Project Appraisal and Option Pricing using Binomial and Black-Scholes-Merton Model

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Certificate of Examination

This is to certify that the dissertation titled **"Project Appraisal and Option Pricing using Binomial and Black-Scholes-Merton Model"** submitted by **Rudra Shekhar** (Reg. No. MS13122) for the partial fulfillment of BS-MS dual degree programme of the Indian Institute of Science Education and Research Mohali, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated : November 30, 2018

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. Kanchan Jain of Panjab university and Dr. Lingaraj Sahu at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgment of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Kanchan Jain

(Thesis Supervisor)

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Abstract

This exposition is the result of a year's study of options and financial derivatives. Derivative trading is an integral part of Indian stock market and with rise in trading volume of stock options, option price calculation has become very significant. In this context Black-Scholes pricing model is used to determine option premium. At fist I have introduced financial market and covered project appraisal. After providing brief summary of options, I have explained binomial option pricing model and then moved to Black-Scholes-Merton model of option pricing. Stochastic calculus is used to develop Black-Scholes differential equation. After this I have outlined all the Greeks present in the model and their interpretation is given to develop trading strategies in Option market. At the end I have tried to establish relevance of Black-Scholes Model in Indian stock market by comparing actual option prices with price calculated from the model. The reason for inconsistency in the result from model is outlined and future aspects of its improvement are discussed.

Chapter 1

Developing Financial Insights

Study of finance acknowledges the pathway of money, its forms and derivatives and observes its route during which it grows and decays and classifies its behaviour accounting for a particular pattern. It broadly encompasses two aspects - how to acquire needed funds and how to efficiently manage them. Financial system derives its concepts from the theories of micro and macro economics.

Financial management refers to planning, controlling, organizing and directing financial activities of an organization. It basically deals with efficient and effective management of funds. Whereas, financial mathematics is about use of mathematical methods, drawing tools from probability, statistics, stochastic process and economic theory to financial problems with an aim to make good decisions in the face of uncertainty based on analyzing financial consequences of risk.

1.1 Cashflows

Cashflow Model - It describes ways of money transaction. Income of money or payments received are called positive cashflow and outgo of money or payments made are termed as negative cashflow. Net cashflow at a given time is difference of income and outgo of money.

Interest Rates - Time value of money suggests that a rupee today is not a rupee tomorrow. Interest rates are studied in time value where capital invested is called principal, denoted by P and total received by the lender after fixed period of time is accumulated value and their difference is called as interest. It is of two types: simple interest and compound interest and accumulated value for both the cases is given by Eq 1.1 and 1.2 respectively:

$$P(1+rt) \tag{1.1}$$

$$C(1+r)^t \tag{1.2}$$

Here, P is the amount deposited for t years at respective rate of interests r.

Present Value (PV) - Present Value or the discounted value of the payment P due at time t is given by:

$$\frac{C}{(1+r)^t} \tag{1.3}$$

It means that an investment of amount given by Eq 1.3, at time 0 will give amount P at time $t > 0$. Effective rates of interest are compound rates that pay interest once per unit time either at the end (in the beginning) while in nominal rate, interest is paid more frequently than once per unit time.

We say nominal rate of interest where interest is paid more frequently than once per unit time. Nominal rate of interest payable p times per period is denoted by $r^{(p)}$. It is also referred as interest rate payable $p$thly or compounded $p$thly. For example, a nominal rate of interest of 8% *pa* convertible quarterly means an interest rate of 2% per quarter. Effective rate (r) is obtained by:

$$1+r = (1+\frac{r^{(p)}}{p})^p \tag{1.4}$$

Force of Interest - Force of interest models continuously paid interest where we consider nominal rate convertible very frequently that is in dt time using,

$$r^{(p)} = p[(1+r)^{1/p} - 1] \tag{1.5}$$

We see that for a fixed value of r if we let p increase, then it approaches a limit and that is called force of interest.

$$\lim_{p\to\infty} r^{(p)} = \delta \tag{1.6}$$

Using Eq 1.5 with an effective rate of 5% pa we get nominal rate of interest convertible $p$thly (eg $r^{(2)} = 0.04949$, $r^{(4)} = 0.04909$, etc.). If we let value of p increase, we obtain the following graph:

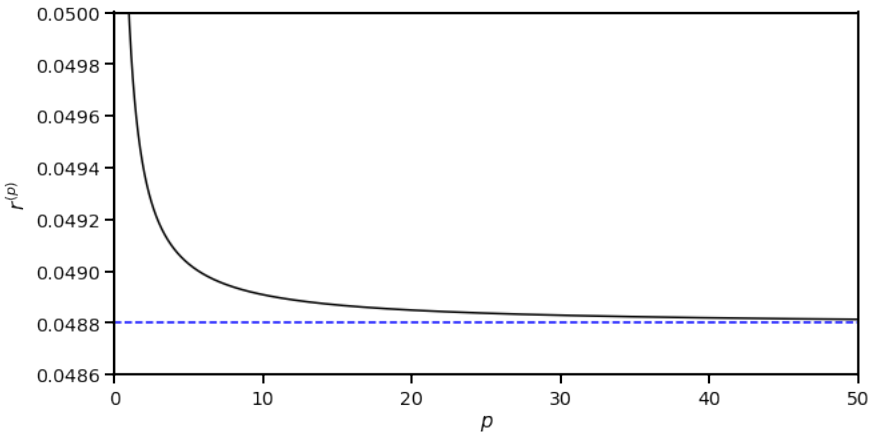


Figure 1.1: Force of Interest

Thus δ is the nominal rate of interest per unit time convertible continuously. Using Euler's law, we find that $1+r=e$.

Accumulation using force of interest are given by:

$$A(n) = e^{\delta t} \tag{1.7}$$

Accumulation and Discounting - Accumulation and Discounting introduced so far were for single payments and now we look at series of payments. For discrete payments, the present value of a series of payments of $p_{t_1}, p_{t_2}...p_{t_n}$ at times $t_1, t_2...t_n$ is given by:

$$\sum_{n}^{j=1} p_{t_j} v^{t_j} \tag{1.8}$$

where, $v = \frac{1}{1+r}$.

For continuous cashflow, let the rate of payment at time t is given by $\rho(t) = M'(t)$ where $M(t)$ is total payment between time 0 and t. PV of continuous cash flow is obtained by:

$$\int_0^T v(t)\rho(t)dt \tag{1.9}$$

where T is the time of last payment. For a general cashflow, the PV of entire cashflow is the sum of both discrete and continuous cashflow.

1.2 Annuity

Here we will get ourselves familiarized with some terms used in financial market. We make payments regularly in our daily life. Suppose we pay monthly to a person who provides us newspaper, such kind of regular payments are called annuity. If payments are made for a definite time period, then it is called annuity certain.

We can classify annuity based on time of payment or amount of payment.

1. Level annuities are those in which each payment is of same amount. If payments increase or decrease by same amount in each interval, then it is called simply increasing or decreasing annuity.

2. If payments are made at the beginning of each time period, then it is called annuity-due or annuity paid in advance and if they are paid at the end of each time period, then it is said to be paid in arrear.

3. Immediate annuity is one in which payment is made during first time interval while in Deferred annuity, no payment is made during initial time interval.

4. Perpetuity is an annuity that is payable forever. For example if we buy stocks of a company, then we get its dividend throughout the life of stock and such payments are called perpetuity.

Equation of Value - It is an equation that equates the present value of all cashflows. It says PV income = PV outgo. All financial transactions are based on this equation. For example premium of a policy bought by an insurance holder is determined by calculating the present value of premiums received with the present value of advantages given to customer or other outgo.

Example 1.1 (An illustration of equation of value) : Suppose Sunil plans to buy a lottery of whic cost is Re 1. Table shown here explains about the money that he could win along with probability of winning and the delay before he receives his prize of lottery.

Prize won (in Rs)	Chances of Winning	Time before Prize won (in days)
5	1 in 10	1
100	1 in 100	1
3,000	1 in 5,000	3
25,000	1 in 50,000	5
2,00,000	1 in 10,00,000	7

Table 1.1: Probability Table for Example 1.1

Here we assume that the rate of interest is 0.1% per day, then the expected present value of the prize is calculated as follows:

$$PV = 5 \times \frac{1}{10}v + 100 \times \frac{1}{100}v + 3,000 \times \frac{1}{5,000}v^3 + 25,000 \times \frac{1}{50,000}v^5 + 2,00,000 \times \frac{1}{10,00,000}v^7$$
$$= \text{Rs } 2.2272$$

1.3 Loan Schedule

Loan is a very common form of transaction used by individuals, private as well public entity to raise funds. Most often we find our self buying things on EMI which increases our purchasing power and this EMI calculation is based on loan scheduling. Loans operate in two ways either as repayment mortgage in which we pay initial borrowed money during the term of loan or endowment mortgage where money taken is repaid when the term ends.

Calculating Capital Outstanding

Consider a simple scenario where Rahul takes a loan from SBI of Rs 3,000 for 3 years, which he will repay in 3 years, after every year where interest rate is 10% pa. According to equation of value of this transaction, Rahul needs to pay Rs 1,170.96 after each year. Here after 1 year he pays interest of Rs 300 and remaining amount is deducted from original capital and next interest is calculated on capital outstanding and this process continues for three years. Here we can easily calculate capital outstanding after each period as it was short term loan but for larger duration this method will become unhandy. So the amount that has been paid or is remaining to be paid is calculated from remaining amount at any particular time which can be calculated by following two ways:

1. By calculating value accumulated by loan and taking its difference with the accumulated value of payments made.
2. By calculating the discounted present value of payments that are to be maid in future.

We can find the loan outstanding at t prospectively or retrospectively.

Prospective Loan Calculation - Let us consider loan transactions at time t, where after the final installment X_t, the loan is exactly repaid. So this amount exactly covers the loan value which remains after $t-1$, and the interest on that value. Here the loan value that remains at time t is the discounted present value at time t of the repayment to be made in future.

Retrospective Loan Calculation - In this case, the loan value remaining at time t is the difference in its value that is accumulated till time t and the accumulated value of repayments that are made.

Example 1.2 (Method of Loan Scheduling) : Ravi takes a loan of Rs 2500 at a rate of $6\frac{1}{2}\%$ which repaid in 10 years, by paying ten installments of equal amount after every year. What is the value of individual installments?

We consider each installment to be of x Rs. In this way he can pay off loan by a 10-year annuity immediate. Discounted cost of annuity at presnt with $6\frac{1}{2}\%$ comes to be Rs 2500.

Hence the value of each installment is Rs 347.7617. Here, we came to know about loan of Rs 2500 at $6\frac{1}{2}\%$ interest which can be paid back by equal installments in 10 year, paid at end of each year where value of each installment is Rs 347.76.

Now we see to a situation to find the amount of the loan that is due after years.

This situation can be tackled using two methods. In first way we considers the retrospective method where we look at payments that are maid in initial years. In second way or prospective method, we look at payments that is to be made in remaining years. Nevertheless , we get same answer using any of the two method.

Using retrospective method, we see that Ravi has made payments of Rs 347.76 at year end, where its accumulated value is: Rs 2456.48

So loan value after years:

$$2500 \cdot (1+r)^6 = 2500 \cdot 1.459142 = 3647.86.$$

and the amount of loan that remains is:

$$3647.86 - 2456.48 = 1191.38.$$

I he prospective method we see that the balance outstanding is the discounted value of future payments. Here Ravi need to pay Rs 347.76 for more years. Here we calculate discounted value of payments to be made, six years after the loan has started. In this case the discounted value of payments to be made is Rs 1191.36. Hence, loan outstanding after years is Rs 1191.36.

Results obtained using the two methods retrospective vary by paisa, which is due to rounding error. Needless to mention the two methods give the same result when we use exact values.

Chapter 2

Project Appraisal

Project appraisal is all about deciding between alternative investment projects and measures investment performance. We take a look at the following criteria for deciding between various projects available to us:

1. Net present value and accumulated profit

2. Internal rate of return

3. Payback period

4. Discounted payback period

Accumulated Value - It is the final amount possessed by the investor at time T when project ends, given the net cash flow at time t earns an interest i. Here i is the rate at which investor can borrow or lend money. Thus it also determines whether we can borrow money to finance our plan or not. Amount accumulated till time T of can be expressed as:

$$A(T) = \sum c_t (1+i)^{T-t} + \int_0^T \rho(t)(1+i)^{T-t} dt \qquad (2.1)$$

Net Present Value - In an investment there is an initial outlay along with other outlays in future and certain receipts. Net cash flow in the project at time t can be positive or negative. The present value of net cash flow at given rate i is called net present value and is denoted by $NPV(i)$, where

$$NPV(i) = \sum c_t(1+i)^{-t} + \int_0^T \rho(t)(1+i)^{-t}dt \tag{2.2}$$

Higher the NPV, the more profitable is the business.

