Group Theoretical Aspects of Asymptotically Strong Supersymmetric GUTs

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Certificate of Examination

This is to certify that the dissertation titled "**Group Theoretical Aspects of Asymptotically Strong Supersymmetric GUTs**" submitted by **Amit Suthar** (Reg. No. MS15053) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. C.S. Aulakh at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. C.S. Aulakh
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Abstract

We recapitulate the basic group theory needed for GUTs. It include the weights, roots, Dynkin diagrams, generalized Gell-Mann matrices for $SU(N)$ and spinorial representations of $SO(10)$.

In the second chapter, we present a quick overview of $SU(5)$ and $SO(10)$ GUTs. For both the GUTs, spontaneous symmetry breaking is discussed at length. In the case of $SU(5)$, exact B,L violating vertices and hence four-Fermi lagrangian is calculated. Then we calculate the decompositions of $SO(10)$ representations under two maximal subgroups $SU(5) \times U(1)$ and G_{PS}.

In third chapter, we present a quick overview of superspace formulation and supersymmetry. It includes the details about how to construct a supersymmetric lagrangian and an instructive example, MSSM (Minimal Supersymmetric Standard Model).

We present a few properties of adjoint type representations $r \times \overline{r}$; especially with *totally symmetric* representations as the base (r) in Chapter 4. We note that the irreducible representations appearing in this particular case have some neat properties. S_2 for all such representations is calculated in closed form. Using these bigger adjoint type multiplets, symmetry breaking of toy models $SU(2), SU(3)$ are presented. Since $SU(5) \to G_{SM}$ also preserves the rank, we can use any adjoint type multiplets for this. We present two non-trivial ways to break this symmetry.

According to a recent work [Aulakh 20], gaugino condensates drive the creation of vevs of chiral supermulitplet in AS gauge theories. This replaces the usual potential driven symmetry breaking by dynamical symmetry breaking. We use this to calculate symmetry breaking vevs for two cases: $SU(2) \to U(1)$ and $SU(5) \to G_{SM}$. Numerical calculations were done to calculate vevs for these two cases. Later on we extend the given framework to include the traceless fields also. The loop equations for such a field are derived from the GKA equations. Numerical calculations were done to calculate vevs for three symmetry breaking patterns: $SU(2) \to U(1)$, $SU(3) \to SU(2) \times U(1)$ and $SU(5) \to G_{SM}$ using traceless 3×3, $6 \times \overline{6}$ and $10 \times \overline{10}$ respectively.

Contents

Introduction

The concept of symmetry is central in theoretical physics. All the properties of the fundamental particles are determined by the way they transform under various symmetries. The spin of a particle is determined by the way it transforms under spacetime rotation, i.e., under the Lorentz group. Our understanding of the interactions of fundamental particles is based on gauge theories. The particles' interactions are determined by the way they transform under the concerned Lie group of the gauge theory. A good understanding of Lie groups and their representations is crucial while studying SM and any Grand unified model. Thus we start with a review of Lie groups and their representations in Chapter 1. We discuss the usual tensor representations, weights, roots, and the spinor representations of $SO(N)$. Thus the representations of the Lorentz group are also covered.

The Standard Model is a gauge theory with gauge group $SU(3) \times SU(2) \times U(1)$. But it could not be the ultimate theory of everything. There are many things that it does not explain, like neutrino masses, dark matter, etc. More over, inspite of many elegant features like cancellation of gauge anomalies, it leaves a room for improvement from an aethetics viewpoint, which are crucial for a theory of everything. So many models were built during the 1970s and '80s, which unify the SM into a gauge theory with a bigger gauge group with a single coupling constant. They come under the broad term: *Grand Unified Theories, GUTs.* Studying GUTs is mostly about studying group theories of bigger groups. After developing the required group theoretical background in Chapter 1, we present an introductory review of the $SU(5)$ and $SO(10)$ GUTs in Chapter 2.

In chapter 3, we present an introductory review of supersymmetry and superspace formulation. Supersymmetry is a spacetime symmetry that can change the spin of a particle. In the supersymmetric paradigm, a fermion and a boson are part of the same supermultiplet. Supersymmetry provides a way out of the notorious *hierarchy problem.* Supersymmetric gauge theories also play a key role in more exotic theories like string theory and supergravity. Without supersymmetry, GUTs make no sense. We shall see that without supersymmetry, the three couplings of SM do not *unify.* Also, without Supersymmetry, GUTs worsen the hierarchy problem. GUTs bring with them another energy scale, generally of the order of $10^{\sim 16}$ GeV. So now the parameters in SM need to be fine-tuned up to an accuracy of one part in $10^{\sim 16}$.

All the GUT gauge groups need to be broken down to the SM gauge group. Usually, the symmetry breaking is achieved by introducing a potential for the Higgs fields. The vacuum state or the ground state is the lowest energy state. So we minimize the potential, and the minimal value of the field gives the vacuum expectation value (vev). This is a purely classical analysis. It does not take into account any quantum loop corrections or anything else. Even then, this concept is widely used for electroweak symmetry breaking in SM and mostly even supersymmetric GUT breaking. Exaggerating it a little bit, this is like having a nice Mercedes with wooden wheels. So it has been a longstanding dream to have a mechanism to replace the potential driven symmetry breaking.

The minimal by parameter counting Susy GUT, generally referred as MSGUT (*Minimal Supersymmetric GUT*) [Aulakh 83], [Clark 82], turns out to be Asymptotically Strong (AS), i.e., the gauge coupling has a Landau pole in UV. This is usually considered as a defect in the GUT. However, recent work [Aulakh 20] provides a new interpretation of the same. It suggests that the AS is not a defect, but rather a way for the model to create its own UV cutoff. In the strongly coupled region, we expect the presence of chiral condensates. In this new framework, the gauge-invariant gaugino condensates cause a chiral supermultiplet to develop vev. This vev lead to the symmetry breaking of the GUT. This framework replaces the usual potential driven mechanism. This is the reason that these gaugino condensates are termed *pleromal*. They create the GUT scale, and they drive the symmetry breaking. Since we can not do any perturbative calculations in the strong coupling region, we resort to the constraints put by Generalized Konishi Anomalies (GKA). Using these, we can calculate various quantities, including full loop corrections. Thus we can calculate the condensate driven vevs.

Since we are interested in the AS case, bigger representations are used. So in chapter 4, we present some novel and interesting aspects of *adjoint type* representations, i.e., $r \times \overline{r}$ representations with arbitrary base r. We calculate S_2 values for a few cases and see how to separate out the irreps in it. We also try to use these bigger representations for a few cases of symmetry breakings of $SU(5)$.

In the last chapter, we summarise the details of pleromal condensate method introduced in [Aulakh 20]. In the original work, a toy model symmetry breaking of $SU(3) \to SU(2) \times U(1)$ is investigated using the novel methods. Here we use the same model to study two more similar symmetry breaking patterns: $SU(2) \to U(1)$ and $SU(5) \to G_{SM}$. We calculate the vevs numerically. Then we extend the framework to include traceless adjoint type representations, i.e., with the singlet removed. Again, we work out all the three cases of symmetry breaking using the traceless representations. Our ultimate goal is to use irreducible representations for the symmetry breaking, and we take one step forward in this thesis.

Chapter 1

Representation of Lie Groups

1.1 Tensor Representations

"A Lie group is a continuously generated group, that contain elements arbitrarily close to the identity, such that the general element can be reached by the repeated action of these infinitesimal elements." [Peskin 95]

A representation of a group is a map from the abstract space in which elements of the group exists to vector space of finite-dimensional matrices.

Consider the transformation law: $\varphi_a \to U_a{}^b \varphi_b$. Here φ_a is an object with one index. When contracted with transformation matrix, U, we get the transformed φ. So it feels natural to define objects with arbitrary number of indices. For example, consider a tensorial object with two indices. It would transform by contracting it with product of two U matrices as follows:

$$\psi'_{ab} = \mathcal{U}_{ab}{}^{cd}\psi_{cd} = U_a{}^c U_b{}^d \psi_{cd} \tag{1.1}$$

However, it turns out that this representation is reducible. The generators in this rep commute with permutation of indices. This essentially means that a symmetric tensor must transform into another symmetric tensor. Thus (anti)symmetric tensors form irreducible representations. The generators in these representations would have the following form:[1]

$$\mathcal{A}_{ab}{}^{cd} = \frac{1}{2} T_{[a}{}^{[c}\delta_{b]}{}^{d]} \tag{1.2}$$

$$\mathcal{S}_{ab}{}^{cd} = \frac{1}{2} T_{\{a}{}^{\{c}\delta_{b\}}{}^{d\}} \tag{1.3}$$

This analysis can be extended to arbitrary number of indices.

$$\psi'_{a_1 a_2 \ldots a_n} = U_{a_1}{}^{b_1} U_{a_2}{}^{b_2} \ldots U_{a_n}{}^{b_n} \psi_{b_1 b_2 \ldots b_n} \tag{1.4}$$

[1] T_a always refer to the generators in fundamental.

These indices need to be symmetrized or antisymmetrized. A representation can be denoted by *Young Tableaux*. It consists of some blocks; each block represents an index. All the indices in a column are antisymmetrized, and those in a row are symmetrized.

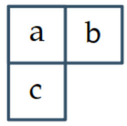


Figure 1.1: A typical Young Tableaux: {[ab}c]

For example, rather than saying that representation is $\{[ab\}c]$, Figure 1.1 can be used.

For SU(N), a column can not have more than N blocks. A column with N blocks is simply a singlet, because this represents a totally antisymmetric tensor.

Upper and lower indices If φ_a transform as $\mathbf{N}$, then an object transforming as $\bar{\mathbf{N}}$ can be stated as φ^a

$$\delta\varphi^a = -(T^*)^a{}_b\,\varphi^b = -(T^T)^a{}_b\,\varphi^b = -\varphi^b\,T_b{}^a \tag{1.5}$$

With this, objects with both lower and upper indices can be created. Note that a lower index can only be summed with an upper index (and vice-versa). Raising and lowering of indices can be done with total antisymmetric invariant tensor.

$$\varphi^a = \epsilon^{a\,i_1 i_2 \ldots i_{N-1}} \varphi_{i_1 i_2 \ldots i_{N-1}} \tag{1.6}$$

So an upper index is equivalent to (N-1) antisymmetrized lower indices.

$C_2(r)$ and $S_2(r)$

Consider the quantity, $T_A T_A$. It can be verified that it commute with all generators. For irreducible representation, such thing must be proportional to identity (otherwise the components with different eigenvalue won't mix, i.e. the representation would be reducible.)

$$T_A T_A = C_2(r)\mathbb{I} \tag{1.7}$$

The constant is characteristic of the representation concerned.

Another such a quantity can be defined as follows:

$$\mathrm{tr}(T_A T_B) = S_2(r)\delta_{AB} \tag{1.8}$$

For $\mathbf{N}$ of SU(N), $S_2(N) = \dfrac{1}{2}$ by convention.

It can be verified that,

$$d(r)C_2(r) = d(G)S_2(r) \tag{1.9}$$

Here $d(r)$ is dimension of representation and $d(G)$ is the dimension of group.

1.2 Cartan Subalgebra and Weights

The set of diagonal generators in a Lie group is called *Cartan subalgebra*. The number of diagonal generators in a group is called the *rank* of the group. *rank of SU(N) is* N−1.

In general, we can choose any linearly independent set of matrices that satisfies the required conditions. A very useful example would be that of U(N). Here we need N^2 matrices for generators. Most convenient and naturally occuring such a set is:

$$(T_{ik})_{ab} = \delta_{ia}\delta_{kb} \qquad (i, k, \ldots, a, b, \ldots \in [1, N]) \tag{1.10}$$

Even though they are not Hermitian, they are used as the generators of U(N) upon a cost that the parameters are no longer real. Unitarity of the matrices is ensured by the constraints on parameters of transformation rather than on generators.

From Eq 1.10, the commutation relations for U(N) turns out to be:

$$[T_{ij}, T_{kl}] = T_{il}\delta_{jk} - T_{kj}\delta_{il} \tag{1.11}$$

The point here is that we have the freedom to choose any convenient form of generators, and for SU(N), such a commonly used basis is *generalised Gell-Mann form.*

Canonical form of generators for SU(N): Generalised Gell-Mann matrices Out of total $N^2 - 1$ generators of SU(N), $N - 1$ are diagonal. The off-diagonal generators come in pairs with each pair being just σ_1 or σ_2 in a certain sector, i.e., a certain row and column. There are ${}^NC_2$ such sectors possible. Thus $2\times({}^NC_2)$ off-diagonal generators just as expected. These are just generalization of Pauli matrices.

The exact form of Cartan Subalgebra for Gell-Mann matrices is as follows: (Cartan generators are denoted by H_m here)

$$\begin{aligned}
H_1 &\equiv T_3 = \frac{1}{2}\,\mathrm{diag}[1 \quad -1 \quad 0 \quad 0 \quad \ldots \quad 0] \\
H_2 &\equiv T_8 = \frac{1}{2\sqrt{3}}\,\mathrm{diag}[1 \quad 1 \quad -2 \quad 0 \quad \ldots \quad 0] \\
&\vdots \\
H_{(N-1)} &\equiv T_{N^2-1} = \frac{1}{\sqrt{2N(N-1)}}\,\mathrm{diag}[1 \quad 1 \quad 1 \quad 1 \quad \ldots \quad (1-N)]
\end{aligned}$$

The factors in front are chosen so as to have the correct normalisation for fundamental, $S_2(N) = \dfrac{1}{2}$

$$[H_m]_{jj} = \frac{1}{\sqrt{2m(m+1)}}\left[\sum_{k=1}^{m}\delta_{kj} - m\;\delta_{j,m+1}\right] \tag{1.12}$$

Weight The set of all the eigenvalues of diagonal generators corresponding to a basis vector of the representation is the *weight* of that vector.

Thus each weight would be a tuple, a vector in $\mathbb{R}^{\mathrm{rank(G)}}$. Total number of basis vectors is the dimension of representation concerned. Thus there are total $d(r)$ weight vectors in $\mathbb{R}^{N-1}$ for SU(N). Since we already have the general form of Cartan subalgebra for fundamental in Eqn 1.12, we also have the weight vectors in fundamental: $[\nu_j]_m = [H_m]_{jj}$

Dynkin Indices Since different representations of SU(3) make different kinds of hexagons, each representation can be denoted by *Dynkin indices*, which are the sizes of two adjacent sides of the hexagon (A triangle is a special case of the hexagon with alternate sides of dimension zero.) Note that the alternate sides would be equal to each other, thus specifying only two adjacent sides for any vertex specifies the whole shape.

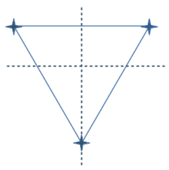


Figure 1.2: $\mathbf{3} \equiv (1,0)$

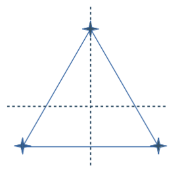


Figure 1.3: $\bar{\mathbf{3}} \equiv (0,1)$

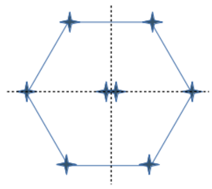


Figure 1.4: $\mathbf{8} \equiv (1,1)$

The concept of Dynkin indices can be extended for bigger groups as well. For example, for SU(4), such indices would be the dimensions of 3 adjacent sides of *football* (the general shape representation in SU(4) make; polyhedron with 20 equilateral triangles faces, look below Eq 1.22 for details). A tetrahedron would be a special case of such a shape with two sides zero. For SU(N), $N-1$ such indices would be needed to depict any representation.

The interesting point is that indices for N would always be $(1,0,\ldots,0)$, and that for adjoint would always be $(1,0,\ldots,0,1)$. This is one of the things that makes the adjoint special.

Weight Vectors of general irreps For any general tensor representation,

$$\mathcal{T}_{i_1 i_2 \ldots i_n}{}^{j_1 j_2 \ldots j_n} \ \psi_{j_1 j_2 \ldots j_n} = \mu(i_1 i_2 \ldots i_n)\psi_{i_1 i_2 \ldots i_n} \tag{1.13}$$

As any representation can be created by tensor products of fundamental representations, weight vector of any representation can be related to weights in fundamental as follows,

$$\mu(i_1 i_2 \ldots i_n) = \sum_{k=1}^{n} \mu_{_N}(i_k) \tag{1.14}$$

Note that even for antisymmetrized indices, weights add up. Another interesting property about weights would be that in any irrep, the sum of all weights is zero.

$$\sum_{i_1 i_2 \ldots i_n} \mu(i_1 i_2 \ldots i_n) = 0 \tag{1.15}$$

It follows from the tracelessness of the generator matrices in all the representations.

Since one upper index is equivalent to N-1 antisymmetrized indices, the weights for conjugate representation would be

$$\mu(\bar{k}) = \mu(i_1 \ldots i_{k-1} i_{k+1} \ldots i_n) = \sum_{i_1 i_2 \ldots i_n} \mu(i_1 i_2 \ldots i_n) - \mu(k) = -\mu(k) \tag{1.16}$$

Relation between S_N and fundamental of SU(N)

As mentioned earlier, $[H_m]_{jj} = [\nu_j]_m$, from Eq 1.12, we have

$$[\nu_j]_m = [H_m]_{jj} = \frac{1}{\sqrt{2m(m+1)}} \left[\sum_{k=1}^{m} \delta_{kj} - m\ \delta_{j,m+1} \right] \tag{1.17}$$

Now with some trouble, it can be shown that

$$\nu_i . \nu_j = \frac{1}{2} \left(\delta_{ij} - \frac{1}{N} \right) \tag{1.18}$$

$$\text{Hence} \qquad i \neq j \qquad |\nu_i - \nu_j| = 1 \tag{1.19}$$

What this implies is that distance between any two weight vectors is unity. In $\mathbb{R}^2$, only an equilateral triangle has this property. In $\mathbb{R}^3$, only a regular tetrahedron has this property. The commonality between these figures is that these shapes respect permutation symmetry between any pair of points i.e., S_N.

For all SU(N), weight diagram for N would be the figure corresponding to S_N group.

Since we have the general structure of weights of fundamental of SU(N) as Eq 1.17, this implies that we have coordinates for all S_N shapes.

A tetrahedron is 4 equilateral triangles joined in one extra dimension. The shape corresponding to S_5 would be 5 regular tetrahedrons joined in one extra dimension. This also follows from:

$$S_N = \bigotimes_{i=1}^{N} S_{N-1} \tag{1.20}$$

1.3 Roots and Dynkin Diagram

Weight vectors corresponding to the adjoint representation are called *roots*. Here is an important property of root vectors: *If α is a root vector, then $n\alpha$ is a root vector if and* *only if* $(n = 0, \pm 1)$. [Georgi 82]

This implies that for every root vector α, $-\alpha$ is also a root. This is because of the fact that the adjoint representation is real. Taking conjugate (of representation) change every α into a $-\alpha$ and adjoint remains unchanged.

Adjoint representation is ${\psi_a}^b$ with a constraint that ${\psi_a}^a = 0$. Now it follows from Eq 1.14 i.e. weights add up:

$$\alpha({\psi_a}^b) = \nu_a - \nu_b \tag{1.21}$$

Here ν_i are the *weights of fundamental.* As proved earlier in totally different context, (Eq 1.19) that all the roots for all SU(N) would have either unit length or zero. Also it is clear that for SU(N), $N-1$ roots would be zero (corresponding to Cartan subalgebra).

Shape of Root vectors in $\mathbb{R}^{N-1}$ Non-zero root vectors of SU(3) would be:

$$\pm(1,0) \qquad \pm\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \qquad \pm\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \tag{1.22}$$

The weight space diagram of root vectors for SU(3) form a hexagon, as shown in Fig 1.2.

For SU(4), the root vectors would form a *'football'* like polyhedron in $\mathbb{R}^3$ with all 20 faces begin equilateral triangles. Total 15 vertices would be arranged as follows: 3 forming a triangle on a plane $\left(z = \frac{2}{\sqrt{6}}\right)$, 6 forming a hexagon on a plane below the former one $(z = 0)$, 3 more forming a triangle on a plane below the hexagon $\left(z = -\frac{2}{\sqrt{6}}\right)$ and remaining 3 at the origin. Here are all the non-zero root vectors of SU(4):

$$\pm\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right) \qquad \pm\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \frac{2}{\sqrt{6}}\right) \qquad \pm(1,0,0)$$
$$\pm\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right) \qquad \pm\left(\frac{1}{2}, \frac{-1}{2\sqrt{3}}, \frac{-2}{\sqrt{6}}\right) \qquad \pm\left(0, \frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{6}}\right)$$

For bigger SU(N), roots would form *higher dimensional footballs.* Again, just like the shapes of S_N, we have the coordinates for all of such shapes.

Positive Roots Out of all weight vectors, some can be called positive weights according to some convention.

If the first non-zero component of the weight vector is positive, then it is termed a positive weight.

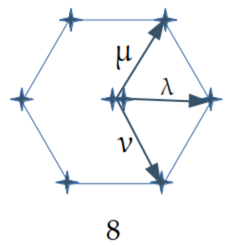
Along the same lines, ordering can also be defined in weight vectors:

$$\alpha > \beta \qquad \Longleftrightarrow \qquad \alpha - \beta > 0 \tag{1.23}$$

Simple Roots *Simple roots are those positive roots that can not be written as the sum of other positive roots.*

Picking out the simple roots is like defining the basis vectors. All the roots can be written as the *linear combination* of simple roots.

A Lie group with rank n would have weight vectors in $\mathbb{R}^n$. Thus we can expect n simple roots. This is clear for the simple example of SU(3). As shown in the figure, there are three positive roots: μ, ν, λ. Two of them are zero. Since $\lambda = \mu + \nu$, λ is not a simple root. Thus for SU(3), μ and ν are the simple roots.



$$\mu = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \qquad \nu = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

Figure 1.5: Positive Roots

Simple Roots for SU(N) in general Using the convention to define the ordering of weight vectors of fundamental, we can find the positive roots as follows:

$$\nu_1 > \nu_2 > \ldots > \nu_N \qquad \Rightarrow \text{Positive Roots:} \quad (\nu_i - \nu_j) \quad i < j$$

(Note, however, that ν_i from Eqn 1.17 are not ordered like this, so we just need to relabel them to fit this order.)

