Stochastic Resetting

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Certificate of Examination

This is to certify that the dissertation titled **"Stochastic Resetting"** submitted by **Himanshu Yadav** (Reg. No. MS15119) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Rajeev Kapri	Dr. Neeraja Sahasrabudhe	Dr. Dipanjan Chakraborty (Supervisor)

Dated: June 14, 2020

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Dipanjan Chakraborty at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Dipanjan Chakraborty
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Notation and Abbreviations

p	$:= p(x,t\|x_0,t_0)$, Transition probability distribution
p_{st}	$:= p_{st}(x)$, Stationary distribution
τ	infinitesimally small time interval, $\tau/t << 1$
$\mathbb{L}^1(\mathbb{R})$	collection of absolutely integrable functions over $\mathbb{R}$
$\langle x\|T_t\|y\rangle$	transition probability/ propagator from y to x in time t
	$= p(x,t\|y,0)$
FPE	Fokker-Planck Equation
$\mathbb{1}_A$	Indicator Function of A
t_{lr}	Time at which the last reset occurred
$E[.]$, $\langle . \rangle$	Expectation Operator
$\eta(t)$	White noise
ξ	Infinitesimal displacement
NESS	Non equilibrium steady state

Abstract

We first develop the understanding of Langevin and Fokker Planck formalisms, path integral formalism and renewal theory. We then shift our focus to the main aim of the project that is to understand the stochastic resetting which can be defined as the abrupt restart of the stochastic process where the time of the next restart follows some waiting time distribution. We start with *Diffusion with Stochastic Resetting* as a first model of study and extend the results of the paper to the general case of initial position not being same as resetting position then to explore the possibility of finding new result we look at several models one after another namely Ornstein Uhlenbeck process with resetting, space dependent resetting, time dependent resetting.

At last we tried to look at: (a) Affect of multiple resetting sites on diffusion with Poissonian resetting and try to understand the changes in the dynamics with respect to the set of resetting points and (b) Random graph evolution with resetting as an example of search algorithm with resetting.

"It is scientific only to say what's more likely or less likely, and not to be proving all the time what's possible or impossible."

- Richard Feynman

"Pure mathematics shall be convinced by definite equations for the given indefinite number system."

- P.S. Jagadeesh Kumar

Contents

Chapter 1

Introduction

Consider an irreducible Markov chain / random walk which can only be null recurrent (if recurrent) or transient. What if we make one of the states positive recurrent? All the states of chain become positive recurrent and thus conceptually new results arise. Stochastic resetting can be thought of as the simplest mechanism to make the states of Markov chain positive recurrent and aperiodic.

In Stochastic resetting, particle evolves accordingly with time but is forced to a particular state time and again abruptly with some probability. The process which has no steady state can be made to have steady state by stochastic resetting, diffusion with stochastic resetting is one such example.

It has attracted a lot of attention recently because it is an analytically approachable NESS problem and many closed form solutions had been found. But, this problem can't be dealt in generality. And, one at a time approach can't exhaust the whole combinatorial space of process-reset pairs. A lot of theory is unexplored yet.

We aim to find a new analytically solvable resetting rate and stochastic process pair.

Sensing of extracellular ligands by single cells, transcription of genetic information by RNA, predator visually searching for prey, effect of catastrophes on population dynamics are only handful of examples where resetting plays an important role. For other numerous applications one can look up [EM11],[RG11],[EM15]. Also, It can be used in the field of algorithm optimization, some examples are permutation routing on the hypercube [MU05], application of random restart to genetic algorithms [GA96].

However, we will mainly be dealing with the affect of random restart on overdamped brownian particle.

It is quite a coincidence that the unexpected incidence of spread of COVID-19 can be seen as an epoch leading to the resetting of many parameters of life.

1.1 Model

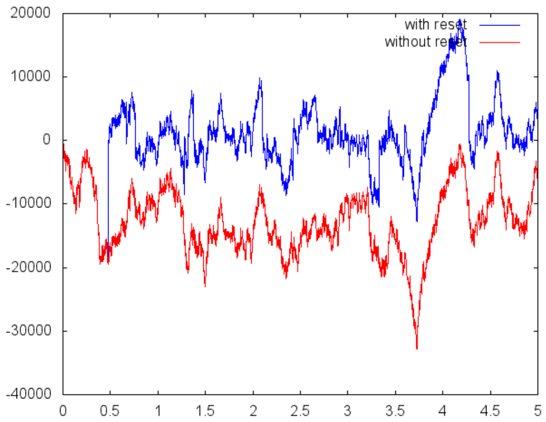


Figure 1.1: Trajectory of a free particle in 1-D with time:(a) red trajectory is the usual trajectory of the particle. (b) blue trajectory is of the particle resetting to origin after exponential waiting time.

For a random variable x(t) whose time time evolution is governed by the differential equation 2.2:

$$\frac{\partial x(t)}{\partial t} = f(x(t)) + \eta(t)$$

In the presence of random resetting its time evolution will be as follows:

$$x(t+\tau) = \begin{cases} x_r & \text{with probability} \quad r(x(t), t - t_{lr})\tau \\ x(t) + f(x(t))\tau + \eta(t)\tau & \text{with probability} \quad 1 - r(x(t), t - t_{lr})\tau \end{cases} \tag{1.1}$$

where, $r(x(t), t - t_{lr})$ is a random resetting rate which is a function of state and time since last reset.

1.2 Philosophical Note

We frequently face a dilemma when we are upto something. The three options that we have in such a situation are:
1. Should we keep proceeding further in the direction we are going in the hope of achieving our goal, even if everything seems against us?
2. Should we give up on what we are doing, calm down and retry to tackle the problem with a fresh approach?
3. As a pessimist give up totally as it seems impossible to achieve the goal.

Let's not consider the third option of being the hopeless pessimist.

Then, we have a trade off between former two choices. First choice of being totally optimistic and second choice of being an optimist evaluating his capabilities time and again before proceeding further, such an individual is more successful in life than former but, if the same individual has low confidence then he will re evaluate himself so frequently which in general leads failure more probably.

What is the point of discussing this? We just want to point out that even on philosophical level resetting plays an important role. Given a set of identical optimistic individuals the successful will be the one who re evaluates his strategy time and again but not too frequently.

Even, reading and understanding the research paper follows the same methodology of going back and forth while reading it.

Though, it is to be noted that we are not going to deal with the problems of such kind. We will be dealing with simple stochastic processes.

Chapter 2

Theory

2.1 Markov Process

A Markov process is a random process whose future is determined only by its current state and not by its past. But,

"There is really no such thing as Markov Process; Rather, there maybe systems whose memory time is too small that, on time scale on which we carry out observations it is fair to regard them as being well approximated Markov Process." [Gra83]

Definition 1. *(Markov Process). A stochastic process x(t) is called Markov if for every* n *and* $t_1 < t_2... < t_n$*, we have* $P(x(t_n) \leq x_n | x(t_{n-1}), ..., x(t_0)) = P(x(t_n) \leq x_n | x(t_{n-1}))$.

For more details and theoretical results on Markov chains see [Nor98].

Chapman Kolmorov Equation

One important property of Markov processes that we will need as well is that Markov processes obey Chapman Kolmogorov Equation:

$$P(x, t|y, s) = \int_{-\infty}^{\infty} p(x, t|z, u)p(z, u|y, s)dz \tag{2.1}$$

where, $t > u > s$.

2.2 Langevin equation to Fokker Planck Equation

Physical phenomena are quite often non deterministic because of the large number of constituents or sources of intrinsic noise. Thus, the resultant of the numerous sources of noise acting on the system simultaneously can be well approximated by a Gaussian (as a consequence of Central Limit Theorem) delta correlated ($\langle \eta(t)\eta(t')\rangle = \delta(t-t')$) stationary noise in the time scale much greater than that of the collision time (memory time).

