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Khovanov Homology

by

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Certificate of Examination

This is to certify that the dissertation titled **Khovanov Homology** submitted by Nilangshu Bhattacharyya (Reg. No. MS15169) for the partial fulfilment of BS-MS dual degree program of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Date: 04 May 2020

Declaration of Authorship

The work presented in this dissertation has been carried out by me under the guidance of **Dr. Mahender Singh** at the Indian Institute of Science Education and Research, Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of expository work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Mahender Singh(Supervisor)
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Nilangshu Bhattacharyya

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Notations and Abbreviations

∂T	boundary of T
	cap cobordism
$Cob^3(\cdot)$	category of smoothings and cobordism
H^r	cohomology of r-th height
$\mathcal{T}^0(k)$	collection of k-ended unoriented tangle diagrams
$\mathcal{T}^0(s)$	collection of $\lvert s \rvert$-ended oriented tangle diagrams
$\mathrm{Kom}(\cdot)$	complexes over a category
Γ	cone of a morphism
	cup cobordism
$\{\cdot\}$	degree shift
$[\![\cdot]\!]'$	formal Khovanov bracket
$4Tu$	four tube relation
$q\mathrm{dim}$	graded dimension
χ_q	graded Euler characteristic
V	graded vector space spanned by $v_{\pm}$
$[\cdot]$	height shift
$\langle\cdot\rangle$	Kauffman bracket polynomial
$[\![\cdot]\!]$	Khovanov bracket
$Kh(\cdot)$	Khovanov cochain complex
$\mathrm{Kob}_{/h}$	Kob modulo homotopy
$\mathrm{Kob}(\cdot)$	$\mathrm{Kom}\Big(\mathrm{Mat}(Cob^3_{/l}(\cdot))\Big)$
$\mathrm{Kom}_{/h}(\cdot)$	$\mathrm{Kom}(\cdot)$ modulo homotopy
$\lvert \cdot \rvert$	length of a string
$\mathrm{Mat}(\cdot)$	matrices over a category
J_K	modified Jones Polynomial
$N(D)$	number of crossings of D
n_-	number of negative crossings
n_+	number of positive crossings
D	planar arc diagram
$Cob^3_{/l}$	quotient of Cob^3 by S, T and $4Tu$ relations

$\mathcal{T}(k)$	quotient of $\mathcal{T}^0(k)$ by Reidemeister moves
$\mathcal{T}(s)$	quotient of $\mathcal{T}^0(s)$ by Reidemeister moves
R1	Reidemeister first move
R2	Reidemeister second move
R3	Reidemeister third move
$)($	saddle cobordism $)(:)(\to \asymp$
$P(s)$	sets making up an oriented planar algebra
$P(k)$	sets making up an un-oriented planar algebra
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rationals
S	the sphere relation
$\lvert\alpha\rvert$	state sum of the state α
TQFT	Topological Quantum Field Theory
T	the torus relation

Abstract

This thesis is an exposition of Khovanov cohomology theory for knots and tangles with a focus on lifting of the construction of Khovanov cochain complex from the local level (for tangles) to the global level (for knots). We review two approaches of the construction of Khovanov cochain complex; one approach is algebraic using graded vector spaces and the other is topological involving cobordisms. Applying a suitable functor (TQFT) on the topological construction sends us to the algebraic set-up. A formal computation of Khovanov cohomology for the figure eight knot and an illustration of a fast computation algorithm developed by Dror Bar-Natan is also included in the thesis.

To my parents...

Chapter 1

Introduction

Knot theory was revolutionised by the fundamental work of Jones [1] and [2], where he defined what is now known as Jones polynomial. No doubt, Jones polynomial has played an important role in the development of knot theory and is a quite strong knot invariant. In the year 2000, Mikhail Khovanov [4] gave a cohomological generalisation of the Jones polynomial, where he recovered the Jones polynomial as the graded Euler characteristic of a cohomology theory. Bar-Natan's paper [7] shows with examples that Khovanov cohomology is a stronger invariant than Jones polynomial. Jones polynomial cannot distinguish $\overline{5}_1$ and 10_{132}, but Khovanov cohomology can.

In Chapter 2, a quick review of Jones polynomial is provided. The original normalised Jones polynomial is a laurent polynomial in the variable $t^{1/2}$, and its value for the unknot is 1. Through a little change in the variables and removing the restriction of its value for the unknot to be 1, we get a modified unnormalized version of the Jones polynomial. To understand why this modification is important for Khovanov cohomology theory, we move to Chapter 3.

In Chapter 3, we give the construction of Khovanov cochain complex as described in [7]. Given a link diagram D, first we construct a cube where each vertex corresponds to smoothing. We then associate a graded vector space $V^{\otimes k}$ to each vertex, where V is a graded vector space over $\mathbb{Q}$ of graded dimension $q + q^{-1}$. We take the arrows of the cube as homogeneous linear maps. These linear maps are defined by taking tensor product of identity maps either with $m : V \otimes V \to V$ or $\Delta : V \to V \otimes V$. Using the cube, we define the Khovanov bracket $[\![D]\!]$, and finally we set $Kh(D) = [\![D]\!][-n_1]\{n_+ - 2n_-\}$, where n_+ and n_- denote the number of positive and negative crossings respectively, $[l_1]$ denotes the operation of height shift by l_1, and $\{l_2\}$ is the operation of degree shift by l_2. If D_1, D_2 are two link diagrams which differ by Reidemeister moves, one can find grading preserving isomorphisms between the cohomologies $H^r(D_1)$, and $H^r(D_2)$ of the cochain complex $Kh(D_1)$ and $Kh(D_2)$, respectively. These cohomologies $H^r(D)$

are graded and we can write them $H^r(D) = \oplus_{j\in\mathbb{Z}} H^{r,j}(D)$. Therefore each graded component $H^{r,j}(D)$, and the graded Poincaré polynomial $\sum_{r,j\in\mathbb{Z}} t^r q^j \dim_{\mathbb{Q}}(H^{r,j}(D))$ are knot invariants as well. In particular, the graded Euler characteristic $\chi_q(Kh(D)) = \sum_{r,j\in\mathbb{Z}} (-1)^r q^j \dim_{\mathbb{Q}}(H^{r,j}(D))$ is knot invariant, and as we will show in Chapter 3, it is equal to the modified unnormalized Jones polynomial of the link.

We found in Khovanov's seminal paper [4] that there is no specific rule or restriction on the signs of maps mentioned, except the only restriction is: each face of the cube should anti-commute. This motivated us to hypothesize that if two anti-commutative cubes differ only by the signs of their maps, then the cochain complexes constructed from these two cubes are isomorphic. This is proved in section 3.6.

Chapter 4 contains a computation of Khovanov cohomology of figure eight knot. Chapter 5 is based on Bar-Natan's approach [6] towards construction of Khovanov cochain complex for tangles. As described in [4, 7], in order to define a Khovanov cochain complex one needs to count the number of simple closed curves in smoothings of a knot diagram. But for a tangle, which is not a knot or a link, smoothing is not necessarily a collection of simple closed curves, rather it is a collection of disjoint arcs (in general). Construction of Khovanov cochain complex involves two pictures: one is topological, made of the smoothings of link diagrams and of cobordisms, and another one is algebraic, which involves graded vector spaces and grading preserving maps. Applying the functor F (1+1 dimensional TQFT) to the topological picture, one can get the algebraic picture. Bar-Natan postponed the application of F to the later stage, and proved the invariance theorem at the level of topological picture. First, he proved the theorem for the local tangles representing the Reidemeister moves, and then he extends the proof to the global invariance under these moves. A $d-$ input planar arc diagram yields an operation of composing smaller tangles to get a bigger tangle. Not only that, one can also talk about the composition (or tensor product) of cochain complexes using the planar arc diagrams. Khovanov bracket is a planar algebra morphism from planar algebra of tangles modulo Reidemeister moves to the planar algebra of formal complexes ($\text{Kob}_{/h}$) modulo cochain homotopy. Hence it is sufficient to prove the invariance theorem of Khovanov cohomology at local level (for tangles), in fact, for the tangles representing the three Reidemeister moves. One can apply F (1+1 dimensional TQFT) to get algebraic Khovanov cohomology theory for tangles. Since the invariance theorem holds at the level of topological picture, applying different functors and choosing different coefficient rings yield different cohomology theories.

The computation of Khovanov cohomology is quite time taking even for knots with a small number of crossings. A search for a faster computational method led us to Bar-Natan's paper [5]. We review this work in Chapter 6.

Chapter 2

Kauffman bracket and Jones polynomial

In this text, we discuss the bracket polynomial, Jones polynomial and their modified versions following Kauffman [3].

2.1 Jones polynomial

Definition 2.1. Given a link diagram D of a link L, the Kauffman bracket $\langle D \rangle \in \mathbb{Z}\,[A^{\pm 1}]$ is defined by the following rules:

- $\langle \bigcirc \rangle = 1$
- $\langle \times \rangle = A\langle \mid\;\mid \rangle + A^{-1}\langle = \rangle$
- $\langle \bigcirc \cup D \rangle = (-A^2 - A^{-2})\langle D \rangle$.

The first rule gives the bracket polynomial of the trivial link diagram of one component. The second rule is the skein relation for reducing a regular diagram with a crossing to two regular diagrams changed only by eliminating this crossing in the two ways illustrated. The third rule gives the bracket polynomial of the disjoint union of a regular diagram D and an unknotted component in terms of the bracket polynomial of D.

This bracket polynomial is invariant under R2 and R3, but not under R1, where R1, R2 and R3 are the three Reidemeister moves.
Convention: Given an oriented link diagram D, the following convention would be followed

throughout the text:

Negative Crossing   Positive Crossing

Let n_+ and n_- denote the number of positive and negative crossings respectively. Then writhe of the knot diagram D, is defined to be the integer $n_+ - n_-$, and it is denoted by $w(D)$. Writhe of a knot diagram is invariant under the R2 and R3 moves but not under the R1 move.

Definition 2.2. Let D be an oriented knot diagram and $w(D)$ be the writhe of D. Consider the laurent polynomial f_K given by

$$f_K(A) = (-A^3)^{-w(D)}\langle D\rangle.$$

We define the normalised Jones polynomial V_K as

$$V_K(t) = f_K(t^{-1/4})$$

The polynomials f_K and $V_K(t)$ are isotopy invariants of knot K, and it can be shown that V_K is a laurent polynomial in the variable $t^{1/2}$ with integer coefficients.
The following bracket polynomial is often called the modified Kauffman bracket polynomial. Using this bracket polynomial we are going to define another knot invariant, which is very similar to the Jones polynomial V_K.

Definition 2.3. Given a link diagram D of a link L, we define another bracket polynomial $\langle D\rangle'$ called the modified Kauffman bracket by the following rules:

- $\langle \bigcirc \rangle' = q + q^{-1}$
- $\langle \times \rangle' = \langle \mid\ \mid \rangle' - q^{-1}\langle = \rangle'$
- $\langle \bigcirc \cup D\rangle' = (q + q^{-1})\langle D\rangle'$.

Definition 2.1 states that the Kauffman bracket for the trivial link diagram with one component is 1. Instead of 1, if we consider this value to be $(-A^2 - A^{-2})$, we will get a new laurent polynomial in the variable A with integer coefficients. Denote this new laurent polynomial by $\langle D\rangle^1$. Further define

$$\langle D\rangle^2 := A^{-N(D)}\langle D\rangle^1,$$

where $N(D)$ denotes the total number of crossings in D.
Now substituting A^2 with $-q^{-1}$ in the expression $\langle D\rangle^2$ we get another laurent polynomial $\langle D\rangle^3$ in the variable q with integer coefficients. It can be easily shown that $\langle D\rangle^3 = \langle D\rangle'$.

Definition 2.4. Let K be an oriented knot/link and f_K be the laurent polynomial as described earlier (see Definition 2.2). We define the modified unnormalized Jones polynomial $J_K \in \mathbb{Z}\left[q^{\pm 1}\right]$ to be the laurent polynomial that one can get by substituting A^2 with $-q^{-1}$ in the expression of $(-A^2 - A^{-2})f_K(A)$.

Proposition 2.1. Let J_K is the unnormalized Jones polynomial as defined in 2.4. Then

$$J_K(q) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle', \tag{2.1}$$

where n_+ and n_- denotes the number of positive and negative crossings respectively.

In this text, we refer to (2.1) as a formula for unnormalized Jones polynomial. Since f_K is an isotopy invariant of K, so is J_K. Note that the unnormalized Jones polynomial for the unknot is $q + q^{-1}$.

2.2 State smoothing and Kauffman bracket

Given a crossing of the form , there are two possible resolutions as follows:

0-resolution **1-resolution**

Definition 2.5. Let D be a regular diagram of a knot/link and its crossings are numbered 1 through $N(D)$. A state is defined to be an $N(D)$-tuple of $\{0,1\}$. Let α be a state of a knot diagram D. We resolve the i-th crossing of D by either 0 or 1-resolution, if the i-th term of α is 0 or 1 respectively. Thus corresponding to each state α, there is a complete resolution of D. After a complete resolution we get nothing but finite union of disjoint simple closed curves in plane. This is referred as the smoothing of D with respect to α. The state sum $|\alpha|$ is defined to be the sum of all the terms in the tuple α.

Proposition 2.2. Given a knot diagram D, let $\langle D \rangle'$ be the modified Kauffman bracket as mentioned in 2.3. Then

$$\langle D \rangle' = \sum_{\alpha} (-1)^{|\alpha|} q^{|\alpha|} (q + q^{-1})^{|\alpha D|}, \tag{2.2}$$

where $|\alpha D|$ denotes the number of disjoint loops in the smoothing of D with respect to α, and $|\alpha|$ denotes its state sum.

By amalgamating this result with the Equation (2.1), we would get the following formula of the unnormalized Jones polynomial

$$J_K(q) = (-1)^{n_-} q^{n_+ - 2n_-} \sum_{\alpha} (-1)^{|\alpha|} q^{|\alpha|} (q + q^{-1})^{|\alpha D|}. \tag{2.3}$$

Chapter 3

Khovanov cohomology

3.1 Graded vector space

Definition 3.1. A $\mathbb{Z}$-graded vector space over a field $\mathbb{F}$ is a vector space which decomposes into direct sum of the form $\bigoplus_{n\in\mathbb{Z}} V_n$, where V_n is a vector space over the field $\mathbb{F}$, and n is the grading/label of V_n. The elements of V_n are called homogeneous elements of degree n. A graded ring is a ring that is direct sum of abelian groups R_i, where $i \in \mathbb{Z}$ such that $R_iR_j \subseteq R_{i+j}$.

Unless otherwise stated, we will only consider $\mathbb{Z}$-graded vector space in this text. Let V and W be two $\mathbb{Z}$-graded vector spaces. A linear map $f : V \to W$ is called grading preserving linear map if $f(V_i) \subseteq W_i$, $\forall\, i \in \mathbb{Z}$. The map f is called homogeneous map of degree j, if $f(V_i) \subseteq W_{i+j}$, $\forall i \in \mathbb{Z}$.

Definition 3.2. Let $W = \bigoplus_{m\in\mathbb{Z}} W_m$ be a $\mathbb{Z}$-graded vector space over a field $\mathbb{F}$, where only finitely many W_m s are non zero.

1. The graded dimension of W, denoted by $q\dim W$ is defined as $\sum_{m\in\mathbb{Z}} q^m \dim W_m$.
2. Let $W = \bigoplus_{m\in\mathbb{Z}} W_m$ be a $\mathbb{Z}$-graded vector space over a field $\mathbb{F}$. A degree shift or grading shift by an integer l is a bijective function on the set of all $\mathbb{Z}$-graded vector spaces, defined as $W \mapsto W\{l\}$, where $W\{l\} = \bigoplus_{m\in\mathbb{Z}} W\{l\}_m$, and $W\{l\}_m = W_{m-l}$.

The following proposition can be verified easily:

Proposition 3.1. If W is a graded vector space over a field $\mathbb{F}$, then the graded dimension $q\dim W\{l\} = q^l\,(q\dim W)$.

Definition 3.3. Let (C, d) be a cochain complex of vector spaces (resp. graded vector spaces) and linear maps (resp. grading preserving maps). Then the height shift of (C, d) by an integer s is a cochain complex $(C[s], d[s])$ defined as, $C[s]^r := C^{r-s}$ and $d[s]^r := d^{r-s}$, where $d^r : C^r \to C^{r+1}$ is a linear map (resp. grading preserving linear map) satisfying $d^{r+1} \circ d^r = 0$.

3.2 Cube over a category

In this section, we provide the definition of cube over a category [4]. Let $\mathcal{I}$ be a finite set, and $|\mathcal{I}|$ its cardinality. Let $r(\mathcal{I})$ be the set of all pairs $(\mathcal{L}, a)$, where $\mathcal{L}$ is a subset of $\mathcal{I}$ and a is an element of $\mathcal{I}$ that does not belong to $\mathcal{L}$. To simplify the notations we would often

- denote singleton $\{a\}$ by a,
- denote a finite set $\{a, b, ..., d\}$ by $ab...d$,
- denote the disjoint union $\mathcal{L}_1 \sqcup \mathcal{L}_2$ of two sets $\mathcal{L}_1, \mathcal{L}_2$ by $\mathcal{L}_1\mathcal{L}_2$, in particular, we denote by $\mathcal{L}a$ the disjoint union of a set $\mathcal{L}$ and singleton $\{a\}$, similarly, $\mathcal{L}ab$ means $\mathcal{L} \sqcup a \sqcup b$, etc.

Definition 3.4. Let $\mathcal{I}$ be a finite set and $\mathcal{B}$ a category. An $\mathcal{I}$-cube V over $\mathcal{B}$ is a collection of objects $V(\mathcal{L}) \in \mathrm{Obj}(\mathcal{B})$ for each subset $\mathcal{L}$ of $\mathcal{I}$, and morphisms

$$\xi_a^V(\mathcal{L}) : V(\mathcal{L}) \to V(\mathcal{L}a)$$

for each $(\mathcal{L}, a) \in r(\mathcal{I})$. Moreover, V is called a commutative $\mathcal{I}$-cube over $\mathcal{B}$ if for each triple $(\mathcal{L}, a, b)$, where $\mathcal{L}$ is a subset of $\mathcal{I}$ and $a \neq b$ are two elements of $\mathcal{I}$ that do not lie in $\mathcal{L}$, the following diagram commutes.

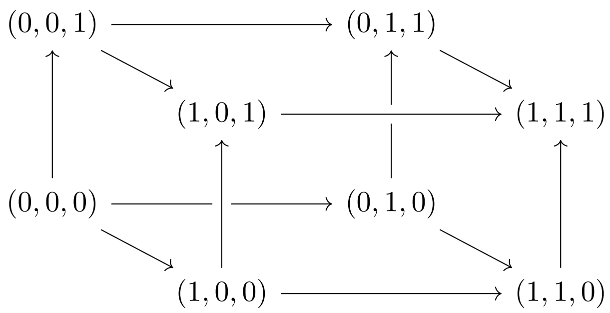
$$\begin{array}{ccc} V(\mathcal{L}) & \xrightarrow{\xi_a^V(\mathcal{L})} & V(\mathcal{L}a) \\ \downarrow{\scriptstyle \xi_b^V(\mathcal{L})} & & \downarrow{\scriptstyle \xi_b^V(\mathcal{L}a)} \\ V(\mathcal{L}b) & \xrightarrow{\xi_a^V(\mathcal{L}b)} & V(\mathcal{L}ab) \end{array}$$

A n-dimensional cube in standard position in $\mathbb{R}^n$ is a collection of

- 2^n vertices with coordinates $(a_1, \ldots, a_n)$, for $a_i \in \{0, 1\}$
- oriented edges between two vertices, whose coordinates differ only at one position, and the orientation is in the direction of the vertex with bigger sum of coordinates.

We often call this standard n-dimensional cube as n-cube and its oriented edges are called arrows.
In general, any I-cube over a category $\mathcal{B}$ can be visualised as a $|I|$-dimensional cube in standard position in $\mathbb{R}^n$, in the following manner. First consider an ordering of the elements of $\mathcal{I}$. Assume that $\mathcal{I} = \{b_i |\, b_i \text{ is the element of } i\text{-th order}\}$. For each $\mathcal{L} \subset \mathcal{I}$, we associate the vertex with coordinate $(a_1, \ldots, a_n)$, where the value of a_i is 1 if $b_i \in \mathcal{L}$ and 0 otherwise. Thus the empty set corresponds to the origin (0,. . . ,0). The following diagram is a 3-dimensional cube in

standard position in $\mathbb{R}^3$.



In this text, we consider a face of a $\mathcal{I}$-cube V over $\mathcal{B}$ to be a sub-collection of objects and morphisms of V, which can be visualised as a 2-cube.

Given two $\mathcal{I}$-cubes V, W over a category $\mathcal{B}$, an $\mathcal{I}$-cube map $\psi : V \to W$ is a collection of maps $\psi(\mathcal{L}) : V(\mathcal{L}) \to W(\mathcal{L})$, for all $\mathcal{L} \subset \mathcal{I}$ such that the following diagram

$$\begin{array}{ccc} V(\mathcal{L}) & \xrightarrow{\psi(\mathcal{L})} & W(\mathcal{L}) \\ \downarrow{\scriptstyle \xi_a^V(\mathcal{L})} & & \downarrow{\scriptstyle \xi_a^W(\mathcal{L})} \\ V(\mathcal{L}a) & \xrightarrow{\psi(\mathcal{L}a)} & W(\mathcal{L}a) \end{array}$$

commutes, for all $(\mathcal{L}, a) \in r(\mathcal{I})$.

Definition 3.5. Let $\mathcal{I}$ be a finite set and $\mathcal{B}$ an additive category. An anti-commutative $\mathcal{I}$-cube V over $\mathcal{B}$ is a collection of objects $V(\mathcal{L}) \in \mathrm{Obj}(\mathcal{B})$ for each subset $\mathcal{L}$ of $\mathcal{I}$, and morphisms

$$\xi_a^V(\mathcal{L}) : V(\mathcal{L}) \to V(\mathcal{L}a)$$

for each $(\mathcal{L}, a) \in r(\mathcal{I})$, such that for each triple $(\mathcal{L}, a, b)$, where $\mathcal{L}$ is a subset of $\mathcal{I}$ and $a \neq b$ are two elements of $\mathcal{I}$ that do not lie in $\mathcal{L}$, the following equality holds.

$$\xi_b^V(\mathcal{L}a) \circ \xi_a^V(\mathcal{L}) + \xi_a^V(\mathcal{L}b) \circ \xi_b^V(\mathcal{L}) = 0.$$

3.3 Construction of Khovanov cochain complex

Definition 3.6. We define V to be a graded vector space generated by two basis $\{v_\pm\}$ over some field $\mathbb{F}$ (in this text, we will consider $\mathbb{Q}$) such that $V = \bigoplus_{i\in\mathbb{Z}} V_i$, where $V_{-1} = \mathbb{F}v_-$, $V_{+1} = \mathbb{F}v_+$ and $V_i = 0$ for $i \notin \{\pm 1\}$. Thus v_+ and v_- are the homogeneous elements of degree ± 1 respectively. Further we define $V^{\otimes n} = \bigoplus_{k\in\mathbb{Z}} V_k^{\otimes n}$, where

$$V_k^{\otimes n} := \mathrm{Span}_{\mathbb{F}}\{v_1 \otimes \cdots \otimes v_n \,|\, v_i = v_\pm,\ \forall\, 1 \leq i \leq n \ \text{ and } \ \deg(v_1) + \cdots + \deg(v_n) = k\}.$$

Unless otherwise stated, we will take $\mathbb{F}$ as the field $\mathbb{Q}$.

