2} \xrightarrow{r_p} \mathbb{Z}_p \to 0$ (refer to Appendix: Bockstein Homomorphisms for more details)

$$\ldots \longrightarrow H^n(Y;\mathbb{Z}_p) \longrightarrow H^n(Y;\mathbb{Z}_{p^2}) \longrightarrow H^n(Y;\mathbb{Z}_p) \xrightarrow{\beta} H^{n+1}(Y;\mathbb{Z}_p) \longrightarrow \ldots$$

Since $\operatorname{Ext}(H_0(Y), \mathbb{Z}_{p^2}) = \operatorname{Ext}(\mathbb{Z}, \mathbb{Z}_{p^2}) = 0$, by the Universal Coefficient Theorem C.7, we obtain $H^1(Y;\mathbb{Z}_{p^2}) \cong \operatorname{Hom}(H_1(Y), \mathbb{Z}_{p^2}) \cong \operatorname{Hom}(\mathbb{Z}_p, \mathbb{Z}_{p^2}) \cong \ker(\mathbb{Z}_{p^2} \xrightarrow{p} \mathbb{Z}_{p^2}) \cong \mathbb{Z}_p$. Similarly, we obtain $H^1(Y;\mathbb{Z}_p) \cong \mathbb{Z}_p$. Thus

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & H^1(Y;\mathbb{Z}_p) & \longrightarrow & H^1(Y;\mathbb{Z}_{p^2}) & \longrightarrow & H^1(Y;\mathbb{Z}_p) & \xrightarrow{\beta} & H^2(Y;\mathbb{Z}_p) & \longrightarrow \ldots \\
& \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow & \\
\ldots \longrightarrow & \mathbb{Z}_p & \longrightarrow & \mathbb{Z}_p & \longrightarrow & \mathbb{Z}_p & \xrightarrow{\beta} & H^2(Y;\mathbb{Z}_p) & \longrightarrow \ldots
\end{array}$$

Since the map $H^1(Y;\mathbb{Z}_p) \to H^1(Y;\mathbb{Z}_{p^2})$ is non-zero, it is an isomorphism. Thus the map induced by r_p is zero. Thus the Bockstein homomorphism $\beta : H^1(Y;\mathbb{Z}_p) \to H^2(Y;\mathbb{Z}_p)$ is injective.

Let $y = \beta(x) \in H^2(Y;\mathbb{Z})$, where $x \in H^1(Y;\mathbb{Z}_p)$ is a generator. Thus, by [11, Corollary 4.7] and [6, Proposition 2.2], the class y has category weight (refer to Definition 3.5.3) at least 2.

Since $\mathbb{Z}_p$ is a field, by [4, Chapter VII, Exercise 7.15, p. 218], it follows that the cross product homomorphism is injective. Thus $y \times \ldots \times y \in H^*(Y^n;\mathbb{Z}_p)$ is non-zero.

Thus, by Proposition 3.5.4, we have

$$\begin{aligned}
\operatorname{cat}(Y^n) &\geq \operatorname{wgt}(y \times \ldots \times y) \\
&= \operatorname{wgt}(\pi_1^*(y) \cup \ldots \cup \pi_n^*(y)) \\
&\geq \operatorname{wgt}(\pi_1^*(y)) + \ldots + \operatorname{wgt}(\pi_n^*(y)) \\
&\geq \operatorname{wgt}(y) + \ldots + \operatorname{wgt}(y) \\
&= 2n.
\end{aligned}$$

Since Y^n is a path-connected covering space of X^n, by Theorem 3.1.6, we have $\operatorname{cat}(Y^n) \leq \operatorname{cat}(X^n)$. Thus $\operatorname{cat}(X^n) \geq 2n$. ■

Let X be a topological space and $\Bbbk$ be a field. Let $H^*(X;\Bbbk)^{\otimes n}$ denote the n-fold tensor product $H^*(X;\Bbbk)$ over $\Bbbk$. Suppose $a \in H^*(X;\Bbbk)$. Let a_i denote the element $1 \otimes \ldots \otimes 1 \otimes a \otimes 1 \otimes \ldots \otimes 1$ in $H^*(X;\Bbbk)^{\otimes n}$ where a is at the i^{th} position and $|a|$ denote the degree of the cohomology class a in $H^*(X;\Bbbk)$.

