Envariance, Born's Rule and Inferences from Correlations

By

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Certificate of Examination

This is to certify that the dissertation titled "**Envariance, Born's Rule and Inferences from Correlations**" submitted by **Pushpinder Singh** (Reg.No. MS15070) for the partial fulfilment of BS-MS dual degree programme of the Institute has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: June 15, 2020

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Manabendra Nath Bera at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Manabandra Nath Bera

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Abstract

The thesis is divided into two main chapters, which are independent pieces of research and establish different results, separately.
In the first chapter, we discuss the article tilted "Quantum Theory cannot consistently describe the use of itself" which, based on a gedankenexperiment designed on Wigner's Friend paradox, questions the universal validity of Quantum Theory and establishes that different observers can draw predictions which are inconsistent with the predictions of Quantum Theory. The basis for predictions are inferences of observers, about other's measurement outcome, from unique correlations. We show, that such inferences based on correlations in an entangled state, possesses some other inconsistencies, which we have discussed.
In the second chapter, we discuss Zurek's work on environment-induced invariance or Envariance. We briefly present his derivation of quantum mechanical Born's Rule, based on swapping of states using envariance. Firstly, we show the problems with Zurek's derivation, which isn't justified even after assuming envariance. Secondly, we show that envariance is not a property of maximally entangled pure states alone, as maximally correlated mixed states, by definition, are also envariant, which arise in the situations having just classical correlation. This establishes that envariance is not a consequence of entanglement, but instead a consequence of correlations which can even be classical. We show, envariance is just another term for a property known as rotational invariance, which is exhibited by maximally correlated mixed and pure states, both; and thus doesn't lead to objective probabilities as such, as claimed by Zurek.
In the third chapter, we reflect upon the results briefly and discuss their implications; and finally discuss some future perspective they have given, to pursue further research on.

1 Wigner's Friend Paradox and Validity of Quantum Theory

In the article titled "Quantum Theory Cannot Consistently Describe the Use of Itself" [2], Frauchiger and Renner questioned the universal validity of quantum theory. They designed a Gedankenexperiment based on Wigner's Friend scenario involving multiple observers, with different levels of observation, who use quantum theory to predict the outcomes of other observers; to show that their conclusions are inconsistent among themselves.

1.1 Wigner's Friend Paradox

Wigner's Friend paradox is a thought experiment proposed by Wigner [3], which exploits different *levels* of observation. The scenario involves an observer F, called friend, performing a measurement on a system, S, inside a laboratory (Figure 1.1). The laboratory, including the system S, the measuring devices D and the observer F, can as a whole be treated as another quantum system, L; which is isolated and is being *indirectly* observed by another observer W outside the laboratory, called Wigner. The act of indirect observation means that W can only observe the composite quantum system L but has no direct access to observe the system S, inside the laboratory. The paradox arises when the two observers try to assign a state to the system, *after* the measurement, based on the information each of them possesses.

Suppose, the observer F measures the polarization of a spin-$\frac{1}{2}$ particle S in z-direction. After measurement, which results in either $z = +\frac{1}{2}$ or $z = -\frac{1}{2}$, F would say that the system S is either in state

$$|\uparrow\rangle_S \quad \text{or} \quad |\downarrow\rangle_S \,, \tag{1}$$

respectively.

On the other hand, the observer W can model the whole laboratory as a big quantum system $L = S \otimes D \otimes F$. For W, being ignorant about F's measurement, the system L is isolated and its evolution is described by linear isometries of the form

$$U_{S\to L}\left|\uparrow\right\rangle_S = \left|\uparrow\right\rangle_S \otimes \left|z=+1/2\right\rangle_D \otimes \left|z=+1/2\right\rangle_F = \left|\uparrow\right\rangle_L,$$

$$\text{and} \qquad U_{S\to L}\left|\downarrow\right\rangle_S = \left|\downarrow\right\rangle_S \otimes \left|z=-1/2\right\rangle_D \otimes \left|z=-1/2\right\rangle_F = \left|\downarrow\right\rangle_L. \tag{2}$$

Now, suppose W knows that the initial state of the system S was

$$\left|+\right\rangle_S = \frac{\left|\uparrow\right\rangle_S + \left|\downarrow\right\rangle_S}{\sqrt{2}},$$

before F performed the measurement. Thus the final state that W would assign to L is

$$\left|+\right\rangle_L = U_{S\to L}\left|+\right\rangle_S = \frac{\left|\uparrow\right\rangle_L + \left|\downarrow\right\rangle_L}{\sqrt{2}}. \tag{3}$$

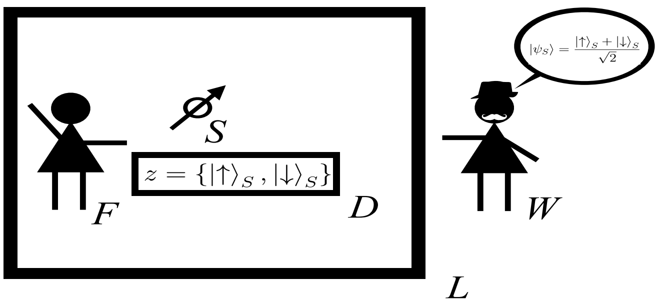

Figure 1.1: Wigner's Friend Scenario: The thought experiment involves an observer F, called friend, performing a measurement on a system, S, inside a laboratory. The laboratory, including the system S, the measuring devices D and the observer F, can as a whole be treated as another quantum system, L; which is isolated and is being observed by another observer W outside the laboratory, called Wigner.

From the perspective of W, being ignorant about F's measurement, the quantum system L is isolated; the evolution of L must be unitary and superposition remains intact, which means W assigns the superposition state in (3) to L, which is a superposition of two macroscopically orthogonal states in (2). *Thus, the two observers F and W assign different pure states based on their level of knowledge, to the same physical system in question, which is paradoxical, but not contradictory as such.*

1.2 Quantum Theory Cannot Consistently Describe the Use of Itself

1.2.1 The Gedankenexperiment

The scenario is an extension of Wigner's Friend paradox, consisting of four observers $F, \bar{F}$, W and $\overline{W}$. The observers F and $\bar{F}$ are located inside two isolated labs L and $\bar{L}$, respectively. The other two observers W and $\overline{W}$ are outside the labs who can perform macroscopic measurements on the labs L and $\bar{L}$, respectively (Figure 1.2). The labs L and $\bar{L}$ are assumed to remain isolated from W and $\overline{W}$ throughout the experiment. All the observers are assumed to be aware of the whole experimental protocol and based on that, each of them can employ quantum theory to infer about the outcomes of other observers.

