Groups acting on the circle

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Certificate of Examination

This is to certify that the dissertation titled **"Groups acting on the circle"** submitted by **Mr. Anjani Gupta** (Reg. No. MS14184) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 26, 2019

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Krishnendu Gongopadhyay at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

Anjani Gupta
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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Krishnendu Gongopadhyay
(Supervisor)

Acknowledgement

I have received a lot of academic and emotional support from various people around me all through the year of my thesis. First of all, I would like to thank Dr. Krishnendu Gongopadhyay, my thesis guide, who personally selected the right paper for me to explore and practiced enough flexibility to even allow me a change of topic, sensing my interest. His proofs and reference suggestions were invaluable to me. I would also like to specifically acknowledge the freedom that he gave to me regarding the pace of the thesis and the digressions that interested me.

I would also like to thank Mr. Manpreet Singh for the academic dialogues that I had with him all the way. And since the emotional support is tantamount to the academic support, I would like to thank my family (Mr. Om Prakash Gupta, Ms. Ranjana Gupta and Ms. Anamika) and friends who gave me the requisite mindset to ruminate on this thesis.

As a starting note for this thesis, I would like to declare that none of the results in here are original work, though some proof techniques may be new. Being an exposition of the survey article by Dr. Etienne Ghys, the source of majority material ahead is his paper and some other lecture notes, as detailed in the bibliography section exhaustively.

Notation

$Im(z)$	Imaginary part of $z \in \mathbb{C}$
$\{x_\alpha\}_{\alpha \in I}$	Sequence indexed by the set I.
Id	Identity map $f(x) = x$
X^c	Complement of the set X.
$supp(f)$	Support of f i.e $\overline{\{x : f(x) \neq x\}}$
$Homeo_+(S^1)$	Group of orientation preserving homeomorphisms of S^1
$\widetilde{Homeo_+}(S^1)$	Group of homeomorphisms of $\mathbb{R}$ with $f(x+1) = f(x) + 1$
$A \sqcup B$	Disjoint union of A and B.
WLOG	Without loss of generality.

Abstract

This thesis aims to study the groups acting on the circle, their properties and the dynamics under their action. The first chapter starts with a basic introduction to the definitions, theorems and some proofs that will come in handy while going through the subsequent chapters. Then some explicit examples of groups acting on the circle are given in the second chapter, to lay the foundation for the more general groups. The third chapter discusses the group of all orientation-preserving homeomorphisms of the circle, which is a big group considering the fact that most of the groups acting continuously on the circle are a subgroup of this group (excluding orientation-reversing groups). In the last chapter, rotation numbers are introduced, which give useful information about the dynamics of the one-generator groups acting on the circle. Finally, the appendix has some general propositions that were used in the proofs, but would have been a digression had they been in the first chapter.

Contents

Chapter 1

Basics

This chapter aims to cover most of the general definitions and theorems that will find context in the later chapters. A few of the lemmas that have very less role in this topic, and rather just serve as a step in the proofs are omitted from here and will find their due role in the appendix.

Definition 1.0.1. A group with a topology on it such that the following conditions are satisfied is called **topological group** :

- $\phi : G \times G \to G$ given by $\phi((g, h)) = gh$ is continuous with respect to product topology.
- $\lambda : G \to G$ given by $\lambda(g) = g^{-1}$ is continuous.

Definition 1.0.2. Let G be a topological group and let X be a topological space. Then an action of G on X is called **continuous action** if the action map $\phi : G \times X \to X$ is continuous.

This is equivalent to having a homomorphism $\lambda : G \to Homeo(X)$ as restriction continuous maps is continuous. Unless stated otherwise, actions would mean continuous actions henceforth.

Definition 1.0.3. Let ϕ_1 and ϕ_2 be two actions of G on X_1 and X_2 respectively. Then these actions are said to be **conjugate** if there exists a homeomorphism $h : X_1 \to X_2$ such that for every $g \in G$, $\phi_2(g) = h\phi_1 h^{-1}$.

A space X is called *homogeneous* under the action of group G if there is only one orbit (i.e the action is transitive).

Proposition 1.0.4. Connected topological groups are generated by any neighborhood of identity.

Proof. Let G be connected and U be a neighborhood of identity. Then assume that $U = U^{-1}$ (if necessary, just set $U \cap U^{-1}$). Let $< U >= S = \{g_1, g_2, ..., g_n \mid g_i \in U; n \in \mathbb{N}\}$.

1. $S \neq \phi$ as $e \in G$.
2. S is open as for any $g \in S$, gU is open (translation is a homeomorphism in topological groups) in S. Ranging g over G, it covers G and hence S is open.
3. S is closed. For any $g \in S^c$, gU is disjoint from S and so S^c is open. Let them have non-empty intersection. Then, $gu.u^{-1} = g \in S$, which is a contradiction (u^{-1} is in the set as we took $U = U^{-1}$).

Therefore, if the group has to be connected, S will have to be the whole group. Or equivalently, any neighborhood of identity generated the whole group. ■

The concept of the orientation of manifolds and orientation preserving maps has general formulations, but it reduces to an easier yet informative format on 1-dimensional manifolds. And in the absence of diffeomorphisms (we are considering only homeomorphisms), the following definition helps.

Definition 1.0.5. Let $p : \mathbb{R} \longrightarrow S^1$ be the fundamental covering map $t \mapsto e^{2\pi it}$. Then f **preserves orientation** if there exists some increasing homeomorphism $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $p \circ g = f \circ p$, i.e. if f can be lifted to an increasing homeomorphism of $\mathbb{R}$.

The motivation for just two types - orientation preserving and orientation reversing, comes from the fact that any homeomorphism of the circle (which is in particular bijective) has to be (strictly) monotonic. Thus it can either be increasing or decreasing. And then we just name the increasing one orientation preserving since if we move monotonically towards ∞ (real line is ordered) on the real line in the domain, the image by increasing homeomorphism too moves in the same direction (order). While for decreasing case, the order in the image gets reversed.

Proposition 1.0.6. $Homeo_+(S^1)$ is a topological group with respect to compact open topology (equivalent to topology of uniform convergence on compact sets).

Proof. Composition is the group operation that we first need to show to be continuous.

$$\circ : Homeo_+(S^1) \times Homeo_+(S^1) \to Homeo_+(S^1)$$

$$(f, g) \mapsto f \circ g$$

Let $S(C, V)$ be open in $Homeo_+(S^1)$. Then $\circ^{-1}(S(C, V)) = \{f, g \in Homeo_+(S^1) \mid f \circ g(C) \subseteq V\}$. Then $f(g(C)) \subset V \implies f \in S(g(C), V)$ and $f(g(C)) \subseteq V \implies g(C) \subseteq f^{-1}(V) \implies g \in S(C, f^{-1}(V))$.

Declare $S(g(C), V) \times S(C, f^{-1}(V)) \subset Homeo_+(S^1) \times Homeo_+(S^1)$. If $h_2, h_1 \in S(g(C), V) \times S(C, f^{-1}(V))$; $h_2(g(C)) \subseteq V$ and $h_1(C) \subseteq f^{-1}(V)$. Then $h_2 \circ h_1(C) \subseteq V$.

Inversion : Let $h \in S(C,U)$.

$$\begin{aligned} h \in S(C,U) \Leftrightarrow & h(C) \subset U \\ \Leftrightarrow & X \setminus U \subseteq X \setminus h(C) \\ \Leftrightarrow & X \setminus U \subseteq h(X \setminus C) \\ \Leftrightarrow & h^{-1}(X \setminus U) \subseteq X \setminus C \\ \Leftrightarrow & h^{-1} \in S(X \setminus U, X \setminus C) \end{aligned}$$

Define the inversion as -

$$inv : Homeo_+(S^1) \to Homeo_+(S^1)$$

$$f \mapsto f^{-1}$$

Let $h \in S(C,U)$. Then, $inv^{-1}(S(C,U)) = inv(S(C,U)) = \{f \in Homeo_+(S^1) \mid f^{-1} \in S(C,U)\} = S(X \setminus U, X \setminus C)$. ■

Proposition 1.0.7. $Homeo_+(S^1)$ is connected.

Proof. First we know that path connected implies connected. Let $f_0, f_1 \in Homeo_+(S^1)$. Lift f_0 to F_0 and f_1 to F_1. Let $p : \mathbb{R} \to S^1$ be given by $t \mapsto^{2\pi it}$.

Define $F_t : tF_0 + (1-t)F_1$, homotopy of paths in $\mathbb{R}$. Then $G_t : p \circ F_t : \mathbb{R} \to S^1$ and $G_t(x+1) = G_t(x) \implies \frac{\mathbb{R}}{p} = S^1 \to S^1$. This shows that $Homeo_+(S^1)$ is path connected and hence connected. ■

Theorem 1.0.8. Let M be a topological manifold and N a smooth manifold. Then N can be given a smooth structure using pullback if there exists a homeomorphism $f : M \to N$.

Proof. Let M and N be as in the statement. As f is a homeomorphism, $f^{-1}(U)$ is open in M for any open set $U \subseteq N$. Let $\{U_\alpha, \varphi_\alpha\}_{\alpha \in I}$ be the smooth structure on N indexed by some set I. Because N is a manifold, around every $x \in N$, pick U_x in its chart and $\varphi_x : U_x \to \mathbb{R}^n$ a homeomorphism. Now $f^{-1}(U_x)$ is open in M and their collection over I would cover M. In addition, $\varphi_x \circ f : f^{-1}(U_x) \to \mathbb{R}^n$ will serve as local homeomorphisms from M. The transition maps,

$$\begin{aligned} & (\varphi_V \circ f) \circ (\varphi_U \circ f)^{-1} : \mathbb{R}^n \to \mathbb{R}^n \\ = & \varphi_V \circ f \circ f^{-1} \circ \varphi_U^{-1} \because \varphi_U \text{ is a homeomorphism} \\ = & \varphi_V \circ \varphi_U^{-1} \end{aligned}$$

are already C^∞. This shows that $\{f^{-1}(U_\alpha), \varphi_\alpha \circ f\}_\alpha$ is a smooth structure on M, pulled back from N via f. ■

Lemma 1.0.9. Let x be a point in S^1. Then iterated rotation of x by an irrational number θ gives an infinite orbit.

Proof. Fix a point $0 \in S^1 = \mathbb{R}/\mathbb{Z}$ and consider its iterated rotation R_θ by an irrational number θ. Assume that after some n iterations, the points get periodic, i.e $R_\theta^n(0) = 0$. Then $n\theta = 1$, which implies $\theta = 1/n$ is rational. A contradiction as θ was chosen to be irrational. Hence, this orbit is infinite. ■

Proposition 1.0.10. The orbit of iterated rotation by an irrational number on the circle is dense in the circle.

Proof. Let the irrational number be θ as before. If we assume that the orbit is not dense, then there exists an open interval I of size say ϵ, in the complement of the orbit.

Let there exist $m, n \in \mathbb{N}$ such that the interval between $m\theta$ and $n\theta$ is of size $d < \epsilon$. Then fixing a starting point $0 \in S^1$, the rotation by $(m-n)\theta$ will give a point at a distance $d < \epsilon$ from 0. Hence on repeated rotation, at-least one point of the orbit will fall in I. But as I was assumed to be having no points of the orbit, there exist no m and n. Hence, the distances between the points of the orbit are at-least ϵ apart. This means that the order of orbit set is at-most $1/\epsilon$ which is finite. But the irrationality of θ will then contradict the previous lemma. Therefore there are no intervals in the complement of the orbit set, and so the orbit is dense in S^1. ■

Theorem 1.0.11. Let G be a locally compact topological group and let its sigma algebra consist of Borel sets. Then there exists a measure λ (called Haar measure) unique upto a multiplicative constant, satisfying the following properties -

1. $\lambda(gS) = g\lambda(S)$ for all g in G and for all Borel sets S.

2. $\lambda(K) < \infty$ for all compact sets K of G.

3. λ is both outer ($\lambda(S) = \inf\{\lambda(U) : S \subseteq U \text{ for open } U\}$) and inner regular ($\lambda(U) = \sup\{\lambda(K) : K \subseteq U \text{ for compact } K\}$) on all the Borel sets S and open sets U.

Also when G is compact, $\lambda(G) < \infty$. Therefore, for compact groups like circle, the Haar measure can be multiplicatively scaled to give a probability measure.