Suppose $a, b \in H^*(X; \Bbbk)$. Due to the graded product on $H^*(X; \Bbbk)^{\otimes n}$, we have

$$a_i b_j = \begin{cases} 1 \otimes \ldots \otimes 1 \otimes a \otimes 1 \otimes \ldots \otimes 1 \otimes b \otimes 1 \otimes \ldots \otimes 1 & \text{if } i \leq j \\ (-1)^{|a||b|}(1 \otimes \ldots \otimes 1 \otimes b \otimes 1 \otimes \ldots \otimes 1 \otimes a \otimes 1 \otimes \ldots \otimes 1) & \text{if } i > j. \end{cases}$$

where a and b are at the i^{th} and j^{th} position respectively. Moreover, we have

$$a_i b_j = (-1)^{|a||b|} b_j a_i.$$

The original statement (partially) and proof of the following lemma (Lemma 3.3 in [6]) is incorrect. This was discovered while reviewing the paper [6] during this thesis. In [1], we have done an exposition of the problem in the proof, given a simple modification to the original statement and then given a direct proof of the Lemma 3.3 in [6].

Lemma 4.8.7. Suppose $a, b \in H^*(X; \Bbbk)$. For $n \geq 2$ we have

$$(b_1 - b_2)(a_1 - a_2)(a_1 - a_3) \ldots (a_1 - a_n) \equiv (-1)^{n+1}(b \otimes a + (-1)^{|a||b|} a \otimes b) \otimes a \otimes \ldots \otimes a$$

modulo terms in the ideal of $H^*(X; \Bbbk)^{\otimes n}$ generated by the elements $a^2 \otimes 1 \otimes \ldots \otimes 1$, $ba \otimes 1 \otimes \ldots \otimes 1$, and $1 \otimes ba \otimes 1 \otimes \ldots \otimes 1$.

Proof. A term in the expression $(b_1 - b_2)(a_1 - a_2)(a_1 - a_3) \ldots (a_1 - a_n)$ will be of the form

$$(-1)^{\epsilon}\, b_j a_{i_1} \ldots a_{i_{n-1}} \tag{4.2}$$

where $j \in \{1, 2\}$, $i_k \in \{1, k+1\}$ for $k \in \{1, \ldots, n-1\}$ and $\epsilon \in \mathbb{Z}$.

If a_1 occurs twice in the expression (4.2), then the term itself is a multiple of $a^2 \otimes 1 \otimes \ldots \otimes 1$. Thus, we can assume a_1 occurs atmost once.

Let us assume $b_i = b_1$. If a_1 comes in the expression (4.2), then the term itself is a multiple of $ba \otimes 1 \otimes \ldots \otimes 1$. Thus, we can assume a_1 doesn't occur in the expression and hence the only expression possible is $(-1)^{n-1} b_1 a_2 a_3 \ldots a_n$. Similarly, if we assume $b_i = b_2$ then the only expression possible is $(-1)^{n-1} b_2 a_1 a_3 \ldots a_n$ since a_1 can occur atmost once.

Therefore, we get

$$(b_1 - b_2)(a_1 - a_2)(a_1 - a_3) \ldots (a_1 - a_n) \equiv (-1)^{n-1}(b_1 a_2 a_3 \ldots a_n + b_2 a_1 a_3 \ldots a_n)$$

modulo terms in the ideal of $H^*(X; \Bbbk)^{\otimes n}$ generated by the elements $a^2 \otimes 1 \otimes \ldots \otimes 1$, $ba \otimes 1 \otimes \ldots \otimes 1$, and $1 \otimes ba \otimes 1 \otimes \ldots \otimes 1$. ■

Theorem 4.8.8. Let X be a path-connected CW complex of finite type. Suppose

$\mathrm{TC}_n(X) = n-1$ for some $n \geq 2$. Then the fundamental group of X is torsion-free, and either X is acyclic or is an odd-dimensional integral homology sphere.