Such a phenomenon can be modeled by Langevin description or by the path probabilities of its trajectory.

Supppose, we have a langevin equation of the form:

$$\frac{\partial x}{\partial t} = \mu f(x(t), t) + g(x(t), t)\eta(t) \tag{2.2}$$

We will be dealing cases where f and g are functions of x (i.e. space) only. However, g will be constant most of the times (namely diffusion constant $\sqrt{2D}$ unless stated), in that case we call the noise to be additive since the strength of the noise added is independent of the random variable and if g is a function of x then we call it multiplicative noise .
η is a delta correlated white noise, function of time only and μ is some constant.

Remark 1. *Langevin equation doesn't guarantee the Gaussianity and Stationarity of the process X inspite of the fact that Gaussian stationary Markov noise is a part of the equation. It just guarantees the Markovian nature of x.*

While doing tedious calculations of integrals of functions of several variables one often treats such cases as calculations of (iterated) integrals of single variable functions as a consequence of **Fubini Theorem**.
Theorem implies the commonly used equality of the type :

$$E\int_a^b X(t)dt = \int_a^b EX(t)dt \tag{2.3}$$

where, E denotes expectation with respect to the underlying probability distribution.

There are several ways to reach to Fokker Planck Equation from Langevin Equation. But, let's just go through the simple way[Ris84],[Gra83]:

We have,

$$p(x,t+\tau|x_0,t_0) = \int_{-\infty}^{\infty} \langle x|T_\tau|x'\rangle p(x',t|x_0,t_0)dx' \tag{2.4}$$

Here,$p(x',t|x_0,t_0)$ denotes the probability of being at x' such that system was in x_0 state at initial time t_0. $\langle x|T_\tau|x'\rangle$ denotes transition probability from x' to x in time τ. Above expression is just a Chapman- Kolmogorov Equation.

By change of variable this can be rewritten as:

$$p(x,t+\tau|x_0,t_0) = \int_{-\infty}^{\infty} \langle x|T_\tau|x-\xi\rangle p(x-\xi,t|x_0,t_0)d\xi \tag{2.5}$$

$\langle x|T_\tau|x-\xi\rangle = \phi(\xi)$ that is simply the probability of picking noise of particular strength so as to reach x. After expanding the above expression around (x,t) assuming τ to be small we get,

$$p(x,t) + \frac{\partial p(x,t)}{\partial t}\tau = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{\partial}{\partial x}\right)^n \langle x+\xi|T_\tau|x\rangle \xi^n p(x,t)d\xi \tag{2.6}$$

$$p(x,t) + \frac{\partial p(x,t)}{\partial t}\tau = \sum_{n=0}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n \frac{M_n(x',t,\tau)}{n!} p(x,t)d\xi \tag{2.7}$$

where,

$$\begin{aligned} M_n(x,t,\tau) &= \langle (x(t+\tau)-x(t))^n\rangle|_{x(t)=x-\xi} \\ &= \int \xi^n \langle x+\xi|T_\tau|x\rangle d\xi \end{aligned} \tag{2.8}$$

assume,

$$M_n(x,t,\tau)/n! = D^{(n)}(x,t)\tau + (\tau^2).$$

Note that coefficient of τ^o vanishes as for $\tau = 0$ the transition probability has initial value as $p(x,t|x',t) = \delta(x-x')$. Thus we have

$$\begin{aligned} \frac{\partial p(x,t)}{\partial t} &= \sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial x}\right)^n D^{(n)}(x',t,\tau)p(x,t) \\ &= L_{KM}(x,t)p(x,t) \end{aligned} \tag{2.9}$$

This is termed as **Kramer- Moyal's Expansion**.

Now to find $\langle \xi^n \rangle$ put $\xi = dx$ in 2.2 and using the moments relation for any normally distributed random variable Y (dW(t)) given by

$$\langle Y^p \rangle = \begin{cases} 0 & \text{when p is odd} \\ \sigma^p (p-1)!! & \text{when p is even} \end{cases} \tag{2.10}$$

We can check that

$$D^{(n)}(x,t) = O(\tau^z) \to 0 \text{ as } z > 0, \forall n > 2 \tag{2.11}$$

And, we get FPE

$$\partial_t p = -\mu \partial_x f(x(t),t) p + D\partial_x^2 p \tag{2.12}$$

2.3 The Fokker-Planck Equation

The theory discussed in this section is from [Ris84].

2.3.1 Purpose

1. To deal with the averages of the macroscopic variables one needs probability distribution which can be found by FPE.
2. FPE can be applied to study the dynamics of the system far from equilibrium if an appropriate time dependent solution is used, example: Laser light.
3. It is a way that gives a deterministic equation for a non- deterministic phenomenon.

2.3.2 Forward FPE

Very frequently we want to study the dynamics of non deterministic system in future based on the present information which is done by Forward FPE. The word forward is often overlooked and by FPE one means Forward FPE implicitly.

Solution of 2.9 for time independent operator $L(x) = -\mu\partial_x f(x) + D\partial_x^2$ can be written as:

$$p(x,t|x',t_0) = e^{L(x)(t-t')}\delta(x-x') \tag{2.13}$$

2.3.3 Backward FPE

Till now we have been dealing with the forward time evolution of the random variable. But, if we want to predict about the past then we use Backward FPE.

Or, the way we derived equation of motion of $p(x,t|x',t_0)$ with respect to x and t where t is a later time i.e. $t > t_0$. We can derive equation of motion of $p(x,t|x',t_0)$ with respect to x' and t_0 implying to the case where we are interested in the stochastic variable $x(t_0)$ at earlier times (t is fixed now unlike t_0 in the former case).

Unlike Forward FPE, reasonable initial condition will be: $\delta(x-y) = p(x,t|y,t)$.

Since,

$$p(x,t|x',t') = \int p(x,t|x'',t'+\tau)p(x'',t'+\tau|x',t')dx'' \tag{2.14}$$

which is just a Chapman Kolmogorov Equation.
And,

$$p(x'',t'+\tau|x',t') = \int \delta(y-x'')p(y,t'+\tau|x',t')dy$$

.
Also, Taylor Expansion of δ function is:

$$\delta(y-x'') = \delta(x'-x''+y-x') = \sum_{n=0}^{\infty} \frac{(y-x')^n}{n!} \left(\frac{\partial}{\partial x'}\right)^n \delta(x-x'')$$

From these we get

$$\begin{aligned} p(x'',t'+\tau|x',t') &= \sum_{n=0}^{\infty} \frac{1}{n!} \int (y-x')^n p(y,t'+\tau|x',t')dy \left(\frac{\partial}{\partial x'}\right)^n \delta(x'-x'') \\ &= \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!} M_n(x',t',\tau) \left(\frac{\partial}{\partial x'}\right)^n\right] \delta(x'-x'') \end{aligned} \tag{2.15}$$

Putting 2.15 in 2.14 gives

$$\begin{aligned}
p(x,t|x',t') - p(x,t|x',t'+\tau) &= \sum_{n=0}^{\infty}\frac{1}{n!}\int (y-x')^n p(y,t'+\tau|x',t')dy \left(\frac{\partial}{\partial x'}\right)^n p(x,t|x',t'+\tau) \\
&= \sum_{n=1}^{\infty}\frac{1}{n!}M_n(x',t',\tau)\left(\frac{\partial}{\partial x'}\right)^n p(x,t|x',t'+\tau) \\
-\frac{\partial p}{\partial t'} &= \sum_{n=1}^{\infty} D^{(n)}(x',t')\left(\frac{\partial}{\partial x'}\right)^n p(x,t|x',t')
\end{aligned} \tag{2.16}$$

For time independent $L^{+}_{KM}(x') = \sum_{n=1}^{\infty} D^{(n)}(x',t')(\partial/\partial_x)^n$ we have solution represented as

$$p(x,t|x',t') = e^{-L^{+}_{KM}(x')(t-t')}\delta(x-x') \tag{2.17}$$

For a white noise acting on the system, equation 2.16 becomes

$$\partial_{t'}p = -\mu f(x)\partial_{x'}p - D\partial^2_{x'}p \tag{2.18}$$

This is Backward FPE which is useful in finding persistence probability and First passage time as described in the next subsection.