Definition 1.0.12. Let (X, Σ, μ) be a measure space. Then $A \in \Sigma$ with $\mu(A) > 0$ is called an **atom** if for all $B \in \Sigma$ such that $B \subset A$, $\mu(B) = 0$.

Definition 1.0.13. A measure μ with no atoms is called **non-atomic** measure.

Lemma 1.0.14. The Lebesgue measure (λ) on $\mathbb{R}^n$ is non-atomic.

Proof. Start with any measurable set A. Let B_r be the box $\{x \in \mathbb{R}^n : |x_k| \leq r \text{ for } k = 1, \ldots, n\}$. Then $\lambda(B_r \cap A)$ turns out to be a continuous function of r in the following way.

Let $r < s$ from $\mathbb{R}$. Then $B_r \cap A \leq B_s \cap A \implies |\lambda(B_s \cap A) - \lambda(B_r \cap A)| = \lambda((B_s \cap A) \setminus (B_r \cap A)) = \lambda((B_s \setminus B_r) \cap A) \leq \lambda(B_s \setminus B_r) = \lambda(B_s) - \lambda(B_r) = (2s)^n - (2r)^n$. Now this implies that for any arbitrary $\epsilon > 0$ in $\mathbb{R}$, we can choose $\delta (= |r_n - r|)$ around r such that $\lambda(B_{r_n}) - \lambda(B_r) < 2^{r_n} - 2^r < \epsilon$. Hence, continuity follows. Now, had A been an atom, $\lambda(B_r \cap A)$ would have taken only the values 0 and $\lambda(A)$ as r progressed to ∞. Whereas continuity implies it will take all the values between 0 and $\lambda(A) \neq 0$, which implies A is not an atom. From the arbitrary choice of A, we now know that λ is non-atomic. ∎

Remark : Continuous function from a set with Borel σ-algebra is measurable. It's so because if we take any open set in the range, by continuity, its inverse image will be open in the domain. But since the domain is based on Borel σ-algebra, this open set is measurable here.

Theorem 1.0.15. Dominated Convergence Theorem : Let (X, Σ, μ) be a measure space and let $f_n : X \to \mathbb{C}$ be a sequence of measurable functions. If f_n is pointwise convergent to an integrable function f such that $f_n(x) \leq f(x)$ for all $x \in X$ and n in the index, then $\lim f_n \to f$ and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

Chapter 2

Some groups acting on the circle

2.1 Different models for the circle

The classification of 1-dimensional manifolds depends only on the number of connected components, whether or not these components are compact, and the cardinality of the boundary points set. The connected 1-dimensional manifolds are classified into the following four, up to homeomorphism :

- $[0, 1]$ - compact with two boundary points
- $[0, 1)$ - non-compact with 1 boundary point
- $(0, 1)$ - non-compact with no boundary points
- S^1 - compact with no boundary points

Since S^1 is the only compact connected 1 manifold, it turns up often! Topologically, it can be constructed in different ways that we build here, which will help us get some 'natural' actions on each of these different constructions.

1. Consider $\mathbb{R}^2$ with the usual topology coming from the Euclidean metric. Then the points satisfying $x^2+y^2 = r^2$ for any fixed $r \in \mathbb{R}$ gives us a circle of radius r. With respect to the **subspace topology that it inherits from** $\mathbb{R}^2$, it can be turned into a smooth manifold by defining stereographic charts. The same can be done by defining $\varphi : \mathbb{R} \to S^1 \subset \mathbb{C}$, where $S^1 = \{z \in \mathbb{C} \text{ such that } |z| = 1)\}$. Here $\varphi(x) = e^{2\pi i x}$.

2. Consider $\mathbb{R}$ with usual topology and quotient it under the identification $x \sim y$ if $(x - y)\ mod\ \mathbb{Z} = 0\ mod\ \mathbb{Z}$. Then $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1. To see the homeomorphism, use the map φ from (1), which will be a homeomorphism had it been injective. Since φ is periodic at an interval of 1, considering both these as groups and φ as a homomorphism, the kernel of φ turns out to be $\mathbb{Z}$. From

first isomorphism theorem, $\mathbb{R}/\mathbb{Z}$ is homomorphic to S^1, which also makes φ a homeomorphism, if quotient by kernel is seen as quotient by subspace. This way, circle can be seen as an abelian group as well. Using Theorem 1.0.8, we get the smooth stucture of $\mathbb{R}/\mathbb{Z}$ as well. Advantage of this model over others is due to the fact that it comes from as a quotient of 1-manifold $\mathbb{R}$ abstractly rather than inheriting topology from $\mathbb{R}^2$

3. Circle as $\mathbb{R}P^1$ (**real projective space** is the collection of all lines passing through the origin). Consider $\mathbb{R}^2 \setminus \{0\}$ with subspace topology from $\mathbb{R}^2$. Then,

$$f : \mathbb{R}^2 \setminus \{0\} \to \frac{\mathbb{R}^2 \setminus \{0\}}{x \sim \lambda x} = \mathbb{R}P^1$$

is a quotient map that can now be restricted to $S^1 \subset \mathbb{R}^2 \setminus \{0\}$. Once this map $f|_{S^1} : S^1 \to \mathbb{R}P^1$ is shown to be injective (which can be done by factoring using relation $x \sim -x$), we get a homeomorphism from S^1. Thus we get $f|_{S^1} : \frac{S^1}{x\sim -x} \to \mathbb{R}P^1$ as the homeomorphism and its well known that $\frac{S^1}{x\sim -x} \cong S^1$ using identification map $g(z) = z^2$ (from universal property after the observation that $g(z) = g(-z)$).

$$\begin{tikzcd} S^1 \arrow[r, "\pi"] \arrow[dr, "z^2"'] & \frac{S^1}{z\sim -z} \arrow[d, "\varphi"] \\ & S^1 \end{tikzcd}$$

Smooth structure : We will denote the line passing through (x, y) as $[x : y]$, which means $(x, y) \sim (x', y')$ if $(x, y) = \lambda(x', y')$ for some $\lambda(\neq 0) \in \mathbb{R}$. The points where $y \neq 0$ give the set $[x/y : 1]$ called the set of **ordinary points** whereas $[1, 0]$ is called the **parallel point**.
Define the following charts on $\mathbb{R}P^1$:

$$U_1 = [x : y] \text{ with } y \neq 0 \text{ and } \varphi_1([x : y]) = x/y := u$$

$$U_2 = [x : y] \text{ with } x \neq 0 \text{ and } \varphi_1([x : y]) = y/x := u'$$

The transition functions here is $u' = 1/u$ which is smooth. Then $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ is an atlas of $\mathbb{R}P^1$. To obtain a diffeomorphism from S^1, define

$$F : S^1 \to \mathbb{R}P^1$$

$$F((x, y)) = \begin{cases} [1 - y : x] = [1 : \frac{x}{1-y} & y \neq 1 \\ [0 : 1] & y = 1 \end{cases}$$

Following are the transition maps, that are clearly smooth :

$$\begin{aligned}
\varphi_1 F \varphi_1^{-1} :& u \mapsto u \\
\varphi_2 F \varphi_1^{-1} :& u \mapsto 1/u \\
\varphi_1 F \varphi_2^{-1} :& u' \mapsto 1/u' \\
\varphi_2 F \varphi_2^{-1} :& u' \mapsto u'
\end{aligned}$$

All these models are homeomorphic (diffeomorphic actually), and hence some transition functions between these models would help us -

$$\begin{aligned}
& t \in \mathbb{R}/\mathbb{Z} \mapsto (\cos(2\pi t), \sin(2\pi t)) \in S^1 \subset \mathbb{R}^2 \\
& t \in \mathbb{R}/\mathbb{Z} \mapsto \tan(\pi t) \in \mathbb{R} \cup \{\infty\} = \mathbb{R}P^1 \\
& t \in \mathbb{R}P^1 \mapsto (\frac{1-s^2}{1+s^2}, \frac{2s}{1+s^2}) \in S^1
\end{aligned}$$

2.2 Projective group

The multiplicative group of invertible matrices over real numbers $GL(2,\mathbb{R})$ acts linearly and hence it gives us a natural action on the lines passing through the origin or $\mathbb{R}P^1$. But observe that $A \in GL(2,\mathbb{R})$ and any scalar multiple λA act in the same manner on $\mathbb{R}P^1$. Therefore, to remove this redundancy, we factor the group by scalar matrices so that $PGL(2,\mathbb{R}) = \frac{GL(2,\mathbb{R})}{\{\lambda I\}}$ acts on $\mathbb{R}P^1$ which is homeomorphic to S^1. The explicit action is,

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, [x:1] \right) \mapsto \frac{ax+b}{cx+d}$$

for ordinary points, while for the parallel point there are two cases - a/c if $c \neq 0$ and ∞ if $c = 0$. Here, the square matrices denote the equivalence classes, i.e elements of $PGL(2,\mathbb{R})$.

This action can also be extended to the disc. Consider $\mathbb{C}P^1$, the complex projective space, which can be identified with the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Then $\mathbb{R}P^1$ sits inside it naturally. Similarly $PGL(2,\mathbb{R})$ is a subgroup of $PGL(2,\mathbb{C})$ which acts on $\mathbb{C} \cup \{\infty\}$ by Moebius transformations. From this, we obtain an action of $PGL(2,\mathbb{R})$ on $\mathbb{C}P^1$ that preserves the circle $\mathbb{R}P^1$.
The complement of the circle ($\mathbb{R}P^1$) in the sphere will have two discs (open hemispheres) which are either preserved of permuted by the elements of $PGL(2,\mathbb{R})$ depending on the sign of their determinants, as the following calculation shows : Let

$T \in PGL(2, \mathbb{R})$ and denote $w = T(z) = \dfrac{az+b}{cz+d}$.

$$\begin{aligned} w &= \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} \\ &= \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2} \end{aligned}$$

Then,

$$\begin{aligned} Im(w) &= \frac{w-\bar{w}}{2\iota} = \frac{(ad-bc)(z-\bar{z})}{2\iota|cz+d|^2} \\ &= \frac{Im(z)(ad-bc)}{|cz+d|^2} \\ &= (\det T)\frac{Im(z)}{|cz+d|^2} \end{aligned}$$

One disc is on the positive side of the sphere where $Im(z)$ will be positive (upper half plane), and it is clearly seen that positive determinant will preserve an element there, where negative determinant will send it to the lower half plane. Denote this upper half plane by $\mathbb{H}$ which is one of the connected components in the complement of $\mathbb{R}P^1$. Thus the set of positive determinant elements of $PGL(2, \mathbb{R})$, namely $PSL(2, \mathbb{R})$ acts on $\mathbb{H}$, extending the action on the boundary of $\mathbb{R}P^1$. This extension is isometric on $\mathbb{H}$ with the Poincaré metric.

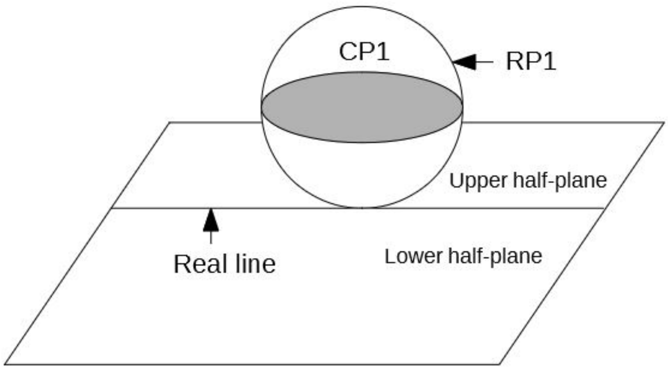

Figure 2.1: Extension of the action of $PGL(2, \mathbb{R})$ to disc.

2.3 Piecewise linear groups

In this chapter, we will use the $\mathbb{R}/\mathbb{Z}$ model of the circle extensively, unless stated otherwise.

Definition 2.3.1. A homeomorphism f of $\mathbb{R}$ is called piecewise linear if there is a sequence of real numbers $\{x_i\}_{i\in\mathbb{Z}}$ such that $\lim_{\pm\infty} x_i = \pm\infty$ and such that the restriction of f to each interval $[x_i, x_{i+1}]$ coincides with an affine map.

If in addition these piecewise linear homeomorphisms satisfy $f(x+1) = f(x)+1$, they induce an orientation preserving homeomorphism of the circle. Their collection forms a group, denoted by $PL_+(S^1)$ and the topology given to them is subspace topology from $Homeo_+(S^1)$.

