Quantization of circuits and study of quantum entanglement in coupled LC-oscillators

Nishant Naresh

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Certificate of Examination

This is to certify that the dissertation titled "Quantization and entanglement in quantum circuits" submitted by Mr Nishant Naresh (Reg. No. MS14189) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Prof Arvind	Dr Samir Kumar Biswas	Dr Mandip Singh
(Committee Member)	(Committee Member)	(Supervisor)

Dated: April 24, 2019

Declaration

The work in this dissertation has been carried out by me under the guidance of Dr. Mandip Singh at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Nishant Naresh

(Candidate)

Dated: April 24, 2019

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Mandip Singh

(Supervisor)

Acknowledgement

I would like to thank Dr. Mandip Singh for his professional guidance and enthusiastic encouragement who gave me the golden opportunity to do this wonderful project on the topic (quantization and entanglement in quantum circuits) and in this whole research I came to know about so many new things and for this I am really thankful to Dr Mandip Singh. I would like to thank my mom, dad and sister for their moral support through out the thesis.

I would like to thank to my friends(Parth Raval, Prateek Meena, Shivangi Vaish, Ritika, Jaideep, Nitish Garg, Nilotpal, Pinku Tung, Vishavdeep, Shubham Mittal) for their support and enthusiastic encouragement.

A special thanks to hostel 7 canteen members and gupta ji.

List of Figures

Notation

$|\Psi\rangle \rightarrow Wavefunction$

$\otimes \rightarrow Tensor\ product$

$\rho \rightarrow Density\ matrix$

$\rho_x \rightarrow Reduced\ Density\ matrix$

$S = -Tr(\rho_x log \rho_x) \rightarrow Entanglement\ entropy$

$|\alpha\rangle \rightarrow Coherent\ state$

$D_{\pm}(\alpha_{\pm}) \rightarrow Displacement\ operator$

$G_{q_x} \rightarrow Propagator(To\ describe\ dynamics)$

$P(Q_1, t) \rightarrow Probability\ distribution\ function$

$\langle Q_1 \rangle \rightarrow Expectation\ value$

Contents

Abstract

In this thesis a study of quantum entanglement in coupled LC-oscillators is presented. Two inductively coupled LC-oscillators are quantized. Ground and excited states of this system are quantum entangled. Entropy of quantum states is calculated. Oscillators in two different states, one is in unperturbed ground state and the second is in coherent state, are also studied. The evolution with one oscillator initially in its ground state, the other in a coherent state is done. Harmonic oscillator propagator is also used to study the dynamics of the system.

Chapter 1

Introduction

1.1 Quantum Entanglement

Why Quantum Entanglement? Why do we need to study quantum entanglement? We live in a world where objects can at will be separated and be treated individually. However, this mindset cannot be kept if we are dealing with world in terms of quantum physics. Entanglement is one of the most fascinating features of quantum physics, and it is at the heart of applications such as quantum computation, teleportation, quantum cryptography etc. It certainly is the key element to the mysteries contained in quantum mechanics and after over seventy years of struggle to understand and interpret this peculiar feature of quantum physics, the term "entanglement" itself hints at an intimate relationship between physical systems, an inseparability of objects, and properties thereof, that in a classical world of everyday life seems inconceivable.

Quantum entanglement of two particles in a quantum state in which measurements made on the quantum state of one particle instantaneously influence measurement outcomes on the other particle.

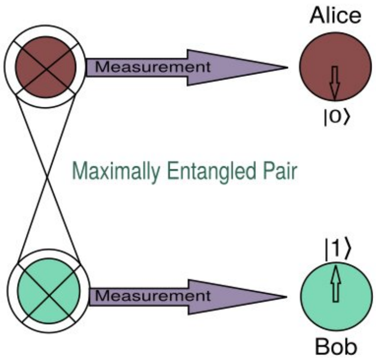

Figure 1.1: Quantum Entanglement

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B) \tag{1.1}$$

In other words, it is entangled and cannot be written as the product of two individual states i.e. $|\Psi\rangle \neq |\Psi\rangle_A \otimes |\Psi\rangle_B$.

According to Quantum Mechanics, if one performs a measurement of the spin of particle one and obtain the result $|\uparrow\rangle_A$, the spin of the second particle is projected onto the state $|\downarrow\rangle_B$. A measurement of quantum state of a particle therefore changes the state of other particle, even though the particles are separated very far from each other. This was a paradox for Einstein, Podolsky and Rosen, since they assumed that the reality should be described by a local theory. They therefore concluded that Quantum Mechanics is an incomplete theory. One can think of alternative hidden variables theories which can be used to explain entanglement. The predictions of local hidden variable theories and Quantum Mechanics differs, and the theories can be tested using Bell inequalities [9]. Now we of course believe that Quantum Mechanics is the correct description of the reality, and we use entanglement to describe non-local and non-classical correlations.

Originally, the notion of entanglement came up in the context of nonclassical correlations between the results of measurements on two distinct parts of a multipartite system. For example, it is possible to prepare two particles in a single quantum state such that when one is observed to be spin-up, the other one will always be observed to be spin-down and vice versa, this despite the fact that it is impossible to predict, according to quantum mechanics, which set of measurements will be observed. As a result, measurements performed on one system seem to be instantaneously influencing other systems entangled with it.

1.1.1 Entanglement of pure states

A pure quantum state can be represented by a vector in a complex Hilbert space with unit length. Thus for each pure state $|\Psi\rangle$ and any basis $|\phi_1\rangle$, ..., $|\phi_n\rangle$ the state $|\Psi\rangle$ can be extended to

$$|\Psi\rangle = \alpha_1 |\phi_1\rangle + \alpha_2 |\phi_2\rangle + ... + \alpha_n |\phi_n\rangle$$

where $\sum_{i=1}^{n} |\alpha_i|^2 = 1$. Now we consider the entanglement of pure states. Let $\{|\phi_i\rangle$, i = 1, 2, ..., n$\}$ and $\{|\chi_j\rangle$, j = 1, 2, ..., m$\}$ be orthonormal bases of n-dimensional Hilbert space $\mathcal{H}_n$ and m-dimensional Hilbert space $\mathcal{H}_m$ respectively. Denote by $\mathcal{H}_{mn}$ a Kronecker product of spaces $\mathcal{H}_n$ and $\mathcal{H}_m$. Thus $\mathcal{H}_{mn} = \mathcal{H}_n \otimes \mathcal{H}_m$ is a nm-dimensional Hilbert space with orthonormal basis $\{|\phi_i\rangle \otimes |\chi_j\rangle$ i, i = 1, 2, ..., n, j = 1, 2, ..., m$\}$, where $|\phi_i\rangle \otimes |\chi_j\rangle = |\phi_i\rangle |\chi_j\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \delta_{ij} |\phi_i\rangle |\chi_j\rangle$ and $\sum_{j=1}^{m} |\delta_{ij}|^2 = 1$.

A pure state $|\Psi\rangle \in \mathcal{H}_{mn}$ is called separable if and only if it can be written as Kronecker product of states $|\phi_i\rangle = \sum_{i=1}^{n} \alpha_i |u_i\rangle \in \mathcal{H}_n$ and $|\chi_i\rangle = \sum_{j=1}^{m} \beta_j |v_j\rangle \in \mathcal{H}_m$

$$|\Psi\rangle = |\phi_i\rangle \otimes |\chi_j\rangle$$

otherwise the given state $|\Psi\rangle$ is entangled.

1.1.2 Entanglement of mixed states

Any pure state $|\Psi\rangle$ can be identified with the density operator expressed as $\rho = |\Psi\rangle\langle\Psi|$. Each density operator for pure state is a projection operator into one-dimensional space thus satisfies the property $\rho^2 = \rho$. Consider statistical mixture of pure states $\{\rho_i = |\Psi_i\rangle\langle\Psi_i|$, i = 1, 2, ..., n$\}$ with probabilities $\{p_i$, i = 1, 2, ..., n$\}$. Then the density operator for the system is expressed as

$$\rho = \sum_{i=1}^{n} p_i \rho_i$$

.

Let $\mathcal{H}_n$ and $\mathcal{H}_m$ be Hilbert spaces. Denote by ρ a density operator of state from $\mathcal{H}_n \otimes \mathcal{H}_m$. Operator ρ is called separable if there exist a sequence $\{p_i\}_{i=1}^n$ of positive real numbers summing to 1, a sequence density operators $\{\rho_i^n\}_{i=1}^n$ corresponding with states from $\mathcal{H}_n$ and a sequence density operators $\{\rho_j^m\}_{j=1}^m$ corresponding with states from $\mathcal{H}_m$ such that

$$\rho = \sum_{i=1}^{n} p_i \rho_i^n \otimes \rho_j^m$$

We can also say that, if the mixed state can be written as a convex combination of kronecker product of density operators then the state is separable.

1.2 Quantization of LC circuit

A well-known LC circuit consists of an inductor and a capacitor cause oscilla ons in the flux of the inductor and the charge of the capacitor. An LC circuit can be quantized using the same methods as for the quantum harmonic oscillator.

