Calculating Scattering Amplitudes Using Modern Techniques

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Certificate of Examination

This is to certify that the dissertation titled **"Calculating Scattering Amplitudes Using Modern Techniques"** submitted by **Gaurav Singh** (Reg. No. MS15177) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Anosh Joseph	Dr. Kinjalk Lochan	Dr. Ambresh Shivaji (Supervisor)

Dated: June 14, 2020

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Ambresh Shivaji at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Gaurav Singh
(Candidate)

Dated: June 14, 2020

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Ambresh Shivaji
(Supervisor)

Acknowledgements

“ Nature is garrulous to the point of confusion;
Let the artist be truly taciturn ”

– Paul Klee

My academic as well as personal journey in IISER has been like a roller-coaster ride. It is definitely an experience I can cherish for my entire life. Lots of people, from faculties to friends, from cleaning staff to mess and canteen workers; everyone has a contribution to my growth in these five years. A paragraph or two can’t really express my gratitude to each and every one of them but still I would like to acknowledge at least those without whom this journey would have been on a completely different track.

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"I feel a very unusual sensation — if it is not indigestion, I think it must be gratitude."

– Benjamin Disraeli

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Abstract

Understanding the nature of fundamental particles and their interactions is a major step towards unravelling the existing mysteries of the universe and to probe new physics. That's why scattering experiments are performed at colliders on such a large scale. What lies at the core of these collider experiments are the gauge-invariant perturbative scattering amplitudes which provide essential information about the cross-sections of these scattering processes. This thesis gives a glimpse about the traditional Feynman diagrammatic approach and its limitations, when it comes to calculate these scattering amplitudes. We review modern techniques like **Trace based Colour-Decompositions** and **Spinor-Helicity Formalism** and discuss, how they have an upper hand over traditional approach, specially while calculating multi-parton or multi-leg scattering amplitudes in the theory of QCD. We see, how symmetries and compact expressions like **Parke-Taylor MHV** amplitudes can make the computation of such complicated scattering amplitudes less intense and more efficient. We further discuss recursive techniques like **BCFW Recursion Relation** that entirely discard the need of Feynman diagrams and just uses the analytic properties of scattering amplitudes to write higher multiplicity amplitudes in terms of lower ones. We give a proof of Parke-Taylor formulae for multi-gluons and also for MHV amplitudes involving a quark-anti quark pair, using these BCFW recursion relations. This thesis just sticks to massless particles and talks about amplitudes only at tree-level. However, these techniques can be used to calculate higher order perturbative corrections that involves loop as well. We use BCFW recursion technique and MHV amplitudes to numerically calculate **Next to Leading Order (NLO)** i.e., one gluon real correction to diphoton production at hadron colliders. We see how this technique can make the calculations less cumbersome and give results with higher accuracy.

Chapter 1

QCD : A Brief Introduction

The aim of this chapter is to review some of the basic concepts from the theory of the Standard Model, in particular, our goal is to understand the fundamentals of perturbative Quantum Chromodynamics (QCD). The idea is just to use and further develop these concepts throughout the thesis, wherever necessary. We do not wish to achieve a broad and detailed understanding of these already well known topics. A rigorous and thorough analysis of these can be found in many sources like the lectures presented by *Benjamin Grinstein at University of California, San Diego* [Gri06], or in various known field theory textbooks, e.g. [PS93].

1.1 Perturbative QCD

The Standard Model is the theory which classifies all known elementary particles that constitute matter and also describes the three fundamental forces of nature, except the gravitational force. It is a gauge theory defined by the local gauge symmetry group $SU(3) \times SU(2) \times U(1)$, all three factors giving rise to the fundamental interactions :

- The gauge group $SU(2) \times U(1)$ describes the unified theory of electromagnetic and weak interactions between leptons and quarks - the Electroweak theory.
- Electroweak symmetry is spontaneously broken into $U(1)$ gauge group describing electromagnetic interactions by Higgs mechanism which generates the masses of charged leptons and quarks.
- The local gauge group $SU(3)$ describes the theory of strong interactions among coloured quarks and gluons - **Quantum Chromodynamics**.

We know that there are many parallels between QCD and QED. One can say that the mathematical structure of QCD is just an extension of QED, see the lectures given at *the University of Frieburg by Christian Schwinn* [Sch15]. The strong interactions between quarks in QCD is described by the exchange of massless spin-1 vector bosons called gluons, similar to the role played by photons in theory of electromagnetic interactions. Unlike photons though, gluons carry charge - colour quantum number, which is responsible for the self interactions among gluons. Both the theories can be considered as gauge theories because for both, Lagrangian is invariant under gauge transformations. However, unlike the QED which is described by the abelian gauge group $U(1)$, QCD is a non-abelian gauge theory with symmetry group $SU(3)$.

If we consider n types of non-interacting quarks, say with masses m_i, then the classical Lagrangian density [Sey10] is given by

$$\mathcal{L} = \sum_{i}^{n} \bar{q}_i^A (i\partial\!\!\!/ - m_i)_{AB} q_i^B, \tag{1.1}$$

where $(i\partial\!\!\!/ - m_i)_{AB}$ is proportional to identity matrix in colour space. The above Lagrangian has a global $SU(N_c)$ symmetry, $N_c = 3$ for QCD,

$$q_A \longrightarrow q'_A = exp(it.\theta)_{AB} q_B. \tag{1.2}$$

Now for $\mathcal{L}$ to have a local transformation invariance,

$$q_A(x) \longrightarrow q'_A(x) = exp(it.\theta(x))_{AB} q_B(x), \tag{1.3}$$

we define the covariant derivative in such a way :

$$D_{\mu,AB} = \partial_\mu \mathbf{1}_{AB} - ig_s(t.A_\mu)_{AB}, \tag{1.4}$$

where A_μ^a are coloured vector fields with the transformation property that allows

$$D'_{\mu,AB} q'_B(x) = \exp(it\cdot\theta(x))_{AB} D_{\mu,BC} q_C(x), \tag{1.5}$$

to give,

$$t\cdot A'_\mu = \exp(it\cdot\theta(x)) t\cdot A_\mu \exp(-it\cdot\theta(x)) + \frac{i}{g_s}\big(\partial_\mu \exp(it\cdot\theta(x))\big)\exp(-it\cdot\theta(x)). \tag{1.6}$$

Now, since we have introduced a new vector field A_μ, an additional kinetic energy term should be added to make the Lagrangian physical.

$$\mathcal{L}_{kinetic} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a, \tag{1.7}$$

where $F^a_{\mu\nu}$ is a non-abelian field strength tensor defined as,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu, \tag{1.8}$$

where g_s is the coupling constant of the strong interaction and is defined as,

$$\frac{g_s^2}{4\pi} = \alpha_s. \tag{1.9}$$

The last term in equation (1.8) makes sure that the equation (1.7) remains invariant under gauge transformations. Finally, we write the Lagrangian for the perturbative QCD,

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \sum_i^n \bar{q}_i^A \left(i\not{D} - m_i\right)_{AB} q_i^B - \frac{1}{2\eta}\left(\partial^\mu A^a_\mu\right)^2 + \mathcal{L}_{\text{ghost}}. \tag{1.10}$$

Here, a or b denotes colour indices from 1 to $N_c^2 - 1$ and A or B denotes colour indices ranging from 1 to N_c. The third term in above equation is the gauge fixing term, with η being a parameter for the choice of gauge.

The matrices t^a are basis set of $N_c^2 - 1$ matrices that generates the group $SU(N_c)$ and are related to the structure constants f_{abc} of the group by the relation,

$$[t^a, t^b] = if^{abc}t^c, \tag{1.11}$$

and satisfy the normalization condition,

$$Tr(t^a t^b) = \frac{1}{2}\delta^{ab}. \tag{1.12}$$

For quantization of a non-abelian gauge theory, one need to put the $\mathcal{L}_{ghost}$ term describing the contribution from unphysical fields. The ghost fields are complex scalar fields but follow Fermi statistics. These are necessary because they cancel off the unphysical degrees of freedom which arise when scattering amplitudes are computed. However, these are beyond the scope of our work and can be ignored because in physical gauges, their contribution always vanishes, [Sey10].

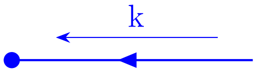
1.2 Feynman rules for QCD

We saw the form of Lagrangian, equation (1.10), for this non-abelian gauge theory. If we simplify it further, we obtain a form with clearly distinguishable interaction terms which can be used to construct the Feynman rules for propagators and vertices.

$$\begin{aligned}\mathcal{L} = \bar{q}(i\not{\partial} - m)q - \frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + g_s A^a_\mu \bar{q}\gamma^\mu t^a q - \\ - g_s f^{abc}\left(\partial_\mu A^a_\nu\right) A^{\mu b} A^{\nu c} - \frac{1}{4} g_s^2 \left(f^{eab} A^a_\mu A^b_\nu\right)\left(f^{ecd} A^{\mu c} A^{\nu d}\right)\end{aligned} \tag{1.13}$$

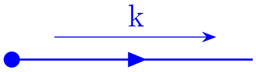
External Lines

⋄ Incoming quark



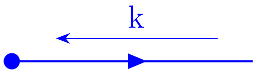
$u(k)$

⋄ Outgoing quark



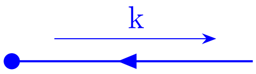
$\bar{u}(k)$

⋄ Incoming antiquark



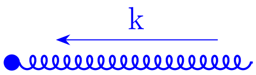
$\bar{v}(k)$

⋄ Outgoing antiquark



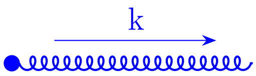
$v(k)$

⋄ Incoming gluon



$\epsilon_\mu(k)$

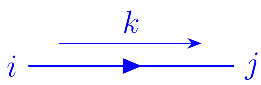
⋄ Outgoing gluon



$\epsilon^*_\mu(k)$

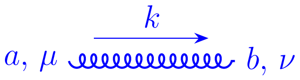
Propagators

⋄ Quark propagator



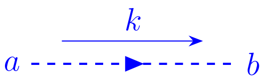
$$i\delta_{ij}\frac{(\not{k}+m)}{k^2-m^2+i\epsilon}$$

◇ Gluon propagator



$$\frac{-i\delta_{ab}}{k^2+i\epsilon}\left[g_{\mu\nu}-(1-\eta)\frac{k_\mu k_\nu}{k^2}\right]$$

◇ Scalar propagator

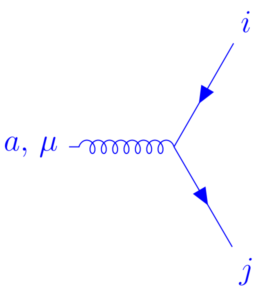


$$\frac{-i\delta_{ab}}{k^2+i\epsilon}$$

$$\eta \text{ fixes the gauge : } \eta = \begin{cases} 1, & \text{Feynman gauge} \\ 0, & \text{Landau gauge} \end{cases}$$

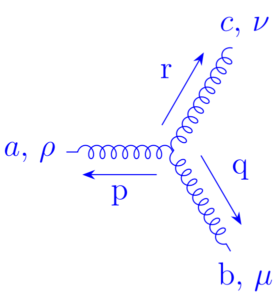
<u>Vertices</u>

◇ Gluon-Quark vertex



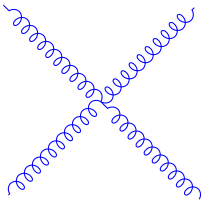
$$ig_s\gamma^\mu t^a_{ji}$$

◇ Three-gluon vertex



$$-g_s f^{abc}[\,(p-q)_\nu\, g_{\rho\mu}+(q-r)_\rho\, g_{\mu\nu}+(r-p)_\mu\, g_{\nu\rho}\,]$$

⋄ Four-gluon vertex



$$-ig_s^2 f^{abe} f^{cde} \left(g_{\rho\nu} g_{\mu\sigma} - g_{\rho\sigma} g_{\mu\nu}\right)$$
$$-ig_s^2 f^{ace} f^{bde} \left(g_{\rho\mu} g_{\nu\sigma} - g_{\rho\sigma} g_{\mu\nu}\right)$$
$$-ig_s^2 f^{ade} f^{cbe} \left(g_{\rho\nu} g_{\mu\sigma} - g_{\rho\mu} g_{\sigma\nu}\right)$$

We will be using these Feynman rules for calculating the scattering amplitudes for physical processes in Quantum Chromodynamics. Even though such rules come in quite handy, still the calculations remain very complex and cumbersome. As we will proceed, we will see what kind of complications one face and how they can be tackled in some efficient and productive ways.

Chapter 2

Modern Amplitude Methods

As stated before, to understand the basic structure of perturbative quantum field theory and to probe new physics, one need to understand scattering amplitudes which make the core of QFT. For a detailed understanding of the same, one can refer to many textbooks and reviews, like [EH14]. The following sections review some of the modern and traditional amplitude calculating methods which are well-known till date. These techniques have been available for several years and are very efficient, specially when it comes to tree-amplitudes. Loop calculations are way more involved and it becomes impossible to perform them analytically and sometimes, even numerical evaluation has its limit. However, we won't be dealing with any loop calculations in this report and will just stick to tree-level amplitudes i.e., we will be comparing traditional and modern techniques, at tree-level only.

2.1 Traditional approach

The well known and the most followed approach, at least in any quantum field theory course, is the traditional Feynman diagrammatic approach where we first derive the Feynman rules from the Lagrangian of a theory like QED, QCD, etc. and using these rules, scattering amplitude of a process is written as a sum of all possible Feynman diagrams. From this amplitude, one can calculate differential cross-section, $\frac{d\sigma}{d\Omega}$ which is related to the amplitude, $\mathcal{M}$ as

$$\frac{d\sigma}{d\Omega} \propto |\mathcal{M}|^2, \tag{2.1}$$

and the cross section, σ by integrating the differential cross-section over the phase space. These are important because they are the observables of interest in any collider physics experiment. However, the objective of this review is not to calculate cross-sections but scattering amplitudes and therefore, we will be sticking to that only except for some cases where we will calculate the differential cross-section as well. So, for future reference (in case needed), we will just itemize the main steps of calculating unpolarised cross-section using the traditional approach :

1. Extracting Feynman rules from the Lagrangian.

2. Writing amplitude as the summation of all possible Feynman diagrams. This means, all possible ways in which a particular scattering process can happen, are incorporated.

3. **Squaring the scattering amplitude** (at the beginning only).

4. Summing analytically over the spin of the external states and averaging over the spin of initial states.

5. Summing over colour quantum number, in case, quarks and/or gluons are involved.

6. Performing spin-sum using the completeness relations of the Dirac spinors,

$$\sum_{s=1,2} u_s(k)\bar{u}_s(k) = \not{k} + m, \tag{2.2}$$

$$\sum_{s=1,2} v_s(k)\bar{v}_s(k) = \not{k} - m. \tag{2.3}$$

7. Using trace technology i.e., identities derived from the Dirac algebra such as,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{2.4}$$

$$Tr[\mathbf{1}] = 4, \tag{2.5}$$

$$Tr[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}. \tag{2.6}$$

   Product of any odd number of the γ^μ matrices have zero traces,

$$Tr[\gamma^\mu] = 0, \qquad Tr[\gamma^\mu\gamma^\nu\gamma^\rho] = 0, \tag{2.7}$$

$$Tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}), \tag{2.8}$$

$$Tr[\gamma^5] = 0, \qquad Tr[\gamma^\mu\gamma^\nu\gamma^5] = 0, \tag{2.9}$$

$$Tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5] = -4i\epsilon^{\mu\nu\rho\sigma}, \tag{2.10}$$

$$Tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.....] = Tr[.....\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu]. \tag{2.11}$$

8. Performing Lorentz algebra, like going into centre of mass (CM) frame to obtain a much simpler expression for the amplitude - the spin-averaged matrix element, $\mathcal{M}$.

9. Converting the expression into a form involving terms with masses and Mandelstam variables.

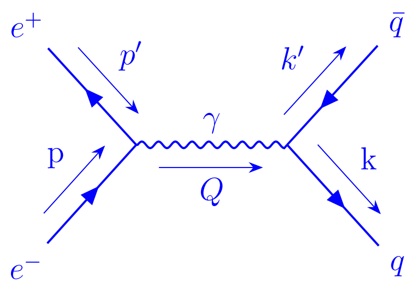


Figure 2.1: $e^-e^+ \longrightarrow q\bar{q}$

10. And finally, calculating the differential cross-section

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \sum_{colour} \frac{1}{64\pi^2 E_{CM}^2} \frac{|\mathbf{K}|}{|\mathbf{P}|} \overline{|\mathcal{M}|^2} \tag{2.12}$$

Let's look at an example to understand this approach more clearly.

2.1.1 An example from QED

$$e^-(p,s) + e^+(p',s') \longrightarrow q(k,r) + \bar{q}(k',r')$$

For this process, the only possible Feynman diagram (lowest order) is Figure 2.1.