1. Closure : Let $f, g \in PL_+(S^1)$ be based on partitions $\{x_i\}_{i\in\mathbb{Z}}$ and $\{y_i\}_{i\in\mathbb{Z}}$ respectively. On composing them (group operation), the new partition becomes $\{x_i\}_{i\in\mathbb{Z}} \cup \{y_i\}_{i\in\mathbb{Z}}$, after reordering them using natural order of $\mathbb{R}$. The new affine maps on each (new) interval are their compositions. i.e $f(x) = mx + c$ and $g(x) = m'x + c'$ give $f \circ g(x) = (mm')x + (mc' + c)$.

2. Identity map of $Homeo_+(S^1)$ is an affine map on single partition.

3. Associativity follows from the fact that these are a subset of $Homeo_+(\mathbb{R})$

4. Inverse : Let $f(x) = mx+c$ be the affine map on any arbitrary interval $[x_i, x_{i+1}]$. Define a new map on this interval as $g(x) = (1/m)x + (-c/m)$. Doing so for each partition will give a piecewise linear homeomorphism which will act as the inverse of f.

Thompson group, a countable subgroup of $PL_+(S^1)$ deserves special mention because it was the first discovery of an *infinite finitely presented simple group.* Finite presentation of Thompson's group $F : < a, b \mid [ab^{-1}, a^{-1}ba] = [ab^{-1}, a^{-2}ba^2] = e_F >$. To define it, first consider the group of piecewise linear homeomorphisms f of $\mathbb{R}$ have the following properties :

- The sequence x_i can be chosen in such a way that x_i and $f(x_i)$ consist of dyadic rational numbers (i.e of the form $p2^q$, $p, q \in \mathbb{Z}$).
- The set of dyadic rational numbers is preserved by f.
- The derivatives of the restrictions of f to (x_i, x_{i+1}) are integer powers of 2.
- $f(x+1) = f(x)$ for all x.

These functions induce homeomorphisms on the circle, the collection of which is called $Thompson's$ group. The following diagram shows an example.

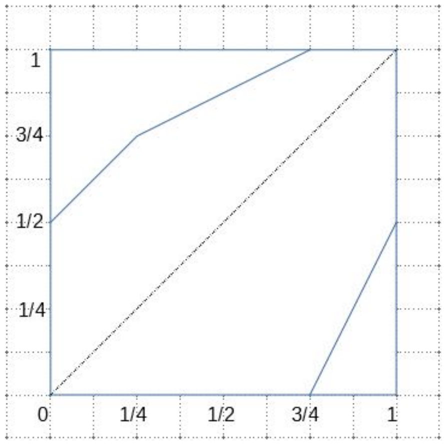

Figure 2.2: An example of an element of Thompson's group.

So, till now we have discussed these two major groups and their actions on the circle. There are many more, but we will now directly pick up on a relatively 'bigger' group $Homeo_+(S^1)$ in the next chapter, that will be built out of homeomorphisms of the real line (and we know them!).

Chapter 3

Homeomorphism group

This chapter aims to discuss the properties of the group $Homeo_+(S^1)$ as any group action on S^1 gives a copy of the group sitting inside $Homeo_+(S^1)$. Hence, information about the group structure would help us know about the possible actions on it.

Let $\widetilde{Homeo_+}(S^1)$ be the group of homeomorphisms of $\mathbb{R}$ which satisfy $\tilde{f}(x+1) = \tilde{f}(x)+1$ for all x. Then each such function defines a homeomorphism from $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ which is orientation preserving by definition (any function such that $\tilde{f}(x+1) = \tilde{f}(x)+1$ is increasing). In the diagram below, π is the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\tilde{f}"] \arrow[d, "\pi"] & \mathbb{R} \arrow[d, "\pi"] \\
S^1 = \mathbb{R}/\mathbb{Z} \arrow[r, "f"] & S^1 = \mathbb{R}/\mathbb{Z}
\end{tikzcd}$$

Thus we get a homomorphism $p : \widetilde{Homeo_+}(S^1) \to Homeo_+(S^1)$, the kernel of which would have all the integral translations of $\mathbb{R}$. Also, using the fact that any orientation preserving homeomorphism of the circle lifts to a homeomorphism of its universal covering space ($\mathbb{R}$) commuting with integral translations, p is seen to be onto. This gives us an exact sequence :

$$0 \to \mathbb{Z} \to \widetilde{Homeo_+}(S^1) \to Homeo_+(S^1) \to 1$$

.

This group turns into a topological group using the topology of uniform convergence, which coincides with the compact-open topology on compact sets (and S^1 is compact).

Though this group is not a Lie group, it shares many properties with finite-dimensional Lie groups. Here, we will attempt to show that $Homeo_+(S^1)$ is sort of an infinite dimensional analog of $PSL(2, R)$ which is a finite dimensional Lie group.

Every Lie group has a unique maximal compact subgroup K up to conjugacy, and the embedding of K in the Lie group is a homotopy equivalence. For $PSL(2, \mathbb{R})$, the maximal compact subgroup is contractible. We show here that such properties are seen in $Homeo_+(S^1)$ as well.

Proposition 3.0.1. Up to conjugacy, the rotation group $SO(2, \mathbb{R})$ is the only maximal compact subgroup of $Homeo_+(S^1)$.

Proof. Begin with a compact subgroup K in $Homeo_+(S^1)$. Then there exists a Haar measure λ' on it using theorem from chapter 1. Also, since S^1 is compact, $\lambda(S^1) < \infty$ and hence by scaling it, a probability measure λ can be defined on it. Now each element k of K sends the Lebesgue measure L on the circle to a probability measure k_*L on the circle as explained below.

Lebesgue measure on the circle : Let S be a measurable set in $S^1 = \mathbb{R}/\mathbb{Z}$. Then take its preimage in $\pi^{-1}(S) \in \mathbb{R}$. Define Lebesgue measure of S as Lebesgue measure (on $\mathbb{R}$) of $\pi^{-1}(S) \cap [0, 1]$.

Now for any $k \in K$, define k_*L (new measure on S^1) as $k_*L(S) = L(k(S))$. Then average over K to get a probability measure μ on S^1 as follows:

$$\mu(S) = \int_K k_*L(S)d\lambda = \int_K L(k(S))d\lambda$$

To see that μ is a probability measure, observe that

$$\mu(S^1) = \int_K k_*L(S^1)d\lambda = \int_K L(k(S^1))d\lambda = \int_k 1d\lambda = 1$$

The last equality follows from the fact that λ is a probability measure. Naturally μ is K invariant as it is averaged over all $k \in K$.

<u>Claim</u> : μ is non-atomic.
<u>Proof</u> : Let A be a measurable set in S^1 such that $\mu(A) > 0$. Then surely A is not countable (in particular singleton), otherwise Lebesgue measure of homeomorphic copy of countable sets will be zero (independent of the homeomorphism) and give μ-measure as well equal to zero. Therefore we can remove a singleton $\{x\}$ from A such that the set $A \setminus \{x\}$ is still measurable (singletons are Borel sets) and $L(k(A \setminus \{x\})) = L(k(A))$ since countable sets are of Lebesgue measure zero. To comprehend this (Lebesgue) integration, fix the set A, and let $\varphi_A(k) := k_*L(A)$. Therefore

we want to integrate the function φ_A as shown below over K :

$$\varphi_A : Homeo_+(S^1) \to \mathbb{R}_{\geq 0}$$

$$k \mapsto k_* L(A) = \varphi_A(k)$$

$$\int_K \varphi_A(k) d\lambda$$

If this integral is greater than zero for A, then we have a non-zero λ-measure subset of K, say S over which $\varphi_A(k)$ is non-zero. Consequently, if we replace A by $A \setminus \{x\}$, then from the definition of measure, $\mu(A \setminus \{x\}) \leq \mu(A)$ because $\varphi_{A\setminus\{x\}}(k) \leq \varphi_A(k)$.

$$\begin{aligned} k(A \setminus \{x\}) \leq& k(A) \\ \implies L(k(A \setminus \{x\})) \leq& L(k(A)) \\ \implies \int_S L(k(A \setminus \{x\}))d\lambda \leq& \int_S L(k(A))d\lambda \\ \implies \int_K L(k(A \setminus \{x\}))d\lambda \leq& \int_K L(k(A))d\lambda \\ \implies 0 \neq \mu(A \setminus \{x\}) \leq& \mu(A) \end{aligned}$$

So, its integral over S is also non-zero and because φ_A is non-negative, the integral over K is also non-zero. So, we have found a proper subset of the arbitrarily chosen set A with non-zero measure, which proves that μ is non-atomic.

<u>Claim</u> : μ is non-zero on non-empty open sets.
<u>Proof</u> : Let A be any non-empty open set of S^1. Then it will be an interval on S^1. Considering that k is a homeomorphism of S^1, $k(A)$ will have non-zero Lebesgue measure for all $k \in Homeo_+(S^1)$. Therefore,

$$\mu(A) = \int_{k \in K} L(k(A))d\lambda \neq 0$$

.

From the above two claims, we can construct an orientation preserving homeomorphism h of S^1. Fix a point x_0 on S^1 and then define $h(x)$ as the unique real point on S^1 such that $\mu([x_0, x]) = L([x_0, h(x)])$. Here any set $[x, y]$ is the interval in the anti-clockwise direction. To check that h is well defined, let there be two images $h(x) \neq h(x)'$ of x. Then $\mu([x_0, x]) = L([x_0, h(x)]) = L([x_0, h(x)'])$. But Lebesgue measure of $[x_0, h(x)]$ and $[x_0, h(x)']$ can't be equal. Such an h exists because $\mu([x_0, x]) \leq 1$ and hence we can find an interval of that same measure since the Lebesgue measure measure on S^1 is also bounded by 1, taking all the values in between.

Also h is injective because for any given $x, y \in S^1$ such that $h(x) = h(y)$, if $x < y$ WLOG, then $[x_0, y] = [x_0, x] \sqcup [x, y]$ and

$$\mu([x_0, x]) = L([x_0, h(x)]) = L([x_0, h(y)]) = \mu([x_0, y])$$

Which gives, $\mu([x_0, x]) = \mu([x_0, y])$

$$\implies \int L(k([x_0, x]))d\lambda = \int L(k([x_0, y]))d\lambda$$

$$\implies \int L(k([x_0, x]))d\lambda = \int L(k([x_0, x]) \sqcup k((x, y]))d\lambda = \int L(k([x_0, y]))d\lambda$$

$$\implies \int L(k([x_0, x]))d\lambda + \int L(k((x, y]))d\lambda = \int L(k([x_0, y]))d\lambda$$

$$\implies \int L(k([]x, y]))d\lambda = 0$$

$$\implies x = y$$

Note : $h(x_0) = x_0$

Claim : h is continuous.

Proof : Let $x_n \to x$. Before we prove this claim, we need to first know that the limit towards ∞ of the integral over K of $\varphi_n : Homeo_+(S^1) \to \mathbb{R}_{\geq 0}$, defined by $\varphi_n(k) = L(k([x_0, x_n]))$ (for a fixed x_0) is the integral of the limit function. But this follows from Lebesgue's Dominated Convergence Theorem (1.0.15) as φ_n converges to φ_x pointwise and is dominated by $L(k(S^1)) = 1$. Hence $\mu([x_0, x_n]) \to \mu([x_0, x])$

Let $x_n \to x$ be a convergent sequence in S^1. Then $h(x_n)$ by definition means the unique point such that $\mu([x_0, x_n]) = L([x_0, h(x_n)])$. Applying $\lim n \to \infty$ to both the sides, (equality is used modulo $\mathbb{Z}$)

$$LHS = \mu([x_0, x_n]) = h(x_n) - x_0$$

$$\implies h(x_n) = x_0 + \mu([x_0, x_n])$$

$$\implies \lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} (x_0 + \mu([x_0, x_n]))$$

$$\implies \lim_{n\to\infty} h(x_n) = x_0 + \mu([x_0, x]) \text{ as discussed in first paragraph.}$$

$$\implies L([x_0, h(x)]) = \lim_{n\to\infty} h(x_n) - x_0$$

$$\implies h(x) - x_0 = \lim_{n\to\infty} h(x_n) - x_0$$

$$\implies h(x_n) = h(x)$$

Claim : h is onto.

