Geometry of Dynamical Systems

Rajesh Kumar Bajiya

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Department of Mathematical Sciences
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Contents

Certificate of Examination

This is to certify that the dissertation titled "Geometry of Dynamical Systems" submitted by Rajesh Kumar Bajiya (Reg. number – MS15173) for the partial fulfilment of BS-MS dual degree program of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Lingaraj Sahu **Dr. Pranab Sardar** **Dr. Soma Maity** (Supervisor)

Date: May 2020

Declaration of Authorship

The work presented in this dissertation has been carried out by me under the guidance of **Dr. Soma Maity** at the Indian Institute of Science Education and Research, Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellow- ship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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Date: May 2020

In my capacity as the supervisor of the candidate′s project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Soma Maity(Supervisor)
Date: May 2020

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Abstract

In this thesis, we look into various aspects of local and global theory of Dynamical Systems. We primarily employ the stable manifold theorem and the Hartman-Grobman theorem. Using these theorems we have determined the qualitative structure of non-linear systems. We have studied the type and the behaviour of hyperbolic and non-hyperbolic critical points of non-linear systems. The stability of the periodic orbits is also determined by the various concepts of dynamical systems thoroughly.

Chapter 1

Introduction

We require proper understanding and knowledge of non-linear systems to solve real-world physical problems, along with utilizing them to work with various biological models. We can solve linear systems very easily by (2.1.1). However, in case of non-linear systems, it is not possible to solve all kinds of equations; but one can analyse local and global behaviour of solutions using existing mathematical techniques. The Stable Manifold theorem (3.1.2) and the Hartman- Grobman theorem (4.1.1) are therefore brought in for such study. These two theorems show that topologically, the local behaviour of the non-linear system $\dot{x} = f(x)$ near an equilibrium point x_0 can be determined by the behaviour of the linear system $\dot{x} = Ax$ near the origin where $A = Df(x_0)$.

However, these theorems are valid only for hyperbolic critical points. What is expected in case of non-hyperbolic critical points? For those non-hyperbolic critical points, we can use the Liapunov method (3.3.2) to observe whether a non-hyperbolic equilibrium point is stable, asymptotically stable, or an unstable critical point.

Poincare-Hopf index theorem is a fundamental theorem in the study of vector fields on surfaces. It relates the index of a vector field and the Euler charateristic of the surface. A partial proof of this theorem may be obtained by studying the global behaviour of the dynamical system originated by the given vector field.

To begin with, In the Chapter 2, we discuss some basic definitions along with some essential preliminary concepts which are related to linear and non-linear systems i.e., stable, unstable, and center subspaces. We also introduce Picard's existence and uniqueness theorem, Maximal interval of existence, the flow of non-linear systems, linearisation, hyperbolic critical points. A proof of Gronawall's lemma is also provided in this chapter which is required to prove the main theorem in Chapter 5.

The prime theorem of this thesis stable manifold theorem is explained in the chapter 3 with the help of several examples. There are also some applications of this theorem which describe the geometry of the non-linear system. Besides the stable manifold theorem, the center manifold (3.2.1) and the Lyapunov stability (3.3.2) are also important concepts to observe the local geometry of the system. The various type of the sectors and non-hyperbolic critical points are also mentioned in chapter 3. Using theorems (3.5.2) and (3.5.3) , a computer program (3.5) is used and run to check whether a non-hyperbolic critical point is a topological saddle, a center or a focus, a node, a critical point with an elliptic domain, a cusp or a saddle-node, given that the dynamical system is in the normal form.

At the end of chapter 3, two exciting types of systems (Gradient system and Hamiltonian systems) are prodded into, (3.6) that arise in a physical problems. We also look at an interesting relationship between Gradient and Hamiltonian systems and the type of critical points of these two systems (see 3.6.5).

Moving towards chapter 4, the Hartman-Grobamn theorem is proved in this chapter. This theorem states that near a hyperbolic critical point x_0, the linear and the non-linear systems are topologically equivalent.

We have defined the time t in some interval let say maximal interval of existence, but if we wide the interval of time to $t \in \mathbb{R}$, then the non-linear system is called the dynamical system and hence we are moving to chapter 5.

In the last chapter of this thesis, the global theory of non-linear dynamical systems is discussed.

Chapter 5 begins with the definition and properties of a dynamical system. The Global existence theorem and its applications are also mentioned there, to show the topological equivalence between a non-linear system and its linearization. Furthermore, this theorem also tells whether the non-linear system is a dynamical system or not. To understand the geometry of a dynamical system or a non-linear system, the limit set, attractors, periodic orbits and limit cycles are explained with the help of examples. The Poincare map (5.5.1) is introduced in this chapter to show the stability of limit cycles (see 5.5.1). For the periodic orbits, the stable manifold and center manifold theorem are explained with the help of some geometrical examples. This thesis, therefore, indulges with the very elegant theorem called the Poincare-index theorem (5.7.1).

Our main reference is the book "Differential Equations and Dynamical Systems" by Lawrence Perko [1].

Chapter 2

Preliminaries

2.1 Definitions and Preliminary Concepts

2.1.1 Linear system

Let A be an $n \times n$ matrix. Consider the following linear system

$$\dot{x} = Ax$$

The solution of this linear system with the initial condition $x(0) = x_0$ is given by

$$x(t) = e^{At}x_0 \;\; \forall t \in \mathbb{R}.$$

2.1.2 Stable, unstable and center subspaces

Let $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$ be an eigenvector of the matrix A corresponding to an eigenvalue $\lambda_j = \alpha_j + i\beta_j$. Then E^s, E^c, E^u are stable, center and unstable subspaces respectively.

$$\begin{aligned} E^s &= Span\,\{\mathbf{u}_j, \mathbf{v}_j | \alpha_j < 0\}, \\ E^c &= Span\,\{\mathbf{u}_j, \mathbf{v}_j | \alpha_j = 0\}, \\ E^u &= Span\,\{\mathbf{u}_j, \mathbf{v}_j | \alpha_j > 0\}. \end{aligned}$$

Theorem 2.1.1. *Let A be an $n \times n$ matrix, then we can write the following*

$$\mathbb{R}^n = E^s \oplus E^c \oplus E^u.$$

Also E^s, E^c, and E^u are invariant under the flow e^{At}.

2.1.3 The fundamental existence and uniqueness theorem

Theorem 2.1.2. *[1] Consider the following initial value problem (IVP).*

$$\begin{aligned} \dot{x} &= f(x) \\ x(0) &= x_0 \end{aligned} \tag{2.1}$$

Let E be an open subset of $\mathbb{R}^n$, which contains x_0 and $f \in C^1(E)$. Then there is a $d > 0$ and a $\delta > 0$ such that for all $a \in U_\delta(x_0)$ the above IVP has a unique solution $u(t, a)$, where $u \in C^1(W)$ and $W = [-d, d] \times U_\delta(x_0) \subset \mathbb{R}^{n+1}$.

2.1.4 Maximal interval of existence

Theorem 2.1.3. *(Maximal interval of existence) [1] Let E be an open subset of $\mathbb{R}^n$, which contains x_0 and $f \in C^1(E)$. Then $\forall x_0 \in E$, there exists a maximal open interval L where the IVP (2.1) has a unique solution $x(t)$ or we can say that if the IVP (2.1) has a solution $u(t)$ on the interval I then for all $t \in I$, $I \subset L$ and $u(t) = x(t)$.*

L is of the form (α, β) and it is called the maximal interval of existence of the IVP (2.1). We need the following lemma to prove the Global existence theorem (5.2.1) and similar theorems in Chapter 5.

Lemma 2.1.4. *Let E be an open subset of $\mathbb{R}^n$ which contains x_0, and $f \in C^1(E)$. Let $[0, \beta)$ be the right maximal interval of solution $x(t)$ of the initial value problem (2.1). If there is a compact set $C \subset E$ such that $\{b \in \mathbb{R}^n | b = x(t) \text{ for some } t \in [0, \beta)\} \subset C$ then $\beta = \infty$. Hence IVP (2.1) has a solution $x(t)$ on $[0, \infty)$.*

2.1.5 Flow of the differential equation

Let E be an subset of $\mathbb{R}^n$ and $f \in C^1(E)$. Let $\phi(t, x_0)$ be the solution of the non-linear system (2.1) for $x_0 \in E$ and defined on its maximal interval $\mathbb{I}(x_0)$. Then for $\forall t \in \mathbb{I}(x_0)$

$$\begin{aligned} &\phi_t : E \to \mathbb{R}^n \\ &\phi_t(x_0) := \phi(t, x_0) \end{aligned}$$

ϕ_t is called the flow of the non-linear differential equation.

2.1.6 Linearization

The following linear system

$$\dot{x} = Ax \tag{2.2}$$

is called the linearisation of the non-linear system (2.1) at x_0 with the matrix $A = Df(x_0)$.

2.1.7 Invariant

Let E be an subset of $\mathbb{R}^n$ and $f \in C^1(E)$. Consider $\phi_t : E \to E$ be the flow of the non-linear system $\dot{x} = f(x)$, $\forall t \in \mathbb{R}$, then a set $W \subseteq E$ is called invariant with respect to the flow ϕ_t if $\phi_t(W) \subseteq W$, $\forall t \in \mathbb{R}$.

2.1.8 Hyperbolic equilibrium point(Critical Point)

A point $x_0 \in \mathbb{R}$ is called a hyperbolic critical point of the non-linear system (2.1), if the eigenvalues of matrix $\mathrm{D}f(x_0)$ have no zero real part.

2.1.9 Gronwall's Lemma

We need the following lemma to prove theorem (5.2.5).

Lemma 2.1.5. *(Gronwall's Lemma) Let $h(t)$ be the continuous real valued function with $h(t) \geq 0$, and*

$$h(t) = c + k \int_0^t h(s) ds$$

for all $t \in [0, a]$ and $c, k \geq 0$, then

$$h(t) \leq ce^{kt} \quad \forall t \in [0, a].$$

Proof. We have

$$h(t) = c + k\int_0^t h(s)ds$$

From the fundamental theorem of calculus

$$\begin{aligned}
&h'(t) = k.h(t) \\
\Rightarrow\ &\frac{h'(t)}{h(t)} \leq k \\
\Rightarrow\ &\frac{d(\log h(t))}{dt} \leq k \\
\Rightarrow\ &\log h(t) \leq k.t + \log h(0) \\
\Rightarrow\ &h(t) \leq h(0)e^{k.t}
\end{aligned}$$

And hence,

$$h(t) \leq ce^{kt} \quad \forall t \in [0, a].$$

□

Chapter 3

The Stable Manifold Theorem and Its Applications

3.1 The Stable Manifold Theorem

This theorem shows that near a hyperbolic critical point x_0, the non-linear system $\dot{x} = f(x)$ has a stable and a unstable manifolds S and U which are tangent to the stable and unstable subspace E^s and E^u of the linearazied system $\dot{x} = Ax$ at x_0, where $A = Df(x_0)$.

To understand this theorem first, we will study the stability of the linear system by a simple example. Then in the stable manifold theorem we will see the stability of non-linear systems. Consider the following 2-dimensional Linear system;

Example 3.1.1.

$$\dot{x} = -2x$$
$$\dot{y} = 3y$$

Let ϕ_t be the solution of this linear system.

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$$

Since A has two real and opposite sign eigenvalues, so this system has a saddle point at the origin. We have

$$x(t) = a_1 e^{-2t}$$
$$y(t) = a_2 e^{3t}$$

Here the stable subspace(E^s) and unstable subspace(E^u) are

$$E^s = span\{(1, 0)\}$$
$$E^u = span\{(0, 1)\}$$

So

$$\lim_{t\to\infty} \phi_t(a) = 0$$

Only when $a \in \mathbb{R}^s$.

Lemma 3.1.1. *Let E be an open subset of $\mathbb{R}^n$ containing the origin. If $f \in C^1(E)$, and $f(0) = Df(0) = 0$, then for any $\epsilon > 0$, $\exists$ a $\delta > 0$ such that $\forall x, y \in N_\delta(0)$, we have*

$$|f(x) - f(y)| \leq \epsilon|x - y|$$

Proof. Given $\epsilon > 0$, $\exists$ a δ such that $\|x\| < \frac{\delta}{2}, \|y\| < \frac{\delta}{2}$ implies that

$$\|x - y\| < \delta$$

Now, $Df(0) = 0$ and Df is continuous. Hence $\lim_{x\to 0} \|Df(x)\| \to 0$.
So from the definition of Df, given $\epsilon' > 0$, $\exists$ a $\delta' > 0$ such that $\|x\| < \delta'$ implies that $\|Df(x)\| < \epsilon'$.
Take $\delta'' = min(\frac{\delta}{2}, \delta')$, $\forall \|x\| < \delta''$, $\|y\| < \delta''$.
Consider

$$\begin{aligned} |\|f(x) - f(y)\| - \|Df(x).(y - x)\|| &\leq \|f(y) - f(x) - Df(x).(y - x)\| \\ &\leq \epsilon\|x - y\| \end{aligned}$$

And hence

$$\begin{aligned} \|f(x) - f(y)\| &\leq (\|Df(x)\| + \epsilon)\|(y - x)\| \\ &\leq (\epsilon' + \epsilon)\|(y - x)\| \\ &= \epsilon''\|(y - x)\|. \end{aligned}$$

□

Theorem 3.1.2. *(The Stable Manifold Theorem)*
Let E be an open subset of $\mathbb{R}^n$ containing the origin, and $f \in C^1(E)$ with $f(0) = 0$. Let ϕ_t be the flow of the non-linear system $\dot{x} = f(x)$. If $Df(0)$ has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part, then $\exists$ a k-dimensional differential manifold S which is tangent to the stable subspace E^s of the linearised system and near the origin it is positively invariant with respect to the flow ϕ_t, i.e. $\forall t \geq 0$, $\phi_t(S) \subset S$ and $\forall x_0 \in S$

$$\lim_{t\to\infty} \phi_t(x_0) = 0$$

also $\exists$ a n-k - dimensional differential manifold U which is tangent to the unstable subspace E^u of the linearised system and near the origin it is negatively invariant with respect to flow ϕ_t i.e. it satisfies $\forall t \leq 0$, $\phi_t(U) \subset U$ and $\forall x_0 \in U$

$$\lim_{t\to-\infty} \phi_t(x_0) = 0.$$

Proof. It is given that $f \in C^1(E)$ and $f(0) = 0$, then we can rewrite the non-linear equation as

$$\dot{x} = Ax + \mathcal{F}(x) \tag{3.1}$$

where $A = Df(0)$ and $\mathcal{F}(x) = f(x) - Ax$.
It is clear that $\mathcal{F} \in C^1(E)$ and also $\mathcal{F}(0) = 0$ and $D\mathcal{F}(0) = 0$.
From lemma (3.1.1), $\mathcal{F}$ satisfies the local lipschitz condition i.e.
for all $\epsilon > 0$ there exists a $\delta > 0$ such that |x| $\leq \delta$, |y| $\leq \delta$ implies that

$$|\mathcal{F}(x) - \mathcal{F}(y)| \leq \epsilon|x - y| \tag{3.2}$$

Now we will diagonalise matrix A; i.e. there exist an invertible matrix B such that

$$C = B^{-1}AB = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$$

Here X is a $k \times k$ block matrix when the real part is negative, and Y is an $n-k \times n-k$ block matrix when the real part is positive.
Let $y = B^{-1}x$ and transform (3.1) as:

$$\dot{y} = Cy + \mathcal{G}(y) \tag{3.3}$$

where $\mathcal{G}(y) = B^{-1}\mathcal{F}(By)$ and $\mathcal{G}(0) = 0$; $D\mathcal{G}(0) = 0$.
Since $\tilde{E}$ is homeomorphic to E as B is an invertible matrix, so $\mathcal{G}(y) \in C^1(\tilde{E})$, where $\tilde{E} = C^{-1}(E)$. Hence, $\mathcal{G}(y)$ satisfies the Lipschitz condition. Now consider

$$P(t) = \begin{bmatrix} e^{Xt} & 0 \\ 0 & 0 \end{bmatrix}, Q(t) = \begin{bmatrix} 0 & 0 \\ 0 & e^{Yt} \end{bmatrix}$$

Clearly, we can see that $\dot{P}(t) = CP$, and $\dot{Q}(t) = CQ$. Also

$$e^{Ct} = P(t) + Q(t) \tag{3.4}$$

The Matrix norm can be defined as:

$$\begin{aligned} \|P(t)\| &\leq \eta e^{-(\alpha+\sigma)t} \qquad \forall t \geq 0 \\ \|Q(t)\| &\leq \eta e^{\sigma t} \qquad\qquad \forall t \leq 0 \end{aligned} \tag{3.5}$$

Where $\alpha = max(\alpha_i)$, $\sigma = min(\sigma_i)$ and η is sufficiently large.
Now Consider the following integral equation;

$$h(t,a) = P(t)a + \int_0^t P(t-s)\mathcal{G}(h(s,a))ds - \int_t^\infty Q(t-s)\mathcal{G}(h(s,a))ds \tag{3.6}$$

It is a solution of the differential equation (3.3), if $h(t,a)$ is a continuous solution of this integral equation.