There are a total of $N(N-1)/2$ positive roots. Out of them, we can identify the simple roots:

$$\nu_i - \nu_{i+1} \qquad i \in [1, N-1] \tag{1.24}$$

These can not be written as the *sum* of other positive roots.

Dynkin Diagram

Root vectors can determine all the properties of the Lie group. Since the simple roots contain all the information about the roots, thus the set of simple roots has all of the information about the group and its algebra encoded in them.

Dynkin diagram is a short-hand notation for writing down the simple roots. Each simple root is depicted by a circle, and pairs of circles are connected by the angle between the simple roots. Dynkin diagram can be used to recreate the full algebra.

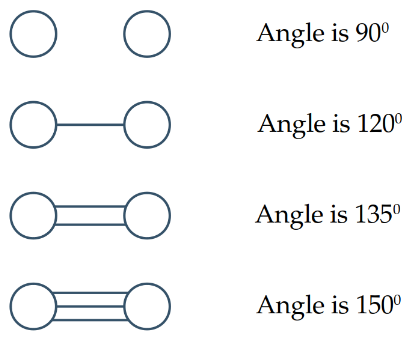


Figure 1.6: Dynkin Diagrams

For SU(N), it follows from (1.18)

$$i < j \qquad (\nu_i - \nu_{i+1}).(\nu_j - \nu_{j+1}) = -\frac{1}{2}\delta_{i+1,j} \tag{1.25}$$

$$\text{Since,} \quad |\nu_i - \nu_{i+1}| = 1 \tag{1.26}$$

$$\Rightarrow \text{For two neighboring simple roots,} \quad \cos\theta = 120^0 \tag{1.27}$$

Thus the Dynkin diagram for SU(N) would look like Figure 1.7 with N circles:

Figure 1.7: Dynkin Diagram for SU(N)

1.4 SO(N)

SO(N) is length preserving rotation in $\mathbb{R}^N$. SO(N) has total of ${}^NC_2$ generators i.e. all possible ways in which a real plane can be chosen for rotation. Generators for SO(N) can be obtained from simple rotation matrix:

$$(L_{ij})_{kl} = -i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \tag{1.28}$$

Note that two indices are used to indicate a generator, and the result is pure imaginary, so that element of SO(N) would turn out to be real. The generators are mere σ_2 in different sectors.

The commutation relations are:

$$[L_{ij}, L_{kl}] = i\left[\delta_{jl}L_{ik} - \delta_{jk}L_{il} + \delta_{il}L_{kj} - \delta_{ik}L_{lj}\right] \tag{1.29}$$

None of the generators defined in Eq 1.28 are diagonal. Though the generators are diagonalizable, we can choose any set to be Cartan subalgebra.

SO(2n)

Cartan Subalgebra For SO(2n), we can not simultaneously diagonlise more than n generators, this can be realized using this unitary transformation:

$$U = v_{2\times 2} \otimes \mathbb{I}_n \qquad \text{where} \qquad v_{2\times 2} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \tag{1.30}$$

However, the convenient form of Eqn 1.28 need not be discarded. We can keep that same form and change the basis vectors from $[1 \quad 0\ldots]$ to $[1 \quad \pm I\ldots]$.

Finally we have the Cartan Subalgebra for SO(2n):

$$[H_m]_{kl} = -i(\delta_{2m-1,k}\delta_{2m,l} - \delta_{2m-1,l}\delta_{2m,k}) \tag{1.31}$$

Weights and Roots Fundamental representation is $2n$ dimensional and rank is n. The eigenvalues of Cartan subalgebra from Eqn 1.31 are ± 1. The following eigenvalue equation shows it explicitly.

$$|\pm e^k\rangle_j = \delta_{j,2k-1} \pm i\delta_{j,2k} \tag{1.32}$$

$$H_m|\pm e^k\rangle = \pm\delta_{km}|\pm e^k\rangle \tag{1.33}$$

Thus weights in fundamental are $\pm$ the unit vectors $[e_k]_m = \delta_{km}$ (Total $2n$ vectors in $\mathbb{R}^n$)

Roots Just like SU(N) case, if $\mu(i)$ are the weights of fundamental, then since adjoint is of the form ${\phi_a}^b$, roots would be $\mu(a) - \mu(b)$. So there would be n zero roots. Non zero roots for SO(2n) would be of form, $[\pm e_i \pm e_j]$ with $i \neq j$

Now its clear that:

$$\begin{aligned} [e_k]_m = \delta_{km} \qquad &\Rightarrow \qquad e_i > e_k \quad \text{for} \quad i < k \\ \text{Thus positive roots are :} \qquad &e_i \pm e_k \quad \text{for} \quad i < k \\ \text{And simple roots are:} \qquad &e_j - e_{j+1} \quad \text{for } j \in [1, n-1] \\ &\text{and } e_{n-1} + e_n \end{aligned}$$

Just to be expected, the number of simple roots is equal to rank i.e., n

Note here that the length of all root vectors is $\sqrt{2}$, unlike those of SU(N) (unit). Non-zero roots of SO(2N) would sit on the vertices of squares on all possible planes centered at the origin. For example, for SO(6), there would be three roots at the origin, and four roots per plane, total 15.

Dynkin Diagram With given simple roots, the angle between consecutive roots turn out to be 120^0. However, both $e_{N-1}-e_N$ and $e_{N-1}+e_N$ share an angle 120^0 with $e_{N-2}-e_{N-1}$. Thus we have a more interesting Dynkin diagram than the boring SU(N) as shown in Fig 1.4.

Figure 1.8: Dynkin Diagram of SO(2N)

SO(2n+1)

Even in this case, there are still only n generators that can be simultaneously diagnolised, hence rank n. Thus the weights in fundamental has one extra null weight than the usual $\pm e_m$. Just as the earlier case, the roots would be all possible combination:

$$\begin{aligned}
\text{Roots :} \quad & [\pm e_i \pm e_j] \quad \text{for} \quad i \neq j \quad \text{and} \quad \pm e_i \\
\text{Positive Roots :} \quad & [e_i \pm e_j] \quad \text{for} \quad i < j \quad \text{and} \quad e_i \\
\text{Thus Simple Roots:} \quad & [e_i - e_{i+1}] \quad \text{for} \quad j \in [1, n-1] \quad \text{and} \quad e_n
\end{aligned}$$

Again, there are a total of n simple roots (same as the rank).

Dynkin Diaram It is pretty straight forward from here. e_n share an angle of 135^0 with $e_{n-1}-e_n$ and rest of the consecutive ones share an angle of 120^0 with each other. Thus we have the diagram as shown in Fig 1.4

Figure 1.9: Dynkin Diagram of SO(2N+1)

Subgroups and Dynkin diagram If H is a subgroup of G, then Dynkin diagram of H would be present as a part of G. For example, all SU(m) are part of SU(n) diagram for $n > m$. Not just that, two Dynkin diagrams equal would mean that both the algebras are identical. For example, diagrams for SO(3) and SU(2) are identical. Similarly, the algebra of SU(4) and SO(6) are identical, and their Dynkin diagrams are also identical. So they hold great importance.

1.5 Clifford Algebra

Consider a set of $2n$ matrices of dimensions 2^n called Γ_i, such that they follow a certain anticommutation rule:

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij} \tag{1.34}$$

Now create another object Σ_{ij} from these matrices:

$$\Sigma_{ij} = \frac{1}{4i}\left[\Gamma_i, \Gamma_j\right] \tag{1.35}$$

Now for no reason lets find commutator of two such objects,

$$[\Sigma_{ij}, \Sigma_{kl}] = i\left[\delta_{jl}\Sigma_{ik} - \delta_{jk}\Sigma_{il} + \delta_{il}\Sigma_{kj} - \delta_{ik}\Sigma_{lj}\right] \tag{1.36}$$

This structure is called *Clifford algebra.*

Comparing Eq 1.36 result with that of Eq 1.29, we find that Σ form a representation of SO(2n). The matrices Σ are 2^n dimensional.

Relation with Clifford algebra of Lorentz group Note that the Clifford algebra for Lorentz group is a special case of this formalism. The anticommutation relations for γ matrices in Lorentz group is the modified metric for Lorentz group i.e. rather than δ_{ij}, its $g_{\mu\nu}$ i.e.

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \tag{1.37}$$

Just like this, the representation theory discussed so far can be extended for groups with modified *geometry*, just like SO(1,3).

Irreducible? Analogous to the γ_5, we can construct an object that anticommute with all of predefined Γ:

$$\Gamma_V = (-i)^n \prod_{i=1}^{2n} \Gamma_i \tag{1.38}$$

Again, analogous to the Lorentz group:

$$\{\Gamma_V, \Gamma_i\} = 0 \qquad \Rightarrow \qquad [\Sigma_{ij}, \Gamma_V] = 0 \tag{1.39}$$

Since Γ_V anticommute with other matrices, it can not be proportional to the identity matrix. So there exists a matrix that commutes with all the generators and is not proportional to the identity matrix implies that the representation concerned is reducible.

The irreducible representations 2^{n-1} dimensional for SO(2n) can be can be seprated out using the projection operators:

$$P_\pm = \frac{1 \pm \Gamma_V}{2} \tag{1.40}$$

This is again analogous to the Lorentz group; Dirac fermions are not the irreducible representation, but the eigenvectors of γ_5 i.e., *left* and *right* handed Weyl fermions are irreducible.

Explicit forms Here is an explicit form for Γ.

$$\begin{aligned}
\Gamma_1 &= \sigma_1 \otimes \ldots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \qquad \text{n times for SO(2n)} \\
\Gamma_2 &= \sigma_1 \otimes \ldots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_2 \\
\Gamma_3 &= \sigma_1 \otimes \ldots \otimes \sigma_1 \otimes \sigma_1 \otimes \sigma_3 \\
\Gamma_4 &= \sigma_1 \otimes \ldots \otimes \sigma_1 \otimes \sigma_2 \otimes \mathbb{I} \\
\Gamma_5 &= \sigma_1 \otimes \ldots \otimes \sigma_1 \otimes \sigma_3 \otimes \mathbb{I} \\
\Gamma_6 &= \sigma_1 \otimes \ldots \otimes \sigma_2 \otimes \mathbb{I} \otimes \mathbb{I} \\
&\vdots \\
\Gamma_{2n} &= \sigma_2 \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I} \otimes \mathbb{I} \\
\Rightarrow \Gamma_V &= \sigma_3 \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I} \otimes \mathbb{I}
\end{aligned}$$

Here the upper (lower) half would have eigenvalues +1(-1). So the irreducible parts are well separated in this. The upper half would not mix with the lower half.

Anyhow in general, Eigenvalue of Γ_V is product of all the weights of that state:

$$\Gamma_V = 2\Sigma_{12}\, 2\Sigma_{34} \ldots\, 2\Sigma_{2n-1,2n} \tag{1.41}$$

$$\Rightarrow \text{Eigenvalue}(\Gamma_V) = \epsilon_1 \epsilon_2 \ldots \epsilon_n \tag{1.42}$$

$SU(n)$ decomposition of $SO(2n)$

Subgroup SO(2n) has a subgroup U(n), this is clear from the fact that $\mathbb{R}^{2n} \sim \mathbb{C}^n$.

SO(2n) preserve $\sum \varphi_i^2$. U(n) that also preserves the same thing, but with a little less freedom. (It can't alter the phases of individual complex numbers).

To see how the SU(n) components are enscribed in the SO(2n) components, consider the following objects:

$$b_k = \frac{1}{2}\left(\Gamma_{2k-1} - i\Gamma_{2k}\right) \qquad ; \qquad b_k^\dagger = \frac{1}{2}\left(\Gamma_{2k-1} + i\Gamma_{2k}\right) \tag{1.43}$$

Now these objects follow certain anticommutation rules:

$$\{b_j, b_k\} = 0 = \{b_j^\dagger, b_k^\dagger\} \qquad ; \qquad \{b_j, b_k^\dagger\} = \delta_{jk} \tag{1.44}$$

Looking at the anticommutation rules, these objects certainly are the creation and annihilation operators of some n fermion fields. So the corresponding number operator should be of some importance:

$$N_{ij} = b_i^\dagger b_j \tag{1.45}$$

$$\Rightarrow \qquad \{N_{ij}, N_{kl}\} = N_{il}\delta_{jk} - N_{kj}\delta_{il} \tag{1.46}$$

This is exactly the algebra of U(N), i.e., Eq 1.11, So this is how the generators of U(n) are placed somewhere in the SO(2n) structure.

Writing Γ_i into the number operators, the form of Γ_V is:

$$\Gamma_V = (-i\Gamma_1\Gamma_2)(-i\Gamma_3\Gamma_4)\dots(-i\Gamma_{2n-1}\Gamma_{2n}) \tag{1.47}$$

$$\Gamma_V = (2N_1 - 1)(2N_2 - 1)\dots(2N_n - 1) \tag{1.48}$$

$N_i \in [0, 1] \Rightarrow (2N_i - 1) \in [-1, +1]$. So it is pretty clear that for odd number of particle states, $\Gamma_V = -1$ and for even number particle states, $\Gamma_V = +1$. Also it is simple exercise in counting to see that all possible states possible with n fermions is 2^n i.e. full reducible spinor.

A very important example is that of SO(10). The full reducible spinor representation is 32 dimensional, which consists of 16 and $\overline{16}$. Now $SU(5) \subset SO(10)$, and as both of these are GUTs; it is very instructive to see which representations of SU(5) are embedded in 16 of SO(10) and how?

Lets say that 16 is the one with Γ_V has eigenvalue +1. That would mean that only even number of particles appear in 16:

$$|\psi\rangle_{16} \equiv |0\rangle + \psi_{ij}\ b_i^\dagger b_j^\dagger\ |0\rangle + \psi_{ijkl}\ b_i^\dagger b_j^\dagger b_k^\dagger b_l^\dagger\ |0\rangle \tag{1.49}$$

Thus we have:

$$16 = 1 + \overline{5} + 10 \tag{1.50}$$

$$\overline{16} = 1 + 5 + \overline{10} \tag{1.51}$$

Self Duality Using the totally antisymmetric tensor $\epsilon_{i_1 \dots i_{2n}}$, any object with a total m vector indices can be converted into another object with $2n - m$ indices. Hence objects with only n vector indices or lesser are linearly independent. An important thing happens when an object has n vector indices precisely. Such an object is dual to itself, so only half of its components are independent. A useful example is that of an object with 5 vector indices for SO(10). There are ${}^{10}C_5$ components expected in it, but self-duality leaves only half of them independent i.e., 126.

Product of Spinor Representations Consider the product of two *left handed* spinors. What this essentially means is that we need to sum up the spinor degrees of freedom somehow and create all possible and valid objects from them. The only way such an object can be created if we place an odd number of Γ_i between the two spinors. Placing just one Γ_i means the fundamental representation. This procedure is just like finding the bilinears for the Lorentz group's Clifford algebra.

For example, consider $16 \otimes 16$ for SO(10). The product would have an object with 5 vector indices (126) and an object with 3 vector indices (120), along with the fundamental.

$$16 \otimes 16 = 10 \oplus 120 \oplus 126 \tag{1.52}$$

Similarly, for the product of a right-handed spinor with that of left-handed, we need to place an even number of Γ_i. Thus there is also a singlet in the product.

For example, consider $\overline{16} \otimes 16$ for SO(10). The product would have an object with two vector indices (45) and an object with four vector indices (210), along with the singlet.

$$\overline{16} \otimes 16 = 1 \oplus 45 \oplus 210 \tag{1.53}$$

Chapter 2

Grand Unified Theories

The Standard Model of Particle Physics is the best available description of interactions of the known elementary particles. It is a gauge theory with three gauge groups; U(1), SU(2), and SU(3). It is a framework which works well and has provided order to a vast wealth of phenomena, yet regarding some aspects require improvements. Another drawback would be that the SM lacks in beauty and aesthetics, which are crucial for a *theory of everything.* Thus new models were introduced, which *unify* the whole clutter of SM. This is the very spirit of reductionism.

Before moving ahead, Table 2 summarises all *observed particles* of SM with their gauge couplings.

Table 2.1: All SM particles and gauge couplings

	q_L		u_R	d_R	l_L		e_R	H	
SM couplings	$\left(3,2,\frac{1}{6}\right)$		$\left(3,1,\frac{2}{3}\right)$	$\left(3,1,-\frac{1}{3}\right)$	$\left(1,2,-\frac{1}{2}\right)$		$(1,1,-1)$	$\left(1,2,\frac{1}{2}\right)$	
	u_l	d_L	u_R	d_R	ν_L	e_L	e_R	h_1	h_2
Y	+1/6		+2/3	-1/3	-1/2		-1	1/2	
T_{3L}	+1/2	-1/2	0	0	+1/2	-1/2	0	+1/2	-1/2
$Q = T_{3L} + Y$	+2/3	-1/3	+2/3	-1/3	0	-1	-1	+1	0

2.1 Gauge Couplings

All the gauge couplings can be rescaled. This is very clear for U(1):

$$D_\mu = \partial_\mu + i g_Y Y B_\mu = \partial_\mu + i \left(\frac{g_Y}{x}\right)(xY)B_\mu$$

g_i never appears alone. It is always accompanied by the generators of the group concerned (for U(1), Y is a generator). Thus there is still freedom to rescale g_i with an appropriate change in the generator as well.

So it would mean that we can just rescale all the gauge couplings so that they match at some energy scales? We can do that, but that would be pointless because the GUT group put constraints on the gauge couplings. As it would be clear later on, for SU(5) and SO(10), the gauge couplings must follow the following constraint: [Nath 16], [Mohapatra 86]

$$\sqrt{\frac{3}{5}}\, g_Y \equiv g_1 = g_2 = g_3 \tag{2.1}$$

The gauge couplings are scale dependent. In the usual $\overline{\mathrm{MS}}$ renormalization scheme, the coupling constants depend on the energy scale (in the leading one loop approximation) as follows: [Srednicki 07], [Peskin 95]

$$\mu \frac{d\alpha_a}{d\mu} = b_a \frac{\alpha_a^2}{2\pi} \qquad \text{where} \qquad \left[\alpha_a = \frac{g_a^2}{4\pi}\right] \tag{2.2}$$

$$\Rightarrow \quad \frac{1}{\alpha_a(\mu)} = \frac{1}{\alpha_a(M_z)} + \frac{b_a}{2\pi} \ln\left[\frac{M_z}{\mu}\right] \tag{2.3}$$

Here b_a depends on the gauge group and the representations in which matter particles are in: [Peskin 95], [Srednicki 07]

$$b_a = -\frac{11}{3} S_2(G_a) + \frac{2}{3} S_2(r)\, d(F) + \frac{1}{3} S_2(r)\, d(S) \tag{2.4}$$

$$\text{Where} \quad \mathrm{Tr}\,(T_A T_B) = S_2(r)\delta_{AB} \tag{2.5}$$

$d(F)$ and $d(S)$ are the dimensionality of (Weyl) fermion and scalar representations.

Now we just need to calculate b_a for SM and see if the relation Eq 2.1 is satisfied at some energy scale.

Generator for U(1) is the hypercharge, thus $S_2(Y) = Y^2$. So we need to sum up all the hypercharge squared. Reading the values of Y from Table 2 and using the equation Eq 2.4, $b_Y = 41/6$. Each quark is counted three times because of the color index. Since we are interested in g_1 rather than g_Y, all the hypercharges get rescaled by a factor of $\sqrt{\frac{3}{5}}$ and thus the b get rescaled by 3/5. Thus we have $b_1 = 41/10$.

For $SU(N)$, $S_2(N) = 1/2$ and $S_2(G) = N$, thus leading to $b_2 = -19/6$ and $b_3 = -7$.

From SM, we have:

$$\alpha_1 = \frac{\alpha}{\cos^2\theta_W} \qquad \alpha_2 = \frac{\alpha}{\sin^2\theta_W} \tag{2.6}$$

Figure 2.1 is the plot we get after plugging in values well known from experiments though they do not intersect at any common point.

Fortunately, supersymmetry come to rescue. For the supersymmetric gauge theories, the beta function is different. There are more particles in the loops, thus the beta functions are

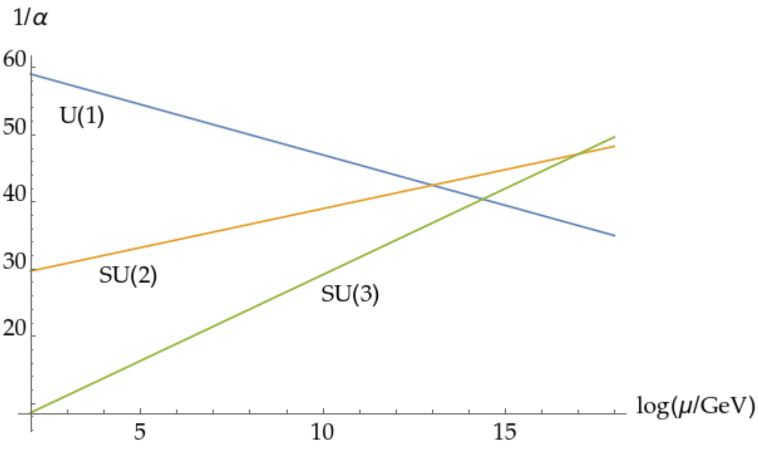


Figure 2.1: Running coupling constants of SM

larger. Avoiding the details, b_a turns out to be $\left(\frac{33}{5}, 1, -3\right)$ for *Minimal Supersymmetric Standard Model* (MSSM).[Martin 98], [Nath 16]. We have assumed that SuSy is exact at M_z. But even this crude and oversimplified calculation gives stunning results as shown in Figure 2.2. All three couplings match within experimental accuracy, around $\mu \sim 5 \times 10^{16}$ GeV. This can not be an accident. Note that putting in more information about SuSy breaking and higher loop corrections only refine this. Figure 2.2 is proof that some sort of GUT exists at high energies or nature has played a prank on us.