Proof : Let $y \in S^1$. Then we need to find an x such that $L([x_0, y]) = \mu([x_0, x])$. But this is possible only if μ takes all the values between 0 and 1 for the sets of type $[x_0, x]$ (x_0 fixed, $x \in S^1$). So pick a point $x \in S^1$ such that $L([x_0, x]) \neq 0$ (any

$x \neq x_0$). Hence, for any $0 \leq c \leq 1$, we need to find y such that $L([x_0, y]) = c$. Let $L([x_0, x]) = c' < c$ WLOG. Then for every increment of the set $[x_0, x]$ to $[x_0, x']$ ($x' > x$), the μ-measure increases because $L(k([x_0, x'])) = L(k([x_0, x] \sqcup (x, x'])) = L(k([x_0, x]) \sqcup k((x, x'])) = L(k([x_0, x])) + L(k([x, x'])) > c' \neq c$. So for all these incremental sets, $\int_K L(k([x_0, x'])) > \int_K L(k([x_0, x]))$ and hence we can monotonically reach c.

Now the inverse, $f(x) = h^{-1}(x)$ will be defined as first finding $L([x_0, x])$ and then picking that unique number such that $L([x_0, x]) = \mu([x_0, f(x)])$ (this can be done as h is shown to be onto). The continuity of the inverse follows in a manner similar to the proof for continuity of h. And the orientation preserving part comes from the fact that $\mu(X) \leq \mu(Y)$ for any $x \subseteq Y$ (for any measure μ). Hence $h \in Homeo_+(S^1)$. Observe that hkh^{-1} preserves the Lebesgue measure for all $k \in K$, because $L(hkh^{-1}(S)) = \mu(kh^{-1}(S)) = \mu(h^{-1}(S)) = L(S)$. The first equality used the definition of h, the second used k-invariance of μ and the third used the definition of h^{-1}. Therefore, K after conjugating by h gives a groups of rotations (these are the only homeomorphisms that preserve Lebesgue measures of all the measurable sets), i.e $SO(2, \mathbb{R})$. Hence, $SO(2, \mathbb{R})$ contains some conjugate of K. Since it is known that $SO(2, \mathbb{R})$ is a compact subgroup of $Homeo_+(S^1)$, it turns out that some conjugate of K is equal to $SO(2, \mathbb{R})$ and not just contained in it. ■

Note : Finite subgroups of $SO(2, \mathbb{R})$ are cyclic groups of rotation from Appendix A.1. Also, we know that any finite subgroup of $Homeo_+(S^1)$ (hence compact) is conjugate to a subgroup of $SO(2, \mathbb{R})$. So, finite subgroups of $SO(2, \mathbb{R})$ are conjugate to cyclic groups of rotations (hence, themselves cyclic).

Proposition 3.0.2. The embedding of $SO(2, \mathbb{R})$ in $Homeo_+(S^1)$ is a homotopy equivalence.

Proof. Observe first that the groups of orientation preserving homeomorphisms of $\mathbb{R}$ is a convex set. Straight line homotopy can be used to show this as then the intermediate maps will as well be orientation preserving homeomorphisms (being strictly increasing till infinity). The explicit map would be $(1-t)\tilde{f}(x) + t\tilde{g}(x)$. Now an element of $Homeo_+(S^1)$ can be written as $\tilde{f}(x) = x + t(x)$, where t is a $\mathbb{Z}$ periodic function. Let the average of t be denoted be $< t > := \int_0^1 t(x)dx$. Then the average of $t_0 := t(x) - < t >$ will be zero and $t(x) = < t > + t_0$ uniquely. Let $0 \leq s \leq 1$ be the parameter now and define a set of functions $\tilde{f}_s$ by $\tilde{f}_s(x) = x + c_s + (1-s)t_0(s)$, where c_s will be the average of the periodic part of each intermediate function. Then $\tilde{f}_0 = \tilde{f}$ and $\tilde{f}_1$ is a translation by $x_0 + c_1$. From the first observation, each of these functions is in $Homeo_+(S^1)$. This gives a retraction of $Homeo_+(S^1)$ on the group of translations of $\mathbb{R}$, which is isomorphic to $\mathbb{R}$. Also $< t(x) + 1 = c_0 + 1 >$ implies that the retraction commutes with integral translations. So, a retraction can be defined from $Homeo_+(S^1) = \widetilde{Homeo}_+(S^1)/\mathbb{Z}$ onto the group of rotations of $\mathbb{R}/\mathbb{Z} = SO(2, \mathbb{R})$. Hence, the homotopy is established. $\blacksquare$

Theorem 3.0.3. The group $Homeo_+(S^1)$ is simple.

Proof. A group G is said to be perfect if $[G, G] = G$. To work out this proof, the following result will be used that will not be proved here - $Homeo_+(S^1)$ is a perfect group.

Let support of a homeomorphism be defined as the closure of the set of points which are not fixed, i.e $supp(f) = \overline{\{x : f(x) \neq x\}}$. Let N be a non-trivial normal subgroup of $Homeo_+(S^1)$ and pick an element f in $Homeo_+(S^1)$ such that $supp(f) \subseteq I$, for some compact interval I in S^1. Pick a non-trivial element $n_0 \in N$.

<u>Claim</u> : $Homeo_+(S^1)$ acts transitively on the set of closed intervals of S^1.
<u>Proof</u> : Let $p : \mathbb{R} \to S^1, p(t) = e^{it}$, be the standard covering. An interval in S^1 is the image $I = p([a, b])$ of an interval $[a, b] \subset \mathbb{R}$ such that $0 < b - a < 2\pi$. The restriction $p_I : [a, b] \to I$ of p is a homeomorphism. Moreover we have $S^1 = I \cup I'$ with $I' = p([b, a + 2\pi])$. Note that $I \cap I' = \{e^{ia}, e^{ib}\}$.

Consider two intervals $I_k = e([a_k, b_k])$. Define "linear" homeomorphisms

$$u : [a_1, b_1] \to [a_2, b_2], u(t) = a_2 + \frac{b_2 - a_2}{b_1 - a_1}(t - a_1)$$

$$u' : [b_1, a_1 + 2\pi] \to [b_2, a_2 + 2\pi], u'(t) = b_2 + \frac{a_2 + 2\pi - b_2}{a_1 + 2\pi - b_1}(t - b_1)$$

These induce orientation preserving homeomorphisms $U : I_1 \to I_2$ and $U' : I_1' \to I_2'$. Now U and U' can be pasted together to an orientation preserving homeomorphisms $h : S^1 = I_1 \cup I_1' \to I_2 \cup I_2' = S^1$. By construction $h(I_1) = I_2$.

<u>Claim</u> : A closed interval C can be chosen in S^1 such that $n_0(C) \cap C = \phi$.
<u>Proof</u> : This is split into the following five cases -

1. Choose $a \in S^1$ such that $f(a) \neq a$ which can always be done as $n_0 \neq Id$ implies at least two points can be found that are not fixed. Then take $b \in [a, f(a)]$ which implies $f(b) > f(a)$ (orientation preserving). It follows that $n_0([a, b]) \cap [a, b] = [f(a), f(b)] \cap [a, b] = \phi$.

2. $f(b) = a$: Since $a \neq b$ by choice, take any closed interval in $[a, b]$, say $[a', b']$. Then $n_0([a', b']) \subseteq [f(a), f(b)]$.

3. $f(b) > a$ but $f(b) < b$: Choose any closed interval $[c, d]$ between $f(b)$ and b (anticlockwise) such that $c \neq f(b)$ and $d \neq b$.

4. $f(b) = b$: Since at least two points are not fixed, this can be avoided.

5. $f(b) > b$ but $f(b) < a$: Choose any point c' such that $f(a) < c' < a$. Then $f^{-1}(c') := c \in [a, b]$. So choose $[a, c]$ going to $[f(a), c'(< a)]$.

Now that such a choice of C is possible, we will conjugate n_0 by some element $k \in Homeo_+(S^1)$ such that $k(I) = C$ (follows from transitivity in the first claim). So now our claim is that $k^{-1}n_0k(I) \cap I = \phi$. Let $i \in I$. Then $k^{-1}n_0k(i) = k^{-1}n_0(k(i))$ implies $n_0(k(i)) \in C^c$ since $k(i) \in C$ and $n_0(C) \cap C = \phi$. So $k^{-1}(n_0(k(i)) \in I^c$ using the fact that $k(I) = C \implies k^{-1}(C^c) = I^c$. Hence, $k^{-1}n_0k(I) \cap I = \phi$. Denote $k^{-1}n_0k$ as $n \in N$ since N is normal and conjugation will leave it within. Which gives us an $n \in N$ such that $n(I) \cap I = \phi$.

Consider the commutator $g = n^{-1}f^{-1}nf$ now. Normality implies $g \in N$. We then have the following claims :

1. $g|_I = f|_I$. Let $i \in I$.

$$\begin{aligned} g(i) =& n^{-1}f^{-1}nf(i) \\ =& n^{-1}f^{-1}n(f(i)) \text{ ... Observe } f(i) \in I \\ =& n^{-1}f^{-1}(n(f(i))) \in I^c \implies n(f(i)) \text{is fixed by } f \text{ and hence } f^{-1} \\ =& n^{-1}(n(f(i))) \\ =& f(i) \text{ for all } i \end{aligned}$$

2. $g|_{n^{-1}(I)} = n^{-1}f^{-1}n|_{n^{-1}(I)}$. Let $i \in I$.

$$\begin{aligned} \text{LHS} = g(i) =& n^{-1}f^{-1}nf(n^{-1}(i)) \text{ ...} n^{-1}(I^c) \cap I^c = \phi \\ =& n^{-1}f^{-1}n(n^{-1}(i)) \text{ ... as } f \text{ is } Id \text{ on } I^c = supp(f)^c \\ =& n^{-1}f^{-1}(i) \\ =& n^{-1}f^{-1}n(n^{-1}(i)) = \text{ RHS} \end{aligned}$$

3. $g|_{(I \cup n^{-1}(I))^c} = Id_{(I \cup n^{-1}(I))^c}$. Let $x \in (I \cup n^{-1}(I))^c = I^c \cap (n^{-1}(I))^c$.

$$\begin{aligned} g(x) =& n^{-1}f^{-1}nf(x) \\ =& n^{-1}f^{-1}(n(x)) \\ =& n^{-1}n(x) \\ =& x \end{aligned}$$

<u>Claim</u> : Let $f_1, f_2 \in Homeo_+(S^1)$ such that $supp(f_1) \subset I$ and $supp(f_2) \subset I$. Then we

can find $n_1, n_2 \in N$ from above such thta $n_1^{-1}(I), n_2^{-1}(I)$ and I are disjoint.
Proof :

1. $n_0(C) \cap C = \phi$ since such a C exist as seen above and $k : S^1 \to S^1$ such that $k(I) = C$. Then $n_1 := k^{-1}n_0k(I) \cap I = \phi$.

2. Let $I' = I \cup seg(A) \cup n_1(I)$, where $seg(A)$ is any interval outside $I \cup n_1(I)$. Then there exists $k' : S^1 \to S^1$ such that $k'(I') = C$. So, define $n_2 := k'^{-1}n_0k'$. This means $n_2(I') \cap I' = \phi$ and hence $n_2(I') \subset I'^c$. Restricting to subsets $n_1(I)$ of I' and I of I', $n_2(I) \cap I = \phi$. Hence, $I, n_1(I)$ and $n_2(I)$ are disjoint, which implies $I, n_1^{-1}(I)$ and $n_2^{-1}(I)$ are disjoint.

Claim : Corresponding $g_1 = n_1^{-1}f_1^{-1}n_1f_1$ and $g_2 = n_2^{-1}f_2^{-1}n_2f_2$ are then in N and their commutator is equal to $[f_1, f_2]$.
Proof : Observe that $g_i|_I = f_i|_I$ implies $g_1^{-1}g_2^{-1}g_1g_2|_I = f_1^{-1}f_2^{-1}f_1f_2|_I = [f_1, f_2]$ on I. Now $[f_1, f_2] = Id$ on I^c as both their supports are in I.
On $I \cup n_1^{-1}(I) \cup n_2^{-1}(I))^c = K$:
Let $x \in K$. Then $g_1^{-1}g_2^{-1}g_1g_2 = g_1^{-1}g_2^{-1}g_1(x) = g_1^{-1}g_2^{-1}(x) = g_1^{-1}(x) = x$, as g agrees wwith Id outside.
On $n_2^{-1}(I)$, we have to show that $[g_1, g_2] = Id$ i.e, $g_1^{-1}g_2^{-1}g_1g_2(n_2^{-1}(i)) = n_2^{-1}(i)$ for all $i \in I$. We know that $g_2|_{n_2^{-1}(I)} = n_2^{-1}f_2^{-1}n_2|_{n_2^{-1}(I)}$. Then, $g_2(n_2^{-1}(i)) = n_2^{-1}f_2^{-1}n_2(n_2^{-1}(i)) = n_2^{-1}f_2^{-1}(i)$. Also we have that $f_2^{-1}(i) \in I \implies n_2^{-1}(f_2^{-1}(i)) \in n_2^{-1}(I)$. Using this, $g_1(n_2^{-1}f_2^{-1}(i)) = Id$ since $n_2^{-1}(I) \cap n_1^{-1}(I) = \phi$.