1.2.2 The Experimental Protocol

Each observer performs a measurement sequentially in time in a particular basis, as follows:

- *At* t_0: $\bar{F}$ performs a measurement on a qubit system R prepared in the state $|\psi_R\rangle = \frac{1}{\sqrt{3}}\,|\uparrow\rangle_R + \sqrt{\frac{2}{3}}\,|\downarrow\rangle_R$. Depending on the outcome r, $\bar{F}$ prepares a spin system S in *either* $|\downarrow\rangle_S$ if $r = |\uparrow\rangle_R$, *or* $|+\rangle_S = \frac{1}{\sqrt{2}}\,|\uparrow\rangle_S + \frac{1}{\sqrt{2}}\,|\downarrow\rangle_S$ if $r = |\downarrow\rangle_R$; and sends it to F.

- *At* t_1: F measures S in $\{|\uparrow\rangle_S\,, |\downarrow\rangle_S\}$ basis, obtaining an outcome z.

- *At* t_2: $\overline{W}$ measures the lab $\bar{L}$ in $\{|+\rangle_{\bar{L}}\,, |-\rangle_{\bar{L}}\}$ basis where $|\pm\rangle_{\bar{L}} = \frac{1}{\sqrt{2}}\,|\uparrow\rangle_{\bar{L}} \pm \frac{1}{\sqrt{2}}\,|\downarrow\rangle_{\bar{L}}$, obtaining an outcome $\bar{w}$, and announces the result to W.

- *At* t_3: W measures the lab L in $\{|+\rangle_L\,, |-\rangle_L\}$ basis where $|\pm\rangle_L = \frac{1}{\sqrt{2}}\,|\uparrow\rangle_L \pm \frac{1}{\sqrt{2}}\,|\downarrow\rangle_L$, obtaining an outcome w.

- *At* t_4: W and $\overline{W}$ compare their outcomes. If $\overline{w} = |-\rangle_{\overline{L}}$ and $w = |-\rangle_L$, the experiment is halted. Else the experiment is repeated, till the halting condition is satisfied.

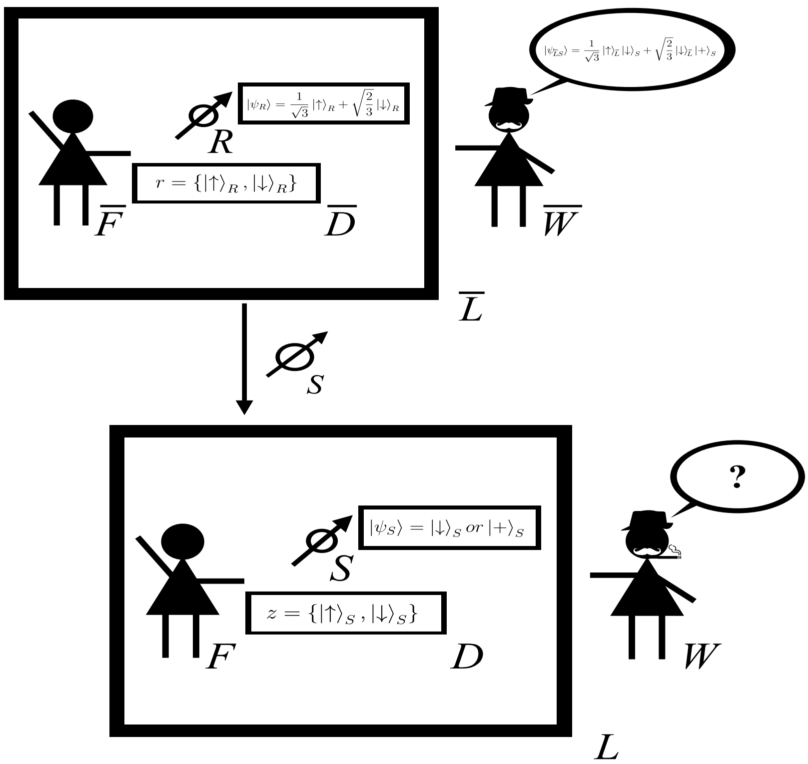

Figure 1.2: The Gedankenexperiment: The setup consists of four observers F, $\overline{F}$, W and $\overline{W}$. The observers F and $\overline{F}$ are located inside two isolated labs L and $\overline{L}$, resp. The other two observers W and $\overline{W}$ are outside the labs who can perform macroscopic measurements on the labs L and $\overline{L}$, resp.

1.2.3 Statements of the Observers

Each observer tries to predict the meausurement outcome of another observer based on their measurement result and the information that they have access to, using the rules of quantum theory.

- **Statement of $\bar{F}$**: If $r = |{\downarrow}\rangle_R$ at t_0, then I am certain that W, upon measurement of L at t_3, would get $|+\rangle_L$.
  Reasoning: If $\bar{F}$ after measuring the system R has got $r = |{\downarrow}\rangle_R$, he prepares the system S in the state $|+\rangle_S$. Since the lab L remains isolated for W, it implies that the later state of the lab L would be

  $$U_{S\to L}^{t_1\to t_3} |+\rangle_S = |+\rangle_L ,$$

  and thus W upon measurement would certainly get $|+\rangle_L$.

- **Statement of F**: If $z = |{\uparrow}\rangle_S$ at t_1, I am certain that W would get $|+\rangle_L$ at t_3.
  Reasoning: Since F knows beforehand that the state he recieves from $\bar{F}$ can *only* be either $|{\downarrow}\rangle_S$ or $|+\rangle_S$, whenever F gets $z = |{\uparrow}\rangle_S$, he is certain that the state he received must have been $|+\rangle_S$. It implies that $\bar{F}$ must have got $r = |{\downarrow}\rangle_R$ at t_0. And since, whenever $\bar{F}$ gets $|{\downarrow}\rangle_R$ he is certain that W would get $|+\rangle_L$, F is certain that W would get $|+\rangle_L$ at t_3.