Now, using the Feynman rules of QED, the matrix element, $\mathcal{M}$ can be computed as,

$$i\mathcal{M} = \bar{v}_{s'}(p')(-ie\gamma^\mu)u_s(p)(\frac{-ig_{\mu\nu}}{Q^2})\bar{u}_r(k)(-ie_q\gamma^\nu)v_{r'}(k') \tag{2.13}$$

$$= \bar{v}_{s'}(p')(-ie\gamma^\mu)u_s(p)\bar{u}_r(k)(\frac{-e_q}{Q^2})(g_{\mu\nu}\gamma^\nu)v_{r'}(k') \tag{2.14}$$

$$= \frac{iee_q}{Q^2}(\bar{v}_{s'}(p')\gamma^\mu u_s(p))(\bar{u}_r(k)\gamma_\mu v_{r'}(k')). \tag{2.15}$$

Taking complex conjugate of the matrix element,

$$-i\mathcal{M}^* = \frac{-iee_q}{Q^2}(\bar{v}_{s'}(p')\gamma^\mu u_s(p))^*(\bar{u}_r(k)\gamma_\mu v_{r'}(k'))^*. \tag{2.16}$$

Now consider,

$$(\bar{v}\gamma^\mu u)^* = (v^\dagger\gamma^\circ\gamma^\mu u)^*. \tag{2.17}$$

Say, we take an element,

$$[\bar{v}_{1i}\gamma^{\mu}_{ij}u_{j1}]^* = [v^{\dagger}_{1k}\gamma^{\circ}_{ki}\gamma^{\mu}_{ij}u_{j1}]^* \tag{2.18}$$

$$= v_{k1}\gamma^{\circ}_{ki}(\gamma^{\mu}_{ij})^* u^*_{j1} \tag{2.19}$$

$$= v_{k1}\gamma^{\circ}_{ki}(\gamma^{\mu})^{\dagger}_{ji}u^{\dagger}_{1j} \tag{2.20}$$

$$= u^{\dagger}_{ij}(\gamma^{\mu})^{\dagger}_{ji}\gamma^{\circ}_{ik}v_{k1} \tag{2.21}$$

$$= u^{\dagger}\gamma^{\circ}\gamma^{\mu}v \tag{2.22}$$

$$= \bar{u}\gamma^{\mu}v. \tag{2.23}$$

Thus,

$$(\bar{v}\gamma^{\mu}u)^* = \bar{u}\gamma^{\mu}v. \tag{2.24}$$

Using the above equation, one can write equation (2.16) as,

$$-i\mathcal{M}^* = \frac{-iee_q}{Q^2}(\bar{u}_s(p)\gamma^{\mu}v_{s'}(p'))(\bar{v}_{r'}(k')\gamma_{\mu}u_r(k)). \tag{2.25}$$

Multiplying equation (2.15) and (2.25), we obtain

$$|\mathcal{M}|^2 = \frac{e^2e_q^2}{Q^4}(\bar{v}_{s'}(p')\gamma^{\mu}u_s(p)\bar{u}_r(k)\gamma_{\mu}v_{r'}(k'))(\bar{u}_s(p)\gamma^{\nu}v_{s'}(p')\bar{v}_{r'}(k')\gamma_{\nu}u_r(k)) \tag{2.26}$$

Now, summing analytically over the spin of the external states and averaging over the spin of initial states,

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\sum_{s}\frac{1}{2}\sum_{s'}\sum_{r}\sum_{r'}|\mathcal{M}|^2 \tag{2.27}$$

$$= \frac{e^2e_q^2}{4Q^4}\sum_{s,s'}(\bar{v}_{s'}(p')\gamma^{\mu}u_s(p)\bar{u}_s(p)\gamma^{\nu}v_{s'}(p'))$$
$$\sum_{r,r'}(\bar{u}_r(k)\gamma_{\mu}v_{r'}(k')\bar{v}_{r'}(k')\gamma_{\nu}u_r(k)). \tag{2.28}$$

Consider,

$$\sum_{s,s'}(\bar{v}_{s'}(p')\gamma^{\mu}u_s(p)\bar{u}_s(p)\gamma^{\nu}v_{s'}(p'))_{11} = \sum_{s,s'}(\bar{v}_{s'}(p'))_{1i}(\gamma^{\mu})_{ij}(u_s(p))_{jk}(\bar{u}_s(p))_{kl}$$
$$(\gamma^{\nu})_{lm}(v_{s'}(p'))_{m1} \tag{2.29}$$

$$= \sum_{s,s'}(v_{s'}(p'))_{m1}(\bar{v}_{s'}(p'))_{1i}(\gamma^{\mu})_{ij}(u_s(p))_{jk}$$
$$(\bar{u}_s(p))_{kl}(\gamma^{\nu})_{lm} \tag{2.30}$$

$$= Tr\left[\sum_{s,s'}(v_{s'}(p')\bar{v}_{s'}(p')\gamma^{\mu}u_s(p)\bar{u}_s(p)\gamma^{\nu})\right]. \tag{2.31}$$

Then, using the completeness relations (2.2) and (2.3),

$$\sum_{s,s'}(\bar{v}_{s'}(p')\gamma^{\mu}u_s(p)\bar{u}_s(p)\gamma^{\nu}v_{s'}(p')) = Tr[(\not{p}' - m_e)\gamma^{\mu}(\not{p} + m_e)\gamma^{\nu}] \tag{2.32}$$

similarly,

$$\sum_{r,r'}(\bar{u}_r(p)\gamma_{\mu}v_{r'}(p')\bar{v}_{r'}(k')\gamma_{\nu}u_r(k)) = Tr[(\not{k} + m_q)\gamma_{\mu}(\not{k}' - m_q)\gamma_{\nu}]. \tag{2.33}$$

Thus, equation (2.28) can be rewritten as,

$$\overline{|\mathcal{M}|^2} = \frac{3e^2e_q^2}{4Q^4}\, Tr[(\not{p}' - m_e)\gamma^{\mu}(\not{p} + m_e)\gamma^{\nu}]\; Tr[(\not{k} + m_q)\gamma_{\mu}(\not{k}' - m_q)\gamma_{\nu}] \tag{2.34}$$

The factor 3 comes up because a quark has three colours. One can see that the above equation has terms involving traces and by using the trace identities wisely, one can simplify the above expression as follows:

Consider,

$$Tr[(\not{p}' - m_e)\gamma^{\mu}(\not{p} + m_e)\gamma^{\nu}] = Tr[\not{p}'\gamma^{\mu}\not{p}\gamma^{\nu} + \not{p}'\gamma^{\mu}m_e\gamma^{\nu} - m_e\gamma^{\mu}\not{p}\gamma^{\nu} - m_e\gamma^{\mu}m_e\gamma^{\nu}] \tag{2.35}$$

$$= Tr[\gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}\gamma^{\nu}p'_{\rho}p_{\sigma} + \gamma^{\rho}\gamma^{\mu}\gamma^{\nu}m_e p'_{\rho} - m_e\gamma^{\mu}\gamma^{\rho}\gamma^{\nu}p_{\rho} - m_e^2\gamma_{\mu}\gamma_{\nu}]. \tag{2.36}$$

The second and third trace terms would become zero because of equation (2.7) and then by using equation (2.6) and (2.8), we obtain

$$Tr[(\not{p}' - m_e)\gamma^{\mu}(\not{p} + m_e)\gamma^{\nu}] = 4(p'^{\mu}p^{\nu} + p'^{\nu}p^{\mu} - g^{\mu\nu}(p.p' + m_e^2)), \tag{2.37}$$

similarly,

$$Tr[(\not{k} + m_q)\gamma_{\mu}(\not{k}' - m_q)\gamma_{\nu}] = 4(k_{\mu}k'_{\nu} + k_{\nu}k'_{\mu} - g_{\mu\nu}(k.k' + m_q^2)). \tag{2.38}$$

Putting these in equation (2.34) and simplifying, one obtain the expression

$$\overline{|\mathcal{M}|^2} = \frac{24e^2e_q^2}{Q^4}[(p.k)(p'.k') + (p.k')(p'.k) + m_q^2(p.p') + m_e^2(k.k') + 2m_e^2m_q^2]. \tag{2.39}$$

Now, since $m_e << m_q$, we can set m_e as zero. Then we go into the centre of mass frame and introduce Mandelstam variables :

$$s = (p + p')^2 = (k + k')^2 = q^2 = 2p.p' = 2k.k' = (2E)^2 = E_{CM}^2, \tag{2.40}$$

$$t = (k - p)^2 = (k' - p')^2 = m_q^2 - 2p.k = m_q^2 - 2p'.k' = m_q^2 - 2(E^2 - E|\mathbf{K}|\, cos\theta), \tag{2.41}$$

$$u = (k' - p)^2 = (k - p')^2 = m_q^2 - 2p.k' = m_q^2 - 2p'.k = m_q^2 - 2(E^2 + E|\mathbf{K}|\, cos\theta) \tag{2.42}$$

where, $|\mathbf{K}| = \sqrt{E^2 - m_q^2}$. Using these, equation (2.39) can be written as

$$\overline{|\mathcal{M}|^2} = \frac{24e^2e_q^2}{s^2}(\frac{t^2}{4} + \frac{u^2}{4} + m_q^2\frac{s}{2}) \tag{2.43}$$

$$= 3e^2e_q^2(1 + \frac{4m_q^2}{s} + (1 - \frac{4m_q^2}{s})cos^2\theta) \tag{2.44}$$

And finally, the differential cross-section can be calculated using equation (2.12),

$$\begin{aligned} \frac{d\sigma}{d\Omega}\bigg|_{CM} &= \frac{1}{64\pi^2E_{CM}^2}\frac{|\mathbf{K}|}{|\mathbf{P}|}\overline{|\mathcal{M}|^2} \\ &= \frac{1}{64\pi^2 s}\sqrt{1 - \frac{4m_q^2}{s}}\,\overline{|\mathcal{M}|^2} \\ &= \frac{3e^2e_q^2}{64\pi^2 s}\sqrt{1 - \frac{4m_q^2}{s}}(1 + \frac{4m_q^2}{s} + (1 - \frac{4m_q^2}{s})cos^2\theta). \end{aligned} \tag{2.45}$$

So, we managed to obtain the result for a scattering process involving quarks, using the traditional approach. There were some laborious calculations involved but still the result wasn't that difficult to obtain. However, one should keep in mind that we considered one of the most simplistic process to begin with. In this case, there was only one possible Feynman diagram (lower order) and therefore, we didn't face much difficulty but there can be many possible arrangements for a particular process. This in indeed the case for many QCD processes involving gluons due to the non-abelian nature of the theory. Also, as the number of external particles increases, one can see a drastic growth in the number of Feynman diagrams. For example, if one consider scattering of gluons [KK89] at tree level,

$$\text{gg} \longrightarrow \text{gg} : \text{Number of diagrams} = 4$$
$$\text{gg} \longrightarrow \text{ggg} : \text{Number of diagrams} = 25$$
$$\text{gg} \longrightarrow \text{ggggg} : \text{Number of diagrams} = 2485$$
$$\vdots$$
$$\text{gg} \longrightarrow \text{gggggggg} : \text{Number of diagrams} = 10525900,$$

it can be noted that as the number of gluons increases, the evaluation becomes uncontrollable and soon it goes away, even from the reach of a computer. On top of that, because this approach squares the amplitude contribution from all diagrams in the very beginning, total number of terms can become too much to handle. That is, if there are N number of Feynman diagrams, then there will be N^2 number of terms in the squared matrix element $|\mathcal{M}|^2$. This can take a lot of time and effort as the calculations become very lengthy and tedious. That is not all though. If polarization vectors or colour quantum numbers are involved, calculations can become more cumbersome. The process that we have considered doesn't even scratch the surface of the problem. We'll see later on, with the help of more complex examples, that despite the final expression of the solution having a simple and compact form - *" the intermediate stages of calculation often explode in an inferno of indices, contracted up and down and in all directions - providing little insight of the physics and hardly any hint of simplicity "* [EH14]. But before going there, we'll introduce some of the modern methods of calculating scattering amplitudes, that may help us to tackle at least some of our problems.

2.2 Trace-based colour decompositions in QCD

In the previous section, we mentioned that presence of colour degrees of freedom increases the complexity of algebra involved in the calculations of multi-parton scattering amplitudes of a non-abelian gauge theory like QCD. This algebra seems to be very tedious because of the structure constants f^{abc} and generators t^a of the underlying symmetric gauge group $SU(3)$ that we encountered in the section 1.2. For the generalisation purpose, we will consider the gauge group $SU(N_c)$, remembering that we can always put $N_c = 3$ in the end. As mentioned earlier, "the generators of the group $SU(N_c)$ in the fundamental representation are traceless and Hermitian $N_c \times N_c$ matrices $(t^a)_i^{\bar{j}}$ where, a = 1,2,..., $N_c^2 - 1$ is the colour index of gluons in the adjoint representation and i = 1,2,..., N_c is the colour index of quarks in the fundamental representation".
Now, for computing the colour factors, we make a slight change in notation of generators by redefining the normalisation as,

$$t^a \equiv \frac{1}{\sqrt{2}} T^a. \tag{2.46}$$

Thus, (1.11) and (1.12) are modified as

$$[T^a, T^b] = i\sqrt{2} f^{abc} T^c, \tag{2.47}$$

$$Tr(T^a T^b) = \delta^{ab}. \tag{2.48}$$

Now if we consider any general QCD Feynman diagram at the tree level, it is evident from the Feynman rules being mentioned in section 1.2 that each quark-gluon-quark vertex contributes a factor of $(T^a)_i^{\bar{j}}$, each pure three-gluon vertex contributes a factor of the structure constant f^{abc}, and more involved and contracted pair of structure constants $f^{abc} f^{cde}$ is the vertex factor contribution for each four-gluon vertex. Furthermore, delta factors like δ_{ab} and $\delta_i^{\bar{j}}$ will be used by the parton (quarks and gluons) propagators for the contraction of many indices [Dix96].

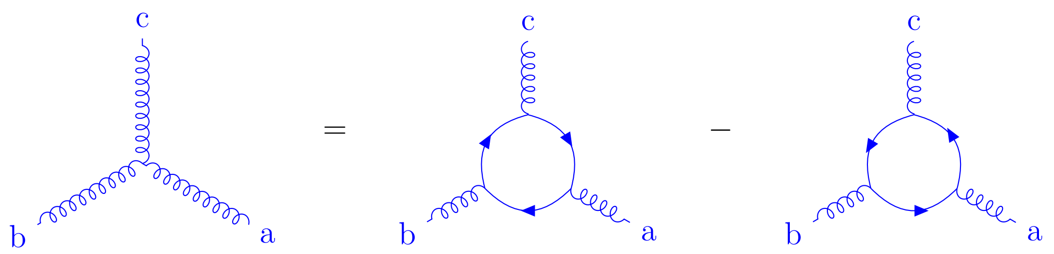


Figure 2.2: **Graphical representation of the identity for eliminating structure constants f^{abc}.**

The main idea is to understand how one can identify all the colour structures appearing in a scattering amplitude and then reorganise these colour degrees of freedom in such a way that one can easily disentangle these colour factors from the kinematics. One thing that can be done is to write all the group theory structure constants f^{abc} in terms of the Lie group generator matrices T^a. For doing the same, we proceed as follows :

We know from the relation (2.47) that

$$[T^a, T^b] = \tilde{f}^{abc} T^c, \tag{2.49}$$

where, we have defined $\tilde{f}^{abc} \equiv i\sqrt{2} f^{abc}$. Multiply both sides of the above equation by T^d,

$$[T^a, T^b]T^d = \tilde{f}^{abc} T^c T^d,$$

then,

$$Tr([T^a, T^b]T^d) = \tilde{f}^{abc} Tr(T^c T^d),$$

using the equation (2.48),

$$Tr([T^a, T^b]T^d) = \tilde{f}^{abc} \delta^{cd},$$

$\Longrightarrow$

$$Tr([T^a, T^b]T^d) = \tilde{f}^{abd},$$

or,

$$\tilde{f}^{abc} \equiv i\sqrt{2} f^{abc} = Tr(T^a T^b T^c) - Tr(T^a T^c T^b). \tag{2.50}$$

This can be represented diagrammatically [Dix13, Dix96], as shown in figure 2.2.