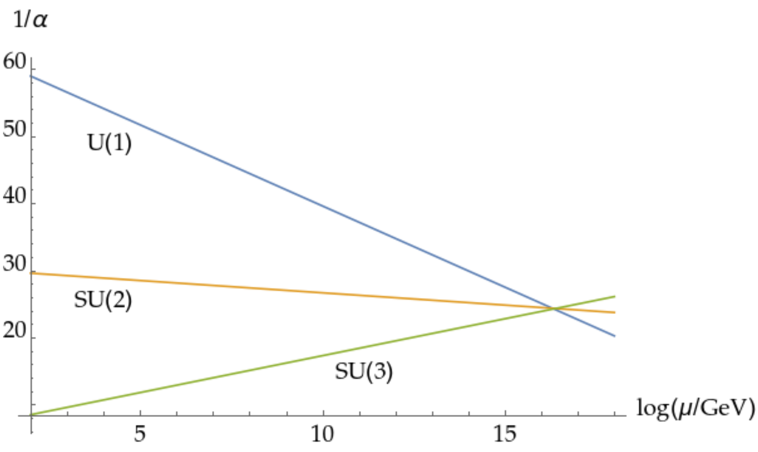


Figure 2.2: Running coupling constants of MSSM

2.2 SU(5) GUT

The standard model group $SU(3) \times SU(2) \times U(1)$ of rank 4, must be a subgroup of the grand unifying group. Thus only Lie groups with rank ≥ 4 can be considered. Simplest such a group is SU(5), therefore the simplest GUT [Georgi 74a].

2.2.1 Spontaneous Symmetry breaking

SU(5) spontaneously break into $SU(3) \times SU(2) \times U(1)$. Rank is preserved in this.

A Higgs field in fundamental representation can not cause a rank preserving symmetry breaking. This can be seen as the term responsible for masses of gauge bosons would be:

$$\Delta\mathcal{L} = -\frac{g^2}{2}|A^a_\mu T^a \langle\phi\rangle|^2 \tag{2.7}$$

For the unbroken generators, $T^a\langle\phi\rangle = 0$. Thus if all generators of Cartan subalgebra have to stay unbroken, then $\langle\phi\rangle = 0$, and there is no symmetry breaking.

Consider a 24-plet Higgs field, that tranform under adjoint representation of SU(5). It is natural to write adjoint in matrix of order N: $[\Phi = \phi^a T^a]$. (T^a are the generators in the fundamental (**5**)). Adjoint, having one upper and one lower index transforms as:

$$\Phi' = U\Phi U^\dagger \qquad\qquad U = \exp\left(-i\theta^a T^a\right) \tag{2.8}$$

It can be shown that the mass term in terms of Φ is

$$\Delta\mathcal{L} = -g^2 \mathrm{tr}\left[[T^a, \Phi][T^b, \Phi]\right] A^a_\mu A^{b\mu} \tag{2.9}$$

Now it is clear that the generators that commute with $\langle\Phi\rangle$ are unbroken. So it should commute with upper left 3×3 block and lower right 2×2 block. So it must be proportional to 3×3 identity matrix in the upper part and 2×2 in the lower two. The following form of traceless diagonal $\langle\Phi\rangle$ ensures all these properties.

$$\langle\Phi\rangle = |\Phi_0| \, \mathrm{diag}\left[-2 \quad -2 \quad -2 \quad 3 \quad 3\right] \tag{2.10}$$

Thus a single mode of ϕ get a vev and breaks $SU(5) \to G_{SM}$. [Mohapatra 86], [Nath 16]

The process of symmetry breaking is also evident from the branchings of adjoint, 24 of SU(5) under G_{SM}:

$$24 = (1, 1, 0) + (1, 3, 0) + (8, 1, 0) + \left(3, 2, -\frac{5}{6}\right) + \left(\overline{3}, 2, \frac{5}{6}\right) \tag{2.11}$$

24 has a singlet of G_{SM}, and when this singlet generates a vev, we have the desired symmetry breaking.

Eq 2.11 explicitly shows how gauge bosons of G_{SM} are embedded in the SU(5). Thus out of 24 gauge bosons of SU(5), $(1,1,0)$ corresponds to $U(1)_Y$, $(1,3,0)$ corresponds to $SU(2)$ and $(8,1,0)$ would be the ones related to color charge. The remaining 12 gauge bosons have the quantum numbers of both leptons and quarks and are called *lepto-quarks*. These new gauge bosons must be super-heavy.

The SM Higgs, corresponding to electroweak symmetry, can be put into a fundamental representation of SU(5) with three new components. We demand that the new triplet gets very high mass, so that it does not interfere with SM. So now there is a 24-plet Φ and a 5-plet H. To achieve the symmetry breaking, we need a potential, which after minimisation gives vev to both the fields. Here is the most general form of potential including both scalars: [Mohapatra 86], [Nath 16]

$$V = M^2 \mathrm{tr}\Phi^2 + \frac{\lambda_1}{2}\,\mathrm{tr}\,\Phi^4 + \frac{\lambda_2}{2}\,(\mathrm{tr}\,\Phi^2)^2 + \mu \mathrm{tr}\Phi^3 + \frac{\lambda_3}{2} H^\dagger H \mathrm{tr}\Phi^2 + \frac{\lambda_4}{2}\,H^\dagger \Phi^2 H + \frac{\lambda}{4}(H^\dagger H)^2 \tag{2.12}$$

The mass of Higgs is well known to be of order 100 GeV. So vev for H should be of the same order. However, we want the new *lepto-quark* gauge bosons to have masses of order 10^{16} GeV. From the most general form of V, this would only be possible if, $10\lambda_3 = \lambda_4$ [Nath 16]. These two arbitrary factors must follow this relation with an accuracy of one part in 10^{16}. This is not impossible but very improbable. This is called the *hierarchy problem* or *fine tuning problem*. If two unrelated numbers can take just any value from the tiniest to the largest, what forced them to be equal? Why is it that the highly improbable is happening?

There is another aspect to the problem as well. Including any GUT, there are now two scales in this framework. Since H is bound to have gauge interaction, H can get an arbitrary large correction to its mass because of the presence of the super-heavy gauge bosons and super-heavy Higgs in the loops. So how is the Higgs able to manage such a low mass?

2.2.2 Generating the generators

We need to construct the generator for hypercharge from the canonical form of generators. It should look like Eq 2.10. So by using $\mathrm{tr} T_a T_b = \delta_{ab}/2$, we have:

$$Y = \frac{1}{2\sqrt{15}} \mathrm{diag}\,[-2, \quad -2, \quad -2, \qquad 3, \qquad 3] \tag{2.13}$$

Forming Y out of canonical generators can be seen as rotating orthogonal vectors in some hyperplane. The diagonal generators in the canonical form that can mix up and form Y are

T_{15} and T_{24} (Eq 1.12)

$$T_{15} = \frac{1}{2\sqrt{6}}\,\mathrm{diag}[1 \quad 1 \quad 1 \quad -3 \quad 0]$$
$$T_{24} = \frac{1}{2\sqrt{10}}\,\mathrm{diag}[1 \quad 1 \quad 1 \quad 1 \quad -4]$$

Constructing Y from these two generators is like rotation in a plane. So we need to look for the generator *orthogonal* to Y. It turns out to be T_{3L}.

$$\begin{bmatrix} -\sqrt{\frac{5}{8}} & -\sqrt{\frac{3}{8}} \\ -\sqrt{\frac{3}{8}} & +\sqrt{\frac{5}{8}} \end{bmatrix} \cdot \begin{bmatrix} T_{15} \\ T_{24} \end{bmatrix} = \begin{bmatrix} Y \\ T_{3L} \end{bmatrix} \tag{2.14}$$

So we have the full gauge fields contracted with generators:[Mohapatra 86]

$$\tilde{A}_\mu \equiv T^a A^a_\mu = \begin{bmatrix} \left(T^{SU(3)}.G_\mu + \sqrt{\frac{3}{5}}\left(-\frac{1}{3}\right)B_\mu\right)_{\alpha\overline{\beta}} & \frac{1}{\sqrt{2}}\tilde{X}^-_{\alpha i\mu} \\ \frac{1}{\sqrt{2}}\tilde{X}^+_{j\overline{\beta}\mu} & \left(T^{SU(2)}.W_\mu + \frac{1}{2}\sqrt{\frac{3}{5}}B_\mu\right)_{ji} \end{bmatrix} \tag{2.15}$$

i, j, k are the SU(2) indices. α, β, γ are the SU(3) indices for objects transforming as **3**. For objects transforming as $\overline{\mathbf{3}}$, upper indices are used.

Relation among Couplings The eigenvalues of Y i.e. the diagonal elements would be the value of Y for particles in fundamental. Thus it is convenient to express Y as:

$$Y = \sqrt{\frac{3}{5}}\,\mathrm{diag}\left[-\frac{1}{3} \quad -\frac{1}{3} \quad -\frac{1}{3} \quad \frac{1}{2} \quad \frac{1}{2}\right] \tag{2.16}$$

Now just as explained earlier, we rescale the coupling constant g to absorb the factor of $\sqrt{\frac{3}{5}}$ and call it g_Y. Here g_5 is the coupling constant of SU(5). It is evident from Eq 2.15 that

$$\sqrt{\frac{5}{3}}\,g_Y = g_5 = g_2 = g_3 \tag{2.17}$$

Weinberg Angle Weinberg angle θ_W is defined as follows:

$$\tan\theta_W = \frac{g_{U(1)}}{g_{SU(2)}} \tag{2.18}$$

From Eq 2.17, it turns out to be:

$$\tan\theta_W = \sqrt{\frac{3}{5}} \qquad \text{and} \qquad \sin^2\theta_W = \frac{3}{8} \tag{2.19}$$

2.2.3 Assigning particle content

Now we can write the decomposition of 5, the fundamental of SU(5) under G_{SM}:

$$5 = \left(3, 1, -\frac{1}{3}\right) + \left(1, 2, \frac{1}{2}\right) \tag{2.20}$$

Comparing this with the SM particles from Table 2:

$$5_a = [d_{\alpha R} \quad l_{iL}{}^c] = [d_\alpha \quad l_i^c]_R \tag{2.21}$$

Note that charge conjugation was used to match the correct values of Y.

$$10 \equiv 5 \times 5_{AS} = \left(\overline{3}, 1, -\frac{2}{3}\right) + (1, 2, +1) + \left(3, 2, +\frac{1}{6}\right) \tag{2.22}$$

Comparing Eq 2.22 with the particles from Table 2, it is evident that,

$$10_{ab} = [q_{\alpha i} \quad u^c_{\alpha\beta} \quad e^c_{ij}]_L \tag{2.23}$$

So all the particles of SM fit nicely in two irreducible representations of SU(5). However, this combination $5 + 10$ is not anomaly free. Though $\overline{5} + 10$ is anomaly free. Note that $\overline{5} + 10$ would have all the particles in the left-handed form.

Now since all the particles in $\overline{5} + 10$ have the right quantum numbers, they give the same SM couplings of all kinds. Since we have an antisymmetric representation, then there is an overcounting during all summations. For example, both ψ_{12} and ψ_{21}, contribute. Thus being precise, we have:

$$10_{ab} = \frac{1}{\sqrt{2}}[q_{\alpha i} \quad u^c_{\alpha\beta} \quad e^c_{ij}]_L \tag{2.24}$$

2.2.4 B,L Violating Sector

The standard Model conserves the baryon number and lepton number (flavor as well). But these symmetries are merely accidental symmetries. However they suffer from quantum anomalies and $B - L$ is the only anomaly free symmetry. There is no good reason why they should be conserved. As GUTs put baryons and leptons in a single representation, they give rise to vertices that do not preserve baryon number or lepton number.

There are 6 unusual, super-heavy *lepto-quark* gauge bosons in SU(5):

$$\tilde{X}^{+}_{\alpha i} = \begin{bmatrix} X^{+4/3}_{\alpha} \\ Y^{+1/3}_{\alpha} \end{bmatrix} \equiv \left(3, 2, +\frac{5}{6}\right) \qquad \tilde{X}^{-}_{\alpha i} = \begin{bmatrix} Y^{-1/3}_{\alpha} & X^{-4/3}_{\alpha} \end{bmatrix} \equiv \left(\overline{3}, 2, -\frac{5}{6}\right) \tag{2.25}$$

Each vertex involving these violate baryon number and lepton number. Here are all such vertices: [Mohapatra 86]

$$\mathcal{L}_{\Delta B \neq 0} = \frac{g}{\sqrt{2}} \overline{d}^{\alpha}_{R} \left(\not{X}^{-}_{\alpha} e_{L}{}^{c} + \not{Y}^{-}_{\alpha} \nu_{L}{}^{c} \right) + \frac{g}{\sqrt{2}} \left[(\overline{u}^{c}{}_{L})_{\delta}\, \epsilon^{\alpha\beta\delta} \left(\not{Y}^{-}_{\alpha} d_{L\beta} + \not{X}^{-}_{\alpha} u_{L\beta} \right) + \left(\overline{d}^{\alpha}_{L} \not{X}^{-}_{\alpha} - \overline{u}^{\alpha}_{L} \not{Y}^{-}_{\alpha} \right) e^{c}{}_{L} \right] + h.c. \tag{2.26}$$

($-ve$ sign in the last term is because of the presence of antisymmetric tensor: $\psi_{ij} = \epsilon_{ij}\, e^{c}{}_{L}$).

$$\mathcal{L}_{\Delta B \neq 0} = \frac{g}{\sqrt{2}} \tilde{X}^{-}_{\alpha i}{}^{\mu} \left[\overline{d}^{\alpha}_{R}\, \gamma_{\mu} (l_{L}{}^{c})^{i} + \epsilon^{\alpha\beta\delta}\, (\overline{u}^{c}{}_{L})_{\delta}\ \gamma_{\mu}\, (q_{L\beta})^{i} + \epsilon^{ij} (\overline{q}^{\alpha}_{L})_{j} \gamma_{\mu} e^{c}{}_{L} \right] + h.c. \tag{2.27}$$

$$= \frac{g}{\sqrt{2}} \tilde{X}^{-}_{\alpha i}{}^{\mu} J_{\mu}{}^{\alpha i} + h.c. \tag{2.28}$$

Eq 2.28 defines the *Lepto-Quark current.* Note that all the vertices conserve $B - L$

Eq 2.27 can be re-written neatly in *supersymmetric notation* i.e. Weyl fermion notation. Here all the right handed particles are accompanied by a dagger †.. Details can be found in [Martin 98]. In this notation, the B and L violating operators are:

$$\mathcal{L}_{\Delta B \neq 0} = \frac{g}{\sqrt{2}} \tilde{X}^{-}_{\alpha i}{}^{\mu} \left[-d^{\dagger\alpha} \overline{\sigma}_{\mu} \overline{l}^{i} + \epsilon^{\alpha\beta\delta} \overline{u}^{\dagger}{}_{\delta} \overline{\sigma}_{\mu} q_{\beta}{}^{i} + \epsilon^{ij} q^{\dagger\alpha}{}_{j} \overline{\sigma}_{\mu} \overline{e} \right] + h.c. \tag{2.29}$$

Effective theory, Four Fermi interactions Not even a single B or L violating process has ever been observed. This should be the case since the lepto-quark gauge bosons are supposed to be super-heavy. In such a case, effective four Fermi interactions would describe the interactions very well at normal energy scales. To get an estimate, if $m_X \sim 10^{16}$ GeV, then from the uncertainty principle, it could borrow this huge energy from vacuum only for a very tiny amount of time, in which it can travel at most up to $\sim 10^{-32} m$. This is not far from the Planck's scale. So at the electroweak energy scales, the new interactions are *local.*

To get the four Fermi Lagrangian, the momentum of gauge bosons is set to zero i.e. propagator has only the inverse of mass term in it. Hence the gauge bosons completely vanish from the lagrangian or get integrated out. What is left is a non-renormalizable current-current interaction lagrangian. We can assume that both the bosons X, Y have equal mass. Here $J^{\mu}{}_{\alpha i}$ is the lepto-quark current defined in Eq 2.28.

$$\mathcal{L}^{\text{eff}}{}_{\Delta B \neq 0} = -\frac{g^2}{2M^2} J^{\dagger\alpha i}_{\mu} J^{\mu}{}_{\alpha i} \tag{2.30}$$

$$= -\frac{g^2}{2M^2}\left[-d^{\dagger\alpha}\overline{\sigma}_\mu \overline{l}^i + \epsilon^{\alpha\beta\gamma}\overline{u}^\dagger{}_\gamma\overline{\sigma}_\mu q_\beta{}^i + \epsilon^{ij}q^{\dagger\alpha}{}_j\overline{\sigma}_\mu\overline{e}\right]\times$$
$$\left[d_\alpha\sigma^\mu\overline{l}^\dagger{}_i - \epsilon_{\alpha\beta\gamma}\overline{u}^\gamma\sigma^\mu q^{\dagger\beta}{}_i - \epsilon_{ij}q_\alpha{}^j\sigma^\mu\overline{e}^\dagger\right] \quad (2.31)$$

There are total 9 terms possible. Only 4 terms out of these are B,L violating.

$$\mathcal{L}^{\text{eff}}{}_{\Delta B\neq 0} = \frac{g^2}{2M^2}\epsilon_{\alpha\beta\gamma}\left[-d^{\dagger\alpha}\overline{\sigma}_\mu\overline{l}^i + \epsilon_{ij}q^{\dagger\alpha j}\overline{\sigma}_\mu\overline{e}\right]\overline{u}^\gamma\sigma^\mu q^{\dagger\beta}{}_i + h.c \quad (2.32)$$

This expression can be simplified using the Fierz rearrangements: [Martin 98]

$$\mathcal{L}^{\text{eff}}{}_{\Delta B\neq 0} = \frac{g^2}{M^2}\epsilon_{\alpha\beta\gamma}\left[(d^{\dagger\alpha}q^{\dagger\beta}_i)(u^\gamma l^i) + \epsilon_{ij}(q^{\dagger\alpha i}q^{\dagger\beta j})(u^\gamma e)\right] + h.c. \quad (2.33)$$

Proton Decay Proton is the lightest baryon, and hence in B, L conserving regime, it is very stable. However, SU(5), with its baryon number violating operators, provides many pathways for the proton to decay. Since in SU(5), $B-L$ is conserved, a proton would decay into an anti-lepton with an optional meson(s).

Exact calculations for proton decay are quite complicated because in IR, the strong force condenses. Low energy physics can not be described using quarks and gluons. Though a naive estimate can be made [Mohapatra 86] [Nath 16]:

$$\Gamma \sim \frac{g_5^4\, m_p^5}{8\pi\, M_X^4} \quad (2.34)$$

Plugging in some numbers to get an estimate: $g_5 \sim 0.6, M_X \sim 10^{16}GeV$, the half-life turns out to be of order $10^{\sim 30}$ yrs. This is far more than the age of the observed universe. Getting a lower bound for the proton lifetime is the goal of a lot of experiments now.

Phenomenological Model: Chiral Lagrangian To handle the difficulty that we can only observe colorless hadrons decaying and not the quarks like the model, a phenomenological approach is adopted. Chiral Lagrangian is one such phenomenological model that treats $(u\ d\ s)$ as a massless triplet. Hence q_L and q_R are uncoupled, and there is a bigger symmetry group: $SU(3)_L \times SU(3)_R$. Since it is known that this symmetry is badly broken, mass terms for quarks would be introduced later on as explicit symmetry breaking terms. For details, consult [Claudson 82]

There are 8 meson fields σ_a. An object Σ is created put of it such that:

$$\Sigma = \exp\left(2i\sigma_a T_a/f\right) \qquad\qquad \Sigma \to L\Sigma R^\dagger$$

$$M = \sigma_a T_a = \begin{bmatrix} \sqrt{\frac{1}{2}}\pi^0 + \sqrt{\frac{1}{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\sqrt{\frac{1}{2}}\pi^0 + \sqrt{\frac{1}{6}}\eta & K^0 \\ K^- & \overline{K}^0 & -\sqrt{\frac{2}{3}}\eta \end{bmatrix} \tag{2.35}$$

Note that these 8 meson fields transform as (adjoint) 8-plet in the chiral symmetry. Similarly, 8 baryon fields (b^a) are introduced. Now the lagrangian can be written in the form of Σ and B, where $B = b^a T^a$. Now to make it realistic, meson mass terms are introduced as explicit symmetry breaking terms.

This phenomenological model can be used to explain strong interactions at low energy scales. Yukawa type interaction terms can be written between baryons and mesons. For example, the free baryons and meson Lagrangian would be of the following form:

$$\mathcal{L}_0 = \frac{1}{8} f^2 \,\mathrm{Tr}\,(\partial_\mu \Sigma)\left(\partial^\mu \Sigma^\dagger\right) + \mathrm{Tr}\,\overline{B}\left(i\not{\partial} - M_B\right) B \tag{2.36}$$

The B, L violating lagrangian from Eq 2.33 can be re-written with baryons and mesons in it rather than quarks by comparing how each one transforms under the chiral symmetry.

2.3 SO(10) GUT

In the last section, we saw how all the standard model particles fit nicely in two representations of SU(5): $\overline{5} + 10$. A reductionist would not see it as a true unification. There should be only one representation that fits in all the particles. SO(10) do this job nicely [Fritzsch 75][Nath 16][Mohapatra 86]. As it was mentioned in Section 1.6, decomposition of the spinor representation of SO(10) under SU(5) is:

$$16 = 1 + \overline{5} + 10 \tag{2.37}$$

All the particles of SM with an additional singlet fits in nicely in 16 of SO(10).

2.3.1 Pati-Salam Unification

It is an experimentally verified fact that nature does not treat left and right-handed particles equally. Left-right symmetric models postulate that the Lagrangian is parity symmetric, but it is the vacuum that does not respect parity. What it essentially means is that the parity-violating standard Model is the left-over of a spontaneously broken parity invariant gauge theory. Such models were one of the first attempts at grand unification.