- **Statement of $\bar{W}$**: If $\bar{w} = |-\rangle_{\bar{L}}$ upon measurement at t_2, I am certain that W would get $|+\rangle_L$ at t_3.
  Reasoning: Since the lab $\bar{L}$ is isolated for $\bar{W}$, it's evolution from t_0 to t_2 must be unitary; that is, the initial state of the system R, $|\psi_R\rangle = \frac{1}{\sqrt{3}} |{\uparrow}\rangle_R + \sqrt{\frac{2}{3}} |{\downarrow}\rangle_R$, isometrically evolves to

  $$U_{R\to\bar{L}\otimes S}^{t_0\to t_2} |\psi_R\rangle = \frac{1}{\sqrt{3}} |{\uparrow}\rangle_{\bar{L}} \otimes |{\downarrow}\rangle_S + \sqrt{\frac{2}{3}} |{\downarrow}\rangle_{\bar{L}} \otimes |+\rangle_S = |\psi_{\bar{L}\otimes S}\rangle .$$

  Consider the event $\bar{w} = |-\rangle_{\bar{L}}$ and $z = |{\downarrow}\rangle_S$. Since $|-\rangle_{\bar{L}} \otimes |{\downarrow}\rangle_S$ is orthogonal to $|\psi_{\bar{L}\otimes S}\rangle$, it implies that

$$p\left((\bar{w}, z) \neq (|-\rangle_{\bar{L}}, |\downarrow\rangle_S)\right) = 1 - p\left((\bar{w}, z) = (|-\rangle_{\bar{L}}, |\downarrow\rangle_S)\right) = 1.$$

Hence, $\overline{W}$ can be certain that $(\bar{w}, z) \neq (|-\rangle_{\bar{L}}, |\downarrow\rangle_S)$. That is, whenever F gets $z = |\downarrow\rangle_S$ at t_1, $\overline{W}$ *cannot* get $\bar{w} = |-\rangle_{\bar{L}}$ at t_2. It implies that whenever $\overline{W}$ gets $\bar{w} = |-\rangle_{\bar{L}}$ at t_2, F must have got $z = |\uparrow\rangle_S$ at t_1. Since, F upon getting $z = |\uparrow\rangle_S$, is certain that W would get $|+\rangle_L$; $\overline{W}$, upon getting $\bar{w} = |-\rangle_{\bar{L}}$, is certain that W would get $|+\rangle_L$ at t_3.

- **Statement of *W***: I am certain that $\overline{W}$ has got $|-\rangle_{\bar{L}}$, then I am certain that I would get $|+\rangle_L$ at t_3.
  Reasoning: Since $\overline{W}$ announces his measurement outcome at t_2 to W, W can be certain $\overline{W}$ has got $|-\rangle_{\bar{L}}$, whenever he does. And since $\overline{W}$, upon getting $|-\rangle_{\bar{L}}$, is certain that W would get $|+\rangle_L$, W is certain that he would get $|+\rangle_L$ at t_3.

The reasoning behind the preceding inferences is based on the *correlations* and the knowledge of the basis in which an observer performs the measurement, to all other observers; which allows them to infer according to the Quantum Theory, as is apparent in the following equations:

$$|\psi_{\bar{L}\otimes L}\rangle = \frac{1}{\sqrt{6}}|\uparrow\rangle_{\bar{L}} \otimes |+\rangle_L - \frac{1}{\sqrt{6}}|\uparrow\rangle_{\bar{L}} \otimes |-\rangle_L + \sqrt{\frac{2}{3}}|\downarrow\rangle_{\bar{L}} \otimes |+\rangle_L,$$
$$\implies \bar{w} = |\downarrow\rangle_{\bar{L}} \implies r = |\downarrow\rangle_R \text{ at } t_0 \implies w = |+\rangle_L \text{ at } t_3\,.$$

$$|\psi_{\bar{L}\otimes S}\rangle = \frac{1}{\sqrt{3}}|\uparrow\rangle_{\bar{L}} \otimes |\downarrow\rangle_S + \frac{1}{\sqrt{3}}|\downarrow\rangle_{\bar{L}} \otimes |\downarrow\rangle_S + \frac{1}{\sqrt{3}}|\downarrow\rangle_{\bar{L}} \otimes |\uparrow\rangle_S,$$
$$\implies z = |\uparrow\rangle_S \text{ at } t_1 \implies \bar{w} = |\downarrow\rangle_{\bar{L}} \implies r = |\downarrow\rangle_R \text{ at } t_0\,.$$

$$|\psi_{\bar{L}\otimes S}\rangle = \sqrt{\frac{2}{3}}|+\rangle_{\bar{L}} \otimes |\downarrow\rangle_S + \frac{1}{\sqrt{6}}|+\rangle_{\bar{L}} \otimes |\uparrow\rangle_S - \frac{1}{\sqrt{6}}|-\rangle_{\bar{L}} \otimes |\uparrow\rangle_S,$$
$$\implies \quad \bar{w} = |-\rangle_{\bar{L}} \text{ at } t_2 \implies z = |\uparrow\rangle_S \text{ at } t_1\,.$$

1.2.4 The Inconsistency

For observers W and $\overline{W}$, the labs L and $\bar{L}$ respectively, are isolated systems. Thus the initial pure state of the system R, remains a pure state and evolves to the final state given by

$$U^{t_0 \to t_3}_{R \to \bar{L} \otimes L} \left|\psi_R\right\rangle = \left|\psi_{\bar{L} \otimes L}\right\rangle = \frac{1}{\sqrt{3}} \left|\uparrow\right\rangle_{\bar{L}} \otimes \left|\downarrow\right\rangle_L + \sqrt{\frac{2}{3}} \left|\downarrow\right\rangle_{\bar{L}} \otimes \left|+\right\rangle_L$$

$$= \sqrt{\frac{3}{4}} \left|+\right\rangle_{\bar{L}} \otimes \left|+\right\rangle_L - \frac{1}{\sqrt{12}} \left|+\right\rangle_{\bar{L}} \otimes \left|-\right\rangle_L - \frac{1}{\sqrt{12}} \left|-\right\rangle_{\bar{L}} \otimes \left|-\right\rangle_L - \frac{1}{\sqrt{12}} \left|-\right\rangle_{\bar{L}} \otimes \left|+\right\rangle_L .$$