Hence, $g_1^{-1}g_2^{-1}(n_2^{-1}(f_2^{-1}(i))) \in n_2^{-1}(I) \implies g_2^{-1}|_{n_2^{-1}(I)} = n_2^{-1}f_2^{-1}n_2|_{n_2^{-1}(I)}$ Therefore,

$$\begin{aligned}
&g_1^{-1}((n_2^{-1}f_2^{-1}n_2)^{-1}(n_2^{-1}f_2^{-1}(i))) \\
=&g_1^{-1}(n_2^{-1}f_2n_2(n_2^{-1}f_2^{-1}(i))) \\
=&g_1^{-1}(n_2^{-1}(i)) \\
=&n_2^{-1}(i)
\end{aligned}$$

Similarly for $n_1^{-1}(i)$. So, we finally get that for $g_1, g_2 \in Homeo_+(S^1)$, $[g_1, g_2] = [f_1, f_2] \in N$ whenever $supp(f_1)$ and $supp(f_2)$ are contained in the same interval I.

Let I_1, I_2, I_3 cover the circle such that $T_i \cap I_j \neq \phi$ for $i, j = 1, 2, 3$ but $I_1 \cap I_2 \cap I_3 = \phi$. The claim is that for any interval I, $G = \{f \in Homeo_+(S^1) | supp(f) \subset I\}$.
Proof :

1. $Id \in G$

2. $supp(f) \subset I \implies supp(f^{-1}) \subset I$ ($\because$ f is bijective)

3. Let $f, g \in G$. Choose $x \in I^c$. Then, $f \circ g(x) = f(g(x)) = f(x) = x$. Therefore $supp(f \circ g) \subset I$.

Let G_1, G_2 and G_3 be subgroups of $Homeo_+(S^1)$ with supports in I_1, I_2 and I_3 respectively, and let $G :=< G_1 \cup G_2 \cup G_3 >$ be the subgroup generated by them.

Claim : If a group G is generated by its subset S, i.e, $G =< S >$, then $[G, G] =< g^{-1}[S, S]g \mid g \in G >$.
Proof : Let $N =< g^{-1}[S, S]g \mid g \in G >$ which is the smallest normal subgroup containing $[S, S]$. THen G/N is abelian as the generators commute. Hence, $[G, G] \subset N$ and also $N \subset [G, G]$. so, $N = G$.

So, $[G, G] =< g^{-1}[G_1 \cup G_2 \cup G_3, G_1 \cup G_2 \cup G_3]g \mid g \in G >$. Union of any two of I_i is not the whole circle, otherwise the endpoints will come in the triple intersection. Therefore $I_i \cup I_j$ is contained in some compact interval. And now using the previous arguments, commutator of elements of G is in N.

If we prove $G = Homeo_+(S^1)$, then our assumed proof that $Homeo_+(S^1) = [Homeo_+(S^1), Homeo_+(S^1)]$ implies $[Homeo_+(S^1), Homeo_+(S^1)] = Homeo_+(S^1) \subset N$. And because the other inequality is true by definition of N, $N = Homeo_+(S^1)$, which means $Homeo_+(S^1)$ is a simple group.

Claim : $G = Homeo_+(S^1)$
Proof :Let $x_{1,2}, x_{2,3}$ and $x_{1,3}$ be in the interior of $I_1 \cap I_2, I_2 \cap I_3$ and $I_1 \cap I_3$ respectively. Let $f \in Homeo_+(S^1)$ be close to the identity so that $f(x_{1,2}), f(x_{2,3})$ and $f(x_{1,3})$ are in interiors of $I_1 \cap I_2, I_2 \cap I_3$ and $I_1 \cap I_3$ respectively. Then we can find commuting (since triple intersection is empty) elements $g_1, g_2, g_3 \in Homeo_+(S^1)$ with supports in $I_1 \cap I_2, I_2 \cap I_3$ and $I_1 \cap I_3$ respectively such that they agree with f on neighborhoods of $x_{1,2}, x_{2,3}$ and $x_{1,3}$. Thus $g_1^{-1} g_2^{-1} g_3^{-1} f = Id$ on neighborhoods of $x_{1,2}, x_{2,3}$ and $x_{1,3}$, which gives us that f is $g_3 g_2 g_1$ on these neighborhoods. Now f is product of g_1, g_2, g_3 implies its a product of elements from G. Hence f is in G. And any connected topological group is generated by any neighborhood of identity, so $G = Homeo_+(S^1)$. ■

To conclude, in this chapter we discussed making of the group $Homeo_+(S^1)$ using homeomorphisms of $\mathbb{R}$ that commute with integral translations, made it into a topological group and concentrated on its relevance. We also discussed how, even though $Homeo_+(S^1)$ is not a Lie group, it behaves quite so, sharing properties with the finite dimensional Lie group $PSL(2, \mathbb{R})$. Existence of a maximal compact subgroup and its homotopy equivalence as an embedding into the space are the properties that have been show to match with those of Lie groups. Also, the simplicity of $Homeo_+(S^1)$

was proved, that makes it more like an analog of simple Lie groups.

Chapter 4

Rotation numbers

4.1 Dynamics of a single homeomorphism

The main invariant for circle homeomorphisms was introduced by Poincare as rotation numbers. Let $\tilde{f}$ be an element of $\widetilde{Homeo_+}(S^1)$, which is the group of homeomorphisms of R that commute with integral translations. Because of the injectivity of $\tilde{f}$, $|\tilde{f}(x) - \tilde{f}(y)| \leq 1$ if $|x - y| \leq 1$.

Lemma 4.1.1. For any two points in $\mathbb{R}$ such that $|x-y| \leq 1$, $|(f(x)-x)-(f(y)-y)| \leq 1$.

Proof. Observe that proving this for $x, y \in [0, 1]$ will suffice since for any $x, y \in \mathbb{R}$, $f(x) - x = f([x] + \{x\}) - [x] - \{x\} = f(\{x\}) + [x] - [x] - \{x\} = f(\{x\}) - \{x\}$, where $[x]$ is the integer part and $\{x\}$ is the fractional part which is obviously in $[0, 1]$. Also, a point to note is that f is a strictly monotonic increasing function, i.e $x - y \geq 0 \Rightarrow f(x) - f(y) \geq 0$.

Let $x, y \in [0, 1]$ such that $y \leq x$ WLOG. Then we know that $0 \leq x - y \leq 1$ and that $0 \leq f(x) - f(y) \leq 1$. Hence the difference of these two can be atmost 1. That is, $(f(x) - f(y)) - (x - y) = (f(x) - x) - (f(y) - y) \leq 1$. Ananlogously, in the cases when $x \leq y$, we can show this quantity to be greater than -1, hence proving that their difference is atmost 1 for any two $x, y \in \mathbb{R}$. ∎

Further, define $T(\tilde{f}) = \tilde{f}(0)$. If $\tilde{f}_1, \tilde{f}_2 \in \widetilde{Homeo_+}(S^1)$, then $T(\tilde{f}_1\tilde{f}_2) = (\tilde{f}_1(\tilde{f}_2(0)) - \tilde{f}_2(0)) + (\tilde{f}_2(0) - 0)$ implies $|T(\tilde{f}_1\tilde{f}_2) - T(\tilde{f}_1) - T(\tilde{f}_2)|$ is bounded by 1 using the previous claim. Such functions are called quasi-homomorphisms, as is formalized below.

Definition 4.1.2. Let Γ be a group. A quasi-homomorphism from Γ to $\mathbb{R}$ is a map $F : \Gamma \to \mathbb{R}$ such that there is a constant D such that for every γ_1, γ_2 in Γ we have $|F(\gamma_1\gamma_2) - F(\gamma 1) - F(\gamma 2)| \leq D$.

Lemma 4.1.3. Let $F : \mathbb{Z} \to \mathbb{R}$ be a quasi-homomorphism. Then, there exists a unique real number τ such that the sequence $F(n) - n\tau$ is bounded.

Before the proof of this lemma, Fekete's lemma would be required which is included in the appendix.

Proof. **(Lemma)** Define the function $f^* : \mathbb{Z} \to \mathbb{R}$ by:

$$f^*(a) = \lim_{n \to +\infty} \frac{f(na)}{n}.$$

To show that the limit exists, fix a and let consider the sequence $a_n = f(na) + D$. Since f is a quasi-homomorphism, $f((m+n)a) \leq f(ma) + f(na) + D$. Then adding D on both the sides, $f((m+n)a) + D \leq f(ma) + D + f(na) + D \implies a_{m+n} \leq a_m + a_n$. Now that a_n is known to be sub-additive, using Fekete's lemma, $\lim \frac{a_n}{n}$ exists which is same as the limit of $\frac{f(a_n)}{n}$. To rule out the possibility of the limit being $-\infty$, observe that $|f(na)| \leq n|f(a)| + (n-1)D$ implies that $\frac{|f(na)|}{n} \leq |f(a)| + (1 - \frac{1}{n})D < |f(a)| + D$, so $\frac{|a_n|}{n}$ is bounded and cannot go to infinity.

We can notice that $f^*(0) = 0$ and $f^*(ka) = kf^*(a)$ for every $k > 0$:

$$f^*(ka) = \lim_{n \to +\infty} \frac{f(nka)}{n} = k \lim_{n \to +\infty} \frac{f(nka)}{nk} = k \lim_{n \to +\infty} \frac{f(na)}{n} = kf^*(a).$$

Note that by induction on $n > 0$ we have,

$$|f(na) - nf(a)| \leq (n-1)D.$$

For $n = 1$, this is trivial, so assume that the inequality holds for n, we have,

$$|f((n+1)a) - (n+1)f(a)| \leq |f((n+1)a) - f(na) - f(a)| + |f(na) - nf(a)|$$

$$\leq D + (n-1)D = nD.$$

Therefore,

$$\frac{|f(na) - nf(a)|}{n} \leq \frac{n-1}{n} D < D,$$

and so,

$$|f^*(a) - f(a)| = |\lim_{n \to +\infty} \frac{f(na)}{n} - f(a)| = \lim_{n \to +\infty} |\frac{f(na)}{n} - f(a)|$$

$$= \lim_{n \to +\infty} \frac{|f(na) - nf(a)|}{n} \leq D.$$

Put $\tau = f^*(1)$. For positive a, we have,

$$|f(a) - a\tau| = |f(a) - af^*(1)| = |f(a) - f^*(a)| \leq D.$$

It remains to prove that $|f(-a) + a\tau|$ is bounded for positive a. By induction on $n > 0$ we see that,

$$|f(0) - f(-na) - nf(a)| \leq nD.$$

For $n = 1$ this is trivial. Assume that the inequality holds for n, we have,

$$|f(0) - f(-(n+1)a) - (n+1)f(a)| \leq |f(0) - f(-na) - nf(a)| + ...$$

$$|f(-na) - f(-(n+1)a) - f(a)| \leq nD + D = (n+1)D.$$

Therefore,

$$-|f(0)| - nD \leq f(0) - nD \leq f(-na) + nf(a) \leq f(0) + nD \leq |f(0)| + nD,$$

so:

$$\frac{|f(-na) + nf(a)|}{n} \leq \frac{|f(0)|}{n} + D,$$

and we have

$$|f^*(-a) + f(a)| = |\lim_{n\to\infty} \frac{f(-na)}{n} + f(a)| = \lim_{n\to+\infty} \frac{|f(-na) + nf(a)|}{n}$$

$$\leq \lim_{n\to+\infty} \frac{|f(0)|}{n} + D = D.$$

Finally,

$$|f(-a) + a\tau| = |f(-a) + af^*(1)| = |f(-a) + f^*(a)|$$

$$\leq |f(-a) - f^*(-a)| + |f^*(-a) + f(a)| + |f^*(a) - f(a)|$$

$$\leq D + D + D = 3D.$$

So $|f(a) - a\tau|$ is bounded by $3D$ for every a.