We will solve this integral equation using the method of successive approximations.
For that consider $h^{(0)}(t,a) = a$, and the following sequence

$$h^{(j+1)}(t,a) = P(t)a + \int_0^t P(t-s)\mathcal{G}(h^{(j)}(s,a))ds - \int_t^\infty Q(t-s)\mathcal{G}(h^{(j)}(s,a))ds$$

We will show that this is a Cauchy sequence.
For that assume the induction hypothesis;

$$\left| h^{(j)}(t,a) - h^{(j-1)}(t,a) \right| \leq \frac{\eta |a| e^{-\alpha t}}{2^{j-1}} \tag{3.7}$$

this holds for $j = 1, \ldots, m$.
From the induction hypothesis $\forall t \geq 0$, we have

$$\begin{aligned} |h^{m+1}(t,a) - h^m(t,a)| &\leq \int_0^t \|P(t-s)\| \epsilon |h^{(m)}(s,a) - h^{(m-1)}(s,a)| ds \\ &+ \int_t^\infty \|Q(t-s)\| \epsilon |h^{(m)}(s,a) - h^{(m-1)}(s,a)| ds \\ &\leq \epsilon \int_0^t \eta e^{-(\alpha+\sigma)(t-s)} \ \frac{\eta |a| e^{-\alpha s}}{2^{m-1}} ds \\ &+ \epsilon \int_0^\infty \eta e^{\sigma(t-s)} \ \frac{\eta |a| e^{-\sigma s}}{2^{m-1}} ds \\ &\leq \frac{\epsilon \eta^2 |a| e^{-\alpha t}}{\sigma 2^{m-1}} + \frac{\epsilon \eta^2 |a| e^{-\alpha t}}{\sigma 2^{m-1}} \\ &< \left(\frac{1}{4} + \frac{1}{4} \right) \frac{\eta |a| e^{-\alpha t}}{\sigma 2^{m-1}} = \frac{\eta |a| e^{-\alpha t}}{2^m}. \end{aligned}$$

Here $\frac{\epsilon\eta}{\sigma} < \frac{1}{4}$, i.e $\epsilon < \frac{\sigma}{4\eta}$.
Hence, (3.7) is true for all j.

$$\begin{aligned} |h^{(j)}(t,a) - h^{(j-1)}(t,a)| &\leq \frac{\eta|a|e^{-\alpha t}}{2^{j-1}} \\ &< \frac{\eta'}{2^{j-1}} \qquad \textit{since } |a| < \frac{\delta}{2\eta} \end{aligned}$$

Now for $n > m > N$,

$$\begin{aligned} |h^{(n)}(t,a) - h^{(m)}(t,a)| &\leq \sum_{j=N}^{\infty} |h^{(j+1)}(t,a) - h^{(j)}(t,s)| \\ &\leq \eta|a| \sum_{j=N}^{\infty} \frac{1}{2^j} = \frac{\eta|a|}{2^{N-1}} \end{aligned} \tag{3.8}$$

Since $|a|$ is bounded, so the above goes to 0 as N goes to ∞.
Hence, we can say that it is a uniformly Cauchy. Also, it implies uniformly convergences. This shows that $h(t,a)$ is differentiable.

$$\lim_{j\to\infty} (h^{(j)}(t,a)) = h(t,a)$$

Satisfies (3.6), and (3.3) also.

$$\begin{aligned} |h^{(j)}(t,a) - h^{(j-1)}(t,a)| &< \frac{\eta|a|}{2^{j-1}} \\ ||h^{(j)}(t,a)| - |h^{(j-1)}(t,a)|| &\leq |h^{(j)}(t,a) - h^{(j-1)}(t,a)| \\ ||a| - |b|| &\leq |a-b| \quad \text{(reverse triangle inequality)} \end{aligned}$$

$$\begin{aligned} |h^{(j)}(t,a)| &< \frac{\eta|a|e^{-\alpha t}}{2^{j-1}} + |h^{(j-1)}(t,a)| \quad \forall j \\ |h^{(j-1)}(t,a)| &< \frac{\eta|a|e^{-\alpha t}}{2^{j-2}} + |h^{(j-2)}(t,a)| \\ |h^{(j)}(t,a)| &< \frac{\eta|a|e^{-\alpha t}}{2^{j-1}} + \frac{\eta|a|e^{-\alpha t}}{2^{j-2}} + |h^{(j-2)}(t,a)| \\ &< \frac{\eta|a|e^{-\alpha t}}{2^{j-1}} + \frac{\eta|a|e^{-\alpha t}}{2^{j-2}} + \frac{\eta|a|e^{-\alpha t}}{2^{j-3}} + |h^{(j-3)}(t,a)| \\ &\vdots \\ &< e^{-\alpha t}\eta|a|\{\frac{1}{2^{j-1}} + \dots\} \\ &< e^{-\alpha t}\eta|a|\{1 + \frac{1}{2} + \dots\} = 2e^{-\alpha t}\eta|a| \end{aligned}$$

$$\begin{aligned} \lim |h^j(t,a)| &< 2e^{-\alpha t}|a| \\ h(t,a) &< 2e^{-\alpha t}|a|. \end{aligned} \tag{3.9}$$

The last $(n-k)$ components of vector a can be taken as 0 since it does not enter in the computation.

$$h\left(t, \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ a_{k+1} \\ \vdots \\ a_n \end{bmatrix}\right) = h\left(t, \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right)$$

$$h\left(0, \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = P(0)\begin{bmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} + 0 - \int_0^\infty Q(-s)\mathcal{G}\left(h\left(s, \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right)\right) ds$$

$$= \begin{bmatrix} (id)_{k\times k} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \int_0^\infty \begin{bmatrix} 0 & 0 \\ 0 & e^{-Ys} \end{bmatrix}\mathcal{G}\left(h\left(s, \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right)\right) ds$$

$$= \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \int * \\ \vdots \\ \int * \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ -\int * \\ \vdots \\ -\int * \end{bmatrix}$$

and hence,

$$\begin{aligned} h_j(0,a) &= a_j \quad \text{for } j = 1, \dots, k \\ &= -\left(\int_0^\infty Q(-s)\mathcal{G}\left(h\left(s, a_1, \dots, a_k, 0\right)\right) ds\right)_j \qquad \text{for } j = k+1, \dots, n. \end{aligned}$$

Now, define the function as follows

$$\begin{aligned} \psi_j : \mathbb{R}^k &\to \mathbb{R} \qquad \forall j \in k+1, \dots, n \\ \psi_j(x_1, \dots, x_k) &:= h_j(0, (x_1, \dots, x_k, 0, \dots, 0)) \\ &= -\left(\int_0^\infty Q(-s)\mathcal{G}\left(h\left(s, x_1, \dots, x_k, 0\right)\right) ds\right)_j \end{aligned}$$

Also, define the following equation

$$\psi_j(x_1,\dots,x_k)_j = x_j \qquad \forall j \in \{k+1,\dots,n\}$$

Now, consider

$$\begin{aligned}\tilde{S} &= \{(y_1,\dots,y_n) \subseteq E| \quad y_j = \psi_j(y_1,\dots,y_k)\} \quad for\ j \in k+1,\dots,n.\\ \tilde{S} &= \{y_1,\dots,y_k,\dots,\psi_{k+1}(y_1,\dots,y_k),\dots,\psi_n(y_1,\dots,y_k)\}\end{aligned}$$

Consider the following disc

$$\mathcal{D} = \{(y_1,..,y_k)|\sqrt{y_1^2+..+y_k^2} < \frac{\delta}{2k}\}$$

So, it is clear that $\tilde{S}$ is a k-dimensional manifold.

Example 3.1.2. *Consider $\mathbb{R}^4$, $n=4$ and $k=2$.*
Let

$$\begin{aligned}\psi_3(x_1,x_2) &= x_1^2 + x_2^2\\ \psi_4(x_1,x_2) &= x_1^3\end{aligned}$$

Define the equations;

$$\begin{aligned}\psi_3(x_1,x_2) &= x_3\\ \psi_4(x_1,x_2) &= x_4\end{aligned}$$

Consider

$$\begin{aligned}\tilde{S} &= \{(x_1,x_2,x_3,x_4) \in \tilde{E}| \quad x_3 = x_1^2+x_2^2, x_4 = x_1^3\}\\ &= \{x_1,x_2,x_1^2+x_2^2,x_1^3\}.\end{aligned}$$

Now, we have

$$\frac{\partial \psi_j}{\partial y_i}(0) = 0$$

for $i=1,\dots,k$ and $j=k+1,\dots,n$.
It means that the differential manifold $\tilde{S}$ is tangent to the stable subspace(E^s) of the linear system.
$y(t)$ is the solution of the equation (3.3) such that $y(0) \in \tilde{S}$.
If $y(0) \in \tilde{S}$, then $y(t) \in \tilde{S} \quad \forall t \geq 0$.
From (3.9), $y(t) = h(t,a)$ when $y(0) = h(0,a)$. So $y(t) \to 0$, as $t \to \infty$.
By replacing $t \to -t$, and from the same method as done above for the stable manifold, we will get an unstable manifold.
This completes the proof of the stable manifold theorem. □

3.2 The Center Manifold Theorem

Theorem 3.2.1. *(The Center Manifold Theorem)[1] Let E be an open subset of $\mathbb{R}^n$ (Contains origin), and $f \in C^\alpha(E)$ for $\alpha \geq 1$. Let ϕ_t be the flow of the non-linear system. Assume that $f(0)=0$ and $Df(0)$ has j eigenvalues with negative real part, k eigenvalues with positive real part, and $n-j-k$ with zero real part. Then,*
(i). There exists $n-j-k$ dimensional center manifold of class C^α, which is tangent to the center

subspace(E^c) of the linearised system near the origin, and this center manifold is invariant with respect to the flow of ϕ_t.
(ii). There exists j dimensional stable manifold of class C^α, which is tangent to the stable subspace(E^s) of the linearised system near the origin, and this stable manifold is invariant with respect to the flow of ϕ_t.
(iii). There exists k dimensional unstable manifold of class C^α, which is tangent to the unstable subspace(E^u) of the linearised system near the origin, and this unstable manifold is invariant with respect to the flow of ϕ_t.

Proof. This theorem can be proved using the same idea used in the proof of the stable manifold theorem. □

Example 3.2.1. *Consider the following system*

$$\dot{x} = -x$$
$$\dot{y} = 2y + x^2$$

We will calculate the required parameters, i.e.

$$A = C = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$
$$\mathcal{F}(x) = \mathcal{G}(x) = \begin{bmatrix} 0 \\ x^2 \end{bmatrix}$$

Let $a = \begin{bmatrix} a_1 \\ 0 \end{bmatrix}$

$$P = \begin{bmatrix} e^{-t} & 0 \\ 0 & 0 \end{bmatrix}$$
$$Q = \begin{bmatrix} 0 & 0 \\ 0 & e^{2t} \end{bmatrix}$$

Using the successive approximations, we have

$$h^{(0)}(t,a) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$h^{(1)}(t,a) = \begin{bmatrix} e^{-t}a_1 \\ 0 \end{bmatrix}$$
$$h^{(2)}(t,a) = \begin{bmatrix} e^{-t}a_1 \\ -\frac{1}{4}e^{-2t}a_1^2 \end{bmatrix}$$
$$h^{(3)}(t,a) = \begin{bmatrix} e^{-t}a_1 \\ -\frac{1}{4}e^{-2t}a_1^2 \end{bmatrix}$$

And hence for all $m \geq 2$, we have

$$h^{(m)}(t,a) = \begin{bmatrix} e^{-t}a_1 \\ -\frac{1}{4}e^{-2t}a_1^2 \end{bmatrix}$$
$$h(t,a) = \begin{bmatrix} e^{-t}a_1 \\ -\frac{1}{4}e^{-2t}a_1^2 \end{bmatrix}$$

So,

$$\begin{aligned} \psi_2(a_1) &= (u(0,a))_{2nd\ component} \\ &= -\frac{1}{4}a_1^2 \end{aligned}$$

So, the stable manifold is written as

$$S : y = -\frac{1}{4}x^2$$

By replacing $t \to -t$, the unstable manifold is written as

$$U : x = -\frac{1}{4}y^2$$

The solution of the system is given by ϕ_t

$$\phi_t = \begin{bmatrix} e^{-t}a_1 \\ -\frac{1}{4}a_1^2(e^{-2t} - e^{2t}) + a_2 e^{2t} \end{bmatrix}$$

As $\phi_t(S) \subset S$ for all $t \geq 0$, and $\phi_t(U) \subset U$ for all $t \leq 0$. So, It is clear that S and U are invariant under the flow ϕ_t.

3.3 Stability and Liapunov Functions

Consider the following non-linear system

$$\dot{x} = f(x) \tag{3.10}$$

The stability of the hyperbolic equilibrium point of (3.10) can be determined by the signs of the real parts of the eigenvalues λ_j of the matrix $Df(x_0)$.

It is tough to determine the stability of non-hyperbolic equilibrium points. Liapunov is the crucial method to assess the stability of non-hyperbolic equilibrium points.

Definition 3.3.1 (*Stable, Unstable, and Asymptotically stable critical point*)**.** *Let ϕ_t be the flow of the non-linear system (3.10), which is defined $\forall t \in \mathbb{R}$. A critical point (x_0) is called stable, if $\forall \epsilon > 0$, $\exists$ a $\delta > 0$ such that $\forall x \in U_\delta(x_0)$ and $t \geq 0$,*

$$\phi_t(x) \in U_\epsilon(x_0)$$

If the critical point is not stable then, it is called an unstable critical point.
x_0 is an asymptotically stable point; if it is stable, and if $\exists$ a $\delta > 0$ such that for all $x \in U_\delta(x_0)$,

$$\lim_{t\to\infty} \phi_t(x) = x_0.$$

From the stable manifold theorem and the Hartman-Grobman theorem, it is clear that any sink of (3.10) is asymptotically stable, and any source or saddle of (3.10) is unstable. So, any hyperbolic equilibrium point of (3.10) is either asymptotically stable or unstable.

Theorem 3.3.1. *[1] If x_0 is a stable critical point of (3.10), then there is no eigenvalue of $Df(x_0)$ which have positive real part.*

Definition 3.3.2. *Let $f, \mathcal{L} \in C^1(E)$, and ϕ_t be the flow of the system (3.10), then the derivative of the function $\mathcal{L}(x)$ along the flow $\phi_t(x)$ is given by*

$$\dot{\mathcal{L}}(x) = \frac{d}{dt}\mathcal{L}(\phi_t(x))\bigg|_{t=0} = D\mathcal{L}(x)f(x)$$

for $x \in E$.