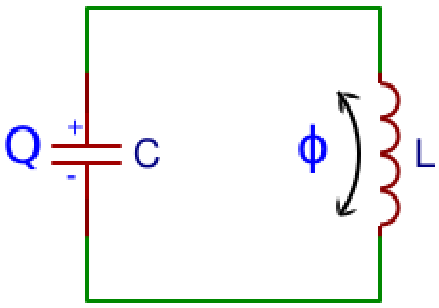

Figure 1.2: LC Circuit

Consider an inductor L and a capacitor C connected in parallel. The Hamiltonian for this circuit is

$$H = \frac{Q^2}{2C} + \frac{C\omega^2\Phi^2}{2} \tag{1.2}$$

where Φ is the flux passing through the inductor and Q is the charge across the capacitor. Take the flux Φ as the variable and write the Hamiltonian in the form

$$H = \frac{P_\Phi^2}{2M_\Phi} + \frac{M_\Phi\omega^2\Phi^2}{2} \tag{1.3}$$

where $\omega = \sqrt{\frac{1}{LC}}$ is the resonant frequency of the circuit.

Similar to one-dimensional harmonic oscillator problem, an LC circuit can be quantized by either solving the Schrödinger equation or using creation and annihilation operators. The energy stored in the inductor can be looked at as a "kinetic energy term" and the energy stored in the capacitor can be looked at as a "potential energy term". The first term represents the energy stored in a capacitor, and the second term represents the energy stored in an inductor. By analogy with the Simple Harmonic Oscillator, define the creation and annihilation operators such that

$$\hat{Q}_k = i\sqrt{\frac{2C\omega_k\hbar}{2}}(\hat{a}_k^\dagger - \hat{a}_k) \qquad \hat{\Phi}_k = \sqrt{\frac{C\omega_k\hbar}{2}}(\hat{a}_k^\dagger + \hat{a}_k)$$

Identifying $\dot{\Phi}C$ as the canonical momentum π and the coordinate as the flux Φ, we are in a position to write the quantum mechanical commutation relation:

$$[\hat{Q}, \hat{\Phi}] = -i\hbar \tag{1.4}$$

The coordinate representation of π is -$\iota\hbar\frac{d}{d\phi}$. Hence the Hamiltonian operator will read:

$$\hat{H} = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) \tag{1.5}$$

The eigen-solutions and energy eigenvalues for the above operator are given by

$$\Psi_n(\Phi) = \frac{1}{\sqrt{2^n n!}}\left(\frac{C\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{C\omega\Phi^2}{2\hbar}} H_n\left(\sqrt{\frac{C\omega}{\hbar}}\Phi\right) \qquad E = \hbar\omega\left(n + \frac{1}{2}\right) \tag{1.6}$$

1.3 Quantization of Two Inductively Coupled LC Circuits

Two inductively coupled LC circuits have a non-zero mutual inductance. This is equivalent to a pair of harmonic oscillators with a kinetic coupling term. These two inductively coupled LC circuits are shown in Figure 1.3, having 2 circuits gives 2 resonant frequencies whose separation depends on the value of the mutual inductance M.

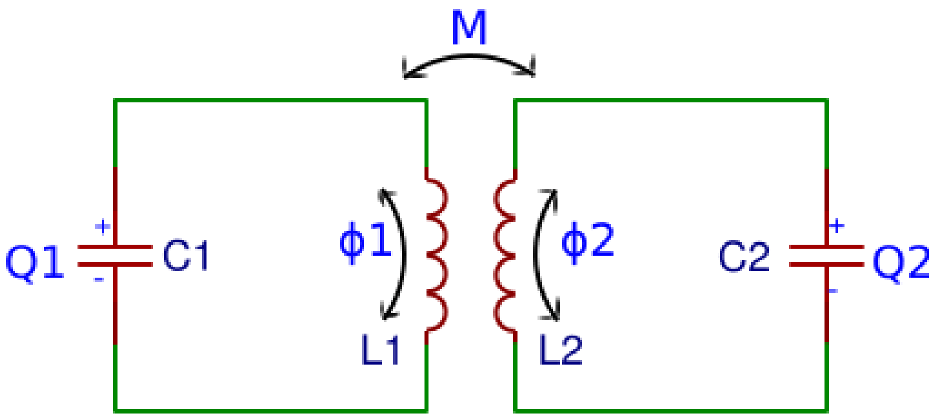

Figure 1.3: Coupled LC Circuits

The Lagrangian for an inductively coupled pair of LC circuits is as follows:

$$L = \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2} + \frac{M\Phi_1\Phi_2}{L_1L_2} - \frac{\Phi_1^2}{2L_1} - \frac{\Phi_2^2}{2L_2} \tag{1.7}$$

As usual, the Hamiltonian is obtained by a Legendre transform of the Lagrangian.

$$\hat{H} = \frac{\hat{Q}_1^2}{2C_1} + \frac{\hat{Q}_2^2}{2C_2} + \frac{\hat{\Phi}_1^2}{2L_1} + \frac{\hat{\Phi}_2^2}{2L_2} + \frac{M\hat{\Phi}_1\hat{\Phi}_2}{L_1L2} \tag{1.8}$$

It is very cumbersome to solve the Schrödinger equation directly so the best way to solve it by decoupling of Hamiltonian by following Transforma ons:

$$\begin{pmatrix} \hat{Q}_X \\ \hat{Q}_Y \end{pmatrix} = \begin{pmatrix} \left(\frac{C_2}{C_1}\right)^{\frac{1}{4}} \cos\beta & \left(\frac{C_1}{C_2}\right)^{\frac{1}{4}} \sin\beta \\ -\left(\frac{C_2}{C_1}\right)^{\frac{1}{4}} \sin\beta & \left(\frac{C_1}{C_2}\right)^{\frac{1}{4}} \cos\beta \end{pmatrix} \begin{pmatrix} \hat{Q}_1 \\ \hat{Q}_2 \end{pmatrix} \tag{1.9}$$

$$\begin{pmatrix} \hat{\Phi}_X \\ \hat{\Phi}_Y \end{pmatrix} = \begin{pmatrix} \left(\frac{C_1}{C_2}\right)^{\frac{1}{4}} \cos\beta & \left(\frac{C_2}{C_1}\right)^{\frac{1}{4}} \sin\beta \\ -\left(\frac{C_1}{C_2}\right)^{\frac{1}{4}} \sin\beta & \left(\frac{C_2}{C_1}\right)^{\frac{1}{4}} \cos\beta \end{pmatrix} \begin{pmatrix} \hat{\Phi}_1 \\ \hat{\Phi}_2 \end{pmatrix} \tag{1.10}$$

After diagonalisation the hamiltonian will read:

$$\hat{H} = \frac{\hat{Q}_X^2}{2C} + \frac{C\omega_X \hat{\Phi}_X^2}{2} + \frac{\hat{Q}_Y^2}{2C} + \frac{C\omega_Y \hat{\Phi}_Y^2}{2} \tag{1.11}$$

where:

$$C = \sqrt{C_1 C_2}, \quad \omega_X = \left[\omega_1^2 \cos^2\beta + \omega_2^2 \sin^2\beta + \tfrac{2M \sin\beta\cos\beta}{CL_1L_2}\right]^{\frac{1}{2}}$$

$$\omega_Y = \left[\omega_1^2 \sin^2\beta + \omega_2^2 \cos^2\beta - \tfrac{2M \sin\beta\cos\beta}{CL_1L_2}\right]^{\frac{1}{2}} \beta = \tfrac{1}{2}\tan^{-1}\left[\tfrac{2M}{CL_1L_2(\omega_1^2-\omega_2^2)}\right]$$

Schrödinger Equation is given by:

$$\hat{H}\hat{\Psi}(\Phi_X, \Phi_Y) = E\hat{\Psi}(\Phi_X, \Phi_Y)$$

$$\tfrac{-\hbar^2}{2C}\left[\tfrac{d^2}{d\Phi_X^2} + \tfrac{d^2}{d\Phi_Y^2}\right]\Psi(\Phi_X,\Phi_Y) + \left[\tfrac{C\omega_X^2\Phi_X^2}{2} + \tfrac{C\omega_X^2\Phi_X^2}{2}\right]\Psi(\Phi_X,\Phi_Y) = E\Psi(\Phi_X,\Phi_Y)$$

Ground state and first excited wave function are as follows:

$$\Psi_{00}(\Phi_X, \Phi_Y) = \sqrt{\frac{C}{\pi\hbar}}(\omega_X\omega_Y)^{\frac{1}{4}} e^{-\frac{C}{2\hbar}[\omega_X\Phi_X^2 + \omega_Y\Phi_Y^2]}$$

$$\Psi_{00}(\Phi_1, \Phi_2) = \sqrt{\frac{C}{\pi\hbar}}(\omega_X\omega_Y)^{\frac{1}{4}} e^{-\frac{C}{2\hbar}\left[\omega_X\left\{\left(\frac{C_1}{C_2}\right)^{\frac{1}{4}}\cos\beta\ \Phi_1 + \left(\frac{C_2}{C_1}\right)^{\frac{1}{4}}\sin\beta\ \Phi_2\right\}^2 + \omega_Y\left\{-\left(\frac{C_1}{C_2}\right)^{\frac{1}{4}}\sin\beta\ \Phi_1 + \left(\frac{C_2}{C_1}\right)^{\frac{1}{4}}\cos\beta\ \Phi_2\right\}^2\right]}$$

Similarly,

$$\Psi_{10}(\Phi_1, \Phi_2) = \sqrt{\frac{2\omega_X C}{\hbar}} \left(K_1\Phi_1 + K_2\Phi_2 \right) \sqrt{\frac{C}{\pi\hbar}} (\omega_X \omega_Y)^{\frac{1}{4}} e^{-\frac{C}{2\hbar}\left[\chi_1 + \chi_2\right]}$$

where $K_1 = \left(\frac{C_1}{C_2}\right)^{\frac{1}{4}} \cos\beta$, $\quad K_2 = \left(\frac{C_2}{C_1}\right)^{\frac{1}{4}} \sin\beta$, $\quad \chi_1 = \omega_X \big\{ \big(\frac{C_1}{C_2}\big)^{\frac{1}{4}} \cos\beta\ \Phi_1 + \big(\frac{C_2}{C_1}\big)^{\frac{1}{4}} \sin\beta\ \Phi_2 \big\}^2$

and $\chi_2 = \omega_Y \big\{ -\big(\frac{C_1}{C_2}\big)^{\frac{1}{4}} \sin\beta\ \Phi_1 + \big(\frac{C_2}{C_1}\big)^{\frac{1}{4}} \cos\beta\ \Phi_2 \big\}^2$

The wave function is separable in terms of Φ_X and Φ_Y variables. However, for the variables Φ_1 and Φ_2, the story is quite different, and can be extended to the issue of entanglement.