2.3.4 First Passage Times of Homogeneous Processes

Suppose, particle is at x at time t=0 contained in the interval (a,b) with absorbing barriers at a and b.
So, the probability that the particle is in (a,b) at time t is:

$$\int_a^b p(y,t|x,0)dy = Q(x,t)$$

Q(x,t) can be defined as the survival probability of the particle.
Let, T be the time when particle leaves (a,b). Then,

$$Prob(T \geq t) = \int_a^b p(y,t|x,0)dy = G(x,t) \tag{2.19}$$

which implies, $Q(x,t) = Prob(T \geq t)$.
By time homogenity of the system we can write $p(y,t|x,0) = p(y,0|x,-t)$. And the

backward FPE can be written as:

$$p_t(y,t|x,0) = A(x)p_x(y,t|x,0) + \frac{1}{2}B(x)p_{xx}(y,t|x,0)$$

Applying operator $\int_a^b dy$ on both side of the equation gives

$$Q_t(x,t) = A(x)Q_x(x,t) + \frac{1}{2}B(x)Q_{xx}(x,t) \tag{2.20}$$

The obvious initial condition is $p(y,0|x,0) = \delta(x-y)$. Hence

$$Q(x,0) = \mathbb{1}_{x\in(a,b)} \tag{2.21}$$

And, since the particle immediately gets absorbed if x=a or b. These conditions lead to $Prob(T \geq t) = 0$ i.e.

$$Q(a,t) = Q(b,t) = 0 \tag{2.22}$$

From 2.19 it follows that

$$\langle f(T)\rangle = -\int_0^\infty f(t)dQ(x,t) \tag{2.23}$$

Therefore,

$$\begin{aligned}\langle T\rangle &= -\int_0^\infty tdQ(x,t) \\ &= -[tQ(x,t)]_0^\infty + \int_0^\infty Q(x,t)dt \\ &= \int_0^\infty Q(x,t)dt\end{aligned} \tag{2.24}$$

Remark 2. $\langle T\rangle = \int_0^\infty e^{-st}Q(x,t)dt|_{s=0} = \int_0^\infty e^{-st}Q(x,t)dt = q(x,s=0)$. *where, Laplace Transform of Q(x,t) is q(x,s).*

Similarly, one can get

$$\langle T\rangle = \int_0^\infty t^{n-1}Q(x,t)dt \tag{2.25}$$

From 2.20 we can derive the differential equation for $\langle T\rangle$ by by integrating that equation over $(0,\infty)$ and noting that $\int_0^\infty \partial_t Q(x,t)dt = G(x,\infty) - G(x,0) = -1$ we get

$$A(x)\langle T\rangle + \frac{1}{2}B(x)\partial_x^2\langle T\rangle = -1 \tag{2.26}$$

with boundary condition $\langle T(a)\rangle = \langle T(b)\rangle = 0$.

2.4 Feynman Kac Formalism

Here, we will be exploiting the fact that we know the probability distribution of white noise, $\eta(t)$ to find the probability distribution of x.

2.4.1 Transform of a Random Variable

It is important to have a look at this before we go further.

Suppose, we have a random variable X and g(X)=Y having probability densities W_x and W_y respectively. We know W_x then, what is W_y ?
We know that probability of the volume element will remain same after transformation. i.e.

$$W_x \prod_{i=1}^{N} dx_i = W_y \prod_{i=1}^{N} dy_i \tag{2.27}$$

where, $X = (x_1, x_2, ..., x_N)$ and $Y = (y_1, y_2, ..., y_N)$.
that implies,

$$W_y = |J| W_x \tag{2.28}$$

where, $J = \left(\frac{\partial x_i}{\partial y_j}\right); 1 \leq i, j \leq N$

2.4.2 Probability of x as a transform of η

The method employed here is from [RG11].

Since, we have

$$\frac{\partial x}{\partial t} = \mu f(x) + \eta(t)$$

To write x as a transform of η it is better to discretize the problem. In a discrete picture, we have $\langle \eta_i \rangle = 0$, $\langle \eta_i \eta_j \rangle = \sigma^2 \delta_{ij}$ and $N\Delta t = T$.
And,$D \equiv \lim_{\sigma\to\infty, \Delta t\to 0} \sigma^2 \Delta t/2$.
For any particular jth time step

$$P(\eta_j) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} exp\left(-\frac{1}{2\sigma^2}\eta_j^2\right) \tag{2.29}$$

$$x_n = x_{n-1} + (\mu\overline{f(x_n)} + \eta_n)\Delta t \tag{2.30}$$

where,$\overline{f(x_n)} = \frac{f(x_n)+f(x_{n-1})}{2}$ (using Stranovich's description)
Since,

$$\eta_n = \frac{x_n - x_{n-1}}{\Delta t} - \mu\overline{f(x_n)} \tag{2.31}$$

Therefore,

$$|J| = \left(\frac{1}{\Delta t}\right)^N \prod_{i=1}^{N}\left(1 - \frac{\Delta t \mu f'(x_i)}{2}\right) \approx \left(\frac{1}{\Delta t}\right)^N exp\left(-\sum_{i=1}^{N}\frac{\Delta t \mu f'(x_i)}{2}\right) \tag{2.32}$$

A particular path realisation will have a particular unique set of noise that is being added at each step. Therefore, the joint probability of noise realisation $\{\eta_i\}_{0\leq i\leq N}$:

$$P(\{\eta_i\}) = \left(\frac{1}{2\pi\sigma^2}\right)^{N/2} exp\left(-\frac{1}{2\sigma^2}\sum_{i=0}^{N}\eta_i^2\right) \tag{2.33}$$

Therefore,

$$P(\{x_i\}) = |J|\left(\frac{1}{2\pi\sigma^2}\right)^{N/2} \prod_{i=1}^{N} exp\left(-\frac{(x_i - x_{i-1} - \mu\bar{f}(x_i)\Delta t)^2}{2\sigma^2(\Delta t)^2}\right) \tag{2.34}$$

So,

$$P(x_N|x_0) = |J|\left(\frac{1}{2\pi\sigma^2}\right)^{N/2} \prod_{i=1}^{N-1}\int_{-\infty}^{\infty} dx_i exp\left(-\sum_{i=1}^{N}\frac{(x_i - x_{i-1} - \mu\bar{f}(x_i)\Delta t)^2}{2\sigma^2(\Delta t)^2}\right) \tag{2.35}$$

Last step is to take limit $\Delta t \to 0$, $N \to \infty$ to jump to the continuous regime.

2.4.3 Feynman Kac Formula

We present it here as described in [Var].

Let, P_x be a Gaussian probability measure on $C[0,\infty)$ such that process started from x (not necesssarily 0).

Suppose, V(x) is a bounded continuous function on R and u is defined as:

$$u(t,x) = E^{P_x}\left[exp\left[\int_0^t V(x(s))ds\right]f(x(t))\right] \tag{2.36}$$

then u obeys :

$$u_t = \frac{1}{2}u_{xx} + V(x)u \tag{2.37}$$

with initial condition $u(0,x) = f(x)$.