Theorem 3.3.2. *(Liapunov Stability) Let E be an open subset of $\mathbb{R}^n$(contains x_0), $f \in C^1(E)$ and also $f(x_0) = 0$. Suppose there is a real valued function $\mathcal{L} \in C^1(E)$ which satisfies $\mathcal{L}(x_0) = 0$ and $\mathcal{L}(x) > 0$ if $x \neq x_0$, then*
(i). If $\dot{\mathcal{L}}(x) \leq 0 \ \ \forall x \in E$, then x_0 is stable.
(ii). If $\dot{\mathcal{L}}(x) < 0 \ \ \forall x \in E - (x_0)$, then x_0 is asymptotically stable.
(iii). If $\dot{\mathcal{L}}(x) > 0 \ \ \forall x \in E - (x_0)$, then x_0 is unstable.
A function $\mathcal{L} : \mathbb{R}^n \to \mathbb{R}$ which satisfies the hypotheses of the above theorem is called a Liapunov function.

Proof. First of all assume that $x_0 = 0$.
Part (i). We want to show that $\forall \epsilon > 0$, $\exists$ a $\delta > 0$ such that for all $x \in U_\delta(0)$ and $t \geq 0$, we have $\phi_t(x) \in U_\epsilon(0)$.
For that, construct a closed ball of radius r, B_r which is subset of E. So for $\epsilon > 0$, and $0 \geq r \geq \epsilon$

$$B_r = \{x \in \mathbb{R}^n | \ \ |x| \leq r\} \subset E.$$

Now, construct $O_\alpha = \{x \in B_r | \ \ \mathcal{L} \leq \alpha\}$ such that O_α lies in the interior of B_r.
Let $\beta = \min_{|x|=r} \mathcal{L}(x)$ and take $0 < \alpha < \beta$

$$O_\alpha = \{x \in B_r | \ \ \mathcal{L} \leq \alpha\}.$$

If a point a is in the boundary, then $\mathcal{L}(a) \geq \beta > \alpha$. So O_α lies in the interior of B_r.
For $x = \phi_0(x) \in O_\alpha$, and $\forall t$,

$$\begin{aligned} \mathcal{L}(\phi_t(x)) - \mathcal{L}(\phi_0(x)) &= \int_0^t \frac{\partial}{\partial s}\mathcal{L}(\phi_s(x))ds \leq 0 \\ \mathcal{L}(\phi_t(x)) &\leq \mathcal{L}(\phi_0(x)) \\ &\leq \alpha. \end{aligned}$$

So, $\phi_t(x) \in O_\alpha$.
Since $\mathcal{L}$ is a continuous function, $\mathcal{L}(0) = 0$, then $\exists$ a $\delta > 0$ such that $|x| < \delta$ implies that $\mathcal{L}(x) < \alpha$.
So,

$$\begin{aligned} x \in U_\delta &\Rightarrow x \in O_\alpha \\ &\Rightarrow \phi_t(x) \in O_\alpha \\ &\Rightarrow \phi_t(x) \in B_r \\ &\Rightarrow \phi_t(x) \in U_\epsilon(0). \end{aligned}$$

This proves the part (i), and hence the origin is a stable.
Part (ii). From the definition, we want to show that $\exists$ a $\delta > 0$ such that $x \in U_\delta(0)$, we have

$$\lim_{t \to \infty} \phi_t(x) = 0.$$

or we can say that
$\exists$ a $\delta > 0$ such that $\forall \epsilon > 0$, $\exists$ a $\tau > 0$ such that for $x \in U_\delta(0)$ and $t > \tau$, we have

$$|\phi_t(x)| < \epsilon.$$

In the previous part (i), we have shown that $\forall \epsilon > 0$ we can construct α such that $O_\alpha \subset U_\epsilon(0)$ i.e.

$$\phi_t(x) \in O_\alpha \Rightarrow \phi_t(x) \in U_\epsilon(0).$$

So, it is sufficient to show that $\forall x \in U_\delta(0)$, we have

$$\lim_{t\to\infty} \mathcal{L}(\phi_t(x)) = 0.$$

This means that $\forall \alpha > 0$, $\exists$ a $\tau > 0$, such that $\forall t > \tau$,

$$\mathcal{L}(\phi_t(x)) < \alpha.$$

i.e.

$$\phi_t(x) \in O_\alpha \subset U_\epsilon(0).$$

It is given that $\mathcal{L}$ is decreasing and bounded below, So

$$\lim_{t\to\infty} \mathcal{L}(\phi_t(x)) = c \geq 0.$$

Assume that $c > 0$, Let

$$O_c = \{x \in B_r | \ \ \mathcal{L}(x) \leq c\}$$

By the continuity of $\mathcal{L}$ and $\mathcal{L}(0) = 0$
$\exists$ a $d > 0$ such that

$$B_d = \{x \in \mathbb{R}^n | \ \ |x| \leq d\} \subset O_c$$

Since

$$\lim_{t\to\infty} \mathcal{L}(\phi_t(x)) = c$$

then $\phi_t(x)$ lies outside B_d or we can say that $\phi_t(x)$ lies in the compact set $d \leq |x| \leq r$,
$\mathcal{L}$ attains its maximum value in this set.
Let

$$\beta = - \max_{d\leq|x|\leq r} \dot{\mathcal{L}} > 0$$

For $t > 0$, we have

$$\begin{aligned}\mathcal{L}(\phi_t(x)) &= \mathcal{L}(\phi_0(x)) + \int_0^t \frac{\partial}{\partial s}\mathcal{L}(\phi_s(x))ds \\ &\leq \mathcal{L}(\phi_0(x)) - \beta t\end{aligned}$$

This shows that

$$\begin{aligned}\mathcal{L}(\phi_t(x)) &< 0 \\ c &< 0.\end{aligned}$$

This is the contradiction of the assumed argument and hence $c = 0$.

$$\lim_{t\to\infty} \mathcal{L}(\phi_t(x)) = 0.$$

Hence x_0 is an asymptotically stable.
Part (iii). By using $t = -t$, and the same steps used in the part (ii), we can get that x_0 is unstable. □

Example 3.3.1. *Consider the system,*

$$\begin{aligned}\dot{x} &= -x + y + xy \\ \dot{y} &= x - y - x^2 - y^3\end{aligned}$$

The origin is an non-hyperbolic equilibrium point of this system, and Liapunov function for this system is

$$\begin{aligned}\mathcal{L}(x) &= x^2 + y^2 \\ \dot{\mathcal{L}}(x) &= 2x(-x + y + xy) + 2y(x - y - x^2 - y^3) \\ &= -2x^2 + 4xy - 2y^2 - 2y^4 \\ &= -2y^4 - 2(x - y)^2 < 0.\end{aligned}$$

Hence, the origin is an asymptotically stable equilibrium point of this system.

3.4 Saddle, Nodes, Foci and Centers

Consider the following non-linear system

$$\dot{x} = f(x) \tag{3.11}$$

Definition 3.4.1. *In the above non-linear system (3.11), a point is considered as the center, if* $\exists$ *a* $\delta > 0$ *such that every solution curve of (3.11) is a closed curve with 0 in its interior, in the deleted neighbourhood* $U_\delta(0) - (0)$.

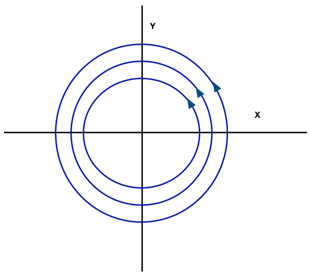

Figure 3.1: Center

Definition 3.4.2. *The stable-focus here is the origin for (3.11), if* $\exists$ *a* $\delta > 0$ *such that for* $0 < r_0 < \delta$ *and* $\theta_0 \in R$*, radius of the solution curve* $r(t, r_0, \theta_0) \to 0$ *and the amplitude of an angle* $|\theta(t, r_0, \theta_0)| \to \infty$ *as* $t \to \infty$.

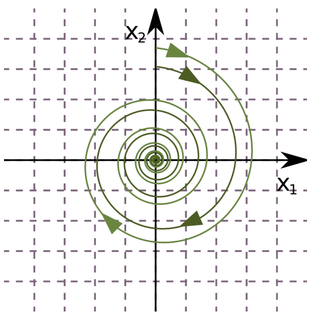

Figure 3.2: Stable Focus

Definition 3.4.3. *For this non-linear system (3.11), the unstable-focus is the origin, if* $\exists$ *a* $\delta > 0$ *such that for* $0 < r_0 < \delta$ *and* $\theta_0 \in R$*, the radius* $r(t, r_0, \theta_0) \to 0$ *and an angle* $|\theta(t, r_0, \theta_0)| \to \infty$ *as* $t \to -\infty$.

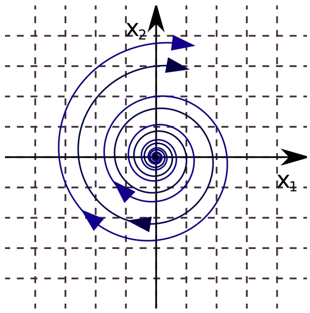

Figure 3.3: Unstable Focus

Definition 3.4.4. *The trajectory of (3.11) is a spiral towards the origin if* $t \to \pm\infty$*, it satisfies* $r(t) \to 0$ *and* $|\theta(t)| \to \infty$.

Definition 3.4.5. *The origin is a stable-node for (3.11), if* $\exists$ *a* $\delta > 0$ *such that for* $0 < r_0 < \delta$ *and* $\theta_0 \in R$*, radius of the solution curve* $r(t, r_0, \theta_0) \to 0$ *and*

$$\lim_{t\to\infty} |\theta(t, r_0, \theta_0)| \quad \textit{exists as } t \to \infty.$$

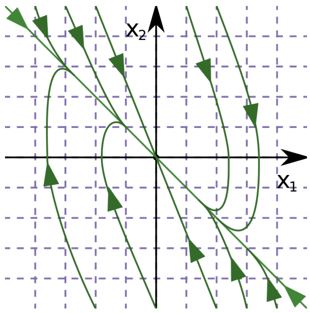

Figure 3.4: Stable Node

Definition 3.4.6. *The origin is said to be an unstable-node for the non-linear system (3.11), if* $\exists$ *a* $\delta > 0$ *such that for* $0 < r_0 < \delta$ *and* $\theta_0 \in R$*,* $r(t, r_0, \theta_0) \to 0$ *and*

$$\lim_{t\to-\infty} |\theta(t, r_0, \theta_0)| \quad \textit{exists as } t \to -\infty.$$

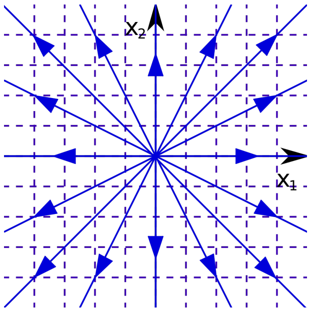

Figure 3.5: Unstable Node

Definition 3.4.7. *The origin is a proper node for (3.11), if it is a node and if every line passing through the origin is tangent to some trajectory of (3.11).*

Definition 3.4.8. *The origin is a (topological) saddle for (3.11), if there exists two trajectories Γ_1, and Γ_2 which approach 0 as $t \to \infty$, two trajectories Γ_3 and Γ_4 which approach 0 as $t \to -\infty$, and if there exists a $\delta > 0$ such that all other trajectories which start in the deleted neighborhod of the origin $N_\delta(0) - \{0\}$ leave $N_\delta(0)$ as $t \to \pm\infty$. The trajectories $\Gamma_1, \ldots, \Gamma_4$ are called separatrices.*

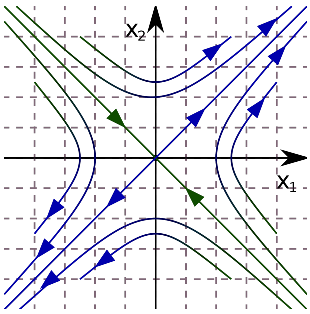

Figure 3.6: Saddle

Theorem 3.4.1. *Consider the linearised system of the non-linear system (3.11)*

$$\dot{x} = Ax \tag{3.12}$$

Let E be an open subset of $\mathbb{R}^2$ (Contains origin), and $f \in C^1(E)$. The origin is a (topological) saddle for the non-linear system (3.11) if and only if the origin is a saddle for the linear system (3.12) with $A = Df(0)$.

Proof. This theorem follows from the Stable Manifold Theorem (3.1.2), and the Hartman-Grobman Theorem (4.1.1). $\square$

Theorem 3.4.2. *[1] Let E be an open subset of $\mathbb{R}^2$ (Contains origin), and $f \in C^2(E)$. Let the origin is a hyperbolic critical point of (3.11). Then,*
(i). The origin is a stable (or unstable) node for the non-linear system (3.11) if and only if it is a stable(or unstable) node for the linear system (3.12).
(ii). The origin is a stable(or unstable) focus for the non-linear system (3.11) if and only if it is a stable(or unstable) focus for the linear system (3.12).

Theorem 3.4.3. *[1] Let E be an open subset of $\mathbb{R}^2$ (Contains origin), and $f \in C^1(E)$ with $f(0) = 0$. If the origin is a center for the linear system (3.12), then the origin may be a focus, a center-focus or a center for the non-linear system (3.11).*

Corollary 3.4.3.1. *[1] Let E be an open subset of $\mathbb{R}^2$ (Containing origin), and f is analytic in E. If the origin is a center for the linear system (3.12), then the origin is either a focus or a center for the non-linear system (3.11).*

Let $x = (x, y)^T$, $f_1(x) = P(x, y)$ and $f_2(x) = Q(x, y)$. The non-linear system (3.11) can be written as

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned} \tag{3.13}$$

Definition 3.4.9. *The system (3.13) is called symmetric with respect to the $x-$axis, if it is invariant under the transformation $(t, y) \to (-t, -y)$; and it is symmetric with respect to the $y-axis$, if it is invariant under the transformation $(t, x) \to (-t, -x)$.*

Theorem 3.4.4. *[1] Let E be an open subset of $\mathbb{R}^2$ (Contains origin), and $f \in C^1(E)$ with $f(0) = 0$. If the non-linear system (3.11) is symmetric with respect to the $x - axis$ or the $y - axis$, and if the origin is a center for the linear system (3.12) with $A = Df(0)$, then the origin is a center for the non-linear system (3.11).*

3.5 Non-Hyperbolic Critical Points in $\mathbb{R}^2$

Definition 3.5.1. *Sector which is topologically equivalent to the sector shown in Fig.(a), (b), (c) are called a hyperbolic sector, parabolic sector, and an elliptic sector, respectively.*

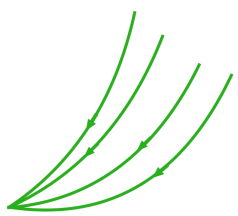

Figure 3.7: (a) Parabolic Sector

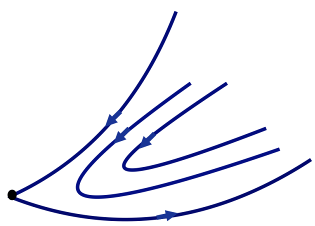

Figure 3.8: (b) Hyperbolic

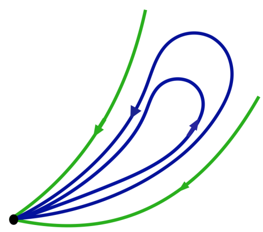

Figure 3.9: (c) Elliptic Sector

Definition 3.5.2. *If the origin contains one elliptic sector, one hyperbolic sector, two parabolic sectors, and four separatrices, then it is called a critical point with an elliptic domain.*

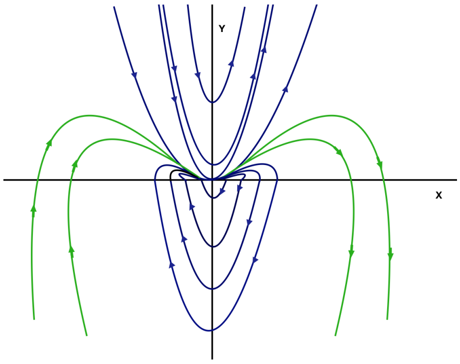

Figure 3.10: Critical point with an elliptic domain

Definition 3.5.3. *If a critical point which consists two hyperbolic sectors and one parabolic sector, as well as three separatrices and the critical point itself, is called a saddle-node.*

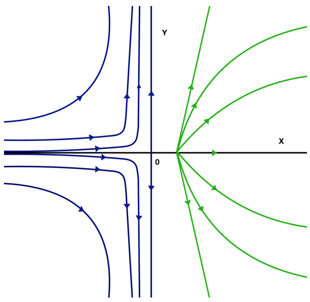

Figure 3.11: Saddle-node

Definition 3.5.4. *If the origin consists of two hyperbolic sectors and two separatrices, then this type of critical point is called a cusp.*

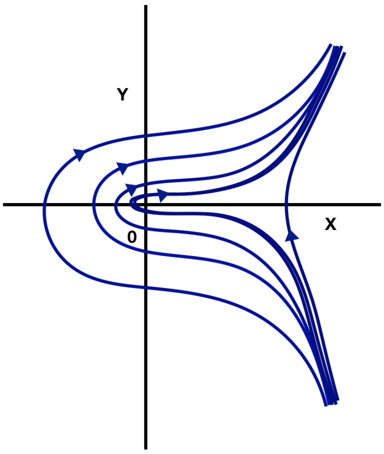

Figure 3.12: Cusp

Consider the planar system

$$\begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned} \tag{3.14}$$

This system can be written as

$$\begin{aligned} \dot{x} &= p_2(x, y) \\ \dot{y} &= y + q_2(x, y) \end{aligned} \tag{3.15}$$

where p_2 and q_2 are analytic in a neighbourhood of the origin and have expansions that begin with second degree terms in x and y.
The next three theorems shows the type and stability of the non-hyperbolic critical point.

