Isoperimetric inequality

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Contents

Certificate of Examination **i**

Declaration of Authorship **ii**

Acknowledgements **iii**

Abstract **iv**

1 Introduction **1**

2 Preliminaries **3**

2.0.1 Curvature . . . 3
2.0.2 Relative Compact Domain . . . 4
2.0.3 Convex Sets . . . 4
2.0.4 Convex Hull and convex combination . . . 4
2.0.5 Linear Combination of convex sets . . . 5
2.0.6 Convex body . . . 7
2.0.7 Affinely Independent . . . 7
2.0.8 Simplex . . . 7
2.0.9 Hyperplane . . . 7
2.0.10 Hypersurface . . . 9
2.0.11 Hahn-Banach Separation Theorem . . . 9

3 Isoperimetric inequality in $\mathbb{R}^n$ **14**

3.1 **Isoperimetric inequality in the Plane($\mathbb{R}^2$)** . . . 14
3.2 Isoperimetric inequality in domains with C^2 boundary . . . 19
3.2.1 Riemannian metric and First Fundamental Form . . . 19
3.2.2 Riemannian Divergence . . . 20
3.2.3 Second Fundamental Form . . . 20
3.2.4 C^k Isoperimetric problem . . . 25
3.2.5 C^k extremal of the Isoperimetric functional . . . 25

4 Isoperimetric inequality in convex Subsets of $\mathbb{R}^n$ 29
4.0.1 Convex Polytopes and Polyhedral sets 29
4.0.2 Convex function 31
4.0.3 Convex body 33
4.0.4 Volume and surface area of a convex body 34
4.0.5 Distribution of volume for B^n 38
4.0.6 Mixed volume for convex body 40
4.0.7 The Brunn-Minkowski Theorem 42

5 Isoperimetric inequality for graph 51
5.0.1 Isoperimetric Number of the graph G 55
5.1 Bounding $\mathrm{E}\,|\lambda_2(G)|$ of random 2d-regular graph 57

Bibliography 61

List of Figures

2.1 Convex set and non-convex set by Oleg Alexandrov(2007) [8] . . . 4
2.2 Convex Hull of a nonconvex body by Scott Davidson(2017) [15] . 5
2.3 Minkowski sum of circle and square in the plane by Allen Chou(2013) [9] 5
2.4 Graph of n-Simplexes for n=2 to 7 by Weisstein(2020) [12] . . . 7
2.5 Half spaces image by Tanoumand(2019) [14] 8
2.6 **Supporting hyperplane** from Wikidots(2019) [11] 8
2.7 Drawing Hyperplane between two domains by Oleg Alexandrov(2008) [10] 9

3.1 Making area to volume ratio larger by Oleg Alexandrov(2007) [13] 15

4.1 96% volume strip in different dimensions by Ball(2002) [3] 40

Certificate of Examination

This is to certify that the titled “Isoperimetric inequality” submitted by Prashant Kumar (Reg. number − MS15114) for the partial fulfilment of BS-MS dual degree program of the institute, has been examined by the thesis committee duly appointed by the institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr Soma Maity (Supervisor)

Dr Lingaraj Sahu **Dr Pranab Sardar**

Date: June 2020

Declaration of Authorship

The work presented in this dissertation has been carried out by me under the guidance of **Dr Soma Maity** at the Indian Institute of Science Education and Research, Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellow-ship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Prashant Kumar (Candidate)
Date: June 2020

In my capacity as the supervisor of the candidate′s project work, I certify that the above statements by the candidate are accurate to the best of my knowledge.

Dr Soma Maity(Supervisor)
Date: June 2020

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Abstract

This dissertation is an exposition of isoperimetric inequality in various spaces with a focus on the evolution of techniques as we explore it in more general spaces. We first focus on differential geometric arguments for Euclidean space hyper-surfaces and review the uniqueness of the solution to C^2 isoperimetric problem and uniqueness of extremal of C^2 isoperimetric functional. We look into convex bodies in $\mathbb{R}$ next and review the popular theorem "Brunn-Minkowski theorem" using convex geometry techniques. From this theorem, as a corollary, isoperimetric inequality for the convex body is proved
We also discuss Isoperimetric inequality for graphs and for 2k-regular graphs, analyze how it relates with the problem of bounding the second eigenvalue.

Chapter 1

Introduction

The literal meaning of 'isoperimetric' is 'having the same perimeter'. Isoperimetric inequality is a geometric inequality which relates the perimeter and the volume of a domain. Classical isoperimetric inequality in the plane is defined as the curve, if any, which maximizes the enclosed area among all the closed curves of a fixed perimeter or among all the closed curves the curve which minimizes the perimeter in the plane surrounding a fixed area.

People have defined Isoperimetric inequalities for various spaces like Euclidean space and Riemannian manifolds. In Euclidean space, sharp isoperimetric inequality is found while in Riemannian manifolds, isoperimetric inequality is not exact but is close enough to get its global information.
Only Isoperimetric inequality in Euclidean spaces, e.g. in $\mathbb{R}^n$ and then specifically in convex subsets of $\mathbb{R}^n$ are discussed here.

In chapter (2), I first discussed some preliminaries defining some basic definitions and terms associated with convex Geometry.

Isoperimetric inequality in $\mathbb{R}^2$ is formulated in the chapter(3), and methodology is generalized for $\mathbb{R}^n$ and then the theorem which discusses the uniqueness and existence of the solution to Isoperimetric Problem which is a disk is demonstrated and proof of it is also described by using classical calculus.
We have further demonstrated the proof of Isoperimetric inequality in $\mathbb{R}^2$ using 2-dimensional divergence theorem.
Next, I have demonstrated Isoperimetric inequality in domains with C^2 boundary. In that, I first started giving standard local calculations pertaining to Hypersurface differential geometry in Euclidean space. In which, together with Riemannian divergence and curvatures, I have defined the Riemannian metric (first fundamental form) and second fundamental form.

Then I have demonstrated some results related to the first variation of domain area and volume and their boundaries and then discussed some standard

local calculations related to differential geometry of Hypersurface in Euclidean space.
Then I have demonstrated that if a domain provides a solution to C^2 Isoperimetric problem, it has to be a disk. After that I strengthened this result, that even if the domain is just an extremal of Isoperimetric functional, it will be a disk.

Firstly, convex sets and their related properties are provided in the chapter (4) and some important theorems related to the convex hull, and convex combination are stated and proved.

I defined convex polytopes and convex polyhedral sets and discussed their properties, and then related theorems are proved. After that, convex functions are defined.
Further volume and surface area of the convex Bodies are discussed after defining convex bodies.
In order to evaluate the volume and surface area for the convex body, I have described the volume and area recursively for polytopes and then established the volume of the arbitrary convex body by approximating it from the polytope sequences.

All convex bodies behave something like an Euclidean ball. 'Where the whole n-ball volume is distributed' is also discussed, and I found that it is situated near the Euclidean ball' s surface.

After that, I have first recursively defined the mixed volume for polytopes then just as I have defined simple volume; I have defined it for the convex body by approximating an arbitrary convex body by a sequence of convex polytopes. Then important characteristics of mixed volume for polytopes and convex bodies are discussed.
Then I have demonstrated the proof of an essential result as a lemma which will be used for proving the well known "The Brunn-Minkowski Theorem", and I got Isoperimetric inequality for convex bodies as a corollary from the Brunn-Minkowski theorem.

Isoperimetric inequalities are also defined for graphs, and in the chapter (5) I have discussed things related to it, e.g. Isoperimetric number for graphs and best Isoperimetric function.
The Isoperimetric numbers $i(G)$ and ikG are defined from the [4] paper, and some fundamental properties of $i(G)$ are discussed. Then, lower bound for i_kG and $i(G)$ in terms of the second eigenvalue of the Laplacian matrix D are proved with the help of bound on the expectation of the second eigenvalue of a random 2k-regular graph.

Chapter 2

Preliminaries

2.0.1 Curvature

For any C^2 path $\omega\colon (\alpha, \beta) \to \mathbb{R}^2$, Its derivative w' is the velocity vector field, and the acceleration vector field is the second derivative w'', assuming that w is an immersion (i.e. w' never vanishes) in the plane. An infinitesimal element of the length of the arc is given $ds = |\omega'(t)|\, dt$.
Unit tangent vector field along ω is $\mathbf{T}(t)$ defined as

$$\mathbf{T}(t) = \frac{\omega'(t)}{|\omega'(t)|} \tag{2.1}$$

and unit normal vector field $\mathbf{N}$ along ω is defined as

$$\mathbf{N} = \tau\mathbf{T} \tag{2.2}$$

where $\tau\colon \mathbf{R}^2 \to \mathbf{R}^2$ is the rotation of $\mathbb{R}^2$ by $\pi/2$ radians, and

Then its curvature κ is defined as

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \tag{2.3}$$

from equation (2.3) we have

$$\frac{d\mathbf{T}}{ds} \cdot \mathbf{N} = \kappa$$

$\mathbf{N}$ and $\mathbf{T}$ are perpendicular to each other so that $\mathbf{N}\cdot\mathbf{T} = 0$ and on differentiating with respect to s we get

$$\mathbf{N} \cdot \frac{d\mathbf{T}}{ds} + \mathbf{T} \cdot \frac{d\mathbf{N}}{ds} = 0$$

so that we get

$$\mathbf{T} \cdot \frac{d\mathbf{N}}{ds} = -\mathbf{N} \cdot \frac{d\mathbf{T}}{ds} = -\kappa$$

and hence finally get

$$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} \tag{2.4}$$

2.0.2 Relative Compact Domain

A relatively compact subspace of topological space X is a sub-set with compact closure.
Every subset of a compact space is relatively compact since closed subsets of a compact space are compact

2.0.3 Convex Sets

A set A in R^n is **convex** if $x, y \in A$ implies that $\lambda x + (1-\lambda)y \in A$ for all $\lambda \in [0,1]$, , i.e. for any x and y in A the closed line segment $[x, y]$ in R^n joining them is contained in A.

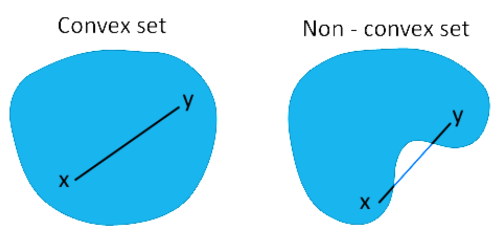

Figure 2.1: Convex set and non-convex set by Oleg Alexandrov(2007) [8]

A convex linear combination of elements $x_1, \ldots, x_k \in \mathbb{R}^n$ is the linear combination $\sum_{j=1}^{k} \lambda_j x_j$ where the coefficients satisfy

$$\sum_{j=1}^{k} \lambda_j = 1, \quad \lambda_j \geq 0 \ \forall \text{ j}$$

If A is convex, then any convex linear combination of points of A lies inside A.

2.0.4 Convex Hull and convex combination

Definition 2.0.1 (Convex Hull)**.** *For a given set $A \subset \mathbb{R}^n$,*
*Intersection of all convex sets containing A is called its **convex Hull** and is denoted by* conv A*.*

Intuitively convex hull of A is the smallest convex set containing A which fills its nonconvex parts and turns out to be a set of all convex combinations of points of A.

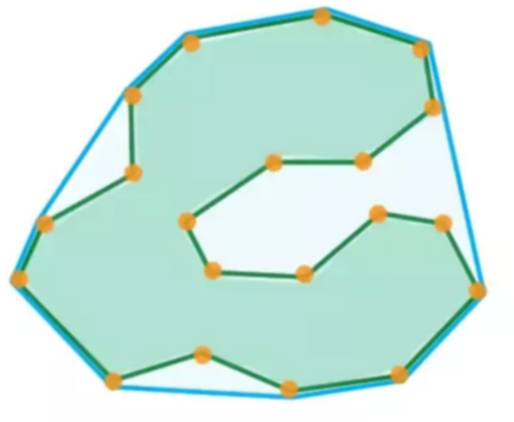

Figure 2.2: Convex Hull of a nonconvex body by Scott Davidson(2017) [15]

Given a family of convex sets $\{A_i : i \in I\}$, their intersection $\bigcap_{i \in I} A_i$ is also a convex set and conv A is the intersection of all convex sets containing A. Hence conv A always exists for a set $A \in \mathbb{R}^n$.
For two points $x, y \in A$ and for $\alpha \in [0, 1]$, $\alpha x + (1-\alpha)y$ is called **convex combination** of x and y, and its generalization for any number of points is as follows:

Let $k \in \mathbb{N}$, let $x_1, \ldots, x_k \in \mathbb{R}^n$, and let $\alpha_1, \ldots, \alpha_k \in [0, 1]$ with $\alpha_1 + \ldots + \alpha_k = 1$, then $\alpha_1 x_1 + \cdots + \alpha_k x_k$ is called a **convex combination** of the points $x_1, \ldots, x_k$.

2.0.5 Linear Combination of convex sets

We define linear combination $\lambda_1 A + \lambda_2 B$ of two sets $A, B \in \mathbb{R}^n$ to be

$$\lambda_1 A + \lambda_2 B := \{\lambda_1 x + \lambda_2 y : x \in A, y \in B\}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$. This type of addition of sets is also called **Minkowski addition**. The Minkowski addition of a circle and a square is illustrated in the following figure.

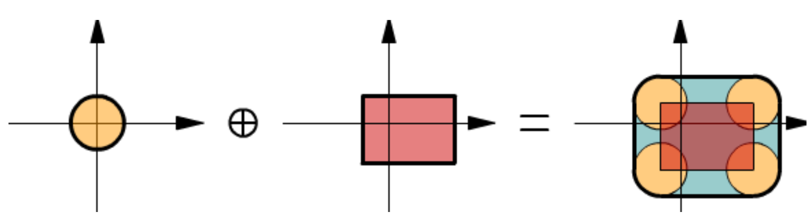

Figure 2.3: Minkowski sum of circle and square in the plane by Allen Chou(2013) [9]

- $\lambda_1 A + \lambda_2 B$ is also a convex set if A and B are convex sets.
- Generally for any set A, $A + A = 2A$ and $A - A = 0$ does not hold, but the case when A is a convex set, it holds. Also for $\lambda_1, \lambda_2 \geq 0$, we also have

$$\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A.$$

We can prove this property for convex sets easily using the convex property.

Proof. First, we see that this property doesn't hold for every set.
For example, take a simple set in $\mathbb{R}^2$, $B = \{(-1,-1),(1,1)\}$ then clearly $B + B \neq 2B$.
Let $a \in \lambda_1 A + \lambda_2 A$ then

$$a = \lambda_1 x + \lambda_2 y \text{ for some } x, y \in A \text{ where } \lambda_1, \lambda_2 \geq 0.$$

$$\text{For } \lambda_1, \lambda_2 \geq 0, \frac{\lambda_1}{\lambda_1 + \lambda_2}, \frac{\lambda_2}{\lambda_1 + \lambda_2} \in (0,1)$$

and also

$$\frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2} = 1$$

so that from convex property

$$\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}x + \frac{\lambda_2}{\lambda_1 + \lambda_2}y\right) \in A.$$

Now

$$\lambda_1 x + \lambda_2 y = (\lambda_1 + \lambda_2)\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}x + \frac{\lambda_2}{\lambda_1 + \lambda_2}y\right) \in (\lambda_1 + \lambda_2)A.$$

So $a \in (\lambda_1 + \lambda_2)A$ and hence

$$\lambda_1 A + \lambda_2 A \subset (\lambda_1 + \lambda_2)A.$$

Other direction is trivial as

$$\text{if } a \in (\lambda_1 + \lambda_2)A \text{ then } a = (\lambda_1 + \lambda_2)x \text{ for some } x \in A$$

and

$$a = (\lambda_1 + \lambda_2)x = \lambda_1 x + \lambda_2 x \in \lambda_1 A + \lambda_2 A$$

Hence

$$\lambda_1 A + \lambda_2 A \supset (\lambda_1 + \lambda_2)A.$$

So that

$$\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A.$$

□

Definition 2.0.2. *K and L are said to be Homothetic, if and only if for some $x \in \mathbb{R}^n, \alpha \in \mathbb{R}$*
$K = \alpha L + x$ or $L = \alpha K + x$.
Note that if one of K, L is a point, then K and L are trivially homothetic.

2.0.6 Convex body

A convex body in $\mathbb{R}^n$ is a nonempty compact convex set with nonempty interior. e.g.convex Polytope in $\mathbb{R}^n$, Cube $[-1, 1]^n$ in $\mathbb{R}^n$, (n)-dimensional regular solid simplex (convex hull of $n + 1$ equally spaced points).

2.0.7 Affinely Independent

Points $x_1, \ldots, x_k \in \mathbb{R}^n$ are called **affinely Independent** if the following vectors $x_2 - x_1, \ldots x_k - x_1$ are linearly independent.

- linearly Independent is defined for vectors in $\mathbb{R}^n$ whereas affinely independence is defined for points in $\mathbb{R}^n$.