2.5 Fokker-Planck Equation vs Schrodinger's Equation

While looking at various models of stochastic resetting we came across [RG11] where Feynman Kac Path Integral Formalism has been adopted to derive the Schrodinger's Equation governing the velocity of $p(x,t)$ unlike rest of the examples [EM11] where Kramer Moyal's Expansion led to FPE or renewal theory has been used [PKE15].

FPE is widely used in Statistical Mechanics while Schrodinger's Equation holds its own renowned place in Physics, especially in Quantum Mechanics. We observed the link between these two apparently unlinked formalisms.

Transform of FPE to Schrodinger's Equation

Consider FPE

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left[V'p + D\frac{\partial p}{\partial x}\right] \tag{2.38}$$

Then, RHS becomes

$$Lp = [V'' + V'\partial_x + D\partial_x^2]p \tag{2.39}$$

Define,

$$\phi(x) = \frac{p(x)}{\sqrt{p_{st}(x)}} \tag{2.40}$$

where, $p_{st}(x) = Ce^{\left(-\frac{V(x)}{D}\right)}$ is the stationary solution and C is the normalisation constant. So,

$$\begin{aligned} p' &= \left(\phi' - \frac{\phi V'}{2D}\right)\sqrt{C}e^{\left(-\frac{V(x)}{D}\right)} \\ p'' &= \left(\phi'' - \frac{\phi' V'}{D} - \frac{\phi V''}{2D} + \frac{\phi V'^2}{4D^2}\right)\sqrt{C}e^{\left(-\frac{V(x)}{D}\right)} \end{aligned} \tag{2.41}$$

Then, equation comes

$$Lp = \left[\left(\frac{V''}{2} - \frac{V'^2}{4D}\right)\phi + D\phi''\right]\sqrt{p_{st}} \tag{2.42}$$

i.e.

$$Lp = \sqrt{p_{st}}H\phi \tag{2.43}$$

We get a new hermitian operator H given as

$$H = \left(\frac{V''}{2} - \frac{V'^2}{4D}\right) + D\partial_x^2 \tag{2.44}$$

Hence, by setting $p(x,t) = \phi(x,t)\sqrt{p_{st}(x)}$ we get

$$\frac{\partial \phi}{\partial t} = H\phi \tag{2.45}$$

2.6 Time dependent resetting and elements of renewal theory

Most of the cases like birth-death process, random walk etc. are dealt by deriving master equation considering constant transition probability rates leading to conditions like: $\omega\{n \to m\} = r\tau$ and so on.. assuming the process to be time homogeneous but it need not be the case. We can have a process where this jump probability rate is a function of time, $r(t)$. But, before going further to the derivation of master equation for such a case let's look at what this $r(t)$ actually mean.

2.6.1 Time dependent rate and waiting time distribution equivalence

Suppose the probability of no reset (epoch) in time τ is

$$P(T > \tau) = 1 - r(\tau)\tau \tag{2.46}$$

Then,$P(T > n\tau)$ is

$$P(T > n\tau) = \prod_{k=1}^{n}(1 - r(k\tau)\tau) = exp(-\sum_{k=1}^{n} r(k\tau)\tau) \tag{2.47}$$

In limit $n \to \infty$ such that $n\tau = t$ we get

$$P(T > t) = exp\left(-\int_0^t r(y)dy\right) \tag{2.48}$$

We can define $R(t) = exp\left(-\int_0^t r(y)dy\right)$ It is easy to check that r(t)=r(constant) corresponds to exponential waiting time.

Also,

$$\begin{aligned} P(t < T \leq T + \tau) &= (1 - e^{-R(t+\tau)}) - (1 - e^{-R(t)}) \\ &= e^{-R(t)} - e^{-R(t+\tau)} \\ &= e^{-R(t)} - (e^{-R(t)} - R'(t)e^{-R(t)}\tau + O(\tau^2)) \\ &= R'(t)e^{-R(t)}\tau = r(t)e^{-R(t)}\tau \end{aligned} \tag{2.49}$$

2.6.2 Elements of renewal theory

In case of time dependent rate r(t) one cannot write Master equation for $p(x,t|x',t')$ as we need to take care of last reset time as well.

For a process with $G(x,t|x_0)$ propagator **Renewal equations** can be written as:

last renewal equation

$$p(x,t|x_0,0) = e^{-R(t)}G(x,t|x_0,0) + \int_0^t dt_{lr}\phi(t_{lr})e^{-R(t-t_{lr})}G(x,t-t_{lr}|x_0,0) \tag{2.50}$$

where the first term is the contribution of reaching x without a single reset in time (0,t) with probability of no reset $e^{-R(t)}$ that is multiplied with $G(x,t|x_0)$ and second term considers those contributions where last reset event happened between t_{lr} and $t_{lr}+dt_{lr}$ whose probability density is $\phi(t_{lr}) = r(t_{lr})e^{-R(t_{lr})}$ which follows from 2.49 and propagator from x_r to x in the remaining time is $e^{-R(t-t_{lr})}G(x,t-t_{lr}|x_r)$.

First renewal equation

$$p(x,t|x_0,0) = e^{-R(t)}G(x,t|x_0,0) + \int_0^t dt' e^{-R(t')}p(x,t-t'|x_0,0) \tag{2.51}$$

Taking Laplace transform we get

$$\begin{aligned} P(x,s|x_0,t) &= Q(x,s|x_0,0) + P(x,s|x_0,0)(-sH(x,s)+1) \\ P(x,s|x_0,0) &= \frac{Q(x,s|x_0,0)}{sH(x,s)} \end{aligned} \tag{2.52}$$

where,

$$\begin{aligned} P(x,s|x_0,0) &= \int_0^\infty dt e^{-st}p(x,t|x_0,0) \\ Q(x,s|x_0,0) &= \int_0^\infty dt e^{-st}e^{-R(t)}G(x,t|x_0,0), \\ H(s) &= \int_0^\infty dt e^{-st}e^{-R(t)} \end{aligned} \tag{2.53}$$

Steady State

Remark 3. *steady state distribution as a function of time is constant therefore, it is necessary for the laplace transform of p in long time limit must be expressible in the form c/s where c is finite. Therefore, as* $t \to \infty$, $\lim_{s\to 0} sP(x,s|x_0,0) < \infty \implies P_{st}(x,s|x_0,t_0) = \lim_{s\to 0}\frac{Q(x,s)}{H(x,s)}$, *which in theory can be inverted to get* $p_{st}(x,t|x_0,0)$.

2.7 Master Equation for the general resetting rate

Till now, cases of space and time dependent resetting have been dealt separately. But, one can derive the master equation for the probability density of reaching (x,t) with no reset and then the theory of renewals can be used.
Here, $\langle x|t_\tau|x-\xi\rangle = (1-r(x-\xi,t)\tau)\phi(\xi)$.

By inserting it in 2.4 , we get

$$p_{\textsf{no res}}(x, t+\tau|x_0, t_0) = \int_{-\infty}^{\infty} (1 - r(x-\xi, t)\tau)\phi(\xi) p_{\textsf{no res}}(x-\xi, t|x_0, t_0) d\xi \qquad (2.54)$$

$$\frac{\partial p_{\textsf{no res}}(x, t|x_0, t_0)}{\partial t} = -\mu \frac{\partial f(x) p_{\textsf{no res}}}{\partial x} + D \frac{\partial^2 p_{\textsf{no res}}}{\partial x^2} - r(x, t) p_{\textsf{no res}} (2.55)$$

where, $t_0 = t_{lr}$.

After obtaining $p_{\textsf{no res}}$ one can use the theory of renewals as discussed in [RG11]. This method can be used for any resetting rate in theory which is not practical.

Chapter 3

Exploring Stochastic Resetting

3.1 Diffusion with stochastic resetting

We will try to understand the approach adopted in [EM11] and give some basic insights and pre requisites for that approach.
Then, we will extend some of the results presented in the paper.