Theorem 3.5.1. *[1] Let the origin is an isolated critical point for the system (3.15). Let $y = \phi(x)$ be the solution of the equation $y + q_2(x, y) = 0$ and the expansion of the function $\psi(x) = p_2(x, \phi(x))$ in a neighbourhood of $x = 0$ is the form*

$$\psi(x) = a_n x^n + \dots$$

where $n \geq 2$ and $a_n \neq 0$. Then
(i). For n odd and $a_n > 0$, the origin is an unstable node.
(ii). For n odd and $a_n < 0$, the origin is a (topological) saddle.
(iii). For n even, the origin is a saddle-node.

The system (3.15) can be written in the "normal" form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= a_k x^k [1 + h(x)] + b_n x^n y [1 + g(x)] + y^2 R(x, y) \end{aligned} \tag{3.16}$$

where $h(x)$, $g(x)$ and $R(x, y)$ are analytic in a neighbourhood of the origin.
$h(0) = g(0) = 0$, $k \geq 2$, $a_k \neq 0$ and $n \geq 1$.

Theorem 3.5.2. *[1] Let $k = 2m + 1$ for $m \geq 1$ in (3.16) and let $\lambda = b_n^2 + 4(m + 1)a_k$.*
(a). If $a_k > 0$, then the origin is a (topological) saddle.
(b). If $a_k < 0$, then the origin is
(i). A focus or a center if $b_n = 0$ and also if $b_n \neq 0$ and $n > m$ or if $n = m$ and $\lambda < 0$.
(ii). An node if $b_n \neq 0$, n is an even number and $n < m$ and also if $b_n \neq 0$, n is an even number, $n = m$ and $\lambda > 0$.
(iii). A critical point with an elliptic domain if $b_n \neq 0$, n is an odd number and $n < m$ and also if $b_n \neq 0$, n is an odd number, $n = m$ and $\lambda \geq 0$.

Theorem 3.5.3. *[1] Let $k = 2m$ for $m \geq 1$ in (3.16).*
Then the origin is
(i). A cusp if $b_n = 0$ and also if $b_n = 0$ and $n > m$.
(ii). A sadle-node if $b_n \neq 0$ and $n < m$.

A computer program using these theorems is performed, which tells about the type of critical point.

```python
import re

#Function to obtain  the system equations and match it with the appropriate regular expression to extract necessary coefficients and powers.
def main():
    print "\nEnter the normal form of the dynamical system:"
    print "x' = y"
    ydot = raw_input("y' = ")

    p = re.compile(r'^(-?)\s*(\d*)[xX](\^\d*)?([\(\[]1\s*\+(.*)[\]\)])?\s*(([\+-])\s*(\d*)[xX](\^\d*)?[yY]([\(\[]1\s*\+(.*)[\]\)])?)?\s*([\+-]\d*.*[yY].*)
    m = p.match(ydot)
    return m

#To check if input is in the normal form of the dynamical system.
mat = main()
while mat == None:
    print '\nError: The equation is not in the normal form.'
    mat = main()

#Extracting coefficients and powers in first term
a_k1 = ''
if mat.group(2) == '':
    a_k1 = '1'
a_k = mat.group(1) + mat.group(2) + a_k1
if mat.group(3) == None:
    k = '1'
else:
    k = mat.group(3)[1:]
h_x_ = mat.group(5)

#Extracting coefficients and powers in second term
if mat.group(6) == None:
    b_n = '0'
    n = 'NA'
    g_x_ = 'NA'
else:
    b_n1 = ''
    if mat.group(8) == '':
        b_n1 = '1'
    b_n = mat.group(7) + mat.group(8) + b_n1
    if mat.group(9) == None:
        n = '1'
    else:
        n = mat.group(9)[1:]
    g_x_ = mat.group(11)

#Converting the 'string' type values to 'int'
a_k, k, b_n = int(a_k), int(k), int(b_n)
if n!= (None or 'NA'):
    n = int(n)
```

Line 1, Column 1 Tab Size: 4 Python

```python
'''
print "\na_k :", a_k
print "k   :", k
print "h_x_:", h_x_
print "b_n :", b_n
print "n   :", n
print "g_x_:", g_x_
'''

#Conditionals corresponding to Theorem 2 and Theorem 3(Nonhyperbolic critical points).
print "\n****************************************"
if k%2 != 0:
    m = (k-1)/2
    lam = b_n**2 +4*(m+1)*a_k
    if a_k>0:
        print 'The origin is a (topological) SADDLE.'
    else:
        if (b_n==0 or (b_n!=0 and (n>m or (n==m and lam<0)))):
            print 'The origin is a FOCUS or a CENTER.'
        elif ((b_n!=0 and (n%2==0)) and (n<m or (n==m and lam>=0))):
            print 'The origin is a NODE.'
        elif ((b_n!=0 and (n%2!=0)) and (n<m or (n==m and lam>=0))):
            print 'The origin is a critical point with an ELLIPTIC DOMAIN.'
else:
    m = k/2
    if (b_n==0 or (b_n!=0 and n>=m)):
        print 'The origin is a CUSP.'
    elif (b_n!=0 and n<m):
        print 'The origin is a SADDLE-NODE.'
print "****************************************\n"
```

Figure 3.13: Computer Program

3.6 Hamiltonian and Gradient Systems

Definition 3.6.1. *Let E be an open subset of $\mathbb{R}^{2n}$, $\mathcal{H} \in C^2(E)$*

$$\mathcal{H} = \mathcal{H}(x,y) \quad \textit{for } x, y \in \mathbb{R}^n$$

A system of the form

$$\begin{aligned} \dot{x} &= \frac{\partial \mathcal{H}}{\partial y} \\ \dot{y} &= -\frac{\partial \mathcal{H}}{\partial x} \end{aligned} \tag{3.17}$$

Where,

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial y} &= \left(\frac{\partial \mathcal{H}}{\partial y_1}, \dots, \frac{\partial \mathcal{H}}{\partial y_n}\right)^T \\ \frac{\partial \mathcal{H}}{\partial x} &= \left(\frac{\partial \mathcal{H}}{\partial x_1}, \dots, \frac{\partial \mathcal{H}}{\partial x_n}\right)^T \end{aligned}$$

Then (3.17) is called the Hamiltonian system with n degree of freedom on E.

Theorem 3.6.1. *Hamiltonian function $\mathcal{H}(x,y)$ remains constant along the trajectories of (3.17).*

Proof. Consider the trajectories $x(t), y(t)$.

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial t} &= \frac{\partial \mathcal{H}}{\partial x} \cdot \dot{x} + \frac{\partial \mathcal{H}}{\partial y} \cdot \dot{y} \\ &= \frac{\partial \mathcal{H}}{\partial x} \cdot \frac{\partial \mathcal{H}}{\partial y} - \frac{\partial \mathcal{H}}{\partial y} \cdot \frac{\partial \mathcal{H}}{\partial x} = 0. \end{aligned}$$

Hence, $\mathcal{H}$ is constant along $x(t), y(t)$. □

Lemma 3.6.2. *Consider the following hamiltonian system*

$$\begin{aligned} \dot{x} &= \mathcal{H}_y(x,y) \\ \dot{y} &= \mathcal{H}_x(x,y) \end{aligned} \tag{3.18}$$

If the origin is a stable focus of the above system, then the origin is not a strict local maximum / strict local minimum of the hamiltonian function $\mathcal{H}(x,y)$.

Proof. Let the origin is a stable focus of (3.18), then by the definition of stable focus (3.4.2) in a polar coordinate, if $\exists$ a $\delta > 0$ such that for $0 < r_0 < \delta$ and $\theta_0 \in R$, the radius of the solution curve $r(t, r_0, \theta_0) \to 0$ and the amplitude of an angle $|\theta(t, r_0, \theta_0)| \to \infty$ as $t \to \infty$.
Now, we can write the above in the Cartesian coordinate.
for $(x_0, y_0) \in N_\epsilon(0) \sim 0$, as $t \to \infty$

$$(x(t, x_0, y_0), y(t, x_0, y_0)) \to (0, 0)$$

by using theorem (3.6.1),

$$\mathcal{H}(x_0, y_0) = \lim_{t \to \infty} \mathcal{H}(x(t, x_0, y_0), y(t, x_0, y_0)) = \mathcal{H}(0,0)$$

So, $\mathcal{H}(x,y) > \mathcal{H}(0,0)$ and $\mathcal{H}(x,y) < \mathcal{H}(0,0)$ is not true for all $(x_0, y_0) \in N_\epsilon(0)$.
The same proof holds if the origin is an unstable focus. Hence, the origin is not strict local maxima and strict local minima for $\mathcal{H}$.

□

Definition 3.6.2. *A critical point of the non-linear system is said to be an non-degenerate critical point, if $Df(x_0)$ has no zero eigenvalues, if not then it will be considered as a degenerate critical point of the system.*

Theorem 3.6.3. *An non-degenerate critical point ($det(A) \neq 0$) of the analytic hamiltonian system (3.18) is either a center or a topological saddle.*
(i). (x_0, y_0) is topological saddle for (3.18) if and only if it is a saddle of the hamiltonian system.
(ii). A strict local maxima or a strict local minima of the function $\mathcal{H}(x, y)$ is a center of (3.18).

Proof. Part (i). Consider (0,0) as a critical point. So,

$$\mathcal{H}_x(0, 0) = \mathcal{H}_y(0, 0) = (0, 0)$$

The linearisation of the system at the origin is given by the

$$\begin{aligned} \dot{x} &= Ax \\ A &= \begin{bmatrix} \mathcal{H}_{yx}(0,0) & \mathcal{H}_{yy}(0,0) \\ -\mathcal{H}_{xx}(0,0) & -\mathcal{H}xy(0,0) \end{bmatrix} \end{aligned} \tag{3.19}$$

$$tr(A) = 0, Det(A) = \mathcal{H}_{xx}(0) \cdot \mathcal{H}_{yy}(0) - \mathcal{H}_{xy}^2(0)$$

Critical point is saddle iff $det(A) < 0$ iff it is saddle for (3.19) iff it is a saddle for the hamiltonian system (3.18) from the theorem (3.4.1).
Part (ii). If we have $tr(A) = 0$ and $det(A) < 0$, then the origin is a center for linear system (3.19), then from the Corollary (3.4.3.1) the critical point is either a center or a focus for (3.18). And hence from the lemma (3.6.2), it can not be a focus. So the origin is a center. □

Definition 3.6.3. *The Newtonian system is the Hamiltonian system with one degree of freedom.*

$$\ddot{x} = f(x)$$

where $f \in C^1(a, b)$.

This equation can be written as a system in R^2,

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= f(x) \end{aligned} \tag{3.20}$$

The total energy for this system is given by

$$\mathcal{H}(x, y) = \mathcal{K}(y) + \mathcal{P}(x)$$

Where $\mathcal{K}(y)$ is the kinetic energy, and $\mathcal{P}(x)$ is the potential energy.

$$\begin{aligned} \mathcal{K}(y) &= \frac{y^2}{2} \\ \mathcal{P}(x) &= -\int_{x_0}^{x} f(s)ds \end{aligned}$$

Theorem 3.6.4. *(i). All the critical points of the system (3.20) lies on the $x-axis$.*
(ii). $(x_0, 0)$ is a critical point of the newtonian system (3.20) iff it is a critical point of $\mathcal{P}(x)$.
(iii). If $(x_0, 0)$ is a strict local maxima of $\mathcal{P}(x)$, then it is a saddle for (3.20).
(iv). If $(x_0, 0)$ is a strict local minima of $\mathcal{P}(x)$, then it is a center for (3.20).
(v). If $(x_0, 0)$ is a horizontal inflection point of $\mathcal{P}(x)$, then it is a cusp for (3.20).

Proof. From the definition of the critical point, we have $y = 0$ and $f(x) = 0$.
Part (i). Since $y = 0$, then it is obvious that the critical points lies on the $x - axis$.
Part (ii). Now, $\mathcal{P}'(x)$ at (0) is same as $f(0)$, which is the critical point of the system (3.20).

$$A = \begin{bmatrix} 0 & 1 \\ f'(x) & 0 \end{bmatrix}$$

Part (iii). If $(x_0, 0)$ is a strict local maxima of the analytic function $\mathcal{P}(x)$, then $\mathcal{P}''(x) < 0$ at $(x_0, 0)$ which is $-f'(x_0)$, so $f'(x_0) > 0$.
$det(A) = -f'(x) < 0$ and $tr(A) = 0$, so by the definition the critical point is a saddle.
Part (iv). If $(x_0, 0)$ is a strict local minima of the analytic function $\mathcal{P}(x)$, then $\mathcal{P}''(x) > 0$, so $f'(x_0) < 0$.
$det(A) = -f'(x) > 0$ and $tr(A) = 0$, so by the definition the critical point is a center.
Part (v). If $(x_0, 0)$ is a horizontal inflection point of the analytic function $\mathcal{P}(x)$, then $det(A) = 0$ as $A \neq 0$, and hence by definition, the critical point is a cusp. □

Example 3.6.1. *Consider the following newtonian system*

$$\ddot{x} + \sin x = 0$$

this equation can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 \\ \mathcal{P}(x_1) &= \int_0^{x_1} \sin t \; dt = 1 - \cos x_1 \end{aligned}$$

The critical points of this system is $(\sin 0, 0) = (\pm\pi, 0)$. *Phase portraits of* $\mathcal{P}(x_1)$ *and* $\mathcal{H}$ *are Fig.*

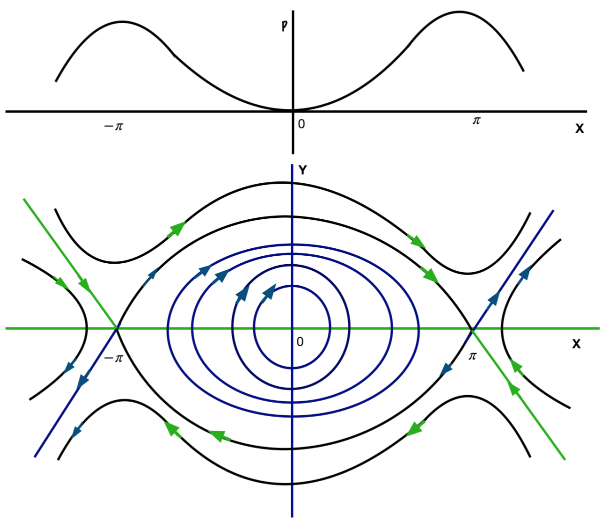

Figure 3.14: Newtonian system

In the phase portrait of $\mathcal{P}$*, 0 is the minima, and* $\pm\pi$ *are the maxima's. Hence 0 is the center, and* $\pm\pi$ *are the saddle critical points of the Newtonian system.*

Definition 3.6.4. *Let E be an open subset of $\mathbb{R}^n$. The following system*

$$\begin{aligned}\dot{x} &= -grad\ G(x) \\ grad\ G &= \left(\frac{\partial G}{\partial x_1}, \dots, \frac{\partial G}{\partial x_n}\right)^T\end{aligned} \tag{3.21}$$

is called the gradient system of E.