2.0.8 Simplex

A **simplex** is the convex hull of affinely independent points, and
a **r-simplex** is the convex hull of $r + 1$ affinely independent points.

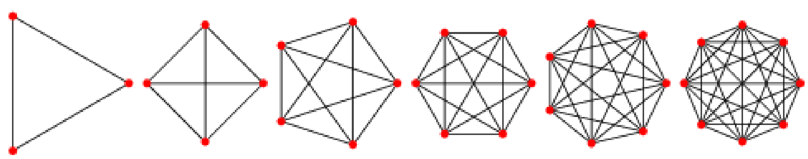

Figure 2.4: Graph of n-Simplexes for n=2 to 7 by Weisstein(2020) [12] (12)

2.0.9 Hyperplane

In general, the word "hyperplane" refers to an $(n - 1)$-dimensional flat in R^n.
A hyperplane is a subspace whose dimension is one less than that of its ambient space.
It is the preimage of a linear function from $\mathbb{R}^n$ to $\mathbb{R}$, i.e.

$$H = \{x \in V : a^T \cdot x = b\} \tag{2.5}$$

where $a \in \mathbb{R}^n$ and b is any other arbitrary constant.
Or simply $H = \{f = b\}$ where f is a linear function from $\mathbb{R}^n$ to $\mathbb{R}$.
It can be written as a linear equation of the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b,$$

where $a_1, a_2, ..a_n \in \mathbb{R}$ and $(a_1, a_2, a_3, ..., a_n)$ is a normal vector to the hyperplane.

Two half-spaces determined by hyperplane H are

$$H^- = \{x \in V : a^T \cdot x \leq b\}$$

and

$$H^+ = \{x \in V : a^T \cdot x \geq b\}.$$

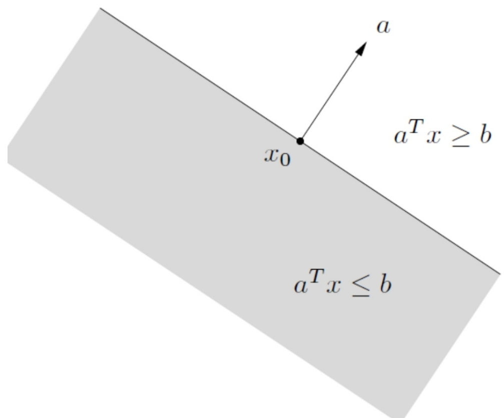

Figure 2.5: Half spaces image by Tanoumand(2019) [14]

Let A, B be subsets of R^n and $H = \{f = b\}$ be a hyperplane. We say that A and B are **separated** by H if A and B lie in different closed half-spaces determined by H.
If neither A nor B intersects H, we say H **strictly separates** A and B.

Definition 2.0.3. *Let A be a subset of R^n which is closed and convex. We say hyperplane $H = \{f = \alpha\}$ is* ***supporting hyperplane*** *of A at $x \in H$ if $A \cap H \neq \emptyset$ and A is contained in any of the two closed half-spaces $\{f \leq \alpha\}, \{f \geq \alpha\}$.*

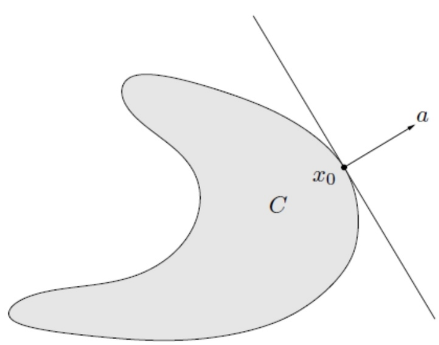

Figure 2.6: **Supporting hyperplane** from Wikidots(2019) [11]

Supporting half-space *of A is half-space which contains A and is bounded by supporting hyperplane of A, and the set $A \cap H$ is called* ***support Set*** *and any of its point $x \in A \cap H$ is called* ***supporting point****.*

2.0.10 Hypersurface

A hypersurface is a manifold of dimension $(n-1)$, embedded in an ambient space of dimension (n), generally an Euclidean space. It is a generalization of the hyperplane, plane curve and surfaces. In $\mathbb{R}^2$ it is a plane curve and, in $\mathbb{R}^3$ it is a surface.

If M and N are differentiable manifolds such that $dim(M) - dim(N) = 1$ and if animmersion is defined as $f\colon N \to M$ then $f(N)$ is a Hypersurface in M. For example

$$x_1^2 + x_2^2 + x_3^2 + \cdots x_n^2 = 1$$

is an $(n-1)$-dimensional hypersurface in $\mathbb{R}^n$.

2.0.11 Hahn-Banach Separation Theorem

Can we always draw a line between given two sets, A and B? If A and B are convex, then we can surely draw a line between them. If they aren't, we could easily run into problems and can easily see it in the following figures.

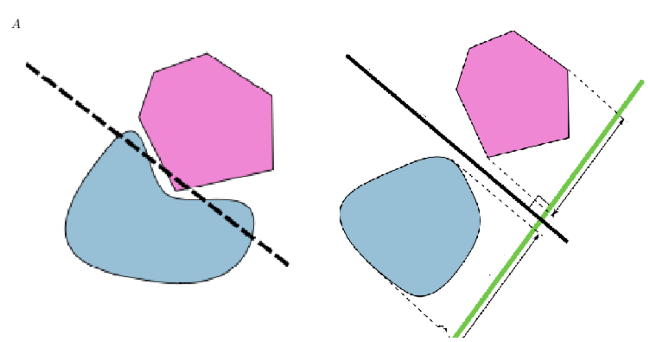

Figure 2.7: Drawing Hyperplane between two domains by Oleg Alexandrov(2008) [10]

Definition 2.0.4. *Let X be a topological vector space. We say $A, B \subset X$ are separated if there is $f \in X^*($ Set of all functional from $X \to \mathbb{C}), f \neq 0,$ and $\alpha \in \mathbb{R}$ such that*

$$A \subset (\operatorname{Re} f)^{-1}((-\infty, \alpha])$$
$$B \subset (\operatorname{Re} f)^{-1}([\alpha, +\infty))$$

And are strictly separated if

$$A \subset (\operatorname{Re} f)^{-1}((-\infty, \alpha))$$
$$B \subset (\operatorname{Re} f)^{-1}((\alpha, +\infty))$$

where $\operatorname{Re} f$ is the real part of f.

And

$(\text{ Re } f)^{-1}((-\infty, \alpha])$ is called a closed half-space.

$(\text{ Re } f)^{-1}((-\infty, \alpha))$ is called an open half-space.

$(\text{ Re } f)^{-1}(\{\alpha\})$ is called a closed affine hyperplane.

Lemma 2.0.1. *Let K be a nonempty closed convex subset of $\mathbb{R}^n$, Then $\exists$ a unique vector in K with the minimum norm.*

Proof. Let $\delta = \inf\{|x| : x \in K\}$ and $\{x_j\}$ be a sequence in K such that $|x_j| \to \delta$.

Since K is convex, so for any two points x_i and x_j in the sequence $\{x_j\}$, from the convex property, their midpoint

$$\frac{x_i + x_j}{2} \in K$$

and from the definition of δ

$$\left|\frac{x_i + x_j}{2}\right| \geq \delta^2$$
$$\implies |x_i + x_j|^2 \geq 4\delta^2.$$

From parallelogram law

$$|x_i - x_j|^2 = 2\,|x_i|^2 + 2\,|x_j|^2 - |x_i + x_j|^2$$
$$\leq 2\,|x_i|^2 + 2|x_j|^2 - 4\delta^2 \to 0$$

as $|x_i|, |x_j| \to \delta$ when $i, j \to \infty$.
So $\{x_j\}$ is a Cauchy Sequence, and so it has its limit point x and $x \in K$ as K is closed and also

$$\lim_{j\to\infty} |x_j| = |x|$$

because map $x \to |x|$ is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$.

This limit x is unique also otherwise

if $\exists y \in K$ such that $|y| = \delta$ then $|x - y|^2 \leq |x|^2 + |y|^2 - 4\delta^2 = 0$
So $x = y$
Hence proved. $\square$

We will be using the above lemma to prove the Hahn-Banach Extension theorem.

Theorem 2.0.2. ***(Hahn-Banach Separation Theorem)***

Let X be a topological vector space, A and B, convex, nonempty subsets of X and A is open then $\exists f \in X^*, \alpha \in \mathbb{R}$ such that $\forall x \in A, \forall y \in B$

$$\operatorname{Re} f(x) < \alpha \leq \operatorname{Re} f(y)$$

and for $F = \mathbb{R}$ it is simply

$$f(x) < \alpha < f(y).$$

Proof. (For $F = \mathbb{R}$ case) Given two disjoint subsets A and B in X

$$\text{define } K = A + (-B) = \{a - b, a \in A, b \in B\}$$

Since A and B are convex, K is also convex and so its closure, $\bar{K}$ is also convex. Hence we can apply the above lemma (2.0.1) for $\bar{K}$ so that we get a unique vector $v \in \bar{K}$ with the minimum norm.
For any $n \in \bar{K}$, as $\bar{K}$ is convex, the line segment

$$v(1-t) + tn = v + t(n - v) \in \bar{K}, \quad 0 \leq t \leq 1$$

and

$$|v|^2 \leq |v + t(n - v)|^2 = |v|^2 + 2t\langle v, n - v\rangle + t^2|n - v|^2.$$

Now for $\quad 0 < t \leq 1$

$$0 \leq 2\langle v, n\rangle - 2|v|^2 + t|n - v|^2$$

and let $t \to 0$ then $\langle n, v\rangle \geq |v|^2$ so for any x and y, $x \in A, y \in B$, we have

$$\langle x - y, v\rangle \geq |v|^2 \qquad \text{as } K = A + (-B).$$

Now if v is nonzero then.

$$\langle x, v\rangle - \langle y, v\rangle \geqslant |v|^2$$

.

$$\langle x, v\rangle - |v|^2 \geqslant \langle y, v\rangle.$$

If for any $y \in B$ the above inequality is valid, then on taking supremum over y in the right side also it will remain valid.

$$\langle x, v\rangle - |v|^2 \geqslant \sup_{y \in B}\langle y, v\rangle.$$

So we have

$$\langle x, v\rangle \geqslant \sup_{y\in B}\langle y, v\rangle + |v|^2.$$

Above inequality is valid for any $x \in A$ so on taking infimum over x on the left side also, it will remain valid
hence

$$\inf_{x\in A}\langle x, v\rangle \geqslant |v|^2 + \sup_{y\in B}\langle y, v\rangle. \tag{2.6}$$

So

$$\langle y, v\rangle \leqslant \sup_{y\in B}\langle y, v\rangle \leqslant \sup_{y\in B}\langle y, v\rangle + |v|^2.$$

Hence finally we do have

$$\langle x, v\rangle \geqslant \sup_{y\in B}\langle y, v\rangle + |v|^2. \tag{2.7}$$

and

$$\langle y, v\rangle \leqslant \sup_{y\in B}\langle y, v\rangle + |v|^2. \tag{2.8}$$

Now if we consider $\alpha = \sup_{y\in B}\langle y, v\rangle + |v|^2$ then we found a vector v whose correspondent functional

$$f = \langle x, v\rangle \quad \forall \text{ x} \in A,\ \text{y} \in B$$

is appropriate functional which satisfies required condition

$$f(x) < \alpha < f(y) \text{ for } \alpha = \sup_{y\in B}\langle y, v\rangle + |v|^2.$$

□

Theorem 2.0.3. *Each closed, convex set in $\mathbb{R}^n$ is the intersection of closed half-spaces.*

Proof. Let $C \subseteq \mathbb{R}^n$ closed, convex set and $\mathcal{H} = \{H : H \supset C\}$, i.e. set of all hyperplanes that contain C
then we have to prove that

$$C = \bigcap_{H\in\mathcal{H}} H.$$

For each $C \subseteq H, H \in \mathcal{H}$ so

$$C \subseteq \bigcap_{H\in\mathcal{H}} H.$$

So the first part is proved.

Now the remaining part is to show that.

$$C \supseteq \bigcap_{H \in \mathcal{H}} H.$$

Or to show that

$$\text{if } x \notin C \text{ then } x \notin \bigcap_{H \in \mathcal{H}} H.$$

Choose a point x outside C, i.e. $x \notin C$ then using Hahn-Banach Theorem (2.0.2), $\exists$ hyperplane A which separates C from x.
From this hyperplane, its one half-space contains C so x will not belong to that half-space of A and so

$$x \notin \bigcap_{H \in \mathcal{H}} H$$

hence proved. □

Chapter 3

Isoperimetric inequality in $\mathbb{R}^n$

The isoperimetric problem is to find the domain which contains the greatest area on considering all bounded domains with a fixed given perimeter.

3.1 Isoperimetric inequality in the Plane($\mathbb{R}^2$)

In $\mathbb{R}$, the discrete measure of the boundary of any bounded open subset of $\mathbb{R}$ is greater than or equal to 2, and equal to 2 only when given bounded open subset is just an open interval. So open interval is the solution to the isoperimetric problem in $\mathbb{R}$.
In $\mathbb{R}^2$, isoperimetric inequality relates the volume and area of a domain.
Disk turns out to be the solution of the isoperimetric problem in $\mathbb{R}^2$.

If the area of the domain is A and the length of its boundary(perimeter) is L then using values of perimeter and area of the disk, as an analytical inequality, Isoperimetric inequality can be written as

$$L^2 \geq 4\pi A \tag{3.1}$$

For a domain with a specified volume, its area can be increased by taking the mirror image of its concave part(first of following figure (3.1) or in the way shown in second following figure (3.1).

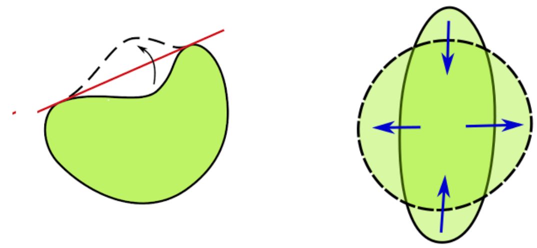

Figure 3.1: Making area to volume ratio larger by Oleg Alexandrov(2007) [13]

Separating Hyperplane

For $\mathbb{R}^n, n \geq 2$, it can be generalized as:

$$\frac{A(\partial\Omega)}{V(\Omega)^{1-1/n}} \geq \frac{A\left(S^{n-1}\right)}{V\left(\mathbb{B}^n\right)^{1-1/n}} \tag{3.2}$$

where Ω is any bounded domain in $\mathbb{R}^n$ and $\partial\Omega$ is its boundary, V denotes the volume of the domain ($n-$measure), and A denotes the area of the domain ($n-1$ measure), B^n is the unit disk in R^n, and S^{n-1}, the unit sphere in R^n.
Let ω_n denote the (n)-dimensional volume of B^n and c_{n-1} the $(n-1)$-dimensional surface area of S^{n-1}.
We have a standard result of ω_n and c_{n-1} for $\mathbb{R}^n$ as following

$$\mathbf{c_{n-1}} = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{3.3}$$

$$\omega_{\mathbf{n}} = \frac{\mathbf{c_{n-1}}}{n} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}. \tag{3.4}$$

where Γ is the standard gamma function.
Together (3.2), (3.3) and (3.4) gives us a simple form of Isoperimetric inequality as following

$$\frac{A(\partial\Omega)}{V(\Omega)^{1-1/n}} \geq n\omega_{\mathbf{n}}^{1/n}. \tag{3.5}$$

Theorem 3.1.1. *(Uniqueness for Smooth Boundaries.)*
Given the area A, let D vary over relatively compact domains in the plane of area A with C^1 boundary, and suppose that the domain Ω and its boundary $\partial\Omega \in C^2$ realizes the minimal boundary length among all such domains D. Then we claim that Ω is a disk.

Proof. since Ω is relatively compact in $\mathbb{R}^2$, there exists a simply connected domain Ω_0 such that

$$\Omega = \Omega_0 \backslash \{ \text{ finite disjoint union of closed topological disks } \}.$$

However, $\Omega_0 = \Omega$ otherwise on adding the topological disks to Ω, will increase its area and decrease the length of the boundary and so Ω will no longer have minimal boundary length.
So $\Omega_0 = \Omega$, and it is bounded by an embedded circle.

We assume that the path Γ is oriented(i.e. it has no self-intersection points, and on travelling through it, its interior always remains on the same side.) and hence its normal vector which is denoted by $\nu = -\mathbb{N}$.