Replacing all the structure constants with T^a matrices using the relation (2.50) will result in a product of large number of traces of the form $\text{Tr}(..T^a T^b T^c..)\text{Tr}(..T^d T^a T^e..)..\text{Tr}(..)$, for multi-gluon amplitudes and also strings of the form $(T^{a_1}...T^{a_m})_i^{\bar{j}}$, if external quarks are involved. If one wishes to simplify these traces and strings of generators, the following Feirz identity comes in handy,

$$\sum_{a=1}^{N_c^2-1}(T^a)_{i_1}^{\bar{j}_1}(T^a)_{i_2}^{\bar{j}_2} = \delta_{i_1}^{\bar{j}_2}\delta_{i_2}^{\bar{j}_1} - \frac{1}{N_c}\delta_{i_1}^{\bar{j}_1}\delta_{i_2}^{\bar{j}_2}. \tag{2.51}$$

As clearly visible, this equation just states that the generators T^a form a complete set of traceless and Hermitian matrices. The graphical representation of the same is shown below :

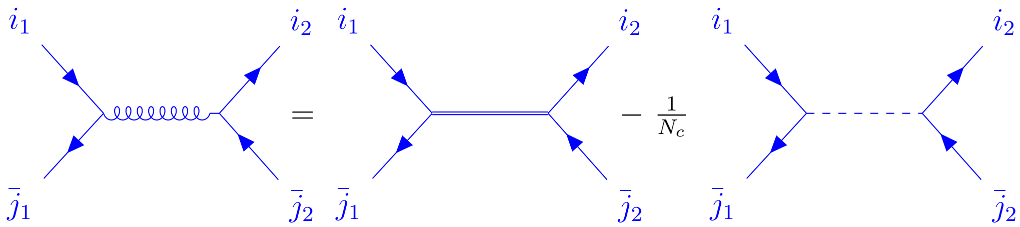


Figure 2.3: **Graphical representation of the Feirz identity.**

Now, let's see how we can simplify our calculations by playing with colour structures using the identity (2.51). Suppose we have the product of structure constants $f^{abc}f^{cde}$, a four-point gluon vertex will always contribute a factor like this. Then from equation (2.50),

$$\begin{aligned} f^{abc}f^{cde} &= -\frac{1}{2}[Tr(T^aT^bT^c) - Tr(T^aT^cT^b)][Tr(T^cT^dT^e) - Tr(T^cT^eT^d)] \\ &= -\frac{1}{2}[Tr(T^aT^bT^c)Tr(T^cT^dT^e) - Tr(T^aT^bT^c)Tr(T^cT^eT^d) \\ &\quad - Tr(T^aT^cT^b)Tr(T^cT^dT^e) + Tr(T^aT^cT^b)Tr(T^cT^eT^d)]. \end{aligned} \tag{2.52}$$

Consider,

$$\begin{aligned} Tr(T^aT^bT^c)Tr(T^cT^dT^e) &= (T^a)_{i_1}^{\bar{j}_1}(T^b)_{\bar{j}_1}^{\bar{j}_2}(T^c)_{\bar{j}_2}^{i_1}(T^c)_{i_2}^{\bar{j}_3}\left(T^d\right)_{\bar{j}_3}^{\bar{j}_4}(T^e)_{\bar{j}_4}^{i_2} \\ &= \left((T^c)_{\bar{j}_2}^{i_1}(T^c)_{i_2}^{\bar{j}_3}\right)(T^a)_{i_1}^{\bar{j}_1}(T^b)_{\bar{j}_1}^{\bar{j}_2}\left(T^d\right)_{\bar{j}_3}^{\bar{j}_4}(T^e)_{\bar{j}_4}^{i_2} \\ &= Tr(T^aT^bT^dT^e) - \frac{1}{N_c}Tr(T^aT^b)Tr(T^dT^e) \\ &= Tr(T^aT^bT^dT^e) - \frac{1}{N_c}\delta^{ab}\delta^{de}, \end{aligned} \tag{2.53}$$

where, we have used the equations (2.48) and (2.51).

Similarly other terms can be written as,

$$Tr(T^aT^bT^c)Tr(T^cT^eT^d) = Tr(T^aT^bT^eT^d) - \frac{1}{N_c}\delta^{ab}\delta^{de}, \tag{2.54}$$

$$Tr(T^aT^cT^b)Tr(T^cT^dT^e) = Tr(T^bT^aT^dT^e) - \frac{1}{N_c}\delta^{ab}\delta^{de}, \tag{2.55}$$

$$Tr(T^aT^cT^b)Tr(T^cT^eT^d) = Tr(T^bT^aT^eT^d) - \frac{1}{N_c}\delta^{ab}\delta^{de}. \tag{2.56}$$

Putting these back in equation (2.52), we obtain

$$f^{abc}f^{cde} = -\frac{1}{2}[Tr(T^aT^bT^dT^e) - Tr(T^aT^bT^eT^d) - Tr(T^bT^aT^dT^e) + Tr(T^bT^aT^eT^d)]. \tag{2.57}$$

From this, one can observe that any tree-level diagram for multi-gluon scattering can be expressed in terms of single traces over product of generators T^{a_i}, with all possible permutations. And using this observation, we define the *colour decomposition/ colour ordering* of n-gluon tree amplitude [Dix13] as ,

$$\mathcal{M}_n^{\text{tree}}(\{k_i, \lambda_i, a_i\}) = g_s^{n-2} \sum_{\sigma \in S_n/Z_n} Tr(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) M_n^{\text{tree}}(\sigma(1^{\lambda_1}), \sigma(2^{\lambda_2}), .., \sigma(n^{\lambda_n})), \tag{2.58}$$

where, $\lambda_i = \pm$ are the helicities of particles with momenta k_i represented as $(1, 2, ..., n)$, g_s is the gauge coupling constant defined by the relation (1.9). S_n is the set of all permutations of n objects, while Z_n is the subset of all cyclic permutations preserving the trace. Thus the sum is done over all (n-1)! non-cyclic permutations σ. This is equivalent to fixing particle 1 and summing over all permutations of the rest.

So, here we have decomposed a QCD amplitude in terms of $SU(N_c)$ colour factors and colour-stripped *partial* amplitudes $M_n^{\text{tree}}(1^{\lambda_1}, ..., n^{\lambda_n})$ that depends only on particular ordering, momenta and helicities of external particles. These independent *partial* amplitudes are more easier to work with rather than the full amplitude because of the following reasons :

(i) They are independent of any colour structures and so these colour-stripped/ordered amplitudes behave like QED amplitudes which in comparison to QCD amplitudes, are easier to work with.

(ii) Because only particular cyclic ordering of gluons contribute, number of singularities are reduced as the factorization poles/channels of these amplitudes must only contain cyclically adjacent momenta [Dix13]. For example, the four-point partial amplitude $A_4^{\text{tree}}(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4},)$ can only have poles in s_{12}, s_{23}, s_{34}, and s_{41} as these are cyclically adjacent and not in s_{13}, or s_{24}, where $s_{ij} = (k_i + k_j)^2$. Clearly, this reduces the number of kinematic variables involved which are more apparent when the number of external legs increase.

(iii) All partial amplitudes are separately gauge-invariant. This is because the colour factors corresponding to each partial/sub-amplitude are independent of each other in any non-abelian gauge group. These single traces form a partial orthogonal basis, [EH14]. And since the full scattering amplitude is gauge-invariant, all partial amplitudes must be gauge-invariant as well.

(iv) It would seem very counter productive that instead of solving one full amplitude, one has to now solve (n-1)! partial amplitudes. However, these sub-amplitudes are related to each other via many relations like,

- **Cyclic Invariance** :

$$M_n^{\text{tree}}(1,2,..,n) = M_n^{\text{tree}}(2,3,..,n,1). \tag{2.59}$$

- **Reflection Symmetry** :

$$M_n^{\text{tree}}(n,n-1,...,2,1) = (-1)^n M_n^{\text{tree}}(1,2,..,n). \tag{2.60}$$

- **Photon Decoupling Equation** :

$$M_n^{\text{tree}}(1,2,..,n)+M_n^{\text{tree}}(2,1,3,..,n)+M_n^{\text{tree}}(2,3,1,..,n)+\ldots+M_n^{\text{tree}}(2,3,..,1,n) = 0. \tag{2.61}$$

There are many other symmetry relations like **parity** or **charge conjugation** that can be used to relate partial helicity amplitudes of n-gluons or when external quark - anti quark pairs are present. Such symmetries make it easier to perform those cumbersome calculations. Actually it has been shown in reference [KK89] that for n-gluon scattering, there are (n-3)! independent partial amplitudes.

Another simple case of tree-level colour decomposition is when an external pair of quark - antiquark is present along with the gluons. For such case, the decomposition [Dix13], is given by

$$\mathcal{M}_n^{\text{tree}} = g_s^{n-2} \sum_{\sigma \in S_{n-2}} (T^{a_{\sigma(3)}} \cdots T^{a_{\sigma(n)}})_{i_2}^{\bar{j}_1} \; M_n^{\text{tree}}(1_{\bar{q}}^{\lambda_1}, 2_q^{\lambda_2}, \sigma(3^{\lambda_3}), .., \sigma(n^{\lambda_n})), \tag{2.62}$$

where the amplitude has been decomposed into a string of generators. For amplitude involving photons, we need to replace the generator $(T^a)_j^i$ by δ_j^i and g_s by $\sqrt{2}eQ_q$ for each gluon to be replaced by a photon.

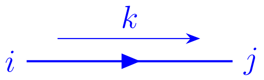
2.2.1 Colour-ordered Feynman rules for QCD

As we have stripped the partial amplitudes of their colour structures, we can give the *colour-ordered* Feynman rules that can be used to construct such sub-amplitudes.

It can be noted that all the group theory structures have been removed and so the rules are purely kinematic. Also, we have fixed $\eta = 1$, i.e. we have chosen the Feynman gauge and as per convention, all momenta are outgoing.

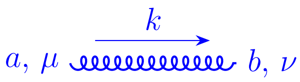
Propagators

⋄ Quark propagator



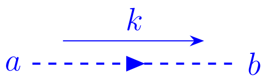
$$i\frac{(\not{k} + m)}{k^2 - m^2 + i\epsilon}$$

⋄ Gluon propagator



$$\frac{-ig_{\mu\nu}}{k^2 + i\epsilon}$$

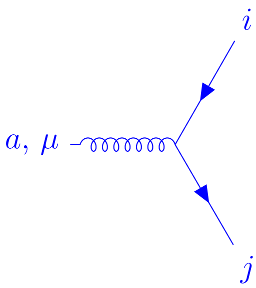
⋄ Scalar propagator



$$\frac{-i}{k^2 + i\epsilon}$$

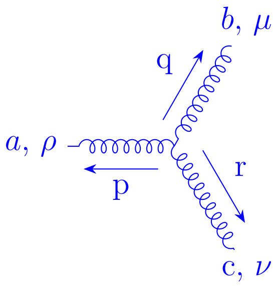
Vertices

⋄ Gluon-Quark vertex



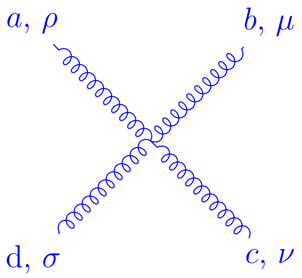
$$\frac{i}{\sqrt{2}}\gamma^{\mu}$$

⋄ Three-gluon vertex



$$\frac{i}{\sqrt{2}}[\,(p-q)_{\nu}\,g_{\rho\mu}+(q-r)_{\rho}\,g_{\mu\nu}+(r-p)_{\mu}\,g_{\nu\rho}\,]$$

⋄ Four-gluon vertex



$$i[g_{\rho\nu}g_{\mu\sigma} - \frac{1}{2}(g_{\rho\mu}g_{\nu\sigma} + g_{\rho\sigma}g_{\mu\nu})]$$

After calculating all the partial amplitudes, to calculate the differential cross-section, the full amplitude must be squared and since the colour quantum number is unobserved, the squared amplitude must be summed over final colours and averaged over initial colours, just like we did with spins in section 2.1. Now, when we will do this process of squaring the amplitude (2.58) and summing it over the colours, we will have terms [MP05], like

$$\sum_{\text{colours}} [Tr(T^{a_1}...T^{a_n})][Tr(T^{b_1}...T^{b_n})]^* = N_c^{n-2}(N_c^2 - 1)(\delta_{\{a\}\{b\}} + O(\frac{1}{N_c^2})) \tag{2.63}$$

where {b} is a permutation of {a}. Earlier, we said that the traces of T^a matrices form an orthogonal basis for the expansion. However, they are orthogonal only at the leading order in powers of N_c. The leading contributions in N_c come when the two permutations are equal (up to cyclic orderings), [MP05]. Using this we can write,

$$\sum_{a_1,..,a_n=1}^{N_c^2-1} \left|\mathcal{M}_n^{\text{tree}}(\{a_i\})\right|^2 = N_c^{n-2}(N_c^2 - 1)\left\{\sum_{\sigma\epsilon S_n/Z_n} \left|M_n^{\text{tree}}(\sigma(1), ..., \sigma(n))\right|^2 + O(\frac{1}{N_c^2})\right\} \tag{2.64}$$

2.3 The Spinor-helicity formalism

We saw in previous section that how colour-ordering can make our life easier to some extent by making the process of calculating scattering amplitudes of the non-abelian gauge theory a bit simpler. This section aims to do the same (actually more) by taking few more steps towards obtaining compact and simpler expressions for tree level amplitudes. It does so by introducing a new set of kinematic variables called *spinor products* which are the Lorentz-invariant contractions of *Weyl spinors* that will serve as the fundamental object in this review of spinor helicity formalism. The reason behind why this formalism has an upper hand over the traditional methods is that most of the quantities that appear in scattering amplitude calculations, like momentum variables, their inner products, Dirac spinors, polarization vectors, etc. can be represented in terms of these Weyl spinors or their inner products. We will see that these spinor products have many useful identities that can make the calculation process somewhat less strenuous.

Another thing to note is that we will be talking about only massless vector bosons like gluons and massless fermions like quarks. Though different flavours of quarks have light and heavy masses, we'll be dealing with light quarks only and it is a fair assumption to consider them massless because they are involved in very high energy processes at colliders. At such energy scale, all fermions can be considered as ultra-relativistic i.e., their momenta are much larger as compared to their individual masses and thus we can treat them as massless particles. Since we are dealing with massless particles only, it would be good to work with amplitudes in which particles are labeled with definite helicity. We have already mentioned one advantage of working with helicity amplitudes - presence of symmetries; the other being that different helicity configurations do not interfere with each other as they are orthogonal and thus squares of all the possible helicity states can be summed incoherently. This definitely reduces the number of terms from the case when all configurations are first added and then squaring of total amplitude is done. Moreover, as we will see later that assigning a fixed helicity to an amplitude can help us to exploit gauge invariance and select an explicit representation for the polarization vectors that may help in reducing the calculation effort.

2.3.1 Spinor variables for massless fermions

We are used to using four-momenta and the corresponding Mandelstam variables as the basic kinematic variables in traditional approaches. However, these four-momenta (k^μ) are Lorentz vectors that are reducible and can be written in terms of a more fundamental (irreducible) representation - the spinor representation which is a two-dimensional representation for massless vectors. Basically we represent massless four-momenta in terms of a pair of two dimensional spinors called *Weyl spinors.*

Let's consider a massless fermion four-momentum k. This formalism basically comes from the massless Dirac spinors of this fermion that are four-component spinors and are solutions of the massless Dirac equation in momentum space

$$\not{k}U(k) = 0. \tag{2.65}$$

This equation has two independent solutions of the form $U_+(k)$ and $U_-(k)$ where, the former is a right-handed spinor while the latter is a left-handed spinor. These solutions are helicity eigenstates and since for massless particles helicity is a conserved Lorentz-invariant quantity, it coincides with chirality. So, these solutions can be written as

$$U_+(k) = \frac{1}{2}(1+\gamma^5)U(k), \qquad U_-(k) = \frac{1}{2}(1-\gamma^5)U(k) \tag{2.66}$$

Negative energy solutions also exist but for light-like momenta they are equivalent to positives ones up to normalisation conventions i.e., $U_\pm = V_\mp$ (crossing symmetry).

The gamma matrices are represented in the Weyl(Chiral) basis, [Con20a] as

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}, \qquad \gamma^5 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}, \tag{2.67}$$

where, $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$. In such a basis, solutions of the Dirac equation take the form

$$U_+(k) = \begin{bmatrix} 0 \\ u_+(k) \end{bmatrix}, \qquad U_-(k) = \begin{bmatrix} u_-(k) \\ 0 \end{bmatrix}. \tag{2.68}$$

Here, $u_\pm(k)$ are two-component spinors in the Weyl representation denoted as,

$$u_+(k) \equiv \lambda_\alpha, \qquad u_-(k) \equiv \tilde{\lambda}^{\dot{\alpha}}, \tag{2.69}$$

where $\{\alpha, \dot{\alpha}\} = 1{,}2$ and the Weyl spinor indices can be raised and lowered using the antisymmetric tensors,

$$\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = -\epsilon_{\dot{\alpha}\dot{\beta}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{2.70}$$

as follows :

$$\begin{aligned} \lambda^\alpha &= \epsilon^{\alpha\beta}\lambda_\beta, \qquad \tilde{\lambda}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\lambda}_{\dot{\beta}} \\ \lambda_\alpha &= \epsilon_{\alpha\beta}\lambda^\beta, \qquad \tilde{\lambda}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^{\dot{\beta}} \end{aligned} \tag{2.71}$$

Now, it's better to give a compact representation of Weyl spinors - the *angle braket*, $| \, \rangle$ and the *square braket*, $| \,]$ notation. Say k_i is the momenta of i^{th} particle, then we first define,

$$|k_i^\pm\rangle \equiv |i^\pm\rangle \,, \tag{2.72}$$

then,

$$|i^+\rangle \equiv (\lambda_i)_\alpha \equiv u_+(k_i) = v_-(k_i) \equiv |i\rangle \,,$$

$$|i^-\rangle \equiv \tilde{\lambda}_i^{\dot{\alpha}} \equiv u_-(k_i) = v_+(k_i) \equiv |i],$$
$$\langle i^+| \equiv (\tilde{\lambda}_i)_{\dot{\alpha}} \equiv \bar{u}_+(k_i) = \bar{v}_-(k_i) \equiv [i|,$$
$$\langle i^-| \equiv \lambda_i^{\alpha} \equiv \bar{u}_-(k_i) = \bar{v}_+(k_i) \equiv \langle i| \,, \tag{2.73}$$

where,

u_+ : Incoming right-handed fermion; v_- : Outgoing left-handed antifermion,
u_- : Incoming left-handed fermion; v_+ : Outgoing right-handed antifermion,
$\bar{u}_+$: Outgoing right-handed fermion; $\bar{v}_-$: Incoming left-handed antifermion,
$\bar{u}_-$: Outgoing left-handed fermion; $\bar{v}_+$: Incoming right-handed antifermion.