Theorem 3.6.5. *[1] An non-degenerate critical point of an analytic gradient system (3.21) on $\mathbb{R}^2$ is either a node or a saddle.*
(i). If (x_0, y_0) is a strict local maximum or minimum of the function $G(x, y)$, then it is respectively an unstable or a stable node for (3.21).
(ii). If (x_0, y_0) is a saddle of the function $G(x, y)$, then it is a saddle of (3.21).

Example 3.6.2. *Consider*

$$G(x_1, x_2) = x_1^2(1 - x_1)^2 + x_2^2$$

then the gradient system is of the form;

$$\begin{aligned}\dot{x}_1 &= -4x_1(x_1 - 1)(x_1 - \frac{1}{2}) \\ \dot{x}_2 &= -2x_2\end{aligned}$$

The critical points of the above system are $(0, 0)$, $(\frac{1}{2}, 0)$ and $(1, 0)$.
Then from the above theorem (3.6.5), $(0, 0)$ and $(1, 0)$ are stable nodes and $(\frac{1}{2}, 0)$ is a saddle point. Fig

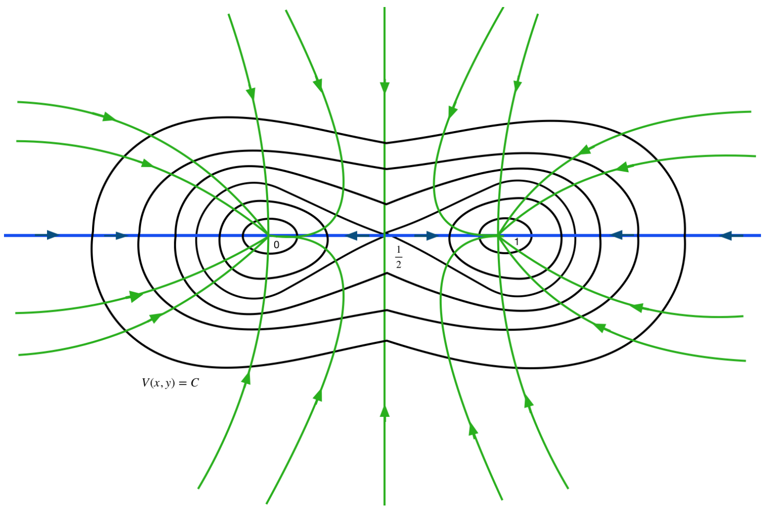

Figure 3.15: The trajectories of the gradient system

Definition 3.6.5. *Consider the planar system as*

$$\begin{aligned} \dot{x} &= U(x,y) \\ \dot{y} &= V(x,y) \end{aligned} \tag{3.22}$$

The system which is orthogonal to the system (3.22) is defined by

$$\begin{aligned} \dot{x} &= V(x,y) \\ \dot{y} &= -U(x,y) \end{aligned} \tag{3.23}$$

If (3.22) is a hamiltonian system with $U = \mathcal{H}_y$ and $V = -\mathcal{H}_x$, then (3.23) is a gradient system. And conversely also holds.

Chapter 4

The Hartman-Grobman Theorem

4.1 The Hartman-Grobman Theorem

We will begin with a couple of definitions that will be used later in this chapter.

Definition 4.1.1 (*Topologically equivalent*)**.** *The following differential equations*

$$\dot{x} = f(x) \tag{4.1}$$

$$\dot{x} = Ax \tag{4.2}$$

are topologically equivalent in a neighbourhood of the origin or we can say that these having the same qualitative structure near the origin, if there is a homeomorphism H from an open set P (containing origin) onto an open set Q (containing origin) which maps trajectories of (4.1) in P onto trajectories of (4.2) in Q. It is also orientation preserving by time i.e. if the trajectory is directed from T_1 to T_2 in P, then its image is directed from $H(T_1)$ to $H(T_2)$ in Q.

Definition 4.1.2 (*Topologically conjugate*)**.** *If H preserves the parametrisation by time, then (4.1) and (4.2) are topologically conjugate in a neighbourhood of the origin.*
let ϕ be the flow of non-linear (4.1) and ψ is the flow of linear system (4.2), then

$$\psi(H(x), t) = H \circ \phi(x, t) \tag{4.3}$$

Example 4.1.1. *Consider the following linear systems*

$$\dot{x}_1 = Ax_1$$
$$\dot{x}_2 = Bx_2$$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}, \; B = \begin{bmatrix} 5 & -3 \\ 1 & 5 \end{bmatrix}$$

Let $H(x_1) = Rx_1$, where

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \; R^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

then $B = RAR^{-1}$.
Let $x_2 = H(x_1) = Rx_1$ or $x_1 = R^{-1}x_2$.

$$\dot{x}_2 = RAR^{-1}x_2 = Bx_2$$

$$e^{Bt} = e^{RAR^{-1}t} = Re^{At}R^{-1}$$

If $x_1(t) = e^{At}x_0$ is the solution of first linear system through x_0, then
$x_2(t) = H(x_1(t)) = Rx_1(t) = Re^{At}x_0 = e^{Bt}Rx_0$ is the solution of second linear system through Rx_0.
So, it is clear that

$$He^{At} = e^{Bt}H$$

Hence this map H preserves the parametrisation by time. This mapping $H(x_1) = Rx_1$ (rotation through $45°$) is homeomorphic. So by definition, the above two linear systems are topologically conjugate.

The Hartman-Grobman theorem shows that near a hyperbolic critical point x_0 the linear and non-linear system have the same qualitative structure.

Theorem 4.1.1. *(The Hartman-Grobman Theorem) Let E be an subset of $\mathbb{R}^n$, and $f \in C^1(E)$. Let ϕ_t be the flow of the non-linear system (4.1). Consider $f(0) = 0$ and the matrix $A = Df(0)$ have no eigenvalue with 0 real part, then the linear and non-linear equations are said to be topologically equivalent.*
i.e. there exists a homeomorphism H of an open set P (containing origin) onto an open set Q (containing origin) such that $\forall x_0 \in P$, there is an open interval $I_0 \subset \mathbb{R}$ (containing 0) such that $\forall x_0 \in P$ and $t \in I_0$,

$$H \circ \phi_t(x_0) = e^{At}H(x_0) \tag{4.4}$$

i.e H maps trajectories of (4.1) near the origin onto trajectories of (4.2) near origin and preserves the parametrisation by time.
Here the flow of the non-linear system is $\phi(x,t)$, and the flow of the linearised system is $\psi(x,t)$, which is simply $e^{At}x$.

Proof. It is given that $f \in C^1(E)$, $f(0) = 0$ and $A = Df(0)$

$$\text{Suppose } A = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$$

Where M is a $k \times k$ block matrix when the real part is negative, N is an $n-k \times n-k$ block matrix when the real part is positive.
Let ϕ_t be the flow of non-linear system. Write the solution as

$$x(t, x_0) = \phi_t(x_0) = \begin{bmatrix} y(t, y_0, z_0) \\ z(t, y_0, z_0) \end{bmatrix} \tag{4.5}$$

$$\text{Where } x_0 = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

$y(t, y_0, z_0)$ have k components and $z(t, y_0, z_0)$ have $n-k$ components.
$y_0 \in E^s$ (stable subspace of A), and $z_0 \in E^u$ (unstable subspace of A).
Now, define the functions as follows:

$$\tilde{Y}(y_0, z_0) = y(1, y_0, z_0) - e^M y_0 \tag{4.6}$$

$$\tilde{Z}(y_0, z_0) = z(1, y_0, z_0) - e^N z_0 \tag{4.7}$$

$$\dot{x} = f(x)$$
$$x(0) = 0$$

This non-linear system has unique solution $x(t) = 0$ for all t. When $f(0) = 0$, then by the flow property $\phi(0,t) = 0$.
Hence from the equations (4.5), (4.6), and (4.7)

$$\tilde{Y}(0) = \tilde{Z}(0) = 0$$
$$Der(\tilde{Y}(0)) = Der(\tilde{Z}(0)) = 0.$$

Since $f \in C^1(E)$ and $e^M y_0, e^N y_0$ are C^∞, then by (4.6) and (4.7) $\tilde{Y}(y_0,z_0), \tilde{Z}(y_0,z_0)$ are continuously differentiable.
Consider a compact set $|y_0|^2 + |z_0|^2 \leq {c_0}^2$, then by the $\epsilon - \delta$ definition of continuity;
For any arbitrary $\epsilon > 0$, $\exists c_0$ such that $|y_0|^2 + |z_0|^2 \leq {c_0}^2$ implies that

$$\|Der(\tilde{Y}(y_0,z_0))\| \leq \epsilon$$

for small c_0 and very small a, we have

$$\|Der(\tilde{Y}(y_0,z_0))\| \leq a \tag{4.8}$$
$$\|Der(\tilde{Z}(y_0,z_0))\| \leq a \tag{4.9}$$

Let $Y(y_0,z_0)$ and $Z(y_0,z_0)$ be the smooth functions. For $0 < a < b$,

$$f(x) = \begin{cases} e^{\left(\frac{1}{x-c_0} - \frac{1}{x-\frac{c_0}{2}}\right)} & if \ \ \frac{c_0}{2} < x < c_0 \\ 0 & otherwise \end{cases}$$

$$F(x) = \frac{\int_x^{c_0} f(t)dt}{\int_{\frac{c_0}{2}}^{c_0} f(t)dt}$$

F is smooth

$$F(t) = \begin{cases} 1 & ; \ \ t \leq \frac{c_0}{2} \\ 0 & ; \ \ t \geq c_0 \end{cases}$$

Now, consider the same on $\mathbb{R}^n$, i.e

$$\psi(x_1 \cdots x_n) = F\left(\sum x_i^2\right)$$
$$\psi = \begin{cases} 1 & ; \ \ \sum x_i^2 \leq \frac{c_0}{2} \\ 0 & ; \ \ \sum x_i^2 \geq c_0 \end{cases}$$

This ψ function is a bump function for the next partition.

$$Y(y_0,z_0) = \begin{cases} \tilde{Y}(y_0,z_0) & ; \ \ |y_0|^2 + |z_0|^2 \leq \left(\frac{c_0}{2}\right)^2 \\ 0 & ; \ \ |y_0|^2 + |z_0|^2 \geq (c_0)^2 \end{cases}$$

$$Z(y_0,z_0) = \begin{cases} \tilde{Z}(y_0,z_0) & ; \ \ |y_0|^2 + |z_0|^2 \leq \left(\frac{c_0}{2}\right)^2 \\ 0 & ; \ \ |y_0|^2 + |z_0|^2 \geq (c_0)^2 \end{cases}$$

From the mean value theorem

$$f'(c) = |\frac{f(x) - f(y)}{x - y}|$$

$$\|Der(Y)(*)\| = \frac{\|Y(y_0,z_0) - y(0)\|}{\|(y_0,z_0) - (0,0)\|}$$

$$\begin{aligned}\|Y(y_0,z_0)\| &\leq a\|(y_0,z_0)\| \\ &\leq a\sqrt{|y_0|^2 + |z_0|^2} \\ &\leq a(|y_0| + |z_0|).\end{aligned}$$

$$\text{as} \quad (y_0^2) + (z_0^2) \leq (y_0 + z_0)^2$$

Hence $\forall (y_0, z_0) \in \mathbb{R}^n$,

$$\|Y(y_0,z_0)\| \leq a(|y_0| + |z_0|) \tag{4.10}$$

$$\|Z(y_0,z_0)\| \leq a(|y_0| + |z_0|) \tag{4.11}$$

Let $D = e^M$ and $E = e^N$, $E^{-1} = e^{-N}$, then we have

$$\|D\| < 1$$
$$\|E^{-1}\| < 1$$

$$\text{For } x = \begin{bmatrix} y \\ z \end{bmatrix} \in \mathbb{R}^n$$

Define the following transformations

$$U(y,z) = \begin{bmatrix} Dy \\ Ez \end{bmatrix}$$
$$V(y,z) = \begin{bmatrix} Dy + Y(y,z) \\ Ez + Z(y,z) \end{bmatrix}$$

$$U(x) = e^A x = \begin{bmatrix} e^M & 0 \\ 0 & e^N \end{bmatrix} x$$

and for $|y|^2 + |z|^2 \leq \left(\frac{c_0}{2}\right)^2$

$$V(x) = \phi_1(x)$$

Lemma 4.1.2. *$\exists$ a homeomorphism H_0 from an open set P(containing origin) onto an open set Q(containing origin) such that*

$$H_0 \circ V = U \circ H_0 \tag{4.12}$$

Proof. We will prove this lemma by using the method of successive approximations.
For $x \in \mathbb{R}^n$, let

$$H(x) = \begin{bmatrix} \phi(y,z) \\ \psi(y,z) \end{bmatrix}$$

Then $H \circ T = L \circ H$ can be written as in the pair of equations

$$D\phi(y,z) = \phi(Dy + Y(y,z), Ez + Z(y,z)) \tag{4.13}$$

$$E\psi(y,z) = \psi(Dy + Y(y,z), Ez + Z(y,z)) \tag{4.14}$$

First, we will solve (4.13), and then it is clear that the same method can be applied to solve the other equation. Define the successive approximations for equation (4.13) by

$$\begin{aligned}\psi_0(y,z) &= z \\ \psi_{k+1}(y,z) &= E^{-1}\psi_k(Dy + Y(y,z), Ez + Z(y,z))\end{aligned} \tag{4.15}$$

From the induction argument for $k = 0, 1, \ldots,$ and for $|y| + |z| \geq 2c_0$; $\psi_k(y, z)$ are continuous and satisfy $\psi_k(y, z) = z$.
We will prove the next inequality with the help of induction i.e for $j = 1, 2, \ldots$

$$|\psi_j(y, z) - \psi_{j-1}(y, z)| \leq \rho\tau^j(|y| + |z|)^\delta \tag{4.16}$$