Parity symmetry would mean the introduction of a right-handed neutrino. Such a field would be a standard model singlet. At the electroweak scale, it would not participate in any process, thus avoiding all attempts of detection. Even if a standard model singlet is produced in some process at some particle accelerator, it would only appear as missing energy. However, in recent times, neutrino oscillations have been observed, which implies that neutrinos are not massless. So either we have to resort to Majorana type mass terms for neutrinos, or we admit the existence of an SM singlet right-handed neutrino. With this, the left-right models were revived. Now neutrino masses find a natural home in SO(10) through the see-saw mechanism.

The Pati-Salam group is a well-known parity symmetric model [Pati 74]. Just like an $SU(2)_L$ for all the left-handed particles, there is an $SU(2)_R$ for the right-handed particles. This means that (ν_R, e_R) ; (u_R, d_R) also form an SU(2) doublet, just like their left-handed counterparts.

One of the many drawbacks of SM is the arbitrary nature of the relation: $Q = T_{3L} + Y$. There is no reasonable justification for the values of Y for all the particles. All the Y of SM, and hence Q are just arbitrary.

The Pati-Salam Model replaces this randomness with a very neat explanation. This is an excellent feature of the Pati-Salam Model. It unifies quarks and leptons in the fundamental representation of $SU(4)$. *Leptons are the fourth color.* The full gauge group is $SU(4) \times SU(2)_L \times SU(2)_R$. Note that the T_{15} of SU(4) would have eigenvalues proportional to $B - L$. This is the only *anomaly free non-gauged conserved* quantity of SM. The gauge coupling is scaled just so that the eigenvalues of T_{15} are $(B - L)/2$.

The hypercharge generator is formed from the T_{15} of $SU(4)$ and the T_{3R} of $SU(2)_R$:

$$Y = T_{3R} + \frac{B - L}{2} \tag{2.38}$$

$$Q = T_{3L} + T_{3R} + \frac{B - L}{2} \tag{2.39}$$

This relation can be checked explicitly by looking closely at Table 2.

In this framework, the lepton number and baryon number are not conserved. But $B - L$ is conserved, a result of a gauge symmetry. The quantum number $B - L$ is gauged and hence conserved now.

Decompostion under G_{SM}: Consider $(4, 2, 1)$. For all the particles in it, $T_{3R} = 0$. After SU(4) is broken, quarks and leptons separate out with $Y = (B - l)/2$:

$$(4, 2, 1) = \left(3, 2, +\frac{1}{6}\right) + \left(1, 2, -\frac{1}{2}\right) \tag{2.40}$$

Consider $(4, 1, 2)$. After the symmetry break back to G_{SM}, the particles with $T_{3R} = +1/2$ separate with $Y = (B - L + 1)/2$ from the particles with $T_{3R} = -1/2$ and $Y = (B - L - 1)/2$.

Since the SU(2) is broken, there are total 4 components:

$$(4,1,2) = \left(3,1,+\frac{2}{3}\right) + \left(3,1,-\frac{1}{3}\right) + (1,1,0) + (1,1,-1) \tag{2.41}$$

$$(\overline{4},1,2) = \left(\overline{3},1,-\frac{2}{3}\right) + \left(\overline{3},1,+\frac{1}{3}\right) + (1,1,0) + (1,1,+1) \tag{2.42}$$

All the particles of standard model and the right handed neutrino find a natural place in the following two representations:

$$(4,2,1) = \begin{pmatrix} u_1 & u_2 & u_3 & \nu \\ d_1 & d_2 & d_3 & e^- \end{pmatrix}_L \tag{2.43}$$

$$(\overline{4},1,2) = \begin{pmatrix} d_1^c & d_2^c & d_3^c & e^c \\ u_1^c & u_2^c & u_3^c & \nu^c \end{pmatrix}_L \tag{2.44}$$

Symmetry Breaking The rank of G_{PS} is 5, while that of G_{SM} is 4. Thus spontaneous symmetry breaking also needs to reduce the rank. Here is the general rule for symmetry breaking.

Consider a symmetry breaking, $G \to H$. This can be realized when a singlet of H that is not a singlet of G develops a vev.

$(\overline{4},1,2)$ contains a singlet of SM. So when it develops a vev, the required symmetry breaking happens, $G_{PS} \to G_{SM}$.

2.3.2 Ammunition for model building

$SU(5) \times U(1)$ and $SU(4) \times SU(2)_L \times SU(2)_R \equiv G_{PS}$ are two interesting maximal subgroups of $SO(10)$. Since both $SU(5)$ and G_{PS} are grand unifying groups, SO(10) is a natural extension of both the models.

Before getting into various symmetry breaking models, we need to gather some ammunition. We need to find decompositions of SO(10) representations under both of the maximal subgroups.

Decompositions under SU(5)×U(1) : Vector 10 of SO(10) is composed of $5+\overline{5}$ under SU(5). We need to find the $U(1)$ charges for both. $U(1)$ generator is constructed by addition of all the diagonal generators. So it follows from Eq 1.31, that 5 has $U(1)$ charge +2. Note that once we diagonalise the Cartan subalgebra in Eq 1.30, 5 and $\overline{5}$ of $U(5)$ appears explicitly.

$U(1)$ for spinor representation is also formed by adding all the diagonal generators. The singlet in 16, with all $(2N_i - 1 = -1)$, gives U(1) charge -5. Similarly, For 10, we have -1, and for $\overline{5}$, we have +3.

Once we have these decompositions of basic representations, we can find decompositions of all others from the following tensor prodcuts:

$$10 \times 10 = 1 + 45_s + 54 \tag{2.45}$$

$$\overline{16} \times 10 = 16 + 144 \tag{2.46}$$

$$16 \times 16 = 10 + 120_s + 126 \tag{2.47}$$

$$\overline{16} \times 16 = 1 + 45 + 210 \tag{2.48}$$

The last two are already mentioned earlier. Eq 2.46 is the new one. 144 is formed by a product of a spinor and a vector. This is equivalent to a product of spin 1/2 and spin 1. The product is a spinor, equivalent to spin 3/2 of the Lorentz group.

Table 2.3.2 contains the decompositions of some relevant representations.

Table 2.2: Decompositions of SO(10) reps under $SU(5) \times U(1)$

10	$= 5[+2] + \overline{5}[-2]$
16	$= 1[-5] + \overline{5}[+3] + 10[-1]$
45	$= 1[0] + 10[+4] + \overline{10}[-4] + 24[0]$
54	$= 15[+4] + \overline{15}[-4] + 24[0]$
120	$= 5[+2] + \overline{5}[-2] + 10[-6] + \overline{10}[+6] + 45[+2] + \overline{45}[-2]$
126	$= 1[-10] + \overline{5}[-2] + 10[-6] + \overline{15}[+6] + 45[+2] + \overline{50}[-2]$
144	$= 5[+7] + \overline{5}[+3] + 10[-1] + 15[-1] + 24[-5] + 40[-1] + \overline{45}[+3]$
210	$= 1[0] + 5[-8] + \overline{5}[+8] + 10[+4] + \overline{10}[-4] + 24[0] + \overline{40}[+4] + 40[-4] + 75[0]$

Decompositions under G_{PS} : $SU(4)$ is isomorphic to SO(6). They have same Dynkin diagrams and hence identical algebra. $SU(2) \times SU(2)$ is identical to SO(4). One can recall this from the representations of the Lorentz group. Dynkin diagrams of both these groups are identical: two disconnected blobs. So $G_{PS} = SO(6) \times SO(4)$. Now decompositions of SO(10) representations under G_{PS} are straight forward.

Vector 10 of SO(10), decomposes trivially to vector 6 of SO(6) and vector 4 of SO(4). Now the vector 6 of SO(6) is the antisymmetric representation 6 of SU(4). Vector 4 of SO(4) translate to (2,2).

Spinor 16 of SO(10), decomposes to spinors of SO(6), 4 and $\overline{4}$. As for SO(4), the two spinors translate to (1,2) and (2,1). So even here, all the particles of SM along with a singlet fit nicely in 16.

Again, using the same tensor products and separating the (anti)symmetric parts, we can find decompositions of SO(10) under $SU(4) \times SU(2)_L \times SU(2)_R$, given in Table 2.3.2.

For even bigger representations, their products and decompositions, look at [Slansky 81],

Table 2.3: Decompositions of SO(10) reps under Pati-Salam group

10	$= (6, 1, 1) + (1, 2, 2)$
16	$= (4, 2, 1) + (\overline{4}, 1, 2)$
45	$= (1, 1, 3) + (1, 3, 1) + (6, 2, 2) + (15, 1, 1)$
54	$= (1, 1, 1) + (1, 3, 3) + (6, 2, 2) + (20, 1, 1)$
120	$= (1, 2, 2) + (6, 3, 1) + (6, 1, 3) + (10, 1, 1) + (\overline{10}, 1, 1) + (15, 2, 2)$
126	$= (6, 1, 1) + (10, 3, 1) + (\overline{10}, 1, 3) + (15, 2, 2)$
144	$= (4, 2, 1) + (\overline{4}, 1, 2) + (4, 2, 3) + (\overline{4}, 3, 2) + (20, 1, 2) + (\overline{20}, 2, 1)$
210	$= (1, 1, 1) + (10, 2, 2) + (\overline{10}, 2, 2) + (6, 2, 2) + (15, 1, 1) + (15, 1, 3) + (15, 3, 1)$

[Aulakh 05].

2.3.3 Spontaneous Symmetry Breaking

There are two schools of thought. SO(10) can either break into $SU(5)$ and then to G_{SM}, or it can break into G_{PS} and then into G_{SM}. Since in both these models, rank also has to be reduced, the symmetry breaking becomes complicated with intermediate energy scales.

Via SU(5)

$SO(10) \to SU(5) \to SU(3) \times SU(2) \times U(1)$

To break SO(10) down to SU(5), a representation of SO(10) is needed that contains a singlet of $SU(5)$ with a U(1) charge. This follows from the general symmetry breaking argument mentioned earlier.

An adjoint type multiplet (multiplets of form $r \times \overline{r}$) can not result in the rank reduction. The singlet in $r \times \overline{r}$ would be a singlet of the GUT group itself.

For Table 2.3.2, we can see that 16 and 126 contain singlets of SU(5) with U(1) charge. Hence if they get a vev, the rank is reduced, and we have only $SU(5)$ symmetry left.

To further break SU(5) to G_{SM}, we need adjoint of SU(5), 24, just as discussed earlier. Looking back at Table 2.3.2, 54 or 210 of SO(10) can be used. These are tensor representations. In general, tensor representations or adjoint type multiplets preserve rank, and the spinor type or vector type representations break rank.

Thus

$$SO(10) \to_{\langle 16 \rangle \text{ or } \langle 126 \rangle} \to SU(5) \to_{\langle 54 \rangle \text{ or } \langle 210 \rangle} \to G_{SM} \tag{2.49}$$

For electroweak symmetry breaking, we need a Higgs in 5 of SU(5). So any representation can be used for this purpose that contains 5, for example, 10.

Look at the decompositions of 144 in Table 2.3.2. It contains a 24[−5]. 24 contains an

SM singlet as seen earlier; it also has the extra U(1) charge. This would mean that it can break the rank as well. Thus instead of so many different representations, only 144 can break SO(10) directly to G_{SM}. It also has 5, that can be used for electroweak symmetry breaking.

Via Pati-Salam Group

$SO(10) \rightarrow SU(4) \times SU(2)_L \times SU(2)_R \rightarrow SU(3)_c \times SU(2)_L \times U(1)_Y \rightarrow SU(3)_c \times U(1)_Q$

Just like the earlier case, we can use any adjoint type multiplet for the first part that preserves rank. For example, 54 and 210 contain singlet of G_{PS}. Note that both are a result of the tensor product. So if singlet in either one of them develops a vev, SO(10) breaks down to G_{PS}.

To obtain further G_{SM}, one needs to reduce the rank. This has already been discussed. Any representation that contain $(\overline{4}, 1, 2)$ can be used to break G_{PS} down to G_{SM}. 16 can be used for this purpose. There are also other representations that can be used for this symmetry breaking. The full supersymmetric theory using $126 + \overline{126}$ representation for $G_{PS} \rightarrow G_{SM}$ has the least number of real parameters in it, thus making it the minimal supersymmetric GUT. [Aulakh 04]

For electroweak symmetry breaking, again, 10 can be used.

Thus the full symmetry breaking pattern is:

$$SO(10) \rightarrow_{\langle 210 \rangle} \rightarrow G_{PS} \rightarrow_{\langle 126 \rangle + \langle \overline{126} \rangle} \rightarrow G_{SM} \rightarrow_{\langle 10 \rangle} SU(3)_c \times U(1)_Q \tag{2.50}$$

Chapter 3

Supersymmetry

3.1 Introduction

Consider a fermion f having Yukawa coupling with a scalar H : $-\lambda_f H f^\dagger f$ and an another scalar S with φ^4 type interaction: $-\lambda_S |H|^2 |S|^2$. The one loop corrections to the mass squared of H will be:

$$\Delta m_H^2 = -\frac{|\lambda_f|^2}{16\pi^2}\Lambda_{UV}^2 + \frac{\lambda_S}{16\pi^2}\Lambda_{UV}^2 + \ldots \tag{3.1}$$

Shifting the mass of Higgs would mean a shift in the vev of Higgs, and hence the whole mass spectrum of SM would be affected. This is the infamous *hierarchy problem* [Georgi 74b].

Looking closely at Eq 3.1, if the couplings are the same for scalar and fermion, then their contribution cancels. Thus corresponding to each complex scalar, there is a Weyl fermion with the same couplings. This is the central idea of *supersymmetry.*

Supersymmetry is the symmetry between fermions and bosons. Supersymmetry generators can change a boson into a fermion and vice-versa. It gives a possible way out of the hierarchy problem since once established hierarchies are protected from destabilization by supersymmetric non-renormalizable theorems. So it is natural to make the GUTs supersymmetric. Since the spin of a particle is determined by the representation of the Lorentz group in which the particle lies. Since supersymmetry changes the spin of a particle, it is spacetime symmetry. *No internal symmetry can change the spin.*[Haag 75]

3.2 Superspace

The notation and other details can be found in [Martin 98].

Points in superspace are labeled by coordinates: $x^\mu, \theta^\alpha, \theta^\dagger_{\dot\alpha}$

A Weyl spinor is a doublet of anticommuting numbers (Grassmann variables). Anticommuting numbers follow the following relations:

$$f(\eta) = f_0 + \eta f_1 \qquad \Rightarrow \qquad \frac{df}{d\eta} = f_1 \qquad \text{and} \qquad \int d\eta f(\eta) = f_1 \tag{3.2}$$

Any general function of η would have only two terms in its power series. It follows from $\eta^2 = 0$. Thus the power series in θ^α would have three terms in it. A general *superfield* would be a function of $(x^\mu, \theta^\alpha, \theta^\dagger_{\dot\alpha})$ and would have the following expansion:

$$S(x, \theta, \theta^\dagger) = a + \theta\xi + \theta^\dagger\chi^\dagger + \theta\theta b + \theta^\dagger\theta^\dagger c + \theta^\dagger\overline{\sigma}^\mu\theta v_\mu + \theta^\dagger\theta^\dagger\theta\eta + \theta\theta\theta^\dagger\zeta^\dagger + \theta\theta\theta^\dagger\theta^\dagger d \tag{3.3}$$

This follows from:

$$\theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta, \quad \theta^\dagger_{\dot\alpha}\theta^\dagger_{\dot\beta} = \frac{1}{2}\epsilon_{\dot\beta\dot\alpha}\theta^\dagger\theta^\dagger, \quad \theta_\alpha\theta^\dagger_{\dot\beta} = \frac{1}{2}\sigma^\mu_{\alpha\dot\beta}\left(\theta^\dagger\overline{\sigma}_\mu\theta\right) \tag{3.4}$$

Note that there are equal number of fermionic and bosonic degrees of freedom in most general superfield. Now integration over the fermionic coordinates can be defined as follows:

$$d^2\theta = -\frac{1}{4}d\theta^\alpha d\theta^\beta\epsilon_{\alpha\beta}, \quad d^2\theta^\dagger = -\frac{1}{4}d\theta^\dagger_{\dot\alpha}d\theta^\dagger_{\dot\beta}\epsilon^{\dot\alpha\dot\beta} \tag{3.5}$$

$$\int d^2\theta\theta\theta = 1, \qquad \int d^2\theta^\dagger\theta^\dagger\theta^\dagger = 1 \tag{3.6}$$

Now the different components of the superfield can be separated by integrating out the θs.

3.2.1 Supersymmetry as translation in superspace

Supersymmetric generators can be written in the following form:

$$\begin{aligned} \hat{Q}_\alpha &= i\frac{\partial}{\partial\theta^\alpha} - \left(\sigma^\mu\theta^\dagger\right)_\alpha \partial_\mu, & \hat{Q}^\alpha &= -i\frac{\partial}{\partial\theta_\alpha} + \left(\theta^\dagger\overline{\sigma}^\mu\right)^\alpha \partial_\mu \\ \hat{Q}^{\dagger\dot\alpha} &= i\frac{\partial}{\partial\theta^\dagger_{\dot\alpha}} - \left(\overline{\sigma}^\mu\theta\right)^{\dot\alpha}\partial_\mu, & \hat{Q}^\dagger_{\dot\alpha} &= -i\frac{\partial}{\partial\theta^{\dagger\dot\alpha}} + \left(\theta\sigma^\mu\right)_{\dot\alpha}\partial_\mu \end{aligned} \tag{3.7}$$

From the general form of generators, they follow the anticommutation identites:

$$\left\{\hat{Q}_\alpha, \hat{Q}^\dagger_{\dot\beta}\right\} = 2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu = -2\sigma^\mu_{\alpha\dot\beta}\hat{P}_\mu \tag{3.8}$$

$$\left\{\hat{Q}_\alpha, \hat{Q}_\beta\right\} = 0, \qquad \left\{\hat{Q}^\dagger_{\dot\alpha}, \hat{Q}^\dagger_{\dot\beta}\right\} = 0 \tag{3.9}$$

The Anticommutator of the supersymmetry generators gives the generator of translation in spacetime, $\hat{P}_\mu$. This explicitly shows that supersymmetry is spacetime symmetry.

The general suersymmetric transformation can be realized as:

$$\begin{aligned}\sqrt{2}\delta_\epsilon S = -i\left(\epsilon\hat{Q} + \epsilon^\dagger\hat{Q}^\dagger\right)S = \left(\epsilon^\alpha\frac{\partial}{\partial\theta^\alpha} + \epsilon^\dagger_{\dot\alpha}\frac{\partial}{\partial\theta^\dagger_{\dot\alpha}} + i\left[\epsilon\sigma^\mu\theta^\dagger + \epsilon^\dagger\overline{\sigma}^\mu\theta\right]\partial_\mu\right)S \\ = S\left(x^\mu + i\epsilon\sigma^\mu\theta^\dagger + i\epsilon^\dagger\overline{\sigma}^\mu\theta, \theta + \epsilon, \theta^\dagger + \epsilon^\dagger\right) - S\left(x^\mu, \theta, \theta^\dagger\right)\end{aligned} \tag{3.10}$$

Thus the general supersymmetric transformation is just translation in superspace:

$$\theta^\alpha \to \theta^\alpha + \epsilon^\alpha \tag{3.11}$$

$$\theta^\dagger_{\dot\alpha} \to \theta^\dagger_{\dot\alpha} + \epsilon^\dagger_{\dot\alpha} \tag{3.12}$$

$$x^\mu \to x^\mu + i\epsilon\sigma^\mu\theta^\dagger + i\epsilon^\dagger\overline{\sigma}^\mu\theta \tag{3.13}$$

3.2.2 Chiral, Vector Superfields

Covariant Derivative $\partial S/\partial\theta^\alpha$ do not transform appropriately under the supersymmetric transformations. So we need to define a derivative that is covariant under supersymmetry. This is just like defining a covariant derivative for gravity, or the gauge theories. In each case, there is some non-trivial manifold associated and the connection term cancels the inhomogenous terms created by the simple derivative.

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\left(\sigma^\mu\theta^\dagger\right)_\alpha\partial_\mu, \quad D^\alpha = -\frac{\partial}{\partial\theta_\alpha} + i\left(\theta^\dagger\overline{\sigma}^\mu\right)^\alpha\partial_\mu \tag{3.14}$$

$$\overline{D}^{\dot\alpha} = \frac{\partial}{\partial\theta^\dagger_{\dot\alpha}} - i\left(\overline{\sigma}^\mu\theta\right)^{\dot\alpha}\partial_\mu, \qquad \overline{D}_{\dot\alpha} = -\frac{\partial}{\partial\theta^{\dagger\dot\alpha}} + i\left(\theta\sigma^\mu\right)_{\dot\alpha}\partial_\mu \tag{3.15}$$

A superfield with the following constraint is called *left-chiral.*

$$\overline{D}^{\dot\alpha}\Phi = 0 \tag{3.16}$$

Similarly a *right-chiral* field can be defined as follows:

$$D^\alpha\Phi = 0 \tag{3.17}$$

The following redefinition is very convenient:

$$y^\mu \equiv x^\mu + i\theta^\dagger\overline{\sigma}^\mu\theta \tag{3.18}$$

In terms of $(y^\mu, \theta, \theta^\dagger)$, the covariant derivative take the following form:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - 2i\left(\sigma^\mu\theta^\dagger\right)_\alpha \frac{\partial}{\partial y^\mu}, \qquad D^\alpha = -\frac{\partial}{\partial\theta_\alpha} + 2i\left(\theta^\dagger\bar{\sigma}^\mu\right)^\alpha \frac{\partial}{\partial y^\mu} \tag{3.19}$$

$$\bar{D}^{\dot\alpha} = \frac{\partial}{\partial\theta^\dagger_{\dot\alpha}}, \qquad \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\theta^{\dagger\dot\alpha}} \tag{3.20}$$

Chiral Superfield Thus the left chiral field takes a neat form:

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \tag{3.21}$$

$$\begin{aligned}\Phi =& \phi(x) + i\theta^\dagger\bar{\sigma}^\mu\theta\partial_\mu\phi(x) + \frac{1}{4}\theta\theta\theta^\dagger\theta^\dagger\partial_\mu\partial^\mu\phi(x) + \sqrt{2}\theta\psi(x) \\ &- \frac{i}{\sqrt{2}}\theta\theta\theta^\dagger\bar{\sigma}^\mu\partial_\mu\psi(x) + \theta\theta F(x)\end{aligned} \tag{3.22}$$

So a chiral superfield contains a chiral Weyl fermion, a complex scalar, and an auxiliary field F. Note that the number of bosonic degrees of freedom is the same as that of fermionic.