3.1.1 Stationary Distribution of Diffusion

There is no stationary distribution for diffusion as:

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2}$$

As, $t \to \infty$

$$D\frac{\partial^2 p}{\partial x^2} = 0$$

$$p = ax + b$$

$p \notin L^1(R)$. Hence, no stationary distribution.

Remark 4. *Diffusion is the approximation of a Simple Symmetric Random Walk in continuous time and space. And, random walk is not positive recurrent, not aperiodic, though irreducible. Thus, stationary distribution doesn't exist.*

3.1.2 Stationary Distribution with resetting

We have:

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} - rp + r\delta(x - x_r) \tag{3.1}$$

As, $t \to \infty$

$$D\frac{\partial^2 p_{st}}{\partial x^2} - rp_{st} = -r\delta(x - x_r) \tag{3.2}$$

From which we get,

$$p_{st}(x, t|x_0) = \frac{1}{2}\sqrt{\frac{r}{D}}e^{(-\sqrt{\frac{r}{D}}|x-x_r|)}$$

Since, $\int_{-\infty}^{\infty} p_{st} = 1$ and $\lim_{r\to\infty} p_{st} = \begin{cases} \infty & x = x_r \\ 0 & else \end{cases}$.

Remark 5. $\lim_{r\to\infty} p_{st} = \delta(x - x_r)$. *Infinite resetting will not let the particle to escape the resetting state therefore, particle will be found at x_r with probability 1.*

Remark 6. $\lim_{r\to 0} p_{st} = 0$. *So, there is no stationary distribution as $r = 0$ corresponds to pure diffusion.*

Remark 7. *Diffusion is null recurrent itself but in the presence of resetting at exponential waiting time we have an average return time to the resetting point $\langle T_{x_r}(x_r)\rangle = \frac{1}{r} < \infty$ implying x_r is positive recurrent and in discrete picture it is aperiodic as well now. Since, state space is irreducible and periodicity and positive recurrence is a class property hence, unique stationary distribution exists.*

Verifying the Stationary Distribution

For the stationary distribution to exist moments $\langle x^n(t)\rangle$ must stabilise as $t \to \infty$. Therefore, To find Stationary Distribution numerically we need to decide upon the time which will be finite physically after which $\langle x^n(t)\rangle$ converges $\forall n$.
Let's calculate stationary first and second moment:

$$\begin{aligned} \lim_{t\to\infty}\langle x(t)\rangle &= \int_{-\infty}^{x_r} x\frac{\alpha}{2}e^{-\alpha(x_r-x)}dx + \int_{x_r}^{\infty} x\frac{\alpha}{2}e^{-\alpha(x-x_r)}dx \\ &= \left(\frac{x_r}{2} - \frac{1}{\alpha}\right) + \left(\frac{x_r}{2} + \frac{1}{\alpha}\right) = x_r \end{aligned} \tag{3.3}$$

$$\begin{aligned} \lim_{t\to\infty}\langle x^2(t)\rangle &= \int_{-\infty}^{x_r} x^2\frac{\alpha}{2}e^{-\alpha(x_r-x)}dx + \int_{x_r}^{\infty} x^2\frac{\alpha}{2}e^{-\alpha(x-x_r)}dx \\ &= \left(\frac{x_r^2}{2} - \frac{x_r}{\alpha} + \frac{1}{\alpha^2}\right) + \left(\frac{x_r^2}{2} + \frac{x_r}{\alpha} + \frac{1}{\alpha^2}\right) = x_r^2 + \frac{2}{\alpha^2} \end{aligned} \tag{3.4}$$

where, $\alpha = \sqrt{\frac{r}{D}}$.

Remark 8. *First moment converges to the resetting point beginning from the starting point, while in the case of diffusion it is the starting point at any time. It can be* *physically deduced in a sense that particle eventually gets reset to x_r so, from that time onward already passed time can be ignored in long time limit and subsequent diffusions between resets will be centered at x_r.*

Let's have a look at the figures below:
As we can see even before 3 units of the time passes both the moments have converged.

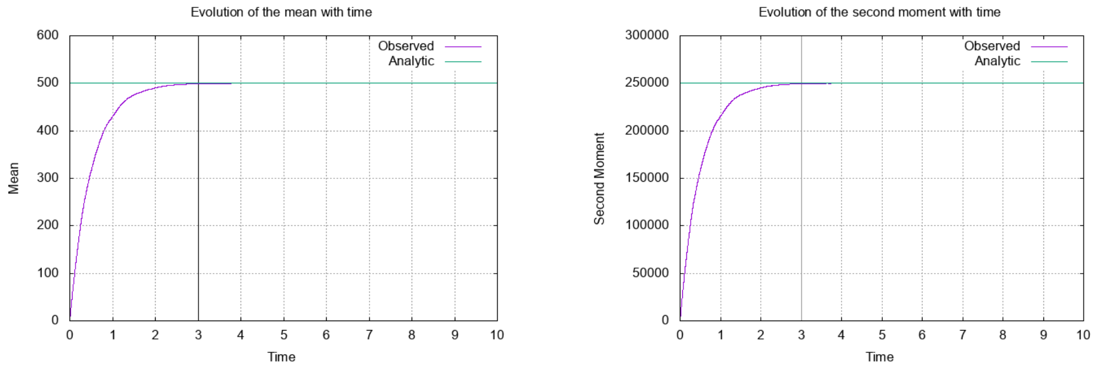


Figure 3.1: Evolution of: (a)first and (b)second moment with time for r=2, D=10, $x_r = 500$, $x_0 = 0$

So, we can sample $x(t)$ for $t > 3$ (for the set of used parameters) in this case to plot stationary distribution.
The stationary distribution for the system is plotted below:

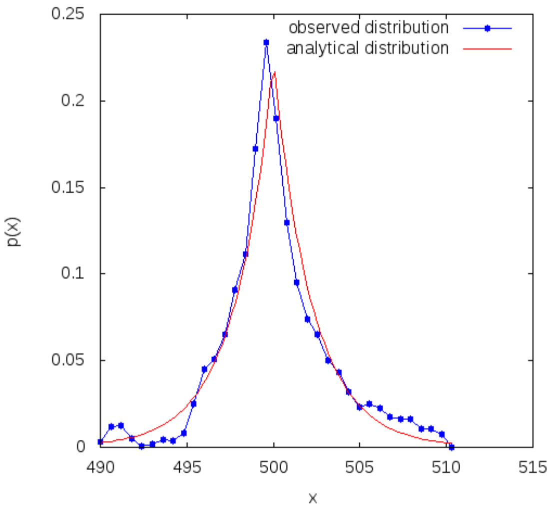


Figure 3.2: Stationary distribution of diffusion with Poissonian resetting for r=2, D=10, $x_r = 500$

3.1.3 Search starting from any position

The result has already been established for the case when the search starts from the reset position itself i.e. when $x_r = x_0$. But, we have extended the result given in [EM11] to the case when $x_r \neq x_0$.