Having defined all the notations and representation of the spinors, we can finally define the Lorentz invariant inner products using equation (2.71) and (2.73). These angle and square braket products are the core of spinor-helicity formalism :

$$\bar{u}_-\left(k_i\right) u_+\left(k_j\right) = \varepsilon^{\alpha\beta}\left(\lambda_i\right)_\alpha \left(\lambda_j\right)_\beta = \langle i_-|j_+\rangle = \langle ij\rangle, \tag{2.74}$$

$$\bar{u}_+\left(k_i\right) u_-\left(k_j\right) = \varepsilon^{\dot{\alpha}\dot{\beta}}\left(\widetilde{\lambda}_i\right)_{\dot{\alpha}} \left(\widetilde{\lambda}_j\right)_{\dot{\beta}} = \langle i_+|j_-\rangle = [ij]. \tag{2.75}$$

Useful Identities

- Consider the four-momentum of a light-like particle and contract it with the Pauli matrices to construct matrix of the form,

$$(\not{k})_{\alpha\dot{\beta}} = k_\mu(\sigma^\mu)_{\alpha\dot{\beta}} = \begin{bmatrix} k_0 - k_3 & -k_1 + ik_2 \\ -k_1 - ik_2 & k_0 + k_3 \end{bmatrix}. \tag{2.76}$$

The determinant of this matrix is zero since the particle is massless. Therefore, rank of this (2×2) matrix is 1 and it can be represented as an outer product of two-component spinors,

$$(\not{k})_{\alpha\dot{\beta}} = \lambda_\alpha(k)\tilde{\lambda}_{\dot{\beta}}(k) = |k\rangle_\alpha \, [k|_{\dot{\beta}}, \tag{2.77}$$

similarly,

$$(\not{k})^{\dot{\alpha}\beta} = \tilde{\lambda}^{\dot{\alpha}}(k)\lambda^{\beta}(k) = |k]^{\dot{\alpha}} \, \langle k|^{\beta} \,. \tag{2.78}$$

The spin-sum completeness relation then can be written as :

$$\not{k} = |k\rangle \, [k| + |k] \, \langle k| \,. \tag{2.79}$$

- It follows that for real-valued momenta k^μ,

$$[i|^{\dot{\alpha}} = (|i\rangle^{\alpha})^*, \qquad \langle i|_\alpha = (|i]_{\dot{\alpha}})^* \tag{2.80}$$

while these angle and square bras and kets are independent of each other for complex-valued momentum [EH14]. Using the above equation (2.80), complex conjugation of spinor products can be related as,

$$[ij] = \langle ji \rangle^* . \tag{2.81}$$

- Anti-symmetry :

$$\langle ij \rangle = -\langle ji \rangle , \quad [ij] = -[ji], \quad [ii] = 0, \quad \langle ii \rangle = 0 \tag{2.82}$$

- Analytic continuation :

$$|-j] = i\ |j], \qquad |-j\rangle = i\,|j\rangle \tag{2.83}$$

- Projection operator :

$$|i^{\pm}\rangle \langle i^{\pm}| = \frac{1}{2}(1 \pm \gamma^5)\not{k}_i \tag{2.84}$$

- Momentum conservation :

$$\sum_{j=1;\ j\neq i,k}^{n} \langle ij\rangle\,[jk] = 0 \tag{2.85}$$

- Schouten identity :

$$\langle ij\rangle \langle kl\rangle - \langle ik\rangle \langle jl\rangle = \langle il\rangle \langle kj\rangle \tag{2.86}$$

- We define the product of angle and square spinor products as following :

$$\langle ij\rangle\,[ji] = Tr(\frac{1}{2}(1-\gamma^5)\not{k}_i\not{k}_j) = 2k_i.k_j = (k_i + k_j)^2 = s_{ij}. \tag{2.87}$$

  We can also write this spinor products [Dix96], in terms of Mandelstam variables,

$$\langle ij\rangle = \sqrt{s_{ij}}e^{i\phi_{ij}}, \qquad [ij] = \sqrt{s_{ij}}e^{-i(\phi_{ij}+\pi)} \tag{2.88}$$

  where ϕ_{ij} is the phase factor that can be defined in terms of light-cone coordinates,

$$\cos\phi_{ij} = \frac{k_i^1 k_j^+ - k_j^1 k_i^+}{\sqrt{|s_{ij}|\,k_i^+ k_j^+}}, \quad \sin\phi_{ij} = \frac{k_i^2 k_j^+ - k_j^2 k_i^+}{\sqrt{|s_{ij}|\,k_i^+ k_j^+}}; \quad k^{\pm} = k^0 \pm k^3. \tag{2.89}$$

Now, consider an angle-square braket of the form

$$[i|\gamma^{\mu}\,|j\rangle = [i|^{\dot{\alpha}}\sigma^{\mu}_{\dot{\alpha}\beta}\,|j\rangle^{\beta} . \tag{2.90}$$

The following identities related to this object come out to be very useful in simplifying the calculations,

- Charge conjugation of current :

$$[i|\gamma^\mu |j\rangle = \langle j|\, \gamma^\mu |i]. \tag{2.91}$$

- Real momentum conjugation :

$$[i|\gamma^\mu |j\rangle^* = [j|\gamma^\mu |i\rangle\, . \tag{2.92}$$

- Feirz rearrangement :

$$\langle i|\, \gamma^\mu |j]\ \ \langle k|\, \gamma_\mu |l] = 2\, \langle ik\rangle\, [jl]. \tag{2.93}$$

- Gordon identity :

$$\langle i|\, \gamma^\mu |i] = [i|\gamma^\mu\, |i\rangle = 2k_i^\mu. \tag{2.94}$$

- Consider an arbitrary four-momentum k^μ, then

$$[i|\not{k}\, |j\rangle = k_\mu [i|\gamma^\mu\, |j\rangle\, . \tag{2.95}$$

  If k^μ is a light-like vector, then we can use the completeness relation (2.79) to simplify the above equation as

$$[i|\not{k}\, |j\rangle = [ik]\, \langle kj\rangle\, . \tag{2.96}$$

2.3.2 Spinor variables for massless vector bosons

We have written almost all the objects (relevant to us) in terms of angle and square spinors except the polarization vectors that appear in Feynman rules for external lines, for spin-1 massless vector particles like gluons. The spinor representation of these massless polarization vectors with fixed helicity is given by

$$\epsilon_+^\mu(k_i, q) = \frac{\langle q|\, \gamma^\mu |i]}{\sqrt{2}\, \langle qi\rangle}, \quad \epsilon_-^\mu(k_i, q) = -\frac{[q|\gamma^\mu\, |i\rangle}{\sqrt{2}[qi]}, \tag{2.97}$$

where, q is a light-like arbitrary reference momentum and k_i is the momentum of i^{th} gluon such that $q.k \neq 0$. The arbitrariness of the reference momentum q is manifested by the gauge invariance of the on-shell scattering amplitude. This means that all scattering amplitudes would be independent of q because changing the reference momentum only shifts the polarisation vector as

$$\epsilon_\pm^\mu(k^\mu) \longrightarrow \epsilon_\pm^\mu(k^\mu) + Ck^\mu. \tag{2.98}$$

To see this, let's assume that the reference momentum q is changed to q'. Then,

$$\begin{aligned}\epsilon_+^\mu(k,q') - \epsilon_+^\mu(k,q) &= \frac{\langle q'|\,\gamma^\mu|k]}{\sqrt{2}\,\langle q'k\rangle} - \frac{\langle q|\,\gamma^\mu|k]}{\sqrt{2}\,\langle qk\rangle} \\ &= \frac{-\langle q'|\gamma^\mu \not{k}|q\rangle + \langle q|\gamma^\mu \not{k}|q'\rangle}{\sqrt{2}\,\langle q'k\rangle\,\langle qk\rangle} \\ &= -\frac{\sqrt{2}\,\langle q'q\rangle}{\langle q'k\rangle\,\langle qk\rangle} \times k^\mu \end{aligned} \tag{2.99}$$

Similarly, we can prove the following properties :

- Normalisation :

$$\epsilon_\pm.(\epsilon_\pm)^* = \epsilon_\pm.\epsilon_\mp = -1, \qquad \epsilon_\pm.(\epsilon_\mp)^* = \epsilon_\pm.\epsilon_\pm = 0. \tag{2.100}$$

- Transversality condition :

$$\epsilon_\pm(k_i,q).k_i = 0, \qquad \epsilon_\pm(k_i,q).q = 0. \tag{2.101}$$

- The completeness relation :

$$\sum_{\lambda=\pm} \epsilon_\lambda^{*\mu}\epsilon_\lambda^\nu = -g^{\mu\nu} + \frac{k^\mu q^\nu + q^\mu k^\nu}{(k\cdot q)}. \tag{2.102}$$

The arbitrariness of the reference momentum q proves to be very useful, as a smart choice of q_i for each gluon can simplify the calculations by making lots of terms to disappear. Now that we have everything in our hand to express any amplitude involving massless fermions and/or vector bosons in terms of spinor products, we'll apply the modern techniques learnt so far to the scattering process whose amplitude has already been calculated using the traditional Feynman diagrammatic approach.

2.4 An example from QCD

$q(k_1) + \bar{q}(k_2) \longrightarrow g(k_3, a) + g(k_4, b)$

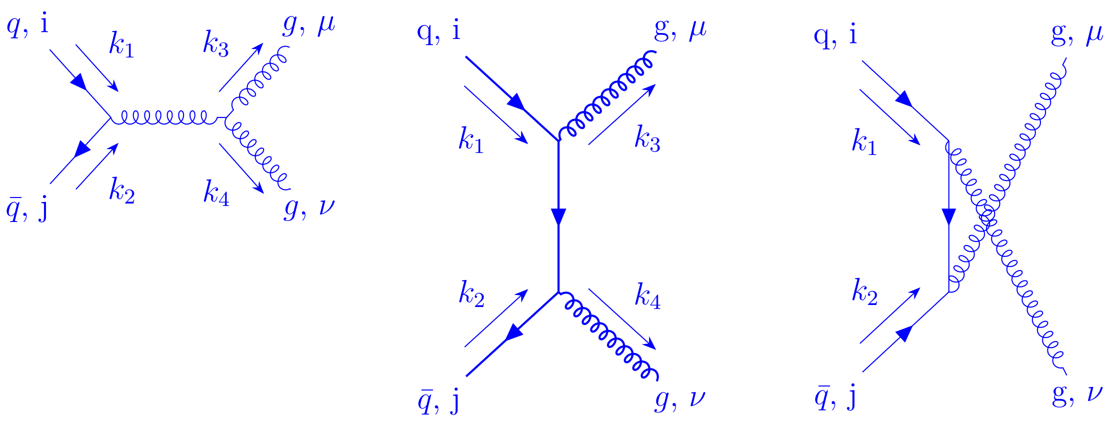


Figure 2.4: $q\bar{q} \longrightarrow gg$

Above figure shows all the possible Feynman diagrams for the scattering process in consideration. The overall amplitude for any arbitrary helicity (λ) configuration can be written using colour-ordered Feynman rules discussed in subsection 2.2.1 and equation (2.62), since a quark-anti quark pair is involved in the process along with two gluons. Before writing that, let's discuss the colour structure contributions from each diagram. These are,

$$f^{abc}(T^c)^i_j, \qquad (T^bT^a)^i_j, \qquad \text{and} \quad (T^aT^b)^i_j,$$

respectively. We can write the first colour structure in terms of the other two using equation (2.47) and thus the full amplitude can be written as

$$\begin{aligned}\mathcal{M}^{a,b}_{j,i}(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}) = g_s^2[(T^bT^a)^i_j\ M_x(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4})\\ + (T^aT^b)^i_j\ M_y(1^{\lambda_1}, 2^{\lambda_2}, 4^{\lambda_4}, 3^{\lambda_3})],\end{aligned} \tag{2.103}$$

where, M_x and M_y are the gauge invariant partial amplitudes independent of each other. M_x is the contribution from diagrams (i + ii) while M_y is the contribution from diagrams (i + iii).

Now, let's start calculating these partial amplitudes one by one. Because there are four particles and each can have $\pm$ helicity, we have 16 possible helicity configurations in total. The expressions for colour stripped amplitudes of an arbitrary helicity configuration for Feynman diagram (i) and (ii) are :

$$
\begin{aligned}
M_{(i)} = & \frac{i}{2(k_1+k_2)^2}\ \bar{v}_\lambda(k_2)\gamma^\rho u_\lambda(k_1)\ [(k_3-k_4)^\rho g^{\mu\nu} + (2k_4+k_3)^\mu g^{\nu\rho} \\
& - (2k_3+k_4)^\nu g^{\rho\mu}]\ \epsilon_{\mu,\lambda}(k_3)\ \epsilon_{\nu,\lambda}(k_4) \\
= & \frac{i}{2(k_1+k_2)^2}\ \bar{v}_\lambda(k_2)\gamma^\rho u_\lambda(k_1)\ [\epsilon_\lambda(k_3).\epsilon_\lambda(k_4)(k_3-k_4)^\rho \\
& + 2\epsilon^\rho_\lambda(k_4)(k_4.\epsilon_\lambda(k_3)) - 2\epsilon^\rho_\lambda(k_3)(k_3.\epsilon_\lambda(k_4))], \qquad (2.104)
\end{aligned}
$$

$$
\begin{aligned}
M_{(ii)} = & \frac{-i}{2(k_1-k_3)^2}\epsilon_{\nu,\lambda}(k_4)\epsilon_{\mu,\lambda}(k_3)\ [\bar{v}_\lambda(k_2)\gamma^\nu \not{k}_1\gamma^\mu u_\lambda(k_1) \\
& - \bar{v}_\lambda(k_2)\gamma^\nu \not{k}_3\gamma^\mu u_\lambda(k_1)] \qquad (2.105)
\end{aligned}
$$

. Then,

Helicity configurations for M_x

- $M_x(1^+,2^+,3^+,4^+)$: For this particular helicity configuration, we have a term like

$$\bar{v}_+(k_2)\gamma^\rho u_+(k_1) = \langle 2\gamma^\rho 1\rangle = 0,$$

in the expressions (2.104) and (2.105). For latter, just use the completeness relation (2.79) to bring it in the above form. Therefore,

$$M_x(1^+,2^+,3^+,4^+) = 0.$$

This actually happens because quarks and anti-quarks are coupled through a vector current and so they must have opposite helicity. This implies that all the configurations, in which they have equal helicity, vanish i.e.,

$$M_x(1^+,2^+,3^-,4^-) = M_x(1^+,2^+,3^+,4^-) = M_x(1^+,2^+,3^-,4^+) = 0.$$

Similarly,

$$
\begin{aligned}
M_x(1^-,2^-,3^-,4^-) &= M_x(1^-,2^-,3^+,4^+) = M_x(1^-,2^-,3^-,4^+) \\
&= M_x(1^-,2^-,3^+,4^-) = 0.
\end{aligned}
$$

One can notice that the last four configurations are related to the first four via parity symmetry. Out of the 16 possible configurations, 8 have already vanished. Now, since quark and anti-quark should have opposite helicity, both gluons must have opposite helicity as well; else, conservation of angular momentum would be violated. Let's verify the same by calculating one of such configuration.

- $M_x(1^+, 2^-, 3^+, 4^+)$: Consider,

$$M_{(ii)}(1^+, 2^-, 3^+, 4^+) = \frac{-i}{2t}\epsilon_{\nu,+}(k_4)\epsilon_{\mu,+}(k_3)\ [\bar{v}_-(k_2)\gamma^\nu \not{k}_1\gamma^\mu u_+(k_1) - \bar{v}_-(k_2)\gamma^\nu \not{k}_3\gamma^\mu u_+(k_1)]$$

Now using equation (2.97) and (2.79) and choosing q_1, q_2 as the reference vectors, we get

$$= \frac{-i}{2t}\frac{\langle q_2|\,\gamma_\nu|4]}{\sqrt{2}\,\langle q_2 4\rangle}\frac{\langle q_1|\,\gamma_\mu|3]}{\sqrt{2}\,\langle q_1 3\rangle}\ \{[2\gamma^\nu 1\rangle[1\gamma^\mu 1\rangle - [2\gamma^\nu 3\rangle[3\gamma^\mu 1\rangle\}$$

We can apply the Feirz arrangement (2.93) to obtain the expression

$$= \frac{-i}{t\,\langle q_2 4\rangle\,\langle q_1 3\rangle}\{\langle q_2 1\rangle\,[42]\,\langle q_1 1\rangle\,[31] - \langle q_2 3\rangle\,[42]\,\langle q_1 1\rangle\,[33]\}.$$

One can now exploit the spinor product relations like [ii] $= \langle ii\rangle = 0$ by choosing the appropriate reference momenta; say, $q_1 = q_2 = k_1$
$\Longrightarrow$

$$M_{(ii)}(1^+, 2^-, 3^+, 4^+) = 0.$$

Now, consider the other diagram ,

$$M_{(i)}(1^+, 2^-, 3^+, 4^+) = \frac{i}{2s}\ \bar{v}_-(k_2)\gamma^\rho u_+(k_1)\ [\epsilon_+(k_3).\epsilon_+(k_4)(k_3 - k_4)^\rho + 2\epsilon_+^\rho(k_4)(k_4.\epsilon_+(k_3)) - 2\epsilon_+^\rho(k_3)(k_3.\epsilon_+(k_4))].$$

First we'll calculate the dot product of polarization vectors,

$$\epsilon_+(k_3, q_1).\epsilon_+(k_4, q_2) = \frac{\langle q_1 q_2\rangle\,[34]}{\langle q_1 3\rangle\,\langle q_2 4\rangle}. \tag{2.106}$$

Again, $q_1 = q_2 = k_1$ since choice of reference momenta must be same for all amplitudes contributing to same helicity configuration. Thus the above equation vanishes leaving the expression,

$$M_{(i)}(1^+, 2^-, 3^+, 4^+) = \frac{i}{2s}\ [2\gamma^\rho 1\rangle\ [2\frac{\langle 1|\,\gamma_\rho|4]}{\sqrt{2}\,\langle 14\rangle}(k_4.\epsilon_+(k_3)) - 2\frac{\langle 1|\,\gamma_\rho|3]}{\sqrt{2}\,\langle 13\rangle}(k_3.\epsilon_+(k_4))].$$

Feirz rearrangement of the outer angle-square braket with the inner ones will give terms like $\langle 11\rangle$, making the above expression zero.
$\therefore$

$$M_x(1^+, 2^-, 3^+, 4^+) = 0.$$

Similarly,

$$M_x(1^+,2^-,3^-,4^-) = M_x(1^-,2^+,3^+,4^+) = M_x(1^-,2^+,3^-,4^-) = 0.$$

This makes the tally of vanishing configurations equal to 12. Let's handle the remaining states one by one.