Proof. Suppose that the torsion part of the fundamental group of X in non-trivial. Then the fundamental group of X has a non-trivial element of finite order. Thus, by Proposition 4.8.6, we have $\mathrm{cat}(X^{n-1}) \geq 2(n-1)$. Since $\mathrm{cat}(X^{n-1}) \leq \mathrm{TC}_n(X)$ and $n \geq 2$, we get a contradiction. Thus the fundamental group of X is torsion-free.

Suppose that $a, b \in \widetilde{H}^*(X;\Bbbk)$ such that they are linearly independent over $\Bbbk$, i.e., $\dim_{\Bbbk}(\widetilde{H}^*(X;\Bbbk)) \geq 2$. By Lemma 4.8.7,

$$(b_1 - b_2)(a_1 - a_2)(a_1 - a_3)\dots(a_1 - a_n) \equiv (-1)^{n+1}(b \otimes a + (-1)^{|a||b|} a \otimes b) \otimes a \otimes \dots \otimes a$$

modulo terms in the ideal of $H^*(X;\Bbbk)^{\otimes n}$ generated by the elements $a^2 \otimes 1 \otimes \dots \otimes 1$, $ba \otimes 1 \otimes \dots \otimes 1$, and $1 \otimes ba \otimes 1 \otimes \dots \otimes 1$. We claim that the term

$$(-1)^{n+1}(b \otimes a + (-1)^{|a||b|} a \otimes b) \otimes a \otimes \dots \otimes a$$

is non-zero since a and b are linearly independent. Suppose $\{\alpha_i\}_{i \in J}$ be a basis of $\widetilde{H}^*(X;\Bbbk)$ over $\Bbbk$. Let $a = \Sigma A_i \alpha_i$ and $b = \Sigma B_j \alpha_j$ where $A_i, B_j \in \Bbbk$. Then

$$\begin{aligned} b \otimes a + (-1)^{|a||b|} a \otimes b &= \sum B_j A_i (\alpha_j \otimes \alpha_i) + (-1)^{|a||b|} \sum A_i B_j (\alpha_i \otimes \alpha_j) \\ &= \sum (A_j B_i + (-1)^{|a||b|} A_i B_j) \alpha_i \otimes \alpha_j. \end{aligned}$$

Since a and b are non-zero, there exists a $k \in J$ such that at least one of A_k or B_k is non-zero. If $b \otimes a + (-1)^{|a||b|} a \otimes b = 0$, then

$$A_k B_i + (-1)^{|a||b|} A_i B_k = 0$$

for all $i \in J$. Moreover,

$$\begin{aligned} \sum (A_k B_i + (-1)^{|a||b|} A_i B_k)\alpha_i &= A_k \sum B_i \alpha_i + (-1)^{|a||b|} B_k \sum A_i \alpha_i \\ &= A_k b + (-1)^{|a||b|} B_k a. \end{aligned}$$

This implies a and b are linearly dependent, which is a contradiction. Thus $b \otimes a + (-1)^{|a||b|} a \otimes b \neq 0$. Extend this argument to show that

$$(-1)^{n+1}(b \otimes a + (-1)^{|a||b|} a \otimes b) \otimes a \otimes \dots \otimes a$$

is non-zero if a and b are linearly independent. Moreover, $a_1 - a_i$ belongs to the kernel of the n-fold cup product homomorphism (4.1), since its image is $a - a = 0$. Thus the n-fold product $(b_1 - b_2)(a_1 - a_2)(a_1 - a_3)\dots(a_1 - a_n)$ also belongs to the kernel of the n-fold cup product homomorphism (4.1). Since it is non-zero, it follows $n \leq \mathrm{nil}(\ker(\cup_n(X)))$. Thus, by Theorem 4.3.2, $\mathrm{TC}_n(X) \geq n$, which is a contradiction. Thus $\dim_{\Bbbk}(\widetilde{H}^*(X;\Bbbk)) \leq 1$. Then, by Proposition 4.8.5, either X is acyclic or X is an integral homology sphere.