Let $\Gamma \colon \mathbf{S}^1 \to \mathbb{R}^2 \in C^2$ be the embedding of the boundary of Ω.
Consider a 1-parameter family $\Gamma_\epsilon \colon \mathbb{S}^2 \to \mathbb{R}^2$ of embeddings:

$$v : (-\epsilon_0, \epsilon_0) \times \mathbf{S}^1 \to \mathbb{R}^2$$

such that $v(\epsilon, t)$ is given by:

$$v(\epsilon, t) = \Gamma_\varepsilon(t) = \Gamma(t) + \Psi(\epsilon, t)\nu(t), \quad \Psi(0, t) = 0 \tag{3.6}$$

is C^1.
We have

$$\frac{\partial v}{\partial \epsilon} = \frac{\partial \Psi}{\partial \epsilon}\nu. \tag{3.7}$$

and

$$\frac{\partial v}{\partial t} = \Gamma' + \left\{ \frac{\partial \Psi}{\partial t}\nu + \Psi\nu' \right\} = \{1 + \kappa\Psi\}\Gamma' + \frac{\partial \Psi}{\partial t}\nu. \tag{3.8}$$

which implies

$$\left| \frac{\partial v}{\partial t} \right| = \left\{ (1 + \kappa\Psi)^2 + \frac{1}{|\Gamma'|^2} \left(\frac{\partial \Psi}{\partial t} \right)^2 \right\}^{1/2} |\Gamma'| \, .$$

Let

$$\phi(t) := \left. \frac{\partial \Psi}{\partial \epsilon} \right|_{\epsilon = 0}.$$

Expanding $\Psi(\epsilon, t)$ around ϵ using Taylor series expansion we get

$$\Psi(\epsilon, t) = \epsilon\phi(t) + o(\epsilon), \quad \frac{\partial \Psi}{\partial \epsilon} = \phi(t) + o(1), \quad \frac{\partial \Psi}{\partial t} = O(\epsilon).$$

On simplifying and ignoring $O^2(\epsilon)$ and its higher-order term, we will get

$$\left| \frac{\partial v}{\partial t} \right| = |\Gamma'| \, \{1 + \epsilon\kappa\phi + o(\epsilon)\}.$$

The area element dA in coordinate (ϵ, t) is given by

$$dA = \left|\frac{\partial v}{\partial \epsilon} \times \frac{\partial v}{\partial t}\right| d\epsilon dt = \phi \left|\Gamma'\right| [1 + o(1)] d\epsilon dt = [\phi + o(1)] d\epsilon ds. \tag{3.9}$$

We have $A(\Omega_\epsilon) = A(\Omega)$ for all ϵ, so for small ϵ

$$0 = A(\Omega_\epsilon) - A(\Omega) = \int_0^\epsilon d\sigma \int_\Gamma [\phi + o(1)] ds.$$

So we have

$$\int_\Gamma \phi ds = 0.$$

Let $L(\epsilon)$ denote the length of Γ_ϵ.

$$L(\epsilon) = \int_{\mathbf{S}^1} \left|\frac{\partial v}{\partial t}\right| dt = \int_{\mathbf{S}^1} \left|\Gamma'\right| \{1 + \epsilon\kappa\phi + o(\epsilon)\} dt = \int_\Gamma \{1 + \epsilon\kappa\phi + o(\epsilon)\} ds.$$

We have $L'(0) = 0$ since Γ has the shortest length.
So

$$0 = L'(0) = \int_\Gamma \kappa\phi ds.$$

Therefore we have

$$\int_\Gamma \kappa\phi ds = 0 \quad \text{for } \forall \phi \in \mathbb{C}^1 : \quad \int_\Gamma \phi ds = 0. \tag{3.10}$$

Now to show that κ is constant(for showing that Γ is a circle), we choose a particular choice of ϕ such that $\int_\Gamma \phi ds = 0$ For any given $\psi\colon \mathbb{S}^1 \to \mathbb{R}$ in C^1 take

$$\phi = \psi - \frac{\int_\Gamma \psi ds}{\int_\Gamma ds} = \psi - \frac{1}{L}\int_\Gamma \psi ds.$$

So that for this ϕ,

$$\int_\Gamma \left(\psi - \frac{1}{L}\int_\Gamma \psi ds\right) ds = \int_\Gamma \psi ds - \left(\frac{1}{L}\int_\Gamma \psi ds\right)\int_\Gamma ds = \int_\Gamma \psi ds - \left(\frac{1}{L}\int_\Gamma \psi ds\right) L = 0.$$

Thus for this particular ϕ, from (3.10)

$$0 = \int_\Gamma \kappa\phi ds = \int_\Gamma \kappa \left(\psi - \frac{1}{L}\int_\Gamma \psi ds\right) ds = \int_\Gamma \left(\kappa - \frac{1}{L}\int_\Gamma \kappa ds\right) \psi ds.$$

Now since ψ is arbitrary C^1 function we have

$$\kappa - \frac{1}{L}\int_\Gamma \kappa ds = 0.$$

So that

$$\kappa = \frac{1}{L}\int_\Gamma \kappa ds = \text{Constant}.$$

Hence Γ is a circle, and Ω is a disk. □

Theorem 3.1.2. ***(Isoperimetric inequality in*** $\mathbb{R}^2$***)*** *Let Ω be a relatively compact domain in $\mathbb{R}^2$, with boundary $\partial\Omega \in C^1$ Then*

$$L^2(\partial\Omega) \geq 4\pi A(\Omega) \tag{3.11}$$

with equality when Ω is a disk.

Proof. Let $x = x^1 e_1 + x^2 e_2$ be a vector field on $\mathbb{R}^2$ with base point $\mathbf{x} = (x^1, x^2)$. We have (2)-dimensional divergence theorem for any vector field $\mathbf{x} \mapsto \xi(\mathbf{x}) \in \mathbb{R}^2$ with support containing cl Ω.

$$\int_\Omega \operatorname{div} \boldsymbol{\xi} dA = \int_{\partial\Omega} \boldsymbol{\xi} \cdot \nu ds \tag{3.12}$$

where ν denote outward unit normal vector along $\partial\Omega$.

For $x = x^1 e_1 + x^2 e_2$ we have $\operatorname{div} \mathbf{x} = 2$ on all Ω.

So from (3.12), we have

$$2A(\Omega) = \int_\Omega \operatorname{div} x dA = \int_{\partial\Omega} x \cdot \nu ds.$$

Using vector-Schwarz inequality we have

$$\int_{\partial\Omega} \mathbf{x} \cdot \nu ds \leq \int_{\partial\Omega} |\mathbf{x}| ds.$$

Furthermore, now using integral Cauchy-Schwarz inequality.

$$\int_{\partial\Omega} |\mathbf{x}| ds \leq \left\{\int_{\partial\Omega} |\mathbf{x}|^2 ds\right\}^{1/2} \left\{\int_{\partial\Omega} 1^2 ds\right\}^{1/2} = L^{1/2}(\partial\Omega) \left\{\int_{\partial\Omega} |\mathbf{x}|^2 ds\right\}^{1/2}.$$

We have

$$|\mathbf{x}|^2 = (x^1)^2 + (x^2)^2, \quad \left|\frac{d\mathbf{x}}{ds}\right|^2 = \left(\frac{dx^1}{ds}\right)^2 + \left(\frac{dx^2}{ds}\right)^2.$$

Applying Wirtinger's inequality to each coordinate function $x^1(s)$ and $x^2(s)$. implies

$$2A(\Omega) \leq L^{1/2}(\partial\Omega) \left\{\int_{\partial\Omega} |\mathbf{x}|^2 ds\right\}^{1/2} \leq L^{1/2}(\partial\Omega) \left\{\frac{L^2(\partial\Omega)}{4\pi^2} \int_{\partial\Omega} |\mathbf{x}'|^2 ds\right\}^{1/2}$$

$$= \frac{L^2(\partial\Omega)}{2\pi}.$$

So we have

$$L^2(\partial\Omega) \geq 4\pi A(\Omega).$$

Equality follows easily for the disk as for the disk of radius r, we have

$$4\pi^2 r^2 = L^2(\partial\Omega) = 4\pi A(\Omega) = 4\pi \cdot \pi r^2 = 4\pi^2 r^2$$

□

3.2 Isoperimetric inequality in domains with C^2 boundary

3.2.1 Riemannian metric and First Fundamental Form

Let Γ denote a $(n-1)$-dimensional Hypersurface in $\mathbb{R}^n$ and is given locally by the C^1 mapping $\boldsymbol{f}\colon A \to \mathbb{R}^n$, where A is an open subset of $\mathbb{R}^n$ and $\boldsymbol{f}$ is of everywhere maximal rank. (i.e. rank of the derivative of $\boldsymbol{f}, d_p\boldsymbol{f} : T_p\boldsymbol{f} \to \mathbb{R}^n$ at point $p \in A$ which is linear, is maximal.)

So $\boldsymbol{f} = \boldsymbol{f}(x)$; and the vectors

$$\frac{\partial \boldsymbol{f}}{\partial x^1}, \frac{\partial \boldsymbol{f}}{\partial x^2}, \cdots, \frac{\partial \boldsymbol{f}}{\partial x^{n-1}}$$

are linearly independent and span the tangent space of Γ at every $\boldsymbol{f}(x)$.
We denote $\mathbf{n}$ to be continuous normal unit vector field along Hypersurface Γ and always take the exterior normal unit vector field when Hypersurface Γ is the boundary of a domain in $\mathbb{R}^n$.

The Riemannian metric of Γ is given locally by the positive definite matrix $G(u)$, where

$$G = (g_{jk}), \quad g_{jk} = \frac{\partial \boldsymbol{f}}{\partial x^j} \cdot \frac{\partial \boldsymbol{f}}{\partial x^k}, \quad j,k = 1, \ldots, n-1. \tag{3.13}$$

It is also called the **First Fundamental Form.**

We use this notation.

$$G^{-1} = \left(g^{jk}\right), g = \det G$$

Also the associated surface area on Γ is given locally by

$$dA = \sqrt{g}dx^1 \cdots dx^{n-1}. \tag{3.14}$$

3.2.2 Riemannian Divergence

Let Γ be a C^2 Hypersurface in $\mathbb{R}^n$ (so that Riemannian metric G is C^1).
For any tangent vector field $\boldsymbol{\xi}$ along Γ, we can write

$$\boldsymbol{\xi} = \sum_{j=1}^{n-1} \xi^j \frac{\partial \boldsymbol{f}}{\partial x^j}.$$

and for this its **Riemannian Divergence** is given by

$$\operatorname{div}_\Gamma \boldsymbol{\xi} = \frac{1}{\sqrt{g}} \sum_{j=1}^{n-1} \frac{\partial(\xi^j \sqrt{g})}{\partial x^j} \tag{3.15}$$

For given any $(n-1)$-dimensional domain $\Lambda \subset \Gamma$, with C^1 boundary $\partial\Lambda$, which is $(n-2)$ - dimension and unit normal exterior vector field ν along $\partial\Lambda$, **Riemannian divergence theorem** is

$$\int_\Lambda \operatorname{div}_\Gamma \zeta dV_{n-1} = \int_{\partial\Lambda} \xi \cdot \nu dV_{n-2} \tag{3.16}$$

3.2.3 Second Fundamental Form

Second Fundamental Form of Γ in $\mathbb{R}^n$ is given locally by

$$\mathcal{B} = (b_{jk}), \quad b_{jk} = \frac{\partial^2 \boldsymbol{f}}{\partial x^j \partial x^k} \cdot \mathbf{n}, \quad j,k = 1,\ldots,n-1 \tag{3.17}$$

$$b_{jk} = \frac{\partial^2 \boldsymbol{f}}{\partial x^j \partial x^k} \cdot \mathbf{n} = \frac{\partial}{\partial x^j}\left(\mathbf{n} \cdot \frac{\partial \boldsymbol{f}}{\partial x^k}\right) - \frac{\partial \mathbf{n}}{\partial x^j} \cdot \frac{\partial \boldsymbol{f}}{\partial x^k} = -\frac{\partial \mathbf{n}}{\partial x^j} \cdot \frac{\partial \boldsymbol{f}}{\partial x^k}$$

as $\boldsymbol{n}$ is an exterior normal vector field and $\frac{\partial \boldsymbol{f}}{\partial x}$ is the basis of tangent space so that

$$\frac{\partial \boldsymbol{f}}{\partial x} \perp \boldsymbol{n}$$

and hence it can also be written as

$$b_{jk} = -\frac{\partial \mathbf{n}}{\partial x^j} \cdot \frac{\partial \boldsymbol{f}}{\partial x^k} \tag{3.18}$$

The **Mean Curvature** H of Γ in $\mathbb{R}^n$ is the trace of matrix $G^{-1}\mathcal{B}$, which is the trace of $\mathcal{B}$ relative to G,

$$\boldsymbol{H} = \operatorname{tr} G^{-1}\mathcal{B}$$

and **Gauss- Kronecker Curvature** is given by

$$\boldsymbol{K} = \det G^{-1}\mathcal{B}$$

Theorem 3.2.1. ***(First variation of volume and area.)*** *Let Ω be a bounded domain in $\mathbb{R}^n$, with C^2 boundary Γ. Given any C^2 time-dependent vector field $X : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ on $\mathbb{R}^n$, let $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ denote the 1-parameter flow determined by X*
Φ_t and X are related by

$$\frac{d}{dt}\Phi_t(x) = X(x,t), \quad \Phi_0 = \mathrm{id}.$$

and

$$\xi(x) = X(x,0), \quad \eta = \xi|\Gamma$$

Then

$$\textbf{\textit{(i)}} \quad \frac{d}{dt}V\left(\Phi_t(\Omega)\right)\Big|_{t=0} = \iint_\Omega \operatorname{div}\xi dv_n = \int_\Gamma \eta\cdot n dA \tag{3.19}$$

$$\textbf{\textit{(ii)}} \quad \frac{d}{dt}A\left(\Phi_t(\Gamma)\right)\Big|_{t=0} = \int_\Gamma \left\{\operatorname{div}_\Gamma \eta^T - H\eta\cdot n\right\} dA = -\int_\Gamma H\eta\cdot n dA \tag{3.20}$$

Where n is chosen to exterior normal vector field and η^T is tangential part of η.

Proof. : **(i)**
If J_ϕ, denotes the Jacobian matrix of Φ_t.Then we have

$$V\left(\Phi_t(\Omega)\right) = \iint_\Omega \det J_\phi(x) dv_n(x) \tag{3.21}$$

so we have

$$\frac{d}{dt}V\left(\Phi_t(\Omega)\right) = \iint_\Omega \left(\frac{d}{dt}\det J_\phi(x)\right) dv_n(x) \tag{3.22}$$

For any differentiable matrix function $t \mapsto \mathcal{A}(t)$, where $\mathcal{A}(t)$ is non singular we have

$$\frac{d}{dt}\det\mathcal{A} = \det\mathcal{A}\cdot\operatorname{tr}\left(\mathcal{A}^{-1}\frac{d\mathcal{A}}{dt}\right) \tag{3.23}$$

Therefore

$$\frac{d}{dt}V\left(\Phi_t(\Omega)\right) = \iint_\Omega \left(\det J_{\phi_t}\right)\cdot\operatorname{tr}\left(J_{\phi_t^{-1}}\frac{d}{dt}J_{\phi_t}\right) dv_n(x)$$

Now J_{ϕ_t} is Jacobian matrix so

$$(J_{\phi_t})_A^B = \frac{\partial \Phi_t^B}{\partial x^A}, \quad A, B = 1, \ldots, n \tag{3.24}$$

and at $t = 0$ we have $\phi_0(x) = x$ or ϕ_0 is identity so

$$\left.\frac{\partial \Phi_t^B}{\partial x^A}\right|_{t=0} = \delta_A^B$$

where δ is Kronecker delta function.