Here we choose sufficiently small $\delta \in (0, 1)$ such that $\tau < 1$ since $e < 1$ i.e

$$\tau = e[2\max(a, b, e)]^\delta$$
$$\rho = \frac{ae(2c_0)^{1-\delta}}{\tau}$$

First we will start with that $j = 1$, we have

$$\begin{aligned} |\psi_1(y, z) - \psi_0(y, z)| &= |E^{-1}\psi_0(Dy + Y(y, z), Ez + Z(y, z)) - z| \\ &= |E^{-1}(Ez + Z(y, z)) - z| \\ &= |E^{-1}Z(y, z)| \\ &\leq \|E^{-1}\||Z(y, z)| \\ &\leq ea(|y| + |z|) \\ &\leq \rho\tau(|y| + |z|)^\delta \end{aligned}$$

Since $Z(y, z) = 0$ for $|y| + |z| \geq 2c_o$. And then assuming that the induction hypothesis holds for $j = 1, \ldots, k$, we have

$$\begin{aligned} |\psi_{k+1}(y, z) - \psi_k(y, z)| &= |E^{-1}\psi_k(Dy + Y(y, z), Ez + Z(y, z)) - E^{-1}\psi_{k-1}(Dy + Y(y, z), Ez + Z(y, z))| \\ &\leq \|E^{-1}\||\psi_k(*) - \psi_{k-1}(*)| \\ &\leq e\rho\tau^k[|Dy + Y(y, z)| + |Ez + Z(y, z)|]^\delta \\ &\leq e\rho\tau^k[|by + 2a(|y| + |z|) + e|z|]^\delta \\ &\leq e\rho\tau^k[2\max(a, b, e)]^\delta[(|y| + |z|)]^\delta \\ &= \rho\tau^{k+1}[(|y| + |z|)]^\delta \end{aligned}$$

Thus, $\psi_k(y, z)$ is a Cauchy sequence of continuous functions which converges uniformly as $k \to \infty$ to a continuous function $\psi(y, z)$.
$\psi(y, z) = z$ for $|y| + |z| \geq 2c_o$. By taking limit in (4.15) which shows that $\psi(y, z)$ is a solution of the equation (4.14).
The equation (4.13) can be written as

$$D^{-1}\phi(y, z) = \phi(D^{-1}y + Y_1(y, z), E^{-1}z + Z_1(y, z)) \tag{4.17}$$

Where the functions Y_1 and Z_1 are defined by the inverse of V (exists when a is very small)

$$V^{-1}(y, z) = \begin{bmatrix} D^{-1}y + Y_1(y, z) \\ E^{-1}z + Z_1(y, z) \end{bmatrix}$$

With $\phi_0(y, z) = y$, and $d = |D| < 1$. The equation (4.14) can be solved precisely the same way by the method of successive approximations. So we obtain the following continuous map

$$H(y, z) = \begin{bmatrix} \phi(y, z) \\ \psi(y, z) \end{bmatrix}$$

Now, we will show that H is a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

$$H\begin{pmatrix} y_1,\dots,y_k \\ z_{k+1},\dots,z_n \end{pmatrix} = \begin{bmatrix} \phi_1(y,z) \\ \vdots \\ \phi_k(y,z) \\ \psi_{k+1}(y,z) \\ \vdots \\ \psi_n(y,z) \end{bmatrix}$$

$$Der(H)\begin{pmatrix} y_1,\dots,y_k \\ z_{k+1},\dots,z_n \end{pmatrix} = \begin{bmatrix} \frac{\partial}{\partial y_1}\phi_1(y,z) & \dots & \frac{\partial}{\partial y_k}\phi_1(y,z) & \frac{\partial}{\partial z_{k+1}}\phi_1(y,z) & \dots & \frac{\partial}{\partial z_n}\phi_1(y,z) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial y_1}\phi_k(y,z) & \dots & \frac{\partial}{\partial y_k}\phi_k(y,z) & \frac{\partial}{\partial z_{k+1}}\phi_k(y,z) & \dots & \frac{\partial}{\partial z_n}\phi_k(y,z) \\ \frac{\partial}{\partial y_1}\psi_{k+1}(y,z) & \dots & \frac{\partial}{\partial y_k}\psi_{k+1}(y,z) & \frac{\partial}{\partial z_{k+1}}\psi_{k+1}(y,z) & \dots & \frac{\partial}{\partial z_n}\psi_{k+1}(y,z) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial y_1}\psi_n(y,z) & \dots & \frac{\partial}{\partial y_k}\psi_n(y,z) & \frac{\partial}{\partial z_{k+1}}\psi_n(y,z) & \dots & \frac{\partial}{\partial z_n}\psi_n(y,z) \end{bmatrix}$$

$$E\psi(y_1,\dots,z_n) = \psi\left(Dy + Y(y,z), Ez + Z(y,z)\right)$$

$$E\begin{bmatrix} \psi_{k+1}(y_1,\dots,z_n) \\ \vdots \\ \psi_n(y_1,\dots,z_n) \end{bmatrix} = \begin{bmatrix} \psi_{k+1}\left(D\begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix} + Y(y_1,\dots,z_n), E\begin{bmatrix} z_{k+1} \\ \vdots \\ z_n \end{bmatrix} + Z(y_1,\dots,z_n)\right) \\ \vdots \\ \psi_n\left(D\begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix} + Y(y_1,\dots,z_n), E\begin{bmatrix} z_{k+1} \\ \vdots \\ z_n \end{bmatrix} + Z(y_1,\dots,z_n)\right) \end{bmatrix}$$

$$\psi^0(y_1,\dots,z_n) = \begin{bmatrix} \psi^0_{k+1}(y_1,\dots,z_n) \\ \vdots \\ \psi^0_n(y_1,\dots,z_n) \end{bmatrix} = \begin{bmatrix} z_{k+1} \\ \vdots \\ z_n \end{bmatrix}$$

$$Der(\psi^0(y_1,\dots,z_n)) = \begin{bmatrix} \frac{\partial}{\partial y_1}z_{k+1} & \cdots & \frac{\partial}{\partial y_k}z_{k+1} & \frac{\partial}{\partial z_{k+1}}z_{k+1} & \cdots & \frac{\partial}{\partial z_n}z_{k+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial y_1}z_n & \cdots & \frac{\partial}{\partial y_k}z_n & \frac{\partial}{\partial z_{k+1}}z_n & \cdots & \frac{\partial}{\partial z_n}z_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\psi^{k+1}(y_1,\dots,z_n) = E^{-1}\psi^k(Dy + Y(y,z), Ez + Z(y,z))$$

For $k = 0$,

$$\psi^1(y_1,\dots,z_n) = E^{-1}\begin{bmatrix} \psi^0_{k+1}\left(D\begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix} + Y(y_1,\dots,z_n), E\begin{bmatrix} z_{k+1} \\ \vdots \\ z_n \end{bmatrix} + Z(y_1,\dots,z_n)\right) \\ \vdots \\ \psi^0_n\left(D\begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix} + Y(y_1,\dots,z_n), E\begin{bmatrix} z_{k+1} \\ \vdots \\ z_n \end{bmatrix} + Z(y_1,\dots,z_n)\right) \end{bmatrix}$$

$$= E^{-1}\begin{bmatrix} E\begin{bmatrix} z_{k+1} \\ \vdots \\ z_n \end{bmatrix} + Z(y_1,\ldots,z_n) \qquad \to 1^{st}\ Co-ordinate \\ \vdots \\ E\begin{bmatrix} z_{k+1} \\ \vdots \\ z_n \end{bmatrix} + Z(y_1,\ldots,z_n) \to (n-k)^{th}\ Co-ordinate \end{bmatrix}$$

$$= E^{-1}E\begin{bmatrix} z_{k+1} \\ \vdots \\ z_n \end{bmatrix} + E^{-1}\begin{bmatrix} Z_{k+1}(y_1,\ldots,z_n) \\ \vdots \\ Z_n(y_1,\ldots,z_n) \end{bmatrix}$$

$$= \begin{bmatrix} z_{k+1} \\ \vdots \\ z_n \end{bmatrix} + E^{-1}Z(y_1,\ldots,z_n)$$

$$Der(\psi^1(y,z)) =$$

$$\begin{bmatrix} \frac{\partial}{\partial y_1}\{E^{-1}Z\}_{1st} & \ldots & \frac{\partial}{\partial y_k}\{E^{-1}Z\}_{1st} & 1+\frac{\partial}{\partial z_{k+1}}\{E^{-1}Z\}_{1st} & \ldots & \frac{\partial}{\partial z_n}\{E^{-1}Z\}_{1st} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial y_1}\{E^{-1}Z\}_{(n-k)} & \ldots & \frac{\partial}{\partial y_k}\{E^{-1}Z\}_{(n-k)} & \frac{\partial}{\partial z_{k+1}}\{E^{-1}Z\}_{(n-k)} & \ldots & 1+\frac{\partial}{\partial z_n}\{E^{-1}Z\}_{(n-k)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 1 \end{bmatrix} + E^{-1}Der(Z(y_1,\ldots,z_n))$$

$$Der(\psi^1(0,\ldots,0)) = \begin{bmatrix} 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 1 \end{bmatrix} + E^{-1}.0$$

$$= \begin{bmatrix} 0 & \ldots & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 1 \end{bmatrix}$$

$$\psi^k(y_1,\ldots,z_n) = \begin{bmatrix} z_{k+1} \\ \vdots \\ z_n \end{bmatrix} + E^{-1}Z\,(y_1,\ldots,z_n) + E^{-2}Z\,(f(y_1,\ldots,z_n)) + \cdots + E^{-k}Z\left(f^{k-1}(y_1,\ldots,z_n)\right)$$

Where

$$f:\mathbb{R}^n \to \mathbb{R}^n$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_k \\ z_{k+1} \\ \vdots \\ z_n \end{bmatrix} \to \begin{bmatrix} D\begin{bmatrix} y_1 \\ \vdots \\ y_k \end{bmatrix} + Y(y_1,\ldots,z_n) \\ E\begin{bmatrix} z_{k+1} \\ \vdots \\ z_n \end{bmatrix} + Z(y_1,\ldots,z_n) \end{bmatrix}$$

$$
\begin{aligned}
f\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix} &= \begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}\\
Der(f(y_1,\dots,z_n)) &= \begin{bmatrix}D & 0\\ 0 & E\end{bmatrix} + \begin{bmatrix}Der(Y(y_1,\dots,z_n))\\ Der(Z(y_1,\dots,z_n))\end{bmatrix}\\
Der(f(0,\dots,0)) &= \begin{bmatrix}D & 0\\ 0 & E\end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
Der(\psi^k(y_1,\dots,z_n)) &= \begin{bmatrix}0 & \dots & 0 & 1 & \dots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \dots & 0 & 0 & \dots & 1\end{bmatrix} + E^{-1}Der(Z(y_1,\dots,z_n))\\
&+ E^{-2}Der(Z(f(y_1,\dots,z_n))).Der(f(y_1,\dots,z_n))\\
&+ \dots + E^{-k}Der(Z(f^{k-1}(y_1,\dots,z_n))).Der(f^{k-1}(y_1,\dots,z_n))\\
Der(\psi^k(0,\dots,0)) &= \begin{bmatrix}0 & \dots & 0 & 1 & \dots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \dots & 0 & 0 & \dots & 1\end{bmatrix} + 0\dots + E^{-k}Der(Z(f\dots f(0,\dots,0)).Der(f^{k-1}(0,\dots,0)))\\
&= \begin{bmatrix}0 & \dots & 0 & 1 & \dots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \dots & 0 & 0 & \dots & 1\end{bmatrix} + 0\dots + E^{-k}Der(Z(0).Der(f^{k-1}(0,\dots,0)))\\
&= \begin{bmatrix}0 & \dots & 0 & 1 & \dots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \dots & 0 & 0 & \dots & 1\end{bmatrix}
\end{aligned}
$$

Now,

$$
\begin{aligned}
\|Der(\psi^{k+1}(y_1,\dots,z_n)) - Der(\psi^k(y_1,\dots,z_n))\| &= \|E^{-(k+1)}Der(Z(f^k(y_1,\dots,z_n))).Der(f^k(y_1,\dots,z_n))\|\\
&\le E^{-(k+1)}\|Der(Z(f^k(y_1,\dots,z_n)))\|.\|Der(f^k(y_1,\dots,z_n))\|\\
&\le E^{-(k+1)}.a'
\end{aligned}
$$

For $N < m < n'$

$$
\begin{aligned}
\|Der(\psi^m(y_1,\dots,z_n)) - Der(\psi^{n'}(y_1,\dots,z_n))\| &= \|Der(\psi^m(y_1,\dots,z_n)) - Der(\psi^{m+1}(y_1,\dots,z_n))\| +\\
&\dots + \|Der(\psi^{n'-1}(y_1,\dots,z_n)) - Der(\psi^{n'}(y_1,\dots,z_n))\|\\
&\le E^{-(m+1)}a' + \dots + E^{-n'}a'\\
&\le \sum_{k=N}^{\infty} E^{-k}a'\\
&\le \frac{E^{-N}a'}{1-E}
\end{aligned}
$$

as $N \to \infty$, the above goes to 0. So ψ_k is uniformly Cauchy and hence uniformly convergence. By the uniform convergence

$$\begin{aligned} \psi' &= \lim \psi_k' \qquad\qquad \forall Points \\ \psi'(0) &= \lim \psi_k' \\ Der(H(0)) &= \begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix} \neq [0] \end{aligned}$$

Hence from the inverse function theorem there exists an neighbourhood around 0, where H is homeomorphic. □

Let

$$U^t(x_0) := e^{At}x_0 \tag{4.18}$$

$$V^t(x_0) := \phi_t(x_0) \tag{4.19}$$

Define:

$$H = \int_0^1 U^{-s}H_0V^s ds \tag{4.20}$$

$$\begin{aligned} U^tH &= \int_0^1 U^{t-s}H_0V^{s-t}ds\ V^t \\ s-t = s' &\implies ds = ds' \\ &= \int_{-t}^{1-t} U^{-s}H_0V^s ds\ V^t \\ &= V^t\left(\int_{-t}^{0} U^{-s}H_0V^s ds + \int_0^{1-t} U^{-s}H_0V^s ds\right) \end{aligned}$$

Since $H_0 = U^{-1}H_0V$, then

$$\begin{aligned} \int_{-t}^{0} U^{-s}H_0V^s ds &= \int_{-t}^{0} U^{-s-1}H_0V^{s+1}ds \\ &= \int_{1-t}^{1} U^{-s}H_0V^s ds \end{aligned}$$

So,

$$\begin{aligned} U^tH &= \int_0^1 U^{-s}H_0V^s ds\ V^t \\ &= HV^t. \end{aligned}$$

Hence

$$\begin{gathered} H \circ V^t = U^tH \\ or \\ H \circ \phi_t(x_0) = e^{At}H(x_0). \end{gathered}$$

□

Chapter 5

Global Theory of Non-linear Dynamical Systems

5.1 Definitions

A dynamical system is a function $\phi(t, x)$, which is defined $\forall t \in \mathbb{R}$ and $x \in G \subset \mathbb{R}^n$. Which describes how the point $x \in G$ moves with respect to time t.

Definition 5.1.1. *A dynamical system on G is a C^1- map.*

$$\phi : \mathbb{R} \times G \to G$$

where G is an open subset of $\mathbb{R}^n$, and if $\phi_t(x) = \phi(t, x)$, then ϕ_t satisfies :

$$\begin{aligned} \phi_0(x) &= x \quad \forall x \in G \\ \phi_t \circ \phi_s(x) &= \phi_{t+s}(x) \qquad \forall s, t \in \mathbb{R},\ x \in G. \end{aligned}$$

From the above condition ϕ_t has C^1 inverse which is ϕ_{-t}. It also satisfies the group property, so this system forms a commutative group under the composition of maps.

Definition 5.1.2. *Let $f \in C^1(E)$ and $h \in C^1(G)$. E and G are open subset of $\mathbb{R}^n$.*
Consider the following equations

$$\dot{x} = f(x) \tag{5.1}$$

$$\dot{x} = h(x) \tag{5.2}$$

these equations are topologically equivalent, if there exists a homeomorphism $H : E \to G$, which maps trajectories of (5.1) onto trajectories of (5.2) and it is also preserves the orientation by time.