Internal Rate of Return - Effective rate of interest at which NPV of the cash flow is equal to zero is known as Internal Rate of Return (IRR). Higher IRR indicates more profitable business.

Here we consider two projects A and B where $NPV_A(i)$ and $NPV_B(i)$ denote net present value using functions and i_A and i_B denote their yields respectively. It is a common notion that an investor chooses a project which has high output, but it is not true in all the cases. A nice way to think is profitability at time T (when later project ends) or the net present value, measured at the risk free rate of interest i at which we can easily borrow or lend money in the market. Thus here A is more suited plan if $NPV_A(i) > NPV_B(i)$. The fact that $i_A > i_B$ may not imply that $NPV_A(i) > NPV_B(i)$ is illustrated in Figure 2.1. Here even i_A is larger than i_B, the $NPV(i)$ functions "cross over" at i'. It follows that $NPV_B(i) > NPV_A(i)$ for any $i < i'$, where i' is the cross-over rate. It is possible that there are more than one cross-over point, in which case the range of interest rates for which project A is more profitable than project B is even more complicated.

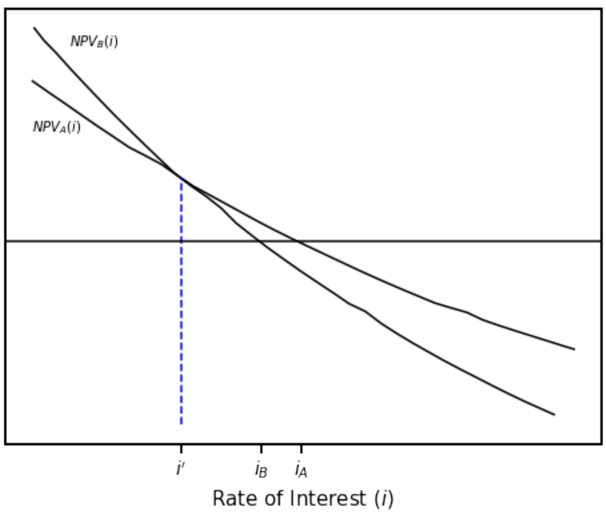


Figure 2.1: Investment Comparison

For simplicity we have assumed here that investor can borrow or lend money at the same rate of interest i. But in real life situations there is higher rate of interest (i_1) on borrowings than the rate (i_2) received on investments. The difference $i_1 - i_2$ between these rates of interest depends on various factors.

Discounted Payback Period (DPP) - The net cash flow changes sign from negative to positive either once or many times during the period or it will always be negative (project is not useful in this case). A project with a lower DPP starts to generate profit more quickly and this is important in project appraisals where one expects a profit as soon as possible. It is the smallest value of t such that $A(t) \geq 0$, where

$$A(t) = \sum_{s \leq t} c_s (1 + j_1)^{t-s} + \int_0^t \rho(s)(1 + j_1)^{t-s} ds \tag{2.3}$$

Here t_1 does not depend on i_2 but only on i_1, which is the rate of interest applicable to the investor's borrowings. Suppose that the project ends at time T. If $A(T) < 0$ then the project has no discounted payback period and is not profitable.

Example 2.2 : Let us consider three projects with following cashflow:
Project P: Initial outlay of Rs 10,000 in return for an annual income of Rs 1000 paid 6-monthly in arrears for 12 years.
Project Q: Initial outlay of Rs 1000 in return for an annual income of Rs 200 at the end of each of next 20 years.
Project R: Initial outlay of Rs 5000 in return for a continuous payment stream of Rs 2000 per annum for 3 years, deferred for 2 years.
If the investor is able to borrow money at the rate of 5 percent per annum, then these projects can be ranked under different measures.

The NPV of three projects at i = 5 are given as follows:

$$NPV_P(i_1) = -1027.26$$

$$NPV_Q(i_1) = 1492.44$$

$$NPV_R(i_1) = 62.55.$$

Thus we can say that Project Q is best using NPV measure and Project P is not profitable at this rate.

Looking at DPP for the three projects:

$DPP(P)$ = 13.96 or 14 years (greater than the term of the project, thus not viable at this rate)

$DPP(Q)$ = 5.90 or 6 years (as the payments are at integer times)

$DPP(R)$ = 4.96 years

Thus Project R is best using DPP measure.

Considering IRR:

$$IRR(P) = 3.05$$

$$IRR(Q) = 19.43$$

$$IRR(R) = 5.38$$

Under IRR measure, project Q is best.

We can conclude that neither of the three measures can be considered in isolation as Project B has greatest IRR and NPV but its DPP is less than Project C. Thus for an investor with unlimited funds, profitability of his investment is of bigger concern and IRR and NPV measures are more suitable and he should invest in Project B while for an investor with limited funds, the time after which he gets a positive return is crucial and so he should go for Project C.

Investment Performance

While investing it seems rational to measure the investment performance of a fund whether be it investment in money market or capital market. Generally we assume that investment performance is measured based on the value of fund on the day we are thinking to sell it, but the value of fund goes up or down as a result of change in these criteria which are:

1. Income generated by the fund (interest payment, dividend etc).
2. Change in market value of assets in the fund.
3. New money added that is the extra money paid into the fund or any money that is not generated by fund itself.

Cashflow in this section refers to new money only.

We can measure investment performance quantitatively using these criterion:

1. Money weighted rate of return
2. Time weighted rate of return
3. Linked internal rate of return

Money Weighted Rate of Return - It measures performance of portfolio of assets. It is the yield of the fund over the period. It is the discount rate at which net present value is zero.

Time Weighted Rate of Return - It measures rate of compounded growth in a portfolio. It is also called geometric mean return as it keeps reinvestment or money withdrawn at specific time intervals into consideration.

Linked Internal Rate of Return - This method aims to approximate Time Weighted Rate of Return by linking Money Weighted rate of return in various intervals.

Chapter 3

Derivatives and Options

Now we will have an in depth look at the world of finance and understand financial instruments. We call Derivative as a financial instrument that obtains its value from an underlying source and its price is determined by movement in price of those assets. The underlying assets can be stock, commodity, currency etc. It is either traded over-the-counter (OTC) or on an exchange. Some common forms of derivatives are Future contracts, Forward contracts, Swaps, Options, Credit derivatives and Mortgage backed security. Here we will be dealing with options as they are the most common and significant form of derivative.

An option is a financial derivative that symbolizes an agreement sold by an option writer to option holder. This agreement between the two parties is called as Option contract. It is of two types, Call option and Put option. In Call option, buyer or holder of the option has the right to buy any security or financial asset at or before a specified time in future at an agreed price but is not obliged to buy. In Put option, seller or owner has the right to sell the security but is not obliged to sell it.

Here the pre-determined price at which exchange of securities will take place is called the strike place and the maximum time till which this contract holds is the expiration time. Buyer of the option is said to have a long position in option contract whereas the seller is said to hold a short position in the contract. Options contract holders can hold the agreement until the expiration date, at which point they can take delivery of the assets mentioned or sell the options contract at any point before the expiration date at the market price of the contract at the time. Based on its nature, option are categorized into European and American options. An European option can only be excercised on its expiry date while an American option can be excercised on any day before its expiry. Call option hopes for the stock price to rise while put option seller expects it to decrease.

Intrinsic Value of the Option - Intrinsic value of an option is the difference of current asset price and its strike price. for a stock,it is given as:

$$\text{Intrinsic Value of Call Option} = \text{Current Price of stock} - \text{Strike Price}$$

whereas

$$\text{Intrinsic Value of Put Option} = \text{Strike Price} - \text{Current Price of stock}$$

It denotes net financial benefit resulting from the immediate exercise of that option. It can't be negative.

Time Value of the Option - The time value of an option is the amount by which the price of any option exceeds its intrinsic value. It is directly related to how much time an option has until it expires as well as the volatility of the stock. It is expressed as :

$$\text{Time Value} = \text{Option Price} - \text{Intrinsic Value}$$

Volatility - Volatility measures dispersion of returns for a given security. It can either be measured by using the standard deviation or variance between returns from same security. Generally higher the volatility, riskier is the security. It is expressed as a percentage coefficient within option-pricing formulas that arises from daily trading activities.

Beta - Beta is a measure of the volatility or associated risk of a security as compared to the market as a whole. It is calculated through regression analysis and it represents the tendency of a security's returns to respond to fluctuations in market. Values of beta can be interpreted as follows

1. Beta value of 1 indicates perfect correlation of market with price of asset.

2. A beta value of less than 1 infers that the market is more volatile than the asset.

3. A beta of greater than 1 indicates that the market is less volatile than that of asset. If beta of stock is 1.4, it's theoretically 40% more volatile than the market.

3.1 Pricing of Options

We have so far dealt with options and its types and are aware that option seller writes an option contract to an option buyer. Here we have also asserted that option is excercised by the owner when he is in a state of profit then is it so that there is no negative cashflow for him? The answer is simple, this is not the case as when an option seller writes an option he charges a small premium just like the premium paid in an insurance policy, which is paid by the option buyer and if option is not excersised then seller has the profit which is equal to option premium. This premium which the buyer pays is termed as option price. Now the question arises how this option price is determined and now we will study about this in detail.

3.1.1 Binomial Option Pricing

Binomial tree is a very popular technique used for pricing options that is based on assumption that stock prices either move up with certain probability by a fixed amount or moves down by a fixed amount with certain probability. Thus at each node it can move to only two different prices. This model explains no arbitrage argument and establishes idea of risk neutral valuation. The idea is to create a portfolio of stocks and bonds so that it exactly replicates the payoff from this option then under no arbitrage principle, the current value of this portfolio must be equal to the the price of call option.

Consider a situation where current price of stock is Rs 50, and after 6 months it will be either Rs 60 or Rs 40. We have a call option to buy the stock for Rs 55 in 6 months. If the stock price after 6 months is Rs 60, the value of the option will be Rs 5. We make a portfolio of the stock and the option so that no uncertainty about the value of the portfolio exists after 6 months. Then we consider that portfolio has no risk, so return earned by it must equal the risk-free interest rate. This helps us to set up the cost of the portfolio and also determine option's price.

We have taken a long position in Δ shares of the stock and a short position in one call option. Value of Δ that makes the portfolio risk less is calculated. When price moves up from Rs 50 to Rs 60, share value is 60Δ and option value is 5, and portfolio value is given by $60\Delta - 5$. When price falls to Rs 40, Δ shares has value given by 40Δ and option value is zero and net portfolio value is 40Δ. We make a riskless portfolio by chosing Δ such that the final value of the portfolio is the same for both cases which implies that $60\Delta - 5 = 40\Delta$ or $\Delta = 1/4$ or 0.25. Thus a riskless portfolio comprises of long position of 0.25 shares and short position of 1 option. Now irrespective of stock price movement, the value of the portfolio is always 10 at the end of the life of the option. Thus Δ is the number of shares necessary to hedge a short position in one option.

Now we know that riskless portfolios must earn the risk-free rate of interest under no arbitrage. Lets take risk-free rate to be 10% per annum. Then value of the portfolio today must be the discounted present value of 10, or

$$10e^{-0.1\times 6/12} = 9.512$$

The price of the stock today is Rs 60 and suppose option price is given by f. Then the value of the portfolio today is

$$50 \times 0.25 - f = 12.5 - f \Rightarrow f = 2.988$$

Thus, under no arbitrage conditions, option value at present should be 2.988. Given its value more than 2.988, it would cost less than Rs 9.512 to set up such portfolio and it would earn more than the risk-free rate. If it value is lower than Rs 2.988, we can borrow money at less than risk-free rate by shorting the portfolio.