Suppose X is an integral homology sphere of even dimension. Then, by Lemma 4.8.7, for any cohomology class $a \in H^*(X;\Bbbk)$ the n-fold product $(a_1-a_2)(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)$ is congruent to

$$(-1)^{n+1}(a \otimes a + (-1)^{|a||a|} a \otimes a) \otimes a \otimes \ldots \otimes a = (-1)^{n+1} 2(a \otimes \ldots \otimes a)$$

modulo terms in the ideal of $H^*(X;\Bbbk)^{\otimes n}$ generated by the elements $a^2 \otimes 1 \otimes \ldots \otimes 1$ and $1 \otimes a^2 \otimes 1 \otimes \ldots \otimes 1$. Thus if $\Bbbk$ if field of characteristic not equal to 2, say $\mathbb{Q}$, and a is fundamental class of X, then the n-fold product $(a_1-a_2)(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)$ is non-zero. Since the n-fold product $(a_1-a_2)(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)$ belongs to the kernel of the n-fold cup product homomorphism (4.1), it follows that $n \leq \mathrm{nil}(\ker(\cup_n(X)))$. Thus, by Theorem 4.3.2, $\mathrm{TC}_n(X) \geq n$, which is a contradiction. Thus either X is acyclic or X is an odd-dimensional integral homology sphere. ■

Appendix

A Variant of Partition of Unity

Theorem A.1. Let $\{U_i\}_{i\in J}$ be an open cover of a paracompact Hausdorff space X. Then there exists a locally finite open cover $\{V_i\}_{i\in J}$ of X such that $\bar{V}_i \subset U_i$ for each $i \in J$ where $\bar{V}_i$ denotes the closure of V_i in X.

Proof. Refer to [9, Chapter 6, Lemma 41.6]. ■

Theorem A.2. Let $\{U_i\}_{i\in J}$ be an open cover of a paracompact Hausdorff space X. Then there exist a system of continuous real-valued functions $\{f_i\}_{i\in J}$ from X satisfying the conditions: (a) $0 \leq f_i \leq 1$; (b) $\text{support}(f_i) \subset U_i$; (c) at each point $x \in X$ there exists $i \in J$ such that $f_i(x) = 1$.

Proof. Suppose $\{U_i\}_{i\in J}$ is an open cover of a normal space X. Then there exists an locally finite open cover $\{V_i\}_{i\in J}$ of X such that $\bar{V}_i \subset U_i$ for each i. Moreover, there exists an locally finite open cover $\{W_i\}_{i\in J}$ of X such that $\bar{W}_i \subset V_i$ for each i. Since $\bar{W}_i$ and $X \setminus V_i$ are closed disjoint subsets of X, by Urysohn's lemma, there exists a continuous function $f_i : X \to [0, 1]$ such that

$$f_i(X \setminus V_i) = 0 \text{ and } f_i(\bar{W}_i) = 1.$$

Therefore we have $f_i^{-1}(0, 1] \subset V_i$. This implies

$$\bar{W}_i \subset \text{support}(f_i) \subset \bar{V}_i \subset U_i.$$

Moreover, for each $x \in X$ there exists $i \in J$ such that $x \in W_i$, since $\{W_i\}_{i\in J}$ is a cover, and hence $f_i(x) = 1$ for some $i \in J$.

Thus any open cover $\{U_i\}_{i\in J}$ of a paracompact Hausdorff space X has a system of continuous real-valued functions $\{f_i\}_{i\in J}$ from X satisfying the conditions: (a) $0 \leq f_i \leq 1$; (b) $\text{support}(f_i) \subset U_i$; (c) at each point $x \in X$ there exists $i \in J$ such that $f_i(x) = 1$. ■