Furthermore

$$\frac{d}{dt}\left(J_\phi\right)_A^B = \frac{\partial}{\partial t}\frac{\partial \Phi_t B}{\partial x^A} = \frac{\partial}{\partial x^A}\frac{\partial \Phi_t^B}{\partial t}$$

and at $t = 0$, which implies

$$\left.\frac{d}{dt}\left(J_\phi\right)_A^B\right|_{t=0} = \frac{\partial \xi^B}{\partial x^A}$$

Since

$$\left.\frac{\partial \Phi_t^B}{\partial x^A}\right|_{t=0} = \delta_A^B$$

so J_{ϕ_t} at $t = 0$, is an Identity matrix and so its inverse, and $\det J_{\phi_0} = 1$

So

$$\left.\frac{d}{dt}V\left(\Phi_t(\Omega)\right)\right|_{t=0} = \iint_\Omega \operatorname{tr}\left(\frac{d}{dt}J_{\phi_t}\right) dv_n(x) = \iint_\Omega \operatorname{div}\xi dv_n \tag{3.25}$$

Now using divergence theorem we get

$$\left.\frac{d}{dt}V\left(\Phi_t(\Omega)\right)\right|_{t=0} = \iint_\Omega \operatorname{div}\xi dv_n = \int_\Gamma \eta \cdot \mathbf{n} dA \tag{3.26}$$

□

Proof. **: (ii)**
Assume the surface Γ is given locally by $\mathbf{x} = \mathbf{x}(u)$, and take

$$\mathbf{y}(u, t) = \Phi_t(\mathbf{x}(u))$$

Denote $\phi = \eta \cdot n$ and the Riemannian metric on $\Phi_t(\Gamma)$ for each fixed t by

$$h_{jk} = \frac{\partial \mathbf{y}}{\partial u^j} \cdot \frac{\partial \mathbf{y}}{\partial u^k}, \quad j,k = 1,\ldots,n-1 \tag{3.27}$$

On Γ with metric $\mathbf{G}$ and $g = det(\mathbf{G})$, the associated surface area is given locally by

$$dA = \sqrt{g} du^1 \cdots du^{n-1} \tag{3.28}$$

So we have

$$\frac{d}{dt} A\left(\Phi_t(\Gamma)\right) = \int_\Gamma \frac{\partial}{\partial t} \sqrt{\det\left(h_{jk}\right)} du^1 \cdots du^{n-1}$$

In calculation of derivative of $\sqrt{\det\left(h_{jk}\right)}$, set $\mathcal{H} = \left(h_{jk}\right), \mathcal{H}^{-1} = \left(h^{jk}\right)$

then

$$\frac{\partial}{\partial t}\sqrt{\det \mathcal{H}} = \frac{1}{2}\{\sqrt{\det \mathcal{H}}\}^{-1} \det \mathcal{H} \cdot \left(\operatorname{tr} \mathcal{H}^{-1} \frac{\partial \mathcal{H}}{\partial t}\right) = \frac{1}{2}\sqrt{\det \mathcal{H}} \sum_{j,k} h^{jk} \frac{\partial h_{kj}}{\partial t}$$

using

$$\frac{d}{dt} \det \mathcal{H} = \det \mathcal{H} \cdot \operatorname{tr}\left(\mathcal{H}^{-1} \frac{d\mathcal{H}}{dt}\right).$$

Now putting $h_{kj} = \frac{\partial \mathbf{y}}{\partial u^k} \cdot \frac{\partial \mathbf{y}}{\partial u^j}$ we get

$$\frac{\partial}{\partial t}\sqrt{\det \mathcal{H}} = \sqrt{\det \mathcal{H}} \sum_{j,k} h^{jk} \frac{\partial \mathbf{y}}{\partial u^k} \cdot \frac{\partial}{\partial u^j} \frac{\partial \mathbf{y}}{\partial t}.$$

At t=0 , for $\eta(u) = (\partial \mathbf{y}/\partial t)(u,0)$, Along Γ, We can write

$$\eta = \sum_{\ell=1}^{n-1} \eta^\ell \frac{\partial \mathbf{x}}{\partial u^\ell} + \phi \mathbf{n}. \tag{3.29}$$

where $\phi = \eta \cdot n$, for t=0, $\mathbf{y}(u,0) = \mathbf{x}(u)$ so $\mathcal{H} = G$ hence we have

$$\left.\frac{\partial}{\partial t}\sqrt{\det \mathcal{H}}\right|_{t=0} = \sqrt{\det G} \sum_{j,k} g^{jk} \frac{\partial \mathbf{x}}{\partial u^k} \cdot \frac{\partial \eta}{\partial u^j}$$

and

$$\frac{\partial \boldsymbol{\eta}}{\partial u^j} = \sum_{\ell=1}^{n-1} \left\{ \frac{\partial \eta^\ell}{\partial u^j} \frac{\partial \mathbf{x}}{\partial u^\ell} + \eta^\ell \frac{\partial^2 \mathbf{x}}{\partial u^j \partial u^\ell} \right\} + \frac{\partial \boldsymbol{\phi}}{\partial u^j} \mathbf{n} + \boldsymbol{\phi} \frac{\partial \mathbf{n}}{\partial u^j}.$$

So since $\mathbf{n}$ is perpendicular to $\partial \mathbf{x}/\partial u^k$ for all $k = 1, \ldots, n-1$

$$\sum_{j,k} g^{jk} \frac{\partial \mathbf{x}}{\partial u^k} \cdot \frac{\partial \eta}{\partial u^j} = \sum_{j,k} g^{jk} \frac{\partial \mathbf{x}}{\partial u^k} \cdot \left(\sum_{\ell=1}^{n-1} \left\{ \frac{\partial \eta^\ell}{\partial u^j} \frac{\partial \mathbf{x}}{\partial u^\ell} + \eta^\ell \frac{\partial^2 \mathbf{x}}{\partial u^\ell \partial u^j} \right\} + \phi \frac{\partial \mathbf{n}}{\partial u^j} \right)$$

$$= \sum_{j,k,\ell} g^{jk} g_{k\ell} \frac{\partial \eta^\ell}{\partial u^j} + \sum_{j,k,\ell} g^{jk} \eta^\ell \frac{\partial \mathbf{x}}{\partial u^k} \cdot \frac{\partial^2 \mathbf{x}}{\partial u^\ell \partial u^j} + \sum_{j,k} \phi g^{jk} \frac{\partial \mathbf{x}}{\partial u^k} \cdot \frac{\partial \mathbf{n}}{\partial u^j}$$

$$= \sum_j \frac{\partial \eta^j}{\partial u^j} + \sum_j \eta^j \frac{1}{2} \mathbf{tr} \left(G^{-1} \frac{\partial G}{\partial u^j} \right) + \sum_{j,k} \phi g^{jk} \frac{\partial \mathbf{x}}{\partial u^k} \cdot \frac{\partial \mathbf{n}}{\partial u^j}$$

$$= \sum_j \frac{1}{\sqrt{g}} \frac{\partial \left(\eta^j \sqrt{g} \right)}{\partial u^j} + \sum_{j,k} \phi g^{jk} \frac{\partial \mathbf{x}}{\partial u^k} \cdot \frac{\partial \mathbf{n}}{\partial u^j}$$

$$= \operatorname{div}_{\boldsymbol{\Gamma}} \eta^T - \phi H.$$

So we have

$$\left. \frac{d}{dt} A \left(\Phi_t(\Gamma) \right) \right|_{t=0} = \int_\Gamma \left\{ \operatorname{div}_r \boldsymbol{\eta}^T - H \boldsymbol{\eta} \cdot \mathbf{n} \right\} dA.$$

Now since Γ is closed and has no boundary Therefore,

$$\int_\Gamma div_\Gamma \eta^T d\mathbf{A} = 0.$$

so

$$\frac{d}{dt} A\left(\Phi_t(\Gamma)\right)|_{t=0} = \int_\Gamma \left\{\mathrm{div}_r\, \eta^T - H\eta \cdot \mathbf{n}\right\} dA = -\int_\Gamma H\eta \cdot \mathbf{n} \Big\} dA. \tag{3.30}$$

□

Following theorem is stated from our reference [2]

Theorem 3.2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ whose boundary, Γ is C^2 and assume that mean curvature of Γ is constant then Ω will be an n-disk in $\mathbb{R}^n$.*

3.2.4 C^k Isoperimetric problem

Let Ω be a bounded domain in R^n, with C^k boundary, $k \geq 1$. We say that Ω is a solution to the C^k Isoperimetric problem if, for any domain D with C^k boundary and volume equal to that of Ω, we have

$$A(\partial D) \geq A(\partial\Omega)$$

3.2.5 C^k extremal of the Isoperimetric functional

We say that Ω is a C^k extremal of the Isoperimetric functional if, for any 1 -parameter family of C^k diffeomorphism $\Phi_t \colon \mathbb{R}^n \to \mathbb{R}^n$ satisfying
$V\left(\Phi_t(\Omega)\right) = V(\Omega)\ \forall\ t$, we have

$$\frac{d}{dt} A\left(\Phi_t(\partial\Omega)\right)\Big|_{t=0} = 0 \tag{3.31}$$

Theorem 3.2.3. *Assume that Ω is a solution to the C^2 Isoperimetric problem with the volume of Ω equal to that of the unit (n)-disk in $\mathbb{R}^n$. Then the mean curvature H of Γ satisfies*

$$-H \leq n - 1$$

on all of Γ.

Proof. Consider the Isoperimetric functional

$$J(D) = \frac{A(\partial D)}{V(D)^{1-1/n}} \tag{3.32}$$

where D varies over bounded domains in R^n having C^2 boundary.

Notations in this proof are used from the above theorem (3.2.1) which are following.
We have ϕ_t, 1-parameter family of diffeomorphism of $\mathbb{R}^n$, with corresponding time-dependent vector field

$$\boldsymbol{X} = \boldsymbol{X}(x, t)$$

and at time zero, $\boldsymbol{X}$ is

$$\boldsymbol{\xi}(\boldsymbol{x}) = \boldsymbol{X}(\boldsymbol{x}, \boldsymbol{0})$$

and restricting $\boldsymbol{\xi}$ on its boundary, we have

$$\eta = \boldsymbol{\xi}|\Gamma.$$

As Ω is a solution to the C^2 Isoperimetric problem, Ω will minimize the isoperimetric functional $J(D)$ so we have

$$\frac{d}{dt} J\left(\Phi_t(\Omega)\right)\Big|_{t=0} = 0$$

Now on differentiating $J(\Phi_t(\Omega))$ with respect to t, we get

$$\frac{d}{dt} J\left(\Phi_t(\Omega)\right)\Big|_{t=0} = -\frac{1}{V(\Omega)^{1-1/n}} \int_\Gamma H\eta \cdot \mathbf{n} dA + \left(\frac{1}{n} - 1\right) \frac{A(\Gamma)}{V(\Omega)^{2-1/n}} \int_\Gamma \eta \cdot \mathbf{n} \;\; dA$$

which implies

$$-\frac{\int_\Gamma H\eta \cdot \mathbf{n} dA}{\int_\Gamma \eta \cdot \mathbf{n} dA} = \frac{n-1}{n} \frac{A(\Gamma)}{V(\Omega)} \tag{3.33}$$

Now that Ω has its $V(\Omega) = \omega_n$, we should have $A(\Gamma) \leq c_{n-1}$ since Ω is the solution of Isoperimetric Problem
recall that

$$\omega_{\mathbf{n}} = \frac{\mathbf{c_{n-1}}}{n}$$

together implies

$$-\frac{\int_\Gamma H\eta \cdot \mathbf{n} d\boldsymbol{A}}{\int_\Gamma \eta \cdot \mathbf{n} d\boldsymbol{A}} \leq \boldsymbol{n} - 1 \tag{3.34}$$

let ϕ be a nonnegative C^∞ function for any $w_0 \in \Gamma$, compactly supported on a neighbourhood of w_0 in $\mathbb{R}^n$. To simplify the above expression, we choose a particular vector field $X(x,t)$

$$X(x,t) = \phi(x)\mathbf{n}_{\omega_0}$$

which is time-independent.
Now picking ϕ with sufficiently small support about w_o we get the left-hand side of expression (3.34) to be
$-H\,(w_0)$

and that finally gives

$$-H(w_0) \leq n-1$$

for any $\omega_0 \in \Gamma$ hence it is proved on all of Γ □

As a corollary of the above theorem, we get that mean curvature H is constant.

Corollary 3.2.1. *Assume that Ω is a solution to the C^2 Isoperimetric problem with the volume of Ω equal to that of the unit (n)-disk in $\mathbb{R}^n$. Then mean curvature of H is a constant which is:*

$$H = \frac{n-1}{n}\frac{A(\Gamma)}{V(\Omega)}$$

Proof. To prove that H is constant, we will be using the same argument which was used in the above proof to prove $-H \leq n-1$,
in the following expression (3.33)

$$-\frac{\int_\Gamma H\eta \cdot \mathbf{n} dA}{\int_\Gamma \eta \cdot \mathbf{n} dA} = \frac{n-1}{n}\frac{A(\Gamma)}{V(\Omega)}$$

So we repeat the same arguments.

Let ϕ be a nonnegative C^∞ function for any $w_0 \in \Gamma$, compactly supported on a neighbourhood of w_0 in $\mathbb{R}^n$. To simplify the above expression, we choose a particular vector field $X(x,t)$,

$$X(x,t) = -\phi(x)\mathbf{n}_{\omega_0}$$

which is time-independent.
Now picking ϕ with sufficiently small support about w_o we get left-hand side of expression (3.33) to be $H\,(w_0)$.
Hence expression (3.33) becomes

$$H(w_0) = \frac{n-1}{n}\frac{A(\Gamma)}{V(\Omega)}.$$

For any $\omega_0 \in \Gamma$ it is true hence it is proved on all points of Γ. □

Remark: Assume that Ω is a solution to the C^2 Isoperimetric problem with the volume of Ω equal to that of the unit (n)-disk in $\mathbb{R}^n$. Then using the above corollary (3.2.1) we get that mean curvature of Γ is constant and then from theorem (3.2.2) we get that Ω will be an (n)-disk in $\mathbb{R}^n$.

Theorem 3.2.4. *Assume Ω is a bounded domain in $\mathbb{R}^n$, with C^2 boundary Γ and* $\mathbf{n}$*, its exterior normal unit vector field along Γ. Assume the mean curvature H of Γ satisfies*

$$-H \leq n-1$$

along all of Γ. Then

$$A(\Gamma) \geq \mathfrak{c}_{\mathfrak{n}-1}$$

with equality if and only if Ω is a disk in $\mathbb{R}^n$

Theorem 3.2.5. *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary that is an extremal of the C^2 Isoperimetric functional. Then $\partial\Omega$ has constant mean curvature.*

Proof. We denote $\phi = \eta \cdot n$, as Ω is an extremal for the Isoperimetric functional, so for any vector field X for which

$$V\left(\Phi_t(\Omega)\right) = \text{ const.} \quad \forall \text{ t}$$

From the first part (3.19) of the above theorem (3.2.1)

$$\int_\Gamma \phi dA = 0$$

and from the second part (3.20)) of the above theorem (3.2.1

$$\int_\Gamma \phi H dA = 0$$

Thus we have

$$\textstyle\int_\Gamma \phi H dA = 0 \ \forall \ \phi \text{ such that } \int_\Gamma \phi dA = 0$$

from here it implies that H is a constant □

Remark: Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary that is an extremal of the C^2 Isoperimetric functional.
Then from the above theorem (3.2.5), we get that mean curvature H is constant and further from theorem (3.2.2) we get that Ω will be an n-disk in $\mathbb{R}^n$.

Chapter 4

Isoperimetric inequality in convex Subsets of $\mathbb{R}^n$

4.0.1 Convex Polytopes and Polyhedral sets

Convex polyhedrons and convex polytopes are a very interesting class of objects.

Definition 4.0.1. *The intersection of finitely many closed half-spaces is called* ***convex Polyhedral Set or Polyhedron*** *where half-spaces are defined as* $\{x : a^T x \leq b\}$*, where* a *is a nonzero vector in* $\mathbb{R}^n$ *and* b *is another vector in* $\mathbb{R}^n$.

- Polyhedral sets are closed as they are the intersection of closed half-spaces.
- Polyhedral sets are convex sets because the intersection of convex sets is also convex. We just need to show that half-spaces are convex.

  Proof. For $a^T x_1 \leq b, a^T x_2 \leq b$

  we have

  $$a^T(\alpha x_1 + (1-\alpha)x_2) = \alpha a^T x_1 + (1-\alpha)a^T x_2 \leq b$$

  for x_1 and x_2 in half-space.

  So half-spaces are convex hence Polyhedron is convex.

  □

- Convex polyhedrons may not be bounded; just one half-space can be a convex polyhedron which is of course unbounded.

Definition 4.0.2. *The convex hull of finitely many points* $x_1, \ldots, x_k \in \mathbb{R}^n$ *is called* ***convex Polytope****. These are bounded in-fact every bounded convex polyhedron is a convex polytope.*

Every convex polytope is a polyhedron, but the reverse is not valid since convex polytope needs to be bounded. If a convex polytope is bounded, then both are equivalent.

- Triangle is the convex hull of 3 distinct points in space, and so is a convex polytope as well as a convex polyhedron as it is the intersection of 3 closed half-spaces.
- The convex hull of 4 distinct points in space is a tetrahedron, convex hull of vertices of a cube is the cube itself.
- For a polytope P, its **vertex** is defined as point $x \in P$ for which $P \backslash \{x\}$ is still convex and P is a convex combination of its vertices which is the next theorem.

Theorem 4.0.1. *Let P be a polytope in $\mathbb{R}^n$, and let $x_1, \ldots, x_k \in \mathbb{R}^n$ be distinct points.*

(a) *If $P = \text{conv}\{x_1, \ldots, x_k\}$, then x_1 is a vertex of P, if and only if $x_1 \notin \text{conv}\{x_2, \ldots, x_k\}$.*

(b) *P is the convex hull of its vertices.*

Proof. **(a)**

If x_1 is a vertex of P then from the definition of vertex, $P \backslash \{x_1\}$ will be convex and $x_1 \notin P \backslash \{x_1\}$. Hence we have $\text{conv}\{x_2, \ldots, x_k\} \subset P \backslash \{x_1\}$.

So $x_1 \notin \text{conv}\{x_2, \ldots, x_k\}$.