Consider the event $\bar{w} = \left|-\right\rangle_{\bar{L}}$ and $w = \left|-\right\rangle_L$. The probability of occurrence of this event given by

$$p\big((\bar{w}, w) = (\left|-\right\rangle_{\bar{L}}, \left|-\right\rangle_L)\big) = \left|\left(\left\langle-\right|_{\bar{L}} \otimes \left\langle-\right|_L\right) \left|\psi_{\bar{L}\otimes L}\right\rangle\right|^2 = \frac{1}{12}.$$

Since, this probability is not exactly zero, W is certain that the event $(\bar{w}, w) = (\left|-\right\rangle_{\bar{L}}, \left|-\right\rangle_L)$ is bound to occur after finitely many rounds.

But, through logical deductions stated in the previous section, W concludes that whenever $\bar{w} = \left|-\right\rangle_{\bar{L}}$, he is certain to get $w = \left|+\right\rangle_L$, that is, he is certain that the event $(\bar{w}, w) = (\left|-\right\rangle_{\bar{L}}, \left|-\right\rangle_L)$ can never occur.

Clearly, the previous two statements taken together are inconsistent.

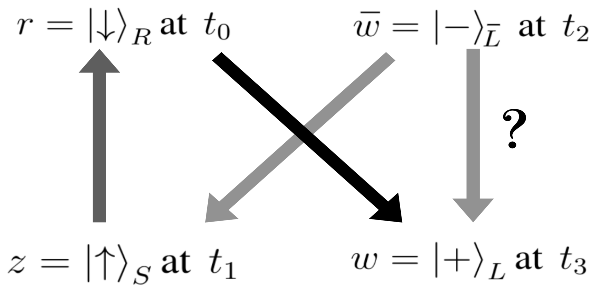

Figure 1.3: The Inconsistency: If $\bar{w} = \left|-\right\rangle_{\bar{L}}$ at t_2, it implies $z = \left|\uparrow\right\rangle_S$ at t_1, which implies $r = \left|\downarrow\right\rangle_R$ at t_0, which in turn implies $w = \left|+\right\rangle_L$ at t_3. Thus, whenever $\bar{w} = \left|-\right\rangle_{\bar{L}}$, it implies $w = \left|+\right\rangle_L$. But, the event $\bar{w} = \left|-\right\rangle_{\bar{L}}$ and $w = \left|-\right\rangle_L$ is also probable, and thus bound to occur after finitely many runs of the experiment.

1.3 Other Inconsistencies

- **In $\overline{W}$'s inference about $\overline{F}$'s measurement outcome:**

  In $\overline{W}$'s inference about $\overline{F}$'s measurement outcome at t_0, that when he gets $\bar{w} = |-\rangle_{\overline{L}}$ at t_2, $\overline{F}$ must have got $r = |\uparrow\rangle_R$ at t_0, is based on the *fact* that he *gets* the outcome $\bar{w} = |-\rangle_{\overline{L}}$ at t_2; which is equivalent to the statement that the state of the system $\overline{L}$ is $|-\rangle_{\overline{L}}$ at t_2.

  It implies that,

$$|-\rangle_{\overline{L}} = \frac{1}{\sqrt{2}} |\uparrow\rangle_{\overline{L}} - \frac{1}{\sqrt{2}} |\downarrow\rangle_{\overline{L}} ,$$

$$= \frac{1}{\sqrt{2}} |\uparrow\rangle_R \otimes |\uparrow\rangle_{\overline{D}} \otimes |\uparrow\rangle_{\overline{F}} + \frac{1}{\sqrt{2}} |\downarrow\rangle_R \otimes |\downarrow\rangle_{\overline{D}} \otimes |\downarrow\rangle_{\overline{F}} .$$

$$\implies \qquad |\psi\rangle \langle\psi|_{\overline{F}} = \frac{1}{2} |\uparrow\rangle \langle\uparrow|_{\overline{F}} + \frac{1}{2} |\downarrow\rangle \langle\downarrow|_{\overline{F}} ,$$

  at t_2 that is, $\bar{w} = |-\rangle_{\overline{L}}$ at t_2 *does not* imply that the state of the observer $\overline{F}$ is $r = |\downarrow\rangle_{\overline{F}}$ at t_2.

  In fact, a measurement by $\overline{W}$ on the composite system $\overline{L}$ changes the state of the subsystems R, $\overline{D}$, and $\overline{F}$; which implies that the measurement by $\overline{W}$ at t_2, *erases* the record of the outcome of the measurement by $\overline{F}$ at t_0.

Thus, after getting the outcome $\bar{w} = |-\rangle_{\overline{L}}$ at t_2, that is, the system $\overline{L}$ being in the state $|-\rangle_{\overline{L}}$; $\overline{W}$, looking at his own measurement outcome, cannot say deterministically if the state of the observer $\overline{F}$ is $|\uparrow\rangle_{\overline{F}}$ or $|\downarrow\rangle_{\overline{F}}$, which are records of $\overline{F}$ having observed the system R in $r = |\uparrow\rangle_R$ or $r = |\downarrow\rangle_R$ at t_0, resp. In other words, $\overline{L}$ cannot say directly from his own state, what the observer $\overline{F}$ inside the lab $\overline{L}$ has observed; which is also consistent with the Wigner's Friend paradox. But this, is inconsistent with his statement based on the inference of observer F, that whenever he gets $\bar{w} = |-\rangle_{\overline{L}}$, he is certain that $\overline{F}$ must have got the outcome $r = |\downarrow\rangle_R$ at t_0.