1.4 Entanglement Measurement

In this section, I am going to give a description of quantum entanglement within the mathematical framework of quantum mechanics and show how the notion of entropy is extended to entanglement entropy, which can be interpreted as a measure of quantum entanglement.

Before defining entanglement entropy, recall that the density matrix corresponding to a composite system consisting of subsystems A and B, can be traced over with respect to one of the subsystems, resulting in the reduced density matrix of the other subsystem, whereas the total density matrix describes a pure state, the reduced density matrix of either subsystem will, in the case that the two subsystems are entangled, be equivalent to the density matrix of a mixed state.

Consider an entangled state which is described by two particles with spin$\frac{1}{2}$, let's say $spin_A$ and $spin_B$, living in subspaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of the complete hilbert space. The state reads

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B) \tag{1.12}$$

Now by comparing this equation to the wavefunction of section 1.1.1 we can define a matrix M as

$$M = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} \tag{1.13}$$

By forming the density matrix $\rho = |\Psi\rangle\langle\Psi|$ and then and tracing out pin_B , we find the reduced density matrix ρ_A of $spin_A$, the elements of which are, in the basis of eigenstates corresponding to $spin_A$, given by the elements of the matrix $MM^\dagger$. This gives

$$\rho_A = MM^\dagger = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{1.14}$$

or it can be written in the form

$$\rho_A = \frac{1}{2}(|\uparrow\rangle_A \langle\uparrow| + |\downarrow\rangle_B \langle\downarrow|) \tag{1.15}$$

A density matrix of this form could never correspond to a pure quantum state, however it could definitely describe a mixture of two pure states, both with probability $\frac{1}{2}$. Here part of the information about subsystem A resides in the degrees of freedom of subsystem B which were traced out and are, therefore, no longer accessible.

Now define an entropy related to the mixed form of the reduced density matrix resulting from the quantum entanglement between different subsystems. It is defined in the same way as the von-Neumann entropy, but with the regular density matrix replaced by the reduced one. When a system in a pure state can be divided into two subsystems A and B that are entangled, the entanglement entropy can be expressed both in terms of ρ_A and ρ_B and is given by

$$S = -Tr(\rho_A \log \rho_A) = -Tr(\rho_B \log \rho_B) \tag{1.16}$$

It states that there can always be found a basis such that the reduced density matrices of both subsystems are diagonal in this basis and have the same eigenvalues. The entanglement entropy in terms of these eigenvalues λ_m simply reads

$$S = -\sum_{m=0}^{\infty} \lambda_m \log \lambda_m$$

Returning to the example, it has to be noted that reduced density matrix ρ_A is of the form of the density matrix which corresponds to a maximally mixed state. It follows that the entanglement entropy of singlet state is S = log 2 and we can infer that this is an example of a maximally entangled state. In general, entanglement entropy is measure for the amount of entanglement between two systems.

Now we are able to to derive the the entanglement entropy corresponding to a system of two coupled LC oscillators.

As a measurement of entanglement, entropy provides one tool that can be used to quantify entanglement, although other entanglement measures exist. Entanglement entropy measures the quantum information content of a quantum state.

The density matrix $\rho = |\Psi\rangle \langle\Psi|$ is symmetric, and we can simply proceed to trace out any one of the sub-systems to obtain the reduced density matrix of the other:

$$\rho_{00}(\Phi_1,\Phi_1^{'}) = \int_{-\infty}^{+\infty} d\Phi_2 \Psi_{00}^{*}(\Phi_1^{'},\Phi_2)\Psi_{00}(\Phi_1,\Phi_2)$$

$$\rho_{00}(\Phi_1,\Phi_1^{'}) = \sqrt{\frac{C}{\pi b\hbar}}(\omega_X\omega_Y)^{\frac{1}{2}} e^{-\frac{C}{2\hbar}\left[(a-\frac{d^2}{8b})(\Phi_1^2+\Phi_1^{'2})-\frac{d^2}{4b}\Phi_1\Phi_1^{'}\right]}$$

$$\rho_{00}(\Phi_1,\Phi_1^{'}) = \sqrt{\frac{C}{\pi b\hbar}}(\omega_X\omega_Y)^{\frac{1}{2}} e^{-[(a_1+a_2)(\Phi_1^2+\Phi_1^{'2})+2a_2\Phi_1\Phi_1^{'}]}$$

Similarly,

$$\rho_{10}(\Phi_1,\Phi_1^{'}) = \frac{2\omega_X C}{\hbar}\left[K_1^2\Phi_1\Phi_1^{'} + \frac{K_2 d}{4b}\left(\frac{K_2 d}{4b} - K_1\right)(\Phi_1+\Phi_1^{'})^2\right]\rho_{00}(\Phi_1,\Phi_1^{'})$$

where: $a = \sqrt{\frac{C_1}{C_2}}(\omega_X \cos^2\beta + \omega_Y \sin^2\beta)$, $b = \sqrt{\frac{C_2}{C_1}}(\omega_X \sin^2\beta + \omega_Y \cos^2\beta)$ and

$d = (\omega_X - \omega_Y)\sin(2\beta)$, $a_1 = \frac{Ca}{2\hbar}$, $a_2 = -\frac{Cd^2}{16b\hbar}$, $K_1 = (\frac{C_1}{C_2})^{\frac{1}{4}}\cos\beta$ and $K_2 = (\frac{C_2}{C_1})^{\frac{1}{4}}\sin\beta$

The entanglement entropies of the system in the von Neumann description are given below:

$$S = -\sum_{m=0}^{\infty} \lambda_m \log\lambda_m$$

$$S_{00} = -\log(1-\zeta) - \frac{\zeta}{1-\zeta}\log\zeta$$

$$S_{10} = -2\log\big(1-\chi^2\big) - \frac{2\chi^2}{1-\chi^2}\log\chi 2 - (1-\chi^2)^2\sum_{\alpha=0}^{\infty}\chi^{2\alpha}\log(\alpha+1)^{\alpha+1}$$

where: $\lambda_m = (1-t)t^m$, $\zeta = \frac{a_2}{a_1+a_2+\sqrt{(a_1+a_2)^2-a_2^2}}$, $\chi = \frac{\sqrt{\omega_Y}\cos\beta-\sqrt{\omega_X}\sin\beta}{\sqrt{\omega_Y}\cos\beta+\sqrt{\omega_X}\sin\beta}$