- $M_x(1^+,2^-,3^+,4^-)$:

$$\begin{aligned}
M_{(i)}(1^+,2^-,3^+,4^-) &= \frac{i}{2s}\ \bar{v}_-(k_2)\gamma^\rho u_+(k_1)\ [\epsilon_+(k_3).\epsilon_-(k_4)(k_3-k_4)^\rho \\
&\quad + 2\epsilon_-^\rho(k_4)(k_4.\epsilon_+(k_3)) - 2\epsilon_+^\rho(k_3)(k_3.\epsilon_-(k_4))] \\
&= \frac{i}{2s}\ [2\gamma^\rho 1\rangle\ [-\frac{\langle q_1 4\rangle\,[3q_2]}{\langle q_1 3\rangle\,[q_2 4]}(k_3-k_4)^\rho - 2\frac{[q_2|\gamma_\rho|4\rangle}{\sqrt{2}[q_2 4]}(k_4.\epsilon_+(k_3)) \\
&\quad - 2\frac{\langle q_1|\gamma_\rho|3]}{\sqrt{2}\,\langle q_1 3\rangle}(k_3.\epsilon_-(k_4))].
\end{aligned}$$

Choose reference momenta, $q_1 = k_4$ and $q_2 = k_3$. Then, first term becomes zero. Also, second and third term become zero because,

$$q.\epsilon_\pm(p,q) = 0.$$

Now consider,

$$\begin{aligned}
M_{(ii)}(1^+,2^-,3^+,4^-) &= \frac{-i}{2t}\epsilon_{\nu,-}(k_4)\epsilon_{\mu,+}(k_3)\ [\bar{v}_-(k_2)\gamma^\nu \not{k}_1\gamma^\mu u_+(k_1) \\
&\quad - \bar{v}_-(k_2)\gamma^\nu \not{k}_3\gamma^\mu u_+(k_1)] \\
&= \frac{i}{2t}\frac{[q_2|\gamma_\nu|4\rangle}{\sqrt{2}[q_2 4]}\frac{\langle q_1|\,\gamma_\mu|3]}{\sqrt{2}\,\langle q_1 3\rangle}\ \{[2\gamma^\nu 1\rangle[1\gamma^\mu 1\rangle - [2\gamma^\nu 3\rangle[3\gamma^\mu 1\rangle\}.
\end{aligned}$$

By using the same choice of reference momenta this becomes,

$$\begin{aligned}
&= \frac{i}{t[34]\,\langle 43\rangle}\{\langle 41\rangle\,[32]\,\langle 41\rangle\,[31] - \langle 43\rangle\,[32]\,\langle 41\rangle\,[33]\} \\
&= \frac{i}{t[34]\,\langle 43\rangle}\{\langle 41\rangle\,[32]\,\langle 41\rangle\,[31]\} \\
&= -i\frac{\langle 14\rangle^2\,[23]}{\langle 13\rangle\,\langle 34\rangle\,[43]}, \qquad \because\ t = s_{13} = \langle 13\rangle\,[31].
\end{aligned}$$

Now multiply top and bottom of the above expression by $\langle 14\rangle\,\langle 24\rangle$, use the momentum conservation relation $\langle 14\rangle\,[43] = -\langle 12\rangle\,[23]$ and anti symmetric properties of spinor products to obtain the final expression ,

$$M_{(ii)}(1^+,2^-,3^+,4^-) = i\frac{\langle 14\rangle^3\,\langle 24\rangle}{\langle 13\rangle\,\langle 34\rangle\,\langle 42\rangle\,\langle 21\rangle},$$

which using equation (2.88), can be written in terms of Mandelstam variables,

$$= \frac{i}{s}\sqrt{\frac{u^3}{t}}e^{i\phi_1},$$

where, $\phi_1 = 3\phi_{14} - \phi_{13} - \phi_{12} - \phi_{34}$. Since $M_i = 0$,

$\implies$

$$M_x(1^+, 2^-, 3^+, 4^-) = i\frac{\langle 14\rangle^3 \langle 24\rangle}{\langle 13\rangle \langle 34\rangle \langle 42\rangle \langle 21\rangle} = \frac{i}{s}\sqrt{\frac{u^3}{t}}e^{i\phi_1}.$$

- $M_x(1^-, 2^+, 3^-, 4^+)$: This relation is related to the previous one by parity. Helicity flip would just change the angle product $\langle\ \rangle$ to square product $[]$ which will result in reversing the sign of phase factor as well. Thus,

$$M_x(1^-, 2^+, 3^-, 4^+) = i\frac{[14]^3[24]}{[13][34][42][21]} = \frac{i}{s}\sqrt{\frac{u^3}{t}}e^{-i\phi_1}.$$

- $M_x(1^+, 2^-, 3^-, 4^+)$: Proceeding in a similar fashion, we obtain

$$M_x(1^+, 2^-, 3^-, 4^+) = i\frac{\langle 13\rangle^3 \langle 23\rangle}{\langle 13\rangle \langle 34\rangle \langle 42\rangle \langle 21\rangle} = i\frac{\sqrt{tu}}{s}e^{i\phi_2},$$

where, $\phi_2 = 2\phi_{13} + \phi_{23} - \phi_{24} - \phi_{12} - \phi_{34}$. And helicity flip of this would lead us to,

- $M_x(1^-, 2^+, 3^+, 4^-)$:

$$M_x(1^-, 2^+, 3^+, 4^-) = i\frac{[13]^3[23]}{[13][34][42][21]} = i\frac{\sqrt{tu}}{s}e^{-i\phi_2}.$$

Now that we have calculated all the vanishing and non-vanishing configurations of M_x, we can easily calculate the same for M_y. All we have to do is to interchange t and u, as M_x and M_y differ only in the exchange of gluon(k_3) with gluon (k_4). Thus, the non-vanishing configurations of M_y has the following expressions :

- $M_y(1^+, 2^-, 4^+, 3^-)$:

$$M_y(1^+, 2^-, 4^+, 3^-) = i\frac{\langle 13\rangle^3 \langle 23\rangle}{\langle 14\rangle \langle 43\rangle \langle 32\rangle \langle 21\rangle} = \frac{i}{s}\sqrt{\frac{t^3}{u}}e^{i\phi_1}.$$

- $M_y(1^-, 2^+, 4^-, 3^+)$:

$$M_y(1^-, 2^+, 4^-, 3^+) = i\frac{[13]^3[23]}{[14][43][32][21]} = \frac{i}{s}\sqrt{\frac{t^3}{u}}e^{-i\phi_1}.$$

- $M_y(1^+, 2^-, 4^-, 3^+)$:

$$M_y(1^+, 2^-, 4^-, 3^+) = i\frac{\langle 14\rangle^3 \langle 24\rangle}{\langle 14\rangle \langle 43\rangle \langle 32\rangle \langle 21\rangle} = i\frac{\sqrt{ut}}{s}e^{i\phi_2}.$$

- $M_y(1^-, 2^+, 4^+, 3^-)$:

$$M_y(1^-, 2^+, 4^+, 3^-) = i\frac{[14]^3[24]}{[14][43][32][21]} = i\frac{\sqrt{ut}}{s}e^{-i\phi_2}.$$

Now we will calculate the square of the full amplitude of a particular helicity state (equation (2.103)), averaged over initial and summed over final colours,

- $\overline{\left|\mathcal{M}(1^+, 2^-, 3^+, 4^-)\right|^2}$:

$$\overline{\left|\mathcal{M}(1^+, 2^-, 3^+, 4^-)\right|^2} = \frac{g_s^4}{9}\sum_{\text{colours}}\left|(T^bT^a)^i_j\ M_x(1^+, 2^-, 3^+, 4^-) + (T^aT^b)^i_j\ M_y(1^+, 2^-, 4^-, 3^+)\right|^2$$

Let's see this term by term :

1st term

$$Tr(T^aT^aT^bT^b)\ \left|M_x(1^+, 2^-, 3^+, 4^-)\right|^2 = \frac{64}{3}\frac{u^3}{ts^2},$$

2nd term

$$Tr(T^aT^aT^bT^b)\ \left|M_y(1^+, 2^-, 4^-, 3^+)\right|^2 = \frac{64}{3}\frac{ut}{s^2},$$

3rd term

$$2\ Tr(T^aT^bT^aT^b)\ \left|M_x^*(1^+, 2^-, 3^+, 4^-)M_y(1^+, 2^-, 4^-, 3^+)\right| = -\frac{16}{3}\frac{u^2}{s^2}.$$

$\therefore$

$$\overline{\left|\mathcal{M}(1^+, 2^-, 3^+, 4^-)\right|^2} = \frac{g_s^4}{9}\ \{\frac{64}{3}(\frac{u^3}{ts^2} + \frac{ut}{s^2}) - \frac{16}{3}\frac{u^2}{s^2}\}.$$

Similarly,

- $$\overline{\left|\mathcal{M}(1^-, 2^+, 3^-, 4^+)\right|^2} = \frac{g_s^4}{9}\ \{\frac{64}{3}(\frac{u^3}{ts^2} + \frac{ut}{s^2}) - \frac{16}{3}\frac{u^2}{s^2}\}.$$
- $$\overline{\left|\mathcal{M}(1^-, 2^+, 3^+, 4^-)\right|^2} = \frac{g_s^4}{9}\ \{\frac{64}{3}(\frac{t^3}{us^2} + \frac{ut}{s^2}) - \frac{16}{3}\frac{t^2}{s^2}\}.$$
- $$\overline{\left|\mathcal{M}(1^+, 2^-, 3^-, 4^+)\right|^2} = \frac{g_s^4}{9}\ \{\frac{64}{3}(\frac{t^3}{us^2} + \frac{ut}{s^2}) - \frac{16}{3}\frac{t^2}{s^2}\}.$$

For calculating the cross-section, we need to sum over all the possible helicity configurations

$$\sum_{\text{Helicity}}\overline{|\mathcal{M}|^2} = \frac{2g_s^4}{9}\left\{\frac{64}{3s^2}\left(\frac{u^3}{t} + ut + \frac{t^3}{u} + ut\right) - \frac{16}{3s^2}\left(u^2 + t^2\right)\right\}$$

$$= \frac{2g_s^4}{9}\left\{\frac{64}{3s^2}\left(\left(u^2 + t^2\right)\left(\frac{t}{u} + \frac{u}{t}\right)\right) - \frac{16}{3s^2}\left(u^2 + t^2\right)\right\}.$$

Now,

$$u + t = -s$$

$\Longrightarrow$

$$u^2 + t^2 = s^2 - 2ut$$

Thus,

$$\begin{aligned}\sum_{\text{Helicity}} \overline{|\mathcal{M}|^2} &= \frac{2g_s^4}{9}\left\{\frac{64}{3}\left(\left(1 - \frac{2ut}{s^2}\right)\left(\frac{t}{u} + \frac{u}{t}\right)\right) - \frac{16}{3}\left(\frac{t^2+u^2}{s^2}\right)\right\} \\ &= \frac{2g_s^4}{9}\left\{\frac{64}{3}\left(\left(\frac{t}{u} + \frac{u}{t}\right) - 2\left(\frac{t^2+u^2}{s^2}\right)\right) - \frac{16}{3}\left(\frac{t^2+u^2}{s^2}\right)\right\} \\ &= \frac{128 g_s^4}{27}\left\{\left(\frac{t}{u} + \frac{u}{t}\right) - \frac{9}{4}\left(\frac{t^2+u^2}{s^2}\right)\right\}\end{aligned}$$

Finally, the differential cross section for the process can be calculated using equation (2.12),

$$\begin{aligned}\frac{d\sigma}{d\Omega}(q\bar{q} \to gg) &= \frac{1}{64\pi^2 s}\sum_{\text{Helicity}} \overline{|\mathcal{M}|^2} \\ &= \frac{8}{27}\frac{\alpha_s^2}{s}\left[\frac{t}{u} + \frac{u}{t} - \frac{9}{4}\frac{t^2+u^2}{s^2}\right]. \qquad (2.107)\end{aligned}$$

Chapter 3

BCFW Recursion Technique

In the previous chapter, we saw how tree-level colour decompositions and spinor-helicity formalism can reduce the complexity of scattering amplitude calculations. However, as the number of external legs increase, it becomes more and more difficult to obtain compact and simplified expressions because possible Feynman diagrams show a rapid growth in number. Here to our rescue, come recursion techniques. As the name 'recursion' suggests, the main idea of such techniques is to use lower-point amplitudes as fundamental starting points to construct higher-point amplitudes in a recursive fashion. As we will see in this chapter, such techniques can manifest the hidden simplicity in the tree-level scattering amplitudes of gluons. One such technique known as - the BCFW recursion technique, was developed by Britto, Cachazo, Feng and Witten in the year 2005. The recursion relations given by them claim to write any tree level amplitude as a sum over terms constructed from products of amplitudes with lesser number of legs multiplied by a Feynman propagator, [CF05].

The main principle behind the derivation of these recursion relations is the exploitation of the idea that tree-level colour-ordered amplitudes are analytic functions of kinematic variables like momenta of external particles, [FW05]. This basically means that poles or branch cuts construct the singularity structure of such amplitudes, allowing the analytical continuation of them to complex momenta in order to reconstruct generic scattering amplitudes from their residues (singularities). And in these singular regions, *"amplitudes factorize into two casually disconnected amplitudes"* [Dix13], with lesser particles, connected by an on-shell propagator. BCFW relations constructs the higher-point amplitudes from *on-shell* gauge invariant objects. For calculating n-point gluon amplitudes, the fundamental building blocks are the on-shell three-point gluon amplitudes whose expressions will be derived in the next section. These three-point interactions actually carry all the relevant information in non-abelian gauge theories like QCD and so working with higher-point vertices seems like a futile exercise. An example of this is that quartic interactions can always be absorbed into cubic ones with no effect whatsoever on the colour structures as they carry the same colour factors as the different channel diagrams that one can construct using three-point gluon vertices.

3.1 Three-gluon amplitudes

It isn't very difficult to calculate three-gluon amplitudes but before going there, one needs to have a look on three-point kinematics of massless particles first. Say, k_1, k_2 and k_3 are light-like momenta such that momentum conservation holds ,

$$k_1^\mu + k_2^\mu + k_3^\mu = 0.$$

Then if these momenta are real, equations (2.81) and (2.87) implies that

$$\langle ij\rangle [ji] = \langle ij\rangle \langle ij\rangle^* = \langle ij\rangle^2 = (k_i + k_j)^2 = k_k^2 = 0, \tag{3.1}$$

$\Longrightarrow$

$$\langle ij\rangle = [ij] = 0, \qquad for\ real\ momenta.$$

However, if these momenta are complex then equation (2.81) is not true but from above relation (3.1), $\langle ij\rangle [ji] = 0$. This implies that either the angular or the square inner product should vanish. Say, $\langle ij\rangle \neq 0$ and from momentum conservation, we have $\langle ij\rangle [jk] = 0$.
$\Longrightarrow$

$$[jk] = 0.$$

Similarly, if we choose the square spinor product to be non-vanishing, we'll have $\langle jk\rangle = 0$. We can itemize this whole massless three-point kinematics as follows :

- $|1] \propto |2] \propto |3] \equiv \tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3$
  $\Longrightarrow$
  $$[12] = [23] = [31] = 0; \qquad all\ \langle ij\rangle \neq 0. \tag{3.2}$$
- $|1\rangle \propto |2\rangle \propto |3\rangle \equiv \lambda_1 \propto \lambda_2 \propto \lambda_3$
  $\Longrightarrow$
  $$\langle 12\rangle = \langle 23\rangle = \langle 31\rangle = 0; \qquad all\ [ij] \neq 0. \tag{3.3}$$

This implies that the expression for a non-vanishing on-shell three-point scattering amplitude will either have a *holomorphic* form (angle product dependence) or *anti-holomorphic* form (square product dependence). Let's verify this.