- **In $\overline{W}$'s prediction about W's measurement outcome:**

  If $\overline{W}$ measures $\bar{w} = |-\rangle_{\bar{L}}$ at t_2, the equation

$$|\psi_{\bar{L}\otimes L}\rangle = \sqrt{\frac{2}{3}}\,|+\rangle_{\bar{L}} \otimes |\downarrow\rangle_L + \frac{1}{\sqrt{6}}\,|+\rangle_{\bar{L}} \otimes |\uparrow\rangle_L - \frac{1}{\sqrt{6}}\,|-\rangle_{\bar{L}} \otimes |\uparrow\rangle_L\,,$$

  implies that $\overline{W}$ can be certain that the state of the system L must be $|\uparrow\rangle_L$ at t_2. Since the evolution of system L from t_2 to t_3 is $\mathbb{I}_L$, it implies that the later state of the system L at t_3 would be,

$$|\uparrow\rangle_L = \frac{1}{\sqrt{2}}\,|+\rangle_L + \frac{1}{\sqrt{2}}\,|-\rangle_L\,.$$

Thus, whenever $\overline{W}$ measures $\bar{w} = |-\rangle_{\bar{L}}$ at t_2, he is certain that the state of the system L at t_3 must be $|\uparrow\rangle_L$, which implies that W upon measurement at t_3 can get either $w = |+\rangle_L$ or $w = |-\rangle_L$, equi-probably. But, this prediction is inconsistent with his earlier prediction based on the inferences of other observers, that whenever he gets $\bar{w} = |-\rangle_{\bar{L}}$ at t_2, he is certain that W must get $w = |+\rangle_L$ at t_3.

1.4 Conclusion

- **A Thought Experiment:**

  Suppose Alice and Bob share a two-qubit entangled state,

$$|\psi_{A\otimes B}\rangle = \frac{1}{\sqrt{3}}\,|\uparrow\rangle_A \otimes |\downarrow\rangle_B + \frac{1}{\sqrt{3}}\,|\downarrow\rangle_A \otimes |\downarrow\rangle_B + \frac{1}{\sqrt{3}}\,|\downarrow\rangle_A \otimes |\uparrow\rangle_B$$

  and say, Alice and Bob have subsystems A and B respectively, and are isolated from each other. Further, Alice and Bob measure the systems A and B in $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis, at time t_1 and t_2, respectively; which is known to both of them.

  If, Alice performs the measurement at t_1 and gets $a = |\uparrow\rangle_A$, then, since

$|\uparrow\rangle_A$ is uniquely correlated to $|\downarrow\rangle_B$, she is certain that when Bob performs a measurement at t_2, he must get $b = |\downarrow\rangle_B$ at t_2. Further, since the two systems are isolated, the time evolution of the state from t_1 to t_2 must be unitary, which implies

$$|\psi_{A\otimes B}\rangle = \sqrt{\frac{2}{3}}\,|+\rangle_A \otimes |\downarrow\rangle_B + \frac{1}{\sqrt{6}}\,|+\rangle_A \otimes |\uparrow\rangle_B - \frac{1}{\sqrt{6}}\,|-\rangle_A \otimes |\uparrow\rangle_B\,,$$

at t_2. Since, $|\downarrow\rangle_B$ is uniquely correlated to $|+\rangle_A$, if Bob gets $b = |\downarrow\rangle_B$ at t_2, he is certain that the state of system A at t_2 must be

$$|+\rangle_A = \frac{1}{\sqrt{2}}\,|\uparrow\rangle_A + \frac{1}{\sqrt{2}}\,|\downarrow\rangle_A\,.$$

Now, if Alice at t_3 decides to make another measurement on her system, A, in the same basis, she would be certain to get the same result over, that is, $a = |\uparrow\rangle_A$, if she has got $a = |\uparrow\rangle_A$ earlier at t_1. But, if she gets $a = |\uparrow\rangle_A$ at t_1, she is certain that Bob must measure $b = |\downarrow\rangle_B$ at t_2. If she relies on Bob's inference, that if he gets $b = |\downarrow\rangle_B$, he is certain that the state of system A must be $|+\rangle_A$; she is certain that the state which she measures at t_3, must be $|+\rangle_A$, and *cannot* say deterministically if she would get $a = |\uparrow\rangle_A$ or $a = |\downarrow\rangle_A$; which contradicts her own statement that if she measures $a = |\uparrow\rangle_A$ at t_1, she must get $a = |\uparrow\rangle_A$ at t_3.

Thus, the predictions in time drawn in such a manner, based on inferences of other observers using unique correlations between entangled systems, lead to statements which are inconsistent, with the observer's own prediction from their measurement outcome. We conclude that, the inconsistency shown by Renner and Frauchiger in [2], is a manifestation of such inferences of different observers about others.

2 On Envariance and Born's Rule

Zurek in [4] used the idea of Environment-assisted invariance or *envariance*, a symmetry of maximally entangled quantum systems, to derive quantum mechanical Born's Rule [1]. Further, he implied that envariance answers the long withstanding question of origin of probabilities and randomness in quantum physics and, quantifies the objective ignorance about the state of a subsystem, present in perfectly entangled quantum systems, as a consequence of perfect knowledge of the composite state.

2.1 Environment-assisted Invariance or Envariance

Given the joint state $|\psi_{SE}\rangle$ of a pair of systems S and E, if the effect of a transformation $U_S = u_S \otimes \mathbb{I}_E$, acting on the system S alone,

$$U_S\,|\psi_{SE}\rangle = \big(u_S \otimes \mathbb{I}_E\big)\,|\psi_{SE}\rangle = |\eta_{SE}\rangle\,,$$

can be *undone* by a transformation $U_E = \mathbb{I}_S \otimes u_E$, acting only on the system E, such that

$$U_E\,|\eta_{SE}\rangle = \big(\mathbb{I}_S \otimes u_E\big)\,|\eta_{SE}\rangle = |\psi_{SE}\rangle\,,$$

then the state $|\psi_{SE}\rangle$ is called *envariant* under U_S (Figure 2.1).