Following are the plots of the ground state entropy with different values of ζ

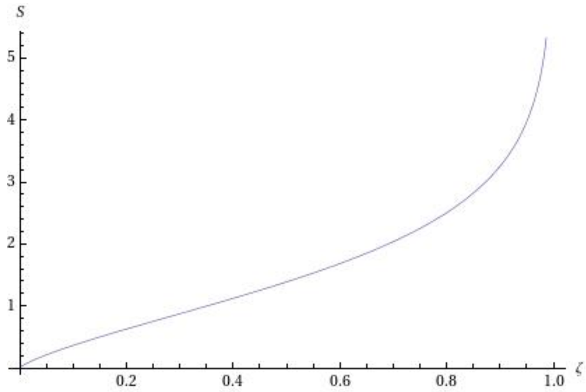

Figure 1.4: S_{00} $\zeta \to 0, 1$

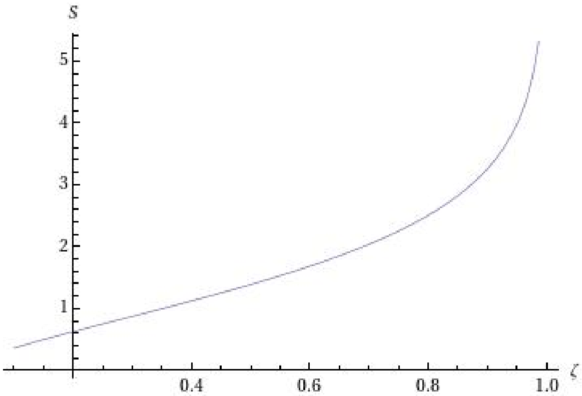

Figure 1.5: S_{00} $\zeta \to 0.1, 1$

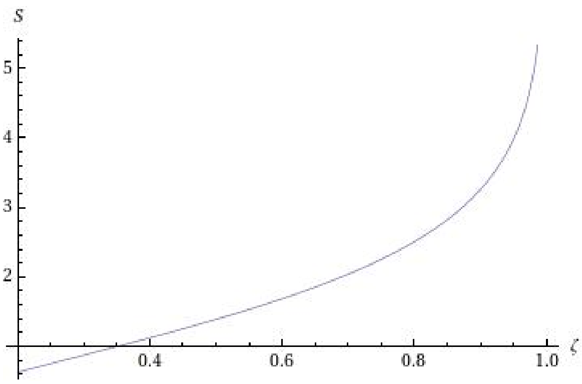

Figure 1.6: S_{00} $\zeta \to 0.2, 1$

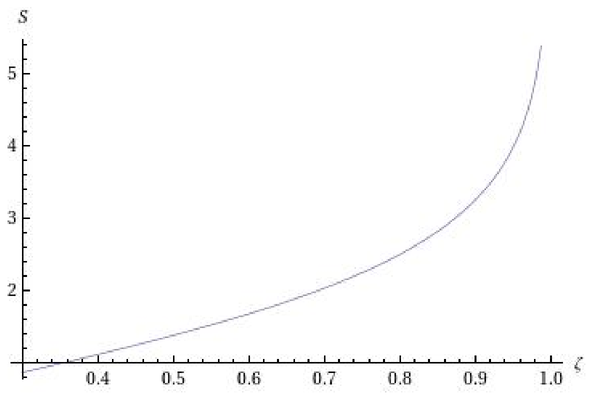

Figure 1.7: S_{00} $\zeta \to 0.3, 1$

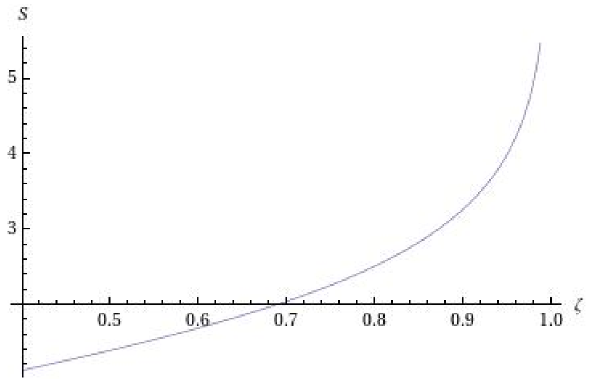

Figure 1.8: S_{00} $\zeta \to 0.4, 1$

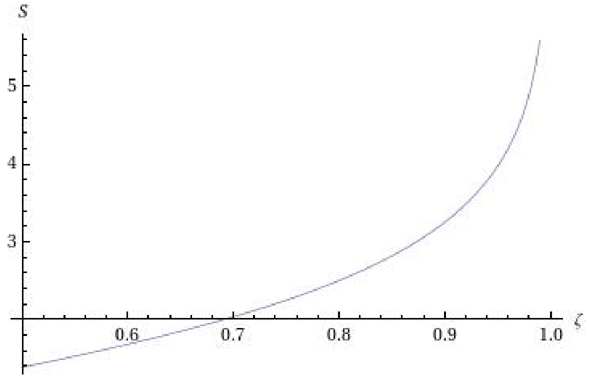

Figure 1.9: S_{00} $\zeta \to 0.5, 1$

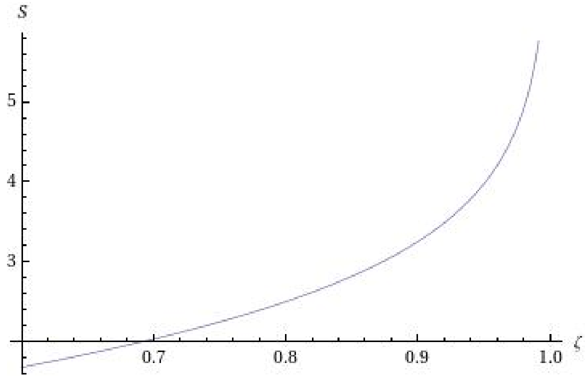

Figure 1.10: S_{00} $\zeta \to 0.6, 1$

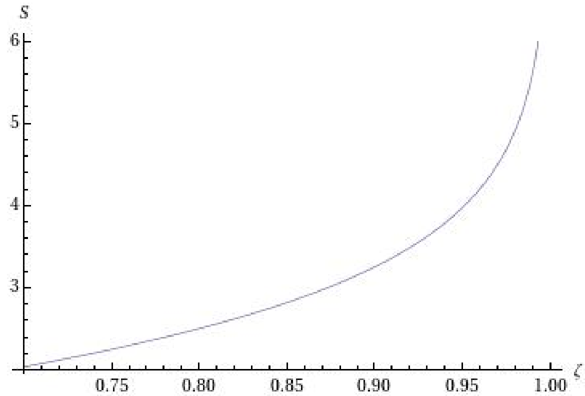

Figure 1.11: S_{00} $\zeta \to 0.7, 1$

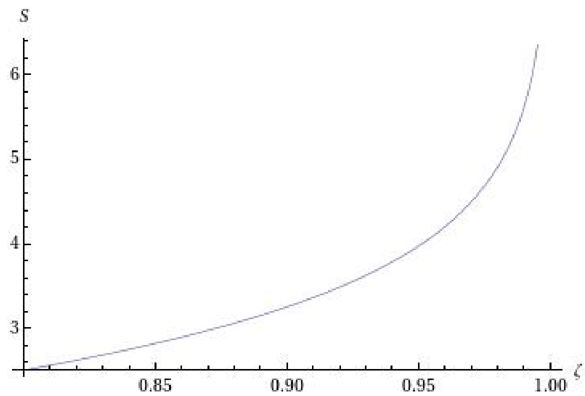

Figure 1.12: S_{00} $\zeta \to 0.8, 1$

Similarly, first excited state entropies are plotted

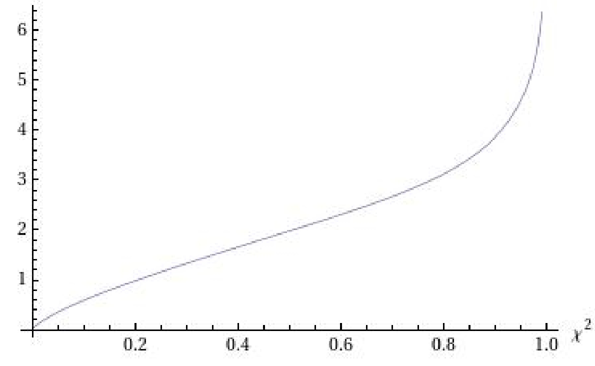

Figure 1.13: $S_{10}\quad \chi^2 \to 0, 1$

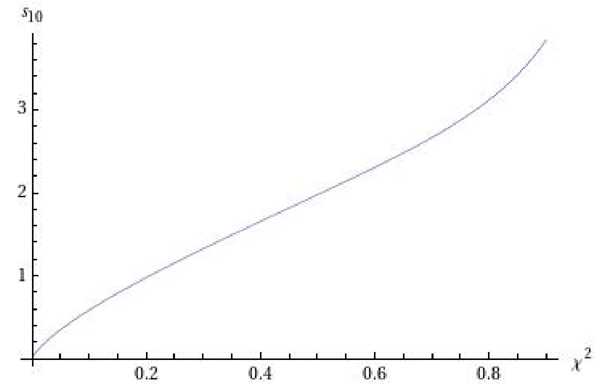

Figure 1.14: $S_{10}\quad \chi^2 \to 0, 0.9$

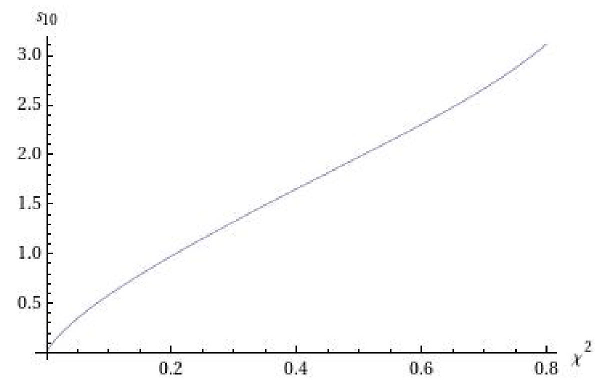

Figure 1.15: $S_{10}\quad \chi^2 \to 0, 0.8$

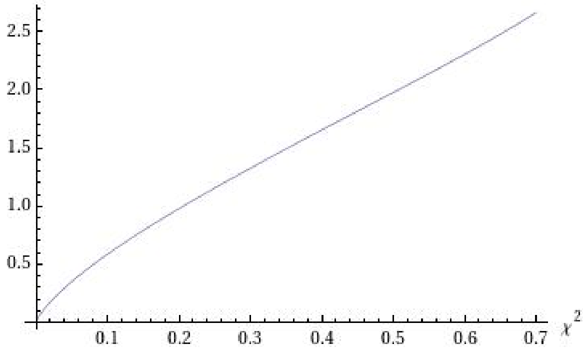

Figure 1.16: $S_{10} \quad \chi^2 \to 0, 0.7$

Plots of the result are shown in the above figure, where ground state is included and where S is displayed as a function of ζ and χ^2. It can be inferred that S increases for increasing χ^2. Moreover, it also increases for other levels. This can be explained by recalling the expression for χ and noting that χ is small when $\omega_X \approx \omega_Y$. This is the case when $M \approx 0$, hence when the coupling between the two original oscillators is small. This should intuitively make sense, since there is simply no entanglement between decoupled systems.