Example 3.6.1. $V^{\otimes 2} = V_{-2}^{\otimes 2} \oplus V_0^{\otimes 2} \oplus V_2^{\otimes 2}$, where $V_{-2}^{\otimes 2} = \text{Span}_{\mathbb{F}}\{v_- \otimes v_-\}$, $V_0^{\otimes 2} = \text{Span}_{\mathbb{F}}\{v_- \otimes v_+, v_+ \otimes v_-\}$, and $V_2^{\otimes 2} = \text{Span}_{\mathbb{F}}\{v_+ \otimes v_+\}$.

Proposition 3.2. The graded dimension of the graded vector space $V^{\otimes n}$ is equal to the laurent polynomial $(q + q^{-1})^n$ for all $n \in \mathbb{N}$.

Proof. From Definition 3.6 and 3.2, $q\dim V = q + q^{-1}$. Now one can apply induction on n and get the desired result. ■

Given an oriented knot diagram D, we are now trying to construct a cochain complex in the following way.

Definition 3.7. Let D be an oriented knot/link diagram with its crossings marked with numbers from the set $\{1,2,\ldots,N(D)\}$, where $N(D)$ denotes the total number of crossings in D. For each state α, we assign a graded vector space $V_\alpha(D) := V^{\otimes|\alpha D|}\{|\alpha|\}$. Now we define Khovanov bracket

$$[\![D]\!]^r := \bigoplus_{\{\alpha:|\alpha|=r\}} V_\alpha(D) = \bigoplus_{\{\alpha:|\alpha|=r\}} V^{\otimes|\alpha D|}\{|\alpha|\}$$

for each integer r. We further define,

$$Kh(D) := [\![D]\!][-n_-]\{n_+ - 2n_-\},$$

where n_+ and n_- denotes the number of positive and negative crossings respectively. Thus for each integer r

$$Kh(D)^r := \bigoplus_{\{\alpha:|\alpha|=r+n_-\}} V^{\otimes|\alpha D|}\{|\alpha| + n_+ - 2n_-\}.$$

Now we want to construct the differential maps $d^r : [\![D]\!]^r \to [\![D]\!]^{r+1}$ such that $d^{r+1} \circ d^r = 0$. Then we can take the differential map for $Kh(D)$ as $d[-n_-]$. The differential will be constructed in the following way.

1. First we take abstract maps (arrows) from a state of state sum r to a state of state sum $r+1$, whenever those two states differ only at one position. Let α_1 and α_2 be two states who differ only at the i-th position and α_2 is the state with bigger state sum. We mark the arrow from α_1 to α_2 by a $N(D)$-tuple β of $\{0, 1, \star\}$, where the j-th position of β is the j-th position of α_1 for $j \neq i$, and the i-th position of β is $\star$. We name this arrow as d_β. If we visualise the states as vertices and arrows as edges, we get a $N(D)$-cube having $2^{N(D)}$ vertices.

2. Now assume that the arrow d_β from α_1 to α_2 associates a linear map from $V^{\otimes|\alpha_1 D|}$ to $V^{\otimes|\alpha_2 D|}$. Then we define the differential map as $d^r = \sum_{|\beta|=r} (-1)^{f(\beta)} d_\beta : [\![D]\!]^r \to [\![D]\!]^{r+1}$, where $|\beta|$ is the number of "1"s in β, and $f(\beta)$ is the number of "1"s before $\star$ in β. The

map d_β is called positive map if $f(\beta)$ is an even number and otherwise it is called negative map. It is easy to check that each face of the cube has odd number of negative maps. Hence to get $d^{r+1} \circ d^r = 0$, it is sufficient to find maps d_β such that each face (2-cube) of the cube commutes. Also, we want all the differential maps d^r to be grading preserving (this requirement is not clear at this moment). This essentially means that we need the maps d_β to be homogeneous maps of degree -1.

3. Consider the smoothings corresponding to α_1 and α_2, which differ only at one position. Without loss of generality assume that α_2 is the state with bigger state sum. Changing the resolution type from 0-resolution to 1-resolution only at one crossing, there are two possible situations that may arise.

   (a) Two simple closed curves may join together, and rest of the loops remain as it is.

   (b) One loop may split into two loops, and rest of the loops remain as it is.

   For each of these above cases, we take $d_\beta : V^{\otimes|\alpha_1 D|} \to V^{\otimes|\alpha_2 D|}$ to be identity on the tensor factors corresponding to the loops which remain as it is.

   (a) For the first case, we take a linear map $m : V \otimes V \to V$. Thus for this case, $d_\beta = m \otimes Id_V^{\otimes(|\alpha_1 D|-2)}$, up to a permutation of the tensor factors.

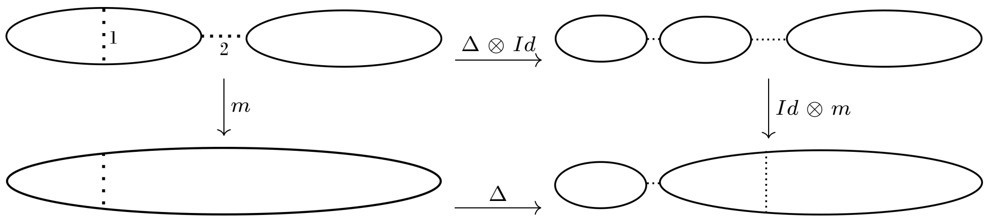
   (b) For the second case, we take a linear map $\Delta : V \to V \otimes V$ to define d_β. Thus for this case, $d_\beta = \Delta \otimes Id_V^{\otimes(|\alpha_1 D|-1)}$, up to a permutation of the tensor factors.

4. As previously stated, we need d_β s to be homogeneous maps of degree -1. This implies that m and Δ have to be homogeneous map of degree -1. Thus we may take

$$\begin{cases} m(v_+ \otimes v_+) = k_1 v_+ \\ m(v_+ \otimes v_-) = k_2 v_- \\ m(v_- \otimes v_+) = k_2 v_- \\ m(v_- \otimes v_-) = 0 \\ \Delta(v_+) = k_3(v_+ \otimes v_- + v_- \otimes v_+) \\ \Delta(v_-) = k_4 v_- \otimes v_-, \end{cases} \tag{3.1}$$

   where $k_1, k_2, k_3, k_4 \neq 0$. When we deal with $V \otimes V$ we don't want the ambiguity about which circle should be considered first and which should be second, that's why we need $m(v_+ \otimes v_-) = m(v_- \otimes v_+)$ and $\Delta(v_+) \in \mathbb{F}(v_+ \otimes v_- + v_- \otimes v_+)$. Also, we need d_β s such that each face of the cube commutes. That's why, we better hope the following relation to be satisfied:

$$\Delta \circ m = (Id \otimes m) \circ (\Delta \otimes Id). \tag{3.2}$$



This gives us the following restriction: $k_1 = k_2$ and $k_3 = k_4$. In particular, we take k_1, k_3 to be 1. Thus

$$\begin{cases} m(v_+ \otimes v_+) = v_+ \\ m(v_+ \otimes v_-) = v_- \\ m(v_- \otimes v_+) = v_- \\ m(v_- \otimes v_-) = 0 \\ \Delta(v_+) = v_+ \otimes v_- + v_- \otimes v_+ \\ \Delta(v_-) = v_- \otimes v_-. \end{cases} \tag{3.3}$$

Now using this m, Δ, we can take d_β such that each face of the cube commutes. Hence we have found differential maps d^r such that $d^{r+1} \circ d^r = 0$. Now assigning cochain spaces as defined in 3.7, we get two cochain complexes $([\![D]\!], d)$ and $(Kh(D), d[-n_-])$ for an oriented knot diagram D.

The below figure is the cube diagram of trefoil knot, where we have written down the graded vector spaces corresponding to each state and their graded dimensions. The red arrows are positive maps and the blue ones are negative maps. The grading shifts by $n_+ - 2n_- = 3$ are written in red color.

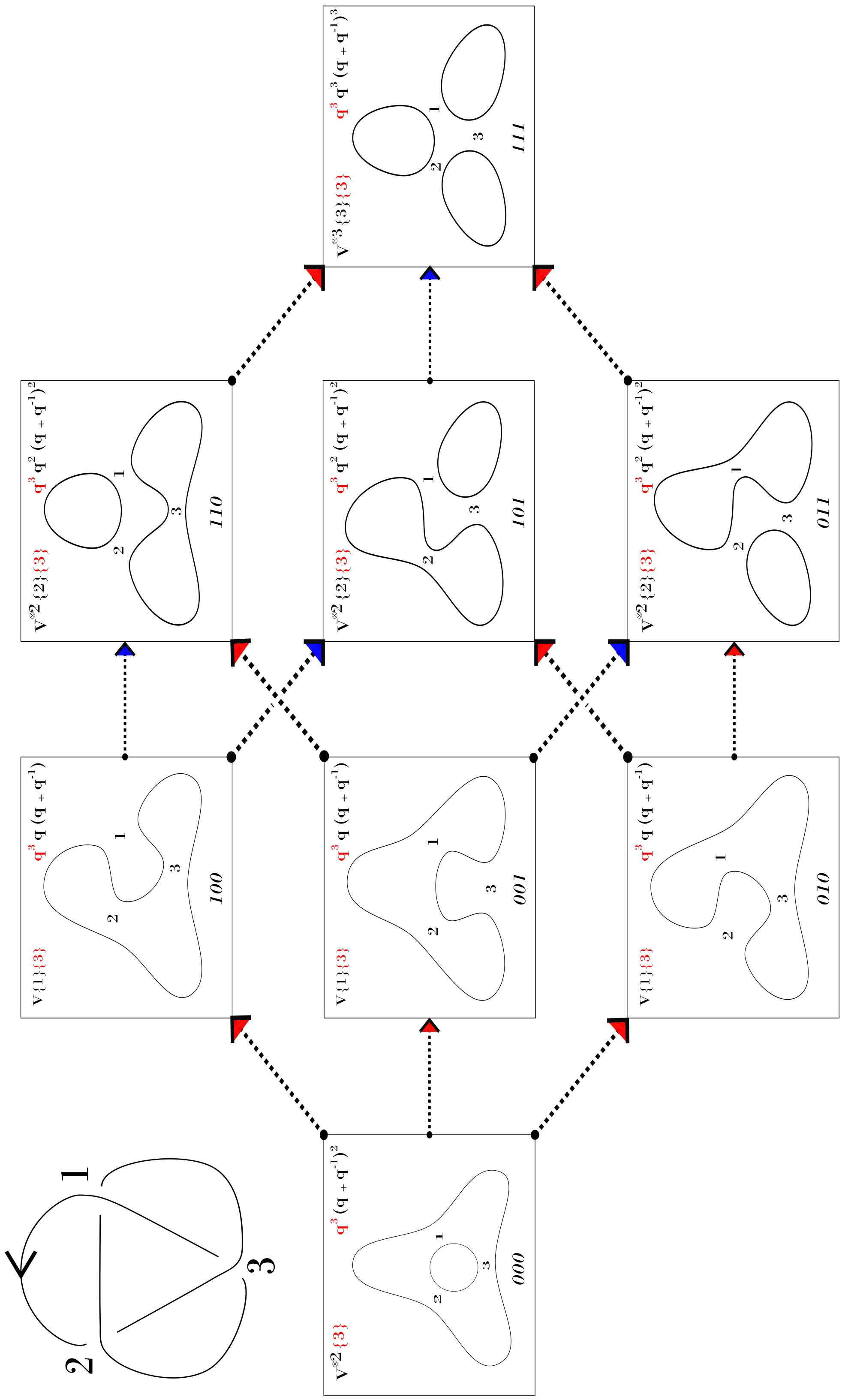


FIGURE 3.1: Cube for trefoil knot

3.4 Graded Euler characteristic of cochain complex

Definition 3.8. Let (C,d) be a cochain complex of graded vector spaces and grading preserving maps. The graded Euler characteristic of (C,d) is defined to be the alternating sum $\sum_r (-1)^r q\text{dim}\, H^r$, where H^r is the r^{th} cohomology of cochain complex (C,d). We denote the graded Euler characteristic of (C,d) by $\chi_q(C)$.

Proposition 3.3. If the degrees of the differential maps of cochain complex (C,d) are zero and all the cochain vector spaces are finite dimensional, then $\chi_q(C,d) = \sum_r (-1)^r q\text{dim}\; C^r$.

Under the hypothesis of the Proposition 3.3, it is true that $\chi_q(C,d)$ only depends on cochain vector spaces. Hence it makes sense to write $\chi_q(C)$.

Proposition 3.4. Let (C,d) be a cochain complex. Then the graded Euler characteristic $\chi_q(C[s])$ is equal to $(-1)^s\chi_q(C)$.

Theorem 3.1. Let D be an oriented diagram of an oriented knot K, then $\chi_q(Kh(D)) = J_K(q)$.

Proof. Definition 3.7 tells that $Kh(D) = [\![D]\!][-n_-]\{n_+ - 2n_-\}$. Now applying Proposition 3.4 we get,

$$\chi_q(Kh(D)) = (-1)^{n_-}\chi_q([\![D]\!]\{n_+ - 2n_-\}).$$

But,

$$\begin{aligned}
q\text{dim}\,[\![D]\!]^r\{n_+ - 2n_-\} &= q\text{dim} \bigoplus_{\{\alpha:|\alpha|=r\}} V_\alpha(D)\{n_+ - 2n_-\} \\
&= q^{n_+-2n_-} \sum_{\{\alpha:|\alpha|=r\}} q\text{dim}\, V_\alpha(D) && \text{[applying Proposition 3.1]} \\
&= q^{n_+-2n_-} \sum_{\{\alpha:|\alpha|=r\}} q\text{dim}\, V^{\otimes|\alpha D|}\{|\alpha|\} && \text{[by definition of } V_\alpha(D)\text{]} \\
&= q^{n_+-2n_-} \sum_{\{\alpha:|\alpha|=r\}} q^{|\alpha|}\, q\text{dim}\, V^{\otimes|\alpha D|} && \text{[applying Proposition 3.1]} \\
&= q^{n_+-2n_-} \sum_{\{\alpha:|\alpha|=r\}} q^{|\alpha|}(q+q^{-1})^{|\alpha D|} && \text{[applying Proposition 3.2]}
\end{aligned}$$

Finally applying the Proposition 3.3, we get

$$\begin{aligned}
\chi_q([\![D]\!]\{n_+ - 2n_-\}) &= \sum_r (-1)^r q\text{dim}\,[\![D]\!]^r\{n_+ - 2n_-\} \\
&= q^{n_+-2n_-} \sum_r (-1)^r \sum_{\{\alpha:|\alpha|=r\}} q^{|\alpha|}(q+q^{-1})^{|\alpha D|} \\
&= q^{n_+-2n_-} \sum_\alpha (-1)^{|\alpha|} q^{|\alpha|}(q+q^{-1})^{|\alpha D|}.
\end{aligned}$$

Finally,

$$\begin{aligned}\chi_q(Kh(D)) =&(-1)^{n_-}\chi_q([\![D]\!]\{n_+ - 2n_-\})\\ =&(-1)^{n_-}q^{n_+-2n_-}\sum_{\alpha}(-1)^{|\alpha|}q^{|\alpha|}(q+q^{-1})^{|\alpha D|}\\ =&J_K(q). \qquad \text{[see Equation (2.3)]}\end{aligned}$$

■

Thus, given a knot K we can think about $\chi_q(K)$, since it is an isotopy invariant of knot.

3.5 Khovanov cohomology

Definition 3.9. Let D be an oriented diagram of an oriented knot/link K, and $H^r(Kh(D))$ denotes the r^{th} cohomology of cochain complex $Kh(D)$. Then, the graded Poincaré polynomial of Khovanov cochain complex for a knot-diagram of K is defined as $\sum_r t^r q\dim H^r(Kh(D))$. We often denote this polynomial as $Kh(K)$.

The above-mentioned definition makes sense if the quantity $\sum_r t^r q\dim H^r(Kh(D))$ remains invariant under Reidemeister moves. The next theorem would ensure that.

Theorem 3.2 (Khovanov [4])**.** The graded dimension of the cohomologies $H^r(K)$ are knot/link invariants, and hence $Kh(K)$, a polynomial in variables t and q, is a knot/link invariant that specializes to the unnormalized Jones polynomial at $t = -1$. The cohomologies are knot invariant up to grading preserving isomorphism.

Theorem 3.2 can be proved independently using the following lemma. We prove it in Chapter 5.

Lemma 3.1. Let C be a cochain complex, and $C' \subset C$ be a sub cochain complex.

- If C' is acyclic, then $H^r(C) \simeq H^r(C/C')$.
- If C/C' is acyclic, then $H^r(C) \simeq H^r(C')$.

Proof of Lemma 3.1. Consider, the short exact sequence,

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C/C' \longrightarrow 0$$

where the inclusion map f and canonical map g are cochain maps. This gives a long exact homology sequence

$$\cdots \longrightarrow H^r(C') \longrightarrow H^r(C) \longrightarrow H^r(C/C') \longrightarrow H^{r+1}(C') \longrightarrow \cdots$$

Hence the lemma follows. ■

3.6 Changing sign in commutative cube

Definition 3.10. Let D be an oriented knot diagram with $N(D)$-crossings, and A be the $N(D)$-cube constructed in the way, as discussed in the Section 3.3. The sign of an arrow is defined to be +1, if it is a positive map and -1, if it is a negative map.

At each face of A, there will be odd number of negative arrows. Let B be an $N(D)$-cube which is same as A, but the sign information of the arrows of B is different from A, in a way such that the number of negative arrows at each face of B be still an odd number. Since all vertices of A and B are identical, we name the vertices of B in the same way as we name the vertices of A (recall that we name the vertices of A with the states of D). Also, we name the arrows of A as d^A_β, and arrows of B as d^B_β. As we have constructed $([\![D]\!], d)$ out of the cube A, we can also construct another co-chain complex $([\![D_B]\!], d_B)$ as follows: $[\![D_B]\!]^r := [\![D]\!]^r$ and $d^r_B := \sum_{|\beta|=r} sign\,(d^B_\beta)\, d^B_\beta$.

Theorem 3.3. With the preceding set-up, $([\![D]\!], d)$ and $([\![D_B]\!], d_B)$ are isomorphic cochain complexes, where the cochain isomorphisms are grading preserving.

Proof. Let v be a vertex of an n-cube X. Consider an (abstract) action of v on X, denoted as $v(X)$, where $v(X)$ is another n-cube which is same as X with the only difference being a change in the sign of the adjacent arrows of v. The action of another vertex v' on $v(X)$ would be written as $v'v(X)$. Thus we have an action of chain of vertices on X. Note that $v'v(X) = vv'(X)$ and $vv(X) = X$. If s be a chain of vertices, then we can exclude those vertices which appear even number of times and get another chain s' such that $s'(X) = s(X)$. Hence, given a chain of vertices s, there exists a chain of distinct vertices $\overline{s}$ such that $s(X) = \overline{s}(X)$.

First we prove this theorem for the case when $N(D) = 2$, i.e. A, B are just faces (2-cubes) and this can be proved by the following steps:

- Let p_A and n_A denotes the number of positive and negative arrows in the cube A, respectively. The tuple (p_A, n_A) can be either (1,3) or (3,1). Now it is easy to check that if we change sign information for odd number of arrows in A, we will get a face, where the number of negative arrows is even. Hence, we can get B from A only by changing sign for even number of arrows.
- Let t be a chain of vertices of A. It can be shown easily that $t(A)$ would differ from A by altering the sign information for even number of arrows in A. Also, one can show that for B there is a chain of vertices t_B such that, $t_B(A) = B$ (when we are writing this we are

also considering the sign information of arrows). We may assume that each vertex in the chain t_B, appears only once.

- Now construct arrows (grading preserving linear maps) from vertex of A to that same corresponding vertex of B, in the following manner. We will take Id_v if v does not appear in t_B or $-Id_v$ if v appears in t_B. It is easy to check that the newly made arrows form a cube map (note that when we say this, the arrows are considered with their signs). Now define ψ^r to be the formal sum of all newly made arrows which are between the vertices of height (state sum) r of the cube A and B. This ψ will give an isomorphism between the cochain complexes $([\![D]\!], d)$ and $([\![D_B]\!], d_B)$.

Now consider the case for $N(D) > 2$. There is a sequence of faces $F_1^A, F_2^A, \ldots, F_{f(D)}^A$, where $f(D)$ is the total number of faces in A, and for each i, F_i^A and F_{i+1}^A are adjacent. We name the faces of B in the same order by $F_1^B, F_2^B, \ldots, F_{f(D)}^B$. As mentioned earlier the only difference between F_i^A and F_i^B is the sign information of the arrows of the face. There will be a chain of distinct vertices s_1, such that, $s_1(F_1^A) = F_1^B$. F_1^A and F_2^A will intersect at two vertices and one arrow. Let v_1, v_2 are those two vertices and d_1^A is the arrow from v_1 to v_2. Let ${d_1}^B$ is the arrow in B, corresponding to d_1^A.

Consider the cases when both v_1 and v_2 appear in s_1 or both of them does not appear in s_1. But, both of these cases arises from two possibilities: either $F_2^A = F_2^B$, or F_2^A differs from F_2^B only by changing sign information for two adjacent arrows. If F_2^A differs from F_2^B only by changing sign information for two adjacent arrows and those two arrows intersect at vertex v, then take a new chain as vs_1 (adding v in s_1) and call it s_1'. But, if $F_2^A = F_2^B$ we would take s_1'=s_1. Now note that $s_1'(F_1^A) = F_1^B$, and $s_1'(F_2^A) = F_2^B$.

Now consider the case when v_1 appears in s_1 but v_2 does not. This case arises from two possibilities: either sign of all arrows of F_2^A differ from F_2^B or sign of only two arrows differ and one of them is d_1^A i.e. sign $(d_1^A) = -$ sign (d_1^B). When sign of all arrows of F_2^A differ from F_2^B, choose the vertex v' which is at the opposite corner of v_1. Now take s_1'= $v's_1$. Note $v's_1(F_1^A) = F_1^B$ and $v's_1(F_2^A) = F_2^B$.

But when sign of only two arrows differ (one of them is d_1^A), then we can choose either one or two vertices of F_2^A other than v_1 and v_2 and consider s_1' by adding those vertices to s_1 such that $s_1'(F_1^A) = F_1^B$ and $s_1'(F_2^A) = F_2^B$.

Note that for all cases, we can find s_1' such that $s_1'(F_1^A) = F_1^B$ and $s_1'(F_2^A) = F_2^B$. Apply this process for third face and so on. If we proceed in this way and keep on adding vertices to s_1 we would get a chain of distinct vertices t_B such that, $t_B(A) = B$.

Finally, using this chain t_B, we define the map between $([\![D]\!], d)$ and $([\![D_B]\!], d_B)$ as we have done for the case $N(D) = 2$. One can check that this will give an grading preserving isomorphism between $([\![D]\!], d)$ and $([\![D_B]\!], d_B)$.