Cup Product

Let R be a commutative ring with identity. The cohomology $H^*(X;R)$ is a graded ring with multiplication defined by the cup product homomorphism

$$\cup : H^*(X;R) \otimes_R H^*(X;R) \to H^*(X;R)$$

given by $\alpha_1 \otimes \alpha_2 \mapsto \alpha_1 \cup \alpha_2$. The cup product has the following property

$$a \cup b = (-1)^{|a|.|b|}(b \cup a)$$

where $|a|$ and $|b|$ denotes the degree of cohomology classes a and b in $H^*(X;R)$ respectively. Let $\cup_n$ denote the ***n*-fold cup product homomorphism**

$$\cup_n : H^*(X;R) \otimes_R \ldots \otimes_R H^*(X;R) \to H^*(X;R)$$

given by $\alpha_1 \otimes \ldots \otimes \alpha_n \mapsto \alpha_1 \cup \ldots \cup \alpha_n$. The n-fold cross product homomorphism becomes a ring homomorphism if we define multiplication in $H^*(X;R) \otimes_R \ldots \otimes_R H^*(X;R)$ by

$$(\alpha_1 \otimes \ldots \otimes \alpha_n).(\beta_1 \otimes \ldots \otimes \beta_n) = (-1)^{\xi}((\alpha_1 \cup \beta_1) \otimes \ldots \otimes (\alpha_n \cup \beta_n)),$$
$$\xi = |\beta_1|(|\alpha_2| + \ldots + |\alpha_n|) + |\beta_2|(|\alpha_3| + \ldots + |\alpha_n|) + \ldots + |\beta_{n-1}||\alpha_n|,$$

where $|\alpha_i|$ and $|\beta_i|$ denotes the degree of the cohomology class α_i and β_i in $H^*(X;R)$ respectively.

Let $\pi_i : X^n \to X$ be the projection map onto the i^{th} factor. Then the map

$$\times_n : H^*(X;R) \times \ldots \times H^*(X;R) \to H^*(X^n;R)$$

given by $(\alpha_1, \ldots, \alpha_n) \mapsto \alpha_1 \times \ldots \times \alpha_n \coloneqq \pi_1^*(\alpha_1) \cup \ldots \cup \pi_n^*(\alpha_n)$ is called the ***n*-fold cross product map**. Since the cup product is distributive, the n-fold cross product map is multilinear. Thus the n-fold cross product map induces a ring homomorphism

$$\times_n : H^*(X;R) \otimes_R \ldots \otimes_R H^*(X;R) \to H^*(X^n;R)$$

given by $\alpha_1 \otimes \ldots \otimes \alpha_n \mapsto \alpha_1 \times \ldots \times \alpha_n \coloneqq \pi_1^*(\alpha_1) \cup \ldots \cup \pi_n^*(\alpha_n)$. This map is called the ***n*-fold cross product homomorphism**. Let $\Delta_n : X \to X^n$ be the n-fold diagonal map. Then

$$\begin{aligned}
\Delta_n^*(\times_n(\alpha_1 \otimes \ldots \otimes \alpha_n)) = \Delta_n^*(\alpha_1 \times \ldots \times \alpha_n) &= \Delta_n^*(\pi_1^*(\alpha_1) \cup \ldots \cup \pi_n^*(\alpha_n)) \\
&= \Delta_n^*(\pi_1^*(\alpha_1)) \cup \ldots \cup \Delta_n^*(\pi_n^*(\alpha_n)) \\
&= (\pi_1 \Delta_n)^*(\alpha_1) \cup \ldots \cup (\pi_n \Delta_n)^*(\alpha_n)
\end{aligned}$$

$$= (\mathrm{id}_X)^*(\alpha_1) \cup \ldots \cup (\mathrm{id}_X)^*(\alpha_n)$$
$$= \alpha_1 \cup \ldots \cup \alpha_n.$$

Thus the n-fold cup product homomorphism $\cup_n$ can be factored as $\cup_n = \Delta_n^* \times_n$.

Universal Coefficient Theorem

Definition C.1. Let H be an abelian group. Then a **free resolution** of H is an exact sequence

$$\ldots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0$$

where each F_n is a free abelian group.

Lemma C.2. Let F and F' be any two free resolutions of an abelian group H. Suppose G is an abelian group. Then $H^n(F;G)$ is isomorphic to $H^n(F';G)$ for all n.