Chapter 2

Coupled ground and coherent state

2.1 Coherent States

Coherent state is the specific quantum state of the quantum harmonic oscillator, often described as a state which has dynamics most closely resembling the oscillatory behavior of a classical harmonic oscillator. It describes the oscillating motion of a particle confined in a quadratic potential well. The coherent state describes a state in a system for which the ground-state wavepacket is displaced from the origin of the system. This state can be related to classical solutions by a particle oscillating with an amplitude equivalent to the displacement.

2.1.1 Quantum Mechanical Definition

Mathematically, a coherent state$|\alpha\rangle$ is defined to be the (unique) eigenstate of the annihilation operator $\hat{a}$ associated to the eigenvalue α.

$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$

Since $\hat{a}$ is not hermitian, α is, in general, a complex number. Writing $\alpha = |\alpha|e^{i\theta}$, $|\alpha|$ and θ

are called the amplitude and phase of the state $|\alpha\rangle$. It means that a coherent state remains unchanged by the annihilation of field excitation or, say, a particle.

2.2 Oscillators in different states

Two coupled LC oscillators constitute an ideally simple model of quantum/classical coupling. First, the system separates into normal modes behaving as independent oscillators, so the evolution of the system from any initial data can be followed exactly. Second, the classical limit of a quantum oscillator is easily described by a coherent state, a Gaussian wave packet of fixed width, the centroid of which follows a classical trajectory. In the present work analysis of the behavior of coupled oscillators with one initially in its quantum ground state, the other initially in such a coherent state is done.

If the system is started with one oscillator in its quantum ground state and the other in a coherent state, for example, if the coupling between them is "turned on" at some initial time, then subsequently the two normal modes evolve one in a coherent state, the other in a modified state termed a displaced squeezed state, for example, in quantum optics. The initially quantum oscillator acquires an oscillating position expectation value—a "beat" between the normal modes.

The wave function for a state with one oscillator initially in its ground state, the other in a coherent state is constructed. From this expectation values of charge and flux are obtained for both the oscillators. Expectation values behave very like those in the identical-oscillator ("symmetric") case,. However, one simple case may be of particular physical importance: that of oscillators with different masses, but equal (uncoupled) frequencies, i.e., that of quantum and classical oscillators interacting "at resonance."

Construct a quantum oscillator coupled to a classical oscillator by a different choice of state:

envision oscillator 1 in its unperturbed ground state, and oscillator 2 in a coherent state with classical amplitude q_0, at time t = 0. Follow the subsequent evolution of this state, and determine the probability distributions, expectation values, for both the oscillators. In effect, oscillator 1 is in its ground state, oscillator 2 is behaving classically with oscillation amplitude q_0 , and the coupling "turns on" at time t = 0.

The Hamiltonian for this system can be written as

$$H = \frac{Q_1^2}{2C_1} + \frac{(Q_2 - q_0)^2}{2C_2} + \frac{\Phi_1^2}{2L_1} + \frac{\Phi_2^2}{2L_2} + \frac{M\Phi_1\Phi_2}{L_1 L2} \tag{2.1}$$

Using the following transformation to decouple the hamiltonian:

$$\begin{pmatrix} \hat{Q}_X \\ \hat{Q}_Y \end{pmatrix} = \begin{pmatrix} \left(\frac{C_2}{C_1}\right)^{\frac{1}{4}} \cos\beta & \left(\frac{C_1}{C_2}\right)^{\frac{1}{4}} \sin\beta \\ -\left(\frac{C_2}{C_1}\right)^{\frac{1}{4}} \sin\beta & \left(\frac{C_1}{C_2}\right)^{\frac{1}{4}} \cos\beta \end{pmatrix} \begin{pmatrix} \hat{Q}_1 \\ \hat{Q}_2 \end{pmatrix} \tag{2.2}$$

$$\begin{pmatrix} \hat{\Phi}_X \\ \hat{\Phi}_Y \end{pmatrix} = \begin{pmatrix} \left(\frac{C_1}{C_2}\right)^{\frac{1}{4}} \cos\beta & \left(\frac{C_2}{C_1}\right)^{\frac{1}{4}} \sin\beta \\ -\left(\frac{C_1}{C_2}\right)^{\frac{1}{4}} \sin\beta & \left(\frac{C_2}{C_1}\right)^{\frac{1}{4}} \cos\beta \end{pmatrix} \begin{pmatrix} \hat{\Phi}_1 \\ \hat{\Phi}_2 \end{pmatrix} \tag{2.3}$$

After diagonalisation the hamiltonian will read:

$$\hat{H} = \frac{\hat{Q}_X^2}{2C} + \frac{C\omega_X \hat{\Phi}_X^2}{2} + \frac{\hat{Q}_Y^2}{2C} + \frac{C\omega_Y \hat{\Phi}_Y^2}{2} \tag{2.4}$$

Define annihilation and creation operator:

$$a_k = \sqrt{\tfrac{C\omega_k}{2\hbar}}\Phi_k + i\sqrt{\tfrac{1}{2C\omega_k\hbar}}Q_k, \qquad a_k^\dagger = \sqrt{\tfrac{C\omega_k}{2\hbar}}\Phi_k - \sqrt{\tfrac{1}{2C\omega_k\hbar}}Q_k$$

And then define the modified annihilation and creation operators a_-, $a_-^\dagger$, a_+ and $a_+^\dagger$ by the equations:

$a_\pm = \frac{1}{\sqrt{2}}(a_1 \pm ia_2), \qquad a_\pm^\dagger = \frac{1}{\sqrt{2}}(a_1^\dagger \mp ia_2^\dagger)$

From the commutation relations among a_1, $a_1^\dagger$, a_2 and $a_2^\dagger$, it can be deduced that

$[a_+, a_+^\dagger] = [a_-, a_-^\dagger] = 1$

Coherent states in this case are the simultaneous eigen states of the two mutually commuting annihilation operators a_+ and a_-. Such a state can be written as

$$|\alpha_+\alpha_-\rangle = |\alpha_+\rangle \otimes |\alpha_+\rangle$$

where

$$a_+ |\alpha_+\rangle = \alpha_+ |\alpha_+\rangle \text{ and } a_- |\alpha_-\rangle = \alpha_- |\alpha_-\rangle$$

Since a_+ and a_- are not Hermitian operators, the eigen values $_+$ and $_-$ are in general complex numbers. Moreover, the eigen vector $|\alpha_+\rangle$ is a vector in the space spanned by the complete set of eigen vectors of $N_+ = a_+^\dagger a_+$. So,

$$|\alpha_+\rangle = \sum_{n_+=0}^{\infty} c_{n_+} |n_+\rangle$$

By applying the annihilation operator a + on both sides of the equation and using the fact that

$$a_+ |n_+\rangle = \sqrt{n_+} |n_+ - 1\rangle$$

it can be shown that

$$c_{n_+} = \frac{\alpha_+^{n_+}}{\sqrt{n_+!}} c_0 \text{ and } |n_+\rangle = \frac{(a_+^\dagger)^{n_+}}{\sqrt{n_+!}}$$

Assuming $|\alpha_+\rangle$ is normalised,

$$c_0 = e^{-\frac{1}{2}|\alpha_+|^2}$$

Now, in order to derive the state vector, assume that at t=0,

$$|\Psi(0)\rangle = |\alpha_+\alpha_-\rangle$$

$$= e^{-\frac{1}{2}|\alpha_+|^2} e^{\alpha_+ a_+} |0_+\rangle \otimes |\alpha_-\rangle$$

$$= e^{-\frac{1}{2}|\alpha_+|^2} \sum_{n_+=0}^{\infty} \frac{(\alpha_+ a_+^\dagger)^{n_+}}{n_+!} |0_+\rangle \otimes |\alpha_-\rangle$$

$$= e^{-\frac{1}{2}|\alpha_+|^2} \sum_{n_+=0}^{\infty} \frac{\alpha_+^{n_+}}{n_+!} |n_+\rangle \otimes |\alpha_-\rangle$$

$$= e^{-\frac{1}{2}|\alpha_+|^2} e^{\alpha_+ a_+} |0_+\rangle \otimes D_-(\alpha_-) |0_-\rangle$$

So, it can be written as

$$|\Psi(0)\rangle = D_+(\alpha_+) |0_+\rangle \otimes D_-(\alpha_-) |0_-\rangle$$

where

$$D_+(\alpha_+) = e^{\alpha_+ a_+^\dagger - \alpha_+^* a_+}, \qquad D_-(\alpha_-) = e^{\alpha_- a_-^\dagger - \alpha_-^* a_-}$$

Use the following equation in writimg displacement operator(D_+, D_-)

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]}$$

The charge coordinate space representation of $|\Psi(0)\rangle$ is given as

$$\Psi(Q_x, Q_y, 0) = \langle Q_x Q_y | \Psi(0)\rangle = \langle Q_x Q_y | e^{\alpha_+ a_+^\dagger - \alpha_+^* a_+ - \alpha_- a_-^\dagger - \alpha_-^* a_-} |\Psi(0)\rangle$$