Generalisation

Consider a stock whose current price is S_0. Option on the stock has price f. This option matures at time T and that during its stock price can either move up from S_0 to a new level, S_0u, or down from S_0 to S_0d. When stock price moves up, the payoff from the option is f_u

and if it moves down payoff from the option is f_d. We again make a portfolio consisting of long position of Δ shares and one short call option. If the price of share rises then value of portfolio is

$$S_0 u\Delta - f_u \tag{3.1}$$

and if it falls then value of portfolio is

$$S_0 d\Delta - f_d \tag{3.2}$$

Equating these two values for risk neutral pricing we have

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d} \tag{3.3}$$

From this Δ refers to ratio of the change in the option price to the change in the stock price. Given risk-free rate of interest is r, the present value of the portfolio is

$$(S_0 u\Delta - f_u)e^{-rT} \tag{3.4}$$

The cost of setting up the portfolio is

$$S_0\Delta - f \tag{3.5}$$

From this we have

$$S_0\Delta - f = (S_0 u\Delta - f_u)e^{-rT} \tag{3.6}$$

$$f = S_0\Delta(1 - ue^{-rT}) + f_u e^{-rT} \tag{3.7}$$

Substituting Δ from Eq 3.3, we get

$$f = S_0(\frac{f_u - f_d}{S_0 u - S_0 d})(1 - ue^{-rT}) + f_u e^{-rT} \tag{3.8}$$

or

$$f = \frac{f_u(1 - de^{-rT}) + f_d(ue^{-rT} - 1)}{u - d} \tag{3.9}$$

or

$$f = e^{-rT}[pf_u + (1-p)f_d] \tag{3.10}$$

where

$$p = \frac{e^{rT} - d}{u - d} \tag{3.11}$$

Risk - Neutral Valuation

This states that while valuing a derivative, we assume that investors are risk-neutral which means investors do not increase the expected return they require from an investment to compensate for increased risk.But in reality, higher the risks investors take, the higher the expected returns they require.

3.1.2 Two Step Binomial

In a two step binomial model, we assume the current stock price to be S_o. The length of every time is $\frac{T}{2}$ and for the intervals from 0 to $\frac{T}{2}$ and from $\frac{T}{2}$ to T, we consider that in every step the stock price can either increase to a value u times itself (with probability P) or decrease to a value that is d times itself (with probability $1-P$), for a non-dividend paying stock. At time T the possible option prices are f_{uu}, f_{dd} and f_{ud}. The two step binomial model comprises of three single step binomial models:

1. f_u, f_{uu} and f_{ud}
2. f_d, f_{ud} and f_{dd}
3. f_0, f_u and f_d

Replacing T with $\frac{T}{2}$, and making use of (1), (2) and (3) in equations 3.10 and 3.11, we get

$$f_u = E^{-rT/2}(pf_{uu} + (1-p)f_{ud}), \tag{3.12}$$

$$f_d = E^{-rT/2}(pf_{ud} + (1-p)f_{dd}), \tag{3.13}$$

$$f_0 = E^{-rT/2}(pf_u + (1-p)f_d), \tag{3.14}$$

with

$$p = \frac{e^{rT/2} - d}{u - d}, \tag{3.15}$$

$$1 - p = \frac{u - e^{rT/2}}{u - d}. \tag{3.16}$$

On simplification, we get

$$f_0 = E^{-rT}[p^2 f_{uu} + 2p(1-p) f_{ud} + (1-p)^2 f_{dd}] \tag{3.17}$$

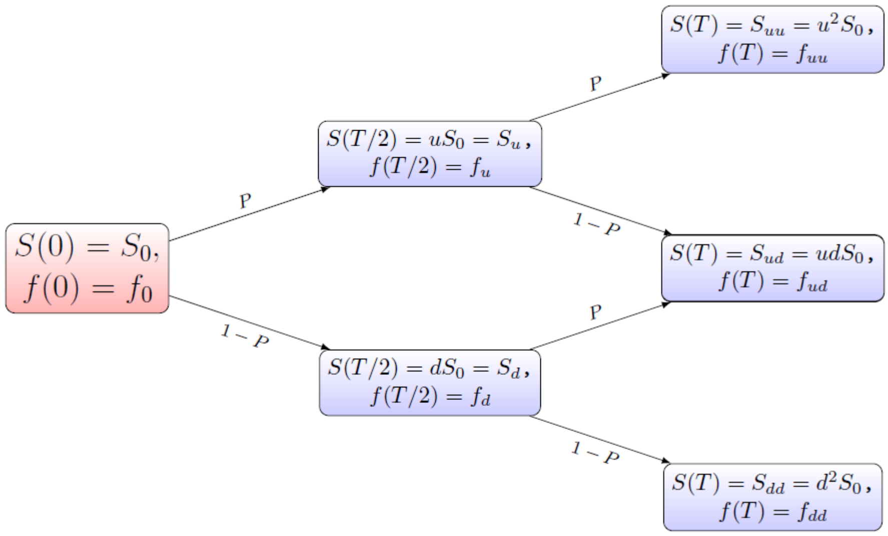


Figure 3.1: 2 - Step Binomial Tree Diagram

3.1.3 N - Step Binomial

Based on our knowledge of the single and double step binomial model, it is easy to deduce the N-step binomial model. Considering the same assumptions as in the one step and the two step binomial model, at time T, the stock price is $S_o u^i d^{N-i}$, with i number of u-jumps and $N - i$ number of d-jumps. Hence, option price at time zero will be,

$$f_0 = e^{-rT}[\sum_{i=0}^{N} \binom{N}{i} p^i (1-p)^{N-i} f_{u^i d^{N-i}}] \tag{3.18}$$

where

$$p = \frac{e^{rt/N} - d}{u - d} \tag{3.19}$$

Simulation of N-Step Binomial Tree Model

A Java program was created to calculate the option price using N - step binomial model. We considered a stock with initial price of Rs 125 (S_0), $u = 1.3$, $d = 0.7$, $r = 0.08\%$ and two different strike prices which are Rs 100 and Rs 150. Volatility of the non-dividend paying stocks are assumed to be 30% and 122 days are left in the option to expire. Having applied the above stated conditions, we got the following set of results for different values of N.

N	European Call Option Price	European Put Option Price
1	22.639	0
2	22.639	0
5	27.789	0.071
10	27.686	0.048
20	27.705	0.066
50	27.706	0.068
75	27.707	0.060
100	27.708	0.069
150	27.707	0.069

Table 3.1: Option Price with strike price of Rs 100

N	European Call Option Price	European Put Option Price
1	0	21.042
2	0	21.042
5	0.526	21.568
10	0.635	21.677
20	0.641	21.683
50	0.643	21.685
75	0.641	21.683
100	0.630	21.679
150	0.643	21.685

Table 3.2: Option Price with strike price of Rs 150

Observations - From table 3.1, we conclude that price of put option is very low or almost negligible if strike price is significantly lower than the current stock price. Whereas, table 3.2 shows that price of call option is very low or almost negligible if strike price is significantly higher than the current stock price.

The code for the above simulation is described in Appendix.

Chapter 4

Way to Black-Scholes-Merton model

Process where value of variable changes over time in an uncertain way is said to be stochastic process. It can be continuous variable where variable can take any value in the range of discrete value where variable can take some discrete set of values. In discrete time process value changes at some particular time and in continuous time process it change continuously at any time.

Markov process is the type of stochastic process where future value of variable is independent of its past and depend only on its present state. We will introduce these ideas in detail later on in the text. First we will start off with the basic ideas mathematically.

4.1 Random Walk

We begin with an object centered at origin and a coin is tossed, with each coin toss i.e. moves either a unit up or down, depending on the outcome whether its head or tail. We denote $\omega = \omega_1\omega_2....$ as the sequence of tosses where each ω_k is the outcome of that particular toss. We denote each step by X_j for a unit up or down where :

$$\mathrm{X}_j = \begin{cases} 1 \text{ if } \omega_j = H, \\ -1 \text{ if } \omega_j = T, \end{cases} \tag{4.1}$$

The process defined by

$$M_k = \sum_{j=1}^{k} X_j, \quad k = 1, 2, ... \tag{4.2}$$

is called as symmetric random walk. It denotes the final position of object after k coin tosses.

Increments of random walk are independent which is the change in position of object between two consecutive times. Their independent nature is observed easily as

$$M_{k_{i+1}} - M_{k_i} = \sum_{j=k_i+1}^{k_{i+1}} X_j \tag{4.3}$$

and each X_j are independent. The expectation of each increment is zero and their variance is

$$Var(M_{k_{i+1}} - M_{k_i}) = k_{i+1} - k_i \tag{4.4}$$

Now we introduce two properties of random walk which are of great importance for this text.

Martingale - This property of random walk states that

$$\mathbb{E}[M_l|\mathcal{F}_k] = M_k \tag{4.5}$$

where $k < l$ which implies that conditional expectation of M_l given its prior information is equal to its present value M_k. Next is the quadratic variation of random walk up to time k and it is defined as follows:

$$[M,M]_k = \sum_{j=1}^{k} (M_j - M_{j-1})^2 = k \tag{4.6}$$

Scaled Random Walk

Now we will talk about scaled random walk where the speed of the object undergoing symmetric random walk is increased and the size of the step is decreased. It is expressed quantitatively as:

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt} \tag{4.7}$$

The limiting case of scaled random walk helps us to present Binomial motion later in the text.

When we make a plot of scaled random walk after fixing time t in it we start observing a pattern. Let's see through it, here we take $n = 100$ and $t = 0.2$, then

$$W^{(100)}(0.2) = \frac{1}{\sqrt{100}} M_{20} \tag{4.8}$$

This is equivalent to 20 independent coin tosses where scaled random walk can take any value from -2.0 to + 2.0. When we plot this information using probability distribution by making a histogram, we obtain nearly a normal distribution and the central limit theorem confirms this distribution.

Central Limit

Fix $t \geq 0$. As $n \to \infty$, the distribution of the scaled random walk $W^{(n)}(t)$ evaluated at time t converges to the normal distribution with mean zero and variance t.

4.2 Brownian Motion

Brownian motion is obtained in the limiting case of scaled random walks $W^n(t)$ as $n \to \infty$ and thus it follows all the property of scaled random walk.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where , there exists a continuous function $W(t)$ of $t \geq 0$ for every $\omega \in \Omega$ and satisfies $W(0) = 0$ and that depends on ω. Then $W(t)$, $t \geq 0$, is a Brownian motion if for all $0 = t_0 < t_1 < ... < t_m$ the increments,

$$W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), ..., W(t_m) - W(t_{m-1}) \tag{4.9}$$

are independent and each of these increments is normally distributed with

$$\mathbb{E}[W(t_{i+1}) - W(t_i)] = 0, \tag{4.10}$$

$$Var[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i \tag{4.11}$$

When a random experiment is performed then the path traced by it is the Brownian motion and $W(t)$ at time t is its value. Thus clearly the value of $W(t)$ varies according to the path resulted from the random experiment.

Since the increments $W(t_1), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1})$ are independent and normally distributed so the random variables $W(t_1), W(t_2), \ldots, W(t_m)$ are jointly normally distributed.

Filtration for Brownian Motion - Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a Brownian motion $W(t)$ of $t \geq 0$, where filtration for the Brownian motion refers to collection of σ-algebras $\mathcal{F}(t)$, $t \geq 0$, that fulfills following three properties:

1. (**Information Accumulates**) For $0 \leq s < t$, all sets in $\mathcal{F}(s)$ are also in $\mathcal{F}(t)$. Information at later time $\mathcal{F}(t)$ is at least same as that of the earlier time $\mathcal{F}(s)$.

2. (**Adaptivity**) For each $t \geq 0$, the Brownian motion $W(t)$ at time t is $\mathcal{F}(t)$-measurable.

3. (**Independence of Future Increments**) For $0 \leq t < u$, the increment $W(u) - W(t)$ is independent of $\mathcal{F}(t)$.

Let $\Delta(t)$, $t \geq 0$, be a stochastic process. We say that $\Delta(t)$ is adapted to the filtration $\mathcal{F}(t)$ if for each $t \geq 0$ the random variable $\Delta(t)$ is $\mathcal{F}(t)$-measurable.

Martingale Property for Brownian Motion - Let $0 \leq s \leq t$ be given. Then

$$\begin{aligned} \mathbb{E}[W(t)|\mathcal{F}(s)] &= \mathbb{E}[(W(t) - W(s)) + W(s)|\mathcal{F}(s)] \\ &= \mathbb{E}[W(t) - W(s)|\mathcal{F}(s)] + \mathbb{E}[W(s)|\mathcal{F}(s)] \\ &= \mathbb{E}[W(t) - W(s)] + W(s) \\ &= W(s) \end{aligned} \tag{4.12}$$

It helps us to ensure that future value of variable is uncertain and is expressed in terms of probability distribution and by virtue of Markov property, probability distribution of future value of variable is independent of path followed by it in the past. Here the quadratic variation is non zero, therefore the paths of Brownian motion are unusual. This leads to the new term volatility in the Black-Scholes-Merton partial differential equation which makes the stochastic calculus different from ordinary calculus.

Let W be a Brownian motion. Then $[W,W](T) = T$ for all $T \geq 0$ almost surely. So the

Brownian motion accumulates quadratic variation at rate one per unit time. Some also refer to it as Weiner process. It is a particular case of stochastic process where the variable z which follows this process has a mean change of zero and variance rate of 1% per year. We can also represent the change Δz during time Δt by

$$\Delta z = \varepsilon\sqrt{\Delta t} \tag{4.13}$$

where ε has a standardized normal distribution ϕ(0,1) From here mean of Δz is zero and its variance is Δt and in this process variance is additive.