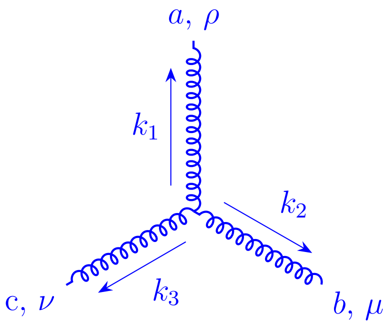


Figure 3.1: **3-gluon vertex**

Suppose we wish to calculate colour-ordered 3-gluon amplitude with the helicity configuration $M(1^-, 2^-, 3^+)$. The three-gluon scattering is represented by the Feynman diagram shown in 3.1.

Then,

$$M(1^-, 2^-, 3^+) = \frac{i}{\sqrt{2}}[\ (k_1 - k_2)^\nu\ g^{\rho\mu} + (k_2 - k_3)^\rho\ g^{\mu\nu} + (k_3 - k_1)^\mu\ g^{\nu\rho}\ ]$$
$$\epsilon_\rho^-(k_1)\epsilon_\mu^-(k_2)\epsilon_\nu^+(k_3)$$
$$= \frac{i}{\sqrt{2}}[\ \epsilon_-(k_1).\epsilon_-(k_2)(k_1 - k_2).\epsilon_+(k_3) + \epsilon_-(k_2).\epsilon_+(k_3)(k_2 - k_3).\epsilon_-(k_1)$$
$$+ \epsilon_+(k_3).\epsilon_-(k_1)(k_3 - k_1).\epsilon_-(k_2)\ ]$$

Now, using equations (2.95), (2.96) and (2.97) , terms inside the square bracket can be written in the form ,

$$\frac{[q_1q_2]\langle 12\rangle(\langle q_31\rangle[13] - \langle q_32\rangle[23]) + [q_23]\langle 2q_3\rangle([q_12]\langle 21\rangle - [q_13]\langle 31\rangle)}{\sqrt{2}[q_11][q_22]\langle q_33\rangle}$$
$$+\frac{[3q_1]\langle q_31\rangle([q_23]\langle 32\rangle - [q_21]\langle 12\rangle)}{\sqrt{2}[q_11][q_22]\langle q_33\rangle},$$

where, q_i are the reference momenta. Now if we will consider the case (3.3), all the terms in the numerator will become zero. Therefore, we choose the case (3.2). This will make the first term vanish. Then applying momentum conservation to write $[q_12]\langle 21\rangle = -[q_13]\langle 31\rangle$ and $[q_23]\langle 32\rangle = -[q_21]\langle 12\rangle$, we obtain

$$M(1^-, 2^-, 3^+) = i\frac{[q_23]\langle 2q_3\rangle[q_13]\langle 13\rangle + [q_13]\langle 1q_3\rangle[q_23]\langle 32\rangle}{[q_11][q_22]\langle q_33\rangle}$$
$$= i\frac{[q_13][q_23](\langle 2q_3\rangle\langle 13\rangle - \langle 1q_3\rangle\langle 23\rangle)}{[q_11][q_22]\langle q_33\rangle}$$
$$= i\frac{[q_13][q_23]\langle 3q_3\rangle\langle 21\rangle}{[q_11][q_22]\langle q_33\rangle},$$

where in the last step, we have used the Schouten identity (2.86). Now multiply the numerator and denominator by $\langle 32\rangle\langle 21\rangle$ to use the momentum conservation just like before. This is done because a scattering amplitude can't depend on reference momenta. Finally we obtain,

$$M(1^-,2^-,3^+) = i\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}. \tag{3.4}$$

As one can see, the expression depends only on angle brakets (like we stated before). Now, we know that reversing the sign of helicity interchanges angle and square brakets. Therefore, a parity reversal will give us

$$M(1^+,2^+,3^-) = -i\frac{[12]^3}{[23][31]}. \tag{3.5}$$

We could have calculated the same by using the other case of 3-point kinematics for this helicity configuration and proceeding just like before. Amplitude of the type (3.5) (two particles with negative helicity) are called ***Maximally Helicity Violating*** or **MHV** amplitudes and of the type (3.4) are called $\overline{\textbf{MHV}}$ amplitudes as they are related to the first by parity symmetry. To understand where this **MHV** term comes from, let's proceed to the next section.

3.2 The Parke-Taylor amplitudes

Consider scattering processes of the type where two gluons collide and scatter into (n-2) gluons. Now in the all outgoing convention, an incoming gluon is related to the outgoing gluon by crossing symmetry i.e., their states with opposite helicity are equal. This means that if we consider all n (≥ 4) particles to be outgoing, then the helicity configuration $M_n^{\text{tree}}(1^+,2^+,3^+,...,n^+)$ is basically equivalent to

$$1^- + 2^- \longrightarrow 3^+ + ... + n^+.$$

Clearly, helicity conservation is being violated here. Same can be said about the configuration $M_n^{\text{tree}}(1^+,2^+,3^-,...,n^+)$ or any configuration like this with the cyclic symmetry. However, amplitudes with such configurations actually vanish i.e.,

$$M_n^{\text{tree}}(1^\pm,2^+,3^+,...,n^+) = 0. \tag{3.6}$$

This happens because for n gluons, the Feynman diagram will contain at most (n-2) three-point vertices and as evident from Feynman rules - that will contribute (n-2) momenta to the expression for amplitude. And there will be n polarization vectors for each gluon. Thus there can be at most (n-2) contractions between momenta and polarizations vectors. This leaves at least one contraction of the form $\epsilon_i.\epsilon_j$. Now using equation (2.97) we can have terms of the form

$$\epsilon_i^+.\epsilon_j^+ \propto \langle q_i q_j \rangle\,, \quad \epsilon_i^-.\epsilon_j^- \propto [q_i q_j], \quad \epsilon_i^+.\epsilon_j^- \propto \langle q_i j \rangle\,[i q_j],$$

depending on the helicity configuration. If we choose all reference momenta to be equal then clearly $M_n^{\text{tree}}(1^+, 2^+, 3^+, ..., n^+)$ vanishes. Similarly for the other case when one of the helicity is flipped (Say for 1), choice of reference momenta can be $q_1 = k_2$ and $q_i = k_1$ for i > 1. And so the result (3.6) is proved. Now if one more helicity is flipped, we'll have configuration of the form $M_n^{\text{tree}}(1^+, 2^+, .., j^-, .., l^-, ..., n^+)$ which is equivalent to

$$1^- + 2^- \longrightarrow 3^+ + .. + j^- + .. + l^- + .. + n^+.$$

We called this type of configuration as the **MHV** amplitude in the previous section and rightly so because this configuration is where one can most violate the helicity and still obtain a non-vanishing amplitude whose general form was given by Parke and Taylor [PT86] in the year 1986,

$$M_{jl}^{\text{MHV}} \equiv M_n^{\text{tree}}(1^+, 2^+, .., j^-, .., l^-, ..., n^+) = i\frac{\langle jl \rangle^4}{\langle 12 \rangle \langle 23 \rangle ... \langle n1 \rangle}. \tag{3.7}$$

The beauty of this expression lies in its simplicity. All we need is just this one formula for writing n-gluon amplitudes with exactly two negative helicity gluons. Similarly for the configuration related to this state via parity, the Parke-Taylor $\overline{\textbf{MHV}}$ amplitude is given by

$$M_{jl}^{\overline{\text{MHV}}} \equiv M_n^{\text{tree}}(1^-, 2^-, .., j^+, .., l^+, ..., n^-) = -i\frac{[jl]^4}{[12][23]...[n1]}. \tag{3.8}$$

If we replace two of the gluons with a quark-antiquark pair, we can have **MHV** amplitudes similar to the Parke-Taylor amplitudes mentioned above. These are :

$$M_n^{\text{tree}}(1^+, 2^+, ..., j_{\bar{q}}^-, ..., k_q^+, ..., n^+) = 0, \tag{3.9}$$

$$M_n^{\text{tree}}(1^-, 2^+, ..., j_{\bar{q}}^-, ..., k_q^+, ..., n^+) = i\frac{\langle 1j \rangle^3 \langle 1k \rangle}{\langle 12 \rangle ... \langle n1 \rangle}. \tag{3.10}$$

These relations are, in fact, consistent with the results we obtained in the section 2.4. However, we'll see in the coming sections that these can be proved using the BCFW recursion technique that we'll learn next.

3.3 The recursion formula

In this section, we will follow the path of BCFW [FW05], to arrive at the recursion relation given by them in [CF05]. We will be mainly sticking to the notation used in reference [Dix13].

Consider a colour-ordered n-gluon tree-level amplitude $M_n^{\text{tree}}(z)$ that depends on a complex variable z. Notice that we have introduced a 'complex' parameter for the first time, the motivation behind which was given in section 3.1. This variable can be used to shift the on-shell momenta of external particles. BCFW used a special type of shift called the $[n,1\rangle$ shift that translates only two gluons legs' (n and 1) momenta. This particular shift is defined as follows :

$$\hat{\tilde{\lambda}}_n = \tilde{\lambda}_n - z\tilde{\lambda}_1, \qquad \hat{\lambda}_n = \lambda_n,$$
$$\hat{\lambda}_1 = \lambda_1 + z\lambda_n, \qquad \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1. \tag{3.11}$$

One can notice that only the right-handed spinor λ_1 and the left-handed spinor $\tilde{\lambda}_n$ has been shifted. The above equations can be written in terms of angle and square braket spinors as follows :

$$|\hat{n}] = |n] - z|1], \qquad |\hat{n}\rangle = |n\rangle\,,$$
$$|\hat{1}\rangle = |1\rangle + z\,|n\rangle\,, \qquad |\hat{1}] = |1]. \tag{3.12}$$

Using this, we can define the following useful spinor product relations :

$$\langle\hat{n}\hat{1}\rangle = \langle n1\rangle\,, \quad [\hat{n}\hat{1}] = [\hat{n}1] = [n1], \quad \langle\hat{n}1\rangle = \langle n1\rangle\,, \quad [\hat{1}n] = [1n]. \tag{3.13}$$

In terms of momenta variables, this shift can be expressed as

$$\hat{\not{k}}_1(z) = (\lambda_1 + z\lambda_n)\tilde{\lambda}_1 = \lambda_1\tilde{\lambda}_1 + z\lambda_n\tilde{\lambda}_1 = |1\rangle\,[1| + z\,|n\rangle\,[1|,$$
$$\hat{\not{k}}_n(z) = \lambda_n(\tilde{\lambda}_n - z\tilde{\lambda}_1) = \lambda_n\tilde{\lambda}_n - z\lambda_n\tilde{\lambda}_1 = |n\rangle\,[n| - z\,|n\rangle\,[1|. \tag{3.14}$$

It is clear from above that $\hat{\not{k}}_1(z) + \hat{\not{k}}_n(z) = \lambda_1\tilde{\lambda}_1 + \lambda_n\tilde{\lambda}_n = \not{k}_1 + \not{k}_n$. This implies that for any value of z, momentum is conserved. Now, one can say that the shifted momenta are not physical as they depend on a complex parameter. However, we can prove that they still are on-shell. Let's say, $\lambda_n\tilde{\lambda}_1 = \not{v}$ is the shifting momenta. Then,

$$(\hat{k}_1)^2 = (k_1 + zv)^2 = (k_1)^2 + 2zv.k_1 + v^2,$$
$$(\hat{k}_n)^2 = (k_n - zv)^2 = (k_n)^2 - 2zv.k_n + v^2. \tag{3.15}$$

Since k_1 and k_n are light-like on-shell momenta, the following conditions should be satisfied for shifted momenta to be on-shell as well,

$$v.k_1 = v.k_n = v^2 = 0 \tag{3.16}$$

One can always find such a v, by allowing it to be a complex vector. For instance, if we choose

$$v^\mu = \frac{1}{2}[1\gamma^\mu n\rangle, \tag{3.17}$$

then,

$$v.k_1 = \frac{1}{4}[1\gamma^\mu n\rangle[1\gamma_\mu 1\rangle \propto [11] = 0.$$

Similarly, other orthogonality relations in equation (3.16) are satisfied as well. Now that shifted momenta are on-shell and satisfy the momentum conservation, we can write the amplitude as a function of these complex momenta,

$$M_n(z) \equiv M_n(\hat{k}_1(z), k_2, ..., k_{n-1}, \hat{k}_n(z)). \tag{3.18}$$

For convenience, we have removed the superscript 'tree'. We can always calculate the physical amplitude by putting $z = 0$ i.e., $M_n(0) = M_n$ is the amplitude with unshifted momenta that we originally wish to calculate.

Now, consider a function of the form $\frac{M_n(z)}{z}$. It has an obvious pole at $z = 0$. Since we are talking about tree-level amplitudes, only poles will construct their singularity or analytic structure as stated before. More specifically, $M_n(z)$ will have only *simple* poles being contributed by propagators in the diagram; different propagators contributing poles at different values of z for generic external momenta, [FW05]. Let's look at our function in the complex plane. The residue of this function at $z = 0$ corresponds exactly to the unshifted physical amplitude. To calculate this, what we can do is consider the integral

$$\frac{1}{2\pi i}\oint_C \frac{M_n(z)}{z} dz, \tag{3.19}$$

where the contour C is a large circle with its origin at simple pole $z = 0$ and assume that the function has no poles at infinity so that $M_n(z) \to 0$ in the limit $z \longrightarrow \infty$. Then we can use this limit and Cauchy's residue theorem [Con20b] to write the integral as

$$\frac{1}{2\pi i}\oint_C \frac{M_n(z)}{z} dz = M_n(0) + \sum_i Res\left[\frac{M_n(z)}{z}\right]\Bigg|_{z=z_i} = 0, \tag{3.20}$$

$\Longrightarrow$

$$M_n = M_n(0) = -\sum_i Res\left[\frac{M_n(z)}{z}\right]\Bigg|_{z=z_i}, \tag{3.21}$$

where, z_i are the other poles contributed by the denominator (propagator) of the amplitude. We need to calculate residue at each such pole but before that we need to find the form of these poles. This can be done by using the factorization property of on-shell tree amplitudes that when such amplitudes factorize into sub-amplitudes with lower number of legs, the intermediate propagator becomes on-shell. Let's look at the denominator of this propagator which has a z dependence now because of the momenta shifts :

$$\begin{aligned}\hat{Q}^2_{1,i}(z_i) &= (\hat{k}_1(z_i) + k_2 + ... + k_i)^2 \\ &= (k_1 + z_i\lambda_n\tilde{\lambda}_1 + k_2 + ... + k_i)^2 \\ &= (Q_{1,i} + z_i\lambda_n\tilde{\lambda}_1)^2 \\ &= Q^2_{1,i} + z_i[1|\not{Q}_{1,i}|n\rangle, \end{aligned} \tag{3.22}$$

where in the last step, we have used the equations (3.16), (3.17) and (2.95). And since the shifted propagator is on-shell,

$\Longrightarrow$

$$Q^2_{1,i} + z_i[1|\not{Q}_{1,i}|n\rangle = 0,$$

or,

$$z_i = -\frac{Q^2_{1,i}}{[1|\not{Q}_{1,i}|n\rangle}. \tag{3.23}$$

Now that we have calculated the poles, next step is to calculate $Res\left[\frac{M_n(z)}{z}\right]$, where $M_n(z)$ can be factorised [Dix13] as ,

$$M_n(z) \xrightarrow{\hat{Q}^2_{1,i}\to 0} M_{i+1}(\hat{k}_1(z), k_2, ..., k_i, -\hat{Q}_{1,i})\frac{i}{\hat{Q}^2_{1,i}(z)}M_{n-i+1}(\hat{Q}_{1,i}, k_{i+1}, ..., k_{n-1}, \hat{k}_n(z)) \tag{3.24}$$

Then,

$$\begin{aligned} Res\left[\frac{M_n(z)}{z}\right]\Bigg|_{z=z_i} &= \mathrm{Lt}_{z\to z_i}\ (z - z_i)\left[\frac{M_n(z)}{z}\right] \\ &= \mathrm{Lt}_{z\to z_i}\ z + \frac{Q^2_{1,i}}{[1|\not{Q}_{1,i}|n\rangle}\left[M_{i+1}(z)\frac{i}{z\hat{Q}^2_{1,i}}M_{n-i+1}(z)\right] \\ &= \mathrm{Lt}_{z\to z_i}\ \frac{z[1|\not{Q}_{1,i}|n\rangle + Q^2_{1,i}}{[1|\not{Q}_{1,i}|n\rangle}\left[M_{i+1}(z)\frac{i}{(z[1|\not{Q}_{1,i}|n\rangle + Q^2_{1,i})z}M_{n-i+1}(z)\right] \\ &= M_{i+1}(z_i)\frac{i}{z_i[1|\not{Q}_{1,i}|n\rangle}M_{n-i+1}(z_i) \\ &= M_{i+1}(z_i)\frac{-i}{Q^2_{1,i}}M_{n-i+1}(z_i). \end{aligned} \tag{3.25}$$

Substituting back equation (3.25) into (3.21), we finally obtain the desired relation,

$$M_n^{\mathrm{tree}}(k_1, ..., k_n) = \sum_{\lambda=\pm}\sum_{i=2}^{n-2} M_{i+1}(\hat{k}_1, k_2, ..., k_i, -\hat{Q}^{-\lambda}_{1,i})\frac{i}{Q^2_{1,i}}M_{n-i+1}(\hat{Q}^{\lambda}_{1,i}, k_{i+1}, ..., k_{n-1}, \hat{k}_n). \tag{3.26}$$

This equation, evaluated at the pole value ($z = z_i$) is known as the BCFW recursion relation [CF05]. Notice that there is a sum over helicity of the intermediate on-shell particle. Since, all particles are considered to be outgoing, helicity appearing on either side of the factorization pole are opposite in signs. The other sum is over all cyclically ordered configurations of gluons with at least two gluons on each sub-amplitude. This implies that the lowest sub-amplitude possible is a 3-point amplitude. So when asked to calculate a colour-ordered n-gluon tree amplitude, we can keep applying this recursion relation until we reach at the basic building block i.e., 3-point amplitude.