Vector superfield Vector superfield is obtained by imposing the condition $V = V^*$. Hence a general vector superfield can be written as:

$$\begin{aligned}V\left(x,\theta,\theta^\dagger\right) =& a + \theta\xi + \theta^\dagger\xi^\dagger + \theta\theta b + \theta^\dagger\theta^\dagger b^* + \theta^\dagger\bar{\sigma}^\mu\theta A_\mu + \theta^\dagger\theta^\dagger\theta\left(\lambda - \frac{i}{2}\sigma^\mu\partial_\mu\xi^\dagger\right) \\ &+ \theta\theta\theta^\dagger\left(\lambda^\dagger - \frac{i}{2}\bar{\sigma}^\mu\partial_\mu\xi\right) + \theta\theta\theta^\dagger\theta^\dagger\left(\frac{1}{2}D + \frac{1}{4}\partial_\mu\partial^\mu a\right)\end{aligned} \tag{3.23}$$

However, most of the fields in it can be supergauged away. Leaving only a vector field, a Weyl fermion, and an auxiliary field D. Even here, a total of fermionic degrees of freedom is the same as bosonic ones. In a specific gauge choice,

$$V_{\text{WZ gauge}} = \theta^\dagger\bar{\sigma}^\mu\theta A_\mu + \theta^\dagger\theta^\dagger\theta\lambda + \theta\theta\theta^\dagger\lambda^\dagger + \frac{1}{2}\theta\theta\theta^\dagger\theta^\dagger D \tag{3.24}$$

Auxiliary fields D_a and F are the auxiliary fields. They do not have any kinetic term in lagrangian. This implies that they can not move in spacetime. Without any interactions, imposing equations of motion would remove them completely. So they exist merely as quantum fluctuations. However, if they have some couplings with the real fields, they become important. Removing them by using their equations of motion impose new types of interactions for the real particles.

3.2.3 Supersymmetric Lagrangian

Since a supersymmetric transformation is merely a translation in superspace, a supersymmetrically invariant action can be generated by integrating over full superspace.

D term Integrating over full superspace give a supersymmetrically invariant quantity. Since imaginary action is ruled out, we need to add the complex conjugate with it to make the $\mathcal{L}$ real.

$$\delta_\epsilon A = 0, \quad \text{for} \quad A = \int d^4x \int d^2\theta \, d^2\theta^\dagger \, V(x, \theta, \theta^\dagger) \tag{3.25}$$

$$\mathcal{L} = \int d^2\theta \, d^2\theta^\dagger \, V(x, \theta, \theta^\dagger) \equiv [V]_D \tag{3.26}$$

F term Since a (left) chiral superfield do not have ($\theta^\dagger$) dependence, integrating over θ is enough to create an invariant quantity. Any superfield can be made a chiral one by setting $\theta^\dagger$ in it to 0.

$$[\Phi]_F \equiv \Phi|_{\theta\theta} = \int d^2\theta \Phi \Big|_{\theta^\dagger=0} = \int d^2\theta d^2\theta^\dagger \delta^{(2)}(\theta^\dagger)\Phi = F \tag{3.27}$$

These two types of terms are related as follows:

$$[V]_D = -\frac{1}{4}[\overline{DD}V]_F \tag{3.28}$$

Simplest SuSy theory: Wess-Zumino model Consider only a chiral superfield Φ with nothing else. Now a vector multiplet can be generated from it as $\Phi^*\Phi$

$$[\Phi^*\Phi]_D = \int d^2\theta d^2\theta^\dagger \Phi^*\Phi = -\partial^\mu\phi^*\partial_\mu\phi + i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi + F^*F \tag{3.29}$$

The terms are the usual kinetic lagrangian for scalar and fermions. Note that the auxiliary field F has no kinetic term and, without any interaction, gives an equation of motion $F = 0$.

Interactions Interactions between chiral supermultiplets can be included using *superpotential*, $W[\Phi]$. In general, it can be any holomorphic function of chiral superfields (treated as complex variables). For renormalizable interactions, terms beyond cubic are not allowed. Here is most general renormalizable superpotential for a few chiral superfields:

$$W(\Phi_i) = \mu_i\Phi^i + \frac{1}{2}M_{ij}\Phi^i\Phi^j + \frac{1}{6}y_{ijk}\Phi^i\Phi^j\Phi^k \tag{3.30}$$

Expanding it up in its component fields and integrating out the auxiliary fields F, one has all possible renormalizable interactions among a collection of Weyl fermions and complex scalars.

3.3 Supersymmetric Gauge theories

The chiral superfields Φ transform as follows under a local gauge symmetry:

$$\Phi_i \to \left(e^{2ig_a\Omega^a T^a}\right)_i^{\ j}\Phi_j, \qquad \Phi^{*i} \to \Phi^{*j}\left(e^{-2ig_a\Omega^a T^a}\right)_j^{\ i} \tag{3.31}$$

The parameters for gauge transformation, Ω_a are chiral superfields and g_a are the gauge couplings. Note that such transformation properties are inspired by the way component fields of Φ transform. The gauge bosons are a part of some vector superfield V. Under the gauge transformation, V transforms as follows:

$$e^V \to e^{i\Omega^\dagger} e^V e^{-i\Omega} \tag{3.32}$$

Where the group indices are summed up with generators to make the expressions neat:

$$V_i^j = 2g_a T_i^{aj} V^a, \qquad \Omega_i^j = 2g_a T_i^{aj}\Omega^a \tag{3.33}$$

For abelian gauge theory, $U(1)$, this simplifies just to

$$V \to V + i(\Omega^* - \Omega) \tag{3.34}$$

These are supergauge transformations. The usual gauge transformations are merely a subset of these. So we can *partially fix the supergauge*, leaving the usual gauge transformations intact. This is what was done to get V in Eq 3.24. This is evident in Eq 3.34. Certain components of Ω can be chosen to get Eq 3.24

Now to construct the lagrangian, $[\Phi^*\Phi]_D$ can not be used, since this quantity is not gauge invariant. Here is the the supergauge invariant lagrangian:

$$\mathcal{L} = \left[\Phi^{*i}\left(e^V\right)_i^{\ j}\Phi_j\right]_D \tag{3.35}$$

Expanding it out and integrating out the auxiliary fields, we get the gauge interaction lagrangian for all the fermions and scalars:

$$\begin{aligned}\left[\Phi^{*i}\left(e^V\right)_i^j\Phi_j\right]_D =& F^{*i}F_i - \nabla_\mu\phi^{*i}\nabla^\mu\phi_i + i\psi^{\dagger i}\bar{\sigma}^\mu\nabla_\mu\psi_i - \sqrt{2}g_a\left(\phi^*T^a\psi\right)\lambda^a - \sqrt{2}g_a\lambda^\dagger\left(\psi^\dagger T^a\phi\right)\\ &+ g_a\left(\phi^*T^a\phi\right)D^a\end{aligned} \tag{3.36}$$

Here, ∇_μ is the gauge covariant derivative to avoid confusion, since D_α is supersymmetric covariant derivative.

Note that the supersymmetric partner of gauge bosons, Weyl fermion *gaugino* λ_a has interactions with the fermions and bosons as well. This is equivalent to say that the gauge interactions have been supersymmetrized.

A crucial part of the story is still misssing. Something equivalent to field strength $F_{\mu\nu}$ is missing and hence the kinetic term of V is missing. A superfield corresponding to field strength can be defined as follows:

$$\mathcal{W}_\alpha = \frac{1}{4}\overline{DD}\left(e^{-V}D_\alpha e^V\right) \tag{3.37}$$

Note that this is a superfield with a spinor index. In the WZ gauge, it takes the following form:

$$(\mathcal{W}^a_\alpha)_{\text{WZ gauge}} = \lambda^a_\alpha + \theta_\alpha D^a + \frac{i}{2}\left(\sigma^\mu\bar{\sigma}^\nu\theta\right)_\alpha F^a_{\mu\nu} + i\theta\theta\left(\sigma^\mu\nabla_\mu\lambda^{\dagger a}\right)_\alpha \tag{3.38}$$

Note that $\mathcal{W}_\alpha$ is a matrix and is obtained by contracting group indices with the generators: $\mathcal{W}_\alpha = 2g_a T_a \mathcal{W}^a_\alpha$

Now the kinetic terms for (gauge) vector superfields can be written using $\mathcal{W}_a$:

$$[\mathcal{W}^{a\alpha}\mathcal{W}^a_\alpha]_F = D^a D^a + 2i\lambda^a\sigma^\mu\nabla_\mu\lambda^{\dagger a} - \frac{1}{2}F^{a\mu\nu}F^a_{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \tag{3.39}$$

Gauginos transform as adjoint under the gauge group. The last term would vanish only in the case of the abelian gauge theory. For non-abelian gauge theory, such a term is sensitive to the gauge manifold's non-trivial geometry and is non-trivial in the presence of instanton backgrounds

Finally all the components can be assembled. Lagrangian for a supersymmetric gauge theory with particles, both scalars and fermions in some representation r with generators T_a, along with any general superpotential interactions among them would be:

$$\mathcal{L} = \frac{1}{4}\left[\mathcal{W}^{a\alpha}\mathcal{W}^a_\alpha\right]_F + \text{c.c.} + \left[\Phi^{*i}\left(e^{2g_a T^a V^a}\right)_i^j \Phi_j\right]_D + \left(\left[W\left(\Phi_i\right)\right]_F + \text{c.c.}\right) \tag{3.40}$$

This lagrangian can be extended in the case of different particles in various representations.

3.4 Minimal Supersymmetric Standard Model

The standard model is the gauge theory with gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$, with a $SU(2)$ doublet Higgs that is used for breaking electroweak symmetry from $SU(2)_L \times U(1)_Y \to U(1)_Q$.

Superpartners To make this supersymmetric, we need to introduce a superpartner for each of the particles from Table 2. The superpartner of quarks and leptons are called *squarks and sleptons.* They have the same quantum numbers as quarks and leptons. The gauginos corresponding to $SU(3), SU(2)$, and $U(1)$ are called *gluinos, winos, and bino*, respectively.

2 Higgs doublets Superpartner of Higgs is called Higgsino. However, corresponding to a Higgs doublet, the Higgsinos would only be left-handed, which leads to non-zero chiral anomalies. Thus for anomalies cancellation, in addition to $\left(1, 2, \frac{1}{2}\right)$ Higgs superfield, a new

Higgs doublet superfield is introduced with quantum numbers: $\left(1, 2, -\frac{1}{2}\right)$. The latter one with couplings $\left(1, 2, -\frac{1}{2}\right)$ is called H_d to tell them apart.

Superpotential There is a need to specify the form of W. The superpotential has to be a polynomial in chiral superfields, should be renormalizable and gauge invariant. The form of W that satisfies all these conditions is (assuming R-parity):

$$W_{\text{MSSM}} = \overline{u}\mathbf{y}_u Q H_u - \overline{d}\mathbf{y}_d Q H_d - \overline{e}\mathbf{y}_e L H_d + \mu H_u H_d \tag{3.41}$$

$\mathbf{y}_i$ are matrices in the family basis. So if they are not diagonal, they cause the mixing of flavor states. Note that the complex conjugate of any superfield is not allowed in W. This is another reason for having a second Higgs doublet. Without it, the mass terms can not be written for all matter particles, since a term like $\overline{u}\mathbf{y}_u Q H_u$ can not be replaced by $\overline{u}\mathbf{y}_u Q H_d^*$

Supersymmetry breaking

If the Susy was exact and unbroken at the electroweak scale, then sleptons and squarks should have been observed by now. They would have exactly the same mass and couplings. Since that is not the case, supersymmetry must be broken at some scale higher than electroweak. There are a lot of postulated ways in which supersymmetry can break, but there is no consensus on the matter yet.

Chapter 4

General $r \times \overline{r}$ type representations

4.1 Introduction

Adjoint type ($r \times \overline{r}$) multiplets hold special significance in the gauge theories, as they are used for rank preserving symmetry breaking. Usually, only *true adjoint* is used (r being the fundamental here). This is because an adjoint type multiplet, $r \times \overline{r}$, with bigger base representation r has $S_2(r \times \overline{r}) = 2S_2(r)d(r)$, which rapidly increases beyond $3S_2(Adj)$ with $d(r)$. This would make the one loop beta function: $b_0 = -3S_2(Adj) + S_2(r \times \overline{r})$ positive resulting in Asymptotic Strength (AS). Success of AF QCD paradigm has functioned as a taboo against the AS field theories. So in all the rank preserving cases, the Higgs representation with small enough $S_2(r_H)$ to preserve AF are used. If we allow the theories to be asymptotically strong, we have many more possibilities for the scalar for symmetry breaking. Note that all of the following analysis is in the context of $SU(N)$. A recent work [Aulakh 20] presents a way in which asymptotically strong theories can find a dynamical way of GUT symmetry breaking as a result of gaugino condensation in UV. Hence a better understanding of $r \times \overline{r}$ for bigger representations as the base would be helpful.

The *true adjoint* of $SU(N)$ can be written as a matrix of order N as $\Phi = \varphi_a T_a$. Such a matrix would transform as follows: [Georgi 82].

$$\Phi_i'^{\,k} = U_i^{\ m}\, \Phi_m^{\ n}\, {U^\dagger}_n^{\ k} \tag{4.1}$$

$$U_i^{\ m} = \delta_i^{\ m} + T_i^{\ m} + O(T^2) \qquad \Rightarrow \qquad \delta\Phi = [T, \Phi] + O(T^2) \tag{4.2}$$

The generators of fundamental here can be replaced with generators of any representation, transformations would still be the same. However, to achieve that, we would have to write the generators as matrices of $d(r)$. So there is a need to introduce new indices to replace the *fundamental indices.* For example, object with two symmetric indices ϕ_{ij} can be labelled with only one new index: ϕ_I. To have the correct index contraction patterns, the new indices must have *multiplicity factors* associated with them. This is clear from the following example of two index symmetric objects:

$$\psi^{ik}\varphi_{ik} = \psi^{11}\varphi_{11} + \psi^{12}\varphi_{12} + \psi^{21}\varphi_{21} + \psi^{22}\varphi_{22} = \psi^{11}\varphi_{11} + 2\psi^{12}\varphi_{12} + \psi^{22}\varphi_{22} \tag{4.3}$$

$$\psi^{I}\varphi_{I} = \psi^{1}\varphi_{1} + \psi^{2}\varphi_{2} + \psi^{3}\varphi_{3} \tag{4.4}$$

$$\varphi_1 \equiv \varphi_{11} \qquad \varphi_3 \equiv \varphi_{22} \tag{4.5}$$

$$\psi^2\varphi_2 = 2\psi^{12}\varphi_{12} \qquad \Rightarrow \qquad \psi^2 \equiv \sqrt{2}\psi^{12} \quad \text{and} \quad \varphi_2 \equiv \sqrt{2}\varphi_{12} \tag{4.6}$$

$$\Rightarrow \psi_I = \sqrt{m(I)}\ \psi_{i_1 i_2 \dots}{}^{j_1 j_2 \dots} \qquad \text{where } m(I) : \text{ Multiplicity of index I} \tag{4.7}$$

Using the new indices, any object with abritrary numbers of indices can be written as a vector and thus all $r \times \overline{r}$ objects can be written as matrices of order $d(r)$. Such objects would have an upper index (new one) and one lower, just like the adjoint. It is easy to see that such objects also transform as Eq 4.2. The only difference would be that the fundamental indices are replaced by the new indices that run from 1 to $d(r)$. [Aulakh 20]

The full matrix $r \times \overline{r}$ would have total $d(r)^2$ components. The full matrix can be separated into various irreducible parts. Each irreducible part would also be a matrix of order $d(r)$, but with fewer independent components and all the group transformations would preserve its respective forms. For example, there would always be a singlet in all $r \times \overline{r}$. Singlet would be the trace of the matrix times the identity matrix. Similarly, there would always be a true adjoint in all the $r \times \overline{r}$ products. It would always have the form $\varphi_a \mathcal{T}_a$, where $\mathcal{T}$ are the generators in r. It can be realized that all the irreps in $r \times \overline{r}$ transform just as the full reducible part i.e., commutator with the generators Eq 4.2. This is obvious for the singlet and the true adjoint.

4.2 Totally symmetric representations as base

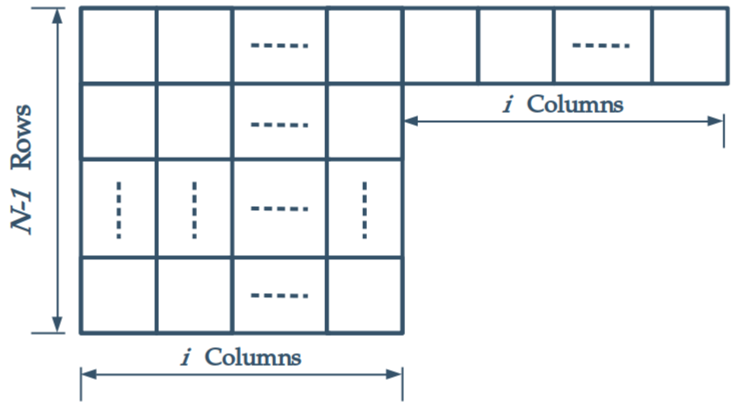
Let us denote totally symmetric representations with m indices for SU(N) as $\mathcal{R}_m[N]$. It follows from simple counting or *Young Tableux* that,

$$\dim\left(\mathcal{R}_m[N]\right) = \begin{pmatrix} N+m-1 \\ m \end{pmatrix} \tag{4.8}$$

Note that $\mathcal{R}_1[N]$ is fundamental.

The full reducible representation $\mathcal{R}_m(= \varphi_{i_1 i_2 \dots i_m}) \times \overline{\mathcal{R}}_m(= \psi^{j_1 j_2 \dots j_m})$ would have objects with m upper indices and m lower indices. Irreducible parts can be formed by contracting one pair at a time. Hence there will be irreps with m upper and lower indices denoted by $\mathcal{A}_m$; $m-1$ upper and lower indices denoted by $\mathcal{A}_{m-1}$ and so on. All the irreducible parts

Figure 4.1: Young Tableux of $\mathcal{A}_i$ representation of SU(N)



would be objects with equal numbers of upper and lower *fundemental* indices.

$$\mathcal{R}_m \times \overline{\mathcal{R}}_m = \bigoplus_{i=0}^{m} \mathcal{A}_i \tag{4.9}$$

Note that $\mathcal{A}_0$ is singlet and $\mathcal{A}_1$ corresponds to the true adjoint. All the $\mathcal{A}_i[N]$ are real representations.

For all SU(N), the weight diagram for N would be the figure corresponding to the S_N (permutation) group. Such a shape would be symmetrical upon the exchange of *any* two vertices. For example, an equilateral triangle in $\mathbb{R}^2$, a regular tetrahedron in $\mathbb{R}^3$ etc. N has Dynkin labels: $(1, 0, \ldots)$. $\mathcal{R}_m$ has the Dynkin labels $(m, 0, \ldots, 0)$[Cheng 84][Slansky 81]. In the weight space, totally symmetric representation has the same shape as the fundamental, though the size is bigger. For example, in SU(3), totally symmetric representations: $3, 6, 10, 15, \ldots$ form equilateral triangles in weight space with sides $1, 2, 3, 4, \ldots$ respectively.

Dynkin labels for $\mathcal{A}_i$ would be:

$$\underbrace{(m, 0, \ldots, 0)}_{\mathcal{R}_m} \times \underbrace{(0, \ldots, 0, m)}_{\overline{\mathcal{R}}_m} = \sum_{i=0}^{i=m} \underbrace{(i, 0, \ldots, i)}_{\mathcal{A}_i} \tag{4.10}$$

Thus it is clear that the $\mathcal{A}_i$ would have the same shape as the *true adjoint* in weight space, though bigger in size. Because of this similarity in Dynkin labels, $\mathcal{A}_i$ can be called the *adjoint type irreps*.

Young Tableux corresponding to $\mathcal{A}_i$ is shown in Figure 4.1. Using hook-length formula[Cheng 84]

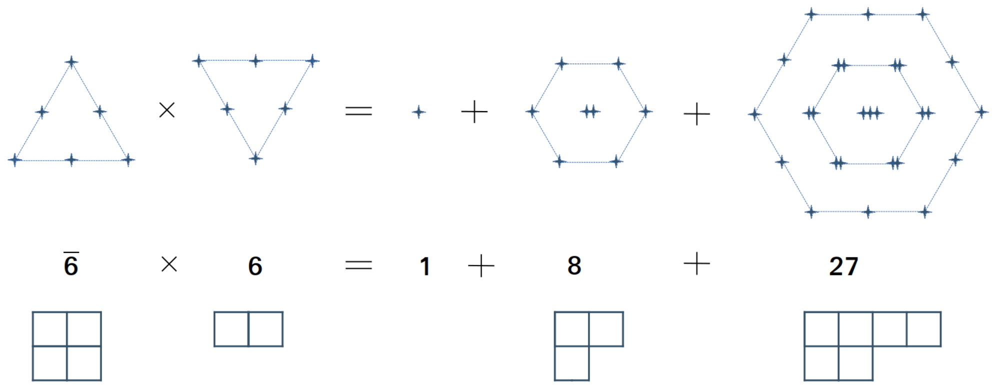


Figure 4.2: $6 \times \overline{6}$ for SU(3)

and some ugly algebra, the dimensions of general $\mathcal{A}_i$ representation can be calculated:

$$\dim(\mathcal{A}_i) = \binom{N+i-2}{i}^2 \frac{N+2i-1}{N-1} \tag{4.11}$$

Here $\binom{j}{k}$ corresponds to the combinations ${}^jC_k$.