■

Chapter 4

Formal computation for figure eight knot

In this chapter, we will calculate the khovanov cohomology for figure eight knot. Since all differential maps of the Khovanov cochain complex are linear maps, they can be represented by matrices. Kernel of such matrix $M_{m\times n}$ is the solution space of the equation $M_{m\times n}X_{n\times 1} = 0_{m\times 1}$. Image of M is its column space.
Denote the figure eight knot diagram as T. The positive maps are colored with red, and the negative maps with blue. First we associate ordered basis for each graded vector space corresponding to each height of $Kh(T)$. Let V be the graded vector space over $\mathbb{Q}$ spanned by v_+ and v_- as defined in Chapter 3.

$$\begin{aligned} V\otimes V\otimes V = {} & \mathbb{Q}(v_+\otimes v_+\otimes v_+)\bigoplus \mathbb{Q}(v_-\otimes v_+\otimes v_+)\bigoplus \mathbb{Q}(v_+\otimes v_-\otimes v_+)\bigoplus \mathbb{Q}(v_+\otimes v_+\otimes v_-)\bigoplus \\ & \mathbb{Q}(v_-\otimes v_-\otimes v_+)\bigoplus \mathbb{Q}(v_-\otimes v_+\otimes v_-)\bigoplus \mathbb{Q}(v_+\otimes v_-\otimes v_-)\bigoplus \mathbb{Q}(v_-\otimes v_-\otimes v_-) \end{aligned}$$

$$\begin{aligned} & V\otimes V\bigoplus V\otimes V\bigoplus V\otimes V\bigoplus V\otimes V = \\ & \qquad \mathbb{Q}^4(v_+\otimes v_+)\bigoplus \mathbb{Q}^4(v_-\otimes v_+)\bigoplus \mathbb{Q}^4(v_+\otimes v_-)\bigoplus \mathbb{Q}^4(v_-\otimes v_-) \end{aligned}$$

$$\begin{aligned} & V\otimes V\otimes V\bigoplus V\bigoplus V\bigoplus V\bigoplus V\bigoplus V = \\ & \qquad \mathbb{Q}(v_+\otimes v_+\otimes v_+)\bigoplus \mathbb{Q}(v_-\otimes v_+\otimes v_+)\bigoplus \mathbb{Q}(v_+\otimes v_-\otimes v_+)\bigoplus \\ & \qquad \mathbb{Q}(v_+\otimes v_+\otimes v_-)\bigoplus \mathbb{Q}(v_-\otimes v_-\otimes v_+)\bigoplus \mathbb{Q}(v_-\otimes v_+\otimes v_-)\bigoplus \\ & \qquad \mathbb{Q}(v_+\otimes v_-\otimes v_-)\bigoplus \mathbb{Q}(v_-\otimes v_-\otimes v_-)\bigoplus \mathbb{Q}^5(v_+)\bigoplus \mathbb{Q}^5(v_-). \end{aligned}$$

Now if we forget about degree shift, then among the above-mentioned vector spaces, the first one is the Khovanov cochain vector space corresponding to heights -2 and 2, second one is the cochain vector space for heights -1 and 1, and the third one corresponds to the height 0.

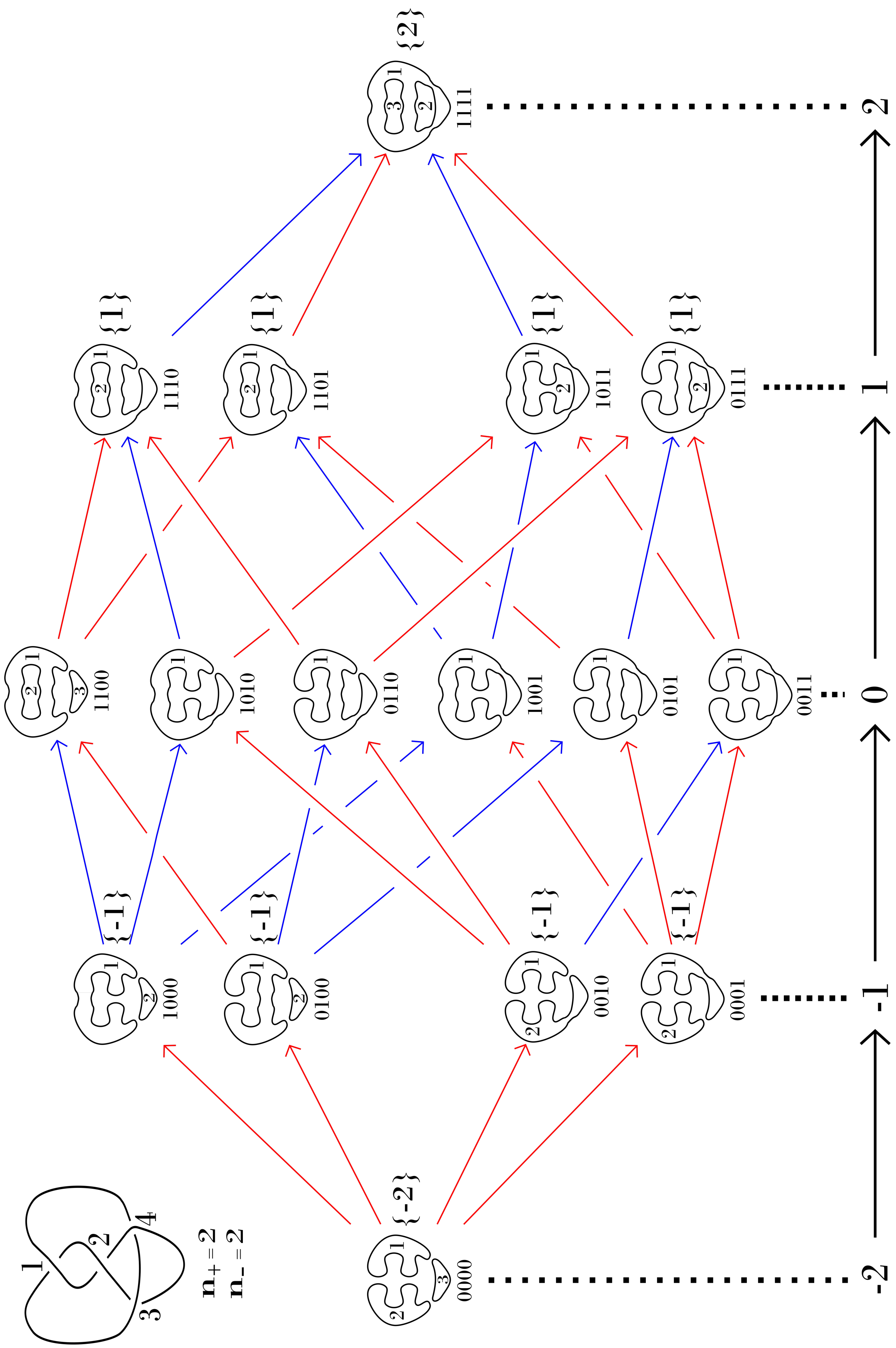


FIGURE 4.1: Cube for figure eight knot

In the above cube diagram, the loops (simple closed curves) are numbered with integers for each smoothing. When we assign graded vector space for each state, we take tensor product of V. Each of these tensor factors V corresponds to each loop. We would take the tensor product in the same order as we numbered the loops. That means the first tensor factor V corresponds to the first loop of the smoothing, and so on.
The differential maps are defined using m and Δ as following.

$$d^{-2}(a_1 \otimes a_2 \otimes a_3) = (m(a_1 \otimes a_2) \otimes a_3,\, m(a_1 \otimes a_2) \otimes a_3,\, a_1 \otimes m(a_2 \otimes a_3),\, m(a_1 \otimes a_3) \otimes a_2)$$

$$\begin{aligned} d^{-1}(b_1 \otimes b_2,\, b_3 \otimes b_4,\, b_5 \otimes b_6,\, b_7 \otimes b_8) = \\ (-\Delta(b_1) \otimes b_2 + \Delta(b_3) \otimes b_4,\, -m(b_1 \otimes b_2) + m(b_5 \otimes b_6), -m(b_3 \otimes b_4) + m(b_5 \otimes b_6), \\ - m(b_1 \otimes b_2) + m(b_7 \otimes b_8), -m(b_3 \otimes b_4) + m(b_7 \otimes b_8),\, -m(b_5 \otimes b_6) + m(b_7 \otimes b_8)) \end{aligned}$$

$$\begin{aligned} d^0(c_1 \otimes c_2 \otimes c_3,\, c_4,\, c_5, c_6,\, c_7,\, c_8) = \\ (m(c_1 \otimes c_3) \otimes u_2 - \Delta(c_4) + \Delta(c_5),\, m(c_1 \otimes c_3) \otimes c_2 - \Delta(c_6) + \Delta(c_7), \\ \Delta(c_4) - \Delta(c_6) + \Delta(c_8),\, \Delta(c_5) - \Delta(c_7) + \Delta(c_8)) \end{aligned}$$

$$\begin{aligned} d^1(d_1 \otimes d_2,\, d_3 \otimes d_4,\, d_5 \otimes d_6,\, d_7 \otimes d_8) = -\Delta(d_1) \otimes d_2 + \Delta(d_3) \otimes d_4 - d_5 \otimes \Delta(d_6) \\ + \Delta_1(d_7) \otimes d_8 \otimes \Delta_2(d_7) \end{aligned}$$

Here $a_i,\, b_j,\, c_k,\, d_l \in V$, for $i, j, k, l \in \mathbb{N}$. $\Delta_1(v)$ denotes the first tensor factor of $\Delta(v)$ and $\Delta_2(v)$ is the second tensor factor where $v \in V$. Now we want to write down the matrix of d^{-2}. For that we just need to know how it acts on the ordered basis.

$$\begin{aligned} &d^{-2}(v_+ \otimes v_+ \otimes v_+) = (v_+ \otimes v_+,\, v_+ \otimes v_+,\, v_+ \otimes v_+,\, v_+ \otimes v_+) \\ &d^{-2}(v_- \otimes v_+ \otimes v_+) = (v_- \otimes v_+,\, v_- \otimes v_+,\, v_- \otimes v_+,\, v_- \otimes v_+) \\ &d^{-2}(v_+ \otimes v_- \otimes v_+) = (v_- \otimes v_+,\, v_- \otimes v_+,\, v_+ \otimes v_-,\, v_+ \otimes v_-) \\ &d^{-2}(v_+ \otimes v_+ \otimes v_-) = (v_+ \otimes v_-,\, v_+ \otimes v_-,\, v_+ \otimes v_-,\, v_- \otimes v_+) \\ &d^{-2}(v_- \otimes v_- \otimes v_+) = (0,\, 0,\, v_- \otimes v_-,\, v_- \otimes v_-) \\ &d^{-2}(v_- \otimes v_+ \otimes v_-) = (v_- \otimes v_-,\, v_- \otimes v_-,\, v_- \otimes v_-,\, 0) \\ &d^{-2}(v_+ \otimes v_- \otimes v_-) = (v_- \otimes v_-,\, v_- \otimes v_-,\, 0,\, v_- \otimes v_-) \\ &d^{-2}(v_- \otimes v_- \otimes v_-) = (0,\, 0,\, 0,\, 0) \end{aligned}$$

Hence the matrix representation of d^{-2} is

$$d^{-2} = \left[\begin{array}{c|c|c|c|c|c|c|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]_{16\times 8}.$$

Consider the linear equation $d^{-2}X = 0$, where $X = (x_1, x_2, \ldots, x_8)^T$. It is just a routine check that the solution space $\operatorname{Ker} d^{-2} = \operatorname{Span}_{\mathbb{Q}}\{(0, 0, .., 1)^T\} = \mathbb{Q}(v_- \otimes v_- \otimes v_-)$. Now let the i-th column vector of d^{-2} be u_i. Hence image space of d^{-2} is $\operatorname{Span}_{\mathbb{Q}}\{u_1, u_2, \ldots, u_8\}$.

$d^{-1}(v_+ \otimes v_+,\ 0,\ 0,\ 0) = (-v_+ \otimes v_- \otimes v_+ - v_- \otimes v_+ \otimes v_+,\ -v_+,\ 0,\ -v_+,\ 0,\ 0)$

$d^{-1}(0,\ v_+ \otimes v_+,\ 0,\ 0) = (v_+ \otimes v_- \otimes v_+ + v_- \otimes v_+ \otimes v_+,\ 0,\ -v_+,\ 0,\ -v_+,\ 0)$

$d^{-1}(0,\ 0,\ v_+ \otimes v_+,\ 0) = (0,\ v_+,\ v_+,\ 0,\ 0,\ -v_+)$

$d^{-1}(0,\ 0,\ 0,\ v_+ \otimes v_+) = (0,\ 0,\ 0,\ v_+,\ v_+,\ v_+)$

$d^{-1}(v_- \otimes v_+,\ 0,\ 0,\ 0) = (-v_- \otimes v_- \otimes v_+,\ -v_-,\ 0,\ -v_-,\ 0,\ 0)$

$d^{-1}(0,\ v_- \otimes v_+,\ 0,\ 0) = (v_- \otimes v_- \otimes v_+,\ 0,\ -v_-,\ 0,\ -v_-,\ 0)$

$d^{-1}(0,\ 0,\ v_- \otimes v_+,\ 0) = (0,\ v_-,\ v_-,\ 0,\ 0,\ -v_-)$

$d^{-1}(0,\ 0,\ 0,\ v_- \otimes v_+) = (0,\ 0,\ 0,\ v_-,\ v_-,\ v_-)$

$d^{-1}(v_+ \otimes v_-,\ 0,\ 0,\ 0) = (-v_+ \otimes v_- \otimes v_- - v_- \otimes v_+ \otimes v_-,\ -v_-,\ 0,\ -v_-,\ 0,\ 0)$

$d^{-1}(0,\ v_+ \otimes v_-,\ 0,\ 0) = (v_+ \otimes v_- \otimes v_- + v_- \otimes v_+ \otimes v_-,\ 0,\ -v_-,\ 0,\ -v_-,\ 0)$

$d^{-1}(0,\ 0,\ v_+ \otimes v_-,\ 0) = (0,\ v_-,\ v_-,\ 0,\ 0,\ -v_-)$

$d^{-1}(0,\ 0,\ 0,\ v_+ \otimes v_-) = (0,\ 0,\ 0,\ v_-,\ v_-,\ v_-)$

$d^{-1}(v_- \otimes v_-,\ 0,\ 0,\ 0) = (-v_- \otimes v_- \otimes v_-,\ 0,\ 0,\ 0,\ 0,\ 0)$

$d^{-1}(0,\ v_- \otimes v_-,\ 0,\ 0) = (v_- \otimes v_- \otimes v_-,\ 0,\ 0,\ 0,\ 0,\ 0)$

$d^{-1}(0,\ 0,\ v_- \otimes v_-,\ 0) = (0,\ 0,\ 0,\ 0,\ 0,\ 0)$

$d^{-1}(0,\ 0,\ 0,\ v_- \otimes v_-) = (0,\ 0,\ 0,\ 0,\ 0,\ 0)$

Hence the matrix representation of d^{-1} is

$$d^{-1} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0
\end{array}\right]_{18\times 16}$$

Finding Ker d^{-1} and Im d^{-1}:

We find the solution of the linear equation $d^{-1}X = 0$, where $X = (x_1, x_2, \ldots, x_{16})^T$. The equation gives us the following relations: $x_1 = x_2 = x_3 = x_4$, $x_5 = x_6$, $x_9 = x_{10}$, $x_{13} = x_{14}$, $x_{11} = x_5 + x_9 - x_7$, and $x_{12} = x_5 + x_9 - x_8$. The general solution of the above-mentioned equation is:

$(x_1, x_1, x_1, x_1, x_5, x_5, x_7, x_8, x_9, x_9, x_5+x_9-x_7, x_5+x_9-x_8, x_{13}, x_{13}, x_{15}, x_{16})^T$, where each x_i is a variable in $\mathbb{Q}$.

Let the i-th column vector of d^{-1} be u'_i. We know that image space of d^{-1} is its column space. Hence, $\operatorname{Im} d^{-1} = \operatorname{Span}_{\mathbb{Q}}\{u'_1, u'_2, \ldots, u'_{16}\}$.

Now we will find a basis of Ker d^{-1}. The general solution indicates that the dimension of Ker d^{-1} is 8. In the general solution if we take $x_1 = 1$ and $x_i = 0$ for $i = 5, 7, 8, 9, 13, 15, 16$ we get one vector in Ker d^{-1}. Now if we take $x_5 = 1$ and rest other x_is to be zero, we would get another vector in Ker d^{-1}, and this vector would be linearly independent of the first solution. Proceeding in this way, we get the basis of Ker d^{-1}.

$$\operatorname{Ker} d^{-1} = \operatorname{Span}_{\mathbb{Q}}\{ \begin{bmatrix} 1\\1\\1\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\1\\1\\0\\0\\0\\0\\1\\1\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\-1\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\-1\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1\\1\\1\\1\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1 \end{bmatrix} \}$$

Note that the first basis vector is u_1, and the second basis vector is u_3. Starting from third to eighth basis element, we name them as $v_1, v_2, \ldots, v_6$ respectively.

Computing H^{-2} and H^{-1}:

We have already calculated that $\operatorname{Ker} d^{-2} = \mathbb{Q}(v_- \otimes v_- \otimes v_-)$, and we know that d^{-3} is a zero map. As a result,

$$H^{-2} = \frac{\operatorname{Ker} d^{-2}}{\operatorname{Im} d^{-3}} = \mathbb{Q}(v_- \otimes v_- \otimes v_-)$$

Since $v_- \otimes v_- \otimes v_- \in V \otimes V \otimes V\{-2\}$, this element is homogeneous element of degree $-3-2=-5$. Hence $q\dim H^{-2} = q^{-5}$.

Now we will calculate H^{-1}. We have seen that $\operatorname{Ker} d^{-1} = \operatorname{Span}_{\mathbb{Q}}\{u_1, u_3, v_1, v_2, \ldots, v_6\}$, and $\operatorname{Im} d^{-2} = \operatorname{Span}_{\mathbb{Q}}\{u_1, \ldots, u_8\}$. The following relations would be helpful to compute homologies.

(i) $\frac{1}{2}(u_6 - u_7 + u_5) = v_5$

(ii) $u_7 - u_6 = v_6 - v_5$

(iii) $\frac{1}{2}(-u_5 + u_6 + u_7) = v_4$

(iv) $u_4 = v_2 + v_3$

(v) $u_2 - u_3 = v_1 + v_2$

$$
\begin{aligned}
H^{-1} = \frac{\operatorname{Ker} d^{-1}}{\operatorname{Im} d^{-2}} &= \frac{\operatorname{Span}_{\mathbb{Q}}\{u_1, u_3, v_1, v_2, \ldots, v_5, v_6\}}{\operatorname{Span}_{\mathbb{Q}}\{u_1, \ldots, u_8\}} \\
&= \frac{\operatorname{Span}_{\mathbb{Q}}\{u_1, u_3, v_1, v_2, \ldots, v_5, v_6 - v_5\}}{\operatorname{Span}_{\mathbb{Q}}\{u_1, \ldots, u_8\}} \\
&= \frac{\operatorname{Span}_{\mathbb{Q}}\{v_1, v_2, v_3\}}{\operatorname{Span}_{\mathbb{Q}}\{u_1, \ldots, u_8\}} && [\text{Since } u_1, u_3, v_4, v_5, v_6 - v_5 \in \operatorname{Im} d^{-2}] \\
&= \frac{\operatorname{Span}_{\mathbb{Q}}\{v_1, v_2, v_2 + v_3\}}{\operatorname{Span}_{\mathbb{Q}}\{u_1, \ldots, u_8\}} \\
&= \frac{\operatorname{Span}_{\mathbb{Q}}\{v_1, v_2\}}{\operatorname{Span}_{\mathbb{Q}}\{u_1, \ldots, u_8\}} && [\text{Since } v_2 + v_3 \in \operatorname{Im} d^{-2}] \\
&= \frac{\operatorname{Span}_{\mathbb{Q}}\{v_1, v_1 + v_2\}}{\operatorname{Span}_{\mathbb{Q}}\{u_1, \ldots, u_8\}} \\
&= \frac{\operatorname{Span}_{\mathbb{Q}}\{v_1\}}{\operatorname{Span}_{\mathbb{Q}}\{u_1, \ldots, u_8\}} && [\text{Since } v_1 + v_2 \in \operatorname{Im} d^{-2}]
\end{aligned}
$$

Now it is easy to check that $v_1 \notin \operatorname{Im} d^{-2}$. If possible let $\sum_{i=1}^{8} \alpha_i u_i = v_1$ for each $\alpha_i \in \mathbb{Q}$. Then a simple calculation will show that $\alpha_i = 0,\ \forall 1 \leq i \leq 8$. So, $H^1 = \operatorname{Span}_{\mathbb{Q}}\{\overline{v_1}\}$, where $\overline{v_1}$ is the coset representative of v_1. But $v_1 = (0,\, 0,\, v_- \otimes v_+ - v_+ \otimes v_-,\, 0,\, 0)$ and this vector lies in $(V \otimes V)\{-1\} \bigoplus (V \otimes V)\{-1\} \bigoplus (V \otimes V)\{-1\} \bigoplus (V \otimes V)\{-1\}$. Therefore, it is a homogeneous element of degree $0 - 1 = -1$ and $q\dim H^{-1} = q^{-1}$.

Now we want to write down the matrix of d^0.

$d^0(v_+ \otimes v_+ \otimes v_+,\, 0,\, 0,\, 0,\, 0,\, 0) = (v_+ \otimes v_+,\, v_+ \otimes v_+,\, 0,\, 0)$

$d^0(v_- \otimes v_+ \otimes v_+,\, 0,\, 0,\, 0,\, 0,\, 0) = (v_- \otimes v_+,\, v_- \otimes v_+,\, 0,\, 0)$

$d^0(v_+ \otimes v_- \otimes v_+,\, 0,\, 0,\, 0,\, 0,\, 0) = (v_+ \otimes v_-,\, v_+ \otimes v_-,\, 0,\, 0)$

$d^0(v_+ \otimes v_+ \otimes v_-,\, 0,\, 0,\, 0,\, 0,\, 0) = (v_- \otimes v_+,\, v_- \otimes v_+,\, 0,\, 0)$

$d^0(v_- \otimes v_- \otimes v_+,\, 0,\, 0,\, 0,\, 0,\, 0) = (v_- \otimes v_-,\, v_- \otimes v_-,\, 0,\, 0)$

$d^0(v_- \otimes v_+ \otimes v_-,\, 0,\, 0,\, 0,\, 0,\, 0) = (0,\, 0,\, 0,\, 0)$

$d^0(v_+ \otimes v_- \otimes v_-,\, 0,\, 0,\, 0,\, 0,\, 0) = (v_- \otimes v_-,\, v_- \otimes v_-,\, 0,\, 0)$

$d^0(v_- \otimes v_- \otimes v_-,\, 0,\, 0,\, 0,\, 0,\, 0) = (0,\, 0,\, 0,\, 0)$

$d^0(0,\, v_+,\, 0,\, 0,\, 0,\, 0) = (-v_+ \otimes v_- - v_- \otimes v_+,\, 0,\, v_+ \otimes v_- + v_- \otimes v_+,\, 0)$

$d^0(0,\, 0,\, v_+,\, 0,\, 0,\, 0) = (v_+ \otimes v_- + v_- \otimes v_+,\, 0,\, 0,\, v_+ \otimes v_- + v_- \otimes v_+)$

$d^0(0,\, 0,\, 0,\, v_+,\, 0,\, 0) = (0,\, -v_+ \otimes v_- - v_- \otimes v_+,\, -v_+ \otimes v_- - v_- \otimes v_+,\, 0)$

$d^0(0,\, 0,\, 0,\, 0,\, v_+,\, 0) = (0,\, v_+ \otimes v_- + v_- \otimes v_+,\, 0,\, -v_+ \otimes v_- - v_- \otimes v_+)$

$d^0(0,\, 0,\, 0,\, 0,\, 0,\, v_+) = (0,\, 0,\, v_+ \otimes v_- + v_- \otimes v_+,\, v_+ \otimes v_- + v_- \otimes v_+)$

$d^0(0,\, v_-,\, 0,\, 0,\, 0,\, 0) = (-v_- \otimes v_-,\, 0,\, v_- \otimes v_-,\, 0)$

$d^0(0,\, 0,\, v_-,\, 0,\, 0,\, 0) = (v_- \otimes v_-,\, 0,\, 0,\, v_- \otimes v_-)$

$d^0(0,\, 0,\, 0,\, v_-,\, 0,\, 0) = (0,\, -v_- \otimes v_-,\, -v_- \otimes v_-,\, 0)$

$d^0(0,\, 0,\, 0,\, 0,\, v_-,\, 0) = (0,\, v_- \otimes v_-,\, 0,\, -v_- \otimes v_-)$

$d^0(0,\, 0,\, 0,\, 0,\, 0,\, v_-) = (0,\, 0,\, v_- \otimes v_-,\, v_- \otimes v_-)$

Hence the matrix representation of d^0 is

$$d^0 = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1
\end{array}\right]_{16\times 18}$$

Finding Ker d^0 and Im d^0: The general solution of the linear equation $d^0X = 0$ is $(0,\ x_2,\ x_3,\ x_3-x_2, x_5,\ x_6,\ x_7,\ x_8,\ x_9,\ x_9-x_3,\ x_{11},\ x_{11}-x_3,\ x_{11}-x_9,\ x_{14},\ x_{14}-x_5-x_7,\ x_{16},\ x_{16}-x_5-x_7,\ x_{16}-x_{14})^T$. Hence Ker d^0 is equal to

$$\mathrm{Span}_{\mathbb{Q}}\{
\begin{bmatrix} 0\\1\\0\\-1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix},
\begin{bmatrix} 0\\0\\1\\1\\0\\0\\0\\0\\0\\-1\\0\\-1\\0\\0\\0\\0\\0\\0 \end{bmatrix},
\begin{bmatrix} 0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\-1\\0\\-1\\0 \end{bmatrix},
\begin{bmatrix} 0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix},
\begin{bmatrix} 0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\-1\\0\\-1\\0 \end{bmatrix},
\begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix},
\begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1\\1\\0\\0\\-1\\0\\0\\0\\0\\0 \end{bmatrix},
\begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\1\\1\\0\\0\\0\\0\\0 \end{bmatrix},
\begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\1\\0\\0\\-1 \end{bmatrix},
\begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\1\\1 \end{bmatrix}
\}.$$

Note that among these 10 above-mentioned basis vectors of $\operatorname{Ker} d^0$, 3rd, 6th, 7th, 8th, 9th, 10th vectors are u'_6, u'_{14}, u'_3, u'_4, u'_7, u'_8 respectively. Let us name the first two basis vectors as v'_1 and v'_2, and the fourth and fifth basis vector as v'_4 and v'_5 respectively.
Consider the matrix of d^0. We name the i-th column vector of d^0 as w_i, $\forall 1 \leq i \leq 18$. So, $\operatorname{Im} d^0 = \operatorname{Span}_{\mathbb{Q}}\{w_1, \ldots, w_{18}\}$.