For the other direction, on assuming that $x_1 \notin \text{conv}\{x_2, \ldots, x_k\}$, and provided that x_1 is not a vertex of P, there exists distinct points $a, b \in P \backslash \{x_1\}$ and $\lambda \in (0,1)$ such that

$$x_1 = (1-\lambda)a + \lambda b$$

As $P = \text{conv}\{x_1, \ldots, x_k\}$, a and b can be written as a convex linear combination of $x_1, \ldots, x_k$ so there should exist $k \in \mathbb{N}, \mu_1, \ldots, \mu_k \in [0,1]$ and $\tau_1, \ldots, \tau_k \in [0,1]$ with

$$\mu_1 + \ldots + \mu_k = 1 \text{ and } \tau_1 + \ldots + \tau_k = 1$$

such that $\mu_1, \tau_1 \neq 1$ and

$$a = \sum_{i=1}^{k} \mu_i x_i, \quad b = \sum_{i=1}^{k} \tau_i x_i \tag{4.1}$$

as $x_1 = (1-\lambda)a + \lambda b$. On putting values of a and b we get

$$x_1 = \sum_{i=1}^{k} \left((1-\lambda)\mu_i + \lambda\tau_i\right) x_i \tag{4.2}$$

that finally gives

$$x_1 = \sum_{i=2}^{k} \frac{(1-\lambda)\mu_i + \lambda\tau_i}{1-(1-\lambda)\mu_1 - \lambda\tau_1} x_i \tag{4.3}$$

where $(1-\lambda)\mu_1 + \lambda\tau_1 \neq 1$

so x_1 can be written as a convex combination of $x_2, \ldots, x_k$ in the last equation, which is a contradiction. □

Proof. **(b)**
For $P = \operatorname{conv}\{x_1, \ldots, x_k\}$ we can remove those points one by one, which are not vertices. This will not change the convex hull and will be equal to P as removed points can be written as a convex combination of other remaining vertex points. If x is a vertex of P but $x \notin \{x_1, \ldots, x_k\}$ then P would be,

$$P = \operatorname{conv}\{x, x_1, \ldots, x_k\}.$$

which implies that x can not be written as a convex combination of $x_1, \ldots x_k$ i.e. $x \notin \operatorname{conv}\{x_1, \ldots, x_k\} = P$ so $x \notin P$ that gives contradiction.

Hence P is the convex hull of its vertices. □

Definition 4.0.3. ***The support function*** $h_A \colon \mathbb{R}^n \to (-\infty, \infty)$ *for a nonempty and convex* $A \subset \mathbb{R}^n$ *is defined as*

$$h_A(u) := \sup_{x \in A} \langle x, u \rangle, \quad u \in \mathbb{R}^n.$$

4.0.2 Convex function

Definition 4.0.4. *For a function* $f \colon \mathbb{R}^n \to (-\infty, \infty]$, *we define* $\operatorname{epi} f$ *as follows*

$$\operatorname{epi} f \colon = \{(x, \alpha) : x \in \mathbb{R}^n, \alpha \in \mathbb{R}, f(x) \leq \alpha\} \subset \mathbb{R}^n \times \mathbb{R} \tag{4.4}$$

Definition 4.0.5. *A function* $f \colon \mathbb{R}^n \to (-\infty, \infty]$ *is called a convex function, if*

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$$

for all $x, y \in \mathbb{R}^n, \alpha \in [0,1]$
f *is called a* ***concave function****, if* $-f$ *is convex.*

If $A \subset \mathbb{R}^n$ is a subset, a function $f \colon A \to (-\infty, \infty)$ is called convex, if the extended function $\tilde{f} \colon \mathbb{R}^n \to (-\infty, \infty]$, given by

$$\tilde{f} := \begin{cases} f & \text{on } A \\ \infty & \text{on } \mathbb{R}^n \setminus A \end{cases}$$

is convex, where A is a convex set. Without loss of generality, we can assume that convex functions are always defined on all of $\mathbb{R}^n$ with this construction.

We define an effective domain of the function where the function is finite.

$$\operatorname{dom} f \colon = \{x \in \mathbb{R}^n : f(x) < \infty\}$$

Lemma 4.0.2. *f is a convex function if and only if* epi f *is a convex subset of* $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$

Proof. First, let epi f is convex,
whenever $(x_1, \beta_1), (x_2, \beta_2) \in \text{epi } f$, i.e. $f(x_1) \leq \beta_1, f(x_2) \leq \beta_2$, for all $\alpha \in [0, 1]$.
We have $\alpha(x_1, \beta_1) + (1-\alpha)(x_2, \beta_2) \in \text{epi } f$ (as epi f is convex subset)

$$\alpha(x_1, \beta_1) + (1-\alpha)(x_2, \beta_2) = (\alpha x_1 + (1-\alpha)x_2, \alpha\beta_1 + (1-\alpha)\beta_2)$$

and

$$(\alpha x_1 + (1-\alpha)x_2, \alpha\beta_1 + (1-\alpha)\beta_2) \in \text{epi } f$$

So that we have

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha\beta_1 + (1-\alpha)\beta_2$$

for all $\beta_1 \geq f(x_1), \beta_2 \geq f(x_2)$ and all $x_1, x_2 \in \mathbb{R}^n, \alpha \in [0, 1]$.

As it is satisfied for all $\beta_1 \geq f(x_1), \beta_2 \geq f(x_2)$, it will also be satisfied for

$$\beta_1 = f(x_1), \beta_2 = f(x_2).$$

For the other side
let's suppose that f is convex, and let

$$(x_1, \beta_1), \ldots, (x_n, \beta_n) \in \text{epi } f$$

for any $\lambda_1, \ldots, \lambda_n \in [0, 1]$ with $\sum \lambda_i = 1$, the point

$$(x, \beta) = \lambda_i \sum (x_i, \beta_i) = \left(\sum \lambda_i x_i, \sum \lambda_i \beta_i\right)$$

has

$$\beta = \sum \lambda_i \beta_i \geq \sum \lambda_i f(x_i) \geq f\left(\sum \lambda_i x_i\right) = f(x)$$

Hence $(x, \beta) \in \text{epi } f$, and epi f is convex.
Hence proved. □

We can generalize the above theorem as following:
f is convex, if and only if

$$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) \leq \alpha_1 f(x_1) + \cdots + \alpha_k f(x_k) \quad (4.5)$$

for all $k \in \mathbb{N}, x_i \in \mathbb{R}^n$, and $\alpha_i \in [0, 1]$ with $\sum \alpha_i = 1$

Definition 4.0.6. *A convex function* $f \colon \mathbb{R}^n \to (-\infty, \infty]$ *is closed if* epi f *is closed. For a convex function* $f \colon \mathbb{R}^n \to (-\infty, \infty]$*, closure of set epigraph of f is denoted by cl epi f. cl epi f is the epigraph of a closed convex function.*

4.0.3 Convex body

Definition 4.0.7 (Convex body). *Convex body in $\mathbb{R}^n$ is a nonempty compact convex set with nonempty interior.*

- Convex polytope in $\mathbb{R}^n$ is a convex body, but polyhedron is not as it need not be compact since it may be unbounded.
- Triangle in the plane is a convex body as it has nonempty interior; however, it is not a convex body if it is considered as a topological subset in space $\mathbb{R}^3$ as then it will have an empty interior.

We denote the space of convex bodies by $\mathcal{K}^n$. For $\mathcal{K}^n$ we are taking convex bodies which have empty interior, i.e. lower-dimensional bodies are also included in $\mathcal{K}^n$. Sum of two convex bodies is a convex body.

$$K, L \in \mathcal{K}^n \Longrightarrow K + L \in \mathcal{K}^n$$

So set $\mathcal{K}^n$ is closed under addition.

Proof. As K and L are convex, for $e, f \in K + L$, let $a, b \in K$ and $c, d \in L$ s.t. $e = a + c$ and $f = b + d$

$$te + (1-t)f = t(a+c) + (1-t)(b+d) = (ta + (1-t)b) + (tc + (1-t)d) \in K + L$$

for all $t \in [0, 1]$.
Hence K+L is convex and as K, L are nonempty and compact K+L will also be nonempty and compact. □

Also, we have

$$K \in \mathcal{K}^n, \alpha \geq 0 \Longrightarrow \alpha K \in \mathcal{K}^n$$

So $\mathcal{K}^n$ is closed under scalar multiplication operation.

Proof. Let K be a convex set, $\alpha \geq 0$ a constant, and $\alpha K := \{\alpha s \mid s \in K\}$.
We want to show that αK is convex, i.e. for any $x, y \in \alpha K$ and $\lambda \in [0, 1]$ that the point $\lambda x + (1-\lambda)y$ lies in αK.

Take any $x, y \in \alpha K$ and $\lambda \in [0, 1]$. By definition of αK, there exist points $a, b \in K$ such that $x = \alpha a$ and $y = \alpha b$.
Since K is convex, we know that $\lambda a + (1-\lambda)b$ lies in K, i.e.

$$\lambda a + (1-\lambda)b = s$$

for some $s \in K$. Multiplying the equation by α, we have

$$\lambda \alpha a + (1-\lambda)\alpha b = cs,$$

which is the same as $\lambda x + (1-\lambda)y = \alpha s$ for some $s \in K$.
Hence, by definition of $\alpha K, \lambda x + (1-\lambda)y$ lies in αK and K is nonempty so αK will also be nonempty and also αK will be compact as K is compact. □

Also, since the reflection $-K$ of a convex body K is again a convex body, $\alpha K \in \mathcal{K}^n$, for all $\alpha \in \mathbb{R}$, and hence it satisfies properties of being a cone so, $\mathcal{K}^n$ is a **convex cone**.

Definition 4.0.8. ***Distance between convex bodies:***
For $K, L \in \mathcal{K}^n$, we define the distance between K and L as

$$d(K, L) := \inf\{\varepsilon \geq 0 : K \subset L + B(\varepsilon), L \subset K + B(\varepsilon)\} \tag{4.6}$$

We will use the results of the following theorem from [4] without proving them here.

Theorem 4.0.3. *For $K \in \mathcal{K}^n$ and $\varepsilon > 0$*

1. $\exists P \in \mathcal{P}^n \mid P \subset K$ *and* $d(K, P) \leq \varepsilon$
   and
   $\exists Q \in \mathcal{P}^n \mid Q \supset K$ *and* $d(K, Q) \leq \varepsilon$
2. $\exists P \in \mathcal{P}^n \mid P \subset K \subset (1 + \varepsilon)P$ *if* $0 \in$ *relint* K

4.0.4 Volume and surface area of a convex body

We can define volume and surface area for a convex body in an elementary sense as it is convex. By the way, its volume can also be defined as Lebesgue measure $\lambda_n(K)$ of K for $K \in \mathcal{K}^n$.
In an elementary sense, first volume and surface area for polytopes are defined recursively on dimension n, and then by approximation, they are defined for arbitrary convex bodies.

Remark- For a convex body K, its support set $K(u), u \in S^{n-1}$ lies in a hyperplane parallel to $u^\perp$ and on translating $K(u)$ to $u^\perp$ we get its orthogonal projection $K(u)|u^\perp$, and we think $K(u)|u^\perp$ as a $(n-1)$-dimensional convex body in $\mathbb{R}^{n-1}$ on identifying $u^\perp$ with $\mathbb{R}^{n-1}$.
For determining the volume of a convex body in $\mathbb{R}^n$ recursively, we assume that we already know the volume in $(n-1)$-dimension and we denote $V^{(n-1)}\left(K(u)|u^\perp\right)$, to the $(n-1)$-dimensional volume of this projection as this projection is a convex body in $\mathbb{R}^{n-1}$
Now let us define volume recursively for polytope first.

Definition 4.0.9. *For a polytope $P \in \mathcal{P}^n$, if $n = 1$, we have $P = [a, b]$ where $a \leq b$, then we define volume $V^{(1)}(P) := b - a$ and surface area $A^{(1)}(P) := 2$.
For $n \geq 2$ and $\dim P \leq n - 2$, there are no facets in the polytope P, hence its volume $V(P) = 0$ and surface area $A(P) = 0$.
We define set A_u to be the set of all $u \in \mathbb{S}^{n-1}$ for which $P(u)$ is a facet of P. Then*

$$V^{(n)}(P) := \begin{cases} \dfrac{1}{n} \displaystyle\sum_{u \in A_u} h_P(u) V^{(n-1)}\left(P(u)|u^\perp\right) & \dim P \geq n-1 \\ 0 & if \dim P \leq n-2 \end{cases} \tag{4.7}$$

and

$$A^{(n)}(P) := \begin{cases} \sum_{u\in A_u} V^{(n-1)}\left(P(u)|u^{\perp}\right) & \textit{if } \dim P \geq n-1 \\ 0 & \textit{if } \dim P \leq n-2 \end{cases} \tag{4.8}$$

Recall that here $h_P(u)$ is the support function of P at point u and $P(u)$ is support set of P at u.

When $\dim P = n-1$, P is like a hyperplane of dimension $n-1$ and hence there are only two support sets for P which are its facets, $P = P(u_0)$ and $P = P(-u_0)$, where u_0 is a normal vector to P. In that case we have

$$V^{(n-1)}\left(P(u_0)|u_0^{\perp}\right) = V^{(n-1)}\left(P(-u_0)|u_0^{\perp}\right)$$

and from the definition of support function, we have $h_P(u_0) = -h_P(-u_0)$, so we find $V(P) = 0$, which matches to Lebesgue measure result also and

$$A(P) = 2V^{(n-1)}\left(P(u_0)|u_0^{\perp}\right). \tag{4.9}$$

Theorem 4.0.4. *For a polytope $P \in \mathcal{P}^n$ volume of P is equal to its Lebesgue measure i.e.*

$$V(K) = \lambda_n(K)$$

Proof. we will be proving this result using induction.
$n = 1$ case is trivial.

Both n-dim volume and n-dim Lebesgue measure are zero for $\dim P \leq n-1$. Assume that the result is true for $(n-1)$-dim volume then for $\dim P = n$ from the definition of volume we have

$$V(P) = \frac{1}{n}\sum_{i=1}^{k} h_P(u_i)\, V^{(n-1)}\left(P(u_i) \mid u_i^{\perp}\right)$$

$P(u_i) \mid u_i^{\perp}$ is an $(n-1)$ dimensional convex polytope so for this its volume will be equal to its lebesgue measure by inductive assumption for dimension $(n-1)$ hence we have

$$V^{n-1}(P(u_i) \mid u_i^{\perp}) = \lambda_{n-1}(P(u_i) \mid u_i^{\perp})$$

we can assume that first m support sets $h_P(u_1), \ldots, h_P(u_m) \geq 0$ and next $k-m$ support sets $h_P(u_{m+1}), \ldots, h_P(u_k) < 0$, without loss of generality and then take the pyramid-shaped polytopes defined by

$$P_i := \operatorname{conv}(P(u_i)\ \{0\}), i = 1, \ldots, k.$$

Then

$$V(P_i) = \frac{1}{n} h_P(u_i) V^{(n-1)}\left(P(u_i) \mid u_i^{\perp}\right), i = 1, \ldots, m,$$

and

$$V(P_i) = -\frac{1}{n} h_P(u_i) V^{(n-1)}\left(P(u_i) \mid u_i^{\perp}\right), i = m+1, \ldots, k$$

Therefore we have

$$V(P) = \sum_{i=1}^{m} V(P_i) - \sum_{i=m+1}^{k} V(P_i) = \sum_{i=1}^{m} \lambda_n(P_i) - \sum_{i=m+1}^{k} \lambda_n(P_i) = \lambda_n(P)$$

above expressions follow by the fact that $V^{(n-1)}\left(P(u_i) \mid u_i^{\perp}\right)$ is $(n-1)$ dimensional and is its support set, hence it is base of the pyramid and height of this pyramid is $h_P(u)$ so for this pyramid we have its volume as

$$\lambda_n(P_i) = \frac{1}{n} \times V^{(n-1)}\left(P(u_i) \mid u_i^{\perp}\right)$$

□

From the above theorem (4.0.4), we can consider the volume of polytope as a Lebesgue measure so all properties of Lebesgue measure will also be satisfied by $V(P)$ and hence we have the following properties for $V(P)$.

Proposition 1.
- *V and A are invariant with respect to rigid motions.*
- $V(\alpha P) = \alpha^n V(P), A(\alpha P) = \alpha^{n-1} A(P)$, *for* $\alpha \geq 0$
- $V(P) = 0$, *if and only if* $\dim P \leq n-1$
- *if* $P \subset Q$, *then* $V(P) \leq V(Q)$ *and* $A(P) \leq A(Q)$ *(Monotone Property)*

Now we will be defining volume and surface area for a convex body.