One last thing that is important to discuss is the assumption that we made in the derivation of relation (3.26) i.e., $M_n(z) \to 0$ as $z \to \infty$. What conditions will allow us to make this assumption? Actually, there's a prescription for it; an overview of which is discussed here. For a rigorous proof of the same, one can have a look at [FW05] and [AHK08]. The amplitude vanishes for large value of z when the helicity configuration for our choice of shift $[n1\rangle$, is $(-,+), (-,-)$ or $(+,+)$. This is because for these choices, the amplitude contributes factors of the $\mathcal{O}(\frac{1}{z})$ while, a contribution of $\mathcal{O}(z^3)$ comes from the choice $(+,-)$. This makes the amplitude diverging and therefore, this configuration isn't a suitable choice for the BCFW relation. Having derived the recursion relation, let's move on to see some of its applications - the first of which would be to derive the Parke-Taylor amplitudes mentioned in section 3.2.

3.4 Proof of Parke-Taylor amplitude by induction

Consider the Parke-Taylor **MHV** amplitude given in equation (3.7),

$$M_{jl}^{\text{MHV}} \equiv M_n^{\text{tree}}(1^+, 2^+, .., j^-, .., l^-, ..., n^+) = i\frac{\langle jl\rangle^4}{\langle 12\rangle\langle 23\rangle ... \langle n1\rangle}.$$

We can shift one of the negative helicity to n^{th} using the cyclic property so that we can stick with our $[n1\rangle \equiv [-+\rangle$ shift. This gives,

$$M_{jn}^{\text{MHV}} \equiv M_n^{\text{tree}}(1^+, 2^+, .., j^-, .., (n-1)^+, n^-) = i\frac{\langle jn\rangle^4}{\langle 12\rangle\langle 23\rangle ... \langle n1\rangle}. \tag{3.27}$$

We aim to prove this relation by induction using the BCFW recursion formula (3.26), which for our convenience can be written in short-hand notation as,

$$M_n^{\text{tree}}(1, 2, ..., n) = \sum_{\lambda=\pm}\sum_{i=2}^{n-2} M_{i+1}(\hat{1}, 2, ..., i, -\hat{Q}_{1,i}^{-\lambda})\frac{i}{Q_{1,i}^2}M_{n-i+1}(\hat{Q}_{1,i}^{\lambda}, i+1, ..., n-1, \hat{n}). \tag{3.28}$$

Now, because of the opposite helicity of $\hat{Q}$ on either side of the propagator, we have a maximum of three negative helicities to be distributed among the two sub-amplitudes. However, for getting a non-vanishing amplitude, at least two gluons should have negative helicity (see section 3.2). This isn't possible when $3 \leq i \leq n-3$ as we'll always have at least one sub-amplitude of the helicity configuration - all plus or all but one minus, which will make the whole term disappear. Therefore, all these middle terms would become zero and we are left with two cases :

- $\underline{i = n-2}$

  For this case, equation (3.28) will become,

$$M_n^{\text{tree}}(1,2,...,n) = \sum_{\lambda=\pm} M_{n-1}(\hat{1},2,...,n-2,-\hat{Q}_{1,n-2}^{-\lambda}) \frac{i}{Q_{1,n-2}^2} M_3(\hat{Q}_{1,n-2}^{\lambda}, n-1, \hat{n}). \tag{3.29}$$

  Now, since n^{th} gluon has one of the two negative helicities, this means that if $j = n-1$, then M_3 will have both the negative helicities and M_{n-1} can only have at most one negative helicity depending upon the helicity of $\hat{Q}$ and so it vanishes. If $j < n-1$, then M_{n-1} will survive only for $\lambda = +$. This implies that we have a three-point amplitude of the form $M_3(+,+,-)$ which from equation (3.5) has an anti-holomorphic form. We saw in section 3.1 that this would be the result of three-point kinematics (3.3). However, since we have shifted the left-handed spinor $[n|$, the 3-point kinematics (3.2) should be followed for this case i.e., all left-handed spinors are proportional to each other. In that scenario though, this anti-MHV three-point configuration M_3 vanishes.

  Either case, $i = n-2$ term in the summation vanishes as well and we are left with only one possibility for i which should give us a non-vanishing result.

- $\underline{i = 2}$

  For this case, equation (3.28) will become,

$$M_n^{\text{tree}}(1,2,...,n) = \sum_{\lambda=\pm} M_3(\hat{1},2,-\hat{Q}_{1,2}^{-\lambda}) \frac{i}{Q_{1,2}^2} M_{n-1}(\hat{Q}_{1,2}^{\lambda}, 3, ..., n-1, \hat{n}). \tag{3.30}$$

  Now, for $j > 2$ and $\lambda = +$, we'll have the following 3-point amplitude,

$$\begin{aligned} M_3(\hat{1}^+, 2^+, -\hat{Q}_{1,2}^-) &= -i \frac{[\hat{1}2]^3}{[2-\hat{Q}_{1,2}][-\hat{Q}_{1,2}\hat{1}]} \\ &= i \frac{[12]^3}{[2\hat{Q}_{1,2}][\hat{Q}_{1,2}1]}. \end{aligned} \tag{3.31}$$

where in the last step, we have used equations (2.83) and (3.12). Now, $\hat{Q}_{1,2} = \hat{k}_1 + k_2$. So,

$$\hat{Q}^2_{1,2} = 0 = 2(\hat{k}_1.k_2) = \langle \hat{1}2 \rangle [2\hat{1}] = \langle \hat{1}2 \rangle [21], \tag{3.32}$$

$\implies$

$$\langle \hat{1}2 \rangle = 0.$$

Then, following the 3-point kinematics (3.3), we have $\langle \hat{1}2 \rangle = \langle 2\hat{Q} \rangle = \langle \hat{Q}\hat{1} \rangle = 0$ and corresponding square products are non-zero. Therefore, M_3 survives. Now, we have to compute M_{n-1} which we will do by induction.

Say, $n = 4$. Then, equation (3.30) becomes,

$$\begin{aligned} M_4^{\text{tree}}(1^+, 2^+, 3^-, 4^-) &= M_3(\hat{1}^+, 2^+, -\hat{Q}^-_{1,2}) \frac{i}{Q^2_{1,2}} M_3(\hat{Q}^+_{1,2}, 3^-, \hat{4}^-) \\ &= -i \frac{[12]^3}{[2\hat{Q}][\hat{Q}1]} \frac{1}{s_{12}} \frac{\langle 34 \rangle^3}{\langle 4\hat{Q} \rangle \langle \hat{Q}3 \rangle} \\ &= -i \frac{[12]^3}{\langle 4\hat{Q} \rangle [\hat{Q}2]} \frac{1}{s_{12}} \frac{\langle 34 \rangle^3}{\langle 3\hat{Q} \rangle [\hat{Q}1]}, \end{aligned} \tag{3.33}$$

now from equation (3.14),

$$\begin{aligned} \hat{\not{Q}}_{1,2}(z_2) &= \not{k}_1 + \not{k}_2 + z_2 \lambda_n \tilde{\lambda}_1 \\ &= |1\rangle [1| + |2\rangle [2| + z_2 |n\rangle [1|, \end{aligned} \tag{3.34}$$

$\implies$

$$\begin{aligned} \langle 4\hat{Q} \rangle [\hat{Q}2] = \langle 4 \hat{\not{Q}}_{1,2} 2] = \langle 4|\not{k}_1 + \not{k}_2|2\rangle + z_2 \langle 44 \rangle [12] = \langle 4|\not{k}_1 + \not{k}_2|2\rangle \\ = \langle 41 \rangle [12] + \langle 42 \rangle [22] = \langle 41 \rangle [12], \end{aligned} \tag{3.35}$$

similarly,

$$\langle 3\hat{Q} \rangle [\hat{Q}1] = \langle 32 \rangle [21]. \tag{3.36}$$

Using these and $s_{12} = \langle 12 \rangle [21]$, equation (3.33) can be modified as

$$\begin{aligned} M_4^{\text{tree}}(1^+, 2^+, 3^-, 4^-) &= -i \frac{[12]^3 \langle 34 \rangle^3}{\langle 41 \rangle [12] \langle 12 \rangle [21] \langle 32 \rangle [21]} \\ &= i \frac{\langle 34 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}. \end{aligned} \tag{3.37}$$

Thus, one can see that the relation (3.27) is true for $n = 4$ gluons. Let's assume that it is true for $4 < k < n$ gluons. Then equation (3.30) can be written as,

$$\begin{aligned} M_{jn}^{\text{MHV}} &= M_3(\hat{1}^+, 2^+, -\hat{Q}_{1,2}^-)\frac{i}{Q_{1,2}^2}M_{n-1}(\hat{Q}_{1,2}^+, 3^+, ..., j^-, ..., \hat{n}^-) \\ &= \frac{i[12]^3}{[2\hat{Q}_{1,2}][\hat{Q}_{1,2}1]}\frac{i}{s_{12}}\frac{i\langle j\hat{n}\rangle^4}{\langle\hat{Q}3\rangle\langle 34\rangle ... \langle n-1,\hat{n}\rangle\langle\hat{n}\hat{Q}\rangle} \\ &= \frac{i[12]^3}{[2\hat{Q}_{1,2}][\hat{Q}_{1,2}1]}\frac{i}{s_{12}}\frac{i\langle jn\rangle^4}{\langle\hat{Q}3\rangle\langle 34\rangle ... \langle n-1,n\rangle\langle n\hat{Q}\rangle} \\ &= -i\frac{[12]^3\langle jn\rangle^4}{\langle 3\hat{Q}\rangle[\hat{Q}1]s_{12}\langle 34\rangle ... \langle n-1,n\rangle\langle n\hat{Q}\rangle\langle\hat{Q}2\rangle\langle 12\rangle[21]}. \end{aligned} \tag{3.38}$$

From (3.34),

$$\begin{aligned} \langle 3\hat{Q}\rangle[\hat{Q}1] &= \langle 32\rangle[21], \\ \langle n\hat{Q}\rangle[\hat{Q}2] &= \langle n1\rangle[12]. \end{aligned} \tag{3.39}$$

Putting these back in equation (3.38), we obtain

$$M_{jn}^{\text{MHV}} = -i\frac{[12]^3\langle jn\rangle^4}{\langle 32\rangle[21]\langle 34\rangle ... \langle n-1,n\rangle\langle n1\rangle\langle 12\rangle}$$

Rearranging, (3.40)

$$= i\frac{\langle jn\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle ... \langle n1\rangle}. \tag{3.41}$$

This completes our induction proof of the Parke-Taylor relation (3.27) for n gluon legs. One can notice how the use of BCFW recursion formula made the proof so simple and neat. Actually the choice of shift for a particular calculation can make the computation simpler, even though each shift would correspond to the same result. In the next section, we'll use the recursion technique to prove the Parke-Taylor like MHV amplitude formula (3.10), following the same procedure but with different choice of reference gluons.

3.5 Proof of MHV amplitude for $q\bar{q} + (n-2)$gluons

Consider the relation (3.10),

$$M_{1j}^{\text{MHV}} \equiv M_n^{\text{tree}}(1^-, 2^+, ..., j_{\bar{q}}^-, ..., k_q^+, ..., n^+) = i\frac{\langle 1j\rangle^3\langle 1k\rangle}{\langle 12\rangle ... \langle n1\rangle}.$$

Once again, we aim to prove this relation by induction using the BCFW recursion relation. For convenience, $n > j > 3$ and we choose our reference gluons to be 1 and 2, so that the shift $[1^-, 2^+\rangle$ is done as follows :

$$|\hat{1}] = |1] - z|2], \qquad |\hat{1}\rangle = |1\rangle\,,$$

$$|\hat{2}\rangle = |2\rangle + z\,|1\rangle\,, \qquad |\hat{2}] = |2]. \tag{3.42}$$

Adapting from equation (3.28), for this particular choice, the relation can be modified as :

$$M_n^{\text{tree}}(1,2,...,n) = \sum_{\lambda=\pm}\sum_{i=4}^{n} M_{n-i+3}(\hat{1},\hat{Q}^{\lambda}_{2,i-1},i,...,n)\frac{i}{Q^2_{2,i-1}}M_{i-1}(-\hat{Q}^{-\lambda}_{2,i-1},\hat{2},3,...,i{-}1). \tag{3.43}$$

We can solve equation (3.43) by dividing it into following three cases :

- **$\underline{4 < i < n}$** :
  All these middle terms vanish because for each i in this case, we'll have at least one sub-amplitude of the form (3.10) or (3.6).

- **$\underline{i = n}$** :
  For $\lambda = -$, $M_{n-1}(+,+,..,-,..,+) = 0$. So this term vanishes. For $\lambda = +$, we'll have a 3-point sub-amplitude,

  $$M_3(\hat{1}^-,\hat{Q}^+,n^+) = -i\frac{[\hat{Q}n]^3}{[\hat{1}\hat{Q}][n\hat{1}]}. \tag{3.44}$$

  Now, since we have shifted the left handed spinor $[1|$, going by the same logic we used in previous section, this term would become zero as well.

- $\underline{i = 4}$: For this case, equation (3.43) can be written as,

  $$M_n^{\text{tree}}(1,2,...,n) = \sum_{\lambda=\pm} M_{n-1}(\hat{1}^-,\hat{Q}^{\lambda}_{2,3},4^+,..,j^-,..,k^+,..,n^+)\frac{i}{Q^2_{2,3}}M_3(-\hat{Q}^{-\lambda}_{2,3},\hat{2}^+,3^+). \tag{3.45}$$

  For $\lambda = -$, $M_3(+,+,+) = 0$. So, other case must surive now for a non-vanishing amplitude. For $\lambda = +$, we get

  $$M_n^{\text{tree}}(1,2,...,n) = M_{n-1}(\hat{1}^-,\hat{Q}^{+}_{2,3},4^+,..,j^-,..,k^+,..,n^+)\frac{i}{Q^2_{2,3}}M_3(-\hat{Q}^{-}_{2,3},\hat{2}^+,3^+). \tag{3.46}$$

Now, we can proceed like we did in previous section and apply induction to this using relation (3.10):

$$M_{1j}^{\text{MHV}} = \frac{i\left\langle \hat{1}j\right\rangle^3 \langle \hat{1}k\rangle}{\langle \hat{1}\hat{Q}\rangle \langle \hat{Q}4\rangle \ldots \langle n\hat{1}\rangle} \frac{i}{\langle 23\rangle [32]} \frac{i[\hat{2}3]^3}{[3\hat{Q}][\hat{Q}\hat{2}]}$$
$$= -i\frac{\left\langle 1j\right\rangle^3 \langle 1k\rangle}{\langle 1\hat{Q}\rangle \langle \hat{Q}4\rangle \ldots \langle n1\rangle} \frac{1}{\langle 23\rangle [32]} \frac{[23]^3}{[3\hat{Q}][\hat{Q}2]}, \tag{3.47}$$

where, we have used relation (3.42) for dealing with hats. Now using equation (3.14),

$$\hat{\not{Q}}_{2,3}(z_4) = \not{k}_2 + \not{k}_3 + z_4\lambda_1\tilde{\lambda}_2$$
$$= |2\rangle\,[2| + |3\rangle\,[3| + z_4\,|1\rangle\,[2|, \tag{3.48}$$

$\implies$

$$\langle 4\hat{Q}\rangle\,[\hat{Q}2] = \langle 43\rangle\,[32], \tag{3.49}$$

$$\langle 1\hat{Q}\rangle\,[\hat{Q}3] = \langle 12\rangle\,[23]. \tag{3.50}$$

Using these equations and rearranging, equation (3.47) becomes,

$$M_{1j}^{\text{MHV}} = i\frac{\left\langle 1j\right\rangle^3 \langle 1k\rangle\,[23]^3}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \ldots \langle n1\rangle\,[32]^2[23]}$$
$$= i\frac{\left\langle 1j\right\rangle^3 \langle 1k\rangle}{\langle 12\rangle \langle 23\rangle \ldots \langle n1\rangle}. \tag{3.51}$$

This completes our proof, by induction, of the **MHV** gluon amplitude with an external pair of quark and anti-quark.

Chapter 4

Application To Diphoton Production At Hadron Colliders

Di-photon production via quark-antiquark annihilation ($\bar{q} + q \to \gamma\gamma$) is one of the most important processes at the hadron colliders. Due to clean final states, the process is a good observable for probing new physics beyond the standard model. A precise theoretical prediction for the process in the standard model is required in order to compare it with the experimental data. In perturbative QCD, precise theoretical predictions can be obtained by calculating higher order QCD corrections. A part of these QCD corrections are obtained by considering real gluon emission processes like: $\bar{q}q \to g\gamma\gamma, gg\gamma\gamma$ etc. In this chapter we will apply the tree level techniques discussed in the previous chapter to calculate scattering amplitude for one gluon emission in di-photon production, and compare it numerically with calculation using publicly available tools.