Computing H^0:
The following relations can be checked very easily.

(i) $v'_1 + v'_2 = u'_2$

(ii) $v'_4 + v'_5 = u'_{10}$

$$\begin{aligned}
H^0 = \frac{\operatorname{Ker} d^0}{\operatorname{Im} d^{-1}} &= \frac{\operatorname{Span}_{\mathbb{Q}}\{v'_1, v'_2, u'_6, v'_4, v'_5, u'_{14}, u'_3, u'_4, u'_7, u'_8\}}{\operatorname{Span}_{\mathbb{Q}}\{u'_1, \ldots, u'_{16}\}} \\
&= \frac{\operatorname{Span}_{\mathbb{Q}}\{v'_1, v'_2, v'_4, v'_5\}}{\operatorname{Span}_{\mathbb{Q}}\{u'_1, \ldots, u'_{16}\}} \\
&= \frac{\operatorname{Span}_{\mathbb{Q}}\{v'_1 + v'_2, v'_2, v'_4, v'_5\}}{\operatorname{Span}_{\mathbb{Q}}\{u'_1, \ldots, u'_{16}\}} \\
&= \frac{\operatorname{Span}_{\mathbb{Q}}\{v'_2, v'_4, v'_5\}}{\operatorname{Span}_{\mathbb{Q}}\{u'_1, \ldots, u'_{16}\}} && [\text{Since } v'_1 + v'_2 \in \operatorname{Im} d^{-1}] \\
&= \frac{\operatorname{Span}_{\mathbb{Q}}\{v'_2, v'_4, v'_4 + v'_5\}}{\operatorname{Span}_{\mathbb{Q}}\{u'_1, \ldots, u'_{16}\}} \\
&= \frac{\operatorname{Span}_{\mathbb{Q}}\{v'_2, v'_4\}}{\operatorname{Span}_{\mathbb{Q}}\{u'_1, \ldots, u'_{16}\}} && [\text{Since } v'_4 + v'_5 \in \operatorname{Im} d^{-1}]
\end{aligned}$$

Now one can prove the following:

(i) $v'_2, v'_4 \notin \operatorname{Im} d^{-1}$

(ii) $v'_2 - \alpha\, v'_4 \notin \operatorname{Im} d^{-1}, \forall \alpha \in \mathbb{Q}$

As a result, the set $\{\overline{v'_2}, \overline{v'_4}\}$ will be a basis of H^0, where $\overline{v'_2}$, $\overline{v'_4}$ are the coset representative of v'_2 and v'_4 respectively. Observe that $v'_2 = (v_- \otimes v_+ \otimes v_+ + v_+ \otimes v_- \otimes v_+,\ 0,\ -v_+,\ 0,\ -v_+,\ 0)$ and $v'_4 = (v_- \otimes v_+ \otimes v_-,\ 0,\ 0,\ 0,\ 0,\ 0)$ and they lie in $V \otimes V \otimes V \bigoplus V \bigoplus V \bigoplus V \bigoplus V \bigoplus V$. Hence v'_2 and v'_4 are homogeneous elements of degree 1 and -1 respectively. So, $q\dim H^0 = (q + q^{-1})$.
Now we want to construct the matrix of d^1.
$d^1(v_+ \otimes v_+,\ 0,\ 0,\ 0) = -v_+ \otimes v_- \otimes v_+ - v_- \otimes v_+ \otimes v_+$
$d^1(0,\ v_+ \otimes v_+,\ 0,\ 0) = v_+ \otimes v_- \otimes v_+ + v_- \otimes v_+ \otimes v_+$
$d^1(0,\ 0,\ v_+ \otimes v_+,\ 0) = -v_+ \otimes v_+ \otimes v_- - v_+ \otimes v_- \otimes v_+$
$d^1(0,\ 0,\ 0,\ v_+ \otimes v_+) = v_+ \otimes v_+ \otimes v_- + v_- \otimes v_+ \otimes v_+$
$d^1(v_- \otimes v_+,\ 0,\ 0,\ 0) = -v_- \otimes v_- \otimes v_+$

$d^1(0,\, v_- \otimes v_+,\, 0,\, 0) = v_- \otimes v_- \otimes v_+$
$d^1(0,\, 0,\, v_- \otimes v_+,\, 0) = -v_- \otimes v_+ \otimes v_- - v_- \otimes v_- \otimes v_+$
$d^1(0,\, 0,\, 0,\, v_- \otimes v_+) = v_- \otimes v_+ \otimes v_-$
$d^1(v_+ \otimes v_-,\, 0,\, 0,\, 0) = -v_+ \otimes v_- \otimes v_- - v_- \otimes v_+ \otimes v_-$
$d^1(0,\, v_+ \otimes v_-,\, 0,\, 0) = v_+ \otimes v_- \otimes v_- + v_- \otimes v_+ \otimes v_-$
$d^1(0,\, 0,\, v_+ \otimes v_-,\, 0) = -v_+ \otimes v_- \otimes v_-$
$d^1(0,\, 0,\, 0,\, v_+ \otimes v_-) = v_+ \otimes v_- \otimes v_- + v_- \otimes v_- \otimes v_+$
$d^1(v_- \otimes v_-,\, 0,\, 0,\, 0) = -v_- \otimes v_- \otimes v_-$
$d^1(0,\, v_- \otimes v_-,\, 0,\, 0) = v_- \otimes v_- \otimes v_-$
$d^1(0,\, 0,\, v_- \otimes v_-,\, 0) = -v_- \otimes v_- \otimes v_-$
$d^1(0,\, 0,\, 0,\, v_- \otimes v_-) = v_- \otimes v_- \otimes v_-$
Hence the matrix representation of d^1 is

$$d^1 = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1
\end{array}\right]_{8\times 16}$$

Finding Ker d^1 and Im d^1:

Let y_i be the i-th column vector of the matrix d^1. Then $\operatorname{Im} d^1 = \operatorname{Span}_{\mathbb{Q}}\{y_{1,16}\}$. It is important to note the following things.

(i) $v_- \otimes v_- \otimes v_-,\ v_+ \otimes v_- \otimes v_-,\ v_- \otimes v_+ \otimes v_-,\ v_- \otimes v_- \otimes v_+ \in \operatorname{Im} d^1$.

(ii) $y_2 + y_3 + y_4 = 2v_- \otimes v_+ \otimes v_+$. This implies $v_- \otimes v_+ \otimes v_+ \in \operatorname{Im} d^1$.

(iii) Since $d^1(0,\, v_+ \otimes v_+,\, 0,\, 0) = v_+ \otimes v_- \otimes v_+ + v_- \otimes v_+ \otimes v_+$, and $v_- \otimes v_+ \otimes v_+ \in \operatorname{Im} d^1$, hence $v_+ \otimes v_- \otimes v_+ \in \operatorname{Im} d^1$.

(iv) Since $d^1(0,\, 0,\, v_+ \otimes v_+,\, 0) = -v_+ \otimes v_+ \otimes v_- - v_+ \otimes v_- \otimes v_+$ and $v_+ \otimes v_- \otimes v_+ \in \operatorname{Im} d^1$, hence $v_+ \otimes v_+ \otimes v_- \in \operatorname{Im} d^1$.

(v) $v_+ \otimes v_+ \otimes v_+ \notin \operatorname{Im} d^1$.

The general solution of the linear equation $d^1 X = 0$ is
$(x_1,\, x_1,\, 0,\, 0,\, x_5,\, x_6,\, x_7,\, x_8,\, x_9,\, x_7 + x_9 - x_8,\, x_5 - x_6 + 2x_7 - x_8,\, x_5 + x_7 - x_6,\, x_{13},\, x_{14},\, x_{15},\, x_{13} + x_{15} - x_{14})^T$, where each x_i is a variable in $\mathbb{Q}$.

Finding the basis elements we can write,

$$\ker d^1 = \mathrm{Span}_{\mathbb{Q}}\{\begin{bmatrix}1\\1\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\0\\0\\0\\1\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\\0\\0\\-1\\-1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\0\\0\\1\\2\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\\0\\-1\\-1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\\1\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\1\\1\end{bmatrix}\}$$

Note that among these 9 above-mentioned basis vectors 1st, 6th, 7th, 8th and 9th vectors are w_1, w_3, w_{15}, w_{17}, and w_{18} respectively. We name the 2nd, 3rd, 4th, and 5th vectors as p_1, p_2, p_3, and p_4 respectively.

Computing H^1:

The following relations are easy to check.

(i) $p_2 + p_3 = -w_{11}$

(ii) $p_1 + p_2 = w_2$

(iii) $p_1 + p_4 = w_{10} - w_3$

$$\begin{aligned} H^1 = \frac{\mathrm{Ker}\, d^1}{\mathrm{Im}\, d^0} &= \frac{\mathrm{Span}_{\mathbb{Q}}\{w_1, p_1, p_2, p_3, p_4, w_3, w_{15}, w_{17}, w_{18}\}}{\mathrm{Span}_{\mathbb{Q}}\{w_1, \ldots, w_{18}\}} \\ &= \frac{\mathrm{Span}_{\mathbb{Q}}\{p_1, p_2, p_3, p_4\}}{\mathrm{Span}_{\mathbb{Q}}\{w_1, \ldots, w_{18}\}} \\ &= \frac{\mathrm{Span}_{\mathbb{Q}}\{p_1, p_2, p_2 + p_3, p_4\}}{\mathrm{Span}_{\mathbb{Q}}\{w_1, \ldots, w_{18}\}} \\ &= \frac{\mathrm{Span}_{\mathbb{Q}}\{p_1, p_2, p_4\}}{\mathrm{Span}_{\mathbb{Q}}\{w_1, \ldots, w_{18}\}} \qquad [\text{Since } p_2 + p_3 \in \mathrm{Im}\, d^0] \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathrm{Span}_{\mathbb{Q}}\{p_1, p_1+p_2, p_4\}}{\mathrm{Span}_{\mathbb{Q}}\{w_1,\ldots,w_{18}\}} \\
&= \frac{\mathrm{Span}_{\mathbb{Q}}\{p_1, p_4\}}{\mathrm{Span}_{\mathbb{Q}}\{w_1,\ldots,w_{18}\}} && [\text{Since } p_1+p_2 \in \mathrm{Im}\, d^0] \\
&= \frac{\mathrm{Span}_{\mathbb{Q}}\{p_1, p_1+p_4\}}{\mathrm{Span}_{\mathbb{Q}}\{w_1,\ldots,w_{18}\}} \\
&= \frac{\mathrm{Span}_{\mathbb{Q}}\{p_1\}}{\mathrm{Span}_{\mathbb{Q}}\{w_1,\ldots,w_{18}\}} && [\text{Since } p_1+p_4 \in \mathrm{Im}\, d^0]
\end{aligned}
$$

Now it is easy to check that $p_1 \notin \mathrm{Span}_{\mathbb{Q}}\{w_1,\ldots,w_{18}\}$. As a result H^1 is one dimensional and it is spanned by $\overline{p_1}$, where $\overline{p_1}$ is the coset representative of p_1. Observe that $p_1 = (v_-\otimes v_+,\ 0,\ v_+\otimes v_-,\ v_+\otimes v_-)$ and it lies in $(V\otimes V)\{1\} \bigoplus (V\otimes V)\{1\} \bigoplus (V\otimes V)\{1\} \bigoplus (V\otimes V)\{1\}$. Hence it is a homogeneous element of degree $0+1=1$. So, $q\dim H^1 = q^1$.

Since we have seen earlier that, among the vectors of the ordered basis of $V\otimes V\otimes V$, all except $v_+\otimes v_+\otimes v_+$ lie in $\mathrm{Im}\, d^1$. Hence $H^2 = \frac{\mathrm{Ker}\, d^2}{\mathrm{Im}\, d^1} = \frac{V\otimes V\otimes V}{\mathrm{Im}\, d^1} = \frac{\mathrm{Span}_{\mathbb{Q}}\{v_+\otimes v_+\otimes v_+\}}{\mathrm{Im}\, d^1} = \overline{v_+\otimes v_+\otimes v_+}$, where $\overline{v_+\otimes v_+\otimes v_+}$ denotes the coset representative of the vector $v_+\otimes v_+\otimes v_+$. But this vector $v_+\otimes v_+\otimes v_+$ actually lies in $(V\otimes V\otimes V)\{2\}$, therefore, it is a homogeneous element of degree $3+2=5$. So, $q\dim H^2 = q^5$.

Graded Poincaré polynomial: The graded Poincaré polynomial of Khovanov cochain complex for any figure eight knot diagram is $t^{-2}q^{-5} + t^{-1}q^{-1} + t^0(q+q^{-1}) + t^1q^1 + t^2q^5$.

Chapter 5

Khovanov Cohomology for Tangles

5.1 Cobordisms

Definition 5.1. A $(n+1)$-dimensional cobordism between two n-dimensional manifolds M and N is a $(n+1)$-dimensional compact manifold W, such that the boundary $\partial W = M \sqcup N$.
A cobordism C between two smoothings (consider one of them as head smoothing and another as tail smoothing) of knots/links is an oriented surface embedded in $\mathbb{R}^2 \times [0,1]$, whose boundary lies in $\mathbb{R}^2 \times \{0,1\}$ and whose top boundary (i.e. the part of the boundary lies in $\mathbb{R}^2 \times \{0\}$ denoted as $\partial_0 C$) is the head smoothing and bottom boundary (i.e. the part of the boundary lies in $\mathbb{R}^2 \times \{1\}$ denoted as $\partial_1 C$) is the tail smoothing. We call this as cobordism from the head smoothing to the tail smoothing.

A tangle is a part of a knot/link diagram bounded inside a disk. Whenever we draw a tangle, we also draw the circle (using dotted line) to show the boundary of the disk. The tangle will intersect the circle at even number of points. These points are called boundary points of the tangle and the collection of these points is called the boundary of the tangle.
Like knot/link, we can also define smoothing of a tangle (corresponding to each state of that tangle) but the only difference is that it will be always inside a disk and the smoothing will intersect the circle (boundary of the disk) at even number points. The collection of these points are called the boundary of the smoothing.

Definition 5.2. A cobordism C' between two smoothings (consider one of them as head smoothing and another as tail smoothing) of tangles having $2k$ boundary points is an oriented surface inside a cylinder such that its boundary is the union of the two smoothings and $2k$ vertical straight lines connecting boundary points of head smoothing and tail smoothing. Note that, such surfaces do not follow the actual definition of cobordism between manifolds.

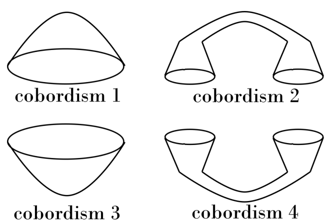
Two cobordisms C_1 and C_2 between two smoothings of knots/links are said to be equivalent if the surfaces C_1 and C_2 are homeomorphic via a homeomorphism that extends the identification $\partial_0 C_1 \cong \partial_0 C_2$ and $\partial_1 C_1 \cong \partial_1 C_2$. Similarly, two cobordisms C_1' and C_2' between two smoothings of tangles are said to be equivalent if they are homoeomorphic via a homeomorphism that extends the identification $\partial_0 C_1' \cong \partial_0 C_2'$, $\partial_1 C_1' \cong \partial_1 C_2'$ and the identification of the corresponding vertical straight lines as well.

5.2 Cob^3 Category

Definition 5.3. $Cob^3(\phi)$ is the category whose objects are collection of simple closed curves in plane (smoothings of knots/links) and the morphisms are cobordisms between two such smoothings up to the equivalence relation mentioned in the section 5.1.
If B is a set of even number of points on a circle, then $Cob^3(B)$ is a category whose objects are smoothings of tangles with boundary B and whose morphisms are cobordisms between such smoothings up to the equivalence relation mentioned in the section 5.1. Similarly $Cob^3(2k)$ is a category whose objects are smoothings of tangles with $2k$-boundary points and whose morphisms are cobordisms between such smoothings up to equivalence, where k is a natural number.

For each above-mentioned case, composition of two morphism $[C_1]$ and $[C_2]$ is defined as: $[C_2] \circ [C_1] = [C]$ where C is the surface that one can get by placing C_2 at the bottom of C_1, where by $[F]$ we represent the equivalence class of the cobordism F. It is easy to show that this composition map is well-defined (since one can extend the homeomorphism using pasting lemma). Clearly this composition map is associative. For a smoothing S, the identity morphism Id_S is $[I]$, where I is the surface $S \times [0,1]$. Hence for each above-mentioned case, Cob^3 is indeed a category.

Remark 5.1. Let's modify the category $Cob^3(\phi)$ by including the empty set as an object and including the cobordisms between empty set and finite disjoint circles as morphisms. One can consider the followings as examples of such cobordisms. The first one is cobordism from empty set to a single loop, second one is from empty set to two loops, third one is from single loop to empty set and fourth one is from two loops to empty set.



Then $Cob^3(\phi)$ will be a monoidal category, where the bi-functor tensor product is defined as follows: for a pair of smoothings S_1 and S_2, $S_1 \otimes S_2$ is the disjoint union of S_1 and S_2 and for two morphisms $[C_1]$ and $[C_2]$, $[C_1] \otimes [C_2]$ is the equivalence class of disjoint union of C_1 and C_2 with appropriate ordering.

Definition 5.4. A pre-additive category is a category C in which the sets of morphisms between any two given objects has the structure of additive abelian group and the composition maps are distributive in the following sense:

$$(f+g) \circ h = f \circ h + g \circ h$$

$$h \circ (f+g) = h \circ f + h \circ g$$

when $f, g, h \in \mathrm{Mor}(C)$ and all composition maps make sense.

Given a category C, we can construct another category C' in a natural way, so that C' be a pre-additive category. If C is pre-additive, then we take $C' = C$. If it is not a pre-additive category, then we extend every $\mathrm{Mor}(O, O')$ by allowing $\mathbb{Z}$-linear combinations of original morphisms to it. Hence, the extended $\mathrm{Mor}(O, O')$ is the free abelian group generated by the original morphisms of $\mathrm{Mor}(O, O')$. We define the composition maps as follows:

$$\sum_i n_i f_i \circ \sum_j m_j g_j = \sum_{i,j} n_i m_j (f_i \circ g_j)$$

where $n_i, m_j \in \mathbb{Z}$ and f_i, g_j are morphisms of the category C such that the composition map makes sense. If $f \in \mathrm{Mor}(O, O')$, we can talk about negative of morphism f. We may consider $-f$ to be $(-1) \times f$. Similarly we can declare the zero morphism $\in \mathrm{Mor}(O, O')$ to be $0 \times f$. From now on we will consider Cob^3 as a pre-additive category.

Definition 5.5. Given a pre-additive category C, now we will define $\mathrm{Mat}(C)$ as follows:

- Objects of $\mathrm{Mat}(C)$ are just "formal direct sum" (possibly empty) $\bigoplus_{i=1}^{n} O_i$. One can think of the formal direct sum as the formal tuple $(O_1, \ldots, O_n)$ of objects O_i of C.
- If $O = (O_1, \ldots, O_n)$, $O' = (O'_1, \ldots, O'_m)$, then morphism $F : O \to O'$ in $\mathrm{Mat}(C)$ will be $m \times n$ matrix $F = [F_{ij}]_{ij}$, where $F_{ij} : O_j \to O'_i$ are morphisms in C.
- Morphisms in $\mathrm{Mat}(C)$ are added using matrix addition.
- Compositions of morphisms in $\mathrm{Mat}(C)$ are defined by a rule modeled on matrix multiplication as following:
$$((F_{ij}) \circ (G_{jk}))_{ik} := \sum_j F_{ij} \circ G_{jk}$$

$\mathrm{Mat}(C)$ is called **the additive closure of** C. For a finite collection of objects $\{O_i\}_{i=1}^n$ in C the tuple $(O_1, \ldots, O_n)$ together with morphisms $\pi_k : (O_1, \ldots, O_n) \to O_k$ and $i_k : O_k \to (O_1, \ldots, O_n)$, where

$$\pi_k = (0, \ldots, Id_{O_k}, \ldots, 0)$$

and

$$i_k = (0, \dots, Id_{O_k}, \dots, 0)^T$$

is a bi-product of objects $O_1, \dots, O_n$, such that $((O_1, \dots, O_n), \pi_k)$ is product and $((O_1, \dots, O_n), i_k)$ is coproduct. If we allow empty collection of formal direct sum we get zero object.

Definition 5.6. Given an additive category C, $\mathrm{Kom}(C)$ is a category whose objects are cochain complexes (of finite length) over C, such as

$$\cdots \longrightarrow \Omega^{r-1} \xrightarrow{d^{r-1}} \Omega^r \xrightarrow{d^r} \Omega^{r+1} \longrightarrow \cdots$$

for which $d^r \circ d^{r-1} = 0$, $\forall r \in \mathbb{Z}$ and morphisms $F : (\Omega_a, d_a) \to (\Omega_b, d_b)$ are cochain morphisms such that the following diagram commutes

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \Omega_a^{r-1} & \xrightarrow{d_a^{r-1}} & \Omega_a^r & \xrightarrow{d_a^r} & \Omega_a^{r+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle F^{r-1}} & & \downarrow{\scriptstyle F^r} & & \downarrow{\scriptstyle F^{r+1}} & & \\
\cdots & \longrightarrow & \Omega_b^{r-1} & \xrightarrow{d_b^{r-1}} & \Omega_b^r & \xrightarrow{d_b^r} & \Omega_b^{r+1} & \longrightarrow & \cdots
\end{array}$$

where all arrows are morphisms in C. Composition of morphisms $F : (\Omega_a, d_a) \to (\Omega_b, d_b)$ and $G : (\Omega_b, d_b) \to (\Omega_c, d_c)$ in $\mathrm{Kom}(C)$ is defined as $(G \circ F)^r := G^r \circ F^r$.