Definition 4.0.10. *For a convex body* $K \in \mathcal{K}^n$, *we define*

$$V_+(K) := \inf_{P \supset K} V(P), \quad V_-(K) := \sup_{P \subset K} V(P)$$

and

$$A_+(K) := \inf_{P \supset K} A(P), \quad A_-(K) := \sup_{P \subset K} A(P)$$

Theorem 4.0.5. *For* $K \in \mathcal{K}^n$, *We have*

$$V_+(K) = V_-(K) \tag{4.10}$$

$$A_+(K) = A_-(K) \tag{4.11}$$

Proof. We will be using Monotone property of volume and surface area from the above property (1) to prove it

$$V_-(K) \leq V_+(K)$$
$$A_-(K) \leq A_+(K)$$

We can assume that $0 \in \text{relint}\, K$ as $V_-(K), V_+(K), F_-(K)$ and $F_+(K)$ are motion invariant so after a suitable transformation, $0 \in \text{relint}\, K$ and from 2nd part of theorem (4.0.3), for $\varepsilon > 0$, we can find a polytope P such that

$$P \subset K \subset (1+\varepsilon)P$$

and then from the above proposition (1) and using definition (4.0.10) of $V_+(K)$ and $V_-(K)$ we have

$$V(P) \leq V_-(K) \leq V_+(K) \leq V((1+\varepsilon)P) = (1+\varepsilon)^n V(P)$$

and

$$A(P) \leq A_-(K) \leq A_+(K) \leq A((1+\varepsilon)P) = (1+\varepsilon)^{n-1} A(P)$$

now since ε is arbitrary, for $\varepsilon \to 0$ we have
$V_+(K) = V_-(K)$ and $A_+(K) = A_-(K)$ □

Definition 4.0.11. *For $K \in \mathcal{K}^n$ its volume $V(K)$ and surface area $A(K)$ are defined as*

$$V(K) =: V_+(K) = V_-(K)$$

and

$$A(K) := A_+(K) = A_-(K)$$

Theorem 4.0.6. *For a convex body $K \in \mathcal{K}^n$, the volume of K is equal to its Lebesgue measure i.e.*

$$V(K) = \lambda_n(K)$$

Proof. Using the definition (4.0.11) of volume of convex body, we get

$$\begin{aligned} V(K) &=: V_+(K) \\ &= \inf_{P \supset K} V(P) \\ &= \inf_{P \supset K} \lambda_n(P) \\ &= \lambda_n(K) \end{aligned}$$

Hence proved. □

Now using the above theorem (4.0.6) all properties of $\lambda_n(K)$ will also get satisfied by $V(K)$ so we have the following properties.

Proposition 2. *For $K \in \mathcal{K}^n$, $V(K)$ and $A(K)$ has following properties*

1. *With respect to rigid motions* V *and* A *are invariant .*
2. *for* $\alpha \geq 0$, $V(\alpha K) = \alpha^n V(K), A(\alpha K) = \alpha^{n-1} A(K)$,
3. $V(K) = 0$, *if and only if* $\dim K \leq n-1$
4. *if* $K \subset L$, *then* $A(K) \leq A(L)$ *and* $V(K) \leq V(L)$

Definition 4.0.12. *We can also define* $A(K)$ *for* $K \in \mathcal{K}^n$ *in an alternative way in terms of derivative of* $V(K)$ *as following:*

$$A(K) = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}(V(K + B(\varepsilon)) - V(K)) \tag{4.12}$$

Convex bodies have a fascinating property that all convex bodies act like Euclidean balls a little bit. In some examples of the convex body, it resembles the Euclidean ball more while in some others it resembles very less.

Cube $[-1,1]^n$ is the simplest example of a convex body. For it, the radius of the largest ball inside cube(i.e. inscribed) and smallest ball covering it(circumscribed) is 1 and $\sqrt{n}$ respectively.
Smallest ball covering the convex body does not fit well to the cube as the dimension grows, i.e. the cube is less and less like a ball as its dimension grows, because the distance of the cube's corner from the origin increases as dimension increases.

The ratio of the radius of inscribed and circumscribed balls is n for regular solid simplex which is the convex hull of $n+1$ equally spaced points, which is even worse than for the cube.

4.0.5 Distribution of volume for B^n

For unit ball

$$B_2^n = \left\{ x \in \mathbb{R}^n : \sum_1^n x_i^2 \leq 1 \right\}$$

from (3.4) its n-dimensional volume ω_n has standard result which is following

$$\omega_{\mathbf{n}} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

which is very small for large n. From Stirling's approximation for $n!$ which is

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

We have

$$\Gamma\left(\frac{n}{2}+1\right) = \left(\frac{n}{2}\right)! \sim \sqrt{2\pi} e^{-n/2} \left(\frac{n}{2}\right)^{(n+1)/2}$$

So approximately ω_n is
$$\left(\sqrt{\frac{2\pi e}{n}}\right)^n$$
Now as the volume of the Euclidean ball of radius r is $\omega_n r^n$, in other terms, we can say that roughly the Euclidean ball of volume 1 has radius

$$r = \sqrt{\frac{n}{2\pi e}} = \quad \omega_n^{\frac{-1}{n}}$$

Which is sufficiently large for a large dimension.
So as the dimension grows, radius required for the Euclidean ball to occupy the volume 1 is sufficiently large and is proportional to $\sqrt{n}$
Now let us see where its all volume concentrates.
We now consider the $(n-1)$-dimensional slice passing through the center of the ball of volume 1 and estimate its $(n-1)$-dimensional volume.
As slice is $(n-1)$-dimensional disk with its radius $\omega_n^{-1/n}$ we have its volume

$$V_{n-1} = \omega_{n-1} r^{n-1} = \omega_{n-1}\left(\frac{1}{\omega_n}\right)^{(n-1)/n}$$

and for large n which turns out to be $\sqrt{e}$ using Stirling's approximation in the same way as above.
Now we try to find the volume of its parallel slice at a distance d from the center which is also a $(n-1)$-dimensional slice of different radius $\sqrt{r^2-d^2}$, so this smaller slices has its volume as following:
For radius r, volume is $\sqrt{e}$, so for radius 1, volume is $\frac{\sqrt{e}}{r^{n-1}}$ and hence for radius $\sqrt{r^2-d^2}$, the volume will be

$$\sqrt{e}\left(\frac{\sqrt{r^2-d^2}}{r}\right)^{n-1} = \sqrt{e}\left(1-\frac{d^2}{r^2}\right)^{(n-1)/2}$$

Now putting approximate value of r which is $\sqrt{n/(2\pi e)}$, we get volume

$$= \sqrt{e}\left(1-\frac{2\pi e d^2}{n}\right)^{(n-1)/2} \approx \sqrt{e}\ e^{\left(-\pi e d^2\right)}$$

So volume distribution approximately follows Gaussian distribution in a single direction, with the variance $\frac{1}{2\pi e}$ and its variance is independent of n, so here we get a very important result that given the fact that volume 1 ball's radius increases as $r = \sqrt{\frac{n}{(2\pi e)}}$, nearly all the volume remains inside a slab of constant width.
For example in the slab

$$\left\{x \in \mathbb{R}^n : -\frac{1}{2} \le x_1 \le \frac{1}{2}\right\}$$

its 96% volume lies, and as n grows, its size grows and about its 96% volume lies more and more about central slice as in the following figure it is shown

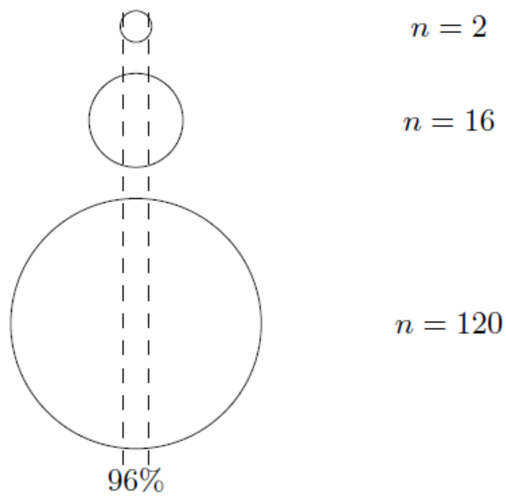

Figure 4.1: 96% volume strip in different dimensions by Ball(2002) [3]

So volume seems to concentrate on the centre of the ball as the volume is concentrated on its $(n-1)$-dimensional slices passing through origin which are all subspace of the ball and all these subspace meet on the origin. But for large n as the slice goes very thin and considering all such slices in each direction of the ball, the volume should lie near the surface of the ball.

So finally we reach to a conclusion that for large dimension objects its measure(here volume) tends to concentrate in places where our low-dimensional intuition considers small.
Hence our intuition in the lower dimension of measure distribution is wrong for higher dimension.

4.0.6 Mixed volume for convex body

First, we define mixed volume for polytopes recursively:
We denote $\mathcal{N}(P_1, P_2, \ldots P_k)$ to be the set of all facets normal of the convex polytope $P_1 + P_2 + \ldots + P_k$

Definition 4.0.13. *We define the mixed volume $V^{(n)}(P_1, \ldots, P_n)$ of $P_1, \ldots, P_n$ recursively for polytopes $P_1, \ldots, P_n \in \mathcal{P}^n$,*

$$\textit{for } n = 1, P_1 = [a, b] \textit{ with } a \leq b,$$

$$V^{(1)}\left(P_1\right) := V\left(P_1\right) = h_{P_1}(1) + h_{P_1}(-1) = b - a, \tag{4.13}$$

and for $n \geq 2$

$$V^{(n)}(P_1, \ldots, P_n) := \frac{1}{n} \sum_{u \in \mathcal{N}(P_1, \ldots, P_{n-1})} h_{P_n}(u) V^{(n-1)}\left(P_1(u), \ldots, P_{n-1}(u)|u^{\perp}\right). \tag{4.14}$$

Theorem 4.0.7. *If* $\left(P_1^{(k)}\right)_{k\in\mathbb{N}}, \ldots, \left(P_n^{(k)}\right)_{k\in\mathbb{N}}$ *are arbitrary approximating*

sequences converging to convex bodies $K_1, \ldots, , K_n \in \mathcal{K}^n$ *respectively. i.e. each sequence of polytopes* $P_j^{(k)}$ *converges to a convex body,* $K_j, j = 1, \ldots, , n,$ *as* $k \to \infty$*, then limit*

$$V\left(K_1, \ldots, , K_n\right) = \lim_{k\to\infty} V\left(P_1^{(k)}, \ldots, , P_n^{(k)}\right) \tag{4.15}$$

exists and does not depend on the choice of approximating sequences $\left(P_j^{(k)}\right)_{k\in\mathbb{N}}$.
Mixed volume of $K_1, \ldots, , K_n$ *is defined to be* $V\left(K_1, \ldots, , K_n\right)$
and the mapping $V\colon \left(\mathcal{K}^n\right)^n \to \mathbb{R}$ *defined by* $(K_1, \ldots, K_n) \mapsto V\left(K_1, \ldots, K_n\right)$ *is called mixed volume.*

Precisely

$$V\left(K_1, \ldots, K_n\right) = \frac{1}{n!} \sum_{l=1}^{n} (-1)^{n+l} \sum_{1\leq r_1 < \cdots < r_l \leq n} V\left(K_{r_1} + \cdots + K_{r_l}\right) \tag{4.16}$$

and, for $m \in \mathbb{N}, K_1, \ldots, K_m \in \mathcal{K}^n$ *and* $\alpha_1, \ldots, \alpha_m \geq 0$

$$V\left(\alpha_1 K_1 + \cdots + \alpha_m K_m\right) = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} \alpha_{i_1} \cdots \alpha_{i_n} V\left(K_{i_1}, \ldots, K_{i_n}\right) \tag{4.17}$$

Above theorem is stated here from [1] without proof.

Proposition 3. *for all* $J \in \mathcal{K}^n$
we have

$$V(J, \ldots, J) = V(J)$$

and

$$nV(J, \ldots, J, B(1)) = A(J)$$

Proof. **(i)**
Assume that given convex body J is just a polytope, so first we prove the required result for n copies of any polytope $P \in \mathcal{P}$ and then approximating any general convex body $J \in \mathcal{K}^n$ by a sequence of polytopes i.e. $P_k \to K$, we will

approximate n copies of J by n copies of the same sequence of polytopes P_k to prove the result for a general convex body J.
So we start with proving the result for P and for that we use induction on n.
For $n = 1$, $V(P) = V(P)$
For $n \geq 2$, assume that the induction step is true so that we have

$$V(\underbrace{P, P, \ldots, P}_{n-1}) = V(P)$$

then for the dimension n, mixed volume for n copies of polytope P would be

$$V(\underbrace{P, P, \ldots, P}_{n}) = V(\{\underbrace{P, P, \ldots, P}_{n-1}, P\}) = V(P, P) = V(P)$$

(from the induction step).
Hence for any polytope $P \in \mathcal{P}^n$ result is proved. Now for any general convex body $J \in \mathcal{K}^n$, taking sequence $P_k \to J$, from above theorem (4.0.7), we have

$$V(\underbrace{J, J, \ldots, J}_{n}) = \lim_{k\to\infty} V(\underbrace{P_k, P_k, \ldots, P_k}_{n}) = \lim_{k\to\infty} V(P_k) = V(\lim_{k\to\infty} P_n) = V(P)$$

from continuity of volume functional.
Hence required result is proved for any general convex body $K \in \mathcal{K}$.

(ii)
Again first we will prove the required result for polytope $P \in \mathcal{P}^n$ then for any general convex body $J \in \mathcal{K}^n$ by approximation of polytopes from inside and from outside.
So let $(P_k)_{k\in\mathrm{N}}$ be a sequence of polytopes with $P_k \to B(1)$. Then,

$$nV\,(P, \ldots, P, P_k) \mapsto nV(P, \ldots, P, B(1))$$

and

$$nV\,(P, \ldots, P, P_k) = \sum_{u\in N(P)} h_{P_k}(u) v(P(u))$$

$$\to \sum_{u\in N(P)} h_{B(1)}(u) v(P(u)) = \sum_{u\in N(P)} v(P(u)) = A(P)$$

Now for arbitrary bodies J, we approximate J, from inside and outside by polytopes and use monotonicity of the surface area measure to get the required result. □

4.0.7 The Brunn-Minkowski Theorem

Brunn-Minkowski Theorem basically says that the function

$$s \mapsto \sqrt[n]{V(sK + (1-s)L)}, \quad s \in [0, 1]$$

is concave for $K, L \in \mathcal{K}^n$. We will get inequalities for mixed volumes, and as a consequence also get the famous Isoperimetric inequality from the concavity of the above function.

We will be using the result of this following lemma further for proving Brunn-Minkowski Theorem

Lemma 4.0.8 (**Power mean inequality**
). *For $a \in (0,1)$ and $r, s, t > 0$*

$$\left(\frac{a}{r} + \frac{1-a}{s}\right)\left[ar^t + (1-a)s^t\right]^{\frac{1}{t}} \geq 1$$

with equality, if and only if $r = s$

Proof. For proving this result, we will be using the fact that the function $x \to lnx$ is strictly concave,
so taking ln of the expression as argument of expression is always positive. we get

$$\ln\left\{\left(\frac{a}{r} + \frac{1-a}{s}\right)\left[ar^t + (1-a)s^t\right]^{\frac{1}{t}}\right\}$$

$$= \frac{1}{t}\ln\left(ar^t + (1-a)s^t\right) + \ln\left(\frac{a}{r} + \frac{1-a}{s}\right)$$

Now using the strict concavity of function $x \to lnx$ we get

$$\frac{1}{t}\ln\left(ar^t + (1-a)s^t\right) + \ln\left(\frac{a}{r} + \frac{1-a}{s}\right) \geq \frac{1}{t}\left(a\ln r^t + (1-a)\ln s^t\right) + a\ln\frac{1}{r} + (1-a)\ln\frac{1}{s}$$

$$= (a\ln r + (1-a)\ln s) + a\ln\frac{1}{r} + (1-a)\ln\frac{1}{s}$$

$$= 0$$

Now, as logarithm is the strict monotone function, we get the required result.

$$\left(\frac{a}{r} + \frac{1-a}{s}\right)\left[ar^t + (1-a)s^t\right]^{\frac{1}{t}} \geq 1$$

clearly equality follows when $r = s$ □

Theorem 4.0.9. ***(Brunn-Minkowski Theorem)*** *For convex bodies $J, K \in \mathcal{K}^n$ and $a \in (0,1)$*

$$\sqrt[n]{V(aJ + (1-a)K)} \geq a\sqrt[n]{V(J)} + (1-a)\sqrt[n]{V(K)} \tag{4.18}$$

with equality, if and only if J and K lie in parallel hyperplanes or J and K are homothetic(see (**??**)

Proof. Based on the dimension of the convex body, we consider four cases.
First Tackle trivial case separately

- J and K lie in parallel hyperplanes
- Both J and K are lower dimensional
- One of them is lower dimensional
- Main case: Both J and K are n dimensional

Based on the dimension of the convex body, we consider four cases.

Case 1 : When J and K lie in parallel hyperplanes. Then $aJ + (1-a)K$ also lies in a hyperplane, so that

$$V(J) = V(K) = 0$$

and also

$$V(aJ + (1-a)K) = 0$$

hence proved for this case.