Proof. Refer to [7, Chapter 3, Lemma 3.1]. ■

Lemma C.3. Let H be an abelian group. Then H has a free resolution of the form

$$0 \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \longrightarrow 0,$$

with $F_i = 0$ for $i \geq 2$.

Proof. Let S be a set of generators for H and F_0 be a free abelian group with basis in one-to-one correspondence with S. Then we have a surjective homomorphism $f_0 : F_0 \to H$ mapping the basis elements to the corresponding generators in S. Moreover, the kernel, say F_1, of the homomorphism f_0 is a free abelian group since the subgroups of free abelian group are free abelian. Thus we have have an exact sequence of the form $0 \to F_1 \xhookrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$ where $f_1 : F_1 \hookrightarrow F_0$ is the inclusion map. ■

Corollary C.4. Let F be any free resolution of an abelian group H. Suppose G is an abelian group. Then $H^n(F;G) = 0$ for all $n \geq 2$.

Proof. It follows from Lemma C.2 and Lemma C.3. ■

Thus the only interesting group left is $H^1(F;G)$. Since it is independent of F and depends only on H and G, let us denote this group by $\mathrm{Ext}(H,G)$.

Proposition C.5. The functor Ext satisfies the following properties:

(a) $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H,G) \oplus \mathrm{Ext}(H',G)$.

(b) $\mathrm{Ext}(H, G) = 0$ if H is free.

(c) $\mathrm{Ext}(\mathbb{Z}_n, G) \cong G/nG$.

where H, H' and G are abelian groups.

Proof. (a) If F and F' are free resolutions of H and H' respectively, then it follows that $F \oplus F'$ is a free resolution of $H \oplus H'$. Since $\mathrm{Hom}(F \oplus F', G) \cong \mathrm{Hom}(F, G) \oplus \mathrm{Hom}(F', G)$, it follows that $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$.

(b) If H is free, then $0 \to H \to H \to 0$ is a free resolution of H. Thus it follows that $\mathrm{Ext}(H, G) = 0$.

(c) Consider the free resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\;n\;} \mathbb{Z} \longrightarrow \mathbb{Z}_n \longrightarrow 0$$

of $\mathbb{Z}_n$. Since the $\mathrm{Hom}(-, G)$ functor is left exact, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}_n, G) \longrightarrow \mathrm{Hom}(\mathbb{Z}, G) \xrightarrow{n} \mathrm{Hom}(\mathbb{Z}, G).$$

Thus, by definition of $\mathrm{Ext}(\mathbb{Z}_n, G)$,

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{Hom}(\mathbb{Z}_n, G) & \longrightarrow & \mathrm{Hom}(\mathbb{Z}, G) & \xrightarrow{n} & \mathrm{Hom}(\mathbb{Z}, G) & \longrightarrow & \mathrm{Ext}(\mathbb{Z}_n, G) & \longrightarrow & 0 \\
 & & & & \downarrow \wr & & \downarrow \wr & & \downarrow & & \\
 & & & & G & \xrightarrow{\quad n \quad} & G & \longrightarrow & \mathrm{Ext}(\mathbb{Z}_n, G) & \longrightarrow & 0
\end{array}$$

is exact. Therefore $\mathrm{Ext}(\mathbb{Z}_n, G) \cong G/nG$. ∎

Proposition C.6. Let G be an abelian group. Then $\mathrm{Hom}(\mathbb{Z}_n, G) \cong \ker(G \xrightarrow{n} G)$.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\;n\;} \mathbb{Z} \longrightarrow \mathbb{Z}_n \longrightarrow 0$$

Since the $\mathrm{Hom}(-, G)$ functor is left exact, we get an exact sequence

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}(\mathbb{Z}_n, G) & \longrightarrow & \mathrm{Hom}(\mathbb{Z}, G) & \xrightarrow{n} & \mathrm{Hom}(\mathbb{Z}, G) \\
 & \downarrow & & \downarrow \wr & & \downarrow \wr \\
0 \longrightarrow & \mathrm{Hom}(\mathbb{Z}_n, G) & \longrightarrow & G & \xrightarrow{\quad n \quad} & G
\end{array}$$

Thus $\mathrm{Hom}(\mathbb{Z}_n, G) \cong \ker(G \xrightarrow{n} G)$. ∎

Proposition C.5 helps to us compute the $\mathrm{Ext}(H, G)$ when H is a finitely generated abelian group. Moreover, if H is a finitely generated abelian group, then $\mathrm{Ext}(H, \mathbb{Z})$ is torsion part of H whereas $\mathrm{Hom}(H, \mathbb{Z})$ is the free part of H.