4.1 One gluon real correction

Consider the process :

$$\bar{q}(k_1) + q(k_2) \longrightarrow g(k_3, a) + \gamma(k_4) + \gamma(k_5)$$

We wish to calculate the square of the full scattering amplitude of this process. We won't be considering any Feynman diagrams here and will just use the smart things that we have discussed so far. One can see that right hand side of the above process contains two photons along with a gluon. Since photon is a colourless particle and we deal with colour-stripped partial amplitudes, we can just treat these photons like any other gluon for the sake of calculating partial amplitudes. So basically, our process just becomes a process with 3 gluons and a quark-anti quark pair. Thus, we can use equation (2.62), to write the full amplitude $\mathcal{M}_5^{\text{tree}}(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}, 5^{\lambda_5})$ for this process as,

$$\begin{aligned}\mathcal{M}_5^{\text{tree}} &= g_s^3 \sum_{\sigma \in S_3} (T^{a_{\sigma(3)}} T^{a_{\sigma(4)}} T^{a_{\sigma(5)}})^i_j M_5^{\text{tree}}(1_{\bar{q}}^{\lambda_1}, 2_q^{\lambda_2}, \sigma_g(3^{\lambda_3}), \sigma_\gamma(4^{\lambda_4}), \sigma_\gamma(5^{\lambda_5})) \\ &= g_s^3 [(T^a T^b T^c)^i_j M_w(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}, 5^{\lambda_5}) + (T^a T^c T^b)^i_j M_x(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 5^{\lambda_5}, 4^{\lambda_4}) \\ &\quad + (T^b T^a T^c)^i_j M_y(1^{\lambda_1}, 2^{\lambda_2}, 4^{\lambda_4}, 3^{\lambda_3}, 5^{\lambda_5}) + (T^c T^b T^a)^i_j M_z(1^{\lambda_1}, 2^{\lambda_2}, 5^{\lambda_5}, 4^{\lambda_4}, 3^{\lambda_3}) \\ &\quad + (T^b T^c T^a)^i_j M_u(1^{\lambda_1}, 2^{\lambda_2}, 4^{\lambda_4}, 5^{\lambda_5}, 3^{\lambda_3}) + (T^c T^a T^b)^i_j M_v(1^{\lambda_1}, 2^{\lambda_2}, 5^{\lambda_5}, 3^{\lambda_3}, 4^{\lambda_4})]\end{aligned} \tag{4.1}$$

Then using the prescription, given after equation (2.62) for processes involving photons, the above equation can be modified as

$$\begin{aligned}\mathcal{M}_5^{tree} &= 2Q_q^2 e^2 g_s \ (T^a)^i_j \ [M_w(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 4^{\lambda_4}, 5^{\lambda_5}) + M_x(1^{\lambda_1}, 2^{\lambda_2}, 3^{\lambda_3}, 5^{\lambda_5}, 4^{\lambda_4}) \\ &\quad + M_y(1^{\lambda_1}, 2^{\lambda_2}, 4^{\lambda_4}, 3^{\lambda_3}, 5^{\lambda_5}) + M_z(1^{\lambda_1}, 2^{\lambda_2}, 5^{\lambda_5}, 4^{\lambda_4}, 3^{\lambda_3}) \\ &\quad + M_u(1^{\lambda_1}, 2^{\lambda_2}, 4^{\lambda_4}, 5^{\lambda_5}, 3^{\lambda_3}) + M_v(1^{\lambda_1}, 2^{\lambda_2}, 5^{\lambda_5}, 3^{\lambda_3}, 4^{\lambda_4})].\end{aligned} \tag{4.2}$$

Now, let's calculate these partial amplitudes . Since five particles are involved, in principle, we can have $2^5 = 32$ possible helicity configurations. However, as we saw in section 2.4, quark and anti-quark can't have the same helicity. Thus, half of the configurations would vanish. Also, from equation (3.9), the following configurations would vanish :

$$\begin{aligned}&M_s(1^-, 2^+, 3^+, 4^+, 5^+) = 0, \quad M_s(1^+, 2^-, 3^+, 4^+, 5^+) = 0, \\ &M_s(1^-, 2^+, 3^-, 4^-, 5^-) = 0, \quad M_s(1^+, 2^-, 3^-, 4^-, 5^-) = 0,\end{aligned} \tag{4.3}$$

where, $s = \{w, x, y, z, u, v\}$. So, we are left with 12 configurations which would be non-vanishing. Since parity is a symmetry of both QED and QCD, we have only six independent configurations. These configurations can be obtained using the result for MHV amplitude given in (3.10) via permutation. Let's write down these independent configurations one by one.

- $$\begin{aligned}M_w(1^-, 2^+, 3^-, 4^+, 5^+) &= i\frac{\langle 13\rangle^3 \langle 23\rangle}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 45\rangle \langle 51\rangle}, \\ M_x(1^-, 2^+, 3^-, 5^+, 4^+) &= i\frac{\langle 13\rangle^3 \langle 23\rangle}{\langle 12\rangle \langle 23\rangle \langle 35\rangle \langle 54\rangle \langle 41\rangle}, \\ M_y(1^-, 2^+, 4^+, 3^-, 5^+) &= i\frac{\langle 13\rangle^3 \langle 23\rangle}{\langle 12\rangle \langle 24\rangle \langle 43\rangle \langle 35\rangle \langle 51\rangle},\end{aligned}$$

$$M_z(1^-,2^+,5^+,4^+,3^-) = i\frac{\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 25\rangle\langle 54\rangle\langle 43\rangle\langle 31\rangle},$$

$$M_u(1^-,2^+,4^+,5^+,3^-) = i\frac{\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 24\rangle\langle 45\rangle\langle 53\rangle\langle 31\rangle},$$

$$M_v(1^-,2^+,5^+,3^-,4^+) = i\frac{\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 25\rangle\langle 53\rangle\langle 34\rangle\langle 41\rangle}. \tag{4.4}$$

•

$$M_w(1^-,2^+,3^+,4^-,5^+) = i\frac{\langle 14\rangle^3\langle 24\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle},$$

$$M_x(1^-,2^+,3^+,5^+,4^-) = i\frac{\langle 14\rangle^3\langle 24\rangle}{\langle 12\rangle\langle 23\rangle\langle 35\rangle\langle 54\rangle\langle 41\rangle},$$

$$M_y(1^-,2^+,4^-,3^+,5^+) = i\frac{\langle 14\rangle^3\langle 24\rangle}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 35\rangle\langle 51\rangle},$$

$$M_z(1^-,2^+,5^+,4^-,3^+) = i\frac{\langle 14\rangle^3\langle 24\rangle}{\langle 12\rangle\langle 25\rangle\langle 54\rangle\langle 43\rangle\langle 31\rangle},$$

$$M_u(1^-,2^+,4^-,5^+,3^+) = i\frac{\langle 14\rangle^3\langle 24\rangle}{\langle 12\rangle\langle 24\rangle\langle 45\rangle\langle 53\rangle\langle 31\rangle},$$

$$M_v(1^-,2^+,5^+,3^+,4^-) = i\frac{\langle 14\rangle^3\langle 24\rangle}{\langle 12\rangle\langle 25\rangle\langle 53\rangle\langle 34\rangle\langle 41\rangle}. \tag{4.5}$$

•

$$M_w(1^-,2^+,3^+,4^+,5^-) = i\frac{\langle 15\rangle^3\langle 25\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle},$$

$$M_x(1^-,2^+,3^+,5^-,4^+) = i\frac{\langle 15\rangle^3\langle 25\rangle}{\langle 12\rangle\langle 23\rangle\langle 35\rangle\langle 54\rangle\langle 41\rangle},$$

$$M_y(1^-,2^+,4^+,3^+,5^-) = i\frac{\langle 15\rangle^3\langle 25\rangle}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 35\rangle\langle 51\rangle},$$

$$M_z(1^-,2^+,5^-,4^+,3^+) = i\frac{\langle 15\rangle^3\langle 25\rangle}{\langle 12\rangle\langle 25\rangle\langle 54\rangle\langle 43\rangle\langle 31\rangle},$$

$$M_u(1^-,2^+,4^+,5^-,3^+) = i\frac{\langle 15\rangle^3\langle 25\rangle}{\langle 12\rangle\langle 24\rangle\langle 45\rangle\langle 53\rangle\langle 31\rangle},$$

$$M_v(1^-,2^+,5^-,3^+,4^+) = i\frac{\langle 15\rangle^3\langle 25\rangle}{\langle 12\rangle\langle 25\rangle\langle 53\rangle\langle 34\rangle\langle 41\rangle}. \tag{4.6}$$

•

$$M_w(1^+,2^-,3^-,4^+,5^+) = i\frac{\langle 23\rangle^3\langle 13\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle},$$

$$M_x(1^+,2^-,3^-,5^+,4^+) = i\frac{\langle 23\rangle^3\langle 13\rangle}{\langle 12\rangle\langle 23\rangle\langle 35\rangle\langle 54\rangle\langle 41\rangle},$$

$$M_y(1^+,2^-,4^+,3^-,5^+) = i\frac{\langle 23\rangle^3\langle 13\rangle}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 35\rangle\langle 51\rangle},$$

$$M_z(1^+,2^-,5^+,4^+,3^-) = i\frac{\langle 23\rangle^3\langle 13\rangle}{\langle 12\rangle\langle 25\rangle\langle 54\rangle\langle 43\rangle\langle 31\rangle},$$

$$M_u(1^+,2^-,4^+,5^+,3^-) = i\frac{\langle 23\rangle^3\langle 13\rangle}{\langle 12\rangle\langle 24\rangle\langle 45\rangle\langle 53\rangle\langle 31\rangle},$$

$$M_v(1^+,2^-,5^+,3^-,4^+) = i\frac{\langle 23\rangle^3\langle 13\rangle}{\langle 12\rangle\langle 25\rangle\langle 53\rangle\langle 34\rangle\langle 41\rangle}. \tag{4.7}$$

•

$$M_w(1^+,2^-,3^+,4^-,5^+) = i\frac{\langle 24\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle},$$

$$M_x(1^+,2^-,3^+,5^+,4^-) = i\frac{\langle 24\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 23\rangle\langle 35\rangle\langle 54\rangle\langle 41\rangle},$$

$$M_y(1^+,2^-,4^-,3^+,5^+) = i\frac{\langle 24\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 35\rangle\langle 51\rangle},$$

$$M_z(1^+,2^-,5^+,4^-,3^+) = i\frac{\langle 24\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 25\rangle\langle 54\rangle\langle 43\rangle\langle 31\rangle},$$

$$M_u(1^+,2^-,4^-,5^+,3^+) = i\frac{\langle 24\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 24\rangle\langle 45\rangle\langle 53\rangle\langle 31\rangle},$$

$$M_v(1^+,2^-,5^+,3^+,4^-) = i\frac{\langle 24\rangle^3\langle 14\rangle}{\langle 12\rangle\langle 25\rangle\langle 53\rangle\langle 34\rangle\langle 41\rangle}. \tag{4.8}$$

•

$$M_w(1^+,2^-,3^+,4^+,5^-) = i\frac{\langle 25\rangle^3\langle 15\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle},$$

$$M_x(1^+,2^-,3^+,5^-,4^+) = i\frac{\langle 25\rangle^3\langle 15\rangle}{\langle 12\rangle\langle 23\rangle\langle 35\rangle\langle 54\rangle\langle 41\rangle},$$

$$M_y(1^+,2^-,4^+,3^+,5^-) = i\frac{\langle 25\rangle^3\langle 15\rangle}{\langle 12\rangle\langle 24\rangle\langle 43\rangle\langle 35\rangle\langle 51\rangle},$$

$$M_z(1^+,2^-,5^-,4^+,3^+) = i\frac{\langle 25\rangle^3\langle 15\rangle}{\langle 12\rangle\langle 25\rangle\langle 54\rangle\langle 43\rangle\langle 31\rangle},$$

$$M_u(1^+,2^-,4^+,5^-,3^+) = i\frac{\langle 25\rangle^3\langle 15\rangle}{\langle 12\rangle\langle 24\rangle\langle 45\rangle\langle 53\rangle\langle 31\rangle},$$

$$M_v(1^+,2^-,5^-,3^+,4^+) = i\frac{\langle 25\rangle^3\langle 15\rangle}{\langle 12\rangle\langle 25\rangle\langle 53\rangle\langle 34\rangle\langle 41\rangle}. \tag{4.9}$$

The other six non-vanishing configurations, as mentioned before, are related to these by parity symmetry and can just be obtained by replacing $\langle ij\rangle$ by $-\ [ij]$.

So, now that we have all the non-vanishing amplitudes in hand, we can calculate the square of full amplitude for a particular helicity configuration using equation (4.1) and then we can add all the helicity configurations summed over final colours and averaged over initial colours and spin,

$$\begin{aligned}\mathbf{R} = \overline{|\mathcal{M}|^2}_{\text{Total}} = \sum_{\text{Helicity}} \frac{1}{2}\frac{8}{9}Q_q^4 e^4 g_s^2\ |&M_w(1^{\lambda_1},2^{\lambda_2},3^{\lambda_3},4^{\lambda_4},5^{\lambda_5}) + M_x(1^{\lambda_1},2^{\lambda_2},3^{\lambda_3},5^{\lambda_5},4^{\lambda_4})\\ &+ M_y(1^{\lambda_1},2^{\lambda_2},4^{\lambda_4},3^{\lambda_3},5^{\lambda_5}) + M_z(1^{\lambda_1},2^{\lambda_2},5^{\lambda_5},4^{\lambda_4},3^{\lambda_3})\\ &+ M_u(1^{\lambda_1},2^{\lambda_2},4^{\lambda_4},5^{\lambda_5},3^{\lambda_3}) + M_v(1^{\lambda_1},2^{\lambda_2},5^{\lambda_5},3^{\lambda_3},4^{\lambda_4})|^2.\end{aligned} \tag{4.10}$$

The factor 1/2! signifies takes care of the fact that the two photons are identical.

We will calculate this numerically for which I have developed a C++ code that can evaluate each partial amplitude helicity configuration at a given phase-space point (see Table : 4.1). Let's have a look at this numerical evaluation process.

4.1.1 Numerical evaluation

The code that I have written aims to calculate all the required angular braket inner products at a given phase-space point and uses these values to evaluate all the non-vanishing helicity configurations (equations (4.4) - (4.9)). It further calculates the sum of all the partial amplitudes appearing in equation (4.2) at a particular helicity configuration and then calculate the square of this sum for each non-vanishing helicity configuration. Finally it performs the sum over all helicities to give the desired output (4.10).

n	E	P_x	P_y	P_z
1	500.00000000000000	353.55339059327372	0.0000000000000000	353.55339059327372
2	500.00000000000000	-353.55339059327372	0.0000000000000000	-353.55339059327372
3	458.57878788544019	-17.005390826698950	379.65366207819869	-256.64843317644414
4	364.06662073681770	62.239353947232971	-347.70430131936706	88.161703698191019
5	177.35459137774205	-45.233963120534021	-31.949360758831560	168.48672947825312

Table 4.1: **The Test Phase-Space Point**

After obtaining the result numerically, we want to cross-check our result with the available result from publicly available tools. The tool that we have used as the reference is **MadGraph5** (MG5), [AFF+14] and the Table : 4.2 shows the result obtained by our numerical computation and already available result from **MadGraph5** for comparison.

Tool	**R**
MG5	$5.7319406091124580 \times 10^{-7}$ GeV^{-2}
My Code	$5.7319406091124590 \times 10^{-7}$ GeV^{-2}

Table 4.2: **Comparison of Matrix element (R) for the phase space points shown in Table : 4.1**

As we can see we have an excellent agreement between the two calculations. This allows us to calculate the total and differential cross-section for diphoton production with one gluon emission.

Chapter 5

Conclusion and Outlook

In this thesis we have barely managed to touch the top layer of modern techniques that are being used to calculate scattering amplitudes. This is because we have restricted ourselves to tree-level techniques only. In section 2.4 we saw an example where we applied the trace-based colour decomposition and spinor-helicity formalism to calculate the differential cross-section. We found out the exact same result (2.107) as obtained through traditional Feynman diagrammatic approach in many textbooks. However we realised through the calculation process that these modern methods are more efficient, specially when more number of external legs are involved. The main highlight of this thesis is the application of BCFW recursion technique and MHV amplitudes to calculate one-gluon real correction to diphoton production at hadron colliders (section 4.1). We have gone beyond the analytic calculation of partial amplitudes and have calculated the matrix element for the process numerically. We have made internal consistency check on our numerical calculation. Our calculation of matrix element has been compared with the results obtained from publicly available tool for calculating scattering amplitude and an excellent agreement has been found between the two.

We can further extend our code to calculate matrix element for two gluon real correction to diphoton production at hadron colliders. In perturbative QCD the two gluon emission process contributes to **Next to Next to Leading Order (NNLO)** correction. This process has 6 particles and so one has to deal with **Next to MHV** (NMHV) amplitudes as well. This isn't a trivial exercise but people have calculated all-gluon NMHV amplitudes and so this should be possible too. It would be an interesting exercise to see the efficiency of BCFW recursion technique while dealing with more or any number of external legs.

Further we would like to bring virtual corrections into the picture by looking at the application of these techniques at loop-level. Also, since we have restricted ourselves to massless particles only, we would like to apply these techniques to ongoing and future high energy collider experiments that mainly involves EW and QCD processes i.e., interactions involving **massive** fermions and gauge bosons.

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