Let $Q(x,t)$ be the persistence or survival probability of the target upto time t with the initial position being x and target be the origin. Then, equation followed by $Q(x,t)$ is

$$\frac{\partial Q}{\partial t} = D\frac{\partial^2 Q}{\partial x^2} - rQ + rQ(x_r,t) \tag{3.5}$$

Q:=Q(x,t)
Boundary condition: $Q(0,t) = 0$.
Initial condition: $Q(x,0) = 1$

Here, **0 is the absorbing target.** After taking Laplace transform of $Q(x,t)$ i.e. $\int_0^\infty e^{-st}Q(x,t)dt = q(x,s)$ we have

$$\begin{aligned} sq - 1 = D\partial_x^2 q - rq + rq(x_r,s) \\ D\partial_x^2 q - (r+s)q = -1 - rq(x_r,s) \end{aligned} \tag{3.6}$$

whose general solution is $q(x,s) = Ae^{\alpha_s x} + Be^{-\alpha_s x} + [1 + rq(x_0,s)]/(r+s)$ with $\alpha_s = \sqrt{(r+s)/D}$. Here, A=0 as q must be finite for large x.
From q(0,s)=0 we get $B = -[1 + rq(x_r,s)]/(r+s)$ and $q(x_r,s)$ We get the Laplace transform q(x,s) of Q(x,t) as:

$$q(x,s) = \frac{(1-e^{-\alpha_s x})(1+rq(x_r,s))}{r+s} \tag{3.7}$$

Now, put $x = x_0$ to find $q(x_0,s) = \frac{1-e^{-ax_r}}{s+re^{-ax_r}}$ and plug $q(x_0,s)$ back in the equation to get

$$q(x,s) = \frac{1-e^{-\alpha_s x}}{r+s}\left(r\frac{1-e^{-\alpha_s x_r}}{s+re^{-\alpha_s x_r}} + 1\right)$$

From this mean first passage time to hit origin starting from x is obtained: $(T_{x_r}(x) = q(x, s=0))$

$$T_{x_r}(x) = \frac{e^{a_0 x_r}(1-e^{-a_0 x})}{r} = T(x_r) + \frac{1-e^{a_0(x_r - x)}}{r} \tag{3.8}$$

where $T(x_r) = \frac{e^{a_0 x_r} - 1}{r}$ is the mean first passage time starting from x_r.

Optimum resetting rate for search time

Now, We aim to find the optimum resetting rate (r_x^*) when we start the search from a given $x > 0$. For that, $\partial T/\partial r = 0$ gives a relation as:

$$\frac{x_r}{x} = \frac{2(e^z - z/2 - 1)}{z} \tag{3.9}$$

where,$z = \sqrt{r/D}x$. From the above relation r_x^* then can be calculated as:
$r_x^* = z^2 D/x^2$. It is clear from the above relation that z hence r_x^* will depend on the ratio of resetting position and the starting position of search ie x_0/x.

And, $r_x^* = z^2 D/x^2$

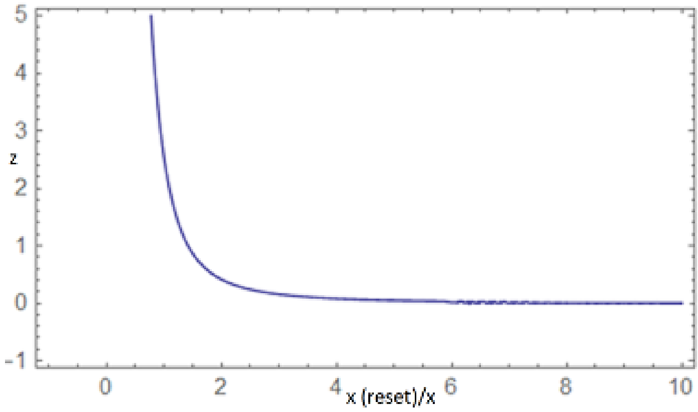


Figure 3.3: The variation of $\sqrt{r_x^*/D}x$ with x_r/x

3.2 Resetting to a set of states

In [EM11] *Diffusion with Poissonian resetting* they have considered only a single resetting site.

Proceeding from that and following [MRES20] we will treat here diffusion with Poissonian resetting to the distribution of resetting sites.

Let $q(x_r)$ be the distribution of resetting sites. Then the probability of reset site being in x_r and $x_r + dx_r$ is $q(x_r)dx_r$. Then, the renewal equation for the process can be written as:

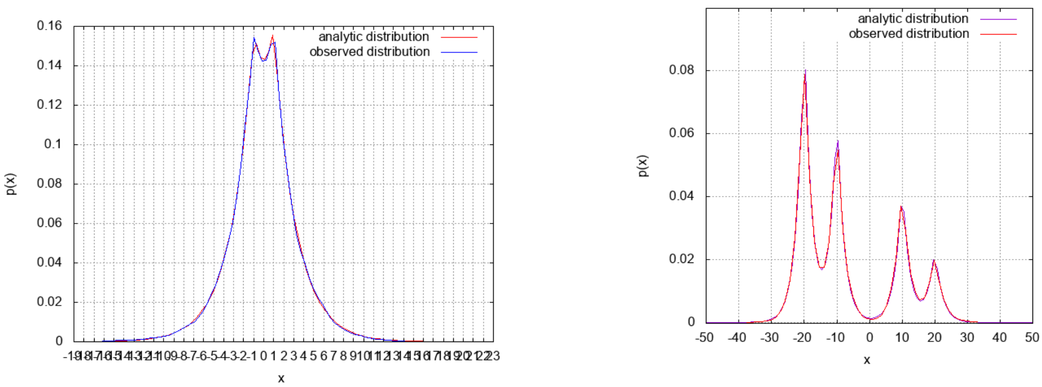
$$p(x,t|x_0) = e^{-rt}G(x,t|x_0) + r\int_0^t dt' e^{-rt'} \int dx_r q(x_r) G(x,t'|x_r). \tag{3.10}$$

As $t \to \infty$

$$p_{st}(x) = \int dx_r q(x_r) p_{st}(x|x_r), \tag{3.11}$$

where $p_{st}(x|x_r)$ is the stationary distribution with reset to fixed point x_r.
For finite N number of reset sites $x_{r_i}, i = 1, 2, ..., N$, each of the site being chosen at reset event with probability P_i then, we have

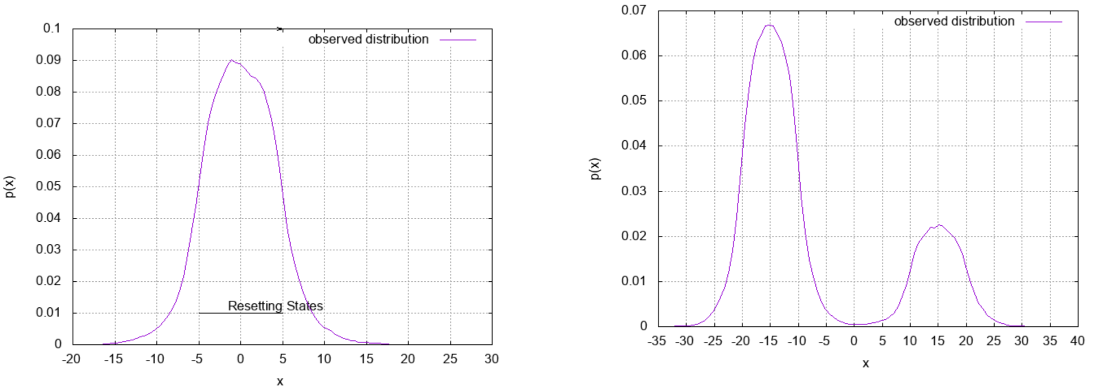
$$p_{st}(x) = \sum_{i=1}^{N} P_i p_{st}(x|x_{r_i}). \tag{3.12}$$



(a) $x_r \in \{1, -1\}$ with uniform site probability

(b) $x_r \in \{-20, -10, 10, 20\}$ with $P_{-20} = 0.4$, $P_{-10} = 0.3$, $P_{10} = 0.2$ and $P_{20} = 0.1$

Figure 3.4: Stationary distributions in the case of multiple isolated reset points for r=2, D=10



(a) $q(x_r) = \begin{cases} 1/10 & x_r \in (-5,5) \\ 0 & else \end{cases}$ (b) $q(x_r) = \begin{cases} 3/40 & x_r \in (-20,-10) \\ 1/40 & x_r \in (10,20) \\ 0 & else \end{cases}$

Figure 3.5: Stationary distributions in the case of reset site distribution for r=2, D=10