Uniqueness : Let there exist $\tau_1, \tau_2 \in \mathbb{R}$ such that $|f(a) - a\tau_1| \leq D$ and $|f(a) - a\tau_2| \leq D'$. Adding these two, we get $|a(\tau_1 - \tau_2)| \leq D + D' = |a||\tau_1 - \tau_2| \leq D$ for all $a \in \mathbb{Z}$. But this means that $|\tau_1 - \tau_2| = 0$ and hence $\tau_1 = \tau_2$.

■

Pick an element $\tilde{f}$ from $\widetilde{Homeo_+}(S^1)$ and consider the subgroup generated by it. This will be isomorphic to $\mathbb{Z}$ and hence we can use the previous lemma to find a unique real number τ corresponding to the quasi-homomorphism $T : \widetilde{Homeo_+}(S^1) \to \mathbb{R}$

(restricted to this one generator subgroup). Note that T was defined as $T(\tilde{f}) = \tilde{f}(0)$ initially. Define this τ such that $T((\tilde{f})^n - n\tau)$ is bounded, as the **translation number** of $\tilde{f}$. Now let $f \in Homeo_+(S^1)$, which will have lifts (by integral translations) in $\widetilde{Homeo_+}(S^1)$ of the type $\tilde{g_k} \circ \tilde{f}$, where $\tilde{g_k}(x) = x + k$ for some integer k. Besides, the translation number of a function composed with integral translation is just an integral translate as shown ahead.

$$\tau(\tilde{g_k} \circ \tilde{f}) = \lim_{n\to\infty} \frac{(\tilde{g_k} \circ \tilde{f})^n(0)}{n}$$

$$= \lim_{n\to\infty} \frac{\tilde{g_k} \circ \tilde{f} \circ \tilde{g_k} \circ ... \circ \tilde{g_k} \circ \tilde{f}(0)}{n}$$

$$= \lim_{n\to\infty} \frac{\tilde{g_k} \circ \tilde{f} \circ \tilde{g_k} \circ ... \circ \tilde{g_k} \circ \tilde{f}(k + \tilde{f}(0))}{n}$$

$$= \lim_{n\to\infty} \frac{\tilde{g_k} \circ \tilde{f} \circ \tilde{g_k} \circ ... \circ \tilde{g_k}((\tilde{f})^2(0)) + k}{n} \quad (\because \tilde{f}(x+k) = \tilde{f}(x) + k \text{ for } k \in \mathbb{Z})$$

$$= \lim_{n\to\infty} \frac{(\tilde{f})^n(0)}{n} + \frac{nk}{n} \text{ (by induction)}$$

$$= \tau(\tilde{f}) + k$$

Consequently, τ mod $\mathbb{Z}$ is well defined and is henceforth defined as the **rotation number** of the homeomorphism $\tilde{f}$ of S^1.

Corollary 4.1.4. The function $\tau : \widetilde{Homeo_+}(S^1) \to \mathbb{R}$ given by $\tau(\tilde{f})$ assigning the translation number to $\tilde{f}$ is a quasi-homomorphism.

Proof. As described before, T is a quasi-homomorphism, i.e $|T(\tilde{f} \circ \tilde{g}) - T(\tilde{f}) - T(\tilde{g})| \leq D$. From 4.1.3, $|T((\tilde{f})^n) - n\tau| \leq 3D$ for all $n \in \mathbb{Z}$.

$$|T((\tilde{f})^n) - n\tau_{\tilde{f}}| \leq 3D$$

$$|T((\tilde{g})^n) - n\tau_{\tilde{g}}| \leq 3D$$

$$|n\tau_{\tilde{f}\circ\tilde{g}} - T((\tilde{f} \circ \tilde{g})^n)| \leq 3D$$

Adding the three and then using triangle inequality, we get,

$$|n\tau_{\tilde{f}\circ\tilde{g}} - T((\tilde{f} \circ \tilde{g})^n) + T((\tilde{f})^n) - n\tau_{\tilde{f}} + T((\tilde{g})^n) - n\tau_{\tilde{g}}| \leq 9D$$

Substituting $n = 1$ and using reverse triangle inequality,

$$|\tau_{\tilde{f}\circ\tilde{g}} - \tau_{\tilde{f}} - \tau_{\tilde{g}}| \leq 9D + |T(\tilde{f} \circ \tilde{g}) - T(\tilde{f}) - T(\tilde{g})|$$

$$|\tau_{\tilde{f}\circ\tilde{g}} - \tau_{\tilde{f}} - \tau_{\tilde{g}}| \leq 10D$$

Also as part of a small observation, τ is a homomorphism on one-generator subgroups since,

$$|T(((\tilde{f})^m)^n) - n(m\tau_{\tilde{f}})| \leq 3D$$

And uniqueness of τ implies that $\tau((\tilde{f})^m) = m\tau(\tilde{f})$. ∎

To get an intuition for these numbers, observe that the translation number of the translation by τ in $\mathbb{R}$ is τ and the rotation number of a rotation by "angle" ρ is indeed ρ as shown. Let $\tilde{f} \in \widetilde{Homeo_+}(S^1)$ be defined as $f(x) = f(x) + r$ where $r \in \mathbb{R}$. Note that $|T((\tilde{f})^n) - nr| = |(\tilde{f})^n(0)) - nr| = |\tilde{f}(0) + nr - nr| = |\tilde{f}(0)|$ which is bounded being a constant. And the uniqueness of the translation number then suggests that $\tau = r$. The claim regarding rotation number hence follows.

While studying iterates of functions (as now, when we look at $(\tilde{f})^n)$), if we have properties that are conjugation invariant, the iterates also continue to be so since $f = h^{-1}gh$ implies$f^n = h^{-1}g^nh$. This gives the motivation for introduction of translation and rotation numbers.

Proposition 4.1.5. The translation number and the rotation number are invariant under conjugation in $\widetilde{Homeo_+}(S^1)$ and $Homeo_+(S^1)$ respectively.

Proof. As τ is a homomorphism on one-generator subgroups, we know that $\tau((\tilde{f})^n) = n\tau(\tilde{f})$ and also that,

$$\tau(\tilde{h}(\tilde{f})^n\widetilde{h^{-1}}) = \tau((\tilde{h}\tilde{f}\widetilde{h^{-1}})^n) = n\tau(\tilde{h}\tilde{f}\widetilde{h^{-1}})$$

By definitions,

$$|\tau(\tilde{h}(\tilde{f})^n\widetilde{h^{-1}}) - \tau(\tilde{h}) - \tau((\tilde{f})^n\widetilde{h^{-1}})| \leq 3D$$

$$|\tau((\tilde{f})^n\widetilde{h^{-1}}) - \tau((\tilde{f})^n) - \tau(\widetilde{h^{-1}})| \leq 3D$$

$$|\tau(0) - \tau(\tilde{h}) - \tau(\widetilde{h^{-1}})| \leq 3D \text{ (0 is the identity in } \widetilde{Homeo_+}(S^1))$$

Adding these,

$$|\tau(\tilde{h}(\tilde{f})^n\widetilde{h^{-1}}) - \tau((\tilde{f})^n) - \tau(0)| \leq 9D$$

From reverse triangle inequality and homomorphism of τ,

$$|n\tau(\tilde{h}\tilde{f}\widetilde{h^{-1}}) - n\tau(\tilde{f})| - |\tau(0)| \leq 9D$$

$$|n||\tau(\tilde{h}\tilde{f}\widetilde{h^{-1}}) - \tau(\tilde{f})| \leq 9D + |\tau(0)|$$

But since n is unbounded ($\in \mathbb{Z}$), and RHS is bounded, we must have that $|\tau(\tilde{h}\tilde{f}\tilde{h^{-1}}) - \tau(\tilde{f})| = 0$, or $\tau(\tilde{h}\tilde{f}\tilde{h^{-1}}) = \tau(\tilde{f})$. The case for rotation number follows trivially. ■

Before moving to the next proposition, we will need a corollary to 4.1.3 which is as stated below.

Corollary 4.1.6. Let $f : \mathbb{Z}^2 \to \mathbb{R}$ be a quasi-homomorphism. Then f differs from any homomorphism from $\mathbb{Z}^2$ to $\mathbb{R}$ by a bounded amount. i.e there exist unique $\tau_1, \tau_2 \in \mathbb{R}$ such that $|f(a,b) - a\tau_1 - b\tau 2| \leq K$ for all $(a,b) \in \mathbb{Z}^2$.

Proof. From the definition of quasi-homomorphism, $|f(a,b) - f(a,0) - f(0,b)| \leq D$. And 4.1.3, gives the following results on one-generator subgroups of $\mathbb{Z}^2$:

$$|f(a,0) - a\tau_1| \leq D'$$

$$|f(0,b) - b\tau_2| \leq D'$$

Adding these three inequalities gives,

$$|f(a,b) - a\tau_1 - b\tau_2| \leq 2D' + D$$

For uniqueness of τ_1 and τ_2, assume the contrary. Then there exist τ_1', τ_2' such that $|f(a,b) - a\tau_1 - b\tau 2| \leq T'$. And we already have that $|f(a,b) - a\tau_1 - b\tau 2| \leq T$. Adding these gives $|a(\tau_1' - \tau_1) + b(\tau_1' - \tau_1)| \leq T + T'$. By setting $a = 0$, we get that $\tau_1' = \tau_1$ and similarly setting $b = 0$, we get $\tau_2' = \tau_2$. ■

Proposition 4.1.7. The translation number is the unique quasi-homomorphism $\tau : Homeo_+(S^1) \to \mathbb{R}$ which is a homomorphism when restricted to one generator subgroups and which takes the value 1 on the translation by 1.

Proof. Using Corollary 1.1.7, if a quasi-homomorphism $\mathbb{Z}^2 \to \mathbb{R}$ is a homomorphism when restricted to one generator groups, it is a homomorphism $\mathbb{Z}^2 \to \mathbb{R}$, as the following argument shows.

$$\begin{aligned} &|f(n(a,b)) - na\tau_1 - nb\tau_2| \leq D \\ \implies &|nf((a,b)) - na\tau_1 - nb\tau_2| \leq D \\ \implies &|n||f((a,b)) - a\tau_1 - b\tau_2| \leq D \\ \implies &f((a,b)) = a\tau_1 + b\tau_2 = g((a,b)) \end{aligned}$$

where g is a homomorphism. Hence, f is a homomorphism on $\mathbb{Z}^2$.

Now assume that the uniqueness part is not true. Then there exists another quasi-homomorphism t satisfying the conditions of the proposition. From the above observation, t is a homomorphism when restricted to the subgroup generated by any element $\tilde{f}$ and integral translations. This subgroup is commutative since if $\tilde{g_k}(x) = x+k(k \in \mathbb{Z})$, $\tilde{f} \circ \tilde{g_k}(x) = \tilde{f}(x+k) = \tilde{f}(x)+k = \tilde{g_k} \circ \tilde{f}(x)$. And any commutative group over two generators is isomorphic to $\mathbb{Z}^2$, which lets us use the previous observation to conclude that t is a homomorphism. Now, consider $r = \tau - t$. Let $\tilde{f} \in \widetilde{Homeo_+}(S^1)$ and $\tilde{g_k}$ the usual integral translation. Then,

$$r(\tilde{f} \circ \tilde{g_k}) = (\tau - t)(\tilde{f} \circ \tilde{g_k}) = \tau(\tilde{f} \circ \tilde{g_k}) - t(\tilde{f} \circ \tilde{g_k})$$

$$= \tau(\tilde{f}) + k - t(\tilde{f}) - t(\tilde{g_k}) = \tau(\tilde{f}) + k - t(\tilde{f}) - k$$
$$= (\tau - t)(\tilde{f})$$

shows that the value of r depends only on the projection of $\tilde{f} \in \widetilde{Homeo_+}(S^1)$ in $Homeo_+(S^1)$, so that we get a quasi-homomorphism $\bar{r} : Homeo_+(S^1) \to \mathbb{R}$ which is a homomorphism on one generator groups. We claim that $\bar{r}$ must be trivial which uses a property of $Homeo_+(S^1)$ that any homeomorphism f in $Homeo_+(S^1)$ can be written as a commutator $[f_1, f_2]$. With the assumption of this result, every quasi-homomorphism ϕ from $Homeo_+(S^1)$ is bounded. We have to bound $|\phi(f)| = |\phi(ghg^{-1}h^{-1})|$.

$$|\phi(ghg^{-1}h^{-1}) - \phi(g) - \phi(h) - \phi(g^{-1}) - \phi(h^{-1})| \leq 3D$$

$$|\phi(0) - \phi(g) - \phi(g^{-1})| \leq D$$
$$|\phi(0) - \phi(h) - \phi(h^{-1}|) \leq D$$

Adding these three equations and using reverse triangle inequality gives,

$$|\phi(ghg^{-1}h^{-1})| \leq 5D + 2|\phi(0)|$$

This tells us that $\bar{r}$ is also bounded and we know that a bounded quasi-homomorphism which is homomorphism on one generator groups is trivial. Hence $\bar{r}$ is zero. ■

Proposition 4.1.8. Let Γ be any subgroup of $Homeo_+(S^1)$. Then there are three mutually exclusive possibilities.