Definition 5.7. Let (Ω_a, d_a) and (Ω_b, d_b) be two cochain complexes in $\mathrm{Kom}(C)$. Let $F, G : \Omega_a \to \Omega_b$ be two morphisms in $\mathrm{Kom}(C)$. We say that F, G are homotopic (often written as $F \sim G$), if there exists backward diagonals $h^r : \Omega_a^r \to \Omega_b^{r-1}$ such that $F^r - G^r = h^{r+1} \circ d_a^r + d_b^{r-1} \circ h^r$, $\forall r \in \mathbb{Z}$.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \Omega_a^{r-1} & \xrightarrow{d_a^{r-1}} & \Omega_a^r & \xrightarrow{d_a^r} & \Omega_a^{r+1} & \longrightarrow & \cdots \\
 & & {\scriptstyle F^{r-1}}\downdownarrows{\scriptstyle G^{r-1}} & \swarrow^{h^r} & {\scriptstyle F^r}\downdownarrows{\scriptstyle G^r} & \swarrow^{h^{r+1}} & {\scriptstyle F^{r+1}}\downdownarrows{\scriptstyle G^{r+1}} & & \\
\cdots & \longrightarrow & \Omega_b^{r-1} & \xrightarrow{d_b^{r-1}} & \Omega_b^r & \xrightarrow{d_b^r} & \Omega_b^{r+1} & \longrightarrow & \cdots
\end{array}$$

Using the fact that C is an additive category (so pre-additive as well) the following proposition can be verified easily,

Proposition 5.1. Consider the homotopy relation "$\sim$" between two morphisms in $\mathrm{Kom}(C)$ as defined above.

- "$\sim$" is an equivalence relation.
- If F, G and H are cochain morphisms so that the compositions $F \circ H$ and $G \circ H$ make sense, then $F \sim G$ implies $F \circ H \sim G \circ H$.

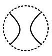
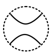
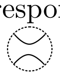
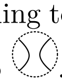
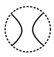
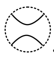
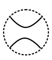
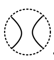
- If F, G and H are cochain morphisms so that the compositions $H \circ F$ and $H \circ G$ make sense, then $F \sim G$ implies $H \circ F \sim H \circ G$.

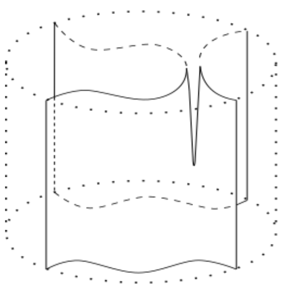
We say that (Ω_a, d_a) and (Ω_b, d_b) in $\text{Kom}(C)$ are cochain homotopic equivalent if there are cochain morphisms $F_1 : \Omega_a \to \Omega_b$ and $F_2 : \Omega_b \to \Omega_a$ such that $F_1 \circ F_2$ and $F_2 \circ F_1$ are cochain homotopic to the Identity morphisms of Ω_b and Ω_a respectively. The morphisms F_1 and F_2 are called homotopy equivalence morphisms. It is easy to verify that this indeed gives an equivalence relation on cochain complexes. The cochain morphisms F_1 and F_2 are called the cochain homotopy equivalence morphisms between (Ω_a, d_a) and (Ω_b, d_b).

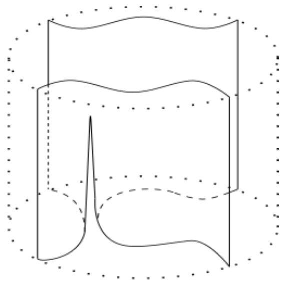
$\text{Kom}_{/h}(C)$ denotes the category which is $\text{Kom}(C)$ modulo homotopies i.e. two objects (Ω_a, d_a) and (Ω_b, d_b) be considered as same object in $\text{Kom}_{/h}(C)$ if they are homotopic equivalent and similarly homotopic cochain morphisms are declared to be same.

5.3 Cube and formal Khovanov bracket for tangles

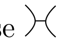
We can define the state, smoothing (0 and 1-smoothings) for a tangle diagram T (having $2k$ boundary points) in the same way as we did for knots/links [see the Section 2.2]. Hence, we can also construct the cube for the tangle diagram T in the same way we did for knots [see the section 3.3]. The vertices of the cube are the smoothings corresponding to states of T. Whenever two states α_1 and α_2 differ at only one position, we draw an arrow in the direction of the state with bigger state sum. We label each arrow by a tuple of $\{0, 1, \star\}$ and define the positive and negative arrows in the same way as we did in the section 3.3, i.e. the arrow is positive if the number of 1s before $\star$ in β is even, and otherwise it is negative. Recall that in chapter 3, we define the arrows using linear maps m and Δ. But in this new set up, we define the arrows using saddle cobordisms.

Definition 5.8. Let α_1 and α_2 are two states of a tangle diagram T which differ only at the k-th crossing. Moreover assume that α_2 is the state with bigger state sum. Consider an open disk neighborhood U of the k-th crossing of diagram T. Note that inside this U, the parts of the smoothings corresponding to α_1 and α_2 can differ only by two ways. They differ either from  to  or from  to . Among the following surfaces the left-surface is defined to be the local saddle cobordism (i.e. inside a cylinder) from  to , and the right-surface is the local saddle cobordism from  to .





Let L_i be the smoothing with respect to α_i, for $i = 1, 2$, and S be a surface totally embedded in $\mathbb{R}^2 \times [0, 1]$, such that inside the region $\overline{U} \times [0, 1]$, it is local saddle cobordism and outside of that region it is $(L_1 \backslash \overline{U}) \times [0, 1]$. The surface S is called the saddle cobordism from L_1 to L_2.

To construct the cube for T, we take the arrow from the smoothing corresponding to α_1 to the smoothing corresponding to α_2 as $\pm[S]$ depending on whether the arrow is positive or negative. We consider $+[S]$ for positive arrow and $-[S]$ for negative arrow. Note that $\pm[S]$ makes sense, if we consider the $Cob^3(2k)$ as a pre-additive category. Thus we get a cube where all vertices are objects of the category $Cob^3(2k)$ and all arrows are morphisms of $Cob^3(2k)$.
Notation: In this text, we use  as a notation to imply saddle cobordism from  to .

Definition 5.9. For a given tangle diagram T, the formal Khovanov bracket $[\![T]\!]'$ is a cochain complex in $\text{Kom}\Big(\text{Mat}(Cob^3(\partial T))\Big)$, where $[\![T]\!]'^{r-n_-}$ is defined to be the formal direct sum of all the smoothings corresponding to the states of state sum r. The differentials are defined in the following manner. Let $\{O_j\}_{j=1}^{n}$ be the collection of smoothings corresponding to the sates of state sum r and $\{O'_i\}_{i=1}^{m}$ be the collection of smoothings corresponding to the states of state sum $r+1$. The differential map d^{r-n_-} from $\bigoplus_{j=1}^{n} O_j$ to $\bigoplus_{i=1}^{m} O'_i$ is defined to be the $m \times n$ matrix $[F_{ij}]_{ij}$, where F_{ij} is the zero morphism when there is no arrow between O_j and O'_i in the cube of T, otherwise it is the saddle morphism (with appropriate sign) associated to the arrow from O_j to O'_i.

Below is the cube of a tangle diagram with 2 crossings. The positive and negative morphisms are colored with red and blue respectively.

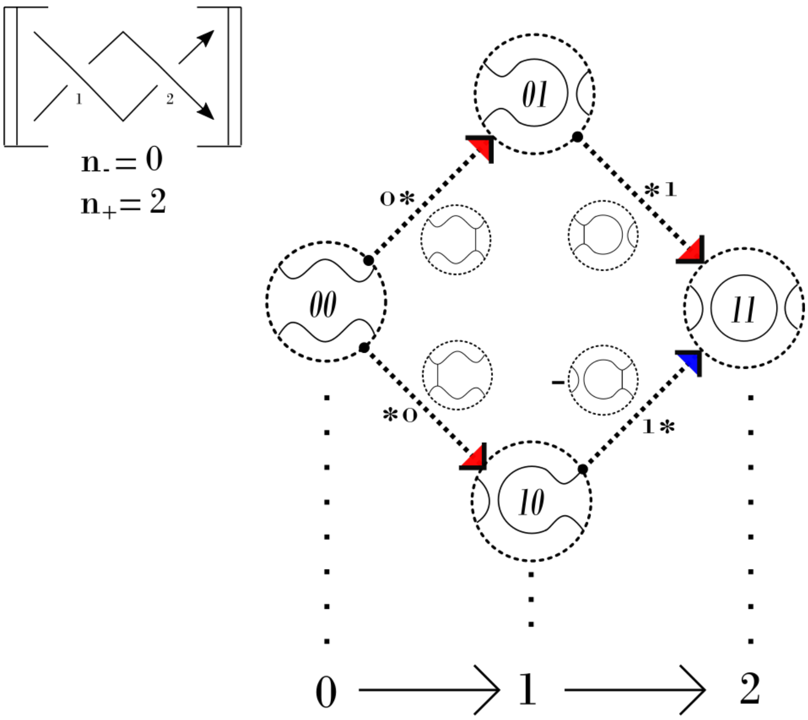


FIGURE 5.1: Cube for a tangle with 2-crossings

Proposition 5.2. For any tangle (or knot/link) diagram T the cochain $[\![T]\!]'$ is a complex in $\text{Kom}\Big(\text{Mat}(Cob^3(\partial T))\Big)$. That is $d^r \circ d^{r-1}$ is 0, $\forall r \in \mathbb{Z}$.

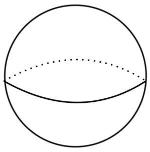
Proof. Consider the cube that can be constructed from the tangle diagram T as mentioned earlier. It is sufficient to show that every face of the cube anti-commutes. But one can easily check that each face of the cube has odd number of negative morphisms (arrows). Hence, if we forget about the signs, we just need to show that each face commutes. Consider the pair of states α_1 and α_2, which differ only at two positions. Without loss of generality, assume that α_2 is the state with bigger state sum. There will be exactly two paths in the cube to go from the vertex corresponding to α_1 to that of α_2, and each path is composition of two arrows. Considering these compositions of cobordisms, it is clear that for both of these paths the corresponding cobordisms are surfaces having two saddle points in two different regions and they are equivalent. ∎

5.4 The invariance theorem

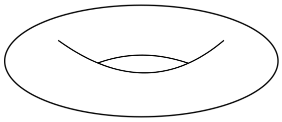
To define $Cob^3_{/l}$ we first define S, T and $4Tu$ relations on the morphisms of Cob^3.

S **relation:**

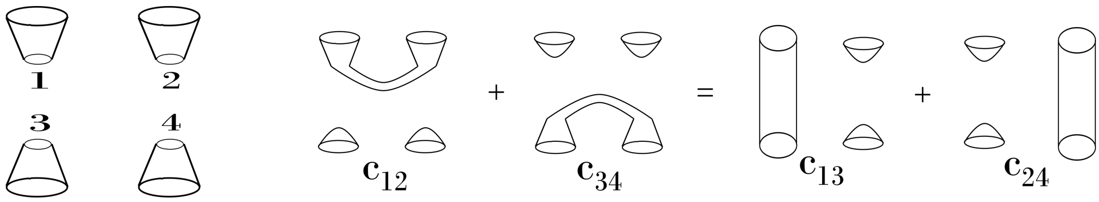
Whenever a cobordism contains a sphere S^2 as one connected component, we consider it as zero morphism.

 $= 0$

T relation:
If a cobordism is disjoint union of a torus and another cobordism C we consider it $2 \otimes C$ or $2C$ (considering that Cob^3 is pre-additive and monoidal category).

 $= 2$

$4Tu$ relation:
Consider a cobordism C and assume that it's intersection with the boundary of certain open ball is union of 4 disks (named as D_1 through D_2). Let C_{ij} be the result of removing that open ball from C and joining D_i and D_j by a tube and capping the other two disks off by attaching disks with them. Then the $4Tu$ relation is $C_{12} + C_{34} = C_{13} + C_{24}$.

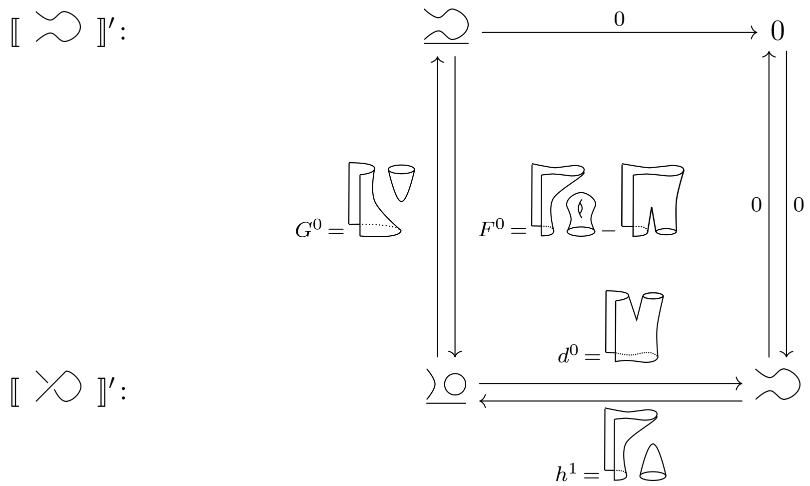


Now we mod out the morphisms of Cob^3 by the three above-mentioned relations. Since these relations are local relations, either we take composition of two cobordism and then apply these relation or, we first apply these relations on each cobordism and then take their composition, we get the same result. The new category $Cob^3_{/l}$ is Cob^3 modulo the local relations S, T, and $4Tu$.

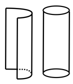
Theorem 5.1 (The Invariance Theorem [6])**.** Given a tangle diagram T, the cochain homotopic equivalence class of the complex $[\![T]\!]'$ regarded in $\text{Kom}_{/h}\Big(\text{Mat}\big(Cob^3_{/l}(\partial T)\big)\Big)$ is an invariant of the tangle T. That is, it does not depend on the ordering of the layers of a cube as column vectors and on the crossings and it is invariant under the three Reidemeister moves.

Proof. In this section, we give the proof only for the local tangles representing the three Reidemeister moves.
Notation: In each of the following cochain complexes the height zero objects will be underlined and we will use the notation  to denote the saddle cobordism from  to . The cup and cap will be denoted by  and  respectively.

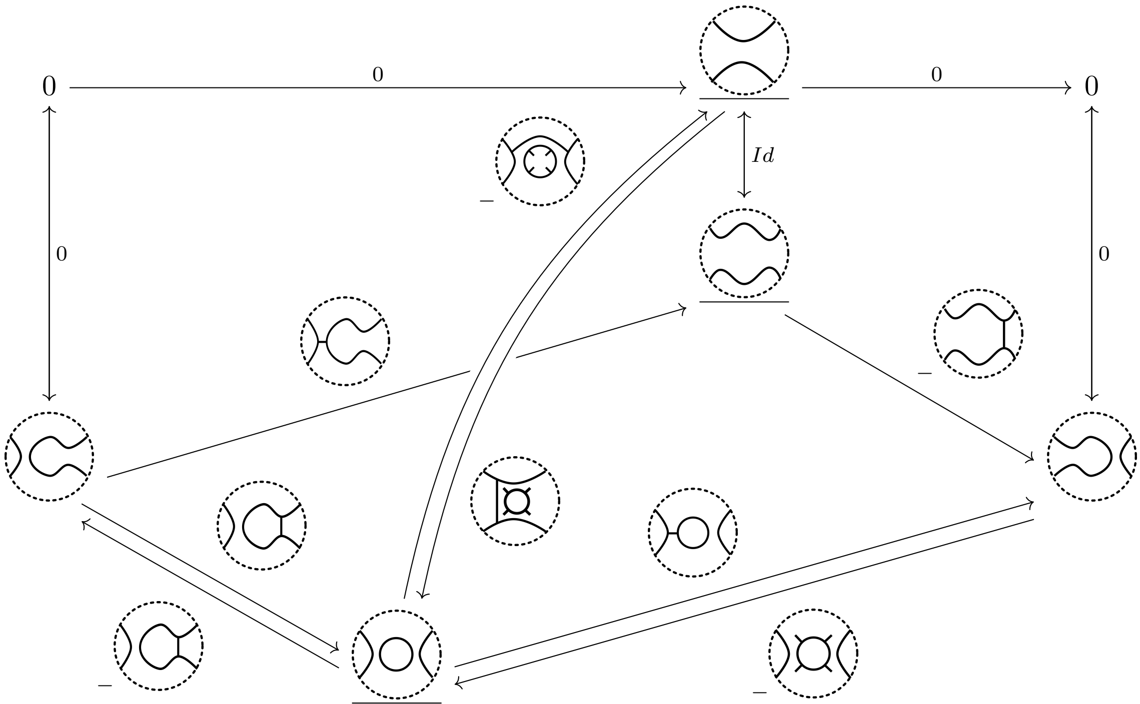
Invariance under R1:

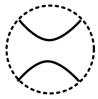
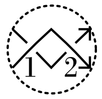


It is easy to check the following:

- $d^0 \circ F^0 = 0$.
- $G^0 \circ F^0 =$  [applying T relation].
-  $- F^0 \circ G^0 = h^1 \circ d^0$ [applying $4Tu$ relation].

Invariance under R2:



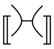
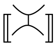
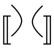
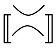
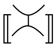
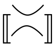
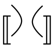
In the above diagram, the top complex is ⟦  ⟧′ and bottom 2- cube (anti-commutative diagram) give us the cochain complex ⟦  ⟧′. The left directional arrows are components of cochain homotopy map h, downward directional arrows are components of cochain homotopy equivalence maps F, upward directional arrows are components of cochain homotopy equivalence maps G, and right directional arrows are components of differential maps d. In this set up the following can be checked:

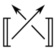
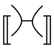
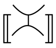
- $d^0 \circ F^0 = 0$.
- $G^0 \circ d^{-1} = 0$.
- $G^0 \circ F^0 = Id$ [applying S relation].
- $Id - F^0 \circ G^0 = h^1 \circ d^0 + h^0 \circ d^1$ [applying $4Tu$ relation].

Invariance under R3:
First we define the following notions, after that we use similar homotopy equivalence morphisms as we did for R2.

Definition 5.10. A morphism of complexes $G : \Omega_a \to \Omega_b$ is said to be strong deformation retract if there is a morphism $F : \Omega_b \to \Omega_a$ and homotopy maps h from Ω_a to itself so that $G \circ F = Id$, $Id - F \circ G = d \circ h + h \circ d$, and $h \circ F = 0$. The morphism F is called *inclusion in the strong deformation retract G*.

Definition 5.11. Let $\psi : (\Omega_0, d_{0)} \to (\Omega_1, d_1)$ be a morphism of cochain complexes. The cone $\Gamma(\psi)$ of ψ is the cochain complex with cochain spaces $\Gamma^r(\psi) = \Omega_0^{r+1} \oplus \Omega_1^r$ and with differentials $\tilde{d^r} = \begin{pmatrix} -d_0^{r+1} & 0 \\ \psi^{r+1} & d_1^r \end{pmatrix}$.
The following two lemmas will be useful to prove the invariance under $R3$ move.

Lemma 5.1. Let ⟦⟧′ and ⟦⟧′ are the saddle morphisms ⟦⟧′ : ⟦⟧′ → ⟦⟧′ and ⟦⟧′ : ⟦⟧′ → ⟦⟧′, then

1. ⟦⟧′ = Γ(⟦⟧′) [1]
2. ⟦⟧′ = Γ(⟦⟧′)

Lemma 5.2. Consider the below diagram of cochain complexes and morphisms.

$$\begin{array}{ccc} \Omega_{0a} & \underset{F_0}{\overset{G_0}{\rightleftarrows}} & \Omega_{0b} \\ \downarrow \psi & & \\ \Omega_{1a} & \underset{G_1}{\overset{F_1}{\rightleftarrows}} & \Omega_{1b} \end{array}$$

If G_0 is a strong deformation retract with inclusion F_0, then the cones $\Gamma(\psi)$ and $\Gamma(\psi F_0)$ are homotopy equivalent. Similarly, if G_1 is a strong deformation retract with inclusion F_1, then the cones $\Gamma(\psi)$ and $\Gamma(F_1\psi)$ are homotopy equivalent.

Proof. Let $h_0 : \Omega^{\star}_{0a} \to \Omega^{\star-1}_{0a}$ be a homotopy for which $Id - F_0 \circ G_0 = d \circ h_0 + h_0 \circ d$ and $h_0 \circ F_0 = 0$. Now consider the below diagram

$$\Gamma(\psi F_0) : \quad \begin{pmatrix} \Omega^{r+1}_{0b} \\ \Omega^{r}_{1a} \end{pmatrix} \xrightarrow{\tilde{d}=\begin{pmatrix} -d_{0b} & 0 \\ \psi F_0 & d_{1a} \end{pmatrix}} \begin{pmatrix} \Omega^{r+2}_{0b} \\ \Omega^{r+1}_{1a} \end{pmatrix}$$

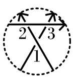
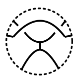
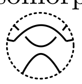
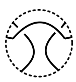
$$\tilde{G_0^r}:=\begin{pmatrix} G_0 & 0 \\ \psi h_0 & Id \end{pmatrix} \uparrow \quad \downarrow \tilde{F_0^r}:=\begin{pmatrix} F_0 & 0 \\ 0 & Id \end{pmatrix} \qquad \qquad \tilde{G_0}^{r+1} \uparrow \quad \downarrow \tilde{F_0}^{r+1}$$

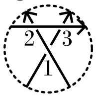
$$\Gamma(\psi) : \quad \begin{pmatrix} \Omega^{r+1}_{0a} \\ \Omega^{r}_{1a} \end{pmatrix} \underset{\tilde{h_0}:=\begin{pmatrix} -h_0 & 0 \\ 0 & 0 \end{pmatrix}}{\overset{\tilde{d}=\begin{pmatrix} -d_{0a} & 0 \\ \psi & d_{1a} \end{pmatrix}}{\rightleftarrows}} \begin{pmatrix} \Omega^{r+2}_{0a} \\ \Omega^{r+1}_{1a} \end{pmatrix}$$

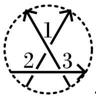
The following can be easily checked

- $\tilde{d} \circ \tilde{F_0}^r = \tilde{F_0}^{r+1} \circ \tilde{d}$ [using $F_0 \circ d = d \circ F_0$].
- $\tilde{d} \circ \tilde{G_0}^r = \tilde{G_0}^{r+1} \circ \tilde{d}$ [using $d_{1a} \circ \psi = \psi \circ d_{0a}$]
- $\tilde{G_0} \circ \tilde{F_0} = Id$ [using $G_0 \circ F_0 = Id$ and $h_0 \circ F_0 = 0$].
- $Id - \tilde{F_0}\tilde{G_0} = \tilde{h_0} \circ \tilde{d} + \tilde{d} \circ \tilde{h_0}$ [using $Id - F_0 \circ G_0 = h_0 \circ d + d \circ h_0$].