As an example of envariance, consider the Schmidt decomposition of $|\psi_{SE}\rangle$:

$$|\psi_{SE}\rangle = \sum_{i=1}^{N} \alpha_i\,|\sigma_i\rangle\,|\epsilon_i\rangle\,. \tag{4}$$

Then, any unitary transformation with eigenstates $|\sigma_i\rangle$,

$$u_s = \sum_{i=1}^{N} e^{i\phi_i}\,|\sigma_i\rangle\,\langle\sigma_i|\,,$$

is envariant, since it can be undone by an unitary

$$u_E = \sum_{i=1}^{N} e^{-i\phi_i} |\epsilon_i\rangle \langle\epsilon_i| \,,$$

acting on the environment alone.

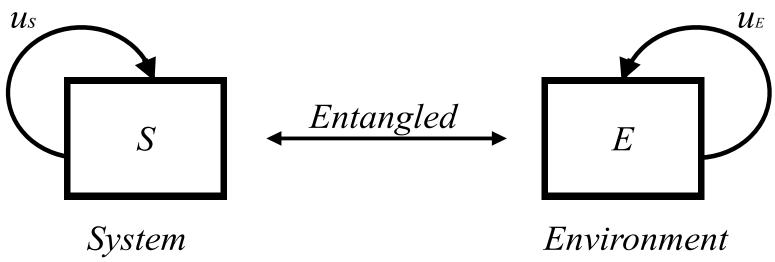

Figure 2.1: Environment Assisted Invariance or *Envariance*: When a transformation, U_S, acting on system S alone can be undone by a transformation, U_E, acting only on the environment E, then the joint state is said to be *envariant* under U_S.

2.2 Review of Zurek's Derivation of Born's Rule

2.2.1 Swapping is Envariant

Given the joint state $|\psi_{SE}\rangle$ expressed in Schmidt decomposition form (4), the unitary

$$u_s(i \leftrightarrow j) = e^{i\phi_{i,j}} |\sigma_j\rangle \langle\sigma_i| + H.c.$$

acting on the system S, *swaps* the states $|\sigma_i\rangle$ and $|\sigma_j\rangle$ along with their coefficients.

This swap can be 'undone' by an unitary,

$$u_e(i \leftrightarrow j) = e^{i(\phi_{i,j}+\phi_i-\phi_j)} |e_i\rangle \langle e_j| + H.c.$$

acting only on the environment E.

If the absolute values of all the coefficients α_i in (4) *are equal*, then the joint state $|\psi_{SE}\rangle$ is envariant under *swapping*, that is

$$\big(u_s(i \leftrightarrow j) \otimes u_e(i \leftrightarrow j)\big) |\psi_{SE}\rangle = |\psi_{SE}\rangle .$$

2.2.2 Swapping and Probabilities

Since, the swap leaves the overall state $|\psi_{SE}\rangle$ unchanged, it implies that the probabilities of any two *envariantly* swappable states must be equal. When all the states can be swapped this way, and if there are N of them, the probability associated with each of them must be

$$p_i = \frac{1}{N},$$

and, the probability of any subset of n orthonormal (thus, mutually exclusive) $\{|\sigma_i\rangle\}$ is

$$\sum_{i=1}^{n} p_i = n/N.$$

2.2.3 Zurek's Approach With a State Having Unequal Coefficients

To illustrate the idea, we focus on the case having only two coefficients, where

$$|\psi_{SE}\rangle = \alpha_0 |\sigma_0\rangle |\epsilon_0\rangle + \alpha_1 |\sigma_1\rangle |\epsilon_1\rangle .$$

Suppose that α_0 and α_1 can be written as:

$$\alpha_0 = e^{i\phi_0}\sqrt{\frac{m}{M}}, \quad \text{and} \quad \alpha_1 = e^{i\phi_1}\sqrt{\frac{M-m}{M}}, \tag{5}$$

which implies that the joint state becomes

$$|\psi_{SE}\rangle = e^{i\phi_0}\sqrt{\frac{m}{M}}\,|\sigma_0\rangle\,|\epsilon_0\rangle + e^{i\phi_1}\sqrt{\frac{M-m}{M}}\,|\sigma_1\rangle\,|\epsilon_1\rangle\,. \tag{6}$$

When there are no m and M for which eq.(5) holds, such an m and M can always be found such that $\sqrt{m/M}$ and $\sqrt{M-m/M}$ are arbitrary close to $|\alpha_0|$ and $|\alpha_1|$ resp.

To convert the joint state (6) to a state with equal coefficients, are needed *two* systems C and E, instead of just the environment E, correlated with the system of interest S. Suppose, C interacts with E such that the joint state $|\psi_{SCE}\rangle$ takes the form

$$|\psi_{SCE}\rangle = e^{i\phi_0}\sqrt{\frac{m}{M}}\,|\sigma_0\rangle\,|C_0\rangle\,|e_0\rangle + e^{i\phi_1}\sqrt{\frac{M-m}{M}}\,|\sigma_1\rangle\,|C_1\rangle\,|e_0\rangle\,,$$

that is, $|C_0\rangle\,|e_0\rangle = |\epsilon_0\rangle$ and $|C_1\rangle\,|e_0\rangle = |\epsilon_1\rangle$.

Assuming, in some orthonormal basis $\{|c_k\rangle\}$, $|C_0\rangle$ and $|C_1\rangle$ can be expressed as

$$|C_0\rangle = \sum_{k=0}^{m-1}\frac{1}{\sqrt{m}}\,|c_k\rangle \quad , \quad |C_1\rangle = \sum_{k=m}^{M-1}\frac{1}{\sqrt{M-m}}\,|c_k\rangle\,,$$

and, C interacts with E such that $|c_k\rangle\,|e_0\rangle \rightarrow |c_k\rangle\,|e_k\rangle$, where $|e_0\rangle$ is the initial state of E and $\langle e_k|e_l\rangle = \delta_{kl}$; $|\psi_{SCE}\rangle$ becomes,

$$|\psi_{SCE}\rangle = \sum_{k=0}^{m-1}\frac{e^{i\phi_0}}{\sqrt{M}}\,|\sigma_0\rangle\,|c_k\rangle\,|e_k\rangle + \sum_{k=m}^{M-1}\frac{e^{i\phi_1}}{\sqrt{M}}\,|\sigma_1\rangle\,|c_k\rangle\,|e_k\rangle\,, \tag{7}$$

$$= \sum_{k=0}^{m-1}\frac{e^{i\phi_0}}{\sqrt{M}}\,|0, c_k\rangle\,|e_k\rangle + \sum_{k=m}^{M-1}\frac{e^{i\phi_1}}{\sqrt{M}}\,|1, c_k\rangle\,|e_k\rangle\,,$$

with all the coefficients having equal absolute values.