- There is a finite orbit.
- All orbits are dense.
- There is a compact Γ invariant subset $K \subseteq S^1$ which is infinite and different from S^1 such that the orbits of points in k are dense in K. This set K is unique, contained in the closure of any orbit and is homeomorphic to a *Cantor* set.

Proof. Consider the collection of compact sets in S^1 which are non-empty and Γ invariant, ordered by inclusion. Trivially this collection is non-empty as S^1 itself is compact and Γ invariant. Therefore, using Zorn's lemma, we can find a minimal set in this collection. Choose such a minimal set from this collection and call it K. Let $k \in K$ and denote its orbit as $\Gamma.k$. Then trivially $\Gamma.k$ is Γ invariant. Now consider k in $\overline{\Gamma.k} \setminus \Gamma.k$. This means there exists a sequence $k_n \in \Gamma.k$ such that $k_n \to k$. Also from continuity of the action, for all $g \in \Gamma$, $g.k_n \to g.k$ (note that $g.k_n \in \Gamma.k$). So, we see that $g.k$ is in the closure of $\Gamma.k$, which means that $\overline{\Gamma.k}$ is Γ invariant, non-empty, closed and contained in K (since K being compact in Hausdorff space is closed and hence $\overline{\Gamma.k}$ still stays in K). From minimality of K, it follows that the closure orbits of elements of K are equal to K. Equivalently, orbits of $k \in K$ are dense in K. The boundary of K, ∂K and the set of limit points of K, K' are also Γ invariant. The proof for these is more or less the same as done above for the closure of orbits. Also, ∂K $(K \setminus interior(K))$ is closed as interior of any set is open and K' is closed by known results. As such, each of them can either be empty, or if non-empty, equal to K, from the minimality of K in the collection of such sets. Following possibilities are hence left :

- K' is empty. In this case, K will be finite, and thus we have found a finite orbit.

  Proof. Let $K \subseteq S^1$ be compact with no limit points but such that it has infinitely many points. Then, around each of these points, we can find a neighborhood such that no other point of K falls in it. This open cover of K will have a finite subcover from compactness, but the disjoint-ness of these neighborhoods will force K to be a finite set. Thus, for any $k \in K$, its orbit will have to be finite since K is Γ invariant. ∎

- ∂K is empty, so that K is the full circle. In this case, all the orbits are dense.

  Proof. We prove this by claiming that any proper closed subset (K is closed being compact in Hausdorff space) of S^1 has at least one boundary point. Then K being non-empty will have to be the other trivial option (S^1). So, first note that non-trivial subsets of S^1 can't be clopen, as such a case would contradict the connectedness of S^1. Let $\partial K = K \setminus interior(K) = \phi$, which implies $interior(K) = K$ $(\phi \neq K \neq S^1)$. But $interior(K)$ is open while K is closed, and we have seen that there can't be any clopen subsets in S^1. So, there must be at least one boundary point of K. ∎

- $K' = K$ and $\partial K = K$, so that K is a compact, perfect set (closed set with every point a limit point) in the circle with empty interior: another definition of Cantor set.

Since K is a minimal set, and a poset can have multiple minimal sets, the uniqueness needs to be proved. It will suffice to show that K is contained in the closure of every orbit, since then all these minimal sets will become comparable (each K is equal to the closure of orbit of any point in it and in turn now contained in any other minimal set's orbits) in the common chain giving one such unique minimal set.

Begin with observing that the complement of K (Cantor set) is a disjoint union of countably many open intervals. Let $x \in K^c$, which will be in some open interval I and let a be the origin of I (I is oriented). This a will be in K as I is open and connected. Pick y to be any point in K. As the orbit of any point of K is dense in K, and K has no isolated point (required as the orbit can be whole of K), there exists a sequence of elements $\gamma_n \in \Gamma$ such that $\gamma_n(a)$ consists of distinct points (K is uncountable being non-empty Cantor set) and converges to y. The claim now is that the intervals $\gamma_n(I)$ are disjoint. First observe that the endpoints of I (say a, b) are in K and as K is Γ invariant, they have to be mapped back in K under Γ-action. Also since each $\gamma \in \Gamma$ gives a homeomorphism, $\gamma(I)$ has to be connected. If K in Γ invariant, then K^c is also so, which means no point of $I \subseteq K^c$ can go in K. This implies that $\gamma(I)$ is again an open interval contained entirely in K^c, sandwiched between two points of K. From orientation-preserving property of γ above statement, it is clear that if $\gamma_i(a) \neq \gamma_j(a)$, then $\gamma_i(b) \neq \gamma_j(b)$ and hence $\gamma_i(I) \cap \gamma_j(I) = \phi$. Consequently these intervals will get smaller and the distance between $\gamma_n(a)$ and $\gamma_n(x)$ will converge to zero, implying that $\gamma_n(x)$ converges to y. Hence, K is contained in the closure of every orbit and uniqueness of K follows. ■

The third case is called the *exceptional minimal set* case and can be reduced to case 2 by introducing the notion of semi-conjugacy.

Definition 4.1.9. Let $\tilde{h} : \mathbb{R} \to \mathbb{R}$ be an increasing, continuous map such that $\tilde{h}(x + 1) = \tilde{h}(x) + 1$. This induces a map $h : S^1 \to S^1$ called **increasing continuous map of degree 1 from the circle to itself**.

Definition 4.1.10. Let Γ be a group and ϕ_1, ϕ_2 be two homomorphisms from Γ to $Homeo_+(S^1)$. We say that ϕ_1 is semi-conjugate to ϕ_2 if there is an increasing continuous map h of degree 1 from the circle to itself such that for every γ in Γ, we have $\phi_2(\gamma)h = h\phi_1(\gamma)$.

Note : Semi-conjugacy is not a symmetric property since h may not be one-to-one (thus not invertible).

Proposition 4.1.11. Let Γ be a group and ϕ be a homomorphism from Γ to $Homeo_+(S^1)$ such that $\phi(\Gamma)$ has an exceptional minimal set K. Then there is a homomorphism $\bar{\phi}$

from Γ to $Homeo_+(S^1)$ such that ϕ is semi-conjugate to $\bar{\phi}$ and $\bar{\phi}(\Gamma)$ has dense orbits on the circle.

Proof. We know that K is a Cantor set and hence it follows that its complement is the countable union of open intervals. Define an identification on S^1 by collapsing the closure of each of these intervals to a point. The quotient space then will be homeomorphic to S^1 as the following map will show). This can be re-framed as saying that we have an increasing continuous map of degree 1 from the circle to itself such that $h(K) = S^1$ and such that the fibers $h^{-1}(x)$ are either points or the closed intervals that are the closures of connected components of the complement of K. Here, h can straightforward be the quotient map. As $\phi(\Gamma)$ acts on the circle and preserves K, it also acts on the quotient circle as follows.

Define action of $\phi(\Gamma)$ on collapsed circle $S^1/\sim = S^1$ by,

$$\varphi : \phi(\Gamma) \times S^1/\sim \to S^1/\sim$$

$$(\phi_g, \bar{s}) \mapsto h\phi_g h^{-1}(\bar{s})$$

To show that this is well defined, if $h^{-1}(\bar{s})$ is a singleton, then there's no ambiguity, and we are done. But if $h^{-1}(\bar{s})$ is a closed interval in S^1, we will have to show that all these points in the closed interval go together into one closure of a connected component of the complement of K. Let the closed interval (fiber of $h^{-1}(\bar{s})$) have end points $a, b \in K$ and rest all the points in K^c. From orientation preserving nature of homeomorphism ϕ_g and $\phi(\Gamma)$ invariance of K (and K^c), the whole closed interval will be sent to a closed interval with end points in K and rest all in K^c. Hence, the quotient map h would then send this whole closed interval to one unique point, and so we are done.

To see the semi-conjugacy, observe the following with reference to the commutative diagram below. Let $\beta \in S^1$. Then $\bar{\phi}_\gamma \circ \lambda(\beta) = \lambda\phi_\gamma\lambda^{-1}(\lambda\beta) = \lambda\phi_\gamma(\beta)$. This step was necessary because h may not be invertible. (Let L=union of the closure of the connected components in K^c)

$$\begin{tikzcd}
\beta \in S^1 \arrow[r, "\lambda"] \arrow[d, "\phi_\gamma"] & \frac{S^1}{L} \arrow[d, "\bar{\phi}_\gamma"] \\
S^1 \arrow[r, "\lambda"] & \frac{S^1}{L}
\end{tikzcd}$$

■

Although the aim here is to discuss the dynamics of large group Γ acting on the circle, we would first start with considering the one-generator groups. In such cases, a homeomorphism f_1 is semi-conjugate to f_1 if the corresponding homomorphism from $\mathbb{Z}$ to $Homeo_+(S^1)$ are semi-conjugate (which makes sense as the whole map is determined just by the image of the generator in one-generator groups). The following result will show the information contained in the rotation numbers.

Theorem 4.1.12. Let f be an element of $Homeo_+(S^1)$. Then f has a periodic orbit if and only if the rotation number $\rho(f)$ is rational, i.e it belongs to $\mathbb{Q}/\mathbb{Z}$. If the rotation number $\rho(f)$ is irrational, then f is semi-conjugate to the rotation on the circle of angle $\rho(f) \in \mathbb{R}/\mathbb{Z}$. This semi-conjugacy is actually a conjugacy if the orbits of f are dense.

Proof. Pick an element $\tilde{f}$ from the lift of f in $Homeo_+(S^1)$. We already know from the section regarding translation numbers that $|\tilde{f}^n(0)-n\tau(\tilde{f})| \leq D$ for some fixed $D \in \mathbb{R}$ and for all $n \in \mathbb{Z}$. Also it is clear that $|\tilde{f}^n(x)-\tilde{f}^n(y)| = |\tilde{f}^n(\{x\}+[x])-\tilde{f}^n(\{y\}+[y])| \leq |\tilde{f}^n(\{x\})-\tilde{f}^n(\{y\})|+|[x]+[y]| \leq 1+[x]+[y]$ (images of points that differ by at-most 1 under an element of $Homeo_+(S^1)$, differ by at-most 1). Substituting $y = 0$ here, gives $|\tilde{f}^n(x)| \leq 1+[x]+D$ using $|\tilde{f}^n(0)| \leq D$ and reverse triangle inequality. Adding the following two inequalities,

$$|\tilde{f}^n(0) - n\tau(\tilde{f})| \leq D$$

$$|\tilde{f}^n(x) - \tilde{f}(0)| \leq 1 + [x] + D$$

gives $|\tilde{f}^n(x) - n\tau(\tilde{f})| \leq 1 + 2D + [x]$ which is of course a constant for any fixed x. Therefore, define $\tilde{h}(x) = \sup_n(\tilde{f}^n(x) - n\tau(\tilde{f}))$. Following are some properties of $\tilde{h}$ that will be used ahead in the proof :

1. $\tilde{h}$ is increasing.
   Let $x < y$ and then consider $\tilde{h}(y) - \tilde{h}(x) = \sup_n(\tilde{f}^n(y) - n\tau(\tilde{f}) - \sup_n(\tilde{f}^n(x) - n\tau(\tilde{f})) \geq 0$ as f is increasing, and supremum brings the equality condition as well.

2. $\tilde{h}(x+1) = \tilde{h}(x) + 1$
   Using the property of f that it commutes with integral translation, $\tilde{h}(x+1) = \sup_n(\tilde{f}^n(x+1) - n\tau(\tilde{f})) = \sup_n(\tilde{f}^n(x) - n\tau(\tilde{f})) + 1 = \tilde{h}(x) + 1$

3. $\tilde{h}(\tilde{f}(x)) = \tilde{h}(x) + \tau(\tilde{f})$
   Computing LHS, $\tilde{h}(\tilde{f}(x)) = \sup_n(\tilde{f}^n(\tilde{f}(x)) - n\tau(\tilde{f})) = \sup_{n+1}(\tilde{f}^{n+1}(x) - (n+1)\tau(\tilde{f})) + \tau(\tilde{f}) = \tilde{h}(x) + \tau(\tilde{f})$.