Theorem C.7. (Universal Coefficient Theorem for Cohomology) Let C be a chain complex of free abelian groups with integral homology groups $H_n(C)$. Suppose G is an abelian group. Then the cohomology groups $H^n(C;G)$ of the cochain complex $\mathrm{Hom}(C,G)$ are determined by the split exact sequences

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C),G) \longrightarrow H^n(C;G) \longrightarrow \mathrm{Hom}(H_n(C),G) \longrightarrow 0$$

for all $n \geq 1$.

Proof. Refer to [7, Chapter 3, Theorem 3.2]. ■

Bockstein Homomorphisms

Let $C_n(X)$ denote the free abelian group with basis elements as the set of singular n-simplices in X. Suppose

$$0 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 0$$

is an exact sequence of abelian groups. Applying the covariant functor $\mathrm{Hom}(C_n(X),-)$, we get

$$0 \longrightarrow \mathrm{Hom}(C_n(X),G) \longrightarrow \mathrm{Hom}(C_n(X),H) \longrightarrow \mathrm{Hom}(C_n(X),K) \longrightarrow 0.$$

Since $C_n(X)$ is free, the sequence above is exact. Thus, by zig-zag lemma, we have a long exact sequence

$$\ldots \longrightarrow H^n(X;G) \longrightarrow H^n(X;H) \longrightarrow H^n(X;K) \xrightarrow{\beta} H^{n+1}(X;G) \longrightarrow \ldots$$

of cohomology groups whose boundary maps $H^n(X;K) \xrightarrow{\beta} H^{n+1}(X;G)$ are called the **Bockstein homomorphisms**.

We are mainly interested in the Bockstein homomorphism associated to the short exact sequence of coefficients

$$0 \to \mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_{m^2} \xrightarrow{r_m} \mathbb{Z}_m \to 0.$$

where r_m denotes the reduction mod m.

Consider the exact sequences $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{\rho} \mathbb{Z}_m \to 0$ and $0 \to \mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_{m^2} \xrightarrow{r_m}$

$\mathbb{Z}_m \to 0$. We have a chain map between them

$$\begin{CD}
0 @>>> \mathbb{Z} @>m>> \mathbb{Z} @>\rho>> \mathbb{Z}_m @>>> 0\\
@. @VV\rho V @VV\varphi V @VV\mathrm{id}_{\mathbb{Z}_m}V @.\\
0 @>>> \mathbb{Z}_m @>m>> \mathbb{Z}_{m^2} @>r_m>> \mathbb{Z}_m @>>> 0
\end{CD}$$

where ρ and φ are the natural quotient maps. This induces a commutative diagram

$$\begin{CD}
\dots @>>> H^n(X;\mathbb{Z}) @>>> H^n(X;\mathbb{Z}) @>>> H^n(X;\mathbb{Z}_m) @>\beta>> H^{n+1}(X;\mathbb{Z}) @>>> \dots\\
@. @VVV @VVV @VV\mathrm{id}^*_{\mathbb{Z}_m}V @VV\rho^*V @.\\
\dots @>>> H^n(X;\mathbb{Z}_m) @>>> H^n(X;\mathbb{Z}_{m^2}) @>>> H^n(X;\mathbb{Z}_m) @>\widetilde{\beta}>> H^{n+1}(X;\mathbb{Z}_m) @>>> \dots
\end{CD}$$

Thus we have $\rho^* \circ \beta = \widetilde{\beta} \circ \mathrm{id}^*_{\mathbb{Z}_m}$, i.e., $\rho^* \circ \beta = \widetilde{\beta}$.

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