Thus, $|\psi_{SCE}\rangle$ is envariant under swaps of the states $|s, c_k\rangle$ of the composite system

SC which can be undone by counterswaps of system E.

From previous section, it implies that

$$p(|\sigma_0\rangle) = \sum_{k=0}^{m-1} p(|0,k\rangle) = \frac{m}{M}, \quad \text{and} \quad p(|\sigma_1\rangle) = \sum_{k=m}^{M-1} p(|1,k\rangle) = \frac{M-m}{M}.$$

Thus, it follows from eq. (6) that,

$$p(|\sigma_0\rangle) = |\alpha_0|^2, \quad \text{and} \quad p(|\sigma_1\rangle) = |\alpha_1|^2,$$

which is the *Born's Rule*.

2.3 Problem with Zurek's Derivation

Zurek's derivation of Born's Rule stems from the fact that the system states $|\sigma_i\rangle$ can be *envariantly swapped*, that is, the swap between two system states $|\sigma_i\rangle$ and $|\sigma_j\rangle$ can be undone by a counterswap between $|\epsilon_i\rangle$ and $|\epsilon_j\rangle$ on the environment, leaving the joint state unchanged. But, the joint state remains unchanged only if all the coefficients have equal absolute values.

To deal with states in which coefficients have unequal absolute values, Zurek used embedding to convert them to a state in which all the coefficients have equal absolute values.

The first problem arises when we try to swap any two system states $|\sigma_i\rangle$ in (7):

- *The resultant state $|\psi_{SCE}\rangle$ in* (7) *after embedding, is not envariant under swaps of system states $|\sigma_i\rangle$, that is, the swap on the system states $|\sigma_i\rangle$ cannot be undone by a counterswap on the composite system CE.*

  Proof: Consider a system S perfectly entangled with an environment E, such that their joint state is

  $$|\psi_{SE}\rangle = \sqrt{\frac{1}{3}}\,|\sigma_0\rangle\,|\epsilon_0\rangle + \sqrt{\frac{2}{3}}\,|\sigma_1\rangle\,|\epsilon_1\rangle\,.$$

To convert it to an even state (having all the coefficients equal), interacts with the system S, another system C, such that the final joint state $|\psi_{SCE}\rangle$ after the interaction takes the form

$$|\psi_{SCE}\rangle = \sqrt{\frac{1}{3}}\,|\sigma_0\rangle\,|c_0\rangle\,|e_0\rangle + \sqrt{\frac{1}{3}}\,|\sigma_1\rangle\,|c_1\rangle\,|e_1\rangle + \sqrt{\frac{1}{3}}\,|\sigma_1\rangle\,|c_2\rangle\,|e_2\rangle\,. \quad (8)$$

The unitary $u_S = |\sigma_1\rangle\langle\sigma_0| + |\sigma_0\rangle\langle\sigma_1|$ acting on the system S, swaps the system states $|\sigma_0\rangle$ and $|\sigma_1\rangle$ and transforms the joint state to

$$|\eta_{SCE}\rangle = \sqrt{\frac{1}{3}}\,|\sigma_1\rangle\,|c_0\rangle\,|e_0\rangle + \sqrt{\frac{1}{3}}\,|\sigma_0\rangle\,|c_1\rangle\,|e_1\rangle + \sqrt{\frac{1}{3}}\,|\sigma_0\rangle\,|c_2\rangle\,|e_2\rangle\,. \quad (9)$$

Note, that the counterswap between states $|c_0\rangle\,|e_0\rangle$ and $|c_1\rangle\,|e_1\rangle$ on the composite system CE, transforms the state $|\eta_{SCE}\rangle$ in (9) to

$$|\phi_{SCE}\rangle = \sqrt{\frac{1}{3}}\,|\sigma_1\rangle\,|c_1\rangle\,|e_1\rangle + \sqrt{\frac{1}{3}}\,|\sigma_0\rangle\,|c_0\rangle\,|e_0\rangle + \sqrt{\frac{1}{3}}\,|\sigma_0\rangle\,|c_2\rangle\,|e_2\rangle\,,$$

which is *not* equal to the pre-swap joint state $|\psi_{SCE}\rangle$ in (8).

Neither, the swap between $|c_0\rangle\,|e_0\rangle$ and $|c_2\rangle\,|e_2\rangle$, restores $|\eta_{SCE}\rangle$ to $|\psi_{SCE}\rangle$.

Thus, the swap $|\sigma_0\rangle \leftrightarrow |\sigma_1\rangle$ on the system S cannot be undone by any counterswap on the composite system CE. It implies, that the even state $|\psi_{SCE}\rangle$ in (8) *is NOT envariant under swaps of the system states $|\sigma_0\rangle$ and $|\sigma_1\rangle$.* Q.E.D

As stated in section 2.3, Zurek claims that the state $|\psi_{SCE}\rangle$ in (7) is envariant under swaps of the states of the composite system SC, which can be undone by a counterswap on the system E. Here, arises the second problem, when we try to swap the states $|s, c_k\rangle$ of the composite system SC:

- *The swaps, on the states $|s, c_k\rangle$ of composite system SC, are non unitary transformations and thus do not conserve norm.*

  Proof: Consider the joint state $|\psi_{SCE}\rangle$ in (8) and the swap between the states $|0, c_0\rangle$ and $|1, c_1\rangle$ of the composite system SC.
  The transformation,

$$u_{SC} = |1, c_1\rangle \langle 0, c_0| + |0, c_0\rangle \langle 1, c_1| + |1, c_2\rangle \langle 1, c_2| \,, \tag{10}$$

  acting on $|\psi_{SCE}\rangle$ in (8), swaps the states $|0, c_0\rangle$ and $|1, c_1\rangle$.