Case 2: When $\dim J \leq n-1$ and $\dim K \leq n-1$, but J and K do not lie in parallel hyperplanes,
for all $a \in (0,1)$,

$$\dim(J+K) = n.$$

and also

$$\dim(aJ + (1-a)K) = n$$

$$V(J) = V(K) = 0 \text{ as } \dim J, K \leq n-1$$

Therefore for all $a \in (0,1)$

$$a\sqrt[n]{V(J)} + (1-a)\sqrt[n]{V(K)} = 0 \leq \sqrt[n]{V(aJ + (1-a)K)}$$

Case 3: When $\dim J \leq n-1$ and $\dim K = n$ (or vice versa)

Then, for $x \in J$, we get

$$ax + (1-a)K \subset aJ + (1-a)K$$

Since adding a point x is translation of $(1-a)K$ and won't affect volume.

$$V(ax + (1-a)K) = (1-a)^n V(K)$$

then

$$V(ax + (1-a)K) = V((1-a)K) \leq V(aJ + (1-a)K)$$

and equality occur, if and only if $J = \{x\}$

Case 3:
Now for the (n)-dimensional case this is the outline of the proof.

- Reduce to $V(J) = V(K) = 1$. Geometrically we are scaling down the objects to unit volume.
- Translate the objects so that their center of gravity is at origin.
- Establish induction base case for $\mathbb{R}$
- Take a hyperplane E_λ at distance λ cutting the convex body J with the cut $J_\lambda = J \cap E_\lambda$.
- Define volume of half-space as a invertible function of λ
- Apply induction step on hyperplane cuts J_β and K_γ for inequality
- Proving for equality case

Reduce to $V(J) = V(K) = 1$

We are allowed to take $V(J) = V(K) = 1$. as for general J, K, we can take

$$\widetilde{J} := \frac{1}{\sqrt[n]{V(J)}} J, \quad \widetilde{K} := \frac{1}{\sqrt[n]{V(K)}} K$$

and

$$\widetilde{a} := \frac{a \sqrt[n]{V(J)}}{a \sqrt[n]{V(J)} + (1-a) \sqrt[n]{V(K)}}$$

Then we have

$$\sqrt[n]{V(\widetilde{a}\widetilde{J} + (1-\widetilde{a})\widetilde{K})} \geq 1$$

Now as $V(\widetilde{J}) = V(\widetilde{K}) = 1$ so

$$\sqrt[n]{V(\widetilde{J})} + (1-a) \sqrt[n]{V(\widetilde{K})} = a + (1-a) = 1$$

hence $\widetilde{J}$ and $\widetilde{K}$ follow the Brunn-Minkowski Theorem, and also J and K are homothetic iff $\widetilde{J}$ and $\widetilde{K}$ are homothetic.

Now we just have to prove theorem for $V(J) = V(K) = 1$,

Shift the center of gravity to origin
We have

$$V(aJ + (1-a)K) \geq 1$$

with equality if and only if J, K are translates of each other.
We define center of gravity for an (n)-dimensional convex body S to be the point $\alpha \in \mathbb{R}^n$ such that

$$\langle c, u \rangle = \frac{1}{V(S)} \int_S \langle x, u \rangle dx$$

for all $u \in S^{n-1}$. By translating J and K we can have their centre of gravity at 0 as the volume is translation invariant, the equality case then just reduces to the claim that $J = K$

Induction base case

Now by induction on n, We prove Brunn-Minkowski Theorem
For $n = 1$, 1 - dimensional volume is linear, so Brunn-Minkowski inequality easily follows
Along with the above conclusion, we also get that equality corresponds to the fact that two convex bodies in $\mathbb{R}$ are homothetic.

Now for $n \geq 2$ assume the Brunn-Minkowski theorem is true in dimension $n-1$

Hyperplane cut on J

We choose a unit vector $u \in S^{n-1}$ and denote the hyperplane E_λ in the direction u with distance(signed) $\lambda \in \mathbb{R}$ from the origin defined by

$$E_\lambda := \{x : \langle\, x, u\rangle = \lambda\},$$

let's denote $J_\lambda = J \cap E_\lambda$ for cut on J by E_λ. Now we have

$$V(J \cap \{x : \langle x, u\rangle \leq \beta\}) = \int_{-h_{K(-u)}}^{\beta} v\left(J_\lambda\right) d\lambda$$

Function for volume after split by the hyperplane for J

The function

$$g\colon\ [-h_J(-u), h_J(u)] \to [0,1], \quad f(a) = V(K \cap \{x : \langle x, u\rangle \leq a\}$$

would be continuous and strictly increasing
where $-h_J(-u), h_J(u)$ are the support function of J at u and $-u$ and are extreme upper point and extreme lower point respectively.

Map $\lambda \mapsto v\left(J_\lambda\right)$ is continuous on $\left(-h_J(-u), h_J(u)\right)$, where v is area functional. g is differentiable function on $(-h_J(-u), h_J(u))$ and

$$g'(\beta) = v\left(J_\beta\right)$$

since g is invertible, inverse function of g, $\gamma\colon [0,1] \to \left[-h_J(-u), h_J(u)\right]$, is also a continuous function and strictly increasing function satisfies

$$\gamma(0) = -h_J(-u), \gamma(1) = h_J(u)$$

$$\gamma'(\tau) = \frac{1}{g'(\gamma(\tau))} = \frac{1}{v\left(J_{\gamma(\tau)}\right)}, \quad \tau \in (0,1)$$

Function for volume after split by the hyperplane for K

Similarly, for K, the function

$$f\colon\ [-h_K(-u), h_K(u)] \to [0,1], \quad g = V(K \cap \{x : \langle x, u\rangle \leq a\}$$

and in the same way we get its inverse function

$$\delta\colon [0,1] \to [-h_K(-u), h_K(u)]$$

with

$$\delta'(\tau) = \frac{1}{vK_{\delta(\tau)}}, \quad \tau \in (0,1)$$

Since

$$aJ_{\gamma(\tau)} + (1-a)K_{\delta(\tau)} \subset (aJ + (1-a)K) \cap E_{a\gamma(\tau)+(1-a)\delta(\tau)}$$

for $a, \tau \in [0,1]$, we obtain from the inductive assumption

Induction step on the cuts J_γ and K_δ

$$\begin{aligned}
&V(aJ + (1-a)K) \\
&= \int_{-\infty}^{\infty} v\left((aJ + (1-a)K) \cap E_\lambda\right) d\lambda \\
&= \int_0^1 v\left((aJ + (1-a)K) \cap E_{a\gamma(\tau)+(1-a)\delta(\tau)}\right) && \times \left(a\gamma'(\tau) + (1-a)(\delta)'(\tau)\right) d\tau \\
&\geq \int_0^1 v\left(aJ_{\gamma(\tau)} + (1-a)K_{\delta(\tau)}\right) && \times \left\{\frac{a}{vJ_{\gamma(\tau)}} + \frac{1-a}{vK_{\delta(\tau)}}\right\} d\tau \\
&\geq \int_0^1 \left\{a^{n-1} v\left(J_{\gamma(\tau)}\right)^{\frac{1}{n-1}} + (1-a) v\left(K_{\delta(\tau)}\right)^{\frac{1}{n-1}}\right\}^{n-1} && \times \left\{\frac{a}{vJ_{\gamma(\tau)}} + \frac{1-a}{vK_{\delta(\tau)}}\right\} d\tau
\end{aligned}$$

(Using Induction step)

Choosing $r := v\left(J_{\gamma(\tau)}\right), s := v\left(K_{\delta(\tau)}\right)$ and $t := \frac{1}{n-1}$,

Applying above lemma on power mean inequality on the above expression written inside integration, we get that expression is ≥ 1, and on the limit of 0 and 1, the whole integrand also gets ≥ 1, which gives the required result.

Equality proof:

Now For equality case, assume that equality occurs so that.

$$V(aJ + (1-a)K) = 1$$

For above expression equal to 1, in our last estimation we must have equality, which means that the integrand is equal to 1, for all τ. and this is the equality case(iff $r = s$) of the same lemma.
So we have

$$v\left(J_{\gamma(\tau)}\right) = v\left(K_{\delta(\tau)}\right), \quad \text{for all } \tau \in [0,1]$$

Hence $\gamma' = \delta'$, so $\gamma - \delta$ is a constant.
Because the centre of gravity of J is at the origin, we get

$$0 = \int_J \langle x, u\rangle dx = \int_{\gamma(0)}^{\gamma(1)} \lambda v\left(J_\lambda\right) d\lambda = \int_{\gamma(0)}^{\gamma(1)} \lambda\gamma'(\lambda) d\lambda = \int_0^1 \gamma(\tau) d\tau$$

where the change of variables $\lambda = \gamma(\tau)$ was used.
In an analogous way,

$$0 = \int_0^1 \delta(\tau) d\tau$$

Consequently,

$$\int_0^1 (\gamma(\tau) - \delta(\tau)) d\tau = 0$$

and therefore $\gamma = \delta$.
In particular, we obtain

$$h_J(u) = \gamma(1) = \delta(1) = h_K(u)$$

since u was arbitrary, $V(aJ + (1-a)K) = 1$ implies $h_J = h_K$, and hence $J = K$
Conversely, when $J = K$

$$V(aJ + (1-a)K) = V(aJ + (1-a)J) = V(J) = 1$$

Hence Proved. □

Corollary 4.0.1. *function* $S(b)\colon b \to \sqrt[n]{V(bJ + (1-b)K)}$ *is concave on* $[0,1]$.

Proof.

$$S(b) := \sqrt[n]{V(bJ + (1-b)K)}$$

We will prove it using the above theorem (4.0.9)
we have to prove that

$$S(ax + (1-a)y) \geq aS(x) + (1-a)S(y)$$

Let $x, y, a \in [0,1]$,

$$S(ax+(1-a)y) = \sqrt[n]{V([ax+(1-a)y]J+[1-ax-(1-a)y]K)}$$

$$= \sqrt[n]{V(a[xJ+(1-x)K]+(1-a)[yJ+(1-y)K])}$$

$$\geq a\sqrt[n]{V(xJ+(1-x)K)}+(1-a)\sqrt[n]{V(yJ+(1-y)K)} \qquad \text{from theorem (4.0.9)}$$

$$= aS(x)+(1-a)S(y)$$

□

Here we are stating the following theorem regarding mixed volume from reference[2] without proof which is a very crucial theorem to prove Isoperimetric Inequality for convex bodies.

Theorem 4.0.10. *For $J, K \in \mathcal{K}^n$*

$$V(J,\ldots,J,K)^n \geq V(J)^{n-1}V(K)$$

with equality, if and only if $\dim J \leq n-2$ *or J and K lie in parallel hyperplanes or J and K are homothetic.*

Now as a corollary of the above theorem we get the famous Isoperimetric inequality for convex body which states that,

Between all convex bodies of a given volume, the balls have the smallest surface area given surface area.

or

Between all convex bodies of given surface area, the balls have the largest volume.

Corollary 4.0.2. ***(Isoperimetric inequality for Convex body)*** *Assume that $K \in \mathcal{K}^n$ is a convex body of dimension n. Then,*

$$\frac{A(K)^n}{V(K)^{n-1}} \geq \frac{A(B(1))^n}{V(B(1))^{n-1}}$$

with equality if and only if K is a ball.

Proof. - On putting $K := B(1)$ in theorem (4.0.10) we get

$$V(K,\ldots,K,B(1))^n \geq V(K)^{n-1}V(B(1))$$

as

$$V(B(1),\ldots,B(1),B(1)) = V(B(1))$$

from properties(3) of mixed volume

We can write it in another form.

$$\frac{n^n V(K,\dots,K,B(1))^n}{n^n V(B(1),\dots,B(1),B(1))^n} \geq \frac{V(K)^{n-1}}{V(B(1))^{n-1}}$$

Now using properties (3) of mixed volume we have

$$nV(K,...,K,B(1)) = A(K), \qquad nV(K,...,K,B(1)) = A(K)$$

So We get

$$\left(\frac{A(K)}{A(B(1))}\right)^n \geq \left(\frac{V(K)}{V(B(1))}\right)^{n-1}$$

And then finally get required Isoperimetric inequality □

As we know volume and area of $B(1)$,

$$V(B(1)) = \kappa_n \qquad \text{and} \qquad A(B(1)) = n\kappa_n,$$

we can rewrite it in terms of κ_n

$$V(K)^{n-1} \leq \frac{1}{n^n \kappa_n} A(K)^n$$

From here for $\mathbb{R}^2$ plane, i.e. $n = 2$ we get Isoperimetric inequality

$$A(K) \leq \frac{1}{4\pi} L(K)^2$$

Where $A(K)$ is the usual area (volume in $\mathbb{R}^2$) and $L(K)$ is the boundary length (surface area in $\mathbb{R}^2$),

Similarly for $n = 3$ we get

$$V(K)^2 \leq \frac{1}{36\pi} A(K)^3$$

Chapter 5

Isoperimetric inequality for graph

Isoperimetric inequality for a graph G is defined similarly to how this is defined for $\mathbb{R}^n$. It involves finding the smallest edge-boundary subgraph among all subgraphs with a given size, which is called Isoperimetric sets.

Let $G(V, E)$ is an undirected graph for vertex set $V(G)$ with number of vertices $|V| = n$ and edge set E with number of edges $|E(G)| = m$.
For $i, j \in V(G)$, $i \sim j$ and $i \nsim j$ indicate that i and j are adjacent(or connected) and are not adjacent to each other respectively. We define d_i to be the degree of i-th vertex.

Definition 5.0.1. *For subsets A and B of the graph G we define the* ***distance*** *between them to the length of the shortest path of edges starting from A and ending in B.*

Definition 5.0.2. ***Edge boundary***$(\partial\Omega)$: *For a graph $G = (V, E)$ with $\Omega \subset V$, edge boundary $\partial\Omega \subset E$ is defined as the subset of edges connecting vertices of Ω with vertices of its complement $\bar{\Omega} = V \backslash \Omega$.*
Number of edges in $\partial\Omega$, i.e.$|\partial\Omega|$ is called the length of edge boundary for edge boundary $\partial\Omega$.

Definition 5.0.3. ***Isoperimetric inequality:*** *For a graph $G = (V, E)$, Isoperimetric inequality is the problem of finding a function $\mathcal{F}$, such that*

$$\forall\, \Omega \subset V, \quad \Omega \neq \emptyset, \quad |\partial\Omega| \geq \mathcal{F}(|\Omega|)$$

Definition 5.0.4. ***Best isoperimetric function:*** *These are functions for which isoperimetric inequality is sharp. Mathematically*

$$\mathcal{F}(k) = \min_{|\Omega|=k} |\partial\Omega|$$

Best isoperimetric functions(or corresponding sharp isoperimetric inequality) are known only for a few classes of graphs which are proved using combinatorial techniques.
Isoperimetric inequality for some trivial graphs is the following.

- For the complete graph K_n inequality generates to $|\partial\Omega| = (n - |\Omega|)|\Omega|$ which is, in fact, equality case
- For the cycle C_n it is $|\partial\Omega| \geq 2$ for $|\Omega| \neq n$.
- For the infinite d-regular tree it is $|\partial\Omega| \geq (d-2)|\Omega| + 2$

Other non-trivial classes for which it is known are some families of Cartesian products of graphs
for e.g.

- n-cube Q_n
- grid $[k]^n$
- lattice $\mathbb{Z}^n$.

Apart from these combinatorial techniques, there are eigenvalue techniques which give good Isoperimetric inequality in general, which we will discuss later for the d-regular graph.

Definition 5.0.5. ***(Adjacency matrix)***
For a graph $G(V,E)$ we define the adjacency matrix A of G to be a $n \times n$ matrix defined by

$$A_{ij} = \begin{cases} 1 & i \sim j \text{ and } i \neq j \\ \\ 0 & i \nsim j \text{ and } i \neq j \end{cases}$$

Definition 5.0.6. ***(Incidence matrix)*** *Incidence matrix is a $n \times m$ matrix B which has rows indexed by the vertices of G(total n vertices) and columns by the edges of G(total m edges).*
We define Incidence matrix B in two ways, first unoriented incidence matrix, defined by

$$B_{ij} = \begin{cases} 1 & \textit{i-th vertex is connected to the edge of j-th vertex.} \\ \\ 0 & \textit{otherwise.} \end{cases}$$

and then for each edge if we assume one vertex of edge to head and other to tail then oriented incidence matrix for undirected graph G is defined by

$$B_{ij} = \begin{cases} 1 & \textit{i-th vertex is the head to the edge of j-th vertex.} \\ \\ -1 & \textit{i-th vertex is the tail to the edge of j-th vertex.} \\ \\ 0 & \textit{otherwise.} \end{cases}$$

Definition 5.0.7. *(**Laplacian matrix**) Let D denote the diagonal matrix D and $D_{jj} = d_j$, and A is the adjacency matrix. We define the Laplacian matrix for a graph G to be*

$$L = D - A$$

It is easy to check that

$$L_{ij} = \begin{cases} d_i & i = j \\ \\ -1 & i \neq j \text{ and } i \sim j \\ \\ 0 & Otherwise \end{cases}$$

Lemma 5.0.1. *Laplacian matrix L is a positive definite matrix and hence all its eigenvalues are real and nonnegative.*

Proof. For B being a positive definite matrix, we must have

$$\lambda = x^T L x \geq 0$$

for any general eigenvalue λ and eigenvector x.
If B is the incidence matrix of G, then we can easily check that

$$L = B^T B$$

and it is independent of orientation of B.
So we have

$$x^T L x = x^T B^T B x = (Bx)^T B x = \|Bx\|^2 \geq 0$$

□

Lemma 5.0.2. *Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$, $\mu_i \geq 0 \, \forall \, i$, be eigenvalues of L then smallest eigenvalue $\mu_1 = 0$*

Proof. For vector $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ we have

$$A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} d \\ \vdots \\ d \end{pmatrix}$$

as the sum of i-th rows of the adjacency matrix is a degree of i-th vertex.