We can generalize this Weiner process where instead of mean 0, we take its mean as a which is also referred as its drift rate and variance rate is variance per unit time. The generalized Weiner process is expressed as:

$$dx = adt + bdz \tag{4.14}$$

We can write the discrete time version of it as

$$\Delta x = a\Delta t + b\varepsilon\sqrt{\Delta t} \tag{4.15}$$

where its mean or drift rate is $a\Delta t$ and standard deviation of Δx is $b\varepsilon\sqrt{\Delta t}$

4.3 It$\hat{o}$ Integral

Now, in this section we will define It$\hat{o}$ integrals to model the value of a portfolio which results from trading assets in continuous time. It$\hat{o}$-Doeblin formula will be used to manipulate these integrals and then we will develop Black-Scholes-Merton differential equation.

We take a positive T and a Brownian motion $W(t)$, $t \geq 0$, together with a filtration, $\mathcal{F}(t)$, $t \geq 0$. Let $\Delta(t)$ be an adapted stochastic process. Let, $\Pi = t_o, t_1, ..., t_n$ be a partition of $[0, T]$. Assume that $\Delta(t)$ is constant in t on each subinterval $[t_j, t_{j+1})$; such process is called a simple process. Then,

$$I(t) = \sum_{j=0}^{k-1} \Delta(t_j)[W(t_{j+1}) - W(t_j)] + \Delta(t_k)[W(t) - W(t_k)] \tag{4.16}$$

In particular, in the case of fine partition,

$$I(t) = \int_0^t \Delta(u)dW(u) \qquad 0 \leq t \leq T \tag{4.17}$$

is defined as It$\hat{o}$'s integral for $\Delta(t)$.

It$\hat{o}$ Process

It is a generalized Weiner process where drift rate a and variance rate b are themselves functions of value of underlying variable x and time t.

$$dx = a(x,t)dt + b(x,t)dz \tag{4.18}$$

Formula for It$\hat{o}$ Processes - Let $W(t)$, $t \geq 0$, be a Brownian motion, and let $\mathcal{F}(t)$, $t \geq 0$, be an associated filtration. An It$\hat{o}$ process is a stochastic process of the form

$$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \Theta(u)du, \tag{4.19}$$

where $X(0)$ is nonrandom and $\Delta(u)$ and $\Theta(u)$ are adopted stochastic processes.

Generally for modelling stock price instead of considering constant expected drift rate we consider expected return which is drift rate divided by stock price. This is because if a person expects 20% return when stock price is Rs 50 then at the same time if a stock is priced at Rs 100, he will expect same 20% return. It is also to be noted that percentage return from a stock is independent of its stock price and standard deviation of change in return is proportional to stock price. Given price of stock be S at time t and μ be expected rate of return and σ be its volatility then we come to a model given by

$$\frac{dS}{S} = \mu dt + \sigma dz \tag{4.20}$$

Example 4.1 (It$\hat{o}$ Processes) : Let us assume, that we have a stock of ONGC of initial price Rs 100, with expected return rate of 25% per annum and 20% per annum volatility. Then the probability distribution of the stock price S_T in the next 1 year is given by

$$lnS_T \sim \phi\left[ln100 + (0.25 - \frac{0.2^2}{2})1, 0.2^2T\right]$$

$$lnS_T \sim \phi[4.735, 0.04]$$

With a 95% probability the value of a normally distributed value lies within 2 standard deviations of its mean. Here, the standard deviation is $\sqrt{0.2} = 0.141$. Therefore, there's a 95% chance that,

$$4.735 - 2 \times 0.141 < lnS_T < 4.735 + 2 \times 0.141$$

or

$$85.88 < S_T < 150.95$$

Hence, there is 95% chance that the price of the stock in the next 1 year will lie between Rs 85.88 and Rs 150.95.

It$\hat{o}$'s Lemma

Until now we have figured that price of option depends on stock price and time and therefore to study this behavior we need to understand It$\hat{o}$'s lemma. It says that if a variable x follows It$\hat{o}$'s process where a and b are function of x and t and dz is a Weiner process then a variable G follows the process

$$dG = (\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2)dt + \frac{\partial G}{\partial x}bdz \tag{4.21}$$

where dz is same Weiner process mentioned before.
Now we take $G = lnS$ and check process followed by lnS using $It\hat{o}$'s lemma. It comes out that G follows the process given by

$$dG = (\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial G}{\partial S}\sigma S dz \tag{4.22}$$

Thus it shows that lnS follows generalized Weiner process and the change in lnS is normally distributed. And thus we say that stock price at time t follows log normal distribution which can be expressed as

$$dG = (\mu - \frac{\sigma^2}{2})dt + \sigma dz \tag{4.23}$$

or

$$lnS_T - lnS_0 \sim \phi[(\mu - \frac{\sigma^2}{2})T, \sigma\sqrt{T}] \tag{4.24}$$

$$lnS_T \sim \phi[lnS_0 + \phi(\mu - \frac{\sigma^2}{2})T, \sigma\sqrt{T}] \tag{4.25}$$

4.4 Black-Scholes-Merton Model

Black-Scholes-Merton (BSM) model assumes that percentage change in the stock price in a short span of time exhibit normal distribution. The asset price has instantaneous mean rate of return and volatility where both the instantaneous mean rate of return and the volatility are allowed to be time-varying and random. Return from a risk less portfolio created by a position in option and stocks should be same as that of return from risk free interest rate r under no arbitrage conditions and this is the base for BSM model. Needless to mention our portfolio remains risk less for a very short interval and it needs to be adjusted regularly to keep it risk less. Here we take stock price as

$$\frac{dS}{S} = \mu dt + \sigma dz \tag{4.26}$$

which is developed before and the price of call option is given by f which depends on both S and t and satisfies

$$df = (\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial f}{\partial S}\sigma S dz \tag{4.27}$$

After some mathematics involved in the derivation which is presented later in the appendix we come to Black-Scholes-Merton (BSM) differential equation given by

$$\frac{\partial f}{T} + rS\frac{\partial f}{S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{S^2} = rf \tag{4.28}$$

Any function $f(s,t)$ that satisfies this differential equation can be the price of derivative without any arbitrage while at other prices there are certain arbitrage available in market.

Let's consider a function e^s, it does not satisfy BSM differential equation and if any derivative with such a price exists then it would create arbitrage in market.

After working through the BSM differential equation we get the price of European call and put option on a stock that pays no dividend which are given by c and p respectively.

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{4.29}$$

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \tag{4.30}$$

$$d_1 = \frac{ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{4.31}$$

$$d_2 = \frac{ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} \tag{4.32}$$

Here $N(x)$ is the cumulative probability distribution of standard normal variable. S_0 is the current stock price, K is strike price, r is the risk free rate, T is time to maturity and σ is the volatility of the stock. We will talk about these later in detail.

If we consider x to be the continuously compounded rate of return then it is given by:

$$x = \frac{1}{T} ln \frac{S_T}{S_0} \tag{4.33}$$

4.4.1 A Look at Volatility

It shows how uncertain we are about the return from the stock. It is calculated by the standard deviation of return expressed as continuous compounding in a year.

Example 4.2 (Calculating Volatility from Historical Data) :

$$u_i = ln \frac{S_i}{S_{i-1}} \qquad for\ i = 1, 2, ..., n$$

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}$$

A plausible set of stock prices for 9 consecutive days is shown in table 4.1. Here,

$$\sum_{i=1}^{n} u_i = 0.0948 \qquad \sum_{i=1}^{n} u_i^2 = 0.0342$$

and the estimate of the standard deviation of the daily return is

$$\sqrt{\frac{0.0342}{8} - \frac{0.0948^2}{9 \times 8}} = 0.0644 \quad \text{or} \quad 6.44\%$$

Day (i)	S_i	S_i/S_{i-1}	$u_i = ln(S_i/S_{i-1})$
0	22.3		
1	24.5	1.098	0.0934
2	23.0	0.938	-0.0640
3	22.7	0.986	-0.0140
4	23.9	1.052	0.0506
5	24.3	1.016	0.0158
6	22.2	0.913	-0.0910
7	23.6	1.063	0.0610
8	21.8	0.923	-0.0801
9	21.4	0.981	-0.0191

Table 4.1: Volatility Calculation

There are 252 trading days in a year and volatility on non - trading day when exchange is closed is negligible as compared to trading day. Thus we calculate volatility per year by

$$\text{Volatility per annum} = \text{Volatility per trading day} \times \sqrt{\text{Number of trading days per annum}}$$

Stocks generally exhibit volatility between 15% to 60%. Option life is also calculated in terms of trading days.

Implied volatility is the actual volatility that is inferred from the option price that is traded in the market. It is calculated by trying different values of sigma through hit and trial method after substituting the values in BSM differential equation and we can get the correct value of sigma with any accuracy.

4.4.2 Application of Model in Stocks Paying Dividend

BSM model read so far deals with option that pays no dividend but now we will consider dividend paying stock and assume that the time and amount of dividend is known to us. Ex - dividend date is the date on which dividend is paid and on this date, prices of stock fall by the amount of dividend. Analyzing European option is simple where we take the discounted value of all dividend to be paid during the life of option which is done at the risk free rate from ex - dividend date. This makes dividend as a risk less part of price and risky part can be calculated using BSM formula taking it to be S_0.

Example 4.3 (Stocks Paying Dividend) : A stock of Coal India pays dividend Re 1 after 3 and 6 months. The current stock price is Rs 100, volatility is 20% per annum, strike price is Rs 120, interest rate is 10% annum and expiry time is 1 year. Value of dividend at present is

$$1 \times e^{-0.1 \times 3/12} + 1 \times e^{-0.1 \times 6/12} = 1.926$$

Using the BSM, the option price can be calculated:

$$d_1 = \frac{ln(\frac{98.074}{120}) + (0.1 + \frac{0.2^2}{2})}{0.2} = -0.4085$$
$$d_2 = \frac{ln(\frac{98.074}{120}) + (0.1 - \frac{0.2^2}{2})}{0.2} = -0.6085$$

Using the table for normal distributed function,

$$N(d_1) = 0.3477 \qquad N(d_2) = 0.2771$$

and the call price comes out to be,

$$98.074 \times 0.3477 - 120e^{-0.1} \times 0.2771 = 4.02$$

From the idea of implementing Black-Scholes model on a dividend paying stock, now we can extend it to other financial elements like foreign currency, future or index as they behave in a similar way and provide dividends in other form. A foreign currency provides return which is the foreign risk free interest rate. Futures provide a dividend yield which is the risk free rate of interest of the country. While index provide a dividend which is the average of the dividend of the stocks on the index.

4.4.3 Simulation of BSM Model

After reading BSM, a sample java code was written for it and I calculated values of call and put option price. The data used for calculation has stock price of Rs 125. I have taken different strike prices from Rs 100 to Rs 150. Time to maturity here is 122 days, that is from July 31 to November 30, volatility is assumed to be 20% and risk free rate of interest is taken to be 8%. Generally risk free rate of interest in India is taken as 91 day treasury bill rate of RBI. After compilation the results which I got are mentioned in the table below,

Stock Price	Strike Price	EC	EP
125	100	27.708	0.069
	105	22.987	0.213
	110	18.464	0.563
	120	10.582	2.4155
	125	7.487	4.189
	130	5.048	6.618
	140	1.981	2.4155
	150	0.644	21.687

Table 4.2: Option Price using BSM

The Java code for this simulation is described in the Appendix.

Chapter 5

Understanding the BSM Model

Until now we are aware with the option prices and various option positions but there are certain risks involved while taking a position in option. Before taking any position in option various risks associated with it should be well understood and the Greeks explain different perspectives of risks involved in an option.

5.1 The Greeks

Any financial institution can hold either naked position or covered position after writing an option. In naked position, after writing call option, institution does nothing and wait for the expiration time to come and hence have their profit or loss that depends on whether option is used or not. In covered position institution buys a stock as soon as it writes the option and hence reduce their risk exposure.

Stop Loss Strategy - It is the case where an institution after writing a call option buys a share as soon as its price rises above the strike price and sells as it falls below its strike price and this is a continuous process. But this process is very expensive if the price of stock crosses strike price many times. So for the purpose of hedging we measure various risk parameters and plan our positions accordingly.