After solving,

$$\Psi(Q_x, Q_y, 0) = \left(\frac{C^2 \omega_1 \omega_2}{\pi^2 \hbar^2}\right)^{\frac{1}{4}}$$

$$e^{-\frac{C}{2\hbar}\left[(\omega_1 \cos^2\beta + \omega_2 \sin^2\beta) Q_x^2 + (\omega_1 \sin^2\beta + \omega_2 \cos^2\beta) Q_y^2 + (\omega_2 - \omega_1)\sin(2\beta) Q_x Q_y - 2\omega_2 Q_0 (Q_x \sin\beta + Q_y \cos\beta) + \omega_2 Q_0^2\right]}$$

The time-dependent wave function for this state is again obtained by applying separate harmonic oscillator propagators G_{q_x} and G_{q_y} for the two normal modes.

$$G_{q_x}G_{q_y}\Psi(Q_x,Q_y,0)=\left(\frac{C^2\omega_1\omega_2}{\pi^2\hbar^2}\right)^{\frac{1}{4}}\left(\frac{C^2\omega_{q_x}\omega_{q_y}}{4i^2\pi^2\hbar^2\sin(\omega_{q_x}t)\sin\left(\omega_{q_y}t\right)}\right)^{\frac{1}{2}}e^{-\frac{C\omega_2^2}{2\hbar}Q_0^2}e^{\frac{iC\omega_{q_x}\cot(\omega_{q_x}t)}{2\hbar}q_x^2}e^{\frac{iC\omega_{q_y}\cot\left(\omega_{q_y}t\right)}{2\hbar}q_y^2}$$

$$\exp\Bigg[-\frac{C}{2\hbar}\big[(\omega_1\cos^2\beta+\omega_2\sin^2\beta-i\omega_{q_x}\cot\omega_{q_x}t)Q_x^2+(\omega_1\sin^2\beta+\omega_2\cos^2\beta-i\omega_{q_y}\cot\omega_{q_y}t)Q_y^2$$

$$+(\omega_2-\omega_1)\sin(2\beta)Q_xQ_y-2\big(\omega_2Q_0\sin\beta-i\omega_{q_x}\tfrac{q_x}{\sin(\omega_{q_x}t)}\big)-2\big(\omega_2Q_0\cos\beta-i\omega_{q_y}\tfrac{q_y}{\sin\left(\omega_{q_y}t\right)}\big)\big]\Bigg]$$

$$G_{q_x}G_{q_y}\Psi(Q_x,Q_y,0)=\left(\tfrac{C^2\omega_1\omega_2}{\pi^2\hbar^2}\right)^{\frac{1}{4}}\left(\tfrac{C^2\omega_{q_x}\omega_{q_y}}{4i^2\pi^2\hbar^2\sin(\omega_{q_x}t)\sin\left(\omega_{q_y}t\right)}\right)^{\frac{1}{2}}e^{-\frac{C\omega_2^2}{2\hbar}Q_0^2}e^{\frac{iC}{2\hbar}[Y^TWY]}e^{-\frac{C}{2\hbar}[X^TBX-A^TX-X^TA]}$$

where

$$X=\begin{pmatrix}Q_x\\Q_y\end{pmatrix}\quad Y=\begin{pmatrix}q_x\\q_y\end{pmatrix}\quad W=\begin{pmatrix}\omega_{q_x}\cot(\omega_{q_x}t) & 0\\ 0 & \omega_{q_y}\cot\left(\omega_{q_y}t\right)\end{pmatrix}\quad A=\begin{pmatrix}\omega_2Q_0\sin\beta-i\omega_{q_x}\frac{q_x}{\sin(\omega_{q_x}t)}\\ \omega_2Q_0\cos\beta-i\omega_{q_y}\frac{q_y}{\sin\left(\omega_{q_y}t\right)}\end{pmatrix}$$

$$B=\begin{pmatrix}\omega_1\cos^2\beta+\omega_2\sin^2\beta-i\omega_{q_x}\cot\omega_{q_x}t & \frac{1}{2}(\omega_2-\omega_1)\sin2\beta\\ \frac{1}{2}(\omega_2-\omega_1)\sin2\beta & \omega_1\sin^2\beta+\omega_2\cos^2\beta-i\omega_{q_y}\cot\omega_{q_y}t\end{pmatrix}$$

Hence, the propagation integral over Q_x and Q_y is a two-dimensional Gaussian integral of the form

$$I=\int e^{-\frac{C}{2\hbar}[X^TBX-A^TX-X^TA]}d^2Q$$

Let $X=Z+B^{-1}A$

$$=\int e^{-\frac{C}{2\hbar}[Z^TBZ-A^TB^{-1}A]}d^2Z$$

$$=e^{\frac{C}{2\hbar}[A^TB^{-1}A]}\int e^{-\frac{C}{2\hbar}[Z^TBZ]}d^2Z$$

$$=\frac{2\pi\hbar}{C|B|}e^{\frac{C}{2\hbar}[A^TB^{-1}A]}$$

The wave function is then

$$\Psi(Y,t) = \left(\frac{C^2\omega_1\omega_2}{\pi^2\hbar^2}\right)^{\frac{1}{4}} \left(\frac{\omega_{q_x}\omega_{q_y}}{i^2\sin(\omega_{q_x}t)\sin\left(\omega_{q_y}t\right)|B|}\right)^{\frac{1}{2}} e^{-\frac{C\omega_2^2}{2\hbar}Q_0^2} e^{\frac{C}{2\hbar}[A^TB^{-1}A+iY^TWY]}$$

A can also be written as

$$A = -iJ(Y+iK) \qquad \text{where } J = \begin{pmatrix} \frac{\omega_{q_x}}{\sin(\omega_{q_x}t)} & 0 \\ 0 & \frac{\omega_{q_y}}{\sin\left(\omega_{q_y}t\right)} \end{pmatrix} \text{and } K = \begin{pmatrix} \frac{\omega_2}{\omega_{q_x}}Q_0\sin(\beta)\sin(\omega_{q_x}t) \\ \frac{\omega_2}{\omega_{q_y}}Q_0\cos(\beta)\sin\left(\omega_{q_y}t\right) \end{pmatrix}$$

Hence,

$$\Psi(Y,t) = \left(\frac{C^2\omega_1\omega_2}{\pi^2\hbar^2}\right)^{\frac{1}{4}} \left(\frac{\omega_{q_x}\omega_{q_y}}{i^2\sin(\omega_{q_x}t)\sin\left(\omega_{q_y}t\right)|B|}\right)^{\frac{1}{2}} e^{-\frac{C\omega_2^2}{2\hbar}Q_0^2} e^{-\frac{C}{2\hbar}[(Y^T+iK^T)JB^{-1}J(Y+iK)-iY^TWY]}$$

To detemine the expectation values, separate the argument of Ψ into real and imaginary parts:

Let $JB^{-1}J = M + iN$

$$\Psi(Y,t) = \left(\frac{C^2\omega_1\omega_2}{\pi^2\hbar^2}\right)^{\frac{1}{4}} \left(\frac{\omega_{q_x}\omega_{q_y}}{i^2\sin(\omega_{q_x}t)\sin\left(\omega_{q_y}t\right)|B|}\right)^{\frac{1}{2}}$$

$$e^{-\frac{C\omega_2^2}{2\hbar}Q_0^2} e^{-\frac{C}{2\hbar}[Y^TMY-K^TNY-Y^TNK-K^TMK+iY^T(N-W)Y+i(K^TMY+Y^TMK)-iK^TNK]}$$

Since the matrix M contains no dependence on the classical amplitude Q_0, normalization of the wave function implies

$$\omega_2^2Q_0^2 = K^T(NM^{-1}N + M)K \quad \textit{and} \quad |M| = \frac{\omega_1\omega_2\omega_{q_x}^2\omega_{q_y}^2}{\sin^2(\omega_{q_x}t)\sin^2\left(\omega_{q_y}t\right)|B|^2}$$

It reduces the wavefunction to the form

$$\Psi(Y,t)=\left(\frac{C^2\omega_1\omega_2}{\pi^2\hbar^2}\right)^{\frac{1}{4}}\left(\frac{\omega_{q_x}\omega_{q_y}}{i^2\sin(\omega_{q_x}t)\sin\big(\omega_{q_y}t\big)|B|}\right)^{\frac{1}{2}}$$

$$e^{\frac{iC}{2\hbar}[K^TNK-(K^TMY+YMK^T)-Y^T(N-W)Y]}\; e^{\frac{C}{2\hbar}[(Y^T-K^TNM^{-1})M(Y-M^{-1}NK)]}$$

The wave function is finite or regular for all Y and t 0; the components of M , M^{-1} , $(V-W)$, and NK contain no vanishing denominators or divergent circular functions.