Special case: SU(3) A pretty neat thing happens if we put $N = 3$ in Eq 4.11:

$$\dim(\mathcal{A}_i[3]) = (i+1)^2 \, \frac{2+2i}{2} = (i+1)^3 \tag{4.12}$$

Dimensions of $\mathcal{A}_i$ would be perfect cubes. Recall that $6, 10, 15$ are a few symmetric representation of SU(3). Here are a few explicit $r \times \overline{r}$ tensor products for symmetric representations as base:

$$3 \times \overline{3} = 1 + 8 \tag{4.13}$$
$$6 \times \overline{6} = 1 + 8 + 27 \tag{4.14}$$
$$10 \times \overline{10} = 1 + 8 + 27 + 64 \tag{4.15}$$
$$15 \times \overline{15} = 1 + 8 + 27 + 64 + 125 \tag{4.16}$$

Thus all these *perfect cube representations* of $SU(3)$ form regular hexagons in weight space. Figure 4.2 shows an explicit example of $\mathcal{R}_2[3] \times \overline{\mathcal{R}}_2[3]$

4.3 S_2 values

It is sufficient to know the exact form of any one generator to calculate the S_2. For all the representations of $SU(N)$, T_3 is the most convenient generator for this purpose. The diagonal entries of T_3 corresponds to the weight components. It is sufficient to find the decompositions under $SU(2)$ of any representation and then adding the S_2 values of all the components. Here is an example of $\mathcal{R}_3[3]$. The $SU(2)$ representations are depicted with a *hat*: $[\hat{\ }]$

$$\overbrace{\begin{array}{cccc}(111) & (112) & (122) & (122)\\ 3/2 & 1/2 & -1/2 & -3/2\end{array}}^{\hat{4}}\quad \overbrace{\begin{array}{ccc}(113) & (123) & (223)\\ 1 & 0 & -1\end{array}}^{\hat{3}}\quad \overbrace{\begin{array}{cc}(133) & (233)\\ 1/2 & -1/2\end{array}}^{\hat{2}}\quad \overbrace{\begin{array}{c}(333)\\ 0\end{array}}^{\hat{1}} \tag{4.17}$$

It follows that

$$S_2(\mathcal{R}_3[3]) = S_2(\hat{4}) + S_2(\hat{3}) + S_2(\hat{2}) + S_2(\hat{1}) \tag{4.18}$$

Note that it is trivial to find $S_2(\hat{m})$ for all the representations of $SU(2)$:

$$S_2(\hat{m}) = \frac{1}{12}m(m^2-1) = \frac{1}{2}\begin{pmatrix} m+1 \\ 3 \end{pmatrix} \tag{4.19}$$

This follows from the known general form of T_3 of $SU(2)$. We just need to find the decompositions of $\mathcal{R}_m[N]$ under $SU(2)$.

Working out a few examples just like Eq 4.17, we find a recursion relation and two terminating conditions as follows:

$$\mathcal{R}_m[N] = \mathcal{R}_m[N-1] \oplus \mathcal{R}_{m-1}[N] \tag{4.20}$$

$$\mathcal{R}_m[2] = \widehat{m+1} \qquad \mathcal{R}_1[N] = \hat{2} + (N-2)\times\hat{1} \tag{4.21}$$

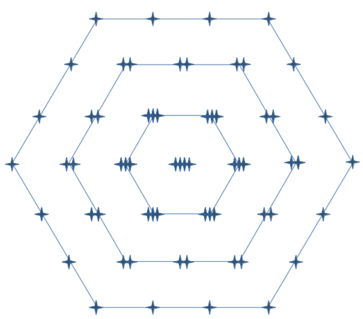


Figure 4.3: **64** of $SU(3)$ with Dynkin labels (3,3)

Now the problem has condensed merely to a problem of counting. Here is the final result:

$$\mathcal{R}_m[N] = \bigoplus_{k=0}^{m} \mathcal{R}_k[2]\,(\times\Theta_{m-k}(N)) \tag{4.22}$$

$\Theta_k(N)$ also has the neat recursion properties as follows:

$$\Theta_k(N) = \sum_{i_1=3}^{N}\sum_{i_2=3}^{i_1}\cdots\sum_{i_k=3}^{i_{k-1}} 1 \qquad \Rightarrow \sum_{j=3}^{n}\Theta_k(j) = \Theta_{k+1}(n); \qquad \Theta_0(j) = 1 \tag{4.23}$$

$$\Theta_k(N) = \binom{N+k-3}{k} \tag{4.24}$$

Now it is straight forward to find $S_2(\mathcal{R}_m[N])$. As we stated earlier that S_2 of full irreducible representation is sum of S_2 of all the $SU(2)$ components.

$$\mathcal{R}_m[N] = \bigoplus_{k=1}^{m+1} \widehat{k}\,(\times\Theta_{m-k+1}(N)) \tag{4.25}$$

$$\Rightarrow S_2(\mathcal{R}_m[N]) = \frac{1}{2}\binom{N+m}{m-1} \tag{4.26}$$

Table 4.1 contains a few values of $S_2(\mathcal{R}_m[N])$, for further reference.

Now from the elemetary group theory, the relation between S_2 values for tensor prodcuts:

$$\sum_{i=0}^{m} C_2(\mathcal{A}_i)\,d(\mathcal{A}_i) = 2C_2(\mathcal{R}_m)\,d(\mathcal{R}_m)^2 = 2S_2(\mathcal{R}_m)\,d(\mathcal{R}_m)\,d(G) \tag{4.27}$$

Using relation Eq 4.8, Eq 4.11 and Eq 4.26, after some manipulation, we get the neat result:

$$C_2(\mathcal{A}_i[N]) = i(N+i-1) \tag{4.28}$$

Table 4.2 contains a few $\mathcal{A}_i$ representations with their S_2 values.

4.4 Separating out $\mathcal{A}_i$ from $\mathcal{R}_2 \times \overline{\mathcal{R}_2}$

$\mathcal{R}_2[N]$ is a widely used representation as it is among the first few non-trivial tensor representations. $\mathcal{R}_2 \times \overline{\mathcal{R}_2}$ contains a singlet, a true adjoint and a $\mathcal{A}_2$. Now we aim to identify the three components explicitly.

Note that $a, b, c, d, \ldots \in [1, N]$ are the fundamental indices and $I, J, K, \ldots \in [1, d(\mathcal{R}_2)]$ refer to the new $\mathcal{R}_2$ base indices.

Table 4.1: $S_2(\mathcal{R}_m[N])$ ($\mathcal{R}_m[N]$: totally symmetric irreps with m indices for SU(N)

	$SU(2)$	$SU(3)$	$SU(4)$	$SU(5)$	$SU(6)$
$\mathcal{R}_1$	$S_2(2) = 1/2$	$S_2(3) = 1/2$	$S_2(4) = 1/2$	$S_2(5) = 1/2$	$S_2(6) = 1/2$
$\mathcal{R}_2$	$S_2(3) = 2$	$S_2(6) = 5/2$	$S_2(10) = 3$	$S_2(15) = 7/2$	$S_2(21) = 4$
$\mathcal{R}_3$	$S_2(4) = 5$	$S_2(10) = 15/2$	$S_2(20) = 21/2$	$S_2(35) = 14$	$S_2(56) = 18$
$\mathcal{R}_4$	$S_2(5) = 10$	$S_2(15) = 35/2$	$S_2(35) = 28$	$S_2(70) = 42$	$S_2(126) = 60$
$\mathcal{R}_5$	$S_2(6) = 35/2$	$S_2(21) = 35$	$S_2(56) = 63$	$S_2(126) = 105$	$S_2(252) = 115$

Table 4.2: $S_2(\mathcal{A}_i[N])$ ($\mathcal{A}_i[N]$: adjoint type irreps in $\mathcal{R}_m \times \overline{\mathcal{R}}_m$ for SU(N)

	$SU(2)$	$SU(3)$	$SU(4)$	$SU(5)$	$SU(6)$
$\mathcal{A}_1$	$S_2(3) = 2$	$S_2(8) = 3$	$S_2(15) = 4$	$S_2(24) = 5$	$S_2(35) = 6$
$\mathcal{A}_2$	$S_2(5) = 10$	$S_2(27) = 27$	$S_2(84) = 56$	$S_2(200) = 100$	$S_2(405) = 162$
$\mathcal{A}_3$	$S_2(7) = 28$	$S_2(64) = 120$	$S_2(300) = 360$	$S_2(1000) = 875$	$S_2(2695) = 1848$
$\mathcal{A}_4$	$S_2(9) = 60$	$S_2(125) = 375$	$S_2(825) = 1540$	$S_2(3675) = 4900$	$S_2(12740) = 13104$

Singlet is trivially:

$$S_I{}^J = \phi_K{}^K \, \delta_I{}^J \qquad \Longleftrightarrow \qquad S_{ab}{}^{cd} = \phi_{mn}{}^{mn} \, \frac{1}{2} \, \delta_a^{(c} \delta_b^{d)} \tag{4.29}$$

The true adjoint is an object that transforms with an upper and a lower index. One pair of indices must be contracted while keeping the final object still symmetric in indices:

$$\mathcal{A}_{1ab}{}^{cd} \sim \phi_{(am}{}^{(cm} \, \delta_{b)}^{d)} \tag{4.30}$$

However this object still contains a singlet part. Subtracting singlet from it and demanding that the final product is traceless, we have:

$$\mathcal{A}_{1ab}{}^{cd} = \phi_{(am}{}^{(cm} \, \delta_{b)}^{d)} - \frac{2}{N} \phi_{mn}{}^{mn} \, \delta_a^{(c} \delta_b^{d)} \tag{4.31}$$

Note that the above expression has fundamental indices. To write it as matrix of order $d(r)$, we need to convert back to new indices.

Now we can separate out $\mathcal{A}_2$ by subtracting adjoint and singlet from the full reducible. The factors are fixed by demanding that $\mathcal{A}_2$ has no singlet ($\mathcal{A}_{2mn}{}^{mn} = 0$) and no adjoint

$(\mathcal{A}_{2mi}{}^{mk} = 0)$:

$$\mathcal{A}_{2ab}{}^{cd} = \phi_{ab}{}^{cd} - \frac{1}{N+2}\mathcal{A}_{1ab}{}^{cd} - \frac{2}{N(N+1)}S_{ab}{}^{cd} \tag{4.32}$$

This method can easily be generalized for bigger representations: $\mathcal{R}_m \times \overline{\mathcal{R}}_m$.

Alternatively, we anyways know that the adjoint would have the form: $\varphi_a \mathcal{T}_a$, $\mathcal{T}_a$ being the generators of $\mathcal{R}_2$. The remaining $\mathcal{A}_2$ can be separated by demanding that there are no modes along the adjoint or singlet i.e., we demand that $\mathcal{A}_2$ modes are *perpendicular* to adjoint and singlet. This will be clear with explicit examples later on.

4.4.1 Example: SU(2)

$\mathcal{R}_2[2] = 3$. Note that 3 is also the adjoint of $SU(2)$ and defining a real vector of $SO(3)$.

$$3 \times 3 = 1 + 3 + 5 \tag{4.33}$$

As we stated earlier that such adjoint type multiplets would be useful for symmetry breaking. Since the only smaller group than $SU(2)$ is $U(1)$, we are interested in this rank preserving symmetry breaking. The *Higgs* field used for this purpose: Φ must be diagonal to achieve this symmetry breaking. So here we only consider diagonal Φ. It also makes the calculations much simpler.

We start with an object Φ transforming reducibly as 3×3:

$$\Phi = \text{diag}\,[\varphi_1, \quad \varphi_2, \quad \varphi_3] \tag{4.34}$$

Singlet is (using Eq 4.29):

$$S_I{}^J = 1_I{}^J = (\varphi_1 + \varphi_2 + \varphi_3)\mathbb{I}_I{}^J \tag{4.35}$$

Adjoint is (using Eq 4.31):

$$(\mathcal{A}_1)_I{}^J = 3_I{}^J = 2(\varphi_1 - \varphi_3)\ \text{diag}\begin{bmatrix}+1 & 0 & -1\end{bmatrix} \tag{4.36}$$

Note that this is proportional to $\mathcal{T}_3$, just as expected.

One way to find the $\mathcal{A}_2[2] = 5$ is simply using Eq 4.32, which gives:

$$(\mathcal{A}_2)_I{}^J = 5_I{}^J = \frac{1}{6}(\varphi_1 - 2\varphi_2 + \varphi_3)\ \text{diag}\begin{bmatrix}+1 & -2 & +1\end{bmatrix} \tag{4.37}$$

Alternatively, we can demand that the modes of 5 would have no overlap with those of 3 and 1 i.e. $\beta.\begin{bmatrix}+1 & 0 & -1\end{bmatrix} = 0$ and $\beta.\mathbb{I} = 0$. Thus we get: $\beta = \begin{bmatrix}+1 & -2 & +1\end{bmatrix}$, which is just same as Eq 4.37.

If any of these 3 modes develops a vev somehow, we get the required symmetry breaking:

$SU(2) \to U(1)$. Note that $S_2(5) = 10$ (Table 4.1, Table 4.2) and since for supersymmetric gauge theories, $b_0 = S_2(r) - 3C_2(G)$, presence of a 5 would make the full theory asymptotically strong.

4.4.2 Example: SU(3)

$\mathcal{R}_2[3] = 6$ and $6 \times \overline{6} = 1 + 8 + 27$ (Figure 4.2)

For now, we are interested only in the diagonal Φ:

$$\begin{array}{llllllll} {\Phi_I}^J = \mathrm{diag} & [\varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \varphi_6] \\ & 11 & 12 & 22 & 13 & 23 & 33 \end{array} \tag{4.38}$$

Fundamental indices associated with each state vector are mentioned. We have arranged the state vectors in this particular order because this would be convenient for symmetry breaking: $SU(3) \to SU(2) \times U(1)$.

Singlet trivial and hence can be skipped. The adjoint ($\mathcal{A}_1$) can be found either by using Eq 4.31 or by simply contracting a field with the generators: $\chi_a \mathcal{T}_a$. $SU(3)$ has rank 2. Thus there would be 2 modes of adjoint on diagonal of Φ out of total 6. It can be verified that in general adjoint would take the following form:

$$\mathcal{A}_1 = \mathrm{diag} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ -\alpha_1 + 2\alpha_2 \\ +\frac{1}{2}\alpha_1 - \alpha_2 \\ -\frac{1}{2}\alpha_1 \\ -2\alpha_2 \end{pmatrix} \tag{4.39}$$

Similarly the general form of $\mathcal{A}_2[N]$ can either be calculated from Eq 4.32 or by simply demanding that our diagonal elements of 27, are perpendicular to those of 8 in $\mathbb{R}^6$, just like in case of $SU(2)$. Thus we get:

$$\mathcal{A}_2 = \mathrm{diag} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ -2\beta_1 - \beta_2 \\ -\beta_2 - 2\beta_3 \\ \beta_1 + \beta_2 + \beta_3 \end{pmatrix} \tag{4.40}$$

Note that out of 6 diagonal modes in $6 \times \overline{6}$, 1 corresponds to the singlet, 2 corresponds to the adjoint, and 3 corresponds to $\mathcal{A}_2 = 27$.

During symmetry breaking of $SU(3)$, we want to preserve $H \equiv SU(2) \times U(1)$. This can be achieved when the structure of H is not tampered with at all. Now look at the decomposition of $6_{SU(3)}$ under H:

$$6 = \hat{3}[+2] + \hat{2}[-1] + \hat{1}[-4] \tag{4.41}$$

Thus the symmetry breaking can only be achieved if Φ transforming as $6 \times \overline{6}$ (or any irrep in it) develops a vev as follows:

$$\Phi = \text{diag}\ [\varphi_1 \quad \varphi_1 \quad \varphi_1 \quad \varphi_2 \quad \varphi_2 \quad \varphi_3] \tag{4.42}$$

It is clear that there are total of 3 modes. It can be seen that one corresponds to adjoint, one corresponds to 27 and last one is a singlet even of $SU(3)$. The last one does not lead to any symmetry breaking. Since we are looking for bigger representations, we want to know the form of 27 that can break $SU(3) \to H$. Comparing Eq 4.42 and Eq 4.40:

$$\phi = \varphi\ \text{diag} \begin{bmatrix} +1 & +1 & +1 & -3 & -3 & +3 \end{bmatrix} \tag{4.43}$$

This is the singlet mode of $SU(2) \times U(1)$ in 27 of $SU(3)$.

4.5 Non-trivial symmetry breaking of SU(5) GUT

Here we move ahead from the toy models and onto an actual GUT. $SU(5) \to G_{SM}$ preserves the rank. Details of $SU(5)$ GUT can be found in [Nath 16][Mohapatra 86]. Hence the formalism developed so far can be used here.

$S_2(200) = 100$ (Table 4.2) would be a large positive contribution to the beta function. So the full GUT becomes *asymptotically strong.*

4.5.1 Using $\mathcal{A}_2[5]$: 200

$\mathcal{R}_2[5] = 15$ and $15 \times \overline{15} = 1 + 24 + 200$.

$$15 = (5 \times 5)_S = \left(6, 1, -\frac{2}{3}\right) + (1, 3, +1) + \left(3, 2, +\frac{1}{6}\right) \tag{4.44}$$

We are interested in using 200 for the symmetry breaking. 200 has only one Standard Model singlet in it. This can be seen by using Eq 4.44 into $15 \times \overline{15}$. [Slansky 81] contains all the relevant decompositions.

Looking at Eq 4.44, we need to identify which state vectors of 15 corresponds to which of the decompositions of 15 under G_{SM}. This can be easily realized by following the same pattern as the standard $SU(5)$. We assign $\alpha, \beta, \ldots \in [1,2,3]$ out of the 5 fundamental indices ($a, b, \ldots \in [1,5]$) to color indices of $SU(3)_C$. $i, j, \ldots \in [4,5]$ were assigned to $SU(2)_L$. $\alpha, \beta \ldots$ indices are associated with $-1/3$ $U(1)_Y$ hypercharge and $i, j, \ldots$ have $+1/2$ $U(1)_Y$ hypercharge. Thus, along these lines we can identify the correlation:

$$\underset{15}{\psi_{(ab)}} = \underset{\left(6,1,-\frac{2}{3}\right)}{\psi_{(\alpha\beta)}} \oplus \underset{\left(3,2,+\frac{1}{6}\right)}{\psi_{(\alpha i)}} \oplus \underset{(1,3,+1)}{\psi_{(ik)}} \tag{4.45}$$

Now we want to preserve G_{SM} while breaking $SU(5)$, we can not tamper with the SM structure inside $15 \times \overline{15}$ of $SU(5)$. Looking at the Eq 4.45 and Eq 4.44, the diagonal elements of $\Phi_{15\times\overline{15}}$ must have the following form:

$$\phi_{15\times\overline{15}} = \text{diag}\ [\beta_1 \mathbb{I}_6 \qquad \beta_2 \mathbb{I}_6 \qquad \beta_3 \mathbb{I}_3] \tag{4.46}$$

Now we want to separate 200; we need to find the explicit form of adjoint (24) embedded in $15 \times \overline{15}$. This can be achieved by Eq 4.31 or by contracting 4 (=rank$_{SU(5)}$) fields with the Cartan generators of $SU(5)$. Then we need to find the pattern in these 15 expressions. The explicit calculations are skipped here. We are only interested in expressions of the form of Eq 4.46. Proceeding just like the toy models, we demand that the modes of 200 have no overlap with that of 24 and that it is traceless. Upon all these constraints, we find the explicit form of G_{SM} singlet in 200 of $SU(5)$:

$$\phi_{200} = \beta \text{diag}\ [\mathbb{I}_6 \qquad -2\mathbb{I}_6 \qquad 2\mathbb{I}_3] \tag{4.47}$$

Note that there is only one SM singlet in 200. If by some mechanism, this mode generates a vev, $SU(5)$ GUT breaks down into G_{SM}.

4.5.2 Using 75

The formalism of this chapter is not confined to be used on symmetric representations only. Here we use it on an antisymmetric two indices representation of $SU(5)$.

In general, the irreducible representations in tensor product $r \times \overline{r}$ for antisymmetric representations are not that interesting. They do not have interesting Dynkin indices or so. For r being representation with 2 antisymmetric indices, $r \times \overline{r}$ has a singlet, an adjoint and one bigger representation. For SU(5):

$$10 \times \overline{10} = 1 + 24 + 75 \tag{4.48}$$

$$\underset{10}{\psi_{[ab]}} = \underset{\left(\overline{3},1,-\frac{2}{3}\right)}{\psi_{[\alpha\beta]}} \oplus \underset{\left(3,2,+\frac{1}{6}\right)}{\psi_{[\alpha i]}} \oplus \underset{(1,1,+1)}{\psi_{[ik]}} \tag{4.49}$$

We can label the state vectors of 10 just according to this decomposition. Now if we want to preserve the G_{SM} during the symmetry breaking, then the $SU(3), SU(2)$ structures should be intact and the the field ϕ should have the following structure:

$$\phi_{10\times\overline{10}} = \text{diag}\,[\beta_1 \mathbb{I}_3 \qquad \beta_2 \mathbb{I}_6 \qquad \beta_3 \mathbb{I}_1] \tag{4.50}$$

Again, we need to write adjoint as a diagonal matrix of order 10, which, just like the earlier case, can be done by contracting 4 fields with 4 diagonal generators. The only difference would be that the state vectors with fundamental indices $(11), (22), \ldots$ will be missing. Now to separate 75, we demand that it is *perpendicular* to the adjoint. Since we are interested in the form of 75 that can be used for symmetry breaking, we find the scalar product of Eq 4.50 with the adjoint and set it to zero, just like the 200 case and find certain constraints and the SM singlet in 75 of $SU(5)$ written as $10 \times \overline{10}$ matrix would have the following form:

$$\phi_{75} = \varphi\,\text{diag}\,[\mathbb{I}_3 \qquad -\mathbb{I}_6 \qquad +3\mathbb{I}_1] \tag{4.51}$$

Note that even here $S_2(75) = 25$ and thus the full GUT would be asymptotically strong.