Delta Hedging - Delta is defined as the rate of change of price of the option with respect to change in the price of underlying asset. Given if delta of a call option on a particular stock is 0.4 then price of option changes by 40% relative to the price change of stock.

$$\Delta = \frac{\partial c}{\partial s} \tag{5.1}$$

Let us assume that price of stock is Rs 50 and that of call option is Rs 5. An investor sells 10 call option that is option on 1000 shares. Then he can make a delta hedge by buying $1000 \times 0.4 = 400$ shares at the same time. Here simultaneous different positions in stock and option balances variation in stock prices and delta of option position is balanced by delta of stock position which is termed as delta neutral valuation. But this delta neutrality remains only for a very short duration and it has to be altered periodically called as rebalancing. This happens as when the stock price rises, delta of the stock also rises. Hedging in which rebalancing is done frequently is termed as Dynamic hedging while hedge that remains as it is after initial hedging is called Static hedging or hedge and forget.

Delta of a European call option on a non - dividend paying stock is

$$\Delta(call) = N(d_1) \tag{5.2}$$

While delta of put is given by

$$\Delta(put) = N(d_1) - 1 \tag{5.3}$$

Delta of a portfolio whose value is and depending on a single asset of whose price is S is given by

$$\frac{\partial \Pi}{\partial S} \tag{5.4}$$

Delta in a long position or call option is taken to be positive and for short position or put option is taken to be negative. Suppose we have a portfolio of long position in 100 call options and delta of 0.5, short positions in 200 call options with delta 0.4 and short position in 500 put option with delta of 0.5, each having different strike price and expiration date. Then delta of whole portfolio is given by $100 \times 0.5 + (-200 \times 0.4) + (-50 \times (-0.5)) = -5$. It suggests this portfolio can be made delta neutral by buying 5 more shares.

Theta - It measures change of portfolios value with respect to time.

$$\Theta(call) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) \tag{5.5}$$

where

$$N'(x) = \frac{1}{2\sqrt{\pi}}e^{-x^2/2} \tag{5.6}$$

$$\Theta(put) = -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2) \tag{5.7}$$

One thing should be taken care of regarding theta and that is in the BSM formula time is quoted in years while in quoting theta it is represented as change in value per day.

Lets consider a stock of Rs 100 and we have strike price of Rs 120, time of 1 year and volatility is 20% and interest rate is 10%. In that case theta of call option is given by the above mentioned formula which comes out to be -4.2199. These values are calculated after using tables of $N(x)$ value of theta is generally negative for an option as if everything else remains same with time then value of option decreases automatically.

Gamma - It measures the rate of change of delta of portfolio with respect to assets price. It is the second partial derivative of portfolio with respect to asset price and thus shows how slowly or fastly delta will change and provides us with a way through to keep portfolio delta neutral.it is given by:

$$\Gamma = \frac{\partial^2 \Pi}{\partial S^2} \tag{5.8}$$

A relationship between change in price of asset, change in portfolio price, theta and gamma in a very short interval of time dt when higher order terms are ignored is

$$\Delta\Pi = \Theta\Delta t + \frac{1}{2}\Gamma\Delta S^2 \tag{5.9}$$

where Θ represents theta of the portfolio.

Suppose there is a delta neutral portfolio of whose gamma is -100 and there is a change in value of asset by Rs 2 in a very short span then value of portfolio will decrease unexpectedly by Rs 400.

Consider a portfolio which is delta neutral and has a given gamma Γ and a traded option has gamma Γ_T. When traded options are added then portfolios new gamma is

$$\omega_T \Gamma_T + \Gamma \tag{5.10}$$

and now to maintain gamma neutrality of portfolio, the position in portfolio should be $-\Gamma/\Gamma_T$. One important consideration here is that while Delta neutrality helps us to adjust with the small movements in stock price, gamma neutrality helps us in rebalancing the hedge when there is large stock price movements.

Calculating gamma of a portfolio:

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}} \tag{5.11}$$

Vega - It measures rate of change of portfolio value with respect to volatility.

$$\mathcal{V} = \frac{\partial \Pi}{\partial \sigma} \tag{5.12}$$

Vega as calculated by BSM, comes out to be -

$$\mathcal{V} = S_0 \sqrt{T} N'(d_1) \tag{5.13}$$

Value of a derivative changes with change in asset price and with the associated volatility, which is not constant. We are required to maintain vega close to zero so that portfolio value is not very sensitive to changes in volatility.

If a portfolio has given vega $\mathcal{V}$ and a traded option with volatility $\mathcal{V}_T$ is given then portfolio can be made vega neutral by taking a position $-\mathcal{V}/\mathcal{V}_T$ in traded option. Making delta neutrality or vega neutrality requires a lot of effort from the traders side in order to ensure balance in his trade.

Rho - It measures the rate of change of portfolios value with respect to change in interest rates.It is given by:

$$\frac{\partial \Pi}{\partial r} \tag{5.14}$$

when we calculate it from BSM equation its value for call and put option comes to be following:

$$rho(call) = KTe^{-rT}N(d_2) \tag{5.15}$$

$$rho(put) = -KTe^{-rT}N(-d_2) \tag{5.16}$$

Consider a one year call option on a stock of Rs 100 with its strike at Rs 120, volatility is 20%, interest is 10% and time of 1 year. Then its rho comes out to be 63.639. This means that for a one percent change in interest rate from 10% to 9%, the value of option increases by $0.01 \times 63.639 =$ Rs 0.63639.

Now after studying all these Greeks I have presented a table with the Greeks calculated for two different strike prices of Rs 100 and Rs 150.

	Strike Price = 100		Strike Price = 150	
Parameters	European Call	European Put	European Call	European Put
Price	27.708	0.069	0.64	21.69
Delta	0.987	-0.013	0.099	-0.901
Lambda	4.451	-23.826	19.189	-5.194
Gamma	0.002	0.002	0.012	0.012
Theta	-0.023	-0.0016	-0.013	-0.019
Vega	0.025	0.025	0.126	0.126
Rho	0.319	-0.0058	0.039	-0.449
Intrinsic Value	25.00	0.00	0.00	25.00
Time Value	2.708	0.069	0.644	-3.313

Table 5.1: Option Pricing with Greeks using BSM

The terminologies mentioned in the table are referred as Greeks and they are an integral part of BSM model and are calculated from it only. They help us to understand market movements during trades. These Greeks are calculated using online Greeks calculator.

5.2 Revisiting Volatility in Detail

Till now we have considered volatility calculations from historical data and moving forward now we will use ARCH and GARCH models to study volatility.

5.2.1 ARCH Method

In models studied so far for calculating volatility we give equal weightage to percentage change in market variable for each pair of day i and $i-1$. We consider u_i as the percentage change in market variable during day i instead of continuously compounded return during day i and S_i as the value of variable at the end of day i and make observations for most recent m days. Thus we have

$$u_i = \frac{S_i - S_{i-1}}{S_{i-1}} \tag{5.17}$$

and from this we can calculate σ_n^2 which is given by

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}^2 \tag{5.18}$$

Here we have assumed mean of percentage change of market variable to be zero. But variance calculated through this way gives equal weightage to each u_i but it is more appropriate to give more weightage to recent data. Thus Engels proposed his ARCH model. Here we define our variables so that more weight is given to recent data. We consider model

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \tag{5.19}$$

where $\alpha_i < \alpha_j$ for every $i > j$ and also sum of weights should be one. This gives an idea that there is a long run average variance rate which should have some weight. Thus we arrive at a model given by

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \tag{5.20}$$

where V_L is the long-run variance rate and γ is its weight. Since weights sum to unity, we have

$$\gamma + \sum_{i=1}^{m} \alpha_i = 1$$

This is the ARCH model and it can be written as

$$\sigma_n^2 = \omega + \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \tag{5.21}$$

where $\omega = \gamma V_L$. Exponentially weighted moving average model is the one in which value of weightage variable changes exponentially. Here $\alpha_{i+1} = \lambda \alpha_i$, where λ is a constant between 0 and 1. Using this we deduce volatility of *n*th day as

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1-\lambda) u_{n-1}^2 \tag{5.22}$$

Here estimate of volatility of a particular day is calculated from the volatility of previous day.

Consider that for a market variable, the estimated volatility for day $n-1$ is 2% per day and its value increased by 3% during day $n-1$. λ is given to be 0.6. Then from equation (5.22), value of daily volatility for day n is 0.0774.

5.2.2 GARCH Model

GARCH (1,1) model was proposed by bollarsev in 1986. In this model variance is calculated using long run average variance rate, volatility on day $n-1$ and return on day $n-1$. This model is represented by

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 \tag{5.23}$$

From this we can see that EWMA model is special case of GARCH(1,1) model, where γ is the weightage of V_L, α is weightage of u_{n-1}^2 and β is the weightage of σ_{n-1}^2. All weights sum to unity, so

$$\gamma + \alpha + \beta = 1$$

when we use $\omega = \gamma V_L$, the GARCH(1,1) model is also written as

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 \tag{5.24}$$

We can interpret GARCH(1,1) model to observe daily volatility . Suppose from our observations we have estimated GARCH(1,1) model as

$$\sigma_n^2 = 0.000004 + 0.15 u_{n-1}^2 + 0.8 \sigma_{n-1}^2 \tag{5.25}$$

This has $\alpha = 0.15$, $\beta = 0.8$, and $\omega = 0.000004$. From this we have $\gamma = 0.05$ and $V_L = 0.00008$. Hence it can be interpreted as volatility of 0.0089 or 0.89% per day.

One particular difference between EWMA method and GARCH(1,1) method is that it not only assign weight to relative return u_i but also to long run average volatility. GARCH model considers that with time variance tends to move toward long run average level of V_L. In this model variance also tend to follow a stochastic process. The parameters ω, α and β are estimated and depending on the value of parameters we decide which method should be opted to calculate volatility. EWMA model is just a case of GARCH(1,1) model when ω is zero and whenever its value is negative it is advisable to use EWMA model.

5.3 Value at Risk

Previously we have covered risk related aspects in terms of delta, gamma, theta and vega and any financial institution takes into account these measures for every market variable. These measures undoubtedly provide good deal of information for the traders of financial institution but they can't provide a measure of total risk exposure of financial institution and therein this Value at Risk (VaR) comes handy. It gives a single number bearing the total risk in portfolio of assets of a financial institution and its comprehensiveness made it useful.

How to Measure VaR

When someone says that he is x percent sure that he will not lose more that v rupees in next n days, then v is the VaR of the portfolio. Usually in our daily life bank regulators ask bank to give their VaR for $x = 99$ and $n = 10$. This means that with 99% surety banks wont lose more than v rupees in 10 days time. This VaR depends on two parameters one is the time of n days and other is the confidence level of $x\%$. n day VaR can me measured from data of a single day by following formula,

$$\text{n-day VaR} = \text{1-day VaR} \times \sqrt{n} \tag{5.26}$$

It can be calculated either by historical simulation or model building approach.

5.3.1 Historical Simulation

In this method we first gather all the market variables that can affect the value of portfolio and collect their data of last 501 days and denote the first day as day 0 and so on. Now we consider various scenarios where value of all market variables will change as it happened between day 0 to day 1 and call this as scenario 1. In scenario 2 they will change as it happened between day 1 and day 2 and thus we have 500 different possible scenarios for the next day. For each scenario price change in portfolio value is calculated between today and tomorrow and thus we have a probability distribution of daily loss. The 99th percentile of distribution is estimated as fifth highest loss and this is the estimate of VaR.

The method of calculating VaR is shown with the help following table (5.1) and (5.2), where N variables which can affect the price of portfolio are listed and their value of last 6 days from day 0 to day 5 is listed in table 5.1. In the table 5.2 there are 5 possible values of each market variable depending on their previous value and for each case we have value of portfolio and the change in value of portfolio. Original value of portfolio is assumed to be 20k. Define υ_i as the value of market variable on day i and assume today is day j then i^{th} scenario assumes that value of market variable tomorrow will be

$$\upsilon_j \frac{\upsilon_i}{\upsilon_{i-1}} \tag{5.27}$$

Day	Variable 1	Variable 2	...	Variable N
0	18.1	0.20	...	520
1	19.2	0.15	...	480
2	17.3	0.25	...	360
3	20.1	0.35	...	265
4	16.4	0.45	...	385
5	18.7	0.50	...	570

Table 5.2: Market Variables

Case Number	Variable 1	Variable 2	. . .	Variable N	Value of Portfolio (Rs in k)	Change (Rs in k)
1	19.83	0.375	. . .	526	19.8	-0.2
2	16.84	0.83	. . .	427	19.20	-0.8
3	21.72	0.7	. . .	419	20.6	0.6
4	15.25	0.64	. . .	828	20.3	0.3
5	15.39	0.44	. . .	615	20.7	0.7

Table 5.3: Expected Values on day $n+1$

Here we have calculated 80th percentile of distribution which is estimated as highest loss in this data set. Thus here our VaR is -0.8.