3.3 Stochastic Resetting of a diffusive particle in harmonic potential

We have a Langevin picture for a diffusive particle in a harmonic potential, $2\kappa x^2$ as

$$dx(t) = -\kappa x(t)dt + dW(t)$$

Then, in the presence of constant (Poissonian resetting) 3.1 becomes

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} + \kappa\frac{\partial xp}{\partial x} - rp + r\delta(x - x_r) \tag{3.13}$$

p_{st} obeys

$$D\frac{\partial^2 p_{st}}{\partial x^2} + \kappa\frac{\partial xp_{st}}{\partial x} - rp_{st} = -r\delta(x - x_r) \tag{3.14}$$

After taking Fourier transform of $p_{st}(x)$ as $\int_{-\infty}^{\infty} e^{-ikx} p_{st}(x) = f(k)$. we get

$$\begin{gathered} -k^2 Df(k) - \kappa k\frac{\partial f(k)}{\partial k} - rf(k) = -re^{-ikx_r} \\ \frac{\partial f(k)}{\partial k} + \left(\frac{r}{\kappa k} + \frac{k^2 D}{\kappa k}\right) f(k) = re^{-ikx_r} \\ \text{Integrating Factor}=e^{\frac{-k^2 D}{2\kappa} - \frac{r}{\kappa}ln|k|}\text{.We get,} \\ f(k) = \frac{r}{\kappa}e^{\frac{k^2 D}{2\kappa} + \frac{r}{\kappa}ln|k|}\left(\int^k e^{-ik'x_r} e^{\frac{k'^2 D}{2\kappa} + \frac{r}{\kappa}ln|k'|} dk'/k' + C\right) \end{gathered} \tag{3.15}$$

Indefinite integral appearing in the expression of $f(k)$ make it cumbersome to invert $f(k)$.

3.4 Space dependent resetting

Suppose, we have a space dependent resetting rate then Using Feymann kac Formalism 2.4 we can find $p_{\text{no reset}}$.
For that multiply 2.34 by $\prod_{i=1}^{N}(1 - r(x_i)\Delta t) \approx \prod_{i=1}^{N} e^{-r(x_i)\Delta t}$.
By the procedure employed in [3] we get to know that master equation followed by $p_{\text{no reset}}(x)$ is:

$$\frac{p_{\text{no reset}}}{\partial t} = D\frac{\partial^2 p_{\text{no reset}}}{\partial x^2} - V(x)p_{\text{no reset}} \tag{3.16}$$

where,

$$V(x) = \frac{\mu^2 f(x)^2}{4D} + \frac{\mu f'(x)}{2} + r(x)[\text{RG11}] \tag{3.17}$$

$V(x)$ looks like a quantum potential and it is to be noted that resetting rate is acing like a potential here. Since, 3.16 is like a Schrodinger's Equation which therefore, resetting rate and potential in which particle is moving can be chosen suitably so that resultant $V(x)$ takes a form of the potential for which solution is known (i.e. αx^2 or 0). Therefore, not much of the pairs can be chosen for getting analytic results.

3.5 Evolution of Random Graph with Random Resetting

In this last example we will try to implement the resetting mechanism on a simple random graph evolution model where we will basically try to see that how resetting can optimize average search time for a desired graph. It is an attempt to demonstrate the effect of random resetting on search algorithms.

Suppose, we have n vertices so the maximum possible number of edges is $\binom{n}{2} = N$. Initially we just have vertices with not a single edge. We decide to throw an edge blindly with probability p and not to do so with probability $1 - p$ at each time step. When we throw an edge it can choose any of the N possible options to form an edge there uniformly at random and if, at later times the position choosen to form an edge is not vacant we will simply replace the edge as we will assume edges are identical. After sufficiently long time, we have a complete graph, K_n then we are stuck so in order to make the process more useful we will just remove all the edges from the system with probability 1, the time we get K_n and will start throwing edges again.

In the setup defined above we want to know what the average search time of a particular graph G^* with k edges on n vertices is.

Mathematical Model

Let, $\{G_t\}_{t\geq 0}$, $(G_t = (V, E_t))$ where V is the set of vertices which is same at all times and E_t is the set of edges at time t) be a Markov chain say RGM with transition matrix P. the entries of P are as follows:
1.when, $0 \leq |E_t| < n$

$$P(|E_{t+1}| = m+1 || E_t| = m) = p - \frac{m}{N}p; P(|E_{t+1}| = m||E_t| = m) = 1 - p + \frac{m}{N}p$$

that implies probability of particular edge e_i being added if it has not been added yet is:

$$P(G + e_i|G) = \frac{p - \frac{m}{N}p}{N - m} = \frac{p}{N}$$

2.$|E_t| = n$. Then,

$$P(|E_{t+1}| = 0||E_t| = n) = 1$$

where, $|E_t|$ is the number of edges at time t and $0 \leq p \leq 1$.
Note that the rest of the not specified transitions have probability 0.

Model with resetting

This Markov process with reset can be described by transition matrix P_r whose entries can be defined as

$$P_r(|E_{t+1}| = j||E_t| = l) = (1 - r)P(|E_{t+1}| = j||E_t| = l) + r\delta_{j0}, \forall j, l \tag{3.18}$$

where, $0 \leq r \leq 1$, and δ_{ab} is a kronecker delta function.

In such a setup, we want to search a graph G^*, the average search time starting from a graph with no edge, $T_0(G^*)$ will vary with r. Suppose we deploy searchers on identical systems parallelly with different r value. Which one will finish the task

of searching G^* first?($G^* = (\{0,1,2,3,4\}, \{(0,1),(1,2),(2,3),(1,3)\})$ and $p = 1/2$ chosen for our purpose)
Let's have a look at the figure below:

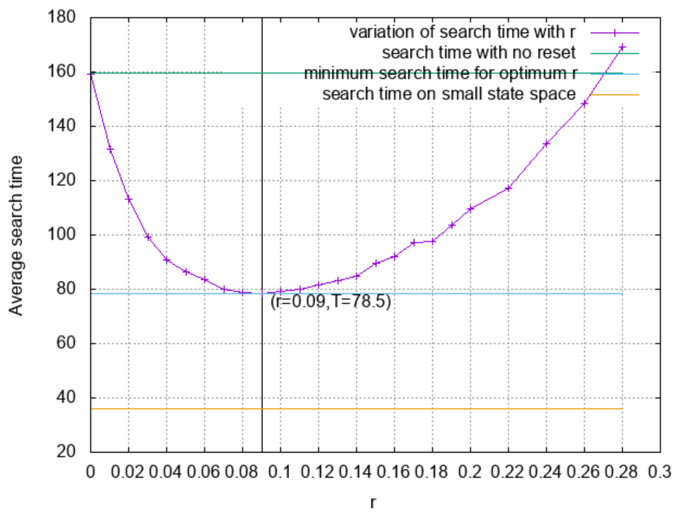


Figure 3.6: The variation of average search time of a graph G^* with resetting probability r

As we can see as r increases initially average search time decreases then again starts increasing with r. For our choice of 5 vertices and a G^* with 4 edges without resetting we get $T_0(G^*) \approx 159.5$. At optimum r value, r^* search time has the least value $T_0(G^*) \approx 78.5$ corresponding to $r^* = 0.09$.

Although if we decide to restart upon reaching any of the graph G with $|E| = |E^*|$ then the average search time decreases drastically to $T_0(G^*) \approx 38.5$.

This empirical result suggests that random resetting can optimize average search time for a wide range of stochastic processes.