Matter Couplings

In the usual $SU(5)$ GUT with 24 as the GUT breaking Higgs [Georgi 74a], the following matter couplings are possible (Note that SM matter particles fit in $\overline{5} + 10$):

$$\overline{5}^a 24_a{}^b 5_b \qquad \overline{10}^{ab} 24_b{}^c 10_{ca} \tag{4.52}$$

If we have a 75, a few possible matter couplings would be:

$$\overline{5}^a \overline{5}^b 75_{ab}{}^{cd} 10_{cd} \qquad \overline{10}^{ab} 75_{ab}{}^{cd} 10_{cd} \tag{4.53}$$

Though the possible superpotential with the first term would be quartic and hence non-renormalizable. In the usual $SU(5)$ with 24, the parts of $\overline{5}$ and 10 never mix. There is no gauge-invariant term like the quartic term mentioned here. Also the latter term $\overline{10}^{ab} 75_{ab}{}^{cd} 10_{cd}$ give more possible interactions than the 24 one. For example, in case of 24, there is no such term as $\overline{10}^{\alpha\beta} 75_{\alpha\beta}{}^{ik} 10_{ik}$, where $\alpha, \beta \in [1, 3]$ are $SU(3)_C$ color indices and $i, k \in [4, 5]$ are $SU(2)_L$ indices.

Note that the case of 200 would not be that convenient since it has symmetric pairs of fundamental indices, and coupling the antisymmetric 10 would give zero.

Chapter 5

Dynamical Symmetry breakings in AS GUTs

5.1 Introduction

The success of AS GUTs as a theory of strong interactions has made AF a generally unquestioned requirement for GUT models. On the other hand, the need to incorporate renormalizable seesaw mechanisms in the context of $SO(10)$ models [Aulakh 83], [Clark 82], which are best suited to describe neutrino mass, led to the emergence of models with UV strong gauge coupling as the Minimal SuSy GUT. So it was proposed that AS is not a defect, but rather the model generates its own UV cutoff in the form of Landau pole Λ_{UV} [Aulakh 20].

Note that for all the gauge theories, there exists a natural scale at which the gauge coupling diverge. This phenomenon of formation of a natural scale in otherwise scale-invariant gauge theories is called *dimensional transmutation.* In AF gauge theories, the physical degrees of freedom condense in the IR as the RG flow into the IR has a Landau pole. Similarly, AS theories has a Landau pole in UV and one expects condensates in UV. A recent work introduces a novel interpretation and calculational framework that shows that the symmetry breaking of any AS gauge theory may be driven by gaugino condensates in UV. [Aulakh 20]

Exact calculations in the strong coupling region need lattice calculations. However, there are certain constraints that the system has to follow even in the strong region. Generalized Konishi Anomalies (GKA) are such constraints that all the SuSy gauge theories follow [Konishi 85], [Cachazo 02]. Using these, we can calculate the vacuum expectation values of fields transforming just like adjoint. Note that such calculations include the loop corrections and are thus exact. We'll use certain forms of superpotential such that classically, the vevs of Φ vanish. Only because of the loop effects, Φ develops a vev and cause spontaneous symmetry breaking. The method can be applied to the symmetry breaking of a large class of AS SuSy GUT models. We discuss three examples, which include symmetry breaking in a toy Susy GUT, $SU(5) \to G_{SM}$.

In [Aulakh 20], the GKA analysis of *pure adjoint* Susy YMH is generalized to the case where Φ is in a large *adjoint type*, i.e. $r \times \overline{r}$ representation. As we saw in the last chapter, the transformation properties of such representations are the same as the *true adjoint*. [Aulakh 20] contains a detailed framework for rank preserving symmetry breaking using full reducible $r \times \overline{r}$ representation. Here we present an extension to that by using *traceless* $r \times \overline{r}$ (without singlet in it).

5.2 Generalized Konishi Anomalies (GKA)

Chiral Ring

A *Chiral Ring*, loosely speaking, is a set of all gauge invariant chiral operators in SYMH theory. Products of these operators in a Susy vacuum factorize and are position independent.

In the chiral ring, we have:

$$[\mathcal{W}^\alpha, \Phi] = 0 \quad \text{mod} \quad \overline{D} \tag{5.1}$$

Since $\mathcal{W}_\alpha$ are two component spinorial superfields, it can be shown that product of more than two $\mathcal{W}_\alpha$ is zero. So the only independent chiral operators are:

$$\text{tr}\Phi^k \qquad \text{tr}\mathcal{W}_\alpha\Phi^k \qquad \text{tr}\mathcal{W}_\alpha\mathcal{W}^\alpha\Phi^k \tag{5.2}$$

$\mathcal{W}_\alpha$ is the gauge field strength superfield ($\mathcal{W}_\alpha = \mathcal{W}_\alpha{}^a T^a$) and Φ is a chiral superfield transforming as $r \times \overline{r}$. Let us define the following convenient objects:

$$R(z) = \kappa\,\text{tr}\left[\frac{\mathcal{W}^\alpha\mathcal{W}_\alpha}{z - \Phi}\right] \qquad T(z) = \text{tr}\frac{1}{z - \Phi} \tag{5.3}$$

When expanded as a power series in z, we get all the relevant chiral operators. Note that not much attention is given to $\text{tr}\mathcal{W}_\alpha\Phi^k$, because this is fermionic in nature and thus its vaccum expectation value vanishes.

$$R(z) = \kappa\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\text{tr}\mathcal{W}^\alpha\mathcal{W}_\alpha\Phi^n = \kappa\sum_{n=0}^{\infty}\frac{R_n}{z^{n+1}} \qquad \Rightarrow R_n = \text{tr}\,\mathcal{W}^\alpha\mathcal{W}_\alpha\Phi^n \tag{5.4}$$

$$T(z) = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\text{tr}\Phi^n = \sum_{n=0}^{\infty}\frac{T_n}{z^{n+1}} \qquad \Rightarrow T_n = \text{tr}\Phi^n \tag{5.5}$$

Note that κ is merely a number $-1/(32\pi^2)$ placed here for later convenience.

R_n and T_n can be calculated from $R(z)$ and $T(z)$ respectively by contour integrals as follows:

$$R_n = \frac{1}{2\pi i}\oint_{c_\infty} \mathrm{d}z\, z^n R(z) \qquad T_n = \frac{1}{2\pi i}\oint_{c_\infty} \mathrm{d}z\, z^n T(z) \tag{5.6}$$

GKA

Konishi Anomalies are Ward identities of chiral symmetry associated with each individual supermulitplet in SYMH. They relate the tree and loop level chiral anomaly contributions. They were intitally introduced by Konishi and Shizuya in [Konishi 85] and generalized by CDSW in [Cachazo 02]. We'll need only the expectation values:

$$\left\langle f_I \frac{\partial W}{\partial \Phi} \right\rangle = \left\langle \kappa \mathcal{W}^{\alpha a} \mathcal{W}^b_\alpha \mathcal{M}^{a}{}^J_I \mathcal{M}^{b}{}^K_J \frac{\partial f(\Phi, \mathcal{W}_\alpha)_K}{\partial \Phi_I} \right\rangle \qquad \kappa = -\frac{1}{32\pi^2} \tag{5.7}$$

Note that the indices I, K are $R = r \times \overline{r}$ representation indices. We can write the same equation in r representation indices, i, k, as follows:

$$\left\langle \mathrm{tr} f(\Phi, \mathcal{W}_\alpha) \frac{\partial W}{\partial \Phi} \right\rangle = \kappa \left\langle \mathcal{W}^{a\alpha} \mathcal{W}^b_\alpha \sum_{i,k} \left[T^a_{(r)}, \left[T^b_{(r)}, \frac{\partial f(\Phi, \mathcal{W}_\alpha)}{\partial \Phi_{ik}} \right] \right]_{ik} \right\rangle \tag{5.8}$$

Here $W(\Phi)$ is the superpotential and $f(\Phi, \mathcal{W}_\alpha)$ is the general variation in the chiral ring.

5.3 Loop Equations

5.3.1 Φ transforming as traceful $r \times \overline{r}$

We can use different sort of variations $f(\mathcal{W}_\alpha, \Phi) \equiv \delta\Phi$, substitute it back in Eq 5.8 and find specific equations.

Justification for using the same equation as adjoint Note that Eq 5.8 was written only for Φ transforming under pure adjoint. We have extended this to any general $r \times \overline{r}$ representation. We saw in the last chapter that general $r \times \overline{r}$ representations transform just like pure adjoint. Here $i, k, m, n, \ldots$ are r representation indices.

$$\Phi'^{\,k}_i = U_i{}^m \Phi_m{}^n U_n{}^k \qquad \Rightarrow \delta\Phi = [T_{(r)}, \Phi] \tag{5.9}$$

So we can use the same equation 5.8 with nested commutators using the generators and indices of representation r.

a: $f(\mathcal{W}_\alpha, \Phi)_{ik} = \left(\frac{1}{z-\Phi}\right)_{ik}$

$$\frac{\mathrm{d}}{\mathrm{d}\Phi_{ik}}\left(\frac{1}{z-\Phi}\right)_{mn} = \left(\frac{1}{z-\Phi}\right)_{mi}\left(\frac{1}{z-\Phi}\right)_{kn} \tag{5.10}$$

Thus the RHS of Eq 5.8 gives us: (Note the extra $-ve$ sign in last two terms because the positions of two fermionic operators got exchanged.)

$$\begin{aligned}\kappa(\mathcal{W}^\alpha\mathcal{W}_\alpha)_{im}\left(\frac{1}{z-\Phi}\right)_{mi}\left(\frac{1}{z-\Phi}\right)_{kk} - \kappa(\mathcal{W}^\alpha)_{im}\left(\frac{1}{z-\Phi}\right)_{mi}\left(\frac{1}{z-\Phi}\right)_{kn}(\mathcal{W}_\alpha)_{nk} \\ + \kappa(\mathcal{W}_\alpha)_{im}\left(\frac{1}{z-\Phi}\right)_{mi}\left(\frac{1}{z-\Phi}\right)_{kn}(\mathcal{W}^\alpha)_{nk} - \kappa\left(\frac{1}{z-\Phi}\right)_{ii}\left(\frac{1}{z-\Phi}\right)_{km}(\mathcal{W}_\alpha\mathcal{W}^\alpha)_{mk}\end{aligned} \tag{5.11}$$

Middle two terms vanish upon taking vev because fermionic fields can not have vev. First and last term add up because $\mathcal{W}_\alpha\mathcal{W}^\alpha = -\mathcal{W}^\alpha\mathcal{W}_\alpha$ and we have $2R(z)T(z)$ on RHS.

$$\begin{aligned}\mathrm{tr}\left[f(\Phi,\mathcal{W}_\alpha)W'(\Phi)\right] &= \mathrm{tr}\left[f(\Phi,\mathcal{W}_\alpha)W'(z)\right] + \mathrm{tr}\left[f(\Phi,\mathcal{W}_\alpha)(W'(\Phi)-W'(z))\right] \\ &= T(z)W'(z) + \frac{1}{4}c(z) \\ \frac{1}{4}c(z) &\equiv \mathrm{tr}\left[\frac{1}{z-\Phi}[W'(\Phi)-W'(z)]\right]\end{aligned} \tag{5.12}$$

Thus we have finally:

$$2R(z)T(z) - T(z)W'(z) - \frac{1}{4}c(z) = 0 \tag{5.13}$$

b: $f(\Phi, \mathcal{W}_\alpha)_{ik} = \left(\frac{\mathcal{W}^\alpha\mathcal{W}_\alpha}{z-\Phi}\right)_{ik}$

$$\frac{\mathrm{d}}{\mathrm{d}\Phi_{ik}}\left(\frac{\mathcal{W}^\alpha\mathcal{W}_\alpha}{z-\Phi}\right)_{mn} = \left(\frac{\mathcal{W}^\alpha\mathcal{W}_\alpha}{z-\Phi}\right)_{mi}\left(\frac{1}{z-\Phi}\right)_{kn} \tag{5.14}$$

Thus RHS gives us: (again taking care of extra $-ve$ sign in last two terms)

$$\begin{aligned}\kappa(\mathcal{W}^\alpha\mathcal{W}_\alpha)_{im}\left(\frac{\mathcal{W}^\beta\mathcal{W}_\beta}{z-\Phi}\right)_{mi}\left(\frac{1}{z-\Phi}\right)_{kk} - \kappa(\mathcal{W}^\alpha)_{im}\left(\frac{\mathcal{W}^\beta\mathcal{W}_\beta}{z-\Phi}\right)_{mi}\left(\frac{1}{z-\Phi}\right)_{kn}(\mathcal{W}_\alpha)_{nk} \\ + \kappa(\mathcal{W}_\alpha)_{im}\left(\frac{\mathcal{W}^\beta\mathcal{W}_\beta}{z-\Phi}\right)_{mi}\left(\frac{1}{z-\Phi}\right)_{kn}(\mathcal{W}^\alpha)_{nk} - \kappa\left(\frac{\mathcal{W}^\beta\mathcal{W}_\beta}{z-\Phi}\right)_{ii}\left(\frac{1}{z-\Phi}\right)_{km}(\mathcal{W}_\alpha\mathcal{W}^\alpha)_{mk}\end{aligned} \tag{5.15}$$

Here the first three terms vanish, because they involve traces of more than two $\mathcal{W}_\alpha$s.

The last term upon using $\mathcal{W}_\alpha \mathcal{W}^\alpha = -\mathcal{W}^\alpha \mathcal{W}_\alpha$ gives $R^2(z)/\kappa$.

As for the LHS, just like the earlier case, we have:

$$\begin{aligned}
\mathrm{tr}\left[f(\Phi, \mathcal{W}_\alpha) W'(\Phi)\right] &= \mathrm{tr}\left[f(\Phi, \mathcal{W}_\alpha) W'(z)\right] + \mathrm{tr}\left[f(\Phi, \mathcal{W}_\alpha)(W'(\Phi) - W'(z))\right] \\
&= \frac{1}{\kappa} R(z) W'(z) + \frac{1}{4\kappa} f(z) \\
\frac{f(z)}{4} &\equiv \kappa \mathrm{tr}\left[\frac{\mathcal{W}^\alpha \mathcal{W}_\alpha}{z - \Phi}[W'(\Phi) - W'(z)]\right] \qquad (5.16)
\end{aligned}$$

Thus we have,

$$R^2(z) - R(z) W'(z) - \frac{1}{4} f(z) = 0 \qquad (5.17)$$

Solving for $T(z)$

From Eq 5.17 and Eq 5.13, we have:

$$R(z) = \frac{1}{2}\left[W'(z) - \sqrt{W'(z)^2 + f(z)}\right] \equiv \frac{1}{2}(W'(z) - y(z)) \qquad (5.18)$$

$$y^2(z) = W'(z)^2 + f(z) \qquad (5.19)$$

$$T(z) = \frac{c(z)}{4(2R(z) - W'(z))} = -\frac{c(z)}{4y(z)} \qquad (5.20)$$

If superpotential $W(z)$ is of order n, from Eq 5.17, we can conclude that $f(z)$ is polynomial of order $n-2$. Thus for the usual cubic superpotenial, $f(z)$ is linear in z. Comparing the coefficients of z^{-n}, we get relation between $R_{n>1}$. In Eq 5.17, we can compare the coefficients of z^n to find the relation between f_i and R_n.

$$W'(z) = \lambda z^2 + mz + \mu \qquad (5.21)$$

$$\Rightarrow f(z) = f_0 + f_1 z = -4\lambda(R_1 + zR_0) - 4mR_0 \qquad (5.22)$$

Similarly, from Eq 5.13, we can infer that $c(z)$ is of order $n-2$.

$$c(z) = -4\left(\lambda T_1 + md(r)\right) - 4\lambda z d(r) \qquad (5.23)$$

From the definition of $y(z)$, Eq 5.19, it is clear that for superpotential of order n, $y^2(z)$ is polynomial of order $(2n-2)$. Now if we consider a contour integral of $T(z)$, since $y(z)$ appears in the denominator, there are total $(2n-2)$ branch points or $n-1$ branch cuts.

Once we have $T(z)$, we can integrate $zT(z)$ and find the vevs as proposed in [Aulakh 20]

$$v_i = \frac{1}{N_i} \frac{1}{2\pi i} \oint_{A_i} \mathrm{d}z\, zT(z) \qquad (5.24)$$

Classical Analysis

The usual way of finding vacuum expectation values is by minimizing the potential ($W'(\Phi) = 0$). Higgs mechanism in SM stands on the usual *semi-classical* way of finding vevs by minimizing the potential (or superpotential). In this method, loop effects can not be taken into consideration since we use a simple tree level (super)potential.

The classical case would be the absence of gaugino condensate and hence, $R(z) \to 0$. In such a case, $y(z) \to W'(z)$. There aren't any branch cuts. But rather there are poles. So essentially, upon *switching on* the loop effects, the classical poles have bifurcated into branch points. One can check that upon setting $R(z) \to 0$, we get the classical vevs back (the ones we get by setting $W'(\Phi) = 0$).

5.3.2 Φ transforming as traceless $r \times \overline{r}$

$r \times \overline{r}$ always contains a singlet and an adjoint, as we saw in the earlier chapter. For example, for $SU(3)$, $6 \times \overline{6} = 1+8+27$. For the traceful case, Φ transforms as a full 1+8+27 reducible representation. The next step would be to make Φ traceless, i.e., taking out the singlet (which is the trace of Φ). So in this section, we consider traceless Φ, i.e., retaining only 8+27 with $r = 6$ for $SU(3)$ case. We need to see how does the loop equations change if Φ is traceless rather than traceful.

The crucial point is that the $f(\Phi, \mathcal{W}_\alpha)$ needs to be changed. Since it is a variation of Φ in the chiral ring, it must also follow all the constraints that Φ follows. Now that our Φ is traceless, $f(\Phi, \mathcal{W}_\alpha)$ also must be traceless. So we just impose tracelessness on the $f(\Phi, \mathcal{W}_\alpha)$ used earlier as follows:

a: $f(\mathcal{W}_\alpha, \Phi)_{ik} = \left(\frac{1}{z-\Phi}\right)_{ik} - \frac{\delta_{ik}}{d(r)}\mathbf{tr}\left(\frac{1}{z-\Phi}\right)$

Note that if we substitute it back in Eq 5.8, RHS would be exactly the same as earlier. This is because the new term in $f(\mathcal{W}_\alpha, \Phi)$ is proportional to identity, which commutes with everything. So RHS is still $2T(z)R(z)$.

Alternatively, we can use Eq 5.7. The extra term in $f(\Phi, \mathcal{W}_\alpha)$ is singlet and when the generator $\mathcal{M}$ acts on it, we get zero.

As for the LHS, we have:

$$\mathrm{tr}\left[f(\Phi, \mathcal{W}_\alpha)W'(\Phi)\right] = \mathrm{tr}\left(\frac{1}{z-\Phi}W'(\Phi)\right) - \mathrm{tr}\left(\frac{1}{z-\Phi}\right)\frac{1}{d(r)}\mathrm{tr}(\mathbb{I}.W'(\Phi)) \tag{5.25}$$

$$= T(z)\left[W'(z) - \frac{1}{d(r)}\mathrm{tr}W'(\Phi)\right] + \frac{c(z)}{4} \tag{5.26}$$

$$\equiv T(z)\omega(z) + \frac{c(z)}{4} \tag{5.27}$$

Here we introduced a new $\omega(z)$. $c(z)$ is the same as earlier:

$$\omega(z) \equiv W'(z) - \frac{1}{d(r)}\mathrm{tr}W'(\Phi) \tag{5.28}$$

Thus we have finally have the new equation to replace Eq 5.13. Note that the only change is that $W'(z)$ is replaced by $\omega(z)$. Eq 5.28 looks like the tracelessness condition imposed on $W'(z)$. Since traceless Φ does not ensure traceless $W'(\Phi)$, it appears as if it also got stripped off of its privilage of having a trace.

$$2R(z)T(z) - T(z)\omega(z) - \frac{1}{4}c(z) = 0 \tag{5.29}$$

b: $f(\mathcal{W}_\alpha, \Phi)_{ik} = \left(\frac{\mathcal{W}^\alpha \mathcal{W}_\alpha}{z-\Phi}\right)_{ik} - \frac{\delta_{ik}}{d(r)}\mathbf{tr}\left(\frac{\mathcal{W}^\alpha \mathcal{W}_\alpha}{z-\Phi}\right)$

Again, the extra term in $f(\mathcal{W}_\alpha, \Phi)$ does not contribute to the RHS of Eq 5.8. RHS still remains, just like the earlier case, $R^2(z)$.