For L we have

$$L \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = (D - A) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} d-d \\ \vdots \\ d-d \end{pmatrix} = 0 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

Now, as all eigenvalues are nonnegative, 0 is the smallest eigenvalue of L. □

For proving this theorem, we first state the following important lemma from the paper (7).

Lemma 5.0.3. *Let $G = (V, E)$ be an undirected graph and $X, Y \subset V(G)$ $X, Y \neq \phi$ and let the distance between X and Y is s. Let E_X and E_Y are sets of edges which are completely inside X and Y respectively. we denote $p = |X|/n, q = |Y|/n$, then we have*

$$n\mu_2 \leqslant \frac{1}{s^2}\left(\frac{1}{p} + \frac{1}{q}\right)(|E| - |E_X| - |E_Y|)$$

Theorem 5.0.4. ***Sharp Isoperimetric inequality for the d-regular graph:)***

For the d-regular graph G with $V(G) = n$, let $\Omega \in V(G)$ is a nonempty subset with its boundary $\partial\Omega$ and its complement subset $\bar{\Omega}$ in $V(G)$ assuming that there are nonzero edges going from Ω to $\bar{\Omega}$.
Then for second smallest eigenvalue μ_2 of the Laplacian matrix L, we have sharp isoperimetric inequality for G as

$$|\partial\Omega| \geq \mu_2 \frac{|\Omega||\bar{\Omega}|}{|V|}$$

Proof. of the above theorem

For the d-regular graph G the d-regular graph is a graph in which the degree of all vertices is d, and hence it is a regular graph.
Let

$$\lambda_1 \geq \lambda_2 .... \geq \lambda_n$$

be the eigenvalues of A, and

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n, \quad \mu_i \geq 0 \; \forall \, i$$

be eigenvalues of L.
Then in general there is no simple relation between λ_is and μ_is
but for the case of d-regular graph, λ_i corresponds to the i-th Laplacian eigenvalue, μ_i and as

$$L = dI - A$$

we have

$$\lambda_i = d - \mu_i$$

In the above lemma we put $p = \frac{|\Omega|}{n}$ and $q = \frac{|\bar{\Omega}|}{n}$, $s = 1$(as there exist some edges from Ω to $\bar{\Omega}$). and

$$|E| - |E_X| - |E_Y| = |E(G)| - |E(\Omega)| - |E_{\bar{\Omega}}| = |\partial\Omega|$$

to get the required form of Isoperimetric inequality for the d-regular graph as following:

$$|V|\mu_2 \leqslant \left(\frac{n}{|\Omega|} + \frac{n}{|\bar{\Omega}|}\right)|\partial\Omega| = |V|\left(\frac{|V|}{|\Omega||\bar{\Omega}|}\right)|\partial\Omega|$$

as $|\Omega| + |\bar{\Omega}| = n$ and so we finally have

$$|\partial\Omega| \geq \mu_2 \frac{|\Omega||\bar{\Omega}|}{|V|}$$

Above inequality is a good approximation of sharp Isoperimetric inequality for the d-regular graph where μ_2 is the second eigenvalue of the graph G (i.e. second smallest eigenvalue of L) □

5.0.1 Isoperimetric Number of the graph G

Isoperimetric number of the graph G is defined as

$$i(G) = \min_{\Omega} \frac{|\partial\Omega|}{|\Omega|}$$

where minimum is taken over all nonempty subsets Ω of V satisfying

$$|\Omega| \leq \frac{1}{2}|V(G)|.$$

we can think of the quantity $\frac{|\partial\Omega|}{|\Omega|}$ as the average boundary degree of X.
The isoperimetric number $i(G)$ can also be defined in the following way.

$$i(G) = \min \frac{|E(X,Y)|}{\min\{|X|,|Y|\}}$$

where minimum runs over all partitions of V into nonempty subsets X and Y such that $V = X \cup Y$.

Here $E(X, Y) = \partial X = \partial Y$ are the edges between X and Y.
From the above definition of the Isoperimetric constant, it can also be understood as a measure of how easy it is to break a large part of the graph.

For getting i(G), we have to find small edge-cut $E(X, Y)$ separating as large as possible subset X from the remaining larger part Y assuming $|X| \leq |Y|$, so in this way, i(G) can also serve as a measure of connectivity of the graph.

Isoperimetric constant for different classes of graphs are following

1. $i(G) = 0$ if and only if G is disconnected.

   Proof. we can choose any part of the graph which is not connected to rest of the graph and will get $i(G) = 0$ □

2. If G is k -edge-connected(require at least k edge to break connectivity) then $i(G) \geq 2k/|V(G)|$.

   Proof. For any chosen subset X of V, we have $|\partial X| \geq k$ and we can take $|X|$ to be $|V(G)|/2$ and hence proved. □

isoperimetric number for some classes of the graph are following

1. For the complete graph $K_n, i\left(K_n\right) = \lceil n/2 \rceil$
2. The cycle C_n has $i\left(C_n\right) = 2/\lfloor n/2 \rfloor$
3. The path P_n on n vertices has $i\left(P_n\right) = 1/\lfloor n/2 \rfloor$.

5.1 Bounding $\mathrm{E}\,|\lambda_2(G)|$ of random 2d-regular graph

Definition 5.1.1. *we define $G_{n,2d}$ to be probability space of random 2d-regular graphs*
constructed in the following way.
We choose d permutations of the numbers $1,\ldots,n$, denoted by $\pi_j, 1 \leq j \leq d$ with each permutation equally likely and construct a directed graph $G=(V,E)$ with vertex set $V=\{1,\ldots,n\}$ and edges

$$E = \left\{ \left(i, \pi_j(i)\right), \left(i, \pi_j^{-1}(i)\right) | j = 1,\ldots,d \quad i = 1,\ldots,n \right\}$$

Although G is directed, we can view it as an undirected graph by replacing each pair of edges $(i,\pi(i)),(\pi(i),i)$ with one undirected edge.

In this way, Each vertex of this graph has its degree 2d, and we denote this probability space of these random graphs by $G_{n,2d}$.

Theorem 5.1.1. (**A**) *For $G \in G_{n,2d}$ we have*

$$\mathbf{E}\,|\lambda_2(G)| \leq 2\sqrt{2d-1}\left(1+\frac{\log d}{\sqrt{2d}}+o\left(\frac{1}{\sqrt{d}}\right)\right)+O\left(\frac{d^{3/2}\log\log n}{\log n}\right)$$

and we have

$$\mathrm{E}|\lambda_2(\mathrm{G})| \leq O(d^{1/2}) \text{ for fixed } d \text{ when } n \to \infty$$

where E denotes the expected value over $G_{n,2d}$ and more generally we have

$$\mathbf{E}\,|\lambda_2(G)|^m \leq \left(2\sqrt{2d-1}\left(1+\frac{\log d}{\sqrt{2d}}+O\left(\frac{1}{\sqrt{d}}\right)\right)+O\left(\frac{d^{3/2}\log\log n}{\log n}\right)\right)^m$$

for any $m \leq 2\lfloor \log n \lfloor \sqrt{2d-1}/2 \rfloor / \log d \rfloor$

(**B**) *As a corollary, For $\beta \geq 1$ we have*

$$|\lambda_2(G)| \geq \left(2\sqrt{2d-1}\left(1+\frac{\log d}{\sqrt{2d}}+O\left(\frac{1}{\sqrt{d}}\right)\right)+O\left(\frac{d^{3/2}\log\log n}{\log n}\right)\right)\beta$$

with probability

$$\leq \frac{\beta^2}{n^2\lfloor\sqrt{2d-1}/2\rfloor \log\beta/\log d}$$

Proof. The standard approach for estimating eigenvalues is to estimate the trace of high power of the adjacency matrix.

Let A be the adjacency matrix of a graph $G \in G_{n,2d}$. Let Π be the alphabet of symbols

$$\Pi = \left\{\pi_1, \pi_1^{-1}, \pi_2, \ldots, \pi_d^{-1}\right\}$$

where $\pi_1, \ldots, \pi_d$ are d permutations from which G was constructed.
We denote Π^k to be the set of all words which are composition of k permutations from Π. for any word, $w = \sigma_1 \ldots, \sigma_k$ of Π^k we define

$$i \xrightarrow{w} j \equiv \begin{cases} 1 & \text{if } w(i) = j \\ 0 & \text{otherwise} \end{cases}$$

and then i, j -th entry of A^k will be

$$\sum_{w \in \Pi^k} i \xrightarrow{w} j$$

What we want is to estimate, the expectation of trace, so we need to take the expectation of the above sum only for $i = j$ In evaluating $i \xrightarrow{w} i$, $\pi\pi^{-1}$ with $\pi \in \Pi$ will fix the vertex, so we can cancel this type of pairs in w.

We define a word $w \in \Pi^k$ to be irreducible if w has no pair of consecutive letters of the form $\pi\pi^{-1}$ and denote the set of irreducible words of length k by Irred $_k$.

For the first position of a word, the first letter has 2d choices, and then its next letter can not be its inverse, so all other k - 1 letters will have only 2d - 1 choices, so Irred $_k$ has its size $2d(2d-1)^{k-1}$.

It turns out that to estimate the second eigenvalue it suffices to get an estimate of the form.

$$\sum_{\boldsymbol{w} \in \text{ Irred }_{\boldsymbol{k}}} i \xrightarrow{w} i = 2d(2d-1)^{k-1}\frac{1}{n} + error$$

for all fixed i with some small error term which will be bounded later.

1/n comes because we expect that words in $Irred_k$ send a fixed vertex to each others with more or less equal probability which is 1/n .

We have the following result as a corollary and will be using this result to estimate the expected sum of the k-th powers of the eigenvalues

For any fixed i, $k \geq 1$, and $d - 2 > \sqrt{2d-1}/2$ (i.e. $d \geq 4$) we have

$$\mathrm{E}\left\{\sum_{w \in \mathrm{Irred}_k} i \xrightarrow{w} i\right\} = 2d(2d-1)^{k-1}\left(\frac{1}{n} + \mathrm{error}_{n,k}\right)$$

where

$$\mathrm{error}_{n,k} \leq (ckd)^c \left(\frac{k^{2\sqrt{2d}}}{n^{1+\lfloor\sqrt{2d-1}/2\rfloor}} + \frac{(2d-1)^{-k/2}}{n}\right)$$

All words in Π^k can be reduced to irreducible words by removing occurrences of $\pi\pi^{-1}$ in the word, and this irreducible word is independent of how the reducing was done.

Let $p_{k,s}$ is the probability that a random word in Π^k reduces to an irreducible word of size s then breaking sum over words in Π_k to sum over irreducible words of size s we get

$$\frac{1}{(2d)^k}\mathbf{E}\left\{\sum_{w\in\Pi^k} i \to i\right\} = p_{k,0} + \sum_{s=1}^{k} p_{k,s}\frac{1}{2d(2d-1)^{s-1}}\mathrm{E}\left\{\sum_{w\in \mathrm{Irred}_s} i \xrightarrow{w} i\right\}$$

since $\sum_s p_{k,s} = 1$, we have

$$\frac{1}{(2d)^k}\sum_i \mathrm{E}\left\{\sum_{w\in\Pi^k} i \xrightarrow{w} i\right\} = 1 + (n-1)p_{k,0} + \sum_{s=1}^{k} np_{k,s}\,\mathrm{error}_{n,s}$$

We are using the following result for $p_{2k}, 2s$ without proof

$$p_{2k,2s} \le \frac{2s+1}{2k+1}\binom{2k+1}{k-s}\left(\frac{1}{2d}\right)^{k-s}\left(1-\frac{1}{2d}\right)^{2s-1}$$

Note that k and s can not have different parity as π and π^{-1} will get vanish in pair

from proof of the above theorem from [6]

it is seen that(Using here as a result)

$$p_{2k,0} \ge \frac{1}{2k+1}\left(\binom{2k+1}{k}\right)\frac{(2d-1)^k}{(2d)^{2k}}$$

It follows that for any graph of degree $2d$,

$$\sum_{i=1}^{n}\lambda_i^{2k} \ge (2d)^{2k}(n-1)p_{k,0} \approx (n-1)2^{2k}(2d-1)^k$$

so that taking $2k$ slightly less than $2\log_d n$ results

$$|\lambda_2| \ge 2\sqrt{2d-1} + O\left(\frac{1}{\log_d n}\right)$$

Now we take

$$k = 2\lfloor \log n \lfloor \sqrt{2d-1}/2\rfloor / \log d\rfloor,$$

so that k is even, and calculate using the simplified bound

$$p_{2k,2s} \le 2^{2k}\left(\frac{1}{2d}\right)^{k-s}$$

It is easy to see that the dominant terms of the summation over s in equation (1) are $s = 1$ and $s = k$, and therefore

$$\mathrm{E}\left\{\sum_{i=2}^{n} \lambda_i^k\right\} \leq n^{1+\frac{\log 2}{\log \alpha}}(ckd)^c k^{2\sqrt{2d}}(2\sqrt{2d}\sqrt{\tfrac{2d}{2d-1}})^k$$

Taking k -th roots, applying Holder's inequality, and noticing that

$$\left(n^{1+\frac{\log 2}{\log d}}\right)^{1/k} = 1 + \frac{\log d}{\sqrt{2d}} + O\left(\frac{1}{\sqrt{d}}\right)$$

and

$$(ckd)^{c/k} k^{2\sqrt{2d}/k} = 1 + O\left(\frac{\log d \log\log n}{\log n}\right)$$

and that

$$k \leq \frac{1}{cd^{3/2}} n^{\frac{1}{c\sqrt{a}}}$$

for

$$\frac{\log n}{\log\log n} \geq c'\sqrt{d}$$

proves theorem(A)

From here it is clear that $\mathrm{E}|\lambda_2(\mathrm{G})| \leq O(d^{1/2})$ as $\frac{\log\log n}{\log n} \to 0$ in limit as n goes to infinity □

Theorem 5.1.2. *Let G be a graph on n vertices and let μ_2 be the second smallest eigenvalue of its difference Laplacian matrix L. Then for every k, $1 \leq k \leq n-1$*

$$i_k(G) \geq \frac{(n-k)\mu_2}{n}$$

and, consequently, $i(G) \geq \mu_2/2$

Proof. for the second smallest eigenvalue μ_2 of D and for an arbitrary $X \subseteq V(G)$ the following relation holds(from paper [4]):

$$\mu_2 \leq |\partial X|\left(\frac{1}{|X|} + \frac{1}{|V\backslash X|}\right)$$

putting $|X| = k$ and $|V\backslash X| = n - k$ and using definition of $i_k(G)$

$$\mu_2 \leq |\partial X|\left(\frac{1}{k} + \frac{1}{n-k}\right) = |\partial X|\left(\frac{n}{k(n-k)}\right)$$

which gives the required result □

Definition 5.1.2. *$F(n,k)$:= max$\{i(G)|G$ is k-regular with n vertices$\}$ and*

$$f(k) := \lim_{n\to\infty} \sup F(n,k)$$

Theorem 5.1.3.

$$f(2k) \geq k - O\left(k^{1/2}\right)$$

Proof. From theorem (5.1.1), we have

$$\mathrm{E}|\lambda_2(\mathrm{G})| \leq O(k^{1/2}),$$

when d is fixed and $n \to \infty$ for the second largest eigenvalue λ_2 of the Adjacency Matrix A.
So second smallest eigenvalue(μ_2 of corresponding the Laplacian Matrix L) will be

$$2k - \lambda_2$$

As $f(k)$ is limsup of $F(n,k)$s and each $F(n,k)$ is maximum over $i(G)$ of k-regular graphs G, so

$$i(G) \leq f(2k)$$

From theorem (5.1.2) we have

$$i(G) \geq \frac{\mu_2}{2} = \frac{2k - \lambda_2}{2}$$

So we have

$$f(2k) \geq \frac{2k - \lambda_2}{2}$$

If we take expectation over $\mathscr{G}_{n,2d}$ on both side we get

$$f(2k) \geq \frac{2k - E(\lambda_2)}{2}$$

and as

$$\mathrm{E}|\lambda_2(\mathrm{G})| \leq O(k^{1/2})$$

we have

$$f(2k) \geq \frac{2k - O(k^{1/2})}{2} \geq k - O(k^{1/2})$$

□

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