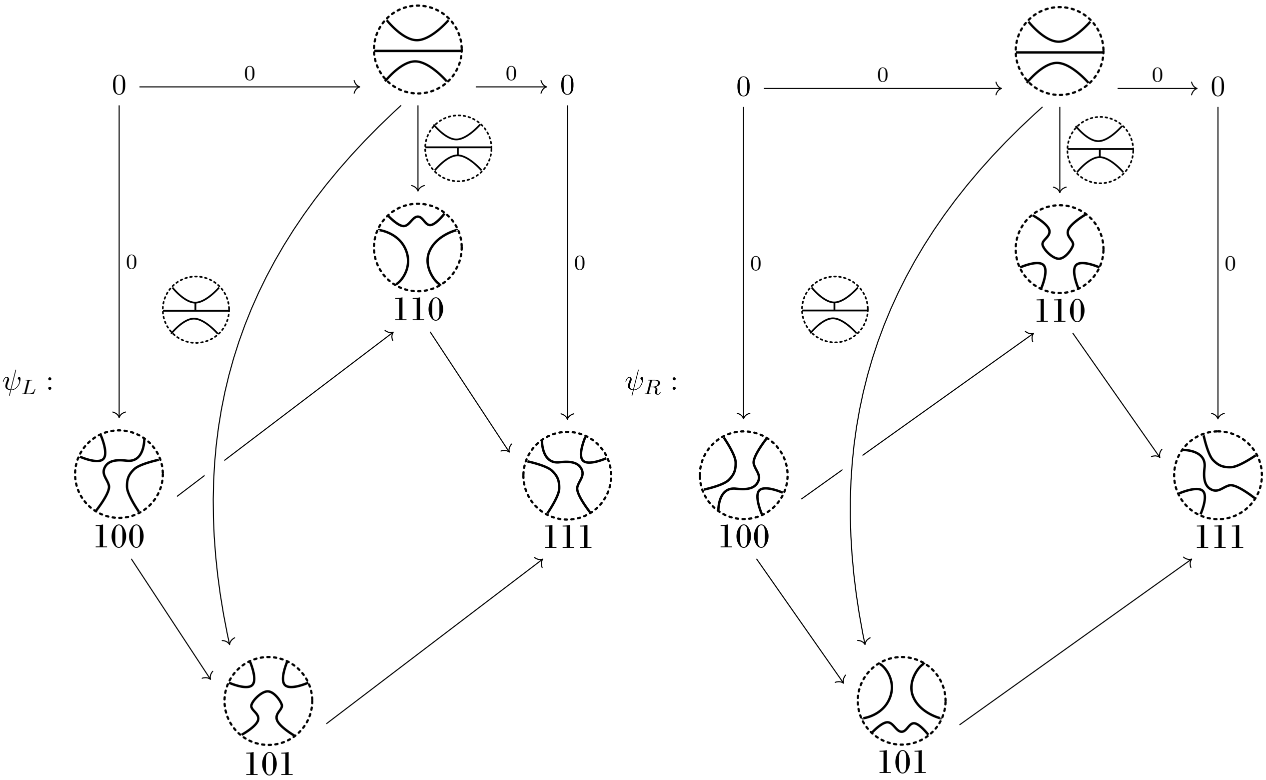
Therefore $\tilde{F_0}$ and $\tilde{G_0}$ are cochain morphisms $\Gamma(\psi \circ F_0) \underset{\tilde{G_0}}{\overset{\tilde{F_0}}{\rightleftarrows}} \Gamma(\psi)$. In fact they are cochain homotopy equivalences, where $\tilde{h_0} : \Gamma(\psi)^{\star} \to \Gamma(\psi)^{\star-1}$ is cochain homotopy. ■

Now we are going to the main part of the proof. It can be shown that up to isomorphism of cochain complexes, $[\![$  $]\!]' = \Gamma(\psi)$, where ψ is the morphism $[\![$  $]\!]' : [\![$  $]\!]' \to$ $[\![$  $]\!]'$. This can be shown directly, applying the idea mentioned in the proof of Theorem 3.3 or one can use the Lemma 5.1.

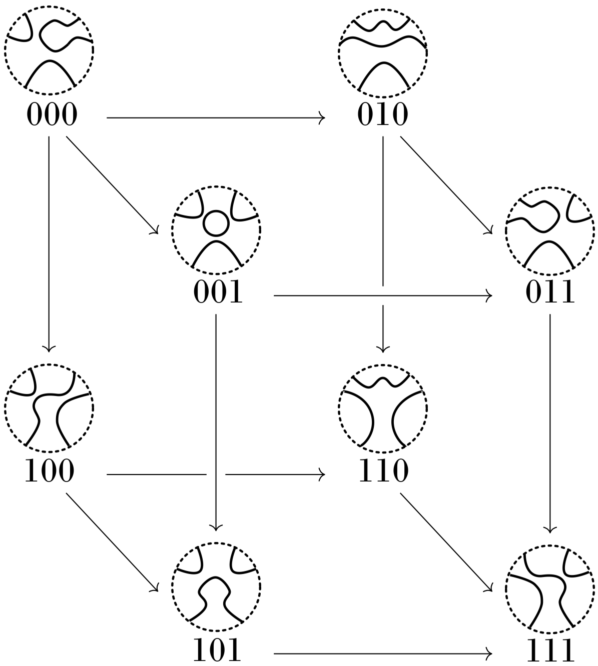
Now consider the following two diagrams. ψ_L is the vertical morphism in the left-sided diagram and ψ_R is the vertical morphism in the right-sided diagram. The bottom 2-cube (face) of the left-sided diagram is a face of the cube for 

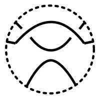
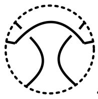
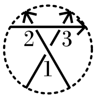
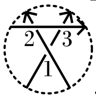
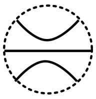
and the bottom 2-cube of right-sided diagram is a face of the cube for 

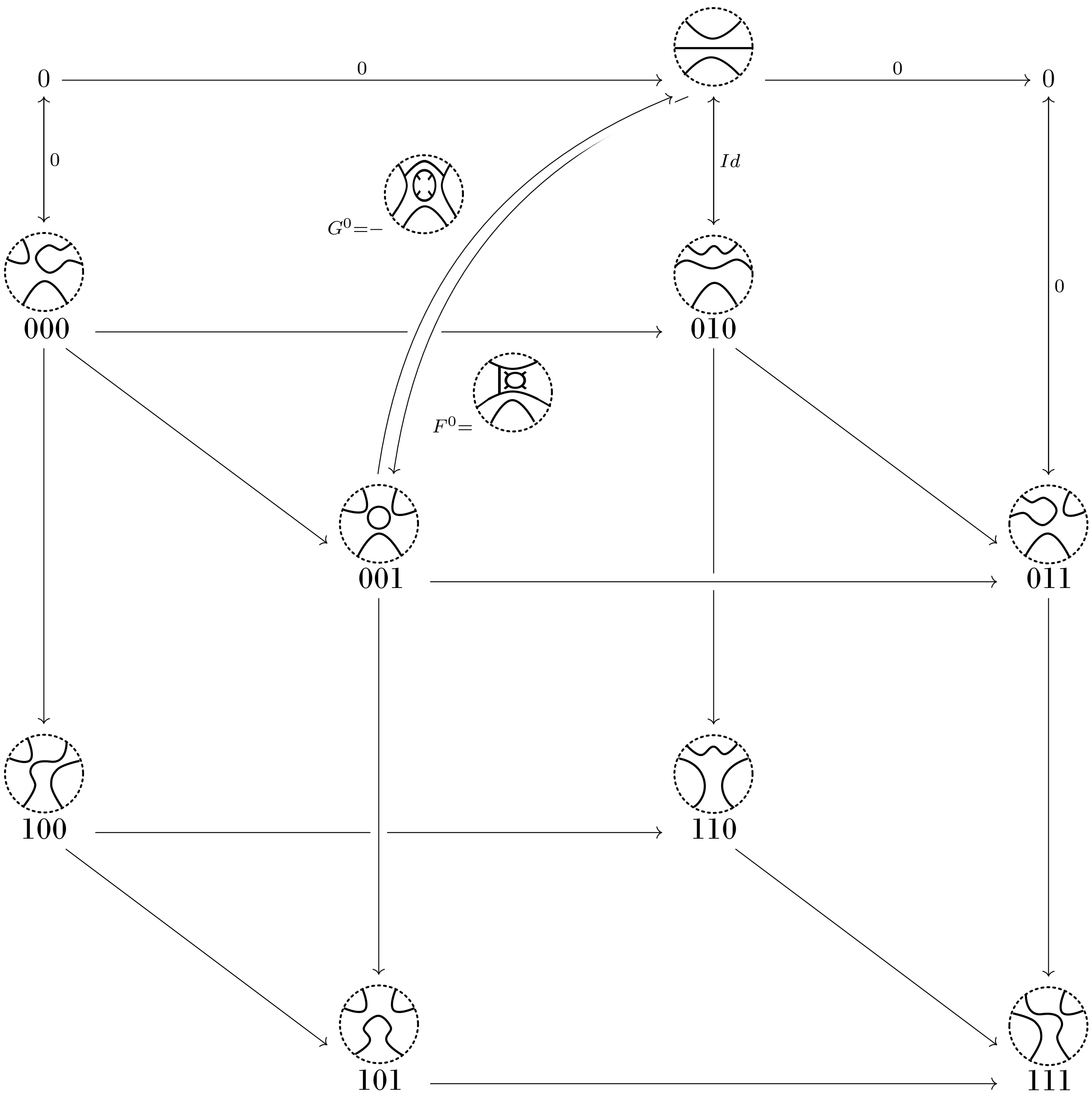
. In each case the state corresponding to each smoothing is written.



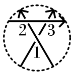
In the diagram below, the vertical arrows are the components of the morphism ψ.

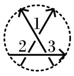


In above diagram, the top layer corresponds to the cube diagram of  and the bottom layer corresponds to the cube diagram of . The vertical arrows from top layer to the bottom layer is the saddle morphism ψ. If we change the signs of the arrows of ψ in a way so that the above diagram becomes an anti-commutative cube, we would get the cube diagram for  (up to isomorphism). The corresponding state information, considering that those are smoothings of , is written in the diagram. Now we attach a layer corresponding to the single object  on top of the cube.



In this above diagram F and G are very similar to the homotopy equivalences that we used to prove invariance under R2. These are cochain homotopy equivalences, and G is strong deformation retract and F is the inclusion. Also observe that $\psi \circ F = \psi_L$. Now applying the Lemma 5.2, we get

$$[\![$$  $$]\!]' = \Gamma(\psi) = \Gamma(\psi \circ F) = \Gamma(\psi_L).$$

Similarly one can get that $[\![$  $]\!]' = \Gamma(\psi_R)$. But up to equivalence relation, ψ_L and ψ_R are same. ■

5.5 Planar algebra

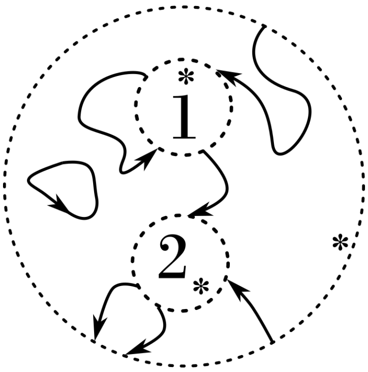
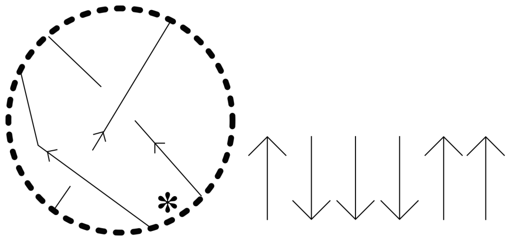


FIGURE 5.2: 2-input oriented planar arc diagram

Definition 5.12. A d-input planar arc diagram is a big output disk with d smaller input disks removed, along with a collection of disjoint embedded oriented arcs that are either closed or begin and end on the boundary. The input disks are numbered 1 through d, and there is a base point ($\star$) marked on each of the input disks as well as on the output disk. Finally this information is considered only up to isotopy. An unoriented planar arc diagram is the same, except the orientation of the arcs is forgotten.

A disk is called based disk if a base point is marked on its boundary.

Definition 5.13. $\mathcal{T}^0(k)$ is defined to be the collection of all k-ended unoriented tangle diagrams (unoriented tangle diagrams in a disk, with k ends on the boundary of the disk) in a based disk (a disk with a base point marked on its boundary). Now for any oriented tangle diagram inside a based disk, we associate a string. Starting from the base point ($\star$), we go anti-clockwise along the boundary of the based disk and note the information whether an arc of the tangle diagram is incoming or outgoing. When an arc is incoming, we use the symbol $\uparrow$ and when it is outgoing, we use ↓. The following diagram is an example to show how a string should be assigned.



If s is a string of ↑ and ↓ with total length $|s|$, then we define $\mathcal{T}^0(s)$ as the collection of $|s|$-ended oriented tangle diagrams in a based disk with incoming/outgoing strands as specified by s. Further $\mathcal{T}(K)$ and $\mathcal{T}(s)$ is defined to be the respective quotient of $\mathcal{T}^0(k)$ and $\mathcal{T}^0(s)$ by the Reidemeister moves (so these are spaces of tangles rather than tangle diagrams).

Let D be a d-input oriented planar arc diagram such that the string corresponding to the i-th input disk is s_i and the string corresponding to the output disk is s. Let $|s_i| = k_i$ and $|s| = k$. Using D, we define the following operations (we use same symbol D to denote the operations):

$$D : \mathcal{T}^0(k_1) \times \cdots \times \mathcal{T}^0(k_d) \to \mathcal{T}^0(k), \text{ and } D : \mathcal{T}(k_1) \times \cdots \times \mathcal{T}(k_d) \to \mathcal{T}(k)$$

If $T_i \in \mathcal{T}^0(k_i)$, then $D(T_1, \ldots, T_d)$ is defined to be the result of placing T_i in the i-th input disk of D (forgetting the orientation of the arcs of D i.e. considering D to be unoriented) in such a way that the base point of T_i coincides with the base point of the i-th input disk of D, for all $1 \leq i \leq d$. Considering D be oriented planar arc diagram, we can similarly define the operations:

$$D : \mathcal{T}^0(s_1) \times \cdots \times \mathcal{T}^0(s_d) \to \mathcal{T}^0(s), \text{ and } D : \mathcal{T}(s_1) \times \cdots \times \mathcal{T}(s_d) \to \mathcal{T}(s).$$

These operations contain the identity operations on $\mathcal{T}^0(k)$ or $\mathcal{T}^0(s)$. We may take radial planar arc diagram like  as identity operation. Also, this operation is compatible (associative) in a natural way. Let D_1, D_2 and D_3 are three planar arc diagrams. $D_1(I \times \cdots \times D_2(I \times \cdots \times D_3 \times \cdots \times I) \times \cdots \times I)$ is the result of first placing D_3 in the j-th hole of D_2 and then placing this new diagram $D_2(I \times \cdots \times D_3 \times \cdots \times I)$ in the i-th hole of D_1 (provided the relevant k/s match). Associative property is obvious in the sense that the above result is the same as first placing D_2 in the i-th hole of D_1 and then placing D_3 in the j-th hole of D_2 which is now sitting inside D_1.

We call a collection $(P(k))$ (resp. $(P(s))$) along with the operations D, defined for each planar arc diagram (resp. oriented planar arc diagram) a planar algebra, provided radial planar arc diagrams act as identities and provided associativity conditions as above hold. Below are some examples of algebra:

Example 5.13.2. The collection $(\mathcal{T}^0(\text{k}))$ and the collection $(\mathcal{T}(\text{k}))$.

Example 5.13.3. The collection $\Big(\mathrm{Obj}(Cob^3_{/l}(k))\Big)$. This is sub-algebra of the collection $(\mathcal{T}(\mathrm{k}))$.

Example 5.13.4. The collection $\Big(\mathrm{Mor}(Cob^3_{/l}(k))\Big)$. The algebra operation is defined in the following manner: let D be a d-input unoriented planar arc diagram, then consider $D \times [0,1]$. $D \times [0,1]$ has d vertical cylindrical holes with vertical curtains connecting those. We can place morphisms of $Cob^3_{/l}$ inside the cylindrical holes. This leads us to get an operation D : $\mathrm{Mor}(Cob^3_{/l}(k_1)) \times \cdots \times \mathrm{Mor}(Cob^3_{/l}(k_d)) \to \mathrm{Mor}(Cob^3_{/l}(k))$ (provided the relevant k match).

For all above-mentioned examples, instead of taking collection over k, if we take collection over s, then we get oriented planar algebras.

Definition 5.14. A morphism ϕ of algebras from $(P^1(k))$ to $(P^2(k))$ is a collection of maps (all denoted by the same symbol) $\phi : P^1(k) \to P^2(k)$ satisfying $\phi \circ D = D \circ (\phi \times \cdots \times \phi)$ for every D. Similarly we could take oriented algebras and define algebra morphisms between them.

Let D be a d-input planar arc diagram with k_i arcs ending on the i-th input disk and k-arcs ending on the outer disk. By extending the algebra structure of $\mathrm{Obj}(Cob^3_{/l})$ and $\mathrm{Mor}(Cob^3_{/l})$ in the usual multi-linear manner, $\mathrm{Obj}(\mathrm{Mat}(Cob^3_{/l}))$ and $\mathrm{Mor}(\mathrm{Mat}(Cob^3_{/l}))$ can be endowed with a algebra structure as follows:

$$D(O_1, \ldots, \bigoplus_{i=1}^{n} O_{ji}, \ldots, O_d) := \bigoplus_{i=1}^{n} D(O_1, \ldots, O_{ji}, \ldots, O_d),$$

where each $O_{ji} \in \mathrm{Obj}(Cob^3_{/l}(k_j))$ and each $O_m \in \mathrm{Obj}(Cob^3_{/l}(k_m))$, for $i \in \{1, \ldots, n\}$ and $m \in \{1, \ldots, j-1, j+1, \ldots, d\}$. Similarly,

$$D(f_1, \ldots, \sum_{i=1}^{n} f_{ji}, \ldots, f_d) := \sum_{i=1}^{n} D(f_1, \ldots, f_{ji}, \ldots, f_d),$$

where each $f_{ji} \in \mathrm{Mor}(Cob^3_{/l}(k_j))$ and each $f_m \in \mathrm{Mor}(Cob^3_{/l}(k_m))$, for $i \in \{1, \ldots, n\}$ and $m \in \{1, \ldots, j-1, j+1, \ldots, d\}$.
Notation: For any natural number k, let $\mathrm{Kob}(k) := \mathrm{Kom}\Big(\mathrm{Mat}(Cob^3_{/l}(k))\Big)$ and similarly, $\mathrm{Kob}_{/h}(k) := \mathrm{Kom}_{/h}\Big(\mathrm{Mat}(Cob^3_{/l}(k))\Big)$

Theorem 5.2. Let D be any planar arc diagram.

1. The collection $(\mathrm{Kob}(k))$ has a natural structure of planar algebra.
2. The operations D on $(\mathrm{Kob}(k))$ send cochain homotopy equivalent complexes to cochain homotopy equivalent complexes and hence the collection $(\mathrm{Kob}_{/h}(k))$ also has natural structure of planar algebra.
3. The formal Khovanov bracket $[\![\cdot]\!]'$ descends to an oriented planar algebra morphism $[\![\cdot]\!]' : (\mathcal{T}(s)) \to (\mathrm{Kob}_{/h}(s))$.

Proof. Here is the brief idea of the proof.

- Let D be a d-input planar arc diagram with k_i arcs ending on the i-th input disk and k-arcs ending on the outer disk. If $(\Omega_i, d_i) \in \text{Kob}(k_i)$ are cochain complexes, we define another cochain complex $(\Omega, d) = D(\Omega_1, \dots, \Omega_d)$ as following:

$$\Omega^r := \bigoplus_{r=r_1+\cdots+r_d} D(\Omega_1^{r_1}, \dots, \Omega_d^{r_d}) \tag{5.1}$$

$$d|_{D(\Omega_1^{r_1},\dots,\Omega_d^{r_d})} := \sum_{i=1}^{d} (-1)^{\sum_{j<i} r_j} D(I_{\Omega_1^{r_1}}, \dots, d_i^{r_i}, \dots, I_{\Omega_d^{r_d}}) \tag{5.2}$$

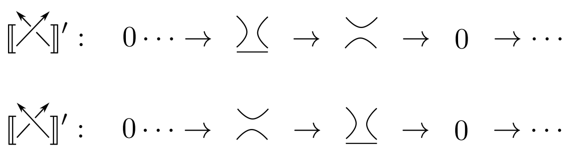
  One can easily check that $d^2 = 0$ and hence $(\Omega, d) \in \text{Kob}(k)$. Thus (Ω, d) can be regarded as tensor product of (Ω_i, d_i).
  Consider the operation D as above. If D_0 is a radial planar arc diagram where the input and output disks have k-ending arcs, and $(\Omega, k) \in \text{Kob}(k)$. Then it is obvious that $D_0(\Omega) = (\Omega, d)$.

- The second assertion also holds, because if $\psi_i : \Omega_{ia} \to \Omega_{ib}$ is a morphism between to cochain complexes, then it induces morphism

$$D(I, \dots, \psi_i, \dots, I) : D(\Omega_1, \dots, \Omega_{ia}, \dots, \Omega_d) \to D(\Omega_1, \dots, \Omega_{ib}, \dots, \Omega_d)$$

  where $\Omega_j \in \text{Kob}(k_j)$, $\forall j \neq i$. It is routine to check that $D(I, \dots, \psi, \dots, I)$ is a cochain morphism from $D(\Omega_1, \dots, \Omega_{ia}, \dots, \Omega_d)$ to $D(\Omega_1, \dots, \Omega_{ib}, \dots, \Omega_d)$. Similarly homotopies at the level of tensor factors induces homotopies at the level of tensor products.

- Let T be a tangle diagram with d crossings, and D be the d-input planar arc diagram obtained from T by deleting a disk neighbourhood of each crossings of T. Let X_i denotes the i-th crossing of T and Ω_i denotes the cochain complex $[\![X_i]\!]'$. Note that,



  where the height zero objects are underlined. It follows from the (5.1), (5.2) and the Definition 5.9, that up to the isomorphism of cochain complexes,

$$[\![T]\!]' = D(\Omega_1, \dots, \Omega_d).$$

  To prove the above equality, we may require the similar idea as mentioned in the proof of Theorem 3.3. Finally,

$$[\![D(X_1, \dots, X_d)]\!]' = [\![T]\!]' = D(\Omega, \dots, \Omega_d) = D([\![X_1]\!]', \dots, [\![X_d]\!]').$$

The general case follows from the associativity of planar algebras involved. ■

5.6 Graded category and Khovanov complex for tangles

Definition 5.15. A graded category is a pre-additive category C with the following additional properties.

- For any $O_1, O_2 \in \text{Obj}(C)$, $\text{Mor}(O_1, O_2)$ are $\mathbb{Z}$-graded abelian groups. The elements of the d-th direct summand/component of this morphism group is called homogeneous element of degree d. Also the composition of two morphisms should satisfy the following: let $O_1, O_2, O_3 \in \text{Obj}(C)$, $g \in$ the j-th direct summand of $\text{Mor}(O_1, O_2)$ and $f \in$ the i-th direct summand of $\text{Mor}(O_2, O_3)$, then $\deg(f \circ g) = \deg(f) + \deg(g) = i + j$. All identity maps should be of degree 0.
- There is a $\mathbb{Z}$-action on $\text{Obj}(C)$ as $m.O = O\{m\}$, called grading shift by m such that $\text{Mor}(O_1\{m_1\}, O_2\{m_2\}) = \text{Mor}(O_1, O_2)$, as abelian group. But grading of morphisms under this action changes. If $f \in \text{Mor}(O_1, O_2)$ and $\deg(f) = d$, then as an element of $\text{Mor}(O_1\{m_1\}, O_2\{m_2\})$, the degree of f is $d + m_2 - m_1$.