The same process is used for real life practical purpose where we take data of all market variables over last 501 days and thus predict it for the next day. After we get original data of next day, we leave behind day 0 and take data of day 501 in our data set of 500 days. The same process is repeated where we estimate 99th percentile of distribution which is estimated as the fifth highest loss in 500 data.

5.3.2 Model Building Approach

We know that volatility per day refers to the standard deviation of percent change in asset price in one day and we are aware that volatility per day is about 6% of annual volatility. We will understand model building approach for single asset case and double asset case with the help of examples.

Single Asset Case - We take example of a portfolio consisting of 50k in shares of TCS and we are interested in finding our loss over next 10 days with confidence level of 99%. We assume volatility of TCS to be 2% per day. Due to volatility, standard deviation of daily change in our portfolio is Rs 1000. Suppose TCS has expected return of 25.2% then its per day return will be 0.1% and return for 10 day period will be 1% and standard deviation of return is $2\sqrt{10}$ or 6.3%.

From property of normal distribution we know that value will not decrease by more than 2.33 SD with 99% surety. Thus our 1 day VaR will be $2.33 \times 1000 = 2330$ and from this 10 day 99% VaR is $2330\sqrt{10}$ which is Rs 7368.1.

After performing same calculations in a portfolio of Rs 70k of HDFC which has a volatility of 1% and return of 20% per annum. We have change in value of portfolio per day to be Rs 700. Its 1 day VaR is Rs 1631 and 10 day VaR is Rs 5157.6.

Double Asset Case - In double asset case we consider portfolio where we have invested our money in two companies, say TCS and HDFC and investment is Rs 50k and 70k. We suppose return on these have bivariate normal distribution with correlation of 0.4. From statistics we know that standard deviation of $X+Y$, when their individual standard deviations and correlation are known is given by

$$\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y} \tag{5.28}$$

Thus here we take σ_X to be Rs 1000 that is associated with TCS and σ_Y to be Rs 700. Change in value of portfolio over a day will be Rs 1243.38. From this we calculate 1 day VaR with 99% surety that is Rs 2897 and 10-day 99% VaR will be 9161.35. From this we come to kmow about benefits of diversification in trading as in TCS we have 10 day VaR of Rs 7368.1 and for HDFC it is Rs 5157.6 but in the portfolio of shares of both of these we have 10 day 99% VaR of rs 9161.35. Thus it helps us in diversifying our risk.

5.4 Criticism of the Model

Black Scholes model has failed to capture the real essence due to some factors which are not hard to find. It should always be considered that neither a model nor people can predict the actual direction of an individual stock or the whole market itself. This model is based on assumption of constant interest rate that is called as risk free rate of interest but this rate is also varying in nature. After the 1987 crash in stock market, the idea of constant volatility also became questionable and many underlying stocks of the option that pay dividend before expiration makes it tough for the model to predict actual prices. And after all this many works are there in financial research that shows that asset returns don't have log normal distribution instead show more resemblance with other distributions. Clark observed that asset returns have finite variance and semi heavy tails while Hull examined then to be leptokurtic. Model ignores market behaviour and assumes that there is complete liquidity in market with continuous trading that doesn't incur any transaction cost. It is also shown that more recent models based on levy process are closer to calculate prices in derivative trading.

Many a times I have talked about market forces that can affect the price of stock but basically what are these forces. I have outlined few of them which are attributed to some news regarding change in policy of government that affect financial sector or some corporate announcement that affect stock prices or various indexes related to financial sector that are released worldwide.

1. Finance ministry lays down Annual Financial Statement or the budget of the country which contains information about taxation on various products. Thus profitability of product changes and hence it influences trader's mind.

2. RBI controls financial market in India by its monetary policy announcement which affects interest rate prevailing in the country.

3. Inflation plays a key role in stock price movement.

4. In India Ministry of Statistics and Program implementation (MOSPI) releases Index of Industrial Promotion (IIP) every month which depicts progress of various industries and thus behaviour of traders changes accordingly.

5. Every listed company have to declare their earning once every quarter and the profitability of the company is directly reflected in the stock prices after the earning announcement.

6. Company may offer dividend or issue bonus to the shareholders which causes movement in stock prices.

7. Company also engages in buyback of shares, rights issue or stock split which alters stock price after such announcement.

8. Credit rating agency rate every company and even Government securities and update their ratings from time to time to monitor progress of organization.

All these factors are around the corner every time and they shape market with combined effort and analyst are always engaged in predicting the impact of these market forces on stock prices.

Chapter 6

Trading Strategy

When we move forward to trade in options, we go with a feeling that either stock prices will move up or down. When you expect prices to rise you take a long position in call option or short position in put option and vice versa. Option contracts expire on last Thursday of every month and there are three expiry dates available for a particular strike which are called as current month, mid month and far month. Suppose its October 3rd, 2018 then for a particular strike price of 100, we have three expiry October 25 (current month), November 29 (mid month) and December 27 (far month). In India options are cash settled which means that two traders on the expiration date just need to pay the difference in amount arising due to option contract and not the actual share is transferred and risk free rate is taken as RBI 91 day treasury bill rate.

Let's revisit some basic ideas. Intrinsic value is the difference between spot price and strike point. If intrinsic value is non-zero number then option is said to be In The Money (ITM), if it is zero then option strike is Out of The Money (OTM), and the strike nearest to spot price is called At The Money (ATM). Break even point is the point at which a call option buyer is at a stage of no profit and no loss and if trade continue to be in his favour then he will start making profit and the same is termed as breakdown point for a call option writer. It is not natural to hold an option till expiry as not only stock price fluctuate but also option premium of a share of given expiry and fixed strike price changes continuously and most sellers engage in trading options, trying to capture fluctuations in option premium.Now the point is why option premium varies. The answer is Greeks, as some of them tend to in-

crease the premium while other tend to decrease it and in turn they are also controlled by market forces.
Some of the common observations are:

1. Call option premium increases with increase in spot value of asset and vice versa

2. Delta varies between 0 to 1 for call option and -1 to 0 for put option. For a portfolio total delta is sum of individual delta.

3. Gamma is always a positive number.

4. Theta is usually a negative number and options lose money on a daily basis owing to theta if all other factors remain same.

When we see the daily return of the stock for a certain period then we notice that their behaviour is very similar to the process observed in Galton board experiment where each balls follow random walk and finally exhibit normal distribution. So we must enrich our understanding of normal distribution in terms of analyzing data. Normal distribution is characterized by two numbers which are its mean and standard deviation (SD). We consider distribution of data in terms of SD around the mean. Within 1 SD of the mean we can expect 68% of the data, 95% of data within 2 SD and we can predict data with 99.5% surety within 3 SD. Normal distribution property of the stock prices give an insight to the trader so that he can choose the right strike price to write an option and collect the premium.

Till now I was talking about trading in terms of call option as it is well observed human nature in which we notice that anxiety spreads faster than avarice and there are great chances that OTM put option which we have written can soon become ATM or ITM. And hence we should prefer to short call options after selecting proper strike price and keeping expiration time in our favour using theta to serve our purpose. When we are writing call options within a week of expiry then we can go with strikes around 1 SD and for duration around 2 weeks, we should consider 2 SD for strike prices. Another thing to decide before trading is our Stop-Loss point or the point where we exit from the trade after we face certain loss and should also choose our risk to reward ratio wisely. Generally people choose Stop Loss (SL) point in terms of absolute numbers or percentage without keeping volatility in mind

but this is an unprofessional approach. Suppose we are trading shares of Indian Oil where we have entered into trade for a period of 5 days with current price at Rs 410, keep a target of Rs 460 and make our SL at Rs 390. Here we have our Risk to Reward ratio of 2.5. We observe our daily volatility of Indian Oil as 1.7%. So there are high chances that we might stop trade in fear of losing money before reaching the right exit time.
Thus we conclude that pre-decided SL point does not keep volatility in consideration which can lead us to immature exit from the trade but volatility based inputs keep all factors in mind and thus serves our purpose.

Volatility always works in favour of option premium as with increase in volatility, there are high chances for premiums to end in money. For volatility to work in favour we should buy call option when volatility is expected to increase and should sell it when volatility is expected to decrease.

6.1 Effect of Time on Options

Here we are going to consider two scenarios and will decide that which strike should be opted depending on what time we are buying the option and what is the time left to expiry.

Consider a scenario where we trade in the first half of the month where spot price of reliance is Rs 500 and we expect it to move 5% from Rs 500 to Rs 525. Here strike price os Rs 500 is ATM. Here if we expect to achieve target within next few days, then we can choose far OTM strikes or strikes which are 2 or 3 strikes away from ATM. We opt for little OTM strike when we hope to achieve target around 15 days, little ITM strike when we expect to achieve target around 25 days and should certainly avoid buying ITM or OTM strikes. And if we expect to achieve target around expiry then we should strictly go for deep ITM strikes.

The same set of observations are valid when we initiate in second half of the month with slight difference in time where we buy deep OTM strike if we expect to achieve target same day, slight OTM for around 5 day period, ITM for around 10 day period and deep ITM when we expect to achieve the desired target around expiration time.

Portfolio Optimization

Portfolio optimization is also worth mentioning as it is an important part of trading strategy. I will explain portfolio optimization through an example. Suppose I invest equally in TCS and JIO of whose expected return are 20% and 30%. Then by simple arithematic my expected return will be 25%. Now I vary the proportion of my investment in both of them and see my returns in all the cases and this is the basis of portfolio optimization which deals with the weights which we give to individual stock in our portfolio so as to maximize our expected return. In stock market, we should choose our portfolio so that it has minimum variance or portfolio which gives maximum return. There is another way to chose our portfolio which is by fixing our variance and then we selecting a portfolio with maximum return. We can use Solver Tool in excel to optimize our portfolio which uses same principle.

After considering all these trading strategies based on interpretation of the Greeks, we move forward to situations where risk is averted through multiple buying and selling of options and patterns associated with it and how they can be interpreted and utilized to maximize our profit. We will talk about general combinations of option and show associated strategies. Strategy for single option and single stock.

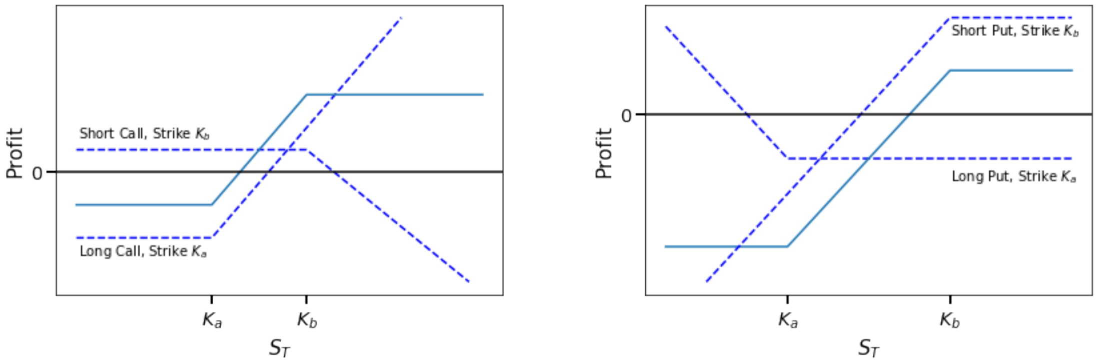
1. **Covered Call** - We make our Portfolio with a long position in a stock and a short position in call option whereby the sharp rise in stock price is adjusted. It's reverse is short position in stock and long position in call option.

2. **Protective Put** - We make our Portfolio by buying a put option and the stock itself and its reverse is short position in put along with short position in stock.

6.2 Spreads

This strategy takes into account two or more options of same type.

Bull Spread - Person who enters this strategy hopes that the price of stock will rise and has to make an initial investment. It is created by buying a call option with a given strike around ATM and selling a call option of same stock with higher strike price that is OTM with same maturity date. There is always a net debit in bull spread using call options and net credit using put options at the beginning. Besides this the maximum profit or the loss of the trader is also capped. Suppose K_a is the ATM strike that is bought and K_b is the OTM strike sold where $K_b > K_a$ then profit made by is represented by the following figure,

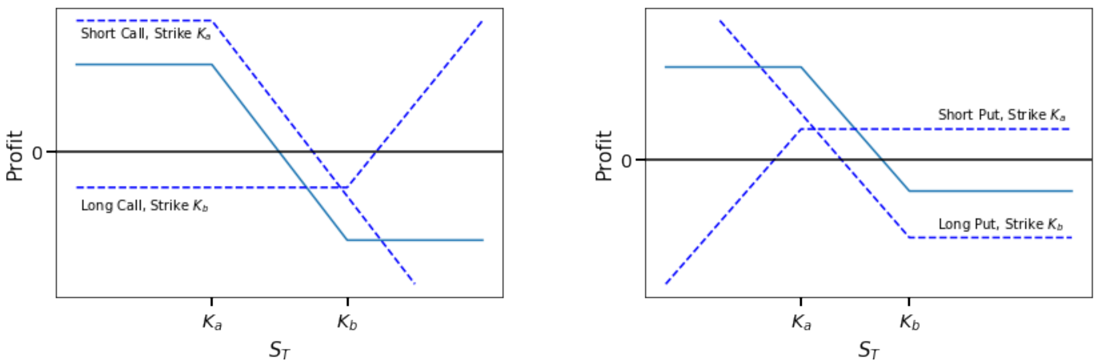


(a) Bull spread profit using call options (b) Bull spread profit using put options

Figure 6.1: Bull Spread

It is not necessary that option which we are buying should be ATM and the one we are selling should be OTM, they can be both OTM, ITM or ATM but the strategy of selecting them so that the one we are buying is at lower strike price and the one we are selling is at higher strike price should be maintained. The choice of perfect strikes depend on time to expiration and Greeks and it has been already explained before.