LHS also follows just like the other traceless case:

$$\mathrm{tr}\left[f(\Phi, \mathcal{W}_\alpha)W'(\Phi)\right] = \mathrm{tr}\left(\frac{\mathcal{W}^\beta \mathcal{W}_\beta}{z-\Phi}W'(\Phi)\right) - \mathrm{tr}\left(\frac{\mathcal{W}^\beta \mathcal{W}_\beta}{z-\Phi}\right)\frac{1}{d(r)}\mathrm{tr}(\mathbb{I}.W'(\Phi)) \tag{5.30}$$

$$= R(z)W'(z) + \frac{f(z)}{4} - \frac{1}{d(r)}R(z)\mathrm{tr}W'(\Phi) \tag{5.31}$$

$$\equiv R(z)\omega(z) + \frac{f(z)}{4} \tag{5.32}$$

So we have another equation to replace Eq 5.17 for traceless case:

$$R^2(z) - R(z)\omega(z) - \frac{1}{4}f(z) = 0 \tag{5.33}$$

Solving for T(z)

Just like the earlier case, we get: (This result appears in [Alday 03] for Φ transforming as *pure adjoint*)

$$T(z) = -\frac{c(z)}{4\chi(z)} \qquad \chi^2(z) = \omega^2(z) + f(z) \tag{5.34}$$

For the usual cubic superpotential, we have: (Note, $T_1 = 0$; $T_0 = d(r)$)

$$\mathrm{tr}W'(\Phi) = \lambda T_2 + mT_1 + \mu d(r) = \lambda T_2 + \mu d(r) \tag{5.35}$$

$$\Rightarrow \omega(z) = W'(z) - \frac{1}{d(r)}\mathrm{tr}W'(\Phi) \tag{5.36}$$

$$= \lambda z^2 + mz + \mu - \frac{1}{d(r)}(\lambda T_2 + \mu d(r)) \tag{5.37}$$

$$\Rightarrow \omega(z) = \lambda z^2 + mz - \frac{\lambda}{d(r)}T_2 \tag{5.38}$$

Note that μ has vanished, because its coefficient in $W(\Phi)$ is T_1.

The central point here is that again, there are $2n-2$ branch points in the integral of $T(z)$. $f(z)$ (Eq 5.22), $c(z)$ (Eq 5.23) remains same.

Earlier there was a relation between R_2 and T_1. But since this time $T_1 = 0$, we have no such relation and R_2 appears as an independent dynamical parameter determined only by the calculations in the strong region. Instead we find a relation between T_2 and R_n:

$$R_2 + \frac{m}{\lambda}R_1 = \frac{1}{d(r)}T_2 R_0 \tag{5.39}$$

Classical Analysis

Classically, we can find the vevs of the traceless Φ by minimizing the superpotential with an additional constraint that $\mathrm{tr}\Phi = 0$. We can use the Lagrange multiplier method or anything else. The final result would be:

$$\Phi = v\begin{bmatrix} N_2\mathbb{I}_{N_1} & -N_1\mathbb{I}_{N_2}\end{bmatrix}\frac{1}{N_2 - N_1} \tag{5.40}$$

Note that any permutation of the diagonal elements shown is acceptable and results in the same subgroup.

Again, if we *turn off* the loop corrections, the branch cuts merge back into poles, and we get the classical vevs (Eq 5.40).

5.4 Numerical Analysis for Traceful case

We are interested in finding the full quantum corrected vevs of Φ. [Aulakh 20] proposes the following expressions for quantum corrected vevs:

$$v_i = \frac{1}{N_i}\frac{1}{2\pi i}\oint_{A_i} \mathrm{d}z\, zT(z) = -\frac{1}{8\pi i N_i}\oint_{A_i}\mathrm{d}z\, z\,\frac{c(z)}{y(z)} \tag{5.41}$$

Note that A_i encircles the branch cut that resulted from the bifurcation of critical point a_i. Note that as $R(z) \to 0$ and thus $y(z) \to W'(z)$, we recover $v_i = a_i$.

We wish to perform this integral numerically. $y(z)$ depends only on the $W'(z)$ and the strong parameters $R(z)$. We can not calculate the latter quantities, so R_i are the input parameters.

We are interested in a simplified case with $W'(z) = \lambda z^2$. The most interesting point regarding this superpotential is that the classical vevs for this are zero. So without loop effects, the symmetry breaking is not possible. For this case, Eq 5.19,Eq 5.23 follow:

$$y^2(z) = \lambda^2 z^4 - 4\lambda(R_1 + zR_0) \tag{5.42}$$

$$c(z) = -4\lambda T_1 - 4\lambda z d(r) \tag{5.43}$$

We can make all these expressions dimensionless (denoted by a *hat*,[^]) using the scale provided by gaugino condensate:

$$R_0 = 2S_2(r)\Lambda^3 \tag{5.44}$$

$$\Rightarrow \hat{R}_1 = \frac{R_1}{{R_0}^{4/3}}; \qquad \hat{f}(z) = \frac{f(z)}{{R_0}^{4/3}}; \qquad \hat{z} = \frac{z}{{R_0}^{1/3}}; \qquad \hat{y}(z) = \frac{y(z)}{{R_0}^{2/3}}; \qquad \ldots \tag{5.45}$$

Hence, we have:

$$\hat{y}(z) = \left[\lambda^2 \hat{z}^4 - 4\lambda(\hat{R}_1 + \hat{z})\right]^{1/2} \tag{5.46}$$

We want to eliminate T_1 in favour of R_1, R_0. This can be realized as follows:

$$\begin{aligned} N_i = \frac{1}{2\pi i}\oint_{A_i} \mathrm{d}z T(z) &= -\frac{1}{8\pi i}\oint_{A_i} \mathrm{d}z \frac{c(z)}{y(z)} \\ &= \frac{1}{2\pi i}\oint_{A_i} \mathrm{d}z \frac{\lambda\left[T_1 + z d(r)\right]}{y(z)} \\ &= \lambda T_1 \left[\frac{1}{2\pi i}\oint_{A_i} \mathrm{d}z\, y(z)^{-1}\right] + \lambda d(r)\left[\frac{1}{2\pi i}\oint_{A_i} \mathrm{d}z\ z y(z)^{-1}\right] \end{aligned} \tag{5.47}$$

Since N_i is dimensionless, RHS is also dimensionless. So we define a set of dimensionless loop integrals as follows:

$$\mathcal{Y}_n = \frac{1}{2\pi i}\oint_{A_1} \mathrm{d}\hat{z}\, \hat{z}^n\, \hat{y}(\hat{z})^{-1} \tag{5.48}$$

Now using Eq 5.47, we can write T_1 in terms of $y(z)$ as follows:

$$N_1 = \lambda \hat{T}_1\, \mathcal{Y}_0 + \lambda d(r)\, \mathcal{Y}_1 \tag{5.49}$$

$$\Rightarrow \hat{T}_1 = \frac{N_1/\lambda - d(r)\, \mathcal{Y}_1}{\mathcal{Y}_0} \tag{5.50}$$

Now, we have:

$$\hat{v}_1 = \frac{1}{2\pi i N_1} \oint_{A_1} \mathrm{d}z\, \lambda \left[\hat{T}_1 \hat{z}\, \hat{y}^{-1}(\hat{z}) + d(r)\, \hat{z}^2\, \hat{y}^{-1}(\hat{z}) \right] \tag{5.51}$$

$$= \frac{1}{N_1} \left[\lambda\, \hat{T}_1 \mathcal{Y}_1 + \lambda\, d(r)\, \mathcal{Y}_2 \right] \tag{5.52}$$

Since T_1 is the trΦ, v_2 also follows from Eq 5.52:

$$T_1 = N_1 v_1 + N_2 v_2 \qquad \Rightarrow \hat{v}_2 = \frac{\hat{T}_1 - N_1 \hat{v}_1}{N_2} \tag{5.53}$$

It is clear that we just need to calculate the integrals $\mathcal{Y}_n$ to find vevs. To find the exact branch cuts, we need to find the zeroes of $y^2(z)$. $y(z)$ can be rewritten in a convenient form using the branch points:

$$y(z) \equiv \lambda \prod_{i=1}^{2n} |z - z_i|^{\frac{1}{2}} \prod_{i=1}^{n} \exp\left[\frac{i}{2} \left[\theta\left(\frac{z - z_{2i-1}}{z_{2i} - z_{2i-1}} \right) + \theta\left(\frac{z - z_{2i}}{z_{2i} - z_{2i-1}} \right) \right] \right] \tag{5.54}$$

Finally the contour integrals around the branch cut A_i running from $z_{2i} \to z_{2i-1}$ is achieved by:

$$\oint_{A_i} \mathrm{d}z\, g(z) = 2 \int_0^1 \mathrm{d}x\, (z_{2i-1} - z_{2i})\, g(x(z_{2i-1} - z_{2i}) + z_{2i}) \tag{5.55}$$

$$\mathcal{Y}_n = \frac{1}{\pi i} \int_0^1 \mathrm{d}x\, (\hat{z}_1 - \hat{z}_2)[(\hat{z}_1 - \hat{z}_2)x + \hat{z}_2]^n\, \hat{y}((\hat{z}_1 - \hat{z}_2)x + \hat{z}_2) \tag{5.56}$$

This follows since the semicircles at the end points of radius ϵ do not contribute to the integral as $\epsilon \to 0$ and thus the contour integral equals sum of two line integrals.

Using Eq 5.55, vevs can be calculated numerically for various values of $\lambda, \hat{R}_1$.

$$R_{0i} = -\frac{1}{4\pi i} \oint_{A_i} \mathrm{d}z\, y(z) \tag{5.57}$$

Solutions such that $R_1 \sim \sum R_{0j} v_j$ may be good candidates for quasi-semiclassical vacuum. [Aulakh 20] proposes the *Semi-classical* parameter as a measure of semi-classicality:

$$\delta_{SC} = \left| \frac{R_1 - \sum_j R_{0j} v_j}{R_1} \right|^2 \tag{5.58}$$

Eq 5.55 can be used to calculate R_{0i}, and hence δ_{SC} can be calculated.

One case $SU(3) \to SU(2) \times U(1)$ has been studied extensively in recent work. So here we study another two symmetry breaking cases using the same mechanism.

5.4.1 Numerical Investigation SU(2)→U(1)

Φ transforms as 3×3. Any diagonal vev of Φ would break $SU(2) \rightarrow U(1)$. An interesting obervation would be that final v_i and δ_{SC} depends on the ratio $N_1/d(r)$. Table 5.1 contains a few points in parameter space with $\delta_{SC} << 1$. Occurance of such points supports the idea that Φ develops vevs purely from loop corrections driven by gaugino condensate in UV. Note that we are using $W'(z) = \lambda z^2$ with minimas $z = 0$. Hence the vevs are purely from quantum loop corrections. All dimensionful quantities are in units of ${R_0}^{1/3} = 4^{2/3}\Lambda$ ($S_2(3) = 2$).

Table 5.1: SU(2)→U(1): Illustrive values obtained by searching $\lambda, \hat{R}_1$ such that $\delta_{SC} << 1$.

N_1	$d(r)$	λ	$\hat{R}_1$	$\hat{T}_1$	$\hat{v}_1$	$\hat{v}_2$	$\hat{R}_0{}^{(0)}$	$\hat{R}_0{}^{(1)}$	δ_{SC}
2	3	-1.582 $-0.295i$	0.247 $+0.374i$	-0.014 $+0.222i$	-0.220 $+0.630i$	0.425 $-1.037i$	0.772 $-0.193i$	0.228 $+0.193i$	7×10^{-6}
1	3	0.456 $+0.012i$	0.595 $+0.995i$	-1.705 $+2.010i$	0.162 $+2.600i$	-0.933 $-0.295i$	0.572 $-0.317i$	0.428 $+0.317i$	0.0004
1	3	0.456 $+0.012i$	0.595 $-1.005i$	-1.721 $-2.030i$	0.038 $-2.630i$	-0.880 $+0.299i$	0.556 $+0.340i$	0.444 -0.340	0.0009
1	3	1.350 $-0.710i$	0.550 $+0.698i$	-1.439 $+1.465i$	-0.733 $+1.817i$	-0.353 $-0.176i$	0.282 $-0.532i$	0.718 $+0.532i$	0.018

Numerical Investigation SU(5)→SU(3)× SU(2)× U(1)

Here we use two indices antisymmetric representation: $\psi_{[ab]}$, 10 as the base representation. From the decomposition of 10 under G_{SM}, it is evident that Φ of the following form would break $SU(5) \rightarrow G_{SM}$:

$$\phi_{10\times\overline{10}} = \text{diag}\ [\beta_1 \mathbb{I}_3 \qquad \beta_2 \mathbb{I}_6 \qquad \beta_3 \mathbb{I}_1] \tag{5.59}$$

Since in a cubic superpotential, only two vevs would appear, either two of β_i have to be equal. we take a specific case, $\beta_3 = \beta_1$ and hence $N_{1,2} = 4, 6$ and calculate a few vevs. Other cases will also be similar just with different values of N_i. We again look for solutions with $\delta_{SC} \ll 1$. For now, Table 5.2 contains a few such cases.

5.5 Numerical Analysis for Traceless case

Just like the traceful case, we are interested in finding the vevs for traceless case:

$$v_i = \frac{1}{N_i}\frac{1}{2\pi i}\oint_{A_i} \mathrm{d}z\, zT(z) = -\frac{1}{8\pi i N_i}\oint_{A_i} \mathrm{d}z\, z\,\frac{c(z)}{\chi(z)} \tag{5.60}$$

Table 5.2: SU(5)$\rightarrow G_{SM}$: Illustrive values obtained by searching $\lambda, \hat{R}_1$ such that $\delta_{SC} << 1$.

N_1	$d(r)$	λ	$\hat{R}_1$	$\hat{T}_1$	$\hat{v}_1$	$\hat{v}_2$	$\hat{R}_0{}^{(0)}$	$\hat{R}_0{}^{(1)}$	δ_{SC}
6	10	-0.500 $-0.325i$	0.871 $+0.570i$	1.259 $+0.291i$	-0.013 $+0.878i$	0.334 $-1.244i$	0.792 $-0.383i$	0.208 $+0.383i$	10^{-17}
6	10	-1.515 $-0.310i$	0.275 $+0.378i$	0.594 $-0.461i$	-0.175 $0.581i$	0.410 $-0.987i$	0.768 $-0.210i$	0.232 $+0.210i$	0.008
4	10	0.500 $-0.350i$	0.585 $+1.005i$	-4.957 $+7.869i$	-0.944 $+1.848i$	-0.197 $+0.080i$	0.292 $-0.548i$	0.708 $0.548i$	0.0007
4	10	0.500 $+0.650i$	1.200 $-0.995i$	-3.893 $-7.514i$	-1.171 $-1.405i$	0.132 $-0.316i$	-0.199 $+0.722i$	$+1.199$ $-0.722i$	0.001

We are interested in a simple calculation: $W'(z) = \lambda z^2$. Note that μ are anyway missing this time. So we have just set $m \to 0$. For this case, we have:

$$c(z) = -4\lambda z d(r) \tag{5.61}$$

$$\omega(z) = \lambda z^2 - \frac{\lambda}{d(r)} T_2 \tag{5.62}$$

$$\Rightarrow \chi^2(z) = \lambda^2 \left(z^2 - \frac{T_2}{d(r)}\right)^2 - 4\lambda(R_1 + zR_0) \tag{5.63}$$

Again we make the expressions dimensionless by using $R_0 = 2S_2(r)\Lambda^3$ as the scale. For $m \to 0$ case, Eq 5.39 reads (dimensionless):

$$d(r)\hat{R}_2 = \hat{T}_2 \tag{5.64}$$

Thus we have:

$$\hat{\chi}^2(z) = \lambda^2 \left(\hat{z}^2 - \hat{R}_2\right)^2 - 4\lambda(\hat{R}_1 + \hat{z}) \tag{5.65}$$

The contour integral for N_1 is:

$$N_i = \frac{1}{2\pi i} \oint_{A_i} \mathrm{d}\hat{z}\hat{T}(\hat{z}) = -\frac{1}{8\pi i} \oint_{A_i} \mathrm{d}z \frac{\hat{c}(\hat{z})}{\hat{\chi}(\hat{z})} \tag{5.66}$$

Though unlike the earlier case, we can not eliminate R_2 in favour of $R_{0,1}$. The branch points themselves depend on R_2. Since R_2 can not be eliminated, it acts as a dynamical variable and thus an input parameter for our numerical investigations. We eliminate T_2 in favour of R_2 from the expression of $\chi(z)$. The result is Eq 5.65. Now we choose appropriate R_2 such that we get the exact values of N_i upon integrating $T(z)$ around the branch cuts. We write the principle value of $\chi(z)$ just like Eq 5.54, with the branch cuts determined by Eq 5.65 and integrate it numerically, following Eq 5.55.

Just like the vevs, we can calculate T_2 by integrating around both the branch cuts separately and adding up the result. Since R_2 would also be known to us, we can check if the results follow Eq 5.64. We can quantify the error as follows:

$$\delta_{R_2T_2} = \left| \frac{d(r)R_2 - \sum T_{2i}}{d(r)R_2} \right|^2 \tag{5.67}$$

5.5.1 SU(2)→U(1)

Earlier, we calculated numerically vevs for Φ, transforming as 3×3 of $SU(2)$. So now for traceless Φ, we only have 3+5. We can set $N_1 = 2$ without loss of generality. Thus we expect the vev pattern to be as follows (follows from Eq 5.40):

$$\Phi = v[-\mathbb{I}_2 \quad 2\mathbb{I}_1] = [-v \quad -v \quad +2v] \tag{5.68}$$

Note that all three permutations are possible and are equally valid.

Table 5.3: SU(2)→U(1): The vevs of Traceless matrix 3×3 containing only 3+5. $d(r) = 3$

λ	$\hat{R}_1$	$\hat{R}_2$	N_1	N_2	$\hat{v}$	$\hat{T}_1$	$\hat{T}_2$	$\delta_{T_2R_2}$
$0.7 + i$	-0.664 $-0.986i$	-0.0736 $+1.325i$	2.0000	1.0000	0.326 $+0.189i$	0.0000	-0.220 $+3.976i$	8E−5
0.7 $+0.5i$	-3.945 $-0.0265i$	-47.711 $-0.0265i$	2.0000 $-0.0001i$	1.0000 $+0.0001i$	-7.059 $-1.021i$	0.0000	-143.133 $-0.079i$	2E−7
$0.3 + I$	-6.979 $-23.157i$	0.0032 $+4.823i$	2.0001	0.9999	0.0788 $+0.0236i$	0.0000	0.0103 $14.469i$	4E−5
0.5 $+0.5i$	-3.510 $-0.0426i$	-38.123 $-0.0426i$	2.0000	1.0000	-5.528 $-1.155i$	0.0000	-114.369 $-0.127i$	5E−6
0.9	-0.539	-1.863	2.0000	1.0000	0.154	0.0000	-5.588	1E−6

5.5.2 SU(3)→SU(2)×U(1)

Here we consider the symmetry breaking of $SU(3)$ using the traceless $6 \times \overline{6}$. Thus Φ contains adjoint and 27. We saw earlier that for it to break down to $SU(2) \times U(1)$, Φ must have the following form

$$\phi_{6\times\overline{6}} = \text{diag } [\beta_1\mathbb{I}_3 \qquad \beta_2\mathbb{I}_2 \qquad \beta_3\mathbb{I}_1] \tag{5.69}$$

For the traceless case, either two must be equal. Let us consider $\beta_1 = \beta_3$. So we can demand $N_1 = 4$ and $N_2 = 2$. There are two other cases possible: $\beta_1 = \beta_2$ with $N_1 = 5$ and $\beta_2 = \beta_3$ with $N_1 = 3$. The whole analysis can be done for these cases as well.

With $N_1 = 4$ and $N_2 = 2$, Eq 5.40 reads:

$$\Phi_{8+27} = v[-1 \quad -1 \quad -1 \quad +2 \quad +2 \quad -1]$$

Table 5.4: SU(3)→SU(2)×U(1): The vevs of Traceless matrix $6 \times \overline{6}$ containing 8+27

λ	$\hat{R}_1$	$\hat{R}_2$	N_1	N_2	$\hat{v}$	$\hat{T}_1$	$\hat{T}_2$	$\delta_{T_2R_2}$
0.7 +0.5i	−3.945 −0.025i	−47.711 −0.026i	4.0000	2.0000	−7.059 −1.021i	0.0000	−286.27 −0.159i	9E−7
0.9	−0.5397 +0.001i	−1.863 −0.001i	4.0000	2.0000	0.154	0.0000	−11.176 −0.001i	7E−6
$0.9 + I$	−4.166 −0.021i	−52.675 −0.022i	4.0000	2.0000	−8.445 −1.451i	0.0000	−316.05 −0.131i	4E−6
0.1 +0.5i	−29.925 −149.52i	0.0026 +17.295i	4.0001	2.0001	0.0459 +0.0092i	0.0000	0.0143 103.77i	1E−5
0.5	−0.383	−1.440	4.0000	2.0000	0.436	0.0000	−8.649	1E−7

All possible permutations are equally valid. Table 5.5 contains a points in parameter space which gives the appropriate results.

5.5.3 SU(5)→SU(3)×SU(2)×U(1)

Here we consider Φ to be in traceless $10 \times \overline{10}$ representation having 24+75. Φ of the form given in Eq 4.50 would break $SU(5) \to G_{SM}$.

We consider a case with $\beta_1 = \beta_3$, so that $N_1 = 6$ and $N_2 = 4$. There are two other cases possible. With $N_1 = 6$ and $N_2 = 4$, Eq 5.40 reads:

$$\Phi_{24+75} = [+3v\mathbb{I}_3 \quad -2v\mathbb{I}_6 \quad +3v\mathbb{I}_1] \tag{5.70}$$

Again, all possible permutations are equally valid. Table 5.5 contains a points in parameter space which gives the appropriate results.

Table 5.5: SU(5)→ G_{SM}: The vevs of Traceless matrix $10 \times \overline{10}$ containing 24+75

λ	$\hat{R}_1$	$\hat{R}_2$	N_1	N_2	$\hat{v}$	$\hat{T}_1$	$\hat{T}_2$	$\delta_{T_2R_2}$
0.1	−1.663	−1.135	6.0000	4.0000	0.658	0.0000	−11.345	1E−6
0.5	−1.299 +0.001i	−1.084 −0.001i	6.0000	4.0000	0.321	0.0000	−10.839 −0.003i	2E−6
0.5 +0.5i	−21.449 −0.023i	−243.422 −0.079i	6.0000	4.0000	−12.702 −1.626i	0.0000	−2434.22 −0.804i	6E−6
0.7	−4.397 0.003i	−11.365 +0.005i	6.0000	4.0000	−0.716 +0.009i	0.0000	−113.653 +0.001i	2E−6
0.9	−5.079	−14.622	6.0000	4.0000	−1.119	0.0000	−146.22	1E−6

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