Chapter 4

Discussion

1. When we have $X = f(Y)$ i.e. in the the form of algebraic transform it is easy to get the probability distribution of x, W_x from W_y as mentioned in 1.3.1. But, when it is not the case different methods have to be adopted. We have encountered a special case of Langevin equation,(2) which is a linear differential equation of X w.r.t. time, t involving the function of ξ. Here, x can't be written as a function of ξ in continuous time and space but, this can be done by discretizing the Langevin equation using Stranovich's scheme with euler method and then reverting back to continuous time and space . then, by the Feynman Kac formula 1.3.4, we get, the master equation.

So, for our purpose Schrodinger's Equation is nothing but a master equation.

2.Another, widely used and simpler method in statistical mechanics to derive the master equation is to use Chapman- Kolmogorov property leading to Kramer- Moyal's Expansion giving rise to Fokker- Planck Equation in the case of white noise. Instead of taking transform here we are taking into account the change in probability in small time.

3.A suitable gauge transform of a path integral can transform Schrodinger's Equation into Fokker Planck's Equation.

4. In order to deal with the case of $r(x, t)$ i.e. space and time dependent resetting one can write the master equation as shown in 2.7 and then can use the theory

of renewals [RG11].

5. As is observed $\delta(x-y)$ is the initial condition for the master equation which is a partial differential equation in 2 independent variables but, $\delta(x-y)$ is not a function but a generalized function therefore, to solve the partial differential equation we take Fourier transform with respect to state variable which converts the initial condition to e^{-iky} so that we have a nice function as an initial condition and we seek bounded non-negative solution. All these requirements suggest to solve the partial differential equation after taking transform.

6.Mathematically, this problem boils down to dealing with the parabolic differential equation of certain type, which limits the capability to obtain solutions because till date only certain special cases have been solved yet.

7. It must be realised that in all the problems we are dealing with, state space is irreducible i.e. one can reach from one state to other. Also, one can start search from any state not necessarily from resetting state in order to get finite search time. This is the motivation of 3.1.

Appendix A

Codes

A.1 Simulation of Diffusion with Poissonian Resetting to set of the set of Reset Sites

We will give a C code snippet here for the case of Diffusion with poissonian resetting to multiple sites:

```
int main()
{

printf("%f",c);
FILE *f2mean, *fmean, *fdist;
fmean=fopen("2cmean.dat","w");
f2mean=fopen("2cavg2.dat","w");
fdist=fopen("2cdistm.dat","w");
double x,g[10001]={0};
double mean[10001]={0};
for(int j=0;j<10000;j++)
{           x=0;
            for( int i=1; i<=1000000;i++)
                    {if(ran(seed2)<R)                \\resetting condition
            \\ probability of sites can be defined accordingly below:
                            { if(ran(seed2)<0.75)
                                    x=-20+10*ran(seed2);
                            else x=10+10*ran(seed2);}
```

```
        else    \\usual euler increment
                x=x+c*dt*gasdev(seed1);

        if((j<=10&&j>=1) && i*dt>100)
                fprintf(fdist,"%lf\n",x);
        else
                {}
        if(i%100==0)
        {mean[i/100]=mean[i/100]+x;
        g[i/100]=g[i/100]+x*x;}
        else {}
        }

}
for(int i=0; i<=10000;i++)
{
fprintf(f2mean,"%f      %lf\n",i*100.0*dt,g[i]/10000);
fprintf(fmean,"%f       %lf\n",i*100.0*dt,mean[i]/10000);
}

fclose(f2mean);
fclose(fdist);
fclose(fmean);
return 0;
}
```

Pseudo-Random Generators used:

1.ran(long idum) and ran2(long idum) are functions with same definition. And, returns Unif(0,1).
2.gasdev(long a) returns $N(0,1)$.
[PTVF92]

A.2 Simulation of Random Graph Evolution with Resetting

Python code snippet for the problem is below:

```
import networkx as nx
#import matplotlib.pyplot as plt
import random

n =5
k=4
p=0.5
E=10 #5C2



# graph evolution function
def res_rgg(G,p,r):
        if random.random()<r or G.number_of_edges()==E:
                G.clear()
                G.add_nodes_from([i for i in range(n)])
        else:
                i=random.randint(0,n-1)
                j=(i+random.randint(1,n-1))%n
                if random.random()<p and ((i,j) not in list(G.edges())
                ) and ((j,i) not in list(G.edges())):
                                G.add_edge(i,j)

                else:
                        G=G
        return G


rate=[0,0.01,0.02,0.03,0.04,0.05,0.06,0.07,0.08,0.09,0.1,0.11,
0.12,0.13,0.14,0.15,0.16,0.17,0.18,0.19,0.20,0.22,0.24,0.26,0.28]
f=open("timeforn.dat","w")
```

```
g1=nx.Graph()
g1.add_nodes_from([i for i in range(n)])
g1.add_edges_from([(0,1),(1,2),(2,3),(1,3)])
for j in range(len(rate)):
        r=rate[j]
        s=0
        for i in range(10000):
                G=nx.Graph()
                G.add_nodes_from([i for i in range(n)])
                while nx.is_isomorphic(G,g1)!= True:
                        s=s+1
                        G=res_rgg(G,p,r)
        f.write("%f     %f\n" %(r, s/10000))
f.close()
```

Appendix B

Some Basics of Markov Chain

Let, $(X_n)_{n\geq 0}$ be a Markov Chain with transition matrix P.

Definition 2. *(Communication). Two states i and j are said to communicate, written as* $i \leftrightarrow j$*, if* $p_{ij}^n > 0, p_{ji}^m > 0$ *for some m,n. In other words,*

$$i \leftrightarrow j \;\; \textit{means} \;\; i \rightarrow j \;\textit{and}\; j \rightarrow i.$$

Communication is an equivalence relation.
Therefore, It divides the state space of the chain into equivalence classes known as communicating classes.

Definition 3. *(Irreducible Markov Chain) A Markov chain is said to be irreducible if all states communicate with each other.*

Definition 4. *(Recurrence and Transience) For any state i, we define*

$$f_{ii} = P(X_n = i, \; \textit{for some}\; n \geq 1 | X_0 = i).$$

State i is recurrent if $f_{ii} = 1$*, and it is transient if* $f_{ii} < 1$*. Recurrence and Transience are class property.*

Definition 5. *(Period) The period of a state i is the largest integer d satisfying the following property:*
$p_{ii}^{(n)} = 0$*, whenever n is not divisible by d. The period of i is shown by d(i). If* $p_{ii}^{(n)} = 0$*,* $\forall n > 0$*, then we let* $d(i) = \infty$*.*

a.If $d(i) > 1$*, we say that state i is periodic.*

b.If $d(i) = 1$, we say that state i is aperiodic.

Period is also a class property.
A Markov Chain is said to be periodic (aperiodic) if all the states of a Markov Chain are periodic (aperiodic).

Theorem 1. *Consider an infinite Markov chain $(X_n)_{n\geq 0}$ where $Xn \in S = \{0, 1, 2, \}$. Assume that the chain is **irreducible** and **aperiodic**. Then, one of the following cases can occur:*
1.All states are transient, and

$$\lim_{n\to\infty} P(X_n = j|X_0 = i) = 0, \forall i, j.$$

2.All states are null recurrent, and

$$\lim_{n\to\infty} P(X_n = j|X_0 = i) = 0, \forall i, j$$

3.All states are positive recurrent. In this case, there exists a limiting distribution, $\pi = [\pi_0, \pi_1, ...]$, where

$$\pi_j = \lim_{n\to\infty} P(X_n = j|X_0 = i) > 0,$$

for all $i, j \in S$. The limiting distribution is the unique solution to the equations

$$\pi_j = \sum_{k=0}^{\infty} \pi_k P_{kj}, for\ j = 0, 1, 2, ..., \sum_{j=0}^{\infty} \pi_j = 1.$$

We also have $r_j = \frac{1}{\pi_j}, \forall j = 0, 1, 2, ...$ where r_j is the mean return time to state j.

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