Remark 5.2. A pre-additive category category C which has only the first property above, can be upgraded to a graded category C' in the following way. We allow artificial $O\{m\}$ for every $m \in \mathbb{Z}$ and define $O\{m_1\}\{m_2\} := O\{m\}$. Then we artificially define and grade the morphisms in such a way that the second property is satisfied.

Remark 5.3. If C is a graded category. In $\text{Mat}(C)$ a morphism (which is a matrix) is considered as homogeneous of degree d if all the entries of it are of degree d. Thus $\text{Mat}(C)$ can also be considered as graded category. A morphism $F : (\Omega_1, d_1) \to (\Omega_2, d_2)$ in $\text{Kom}(\text{Mat}(C))$ is considered to be homogeneous of degree d, if for each height r, F^r is homogeneous of degree d. As a result, we can consider $\text{Kom}(\text{Mat}(C))$ as graded category too.

Definition 5.16. Let $[C] \in \text{Mor}(Cob^3(B))$ be an equivalence class of cobordism, we define $\deg([C]) := \chi(C) - \frac{1}{2}|B|$, where $\chi(C)$ is the Euler characteristic of the surface C.

Proposition 5.3. $\deg([C_2] \circ [C_1]) = \deg([C_1]) + \deg([C_2])$

Proof. $\chi(C_2 \circ C_1) = \chi(C_1) + \chi(C_2) - \chi(C_1 \cap C_2) = \chi(C_1) + \chi(C_2) - \frac{1}{2}|B|$, since $C_1 \cap C_2$ is nothing but $\frac{1}{2}|B|$ disjoint arcs. $\deg([C_2] \circ [C_1]) = \chi(C_2 \circ C_1) - \frac{1}{2}|B| = \chi(C_1) + \chi(C_2) - |B|$, but this quantity is exactly equal to $\deg([C_1]) + \deg([C_2])$. ■

Remark 5.4. Recall that if we consider Cob^3 as a pre-additive category, any morphism group $\text{Mor}(O_1, O_2)$ is the free abelian group generated by the set: $\{[C] \mid C$ is a cobordism from O_1 to O_2 $\}$.

let $[C] \in \mathrm{Mor}(Cob^3)$ and n be an integer. If we define $\deg(n[C]) := \deg([C])$, then Cob^3 can be considered as graded category due to the Proposition 5.3. Since S, T, $4Tu$ relations are degree homogeneous, $Cob^3_{/l}$ can be considered as graded category too. As per our previous observations, we can similarly consider $\mathrm{Mat}(Cob^3_{/l})$, $\mathrm{Kom}(\mathrm{Mat}(Cob^3_{/l}))$, and $\mathrm{Kom}_{/h}(\mathrm{Mat}(Cob^3_{/l}))$ as graded categories.

Definition 5.17. Let T be a tangle diagram with n_+ positive crossings and n_- negative crossings. Let $Kh(T)$ be the complex whose chain spaces are

$$Kh^r(T) := [\![T]\!]'\{r + n_+ - n_-\}$$

and whose differentials are the same as those of $[\![T]\!]'$:

$$[\![T]\!]' : \quad \cdots \longrightarrow [\![T]\!]'^{(-n_-)} \longrightarrow \cdots \longrightarrow [\![T]\!]'^{(n_+)} \longrightarrow \cdots$$

$$Kh(T) : \quad \cdots \longrightarrow [\![T]\!]'^{(-n_-)}\{n_+ - 2n_-\} \longrightarrow \cdots \longrightarrow [\![T]\!]'^{(n_+)}\{2n_+ - n_-\} \longrightarrow \cdots$$

Theorem 5.3. Let T be an oriented tangle diagram.

1. All differentials in $Kh(T)$ are of degree 0.
2. If T_1 and T_2 are tangle diagrams which differ by Reidemeister moves, then there is a homotopy equivalence $F : Kh(T_1) \to Kh(T_2)$ with deg $(F) = 0$.
3. Like $[\![\cdot]\!]$, Kh descends to an oriented planar algebra morphism $(\mathcal{T}(s)) \to (\mathrm{Kob}_{/h}(s))$.

Proof. Here is the brief idea of the proof.

- The first assertion follows from the fact that the differential maps are nothing but saddle morphisms, and

  deg  $= -1.$

- The second assertion follows from the Theorem 5.1. Note that the homotopy equivalence morphisms in the proof of invariance under R1 and R2 are degree-0 morphisms.

- Let D be a planar arc diagram, and $O_i \in Cob^3_{/l}(k_i)$ for all $1 \le i \le d$. Let's define the operation

  $$D(O_1\{n_1\}, \dots, O_d\{n_d\}) := D(O_1, \dots, O_d)\{n_1 + \cdots + n_d\}$$

  where each $n_i \in \mathbb{Z}$. As a result if $(\Omega_i, d_i) \in \mathrm{Kob}(k_i)$, then

  $$D(\Omega_1\{n_1\}, \dots, \Omega_d\{n_d\}) = D(\Omega_1, \dots, \Omega_d)\{n_1 + \cdots + n_d\}.$$

  Finally the third assertion follows from the Theorem 5.2

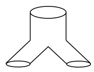
∎

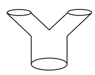
5.7 Applying TQFT

Consider any abelian category A such as the category of $\mathbb{Z}$-modules or $\mathbb{Q}$-vector spaces. Let F be a functor from the pre-additive category $Cob^3_{/l}$ to A, such that for any two morphism $f_1, f_2 \in \mathrm{Mor}(Cob^3_{/l})$, $F(f_1+f_2) = F(f_1) + F(f_2)$. Hence F sends zero morphism to the zero morphism in the abelian category A. Now for $O_i \in \mathrm{Obj}(Cob^3_{/l})$, we define $F(\bigoplus_i O_i) := \bigoplus_i F(O_i)$ (in L.H.S. we have considered formal direct sum while in R.H.S. we have taken direct sum in the abelian category A). Also for any matrix $[a_{ij}]_{ij} \in \mathrm{Mor}\big(\mathrm{Mat}(Cob^3_{/l})\big)$, we define $F([a_{ij}]_{ij}) := \big[F(a_{ij})\big]_{ij}$. In this way we can extend the functor $F : Cob^3_{/l} \to A$ to a functor $F : \mathrm{Mat}(Cob^3_{/l}) \to A$. Also if $(\Omega, d) \in \mathrm{Kom}\big(\mathrm{Mat}(Cob^3_{/l})\big)$ then $F(d) \circ F(d) = F(d \circ d) = F(0) = 0$. Hence we can extend F to another functor $F : \mathrm{Kob} \to \mathrm{Kom}(A)$.

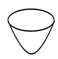
Let T_1, T_2 be two tangle diagrams which differ by Reidemeister moves. Due to the Theorem 5.3, $Kh(T_1)$ and $Kh(T_2)$ are cochain homotopic equivalent. Let $G_1 : Kh(T_1) \to Kh(T_2)$ and $G_2 : Kh(T_2) \to Kh(T_1)$ are two cochain morphisms such that $G_1 \circ G_2$ and $G_2 \circ G_1$ are cohain homotopic equivalent to identities, then applying the functor F, one can easily check that $F(G_1) : F\big(Kh(T_1)\big) \to F\big(Kh(T_2)\big)$ and $F(G_2) : F\big(Kh(T_2)\big) \to F\big(Kh(T_1)\big)$ are cochain morphisms such that $F(G_1) \circ F(G_2)$ and $F(G_2) \circ F(G_1)$ are cochain homotopic to identities.
Now let A be a graded abelian category. For example, we may take the category of $\mathbb{Z}$-graded vector spaces over field $\mathbb{Q}$. If the functor F is degree preserving, then $F(G_1)$ and $F(G_2)$ are degree-0 morphisms between $F\big(Kh(T_1)\big)$ and $F\big(Kh(T_2)\big)$, such that $F(G_1) \circ F(G_2)$ and $F(G_2) \circ F(G_1)$ are cochain homotopic to identities. Therefore, the cohomologies $H\Big(F\big(Kh(T_1)\big)\Big)$ and $H\Big(F\big(Kh(T_2)\big)\Big)$ are isomorphic with a grading preserving isomorphism. Hence the cohomology $H\big(F(Kh(T)\big)$ is a graded invariant of tangle T.

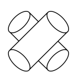
Due to the Remark 5.1, $Cob^3_{/l}(\phi)$ is a monoidal category. Note that the morphisms in this category can be constructed by taking composition and tensor product of the following generator cobordisms:

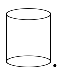
C^1_2 = pair of pants= 

C^2_1 = upside down pair of pants = 

C^0_1 = cap = 

C^1_0 = cup = 

C^2_2 = permutation = 

C^1_1 = identity cylinder = .

Definition 5.18. Let A be the category of graded vector spaces over $\mathbb{Q}$ (or we can take the category of graded $\mathbb{Z}$ modules). Let V be the vector space generated by two basis $\{v_\pm\}$ with deg $(v_\pm) = \pm 1$. The TQFT F is defined by, $F(\bigcirc) := V$ and by $F(C^1_2) = \Delta : V \to V \otimes V$,

$F(C_1^2) = m : V \otimes V \to V$, $F(C_1^0) = \epsilon : \mathbb{Q} \to V$, $F(C_0^1) = \eta : V \to \mathbb{Q}$, $F(C_2^2) = perm : V \otimes V \to V \otimes V$, $F(C_1^1) = Id : V \to V$, where these maps are defined as follows:

$$F(C_2^1) = \Delta : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$$

$$F(C_1^2) = m : \begin{cases} v_+ \otimes v_+ \mapsto v_+ \\ v_+ \otimes v_- \mapsto v_- \\ v_- \otimes v_+ \mapsto v_- \\ v_- \otimes v_- \mapsto 0 \end{cases}$$

$$F(C_1^0) = \epsilon : \begin{cases} 1 \mapsto v_+ \end{cases}$$

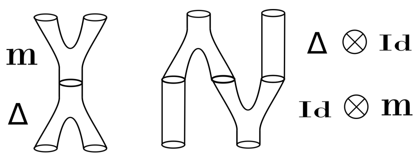
$$F(C_0^1) = \eta : \begin{cases} v_+ \mapsto 0 \\ v_- \mapsto 1 \end{cases}$$

$$F(C_2^2) = perm : \begin{cases} u \otimes v \mapsto v \otimes u \end{cases}$$

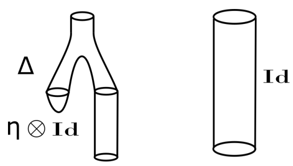
$$F(C_1^1) = Id : \begin{cases} v \mapsto v \end{cases} .$$

Proposition 5.4. The TQFT F is well defined and degree-respecting. It descends to a functor $Cob^3_{/l}(\phi) \to A$.

Proof. Let a morphism $[C]$ in $Cob^3(\phi)$ be constructed in two different ways by above-mentioned six generators. Now to prove that F is well defined, first we need to show that the two ways of taking the operations in the category A gives the same linear maps. Note that in the figure below, the two cobordisms are in same equivalent class.



We need to show that $\Delta \circ m = (Id \otimes m) \circ (\Delta \otimes Id)$. But this holds as mentioned earlier [see Section 3.3 and Equation 3.2]. Similarly for the following pair:



$(\eta \otimes Id) \circ \Delta = Id$, when we consider the identification of $\mathbb{Q} \times V \cong V$ by the natural map $r \otimes v \mapsto rv$, $\forall r \in \mathbb{Q}$ and $v \in V$.
Now we check that F is degree-respecting. This true because

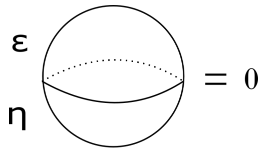
$$\deg(C_1^2) = \deg(C_2^1) = \deg(\Delta) = \deg(m) = -1$$

$$\deg(C_1^1) = \deg(C_0^1) = \deg(\epsilon) = \deg(\eta) = 1$$

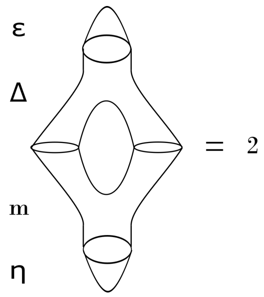
$$\deg(C_2^2) = \deg(C_1^1) = \deg(perm) = \deg(Id) = 0$$

To complete the proof we need to verify that F satisfies S, T, and $4Tu$ relation.

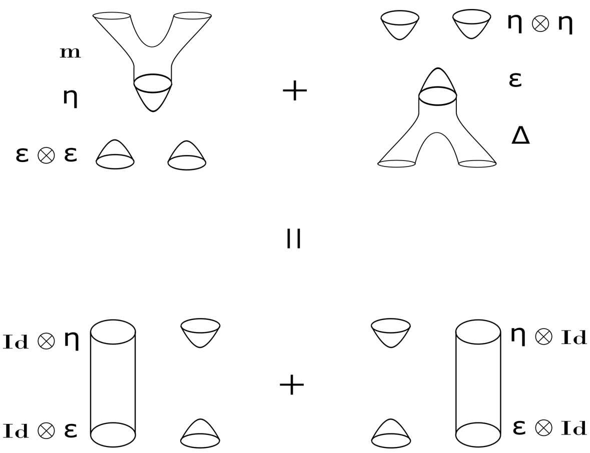
- S Relation: We can check that $\eta \circ \epsilon$ is a linear map $\mathbb{Q} \to \mathbb{Q}$ sending $r \mapsto 0$, for any $r \in \mathbb{Q}$.



- T Relation: We can check that $\eta \circ m \circ \Delta \circ \epsilon$ is a linear map $\mathbb{Q} \to \mathbb{Q}$ sending $r \mapsto 2r$, for any $r \in \mathbb{Q}$.



- 4Tu Relation: We can check that $(\epsilon \otimes \epsilon) \circ \eta \circ m + \Delta \circ \epsilon \circ (\eta \otimes \eta) = (Id \otimes \epsilon) \circ (Id \otimes \eta) + (\epsilon \otimes Id) \circ (\eta \otimes Id)$, when we consider the identification $\mathbb{Q} \otimes \mathbb{Q} \cong \mathbb{Q}$ by the natural map $r \otimes r' \mapsto rr'$, $\forall r, r' \in \mathbb{Q}$.



■

5.8 Tautological functor

Definition 5.19. Let B be a set of even number of points in a circle, and $\mathbb{Z}$Mod denotes the category of graded $\mathbb{Z}$ modules and $\mathbb{Z}$-linear maps. Then the tautological functor F_O : $Cob^3_{/l}(B) \to \mathbb{Z}\text{Mod}$ is defined on objects by $F_O(O') := \text{Mor}(O, O')$ and on morphisms in the following way: if $f \in \text{Mor}(O_1, O_2)$ is a morphism in $Cob^3_{/l}(B)$, we define,

$$F_O(f) : \text{Mor}(O, O_1) \to \text{Mor}(O, O_2)$$
$$g \mapsto f \circ g$$

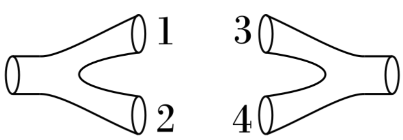
where $O, O', O_1, O_2 \in \text{Obj}(Cob^3_{/l}(B))$.

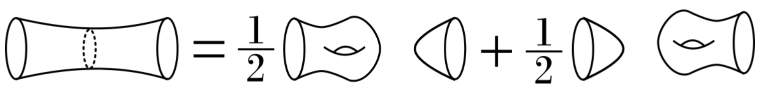
In case of $Cob^3_{/l}(\phi)$, we usually take $O = \phi$ and consider the tautological functor F_ϕ. Note that we are considering $Cob^3_{/l}(\phi)$ as a pre-additive category, and also considering that the empty set ϕ is an object in $Cob^3_{/l}(\phi)$. Note that applying S, T and $4Tu$ relations we get: the elements of $\text{Mor}(\phi, \phi)$ are integers. Hence, we take $\text{Mor}(\phi, \phi)$ to be $\mathbb{Z}$ and for any $f \in \text{Mor}(\phi, O)$, $g \in \text{Mor}(O, \phi)$, $f \circ m = m \times f$ and $m \circ g = m \times g$, where $m \in \text{Mor}(\phi, \phi)$.

Original Khovanov cohomology theory as described in [4, 7]:
Consider any extension of the ring $\mathbb{Z}$ where 2 has inverse, for example the ring $\mathbb{Z}[1/2]$. Let A be the category of $\mathbb{Z}[1/2]$-modules. We want to define a new functor $F_1 : Cob^3_{/l}(\phi) \to A$. Let $O \in \text{Obj}(Cob^3_{/l}(\phi))$, and $R(\phi, O)$ be the free $\mathbb{Z}$-module generated by the equivalence classes of cobordisms from ϕ to O having genus greater than 1, modulo S, T and $4Tu$ relations. On the objects of $Cob^3_{/l}(\phi)$ we define the reduced (reduced) tautological functor F_1 in the following way:

$$F_1(O) := \mathbb{Z}[1/2] \otimes_{\mathbb{Z}} \frac{\text{Mor}(\phi, O)}{R(\phi, O)},$$

which is a $\mathbb{Z}[1/2]$-module. On the morphisms of $Cob^3_{/l}(\phi)$, it is defined in the following way: if $f \in \text{Mor}(O_1, O_2)$, and $g \in \text{Mor}(\phi, O_1)$ are morphisms in $Cob^3_{/l}(\phi)$ and let $\overline{g}$ is coset representative of g in $\frac{\text{Mor}(\phi, O_1)}{R(\phi, O_1)}$, then $F_1(f)(r \otimes \overline{g}) := r \otimes \overline{(f \circ g)} \in F_1(O_2)$, for any $r \in \mathbb{Z}[1/2]$.
Since 2 has inverse in $\mathbb{Z}[1/2]$, the $4Tu$ relation gives rise to the following **neck-cutting** relation.





Whenever there is a tube inside a cobordism we can replace it with a handle localised at one side of the tube and a disk localised at the another side. As a result, if we take $[C]$ in $\mathbb{Z}[1/2] \otimes_{\mathbb{Z}} \text{Mor}(\phi, O)$ and repeatedly cut tubes inside C, we get cobordisms whose every connected components have at most one boundary S^1. But if we also apply the relation that the surfaces having genus greater that 1 are taken to be 0, we can reduce C to a $\mathbb{Z}[1/2]$-linear combination of cobordisms whose connected components can either be torus or sphere (when the connected component has no boundary), or it can be a disk or a torus with a hole (when the connected component has one boundary circle). Finally if we apply S and T relations, we get that in $\mathbb{Z}[1/2] \otimes_{\mathbb{Z}} \frac{\text{Mor}(\phi,O)}{R(\phi,O)}$, C can be considered as $\mathbb{Z}[1/2]$-linear combination of cobordisms whose connected components are either disk or a torus with a hole.
Thus if O is just the collection of n disjoint simple closed curves, then we get $F_1(O) = V^{\otimes n}$, where V is a $\mathbb{Z}[1/2]$-module generated by $v_+ =$  and $v_- = \frac{1}{2} \times$ .

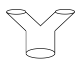
Proposition 5.5. Let A be a category of $\mathbb{Q}$-vector spaces. Instead of $\mathbb{Z}[1/2]$, we take the coefficient ring to be $\mathbb{Q}$, i.e. we define the (reduced) tautological functor $F_1 : Cob^3_{/l}(\phi) \to A$ as

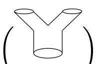
$$F_1(O) := \mathbb{Q} \otimes_{\mathbb{Z}} \frac{\text{Mor}(\phi, O)}{R(\phi, O)},$$

for $O \in \text{Obj}(Cob^3_{/l}(\phi))$. Then F_1 will be the same functor TQFT F of Definition 5.18.

Proof. Here is the abbreviated proof.

- we have already seen, that $F_1(n \text{ disjoint loops}) = V^{\otimes n}$.
- Note that,

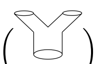
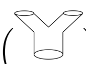
 $\in$ Mor(two disjoint loops, single loop).

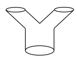
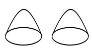
$F_1\big($  $\big)$: $F_1($ two disjoint loops$) \to F_1($single loop$)$.

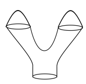
But,

$$F_1(\text{two disjoint loops}) = V \otimes V, \;\; F_1(\text{single loop}) = V.$$

Now,

$F_1\big($  $\big) \circ (v_+ \otimes v_+) = F_1\big($  $\big) \circ \big($  $\big)$

$=$  $\circ$ 

$=$ 

$= v_+$

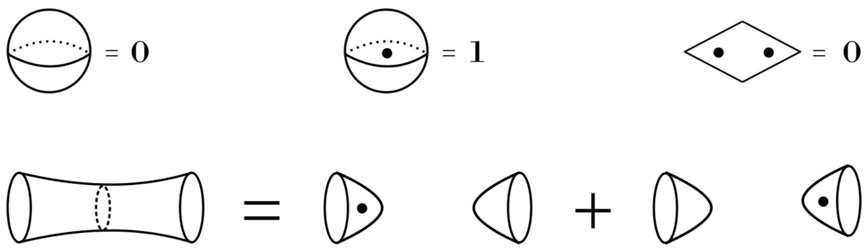
$= m(v_+ \otimes v_+)$.

Similarly we prove for rest of the cases.

■

5.9 Dotted cobordism

We would extend the category Cob^3 to a new category $Cob^3_\bullet$. The objects in this new category are same as Cob^3, and the morphisms are same too, except now we allow dots on a cobordism and these dots can be moved freely on each connected component of the cobordism. $Cob^3_{\bullet/l}$ is the category $Cob^3_\bullet$ modulo the local relations as follows:



The S, T and $4Tu$ relations follow from the above-mentioned relations. Now this construction of the category $Cob^3_{\bullet/l}$ gives us one facility: if we apply tautological functor F_ϕ (as defined in the Definition 5.19) on $Cob^3_{\bullet/l}(\phi)$ and consider $v_+ =$  and $v_- =$ , we get the original Khovanov cohomology theory regardless of any restriction on the coefficient ring.

Chapter 6

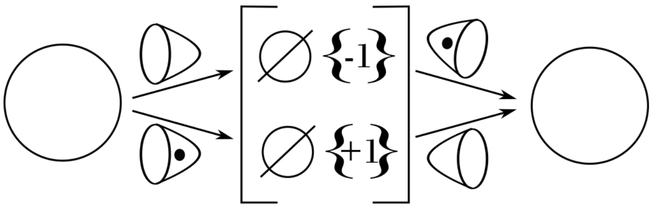
Fast Computations

6.1 Delooping and Gaussian elimination

In this chapter, we compute Khovanov cohomology for figure eight knot using a faster algorithm due to Bar-Natan [5]. This algorithm requires applying the following two Lemmas repeatedly.

Lemma 6.1. Let S be an object in $Cob^3_{\bullet/l}$ such that it has a simple closed curve t as one connected component. Let S' be another object in $Cob^3_{\bullet/l}$, which is the result of removing t from S. Then S is isomorphic to $S'\{+1\} \oplus S'\{-1\}$ in the category $\mathrm{Mat}(Cob^3_{\bullet/l})$.

Proof. Here we are only providing the proof for the case when S is just single loop. In this case the isomorphisms are as follows:



We can easily extend the proof for general S. ■

Bar-Natan [5] called this Lemma 6.1 as delooping lemma.