If ϕ_t is the flow on E defined by the equation (5.1). And (5.2) is the dynamical system ψ_t on G, then (5.1) and (5.2) are topologically equivalent if and only if there is a homeomorphism $H : E \to G$ and for each $x \in E$ there is a C^1 function $t(x, \tau)$ defined $\forall \tau \in \mathbb{R}$ such that $\frac{\partial t}{\partial \tau} > 0$ and $\forall x \in E$, $\forall \tau \in \mathbb{R}$,

$$H \circ \phi_{t(x,\tau)}(x) = \psi_\tau \circ H(x)$$

5.2 Global Existence Theorem

Theorem 5.2.1. *Let $f \in C^1(\mathbb{R}^n)$, and $x_0 \in \mathbb{R}^n$, the initial value problem*

$$\begin{aligned} \dot{x} &= \frac{f(x)}{1+|f(x)|} \\ x(0) &= x_0 \end{aligned} \tag{5.3}$$

has a unique solution $x(t)$ defined $\forall t \in \mathbb{R}$, which means that (5.3) is a dynamical system on $\mathbb{R}^n$. Also (5.3) is topologically equivalent to (5.1) on $\mathbb{R}^n$.

Proof. Time t along the solution $x(t)$ of (5.1) is rescaled by the below formula:

$$\tau = \int_0^t (1+|f(x(s))|)\, ds \tag{5.4}$$

Lemma 5.2.2. *τ is increasing with respect to the time t.*

Proof. Let $t_1 > t_2$, then

$$\begin{aligned} \tau_1 - \tau_2 &= \int_0^{t_1} (1+|f(x(s))|)\, ds - \int_0^{t_2} (1+|f(x(s))|)\, ds \\ &= \int_{t_2}^{t_1} (1+|f(x(s))|)\, ds \\ &= (t_1 - t_2) + \int_{t_2}^{t_1} |f(x(s))| ds > 0 \\ \tau_1 &> \tau_2 \end{aligned}$$

Hence τ is increasing with respect to the time t. Also,

$$\frac{\partial \tau}{\partial t} > 0.$$

□

Let ϕ_t be the flow of f. Then $\frac{d\phi_t()}{dt} = f(\phi_t(x))$. Therefore,

$$\begin{aligned} \frac{d\phi_t(x)}{d\tau} &= \frac{\frac{d\phi_t(x)}{dt}}{\frac{d\tau}{dt}} \\ &= \frac{f(\phi(x,t))}{1+|f(\phi(x,t))|} \end{aligned}$$

i.e $\phi_t(x)$ is the solution of (5.3). Now, if $t = 0$ then $\tau = 0$. And hence (unique solution),

$$\begin{aligned} \phi_t(x)|_{\tau=0} &= \phi_t(x)|_{t=0} \\ \phi_t(x) &= \psi_\tau(x). \end{aligned}$$

Define the identity map

$$\begin{aligned} &H : \mathbb{R}^n \to \mathbb{R}^n \\ &x :\to x \end{aligned}$$

Now,

$$H \circ \phi_t(x) = \psi_\tau \circ H(x)$$
$$\phi_t(x) = \psi_\tau(x)$$

From the definition, these two equations (5.1) and (5.3) are topologically equivalent.
Now, we will prove that (5.3) is dynamical system, i.e if $x(t)$ is the solution of (5.3), then it is defined for all t.

Lemma 5.2.3. *$f(x) \in C^1(\mathbb{R}^n)$, then $\frac{f(x)}{1+|f(x)|} \in C^1(\mathbb{R}^n)$.*

Proof. Let

$$F(x) = \frac{f(x)}{1+|f(x)|}$$

Case 1: Consider $f : \mathbb{R}^n \to \mathbb{R}$.
Let $f(a_1, a_2, \dots, a_n) = 0$

$$\begin{aligned}
\frac{\partial F|_{(a_1,\dots,a_n)}}{\partial x_1} &= \lim_{t\to 0} \frac{F(a_1+t, a_2, \dots, a_n) - F(a_1, \dots, a_n)}{t} \\
&= \lim_{t\to 0} \frac{f(a_1+t,\dots,a_n)[1+|f(a_1,\dots,a_n)|] - f(a_1,\dots,a_n)[1+|f(a_1+t,\dots,a_n)|]}{t(1+|f(a_1,\dots,a_n)|)(1+|f(a_1,\dots,a_n)|)} \\
&= \frac{f'(a_1,\dots,a_n)}{[1+|f(a_1,\dots,a_n)|]^2} + \lim_{t\to 0} \frac{|f(a_1,\dots,a_n)|[f(a_1+t,\dots,a_n)] - |f(a_1+t,\dots,a_n)[f(a_1,\dots,a_n)]}{t(1+|f(a_1,\dots,a_n)|)(1+|f(a_1,\dots,a_n)|)} \\
&= \frac{f'(a_1,\dots,a_n)}{[1+|f(a_1,\dots,a_n)|]^2} + 0 \\
&= f'(a_1,\dots,a_n) \qquad \{exists.\}
\end{aligned}$$

For non-zero case,

$$\frac{\partial F(x)}{\partial x_1} = \begin{cases} \dfrac{\frac{\partial f(x)}{\partial x_1}}{[1-f(x)]^2} & ; \ f(x) < 0 \\ \dfrac{\frac{\partial f(x)}{\partial x_1}}{[1+f(x)]^2} & ; \ f(x) > 0 \end{cases}$$

As $x \to (a_1, \dots, a_n)$

$$\frac{\partial F(x)}{\partial x_1} = f'(a_1, \dots, a_n).$$

And hence, $\frac{\partial F(x)}{\partial x_i}$ are continuous and exists so $F \in C^1$.
Let $f(x_0) \neq 0$, i.e if $f(x_0) > 0$, then $\exists$ a δ neighbourhood of x_0 $\mathbb{B}(x_0, \delta)$ such that $\forall x \in \mathbb{B}(x_0, \delta)$, $f(x) > 0$.
And if $f(x_0) < 0$, then $\exists$ a δ neighbourhood of x_0 $\mathbb{B}(x_0, \delta)$ such that $\forall x \in \mathbb{B}(x_0, \delta)$, $f(x) < 0$. Hence, $f(x)$ is continuous.
Case 2: Consider

$$f : \mathbb{R}^n \to \mathbb{R}^m \quad m \geq 2$$
$$(x_1, \dots, x_n) \to \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix}$$

$$F(x) = \left(\frac{f_1(x)}{1+\sqrt{f_1^2(x)+\cdots+f_m^2(x)}}, \frac{f_2(x)}{1+\sqrt{f_1^2(x)+\cdots+f_m^2(x)}}, \ldots, \frac{f_m(x)}{1+\sqrt{f_1^2(x)+\cdots+f_m^2(x)}} \right)$$

Let $\quad G_1(x) = \dfrac{f_1(x)}{1+\sqrt{f_1^2(x)+\cdots+f_m^2(x)}}$

As from the above calculation,

$$\begin{aligned}
\frac{\partial G_1(x)}{\partial x_1} &= \frac{\frac{\partial f_1(a_1,\ldots,a_n)}{\partial x_1}}{[1+\sqrt{f_1^2(a_1,\ldots,a_n)+\cdots+f_m^2(a_1,\ldots,a_n)}]^2} \\
&+ \frac{\lim_{t\to 0}\sqrt{f_1^2(a_1,\ldots,a_n)+\cdots+f_m^2(a_1,\ldots,a_n)}f_1(a_1+t,\ldots,a_n)}{t(1+|f(a_1,\ldots,a_n)|)(1+|f(a_1,\ldots,a_n)|)} \\
&- \frac{\lim_{t\to 0}\sqrt{f_1^2(a_1+t,\ldots,a_n)+\cdots+f_m^2(a_1,\ldots,a_n)}f_1(a_1,\ldots,a_n)}{t(1+|f(a_1,\ldots,a_n)|)(1+|f(a_1,\ldots,a_n)|)} \\
&+ \frac{\lim_{t\to 0}\sqrt{f_1^2(a_1,\ldots,a_n)+\cdots+f_m^2(a_1,\ldots,a_n)}f_1(a_1,\ldots,a_n)}{t(1+|f(a_1,\ldots,a_n)|)(1+|f(a_1,\ldots,a_n)|)} \\
&- \frac{\lim_{t\to 0}\sqrt{f_1^2(a_1,\ldots,a_n)+\cdots+f_m^2(a_1,\ldots,a_n)}f_1(a_1,\ldots,a_n)}{t(1+|f(a_1,\ldots,a_n)|)(1+|f(a_1,\ldots,a_n)|)} \\
\text{When norm=0, then } &= \frac{\partial f_1(a_1,\ldots,a_n)}{\partial x_1} \\
\text{When norm is not 0, then } &= \frac{\partial f_1(a_1,\ldots,a_n)}{\partial x_1} + \frac{\frac{\partial f_1|_{(a_1,\ldots,a_n)}}{\partial x_1}\sqrt{f_1^2+\cdots+f_m^2}}{[1+\sqrt{f_1^2(a_1,\ldots,a_n)+\cdots+f_m^2(a_1,\ldots,a_n)}]^2} \\
&- \frac{\frac{\partial(\sqrt{f_1^2+\cdots+f_m^2})|_{(a_1,\ldots,a_n)}}{\partial x_1}f_1(a_1,\ldots,a_n)}{[1+\sqrt{f_1^2(a_1,\ldots,a_n)+\cdots+f_m^2(a_1,\ldots,a_n)}]^2}
\end{aligned}$$

as $\lim x \to (a_1,\ldots,a_n)$, the above expression is same as when norm is 0.
Hence, $\frac{\partial G_1(x)}{\partial x_i}$ are continuous and exists. Since, all $G_1(x),\ldots,G_m(x)$ are symmetric so all are C^1. And hence $F(x)$ is $C^1(\mathbb{R}^n)$. □

Let $x(t)$ be the solution of initial value problem (5.3) on the maximal interval (α,β). $x(t)$ satisfies the following integral equation

$$x(t) = x_0 + \int_0^t \frac{f(x(s))}{1+|f(x(s))|}ds$$

$\forall t \in (\alpha,\beta)$
from the above integral equation

$$|x(t)| \le |x_0| + \int_0^{|t|} ds = |x_0| + t$$
$$|x(t)| \le |x_0| + \beta$$

if $t \in [0,\beta]$.

Since the solution of (5.3) is contained in the compact set K

$$S = \{\phi_t(x_0)| \quad t \in [0,\beta)\} \subset K$$
$$K = \{x \in \mathbb{R}^n|\ |x| \le |x_0| + \beta\}$$

Then from the Lemma (2.1.4), $\beta = \infty$.
Now take $t \to -t$,

$$S' = \{\phi_{-t}(x_0) | \ \ t \in [0, -\alpha)\} \subset K'$$
$$K' = \{x \in \mathbb{R}^n | \ |x| \leq |x_0| + \alpha\}$$

Then by the same Lemma (2.1.4), $-\alpha = \infty$.
$\Rightarrow (\alpha, \beta) = (-\infty, \infty)$. So the maximal interval of existence of $x(t)$ of the IVP (5.3) is $(-\infty, \infty)$ and hence, it is a dynamical system on $\mathbb{R}^n$. □

Theorem 5.2.4. *[1] Let E be an open subset of $\mathbb{R}^n$ and $f \in C^1(E)$. Then there exists a function $F \in C^1(E)$ such that*

$$\dot{x} = F(x) \tag{5.5}$$

which defines a dynamical system on E and it is topologically equivalent to (5.1) on E.

Theorem 5.2.5. *Let $f \in C^1(\mathbb{R}^n)$ If $f(x)$ satisfies the global Lipschitz condition*

$$|f(x) - f(y)| \leq K|x - y| \tag{5.6}$$

$\forall x, y \in \mathbb{R}^n$ then for $x_0 \in \mathbb{R}^n$, the IVP (5.1) has a unique solution $x(t)$ which is defined $\forall t \in R$.

Proof. Let $x(t)$ be the solution of the IVP (5.1) on its maximal interval (α, β). We know the fact

$$\frac{d|x(t)|}{dt} \leq |\dot{x}(t)|$$

by using the triangle inequality

$$\begin{aligned}
\frac{d}{dt}|x(t) - x_0| &\leq |\dot{x}(t)| = |f(x(t))| \\
&\leq |f(x(t)) - f(x_0)| + |f(x_0)| \\
&\leq K|x(t) - x_0| + |f(x_0)|
\end{aligned}$$

If $\beta < \infty$, then the function $g(t) = |x(t) - x_0|$ satisfies

$$\begin{aligned}
g(t) &= \int_0^t \frac{dg(s)}{ds} ds \\
&\leq \int_0^t [K|x(t) - x_0| + |f(x_0)|] ds \\
&= K \int_0^t [|x(t) - x_0|] ds + |f(x_0)| t \\
&\leq K \int_0^t g(s) ds + |f(x_0)| \beta
\end{aligned}$$

$\forall t \in (0, \beta)$.
Then from the Gronwall's lemma (2.1.5), we can write;

$$|x(t) - x_0| \leq \beta |f(x_0)| e^{K\beta} \ \ \forall t \in [0, \beta).$$

Then the trajectory of (5.1) through the point x_0 at time $t = 0$ is contained in the compact set C.

$$C = \{x \in \mathbb{R}^n | \ \ |x - x_0| \leq \beta |f(x_0)| e^{K\beta}\} \subset \mathbb{R}^n$$

From the Lemma (2.1.4), $\beta = \infty$. Also $\alpha = -\infty$.
Thus $\forall x_0 \in \mathbb{R}^n$, the maximal interval of existence of the solution $x(t)$ is $(-\infty, \infty)$. □

5.3 Limit set and Attractors

Consider the function

$$\phi(., x) : \mathbb{R} \to E$$

The solution curve, trajectory, orbit of (5.1) through the point x_0 is

$$\tau_{x_0} = \{x \in E | \ x = \phi(t, x_0), t \in \mathbb{R}\}$$

$\tau^+_{x_0}$, $\tau^-_{x_0}$ are the positive and negative half trajectory respectively

$$\tau^+_{x_0} = \{x \in E | \ x = \phi(t, x_0), t \geq 0\}$$
$$\tau^-_{x_0} = \{x \in E | \ x = \phi(t, x_0), t \leq 0\}$$
$$\tau_{x_0} = \tau^+_{x_0} + \tau^-_{x_0}$$

Definition 5.3.1. *A point p is a ω-limit point of trajectory $\phi(., x)$ of (5.1), if there is a sequence $t_n \to \infty$ such that*

$$\lim_{n\to\infty} \phi(t_n, x) = p$$

and if there is a sequence $t_n \to -\infty$ such that

$$\lim_{n\to\infty} \phi(t_n, x) = q$$

then the point $q \in E$ is called the α-limit point.

Definition 5.3.2. *Set of all ω-limit points of τ is called ω-limit set of τ and it is written by $\omega(\tau)$.*
Set of all α-limit points of τ is called α-limit set of τ and it is written by $\alpha(\tau)$.
The set of all limit points of τ, i.e. $\alpha(\tau) \cup \omega(\tau)$ is called limit set of τ.

Example 5.3.1. *Consider* $\sin\left(\frac{1}{x}\right)$.