The third case after modulo integers changes in the following manner :

$$\begin{aligned}\tilde{h}(\tilde{f}(x)) =& \tilde{h}(\tilde{f}(\{x\} + [x])) \\ =& \tilde{h}(\tilde{f}(\{x\})) + [x] \\ =& \tilde{h}(\tilde{f}(\{x\})) mod\ \mathbb{Z}\end{aligned}$$

Using same idea on RHS, and that $\rho(f) = \tau(\tilde{f}\ mod\ \mathbb{Z})$, we see that equation 3 reduces to $h(f(x)) = (h(x) + \rho(f))\ mod\ \mathbb{Z}$. From this information, if h is continuous (increasing doesn't imply continuous) then from the following argument, h acts a semi-conjugacy between f and the rotation map by angle $\rho(f)$, denoted $R_{\rho(f)}$. To show the semi-conjugacy, we need to show that $R^n_{\rho(f)}h = hf^n$. Pick any $x \in \mathbb{R}/\mathbb{Z}$. Then LHS$=R^n_{\rho(f)}h(x) = (n\rho(f) + h(x))\ mod\ \mathbb{Z} = h(x) + \rho(f^n)$ =RHS. And then, since by definition semi-conjugacy needs the map to be continuous, and we have assumed that h is continuous, we are done. Now we look at the structure of the (in-fact any) increasing function $\tilde{h} : \mathbb{R} \to \mathbb{R}$. The fibers $h^{-1}(x)$ are either empty, singletons of intervals. Since real line can have only countably many intervals, so are these intervals in cardinality (at most). Union of the interior of these intervals is defined to be the set $Plat(\tilde{h})$ and it is empty if and only if $\tilde{h}$ is injective (since then $h^{-1}(x)$ will be a singleton). Now consider the image $\tilde{h}(\mathbb{R})$ which is the complement of the union of at most countably many disjoint intervals (disjoint because if they intersect, the value in their union remains constant and hence they were the same, to begin with). Define the union of the interior of these intervals to be the set $Jump(\tilde{h})$, which is empty if and only if $\tilde{h}$ is continuous and onto (it is already onto from property (2) above). These sets are open in $\mathbb{R}$ and invariant under integral translation (from the property (2)), so that they will define open sets in the quotient $\mathbb{R}/\mathbb{Z}$. Analogous to (3), observe the following:

$$\begin{aligned}\tilde{h}(\tilde{f}^{-1}(x)) =& \sup_n(\tilde{f}^n(\tilde{f}^{-1}(x)) - n\tau(\tilde{f})) \\ =& \sup_n(\tilde{f}^{n-1}(x) - (n-1)\tau(\tilde{f})) - \tau(\tilde{f}) \\ =& \tilde{h}(x) - \tau(\tilde{f})\end{aligned}$$

Using this result, $Plat(\tilde{h})$ is invariant under $\tilde{f}$ as follows : Let $x \in Plat(\tilde{h})$, i.e there exists an interval I such that $x \in I$ and $\tilde{h}(I)$ is a constant. As f is a homeomorphism, this I goes again to an interval, where for invariance, we would want $\tilde{h}$ to be constant over $\tilde{f}(I)$. Assume contrary by allowing $x, y \in \tilde{f}(I)$ such that $\tilde{h}(x) \neq \tilde{h}(y)$. Then

$\tilde{f}^{-1}(x), \tilde{f}^{-1}(y) \in I$, which implies,

$$\begin{aligned} \tilde{h}(\tilde{f}^{-1}(x)) =& \tilde{h}(\tilde{f}^{-1}(y) \\ \tilde{h}(x) - \tau(\tilde{f}^{-1}) =& \tilde{h}(y) - \tau(\tilde{f}^{-1}) \\ \tilde{h}(x) =& \tilde{h}(y) \end{aligned}$$

which contradicts the fact that $x, y \in I$ had to have same image. Also, $Jump(\tilde{h})$ is invariant under translation by $\tau(\tilde{f})$ by the following argument. An element of $\tilde{h}$ is characterized by $\tilde{h}^{-1}(x) = \phi$. Let $\tilde{h}^{-1} \neq \phi$. Then there exists p such that

$$\begin{aligned} &\tilde{h}(p) = x + \tau(\tilde{f}) \\ &\implies \tilde{h}(p) - \tau(\tilde{f}) = x \\ &\implies \sup_n(\tilde{f}^n(p) - n\tau(\tilde{f})) - \tau(\tilde{f}) = x \\ &\implies \sup_{n+1}(\tilde{f}^{n+1}(\tilde{f}^{-1}(p)) - (n+1)\tau(\tilde{f})) = x \\ &\implies \tilde{h}(\tilde{f}^{-1}(p)) = x \end{aligned}$$

This contradicts the fact that $\tilde{h}^{-1}(x) = \phi$ and so the result follows. Once we know that the set is invariant under translation by $\tau(\tilde{f})$, it is clear that this will be an open set in the circle, invariant under rotation by $\rho(f)$.

Begin with $\tau(\tilde{f})$ being irrational. In such a case, all the orbits of rotation by angle $\rho(f)$ are dense (details in the appendix). As such if these orbits are dense in S^1 and contained in $Jump(\tilde{h})$ (which will happen to orbit of any point from $Jump(\tilde{h})$ as it is invariant under rotation by $\rho(f)$), then $Jump(\tilde{h}) \neq \mathbb{R}$ will also have to be dense in S^1. This means that the closure of intervals that make up $Jump(\tilde{h})$ is $\mathbb{R}$ and hence there can't be any intervals in $Jump(\tilde{h})$ of which this set was made. Hence, $Jump(\tilde{h}) = \phi$, and $\tilde{h}$ is continuous. The function $\tilde{h}$ hence defines semi-conjugacy between f and $R_{\rho(f)}$. If the orbits of f are given to be dense, then $Plat(\tilde{h})$ is empty and $\tilde{h}$ is injective. So, the semi-conjugacy becomes an actual conjugacy as inverses do exist now.

<u>Claim</u> : Let $f^n(h(x)) = h(g^n(x))$ (where h is the semi-conjugacy). Say, p has finite orbit under g. Then $h(p)$ has finite orbit under f.
Proof : p has finite orbit under g implies that the set $\{g^n(p)\}_n$ has finite order. This means that that $f^n(h(p))$ has finite order, so $h(x)$ has finite orbit under f.

Using the above claim, we know that if a point x has periodic orbit under f, it is finite and hence $h(x)$ has finite orbit under $R_{\rho(f)}$. But given $\rho(f)$ is irrational, any orbit under it will be dense in S^1 (hence infinite) using 1.0.10. And we know that finite orbits can't be dense in S^1. So, f can't have periodic orbits.

Rational rotation numbers are left to be examined now. Let $\tau(\tilde{f}) = p/q$ a rational number. Define $\tilde{l} = \tilde{f}^q - p$, which will share the properties of f.

$$\tau(\tilde{l}) = \lim_{n\to\infty} \frac{\tilde{l}^n(0)}{n} = \lim_{n\to\infty} \frac{\tilde{f}^{nq}(0) - np}{n} = \lim_{n\to\infty} \frac{\tilde{f}^{nq}(0)}{n} - \frac{np}{n} = \lim_{n\to\infty} q\frac{\tilde{f}^{nq}(0)}{nq} - p$$

$$= q\tau(\tilde{f}) - p = p - p = 0$$

Hence, the translation number of $\tilde{l}$ is 0. And now using calculation from the first paragraph of this proof, $\tilde{l}^n(x) - n\tau(\tilde{l})$ is bounded. Setting $\tau(\tilde{l}) = 0$, its evident that the orbit of every element under $\tilde{l}$ is bounded. Now our aim is to find an integral translation g_k (since that will still leave it in the lift of f) that will ensure that we are able to find a k such that $\tilde{f}^q \circ g_k(x) \geq x$ for at least one $x \in \mathbb{R}$. To see that such a k exists, let $\tilde{f}^q(x) - p < x$ for all $x \in \mathbb{R}$. Then, $\tilde{f}^q \circ g_k(x) - (p + qk) < x$ is true for $k = 0$. By adjusting value of k and using Archimedean property, for any fixed x, we can find a k that will satisfy the need. Keeping this in mind, $\tilde{l}(x) < x$ or $\tilde{l}(x) \geq x$. From previous argument, we can always reduce case 1 to case 2, where now the new $\tilde{l}(x) = \tilde{f}^q \circ\ g_k - (p + qk)$. And increasing property of $\tilde{l}$ shows that,

$$x < \tilde{l}(x) < \tilde{l}^2(x) < \tilde{l}^3(x) < \tilde{l}^4(x) < .....$$

In addition we also have that every orbit is bounded, which tells us that the above sequence will converge. i.e,

$$\tilde{l}^n(x) \to u = \sup_n(\tilde{l}^n(x))$$

$$\implies \tilde{l}^{n+1}(x) \to \tilde{l}(u) \text{ (from continuity of } \tilde{l})$$

Uniqueness of limit will hence imply that $u = f(u)$. This fixed point of $\tilde{l}$ will be a fixed point of its projection in $Homeo_+(S^1)$, namely f^q, which gives us a periodic orbit of f. ■

Hence, the dynamics of a homeomorphism $f \in Homeo{+}(S^1)$ can be described precisely as follows. Let $\rho(f)$ be irrational, then the only possibilities are -

- f is conjugate to the rotation of angle $\rho(f)$ if all the orbits are dense.
- Since there can't be any finite orbit, there can be an exceptional minimal set $K \subset S^1$. Then f is semi-conjugate to the rotation of angle $\rho(f)$

Thus, we have seen till now the way in which the single generator groups act on the circle. The dynamics of the orbits under their action were also discussed using the rotation number. Furthermore, the dynamics of larger groups acting on the circle can also be discussed, but this is where we will conclude by giving reference to the content in the bibliography.

Appendix A

Proposition A.1. Finite subgroups of $SO(2, \mathbb{R})$ are cyclic.

Proof. Let G be a finite subgroup of $SO(2, \mathbb{R})$. Then we know that it contains elements that rotate the plane by some θ degrees anti-clockwise. Let S be the set of degrees by which the elements of G rotate the plane. Then by Well Ordering principle, there exists a minimal non-zero degree, say θ, by which the plane is rotated. Using, Euclidean division algorithm, any other rotation say ϕ can be factored as $\phi = q\theta + r$ where $0 \leq r < q$. Using multiplication operation of $SO(2, \mathbb{R})$, $R_\phi = R_\theta^q R_r$. Then $R_r = R_\theta^{-q} R_\phi$ belongs to G from closure of the binary operation of the subgroup. But R_r is then a smaller rotation that R_θ which will contradict the minimality of R_θ unless $r = 0$. So, G is a cyclic subgroup generated by R_θ of order q. ■

Theorem A.2. (**Fekete's**) For every subadditive sequence $\{a_n\}_{n=1}^{\infty}$, the limit $\lim_{n\to\infty} \dfrac{a_n}{n}$ exists and is equal to $\inf \frac{a_n}{n}$. The limit may be $-\infty$.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a real sub-additive sequence, i.e $a_{i+j} \leq a_i + a_j \ \forall i, j \geq 1$. Denote $I = inf_{n\geq 1}\dfrac{a_n}{n}$. By the definition of infimum, for any $\epsilon \geq 0$, there exists an n such that $a_n \leq n(I + \epsilon)$. Set $b = \max_{1\leq i<n} a_i$. If $m \geq n$, let $m = qn + r$ with $0 \leq r < n$. Then it follows from sub-additivity that

$$a_{nq+r} = a_{n+n+\ldots+n+r} \leq \underbrace{a_n + a_n + \ldots + a_n + a_r}_{qtimes} \leq qa_n + b$$

Hence,

$$\frac{a_m}{m} \leq \frac{qa_n}{m} + \frac{b}{m} < \frac{qn(I+\epsilon)}{m} + \frac{b}{m}$$
$$\to I + \epsilon$$

as $m \to \infty$ since $\frac{qn}{m} \to 1$ as $m \to \infty$. ■

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