Bear Spread - Person entering this strategy hopes that the price of stock will fall and buys a put with a given striken price K_b and sells a put with lower strike price K_a. Bear put offers net debit initially while bear call offers net credit. Payoff from bear spread is represented as follows:



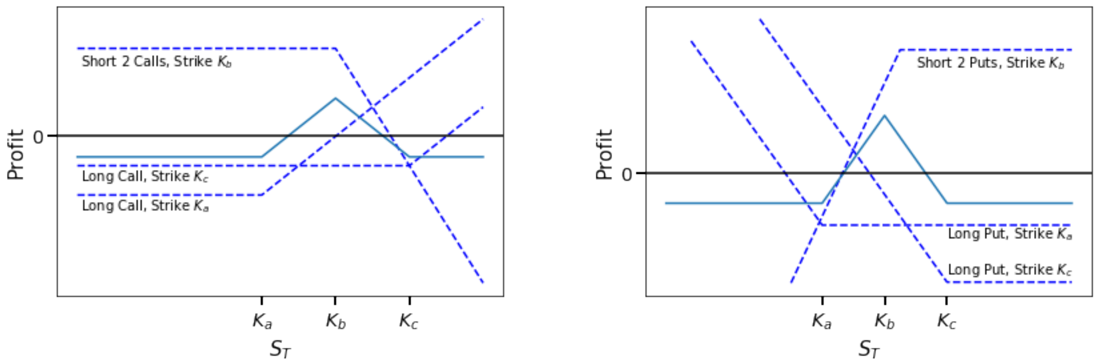
(a) Call option profit (b) Put option profit

Figure 6.2: Bear Spread

Bear put should be implemented when one expects volatility to increase in second half of the month.

Box Spread - When bull call spread and bear put spread of strike prices K_a and K_b are used in combination then it gives rise to box spread. Basic idea here is to buy an ITM call and put and sell an OTM call and put. Payoff of this spread is always $K_b - K_a$.

Butterfly Spread - This spread is created by buying a call option with strike prices K_a and K_c and selling two call options with strike price K_b that is between K_a and K_c. This strategy requires little upfront investment and is best suited for those who expect that stock price is not much volatile. This strategy is profitable when stock price remains close to K_b. It can also be created by buying two put option each with different strike price and selling two put option with same intermediate strike price. Payoff from this spread is represented by following diagram:



(a) Call options

(b) Put options

Figure 6.3: Profit using Butterfly Spread

Till now we were considering options with same expiration date but now we dive to cases where options have different expiration date and same strike price. It is obvious to keep in mind that options which expire later are usually more costly.

Calendar Spread - It is created when we sell a call option with a strike price and buy another call option which has more time in its expiry, price of both the option being same. If the price of stock at the expiration time of early maturing option is close to its strike price, then a profit is made. With put options it can be created by buying put option which has expiration date after the and put option which is being sold. Some traders opt for reverse calendar spread where early maturing option is bought and later maturing is sold.

Diagonal Spread - Cases where both the strike price and expiration date are different and person takes a long position in one call and short position in another call comes under diagonal spread.

6.3 Combinations

In spreads we were taking different positions in call or put but now we will take positions simultaneously in call and put of the same stock and it gives rise to combinations.

Straddle - Most simple combination is straddle in which we buy a call and a put with same strike price and same expiration date. This is called straddle purchase. Similarly we have straddle write in which we sell a call and a put with same strike price and expiration date. If the volatility of stock is high then there is a good profit in straddle but if stock price at expiration time remains close to strike price then straddle incurs a loss.

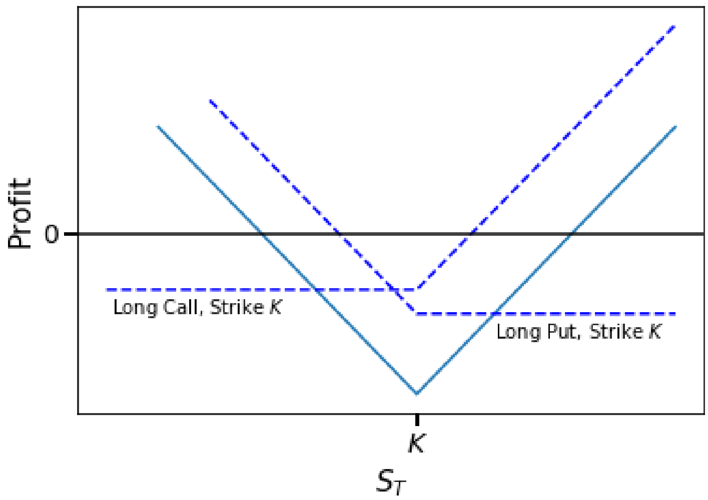
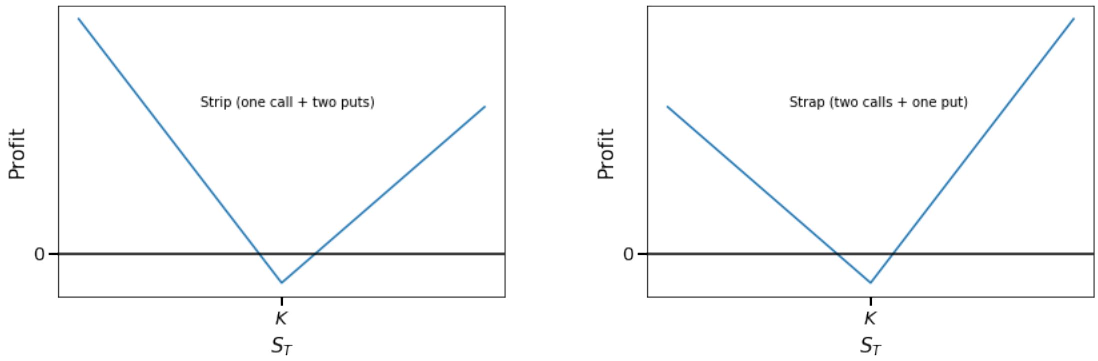


Figure 6.4: Straddle

Strips and Straps - Strip is observed with long position in one call and two put all having same expiration date and strike price with the expectation that price is highly volatile and will go down while strap is observed with long position in two call and put with same strike price and expiration date, considering high volatility and expecting prices to rise.



(a) Strip (b) Strap

Figure 6.5: Profit Pattern from Strips and Straps

Strangle - When we buy a call and a put where strike price C of call is higher than the strike price P of put, both having same expiration date then strangle is created.

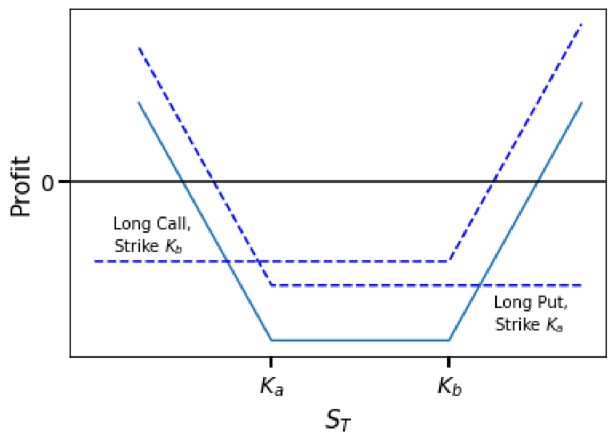


Figure 6.6: Strangle

Condor Spread - Other complex trading strategy involves condor spread which is the linear combination of four call options. Butterfly is the specific case of condor where strike price of second and third call are same.

Shout Option - This is the one in which holder shouts to the writer before option expires and strike price is matched with current price of the asset.

This was all about the basic trading strategy related to options and the whole world of option trading is now open to dive in.

Chapter 7

Empirical Analysis of BSM in Indian Stock Market

In this chapter I have presented my work where I have done data analysis of option premium of call and put options of various companies, trading in different price ranges. Option prices calculated through BSM model are compared with actual market prices of the market. My work here is diversified in nature and I have varied parameters in different set of observations to study their impact on option prices. For some companies I have kept same expiration date and varied strike prices in the range from Out of the money to In the money to see changes based on different strike prices while in other case I have kept strike price same and varied expiration date to observe diversity created by different expiration date. In a general comparison table with one strike price price and one expiration date I have tried to observe the difference between the predicted premium price. I have used paired t-test to compare option prices and the companies are selected from the core sectors of Indian economy.

In the beginning I have listed the common steps involved in various observation sets:

1. I collected data of stock prices and strike prices of the listed companies from NSE website. Closing prices of the stock are used for calculation of option price.

2. I calculated volatility using daily log returns using following steps:

$$\text{Daily Return} = ln\Big(\frac{\text{closing price of a particular day}}{\text{closing price of previous day}}\Big)$$

$$\text{Daily Standard Deviation (SD)} = (\text{Daily variance return})^{0.5}$$

$$\text{Historical Volatility} = \text{Daily SD} \times (252)^{0.5}$$

Volatility is also cross checked from website of NSE and India vix.

3. The risk free rate of interest used in this study is the 91 day T-bill rate of RBI which is rounded to 7%.

4. I compared option price calculated from Black Scholes model with the actual option price of the market.

The main objective was to check whether there is significant difference between option prices using BSM model and actual option price present in the market as option pricing is equally important for writer as well as buyer. In market option may be over priced or under priced. Thus it is advised to see expected option price while making an option contract.

Procedure

- I have taken data of various companies with option values for 10 days starting from 30th October to 13th November and used volatility of stock market on these days and performed analysis on various samples in different ways.

- I have used paired sample t-test for finding the difference. It compares mean of two data set and checks whether their mean difference is significant or not.

- Null Hypothesis (H_0) says that no significant difference is observed between the mean of two set of values.

- Alternate Hypothesis states that significant price difference is observed between the mean.

- As per this test, we can say with 95% that null hypothesis is accepted if p-value is greater than 0.05 and reject it if it's less than 0.05.

TYPE I

In this case I have used data of three companies with three strike prices for each of these companies with same expiration date that is of 29th November. The three companies considered are:

1. Tata Motors with strike prices Rs 150, Rs 175 and Rs 200.

2. Sun Pharma with strike prices of Rs 500, Rs 550 and Rs 600.

3. Reliance with strike prices of Rs 1080, Rs 1100 and Rs 1120.

Observations

The statistics of stock price and corresponding option prices for both call and put options are taken. The table shown here gives result that was obtained after paired t-test was done between the two set of option prices.