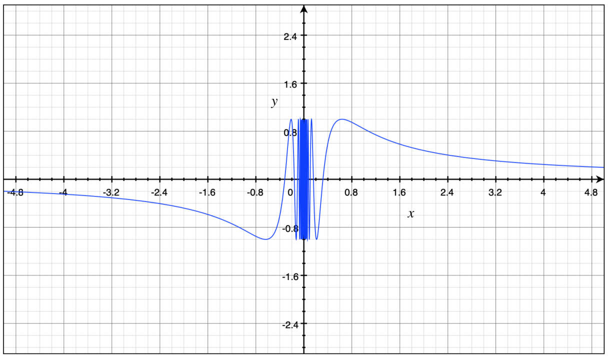

Figure 5.1: Graph of $\sin\left(\frac{1}{x}\right)$

In this graph, consider all the points which are intersecting x-axis. The subsequence of these points goes to the origin, hence the origin is an ω-limit point. Similarly, the line joining form (0,-1) to (0,1) are also an ω- limit points. The collection of all these limit points is an ω-limit set.

Theorem 5.3.1. *(i). $\alpha(\tau), \omega(\tau)$ are closed subsets of E.*
If τ is in a compact subset of $\mathbb{R}^n$, then
(ii). $\alpha(\tau), \omega(\tau)$ are non empty.
(iii). $\alpha(\tau), \omega(\tau)$ are connected subsets of E.
(iv). $\alpha(\tau), \omega(\tau)$ are compact subsets of E.

Proof. Part (i). First of all, from the definition of ω-limit set we have $\omega(\tau) \subset E$. Now, to show that $\omega(\tau)$ is closed subset of E, consider a sequence of points $\{s_n\}$ in $\omega(\tau)$ with $\lim_{n\to\infty} s_n = p \in \mathbb{R}^n$. We show that $p \in \omega(\tau)$.
Let $\tau = \{\phi_t(x_0)|t \in \mathbb{R}\}$. Since $s_n \in \omega(\tau)$, for $1, 2, \ldots$ there is a sequence $t_k^{(n)} \to \infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty} \phi(t_k^{(n)}, x_0) = s_n$$

Assume $t_k^{(n+1)} > t_k^{(n)}$. For all $n \geq 2$, there is a sequence of integers $N(n) > N(n-1)$ such that for $k \geq N(n)$

$$|\phi(t_k^{(n)}, x_0) - s_n| < \frac{1}{n}$$

Consider

$$\begin{aligned} |\phi(t_{N(n)}^{(n)}, x_0) - p| &\leq |\phi(t_{N(n)}^{(n)}, x_0) - s_n| + |s_n - p| \\ &\leq \frac{1}{n} + |s_n - p| \end{aligned}$$

as $n \to \infty$ we have $|\phi(t_{N(n)}^{(n)}, x_0) - p|$ goes to 0. Hence $p \in \omega(\tau)$.

Part (ii). The sequence of points $\phi(t_n, x_0) \in C$(compact set) contains a convergent subsequence which converges to a point in $\omega(\tau) \subset C$, so $\omega(\tau)$ is non-empty.

Part (iii). To show that $\omega(\tau)$ is connected for that suppose $\omega(\tau)$ is not connected, so there exists two nonempty disjoint closed set P and Q such that $\omega(\tau) = P \cup Q$. Distance from P to Q is δ and it is given by

$$d(P, Q) = \inf_{x\in P, y\in Q} |x - y|$$

Since P and Q are ω-limit points of τ, so it is clear that there are arbitrarily large t such that $\phi(t, x_0)$ are inside $\frac{\delta}{2}$ of P and there are arbitrarily large t such that the distance of $\phi(t, x_0)$ from P is greater than $\frac{\delta}{2}$. Since distance function is always a continuous function, So there should be a sequence $t_n \to \infty$ such that

$$d(\phi(t_n, x_0), P) = \frac{\delta}{2}$$

It is given that the trajectory is contained in a compact set so there is a subsequence which is converging to a point $p \in \omega(\tau)$ with $d(p, P) = \frac{\delta}{2}$.
Now,

$$d(p, Q) > d(P, Q) - d(p, P) = \frac{\delta}{2}$$

which shows that $p \notin P$ and $p \notin Q$, also $p \notin \omega(\tau)$. It is contradiction of the assumed argument, hence $\omega(\tau)$ is connected.

Part (iv). Let C be a compact set, If $\tau \subset C$ and $p \in \omega(\tau)$, then $p \in C$. And hence $\omega(\tau) \subset K$, since closed subset of a compact set is compact therefore $\omega(\tau)$ is compact.
Using the same steps it is clear that $\alpha(\tau)$ is a closed subset of E, nonempty, connected and compact subset of E. ☐

Theorem 5.3.2. *If p is an ω-limit point of a trajectory τ of (5.1), then all other points of the trajectory $\phi(., p)$ of (5.1) passing through the point p are also ω-limit point of τ or we can say that if $p \in \omega(\tau)$ then $\tau_p \in \omega(\tau)$, and similarly if $p \in \alpha(\tau)$, then $\tau_p \in \alpha(\tau)$.*

Proof. This theorem can be proved by using the continuity with respect to initial conditions, and property of dynamical systems. ☐

Corollary 5.3.2.1. *$\omega(\tau)$ and $\alpha(\tau)$ are invariant with respect to the flow ϕ_t of (5.1).*

Proof. It is merely the use of the above theorem. □

Definition 5.3.3. *A closed invariant set $\mathcal{A}$ which is contained in the set E, is called an attracting set of (5.1) if there exists a neighbourhood N of $\mathcal{A}$ such that $\forall x \in N$, $\phi_t(x) \in N$ for all $t \geq 0$ and $\phi_t(x) \to \mathcal{A}$ as $t \to \infty$.*
An attracting set which contains a dense orbit is called an attractor.

Example 5.3.2. *Consider the system*

$$\dot{x} = -y + x(1 - x^2 - y^2)$$
$$\dot{y} = x + y(1 - x^2 - y^2)$$

The above system can be written in the polar coordinates as

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = 1$$

The Phase portrait of this system is

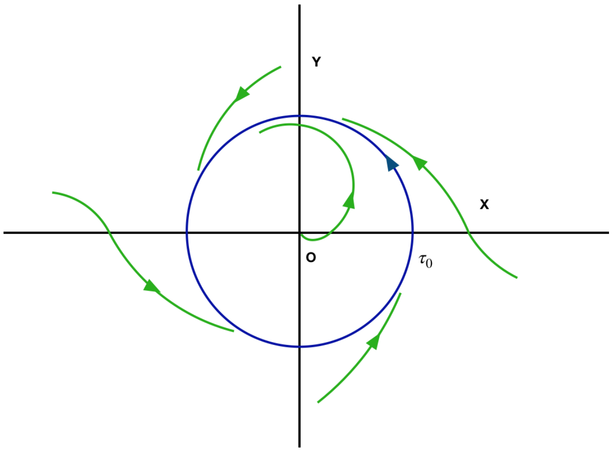

Figure 5.2: An Attractor (τ_0)

In this phase diagram, If we take the neighbourhood (N) around τ_0, then all the other trajectories starting from the point $x \in N$ are approaching τ_0 as $t \geq 0$, and hence τ_0 is an attracting set. It also contains the dense orbit, so τ_0 is an attractor.

5.4 Periodic Orbits and Limit Cycles

Definition 5.4.1. *A cycle or we can say a periodic orbit of a system is a closed solution curve, which is not a critical point of that system.*

Definition 5.4.2. *A periodic orbit τ is said to be stable periodic orbit if for each $\epsilon > 0$ there is a neighbourhood N of τ, such that $\forall x \in N$,*

$$d(\tau_x^+, \tau) < \epsilon$$

in other words if $\forall x \in N$ and $t \geq 0$,

$$d(\phi(t,x), \tau) < \epsilon.$$

If it is not stable, then the periodic orbit is called an unstable orbit.

Definition 5.4.3. *The periodic orbit is said to be asymptotically stable, if the cycle is stable and if* $\forall x \in U$*,*

$$\lim_{t\to\infty} d(\phi(t,x),\tau) = 0.$$

Periodic orbit is asymptotically stable only when the following holds

$$\int_0^{\lambda} \bigtriangledown . f(\eta(t))dt \leq 0.$$

Where λ is the Period of the periodic orbit.

Definition 5.4.4. *A limit cycle τ is a cycle of the system which is α or ω- limit set of some trajectory which is not the τ itself.*

Definition 5.4.5. *A cycle is called a stable limit cycle or ω- limit cycle, if the cycle τ is the ω-limit set of every trajectory in the neighbourhood of τ.*

Definition 5.4.6. *A cycle is called an unstable limit cycle or α- limit cycle, if the cycle τ is the α-limit set of every trajectory in the neighbourhood of τ.*

Definition 5.4.7. *A cycle is called a semi-stable limit cycle, if the cycle τ is the ω- limit set of one trajectory other than τ and the α- limit set of another trajectory other than τ.*

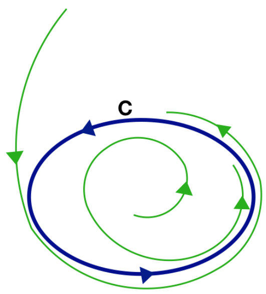

Figure 5.3: Stable Limit Cycle

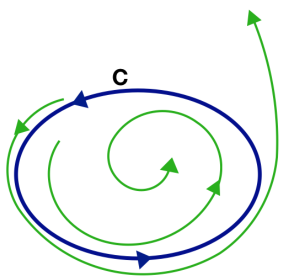

Figure 5.4: Unstable Limit Cycle

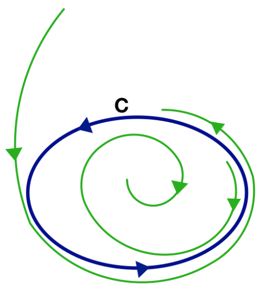

Figure 5.5: Semi-Stable

Example 5.4.1. *Consider the system*

$$\dot{x} = \alpha x - y - \alpha x(x^2 + y^2)$$
$$\dot{y} = x + \alpha y - \alpha y(x^2 + y^2)$$

Where α is a parameter.
Now transform to radial co-ordinates, it can be seen that the periodic orbit lies on a circle with $|r| = 1$*. For any* $\alpha > 0$*,*

$$\dot{r} = \alpha r(1 - r^2)$$
$$\dot{\theta} = 1$$

This periodic orbit has a stable limit cycle for $\alpha > 0$, an unstable limit cycle for $\alpha < 0$, and it has infinite number of periodic orbits and no limit cycles for $\alpha = 0$.

5.5 The Poincaré Map

Consider the following system

$$\dot{x} = f(x) \tag{5.7}$$

Let τ is a periodic orbit of the non-linear system through x_0 and Ω is a hyperplane which is perpendicular to the orbit at x_0. Then near x_0, any point $x \in \Omega$, at $t = 0$ $\phi_t(x)$ will cross Ω again at a point $P(x)$. Then the mapping

$$x \to P(x) \tag{5.8}$$

is called a poincaré map.

Definition 5.5.1. *Let E be an open subset of $\mathbb{R}^n$ and $f \in C^1(E)$. Let $\phi_t(x_0)$ be the periodic solution also the cycle τ is contained in E. Let Ω be the hyperplane which is orthogonal to τ at x_0. Then there is a $\delta > 0$ and a unique function $\mu(x)$ which is C^1 and for $x \in N_\delta(x_0)$ such that*

$$\mu(x_0) = \lambda$$

and

$$\phi_{\mu(x)}(x) \in \Omega$$

for $x \in N_\delta(x_0) \cap \Omega$

$$P(x) = \phi_{\mu(x)}(x)$$

is called the poincaré map for τ.

The next theorem tells about the stability of the limit cycle.

Theorem 5.5.1. *[1] Let E be an open subset of $\mathbb{R}^n$ and $f \in C^1(E)$. Let $\eta(t)$ be the periodic solution of period λ. The derivative of the poincaré map along Ω is given by*

$$P'(0) = e^{\int_0^\lambda \triangledown . f(\eta(t))dt}$$

Now, $\eta(t)$ is a stable limit cycle if

$$\int_0^\lambda \triangledown . f(\eta(t))dt < 0.$$

It is an unstable limit cycle if

$$\int_0^\lambda \triangledown . f(\eta(t))dt > 0.$$

and semi-stable if

$$\int_0^\lambda \triangledown . f(\eta(t))dt = 0.$$

5.6 The Stable Manifold Theorem for Periodic Orbits

Consider the following non-linear system

$$\dot{x} = f(x)$$

Let E be an open subset of $\mathbb{R}^n$ and $f \in C^1(E)$.
Let this system has a periodic orbit of period λ.

$$\tau : x = \eta(t) \quad 0 \leq t \leq \lambda$$

The linearisation of the non-linear system about τ is

$$\dot{x} = A(t)x$$

or we can write

$$\dot{\phi} = A(t)\phi$$

Where

$$A(t) = Df(\eta(t)).$$

The fundamental matrix solution is given by

$$\phi(t) = N(t)e^{Ct}$$

Where $N(t)$ is a non-singular matrix and C is a constant matrix.
Using the conditions at $t = 0$, $\phi(0) = I$, $N(0) = I$.

$$\phi(t) = e^{Ct}$$

The eigenvalues (e_j) of the constant matrix C are called the characteristic exponents of $\eta(t)$.

5.6.1 The stable manifold theorem for periodic orbits

Let E be an open subset of $\mathbb{R}^n$ and $f \in C^1(E)$ contains a periodic orbit of period λ.
Let ϕ_t be the flow of the non-linear system and

$$\eta(t) = \phi_t(x_0)$$

If k characteristic exponents of $\eta(t)$ has negative real part and $n - k$ has positive real part where $0 \leq k \leq n - 1$, then there exists a $k + 1$ dimensional differential stable manifold (S) which is positively invariant with respect to flow ϕ_t. And there exists an $(n - k)$ dimensional differential unstable manifold (U) which is negatively invariant with respect to flow ϕ_t.

5.6.2 The center manifold theorem for periodic orbits

Let E be an open subset of $\mathbb{R}^n$ and $f \in C^g(E)$ with $g \geq 1$ contains a periodic orbit of period λ.
Let ϕ_t be the flow of the non-linear system and

$$\eta(t) = \phi_t(x_0)$$

If k characteristic exponents of $\eta(t)$ has negative real part, j has positive real part and $m = n-k-j$ have zero real part, then there exists a m dimensional center manifold (C) which is invariant with respect to flow ϕ_t.

We can prove the stable and center manifold theorems for periodic orbits by the same method used in the proof of the stable manifold theorem (3.1.2).

Example 5.6.1. *Geometrical example of Periodic orbit with three dimensional stable manifold.*

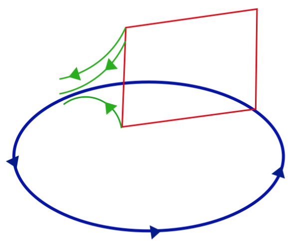

Figure 5.6: Periodic orbit with stable manifold

Example 5.6.2. *Geometrical example of Periodic orbit with two dimensional stable and unstable manifolds.*

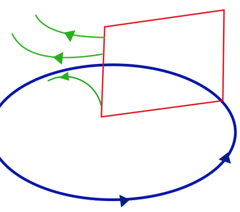

Figure 5.7: Periodic orbit with stable and unstable manifolds

Example 5.6.3. *Geometrical example of A periodic orbit with two dimensional stable and center manifolds.*

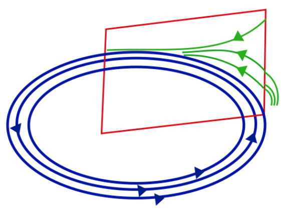

Figure 5.8: Periodic orbit with stable and center manifolds

5.7 The Poincaré- Index Thorem

Theorem 5.7.1. *[1] Consider a two dimensional surface σ which is relative to any C^1 vector field f on σ with at most a finite number of critical points and it is independent of the vector field f, then the index $Ind_f(\sigma)$ is equal to the Euler-Poincaré characteristic of σ; i.e.*

$$Ind_f(\sigma) = \chi_\sigma.$$

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