The probability distribution is given by:

$$P(Y,t)=|\Psi(Y,t)|^2$$

$$=\left(\frac{C^2|M|}{\pi^2\hbar^2}\right)^{\frac{1}{2}}e^{\frac{-C}{\hbar}[(Y-M^{-1}NK)^TM(Y-M^{-1}NK)]}$$

To obtain the reduced probability distribution for the single oscillator coosdinate Q_1, the inverse transformation is needed:

$$Let \quad = OQ$$

$$Y=\begin{pmatrix}\left(\frac{C_2}{C_1}\right)^{\frac{1}{4}}\cos\beta & \left(\frac{C_1}{C_2}\right)^{\frac{1}{4}}\sin\beta\\ -\left(\frac{C_2}{C_1}\right)^{\frac{1}{4}}\sin\beta & \left(\frac{C_1}{C_2}\right)^{\frac{1}{4}}\cos\beta\end{pmatrix}\begin{pmatrix}Q_1\\ Q_2\end{pmatrix}$$

Probability distribution now can be expressed as:

$$P(Q,t)=\left(\frac{C^2|M|}{\pi^2\hbar^2}\right)^{\frac{1}{2}}e^{\frac{-C}{\hbar}[(Q-O^{-1}M^{-1}NK)^T(O^TMO)(Q-O^{-1}M^{-1}NK)]}$$

Reduction to $P(Q_1,t)$:

$$P(Q_1,t)=\left(\frac{C^2|M|}{\pi^2\hbar^2}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}e^{\frac{-C}{\hbar}[(Q-O^{-1}M^{-1}NK)^T(O^TMO)(Q-O^{-1}M^{-1}NK)]}dQ_2$$

Hence integrating $P(Q,t)$ over Q_2 yields

$$P(Q_1,t) = \left(\frac{C^2|M|}{\pi^2\hbar^2}\right)^{\frac{1}{2}} e^{-\frac{C|M|}{\hbar(O^tMO)}[Q_1-(O^{-1}M^{-1}NK)^2]}$$

Moreover, the above distribution is normalized and gaussian distribution for Q_1.

The expectation value for the position of oscillator 1 in this state can be read off of $P(Q_1,t)$.

$$\langle Q_1\rangle = \left(\frac{C^2|M|}{\pi^2\hbar^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{C|M|}{\hbar(O^tMO)}[Q_1-(O^{-1}M^{-1}NK)^2]}$$

we know that

$$\int_{\infty}^{\infty} xe^{-(x-a)^2}dx = \int_{\infty}^{\infty}(a+t)e^{-t^2}dt$$

where $x-a=t$

$$= a\int_{\infty}^{\infty} e^{-t^2}dt + \int_{\infty}^{\infty} te^{-t^2}dt$$

$$= a\sqrt{\pi}$$

After solving:

$$\langle Q_1\rangle = (O^{-1}M^{-1}NK)$$

$$\langle Q_1\rangle = Q_0\sin(\beta)\cos(\beta)[\cos(\omega_x t) - \cos(\omega_y t)]$$

Similarly, it can be solved for oscillator 2

$$\langle Q_2\rangle = Q_0[\sin^2(\beta)\cos(\omega_x t) + \cos^2(\beta)\cos(\omega_y t)]$$

Hence, the charge and flux coordinates are given by

$$\langle Q \rangle = \begin{pmatrix} Q_0 \sin(\beta)\cos(\beta)[\cos(\omega_x t) - \cos(\omega_y t)] \\ Q_0[\sin^2(\beta)\cos(\omega_x t) + \cos^2(\beta)\cos(\omega_y t)] \end{pmatrix}$$

$$\langle \Phi \rangle = \begin{pmatrix} \frac{-Q_0^2}{2L_1\omega_1^2}\sin(2\beta)[\omega_x \sin(\omega_x t) - \omega_y \sin(\omega_y t)] \\ \frac{-Q_0^2}{2L_2\omega_2^2}[\omega_x \sin^2(\beta)\sin(\omega_x t) - \omega_y \cos^2(\beta)\sin(\omega_y t)] \end{pmatrix}$$

Following are the plots of the expectation values:

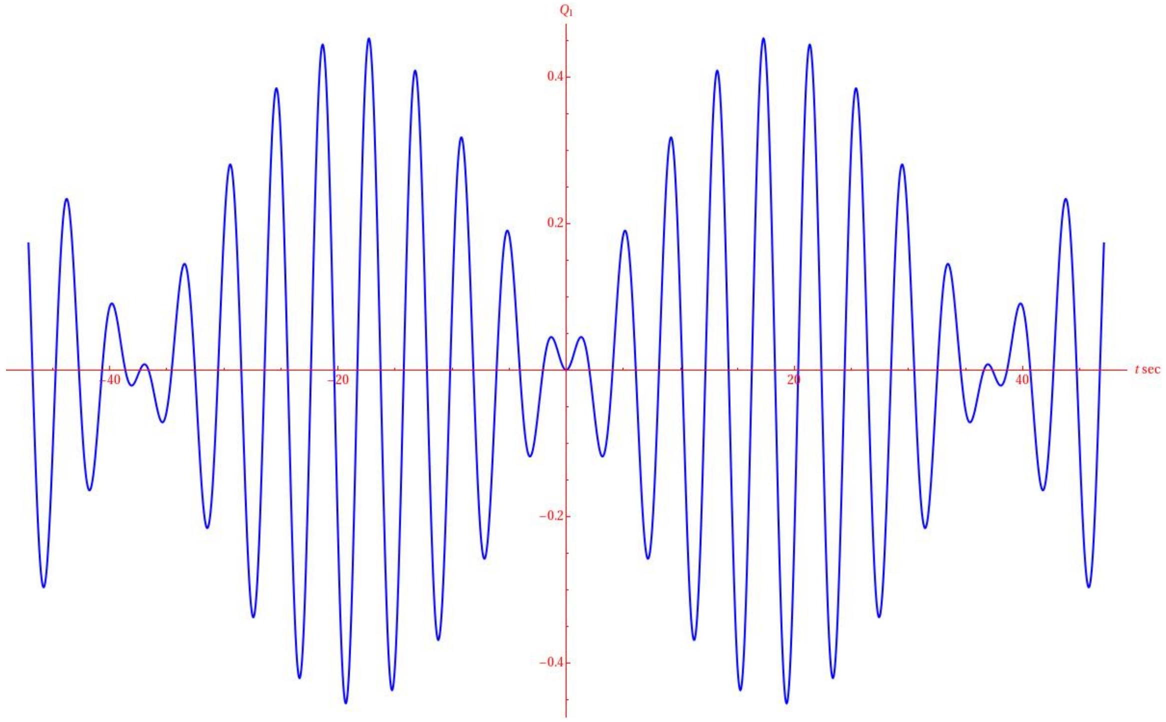

Figure 2.1: $\langle Q_1 \rangle$

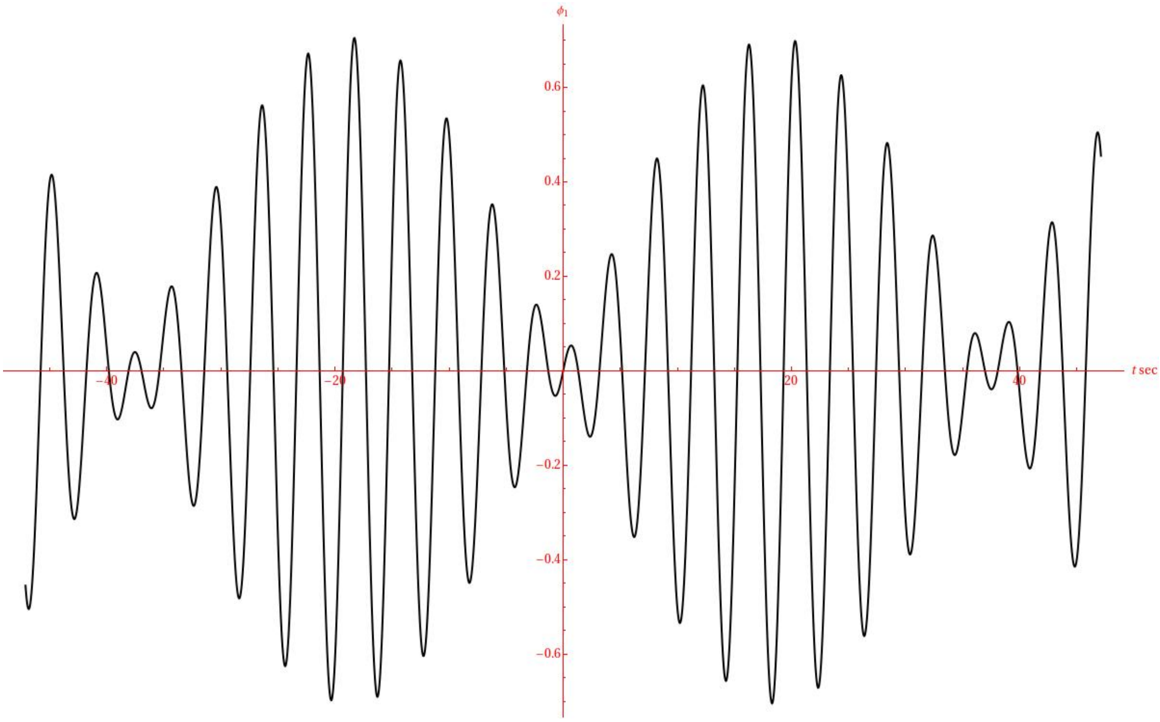

Figure 2.2: $\langle \Phi_1 \rangle$

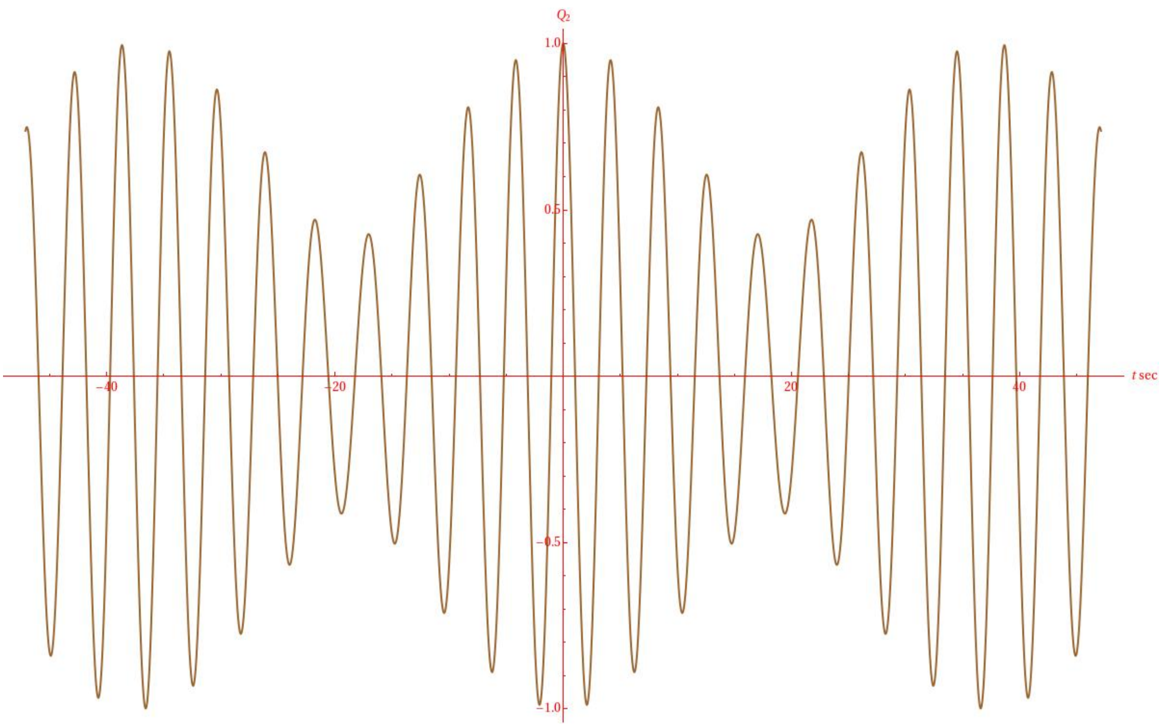

Figure 2.3: $\langle Q_2 \rangle$

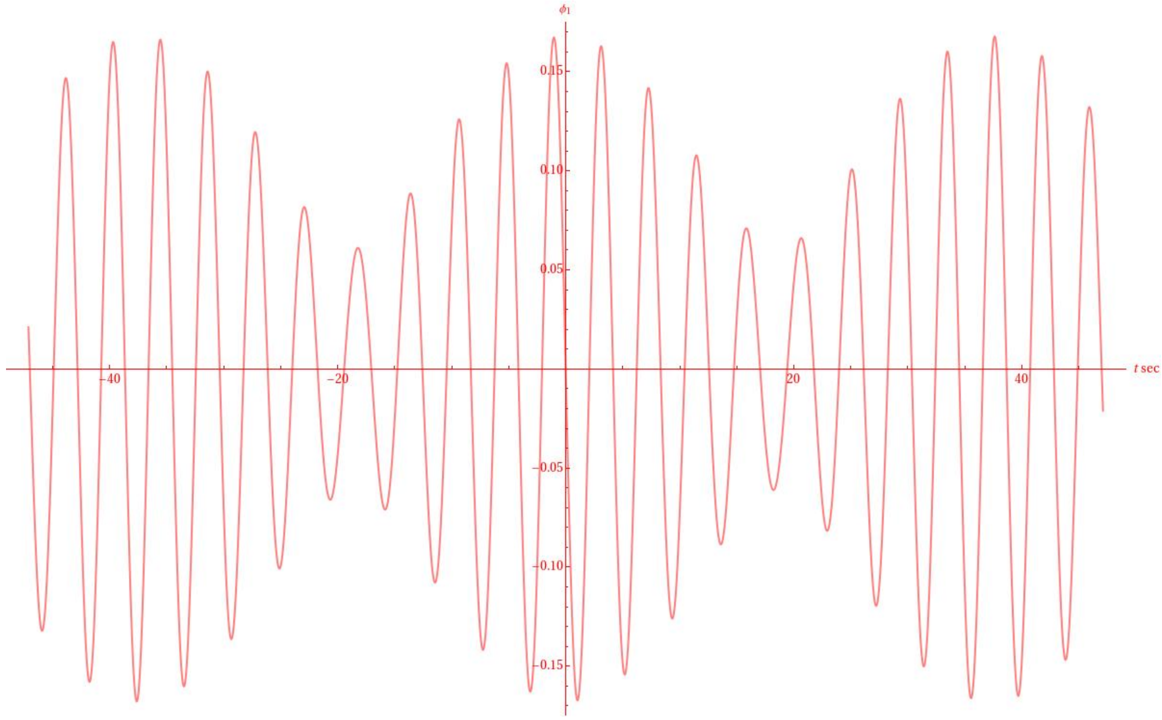

Figure 2.4: $\langle \Phi_2 \rangle$

2.3 Realization of these circuits

Coupling qubits is essential for implementing 2-qubit gates. Coupling two qubits may be achieved by connecting them to an intermediate electrical coupling circuit. The circuit might be a fixed element, such as a capacitor, or controllable, such as a DC-SQUID. In the first case, decoupling the qubits (during the time the gate is off) is achieved by tuning the qubits out of resonance one from another, i.e. making the energy gaps between their computational states different. This approach is inherently limited to allow nearest-neighbor coupling only, as a physical electrical circuit is to be lay out in between the connected qubits. Another method of coupling two or more qubits is by coupling them to an intermediate quantum bus. The quantum bus is often implemented as a microwave cavity, modeled by a quantum harmonic oscillator. Coupled qubits may be brought in and out of resonance with the bus and one with the other, hence eliminating the nearest-neighbor limitation. The formalism used to describe this coupling is cavity quantum electrodynamics, where qubits are analogous to atoms interacting with optical photon cavity, with the difference of GHz rather than THz regime of the electromagnetic radiation.

Under some approximations coupled rf-Squid collapses to coupled LC oscillators, it can be deduced that coupled LC oscillators is a particualr case of coupled rf-Squid.

In superconducting quantum computing, flux qubits are loops of superconducting metal interrupted by a number of Josephson junctions, functioning as quantum bits. The junction parameters are engineered during fabrication so that a persistent current will flow continuously when an external magnetic flux is applied. As only an integer number of flux quanta are allowed to penetrate the superconducting ring, clockwise or counter-clockwise currents are developed in the loop to compensate (screen or enhance) a non-integer external flux bias. When the applied flux through the loop area is close to a half integer number of flux quanta, the two lowest energy eigenstates of the loop will be a quantum superposition of the clockwise and counter-clockwise currents. The two lowest energy eigenstates differ only by the relative quantum phase between the composing current-direction states. Higher energy eigenstates correspond to much larger persistent currents, that induce an additional flux quantum to the qubit loop, thus are well separated energetically from the lowest two eigenstates. This separation, known as the ”qubit non linearity” criteria, allows operations with the two lowest eigenstates only, effectively creating a two level system.

Chapter 3

Summary & Conclusions

3.1 Concluding Remarks

In chapter 1 entanglement entropy in a system of coupled harmonic oscillators is examined. After reviewing the phenomenon of quantum entanglement and the associated entanglement entropy within a quantum mechanical framework, it has been shown that how the entanglement entropy between the oscillators in the ground state of the composite system could be derived by integrating the density matrix over the position coordinates of one of the oscillators and determining the eigenvalues of the resulting reduced density matrix. The entanglement entropy S is expressed as a function of the parameter ζ and χ, which, in turn, solely depended on the ratio between the two eigenfrequencies of the system in such a way that it increased for larger coupling between the two original oscillators.

The obtained expression for the ground state is such that it could be acted on by creation operators to form excited states. Excited states are considered that are linear combinations of the Hamiltonian eigenstates, for these turned out to have an interesting and elegant form in the basis which were studied. More precisely, acted with a creation operator that caused only the states forming the basis in one of the two subspaces, to be excited to the next level. The corresponding reduced density matrix could be evaluated analytically in diagonal form, from entanglement entropy S is evaluated in the first excited state. S turned out to increase with χ and its curve depicting the χ , hence on the coupling strength, remained of the same form.

In chapter 2, the interaction of two quantum LC oscillators are studied with one oscillator initially in its unperturbed ground state, the other initially in a coherent state incorporating classical behavior. The subsequent evolution of the wave function is calculated exactly, using ordinary harmonic-oscillator propagators for the normal modes of the system. The reduced probability distribution for the charge of the initially quantum oscillator—a Gaussian distribution with time-dependent expectation value, its charge and flux expectation values, all follow from this wave function. The expectation values can be characterized as a "beat" amplitude between the normal modes. The quantum character of the oscillator, can be quite complicated, for oscillators with equal unperturbed frequencies, for example, at resonance, this behavior can be described as a time-dependent quantum squeezing.

3.2 Future Outlook

In this thesis entanglement entropy(ground, first excited state), oscillators in two different states are explored but there are other physical factors which could be explored exactly and analytically such as entanglement entropy for different states, decoherence etc. The corresponding reduced density matrix could be evaluated analytically in diagonal form, from which we numerically evaluated entanglement entropy S in the excited states. Further work could examine how entanglement entropy S would behave for excited eigenstates. How the derivation of the entanglement entropy in charge(momentum) space seems to be less tedious than that in Fock space. How the loss in the system can lead to the decoherence? how the derivation of the entanglement entropy in position space seems to be less tedious than that in Fock space.

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