Parameters	Reliance (Call)			Reliance (Put)		
Strike Price	$K_1 = 1080$	$K_2 = 1100$	$K_3 = 1120$	$K_1 = 1080$	$K_2 = 1100$	$K_3 = 1120$
Mean Difference	12.1	12.3	11.4	13.7	13.1	12.5
t - stat	3.04	4.19	5.37	2.37	1.93	1.58
df	17	18	18	17	18	18
p - value	3.7e-3	2.7e-4	2.0e-5	1.4e-2	3.4e-2	6.5e-2
Null Hypo	Rejected	Rejected	Rejected	Rejected	Rejected	Accepted

Table 7.1: Paired t-test result table for Reliance

Parameters	Sun Pharma (Call)			Sun Pharma (Put)		
Strike Price	$K_1 = 500$	$K_2 = 550$	$K_3 = 600$	$K_1 = 500$	$K_2 = 550$	$K_3 = 600$
Mean Difference	64.4	6.6	8.9	2.2	8.4	9.2
t - stat	6.76	1.36	6.16	3.36	5.48	2.34
df	12	18	17	9	13	17
p - value	9.9e-6	9.4e-2	5.1e-6	4.2e-3	5.2e-5	1.6e-2
Null Hypo	Rejected	Accepted	Rejected	Rejected	Rejected	Rejected

Table 7.2: Paired t-test result table for Sun Pharma

Parameters	Tata Motors (Call)			Tata Motors (Put)		
Strike Price	$K_1 = 150$	$K_2 = 175$	$K_3 = 200$	$K_1 = 150$	$K_2 = 175$	$K_3 = 200$
Mean Difference	0.4	3.3	3.3	1.0	4.2	4.0
t - stat	-0.20	1.00	4.73	0.91	3.03	1.06
df	17	16	13	9	11	18
p - value	4.2e-1	1.6e-1	1.9e-4	1.9e-1	5.7e-3	1.5e-1
Null Hypo	Accepted	Accepted	Rejected	Accepted	Rejected	Accepted

Table 7.3: Paired t-test result table for Tata Motors

Interpreting the Data

The results of the t-test shows that there is inconsistency in theoretical value of option price calculated from Black-Scholes model but markt value of subsequent days can be predicted as the direction of change is similar. In case of Reliance, null hypothesis is rejected for call option for all the strike prices which means that there is significant price difference while hypothesis is accepted for put option with highest strike price which suggests that there is insignificant price difference. On Using three different strike prices, keeping all other factors same, I was able to track price variations based on strike prices and an important outcome was that in case of stocks of lower stock price and within them stocks with lower strike price which has low premium and steady variations in their premium are easier to predict and gives consistency in result. It is also observed that when volatility is high, there seems to be more difference between the option price using BSM and actual option price.

TYPE II

Here I have used data of three companies which are Indian Oil (petrochemical giant), ONGC and Just Dial about which we have heard many advertisements. In this case I have chosen strike price for all the company which is In the money and varied expiration dates. Strike price for Indian Oil, ONGC and Just Dial are Rs 135, Rs 155 and Rs 500 respectively.

Observations

Data obtained after analyzing is presented in the table below.

Parameters	Indian Oil (Call)			Indian Oil (Put)		
Expiration Date	29th Nov '18	27th Dec '18	31st Jan '19	29th Nov '18	27th Dec '18	31st Jan '19
Mean Difference	2.3	14.5	17.2	2.4	0.5	16.4
t - stat	0.96	14.56	18.06	3.03	-1.82	47.40
df	18	9	9	15	9	9
p - value	1.7e-1	7.3e-8	1.1e-8	4.2e-3	5.1e-2	2.1e-12
Null Hypo	Accepted	Rejected	Rejected	Rejected	Accepted	Rejected

Table 7.4: Paired t-test result for Indian Oil

Parameters	ONGC (Call)			ONGC (Put)		
Expiration Date	29th Nov '18	27th Dec '18	31st Jan '19	29th Nov '18	27th Dec '18	31st Jan '19
Mean Difference	1.8	21.3	14.3	2.6	1.0	14.8
t - stat	3.54	105.21	71.05	2.52	-4.33	44.24
df	18	9	9	14	18	9
p - value	1.1e-3	1.6e-15	5.4e-14	1.2e-2	2.0e-4	3.8e-12
Null Hypo	Rejected	Rejected	Rejected	Rejected	Rejected	Rejected

Table 7.5: Paired t-test result for ONGC

Parameters	Just Dial (Call)			Just Dial (Put)		
Expiration Date	29th Nov '18	27th Dec '18	31st Jan '19	29th Nov '18	27th Dec '18	31st Jan '19
Mean Difference	15.0	41.0	39.0	15.4	29.9	87.4
t - stat	5.29	23.02	21.20	8.60	27.16	86.08
df	17	9	9	18	9	9
p - value	3.0e-5	1.3e-9	2.7e-9	4.2e-8	3.0e-10	9.7e-15
Null Hypo	Rejected	Rejected	Rejected	Rejected	Rejected	Rejected

Table 7.6: Paired t-test result for Just Dial

Interpreting the Data

Here for Indian Oil null hypothesis is accepted for near expiration call option and middle expiration put option while rejected in other cases. While for ONGC and Just Dial, hypothesis is rejected which means that there are irregularity in price of the model and it is not consistent with market prices. It can also be concluded that the mean price difference between the models varies according to expiration date and the model predicts option value more accurately for expiration date which is most closer. Observed difference is maximum for option which has longer maturity.

TYPE III (Using Black Model)

Here I have chosen three companies out of which two are 'too big to fail' Indian banks as per RBI report. This term suggest that if these company collapse then Indian Economy will crash. It is also refereed as Domestic Systemically Important Banks ($DSIB_S$). This term became very famous in financial market after US economy crashed in 2008. The third company chosen is TCS which is second biggest company of India in terms of market capitalization. Shares of these companies are called blue chip stocks and the reason for choosing such prominent companies is that, here I am not only comparing actual market option price with only option price using Black-Scholes model but also with option price calculated using Black model.

Black model uses idea of forward price. Forward price (F) is also a derivative which can be traded and it is the expected price of a stock on a future date and these values are also listed on NSE website and I have used the same source to gather future price data. Fischer Black one of the co-author of Black-Scholes model (1973) tried to improve upon the model accuracy in 1976 by using forward price instead of spot price used in BSM formula. He observed that forward price considers irregularity in market and also incorporates cost of carry. He proposed to use discounted present value of future price (Fe^{-rt}) instead of current stock price (S) in the formula for calculating option price using BSM method. Various studies are going around to check whether this method of using future prices is more efficient than regular BSM model or not and I have tried to establish the same.

Strike prices of TCS, SBI and HDFC are Rs 1900, Rs 250 and Rs 2100 respectively, all expiring on 29th November. I have calculated call and put option prices using both BSM model and Black model and compared them individually with actual option prices.

Observations

The set of observations obtained from t-test are mentioned in tables to follow.

Parameters	TCS (Call)		TCS (Put)	
Model	BSM Model	Black Model	BSM Model	Black Model
Mean Difference	11.4	12.6	13.5	12.7
t - stat	2.08	2.49	2.89	2.69
df	17	18	17	18
p - value	2.6e-2	1.1e-2	5.0e-3	7.4e-3
Null Hypo	Rejected	Rejected	Rejected	Rejected

Table 7.7: Paired t-test result for TCS

Parameters	SBI (Call)		SBI (Put)	
Model	BSM Model	Black Model	BSM Model	Black Model
Mean Difference	1.5	1.8	2.0	2.0
t - stat	0.19	0.30	2.70	2.69
df	18	18	9	9
p - value	4.2e-1	3.8e-1	1.2e-2	1.2e-2
Null Hypo	Accepted	Accepted	Rejected	Rejected

Table 7.8: Paired t-test result for SBI

Parameters	HDFC (Call)		HDFC (Put)	
Model	BSM Model	Black Model	BSM Model	Black Model
Mean Difference	2.0	2.1	3.2	4.6
t - stat	1.73	2.08	-0.49	-0.66
df	17	16	18	18
p - value	5.1e-2	2.6e-2	3.1e-1	2.6e-1
Null Hypo	Accepted	Rejected	Accepted	Accepted

Table 7.9: Paired t-test result for HDFC

Interpreting the Data

The first striking thing is that in most of the cases future prices are under priced. They are lower than expected fair values while in some cases future prices are even lesser than spot price. Null hypothesis is rejected for TCS. In SBI there is no significant price difference for call option but hypothesis is rejected for put options. In case of HDFC hypothesis is accepted for call option using BSM model but rejected using Black model while it is accepted in put option using both models. It is also observed that for put options Black model seems to be more accurate than BSM model as greater consistency and accuracy in price is noticed. It gave less error than BSM method and thus it can be said that Black model is more fitting than BSM model for put options.

Chapter 8

Conclusion

In this thesis, I have presented project appraisal and studies option pricing using Binomial and Black-Scholes model. Trading decisions based on option prices in stock exchanges across the globe are made, keeping price calculated from Black-Scholes model. So here I have tried to establish the relevance of Black-Scholes model in Indian stock market by comparing actual option price with the price calculated using the model. I found out that model shows inconsistency in calculating option prices which can be due to various force driving the financial market. Difference between predicted and actual option price depend upon the moneyness of the option and expiration dates. Results are more coherent for In the Money options with near expiration dates. Attempts have been made to improve upon the model and it can be further improved by addressing assumptions taken in the Black-Scholes model. Black model seems to be more relevant for pricing of put options.In spite of some loop holes this model still is used by all the analyst in the market for deciding trading behaviour and various models which come up and try to imitate market behaviour are also based on Black-Scholes model.

Further I intend to study other models which can predict market behaviour in a much better way and develop professional trading strategies in option market.

Appendix

1. A Java program to calculate option price using Black-Scholes Model.

```
import java.util.Scanner;

importorg.apache.commons.math3.distribution.NormalDistribution;

public class BlackScholes
{
  public void main (strings[] arg)
  {
    Scanner scan = new Scanner ( system .in);
    system.out.println("Price of stock :");
    double st = scan.nextDouble();
    system.out.println("Strike price of stock :");
    double str = scan.nextDouble();
    system.out.println("Volatility of stock :");
    double V = scan.nextDouble();
    system.out.println("Time to maturity:");"
    double T = scan.nextDouble();
    system.out.println("Risk free rate of interest:");
    double T = scan.nextDouble();
    double D1 = (Math.log(st/str)+T*(R+(V*V/2)))/(V*Math.sqrt(T));
```

```
    double D2 = (Math.log(st/str)+T*(R-(V*V/2)))/(V*Math.sqrt(T));

    NormalDistribution n = new NormalDistribution(0,1);

    double D3C = n.CumulativeProbability(D1);

    double D4C = n.CumulativeProbability(D2);

    double D3P = n.CumulativeProbability(-D1);

    double D4p = n.CumulativeProbability(-D2);

    double callPrice = st*D3C-str.Math.Pow(Math.E,-R*T)*D4C;

    double putprice = st*D3P+str.Math.Pow(Math.E,-R*T)*D4P;

    system.out.println(Price of European Call Option is:"+ callPrice);

    system.out.println(Price of European Put Option is:"+ putPrice);

    scan.close();

  }

}
```

The given program when run on java compiler, accepts input from the user and gives the following output on display screen:

```
Price of Stock:  125

Strike price of stock:  100

Volatility of stock:  20%

Time to maturity:  0.33

Risk free rate of interest:  8%

Price of European Call Option is:  27.708

Price of European Put Option is:  0.069
```

2. Sample code for calculating option price by N-step Binomial Method.

```
public class Binomial
  {
  public void nodeprice(double[][]nodes)
    {
    int l = nodes.length-1;
    int s = stockprice
    double u = risefactor
    double d = downfactor
    for(int i=0;i<=l;i++)
      {
      for(int j=0;j<=i;j++)
        {
        nodes[i][j] = s*Math.pow(u,j)*Math(d,i-j);
        }
      }
    }
  public void optionvalue(double[][]price)
    {
    int l = price.length-1;
    int n = nodeprice;
    int k = strikeprice
    for(int i=0;i<=last;i++)
      {
      if(calloption)
```

```
        price[l][j] = Math.max(n[l][j]-k,0);
      else
        price[l][j] = Math.max(k-n[l][j],0);
      }
    }
```

3. Deriving the BSM differential equation.

The process followed by stock price is already developed in the text which is given by

$$\frac{dS}{S} = \mu dt + \sigma dz \tag{8.1}$$

and if we suppose f to be the price of call option which is a function of S and t, then it must satisfy

$$df = (\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial f}{\partial S}\sigma S dz \tag{8.2}$$

Moreover, Weiner process of f and S are same so we can construct our portfolio wisely so as to eliminate Weiner process. We can choose a portfolio in which we take a short position in one derivative and long position in shares $\partial f/\partial S$. Then the value of our portfolio will be given by Π where

$$\Pi = -f + \partial f \partial S S \tag{8.3}$$

Change in the value of portfolio in time Δt is given by:

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S}\Delta S \tag{8.4}$$

If we substitute the discrete versions of equation 8.1 and equation 8.2 in equation 8.4 we have

$$\Delta\Pi = \Big(-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\Big)\Delta t \tag{8.5}$$

Now here we see that this equation does not include Δz hence this portfolio remains riskless for time Δt and should earn risk free rate of interest without arbitrage opportunities. Thus it gives:

$$\Delta\Pi = r\Pi\Delta t \tag{8.6}$$

Using equation 8.3 and equation 8.5 in equation 8.6 we have

$$\Big(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\Big)\Delta t = r\Big(f - \frac{\partial f}{\partial S}S\Big)\Delta t \tag{8.7}$$

which is modified into

$$\frac{\partial f}{T} + rS\frac{\partial f}{S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{S^2} = rf \tag{8.8}$$

This is the BSM differential equation.

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