Lemma 6.2. If $\phi : b_1 \to b_2$ is an isomorphism (in an additive category K), then the four term complex segment in $\mathrm{Mat}(K)$

$$\cdots \begin{bmatrix} C \end{bmatrix} \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix} b_2\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} \begin{bmatrix}F\end{bmatrix}\cdots \tag{I}$$

is isomorphic to the following complex segment:

$$\cdots\begin{bmatrix} C \end{bmatrix} \xrightarrow{\begin{pmatrix} 0 \\ \beta \end{pmatrix}} \begin{bmatrix} \color{red}{b_1} \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix} \color{red}{\phi} & 0 \\ 0 & \epsilon - \gamma\phi^{-1}\delta \end{pmatrix}} \begin{bmatrix} \color{red}{b_2} \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix} 0 & \nu \end{pmatrix}} \begin{bmatrix} F \end{bmatrix}\cdots \tag{II}$$

both of these complexes I and II are cochain homotopy equivalent to the following (simpler) complex segment:

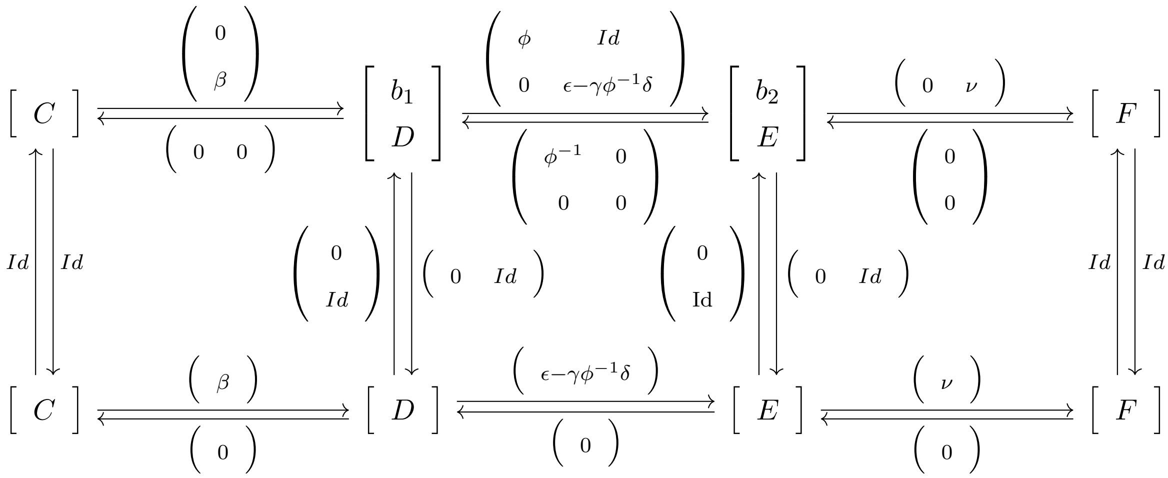
$$\cdots\begin{bmatrix} C \end{bmatrix} \xrightarrow{\begin{pmatrix} \beta \end{pmatrix}} \begin{bmatrix} D \end{bmatrix} \xrightarrow{\begin{pmatrix} \epsilon - \gamma\phi^{-1}\delta \end{pmatrix}} \begin{bmatrix} E \end{bmatrix} \xrightarrow{\begin{pmatrix} \nu \end{pmatrix}} \begin{bmatrix} F \end{bmatrix}\cdots \tag{III}$$

where C, D, E and F are arbitrary columns of objects (i.e. formal direct sum of objects) in K, and α, β, δ, γ, ϵ, μ, ν are morphisms in $\mathrm{Mat}(K)$ i.e. they are all matrices of morphisms in K. b_1 and b_2 are individual objects of K, but they can equally be considered as columns of objects in K provided (the morphism matrix) ϕ remains invertible.

Proof. The isomorphism between I and II is as following:

$$\begin{array}{ccccccc}
\begin{bmatrix} C \end{bmatrix} & \xrightarrow{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} & \begin{bmatrix} b_1 \\ D \end{bmatrix} & \xrightarrow{\begin{pmatrix} \phi & \delta \\ \gamma & \epsilon \end{pmatrix}} & \begin{bmatrix} b_2 \\ E \end{bmatrix} & \xrightarrow{\begin{pmatrix} \mu & \nu \end{pmatrix}} & \begin{bmatrix} F \end{bmatrix} \\
Id \uparrow\downarrow Id & & \begin{pmatrix} \phi^{-1} & -\phi^{-1}\delta \\ 0 & \mathrm{Id} \end{pmatrix} \uparrow\downarrow \begin{pmatrix} \phi & \delta \\ 0 & Id \end{pmatrix} & & \begin{pmatrix} \phi^{-1} & 0 \\ \gamma\phi^{-2} & \mathrm{Id} \end{pmatrix} \uparrow\downarrow \begin{pmatrix} \phi & 0 \\ -\gamma\phi^{-1} & \mathrm{Id} \end{pmatrix} & & Id \uparrow\downarrow Id \\
\begin{bmatrix} C \end{bmatrix} & \xrightarrow[\begin{pmatrix} 0 \\ \beta \end{pmatrix}]{} & \begin{bmatrix} b_1 \\ D \end{bmatrix} & \xrightarrow[\begin{pmatrix} \phi & 0 \\ 0 & \epsilon - \gamma\phi^{-1}\delta \end{pmatrix}]{} & \begin{bmatrix} b_2 \\ E \end{bmatrix} & \xrightarrow[\begin{pmatrix} 0 & \nu \end{pmatrix}]{} & \begin{bmatrix} F \end{bmatrix}
\end{array}$$

The cochain homotopies and cochain homotopy equivalence maps between II and III can be given as follows:



where the left directional arrows are cochain homotopy maps, downward directional and upward directional maps are cochain homotopy equivalences between II and III. ■

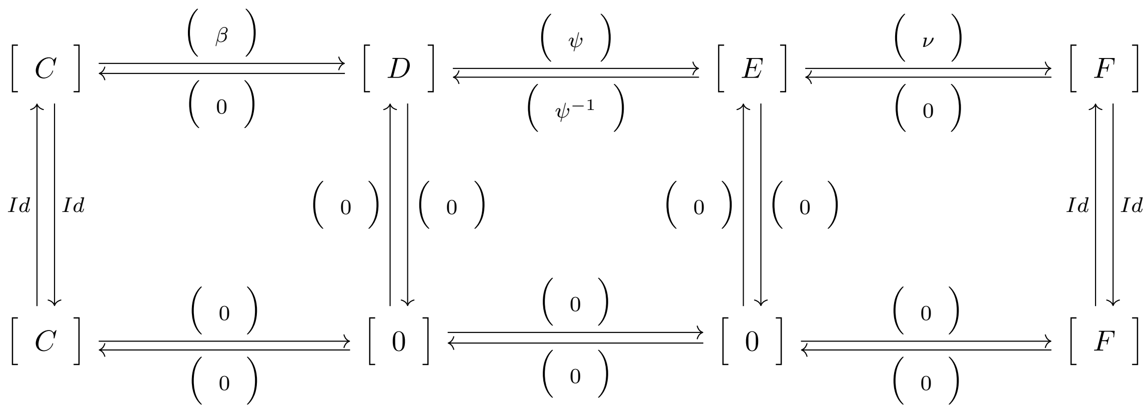
Remark 6.1. Let K be a nice abelian category like the category of graded vector spaces over $\mathbb{Q}$ and grading preserving maps, and C, D, E, F are columns of objects in K, $\psi : D \to E$ is an isomorphism in $\mathrm{Mat}(K)$, and β and γ are morphisms in $\mathrm{Mat}(K)$, then the following complex

$$\cdots\left[\,C\,\right] \xrightarrow{\left(\,\beta\,\right)} \left[\,D\,\right] \xrightarrow{\left(\,\psi\,\right)} \left[\,E\,\right] \xrightarrow{\left(\,\nu\,\right)} \left[\,F\,\right]\cdots \qquad \text{(IV)}$$

is cochain homotopic equivalent to

$$\cdots\left[\,C\,\right] \xrightarrow{\left(\,0\,\right)} \left[\,0\,\right] \xrightarrow{\left(\,0\,\right)} \left[\,0\,\right] \xrightarrow{\left(\,0\,\right)} \left[\,F\,\right]\cdots \qquad \text{(V)}$$

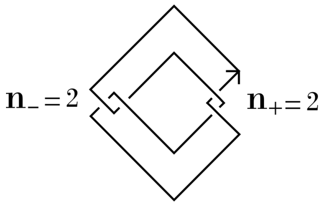
Proof. The homotopy equivalence between IV and V and cochain homotopy maps are as follows



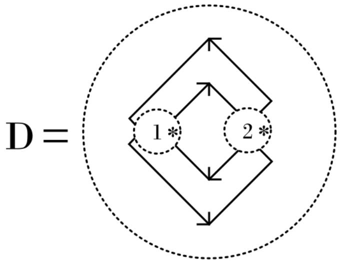
where the left directional arrows are cochain homotopy maps, downward directional and upward directional maps are cochain homotopy equivalences between IV and V. ■

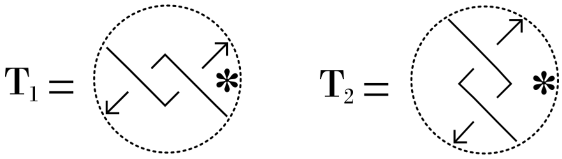
6.2 Computation for figure eight Knot

We now calculate the Khovanov cohomology for the following diagram T of figure eight knot.

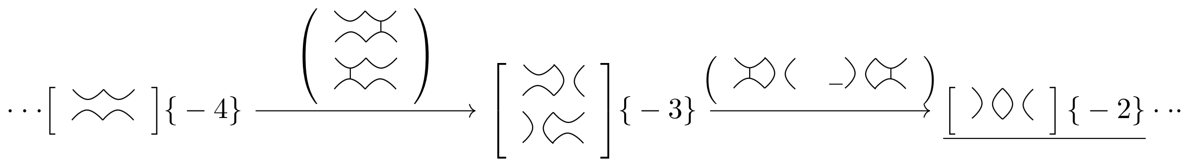


Consider the following 2-input planner arc diagram D and the tangles T_1 and T_2.

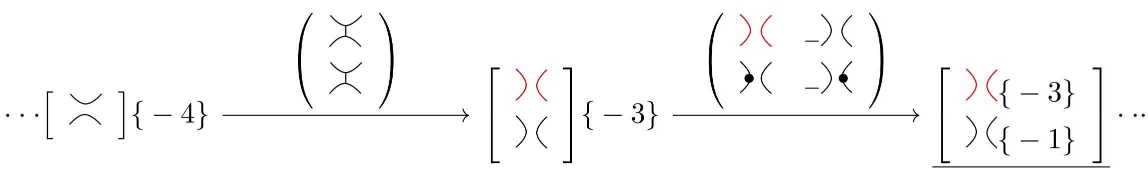




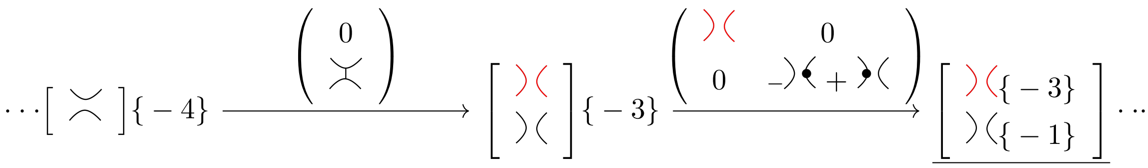
We can take figure eight knot as $D(T_1, T_2)$. As per Theorem 5.3, $Kh(T)$ is cochain homotopic equivalent to $D\big(Kh(T_1), Kh(T_2)\big)$.
In each of the following cochain complexes the height zero object is underlined and we use the notation to denote the saddle cobordism from to . Also for simplicity, we use notation to denote the identity morphism of .
$Kh(T_1)$:



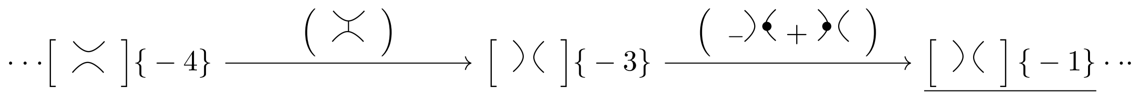
Applying the Lemma 6.1 we get that $Kh(T_1)$ is isomorphic to the following complex.
$Kh(T_1)^1$:



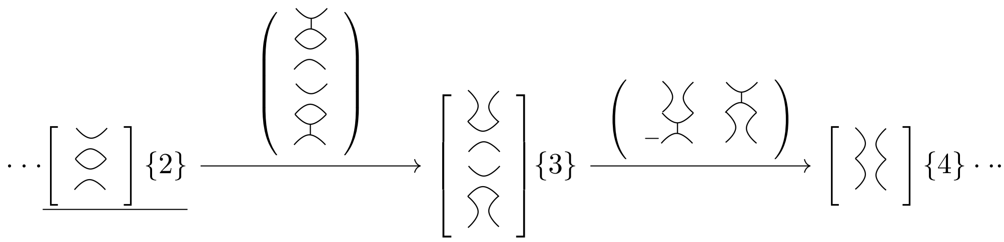
Applying the Lemma 6.2 we get that the cochain complex $Kh(T_1)^1$ is isomorphic to the following complex.
$Kh(T_1)^2$:



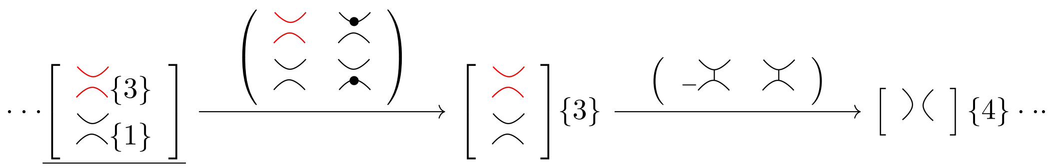
And this $Kh(T_1)^2$ is cochain homotopic equivalent to
$Kh(T_1)^3$:



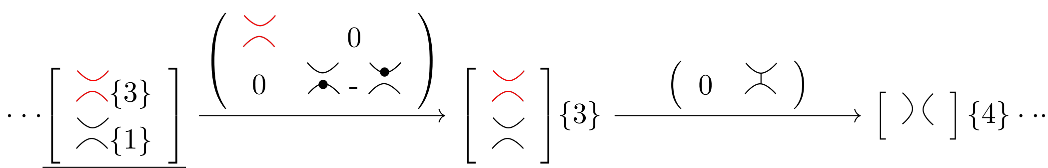
Now we simplify $Kh(T_2)$ in the same way as we have done for $Kh(T_1)$. The steps are as follows:
$Kh(T_2)$:



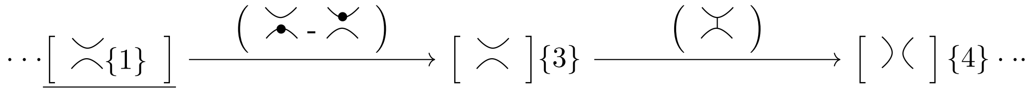
Applying the Lemma 6.1, the cochain complex $Kh(T_2)$ be isomorphic to the following cochain complex,
$Kh(T_2)^1$:



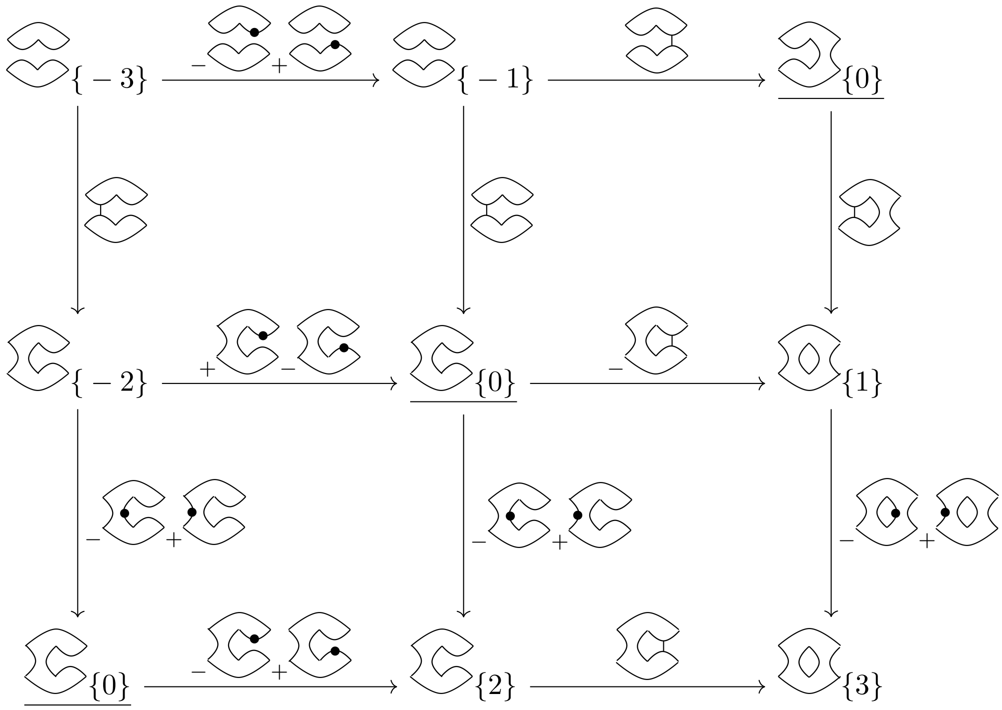
Now applying the Lemma 6.2 we get that the cochain complex $Kh(T_2)^1$ is isomorphic to the following cochain complex,
$Kh(T_2)^2$:



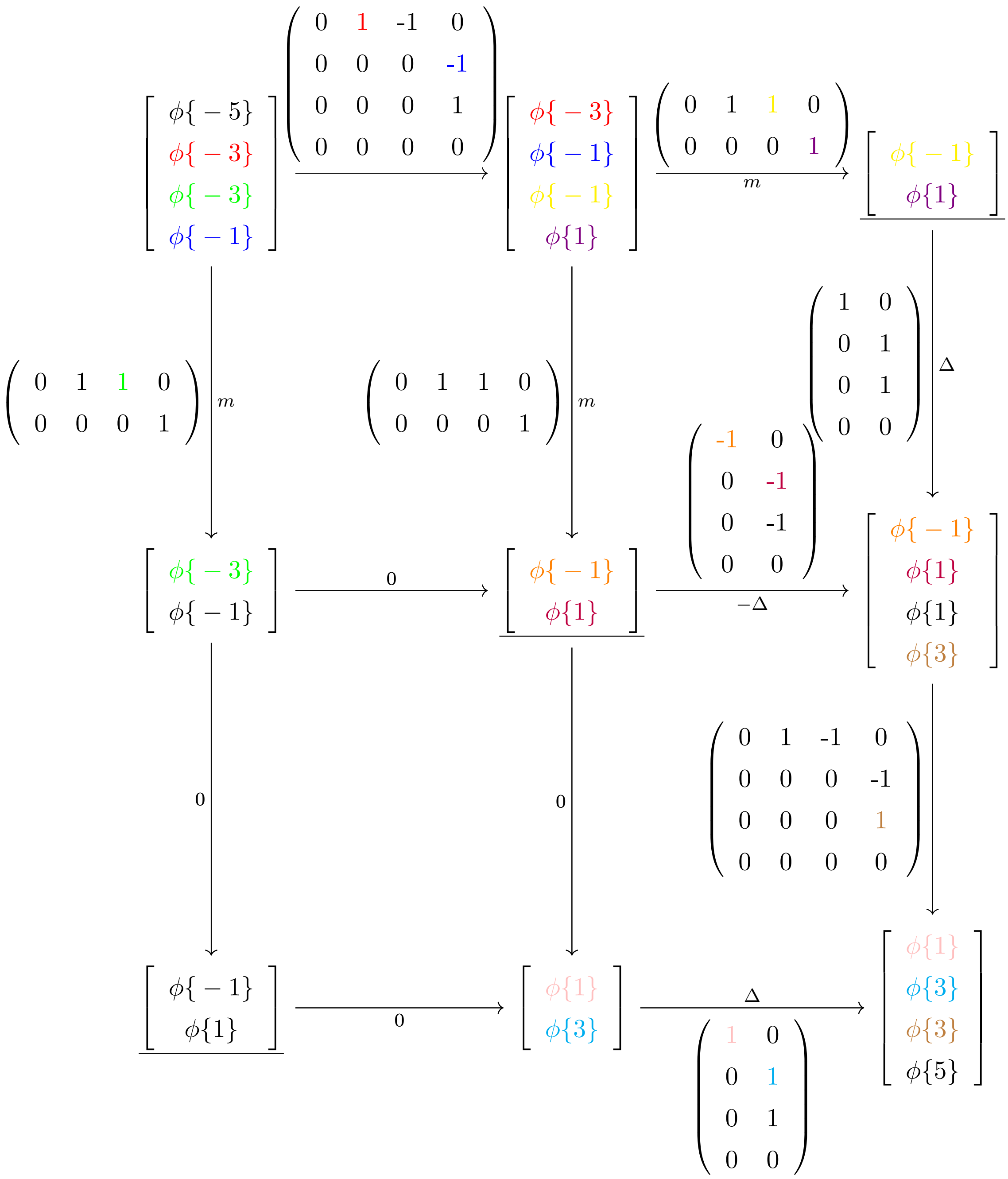
And this complex is cochain homotopic equivalent to the following complex,
$Kh(T_2)^3$:



Now instead of $D(Kh(T_1), Kh(T_2))$, we consider $D(Kh(T_1)^3, Kh(T_2)^3)$ since they are cochain homotopic equivalent. The following anti-commutative diagram is the tensor product of $Kh(T_1)^3$ and $Kh(T_2)^3$.



Since one can move the dots anywhere around the cobordisms, all the four morphisms of the lower left part (2-cube) are zero. Applying the Lemma 6.1, we can remove all loops. Thus we get anti-commutative diagram where all objects are (degree-shifted) ϕ and morphisms are matrices, such that each entries are $\mathbb{Z}$ linear combinations of cobordisms between empty object ϕ. But in the category $Cob^3_{\bullet/l}$ if we apply the local relations, these entries are nothing but integers. Hence after applying the Lemma 6.1, we get the following anti-commutative diagram.



Here we have used same special color for the objects which are isomorphic (we have also marked the isomorphism with the same special color). Now note that if we apply TQFT or tautological functor (as mentioned in Section 5.7 and 5.8) to reach at the category of graded vector spaces, the object ϕ would correspond to $\mathbb{Q}$ (as $\mathbb{Q}$ vector space). Any non-zero entry a of the matrix morphism of the above anti-commutative diagram would correspond to a linear map $x \mapsto ax$, $\forall x \in \mathbb{Q}$. Hence, any nonzero entry would correspond to an linear isomorphism over $\mathbb{Q}$. If we apply the Lemma 6.2 again and again (to simplify the diagram by removing the isomorphic objects), the process would stop only when all matrices are 0 (see Remark 5.1).

Hence, we get the final simplified complex as follows:

$$\phi\{-5\} \xrightarrow{0} \phi\{-1\} \xrightarrow{0} \underline{\begin{bmatrix} \phi\{-1\} \\ \phi\{1\} \end{bmatrix}} \xrightarrow{0} \phi\{1\} \xrightarrow{0} \phi\{5\} \ .$$

After applying the functor (as mentioned above) we get:

$$\mathbb{Q}\{-5\} \xrightarrow{0} \mathbb{Q}\{-1\} \xrightarrow{0} \underline{\mathbb{Q}\{-1\} \oplus \mathbb{Q}\{1\}} \xrightarrow{0} \mathbb{Q}\{1\} \xrightarrow{0} \mathbb{Q}\{5\} \ .$$

Hence the graded Poincaré polynomial of $Kh(T)$ is

$$\sum_r t^r qdim(H^r(Kh(T))) = t^{-2}q^{-5} + t^{-1}q^{-1} + (q + q^{-1})t^0 + t^1q^1 + t^2q^5.$$

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