  It implies that,

$$(u_{SC})(u_{SC}^{\dagger}) = |0, c_0\rangle \langle 0, c_0| + |1, c_1\rangle \langle 1, c_1| + |1, c_2\rangle \langle 1, c_2| \,,$$

$$= |\sigma_0\rangle \langle \sigma_0| \otimes |c_0\rangle \langle c_0| + |\sigma_1\rangle \langle \sigma_1| \otimes (|c_1\rangle \langle c_1| + |c_2\rangle \langle c_2|) \,.$$

  Since, $|\sigma_0\rangle \langle \sigma_0| + |\sigma_1\rangle \langle \sigma_1| = \mathbb{I}$, it implies that $(u_{SC})(u_{SC}^{\dagger}) = \mathbb{I}$ *only* if

$$|c_0\rangle \langle c_0| = \mathbb{I}, \qquad \text{and} \qquad |c_1\rangle \langle c_1| + |c_2\rangle \langle c_2| = \mathbb{I}.$$

  And since, $|c_k\rangle$ forms a complete orthonormal basis i.e.

$$|c_0\rangle \langle c_0| + |c_1\rangle \langle c_1| + |c_2\rangle \langle c_2| = \mathbb{I},$$

  it follows that,

$$(u_{SC})(u_{SC}^{\dagger}) \neq \mathbb{I}.$$

  Thus, the swaps between the states $|s, c_k\rangle$ of the composite system SC, are non unitary. Q.E.D

Zurek's approach to use envariance for swapping the states is valid only for *maximally entangled* states, in which all the coefficients have equal absolute values.

For the states having unequal coefficients, the idea of converting them to an even state using embedding, sure converts them to a state in which all the coefficients are equal. But, the resulting states, are no more envariantly swappable and the arguments leading to probabilities and Born's Rule, do not follow. Thus, Zurek's derivation of Born's Rule is limited *only* to maximally entangled pure states.

2.4 Born's Rule Using Isometries

Here, we propose an alternate derivation of Born's Rule following the line of thought of Zurek. We show, how we can use isometries to convert a state with unequal coefficients to an even state. The basis states in the resulting state after isometric transformation are mutually orthonormal, and allows us to swap any two states using unitary transformations, which leads to equal probabilities and thus Born's Rule.

2.4.1 Isometries

An *isometry* $\mathcal{S} : \mathbb{V} \to \mathbb{W}$ is a *norm-conserving* transformation, which translates to

$$\mathcal{S}^{\dagger}\mathcal{S} = \mathbb{I}_V,$$

where $\mathbb{V}$ and $\mathbb{W}$ input and output Hilbert spaces resp. and $\dim(\mathbb{V}) < \dim(\mathbb{W})$.

Since, isometry is a mapping from a given Hilbert space to a larger one, it physically means to increase the number of accessible *dimensions*. Such a transformation can be physically implemented by attaching with the system an *ancilla*, which contributes the extra dimensions and extends the Hilbert space of the system.

2.4.2 Converting a State with Unequal Coefficients to an Even State Using Isometries

For simplicity, consider the two dimensional case where the state of the system S can be written as in (6). To extend this 2 dim Hilbert space of the system to a M dim Hilbert space, we attach with the system an ancilla A, in the initial state $|0\rangle_A$, having a M dim Hilbert space, such that the joint state, $|\psi_{SA}\rangle$, of system and ancilla becomes

$$|\psi_{SA}\rangle = e^{i\phi_0}\sqrt{\frac{m}{M}}\,|\sigma_0\rangle\,|0\rangle_A + e^{i\phi_1}\sqrt{\frac{M-m}{M}}\,|\sigma_1\rangle\,|0\rangle_A\,.$$

The joint state $|\psi_{SA}\rangle$ under unitary evolution, U, evolves to a final state $|\Psi_{SA}\rangle$ such that

$$|\Psi_{SA}\rangle = U\,|\psi_{SA}\rangle = e^{i\phi_0}\sqrt{\frac{m}{M}}\,|\sigma_0\rangle\,|\psi_A\rangle + e^{i\phi_1}\sqrt{\frac{M-m}{M}}\,|\sigma_1\rangle\,|\phi_A\rangle\,, \tag{11}$$

where

$$|\psi_A\rangle = \sum_{i=0}^{m-1}\frac{1}{\sqrt{m}}\,|\eta_i\rangle \quad , \quad |\phi_A\rangle = \sum_{i=m}^{M-1}\frac{1}{\sqrt{M-m}}\,|\eta_i\rangle\,,$$

and, $$U = |0\rangle\langle 0|_S \otimes U^A_{|0\rangle_A\longrightarrow|\phi\rangle_A} + |1\rangle\langle 1|_S \otimes U^A_{|0\rangle_A\longrightarrow|\psi\rangle_A}.$$

This implies, that the final joint state $|\Psi_{SA}\rangle$ becomes

$$|\Psi_{SA}\rangle = \sum_{i=0}^{m-1}\frac{e^{i\phi_0}}{\sqrt{M}}\,|\sigma_0\rangle\,|\eta_i\rangle + \sum_{i=m}^{M-1}\frac{e^{i\phi_1}}{\sqrt{M}}\,|\sigma_1\rangle\,|\eta_i\rangle\,. \tag{12}$$

The basis states in (12) belong to $\mathbb{H}_S \otimes \mathbb{H}_A$ which is $2M$ dim, where $\mathbb{H}_S$ and $\mathbb{H}_A$ are the Hilbert spaces of the system and the ancilla, respectively. There are M more dimensions than needed. So, forgetting the *redundant* M dimensions, gives

$$|\Phi\rangle\langle\Phi| = \mathrm{Tr}\left(|\Psi_{SA}\rangle\langle\Psi_{SA}|\right), \tag{13}$$

where,

$$|\Phi\rangle = \sum_{i=0}^{m-1}\frac{e^{i\phi_0}}{\sqrt{M}}\,|0,i\rangle + \sum_{i=m}^{M-1}\frac{e^{i\phi_1}}{\sqrt{M}}\,|1,i\rangle\,, \tag{14}$$

$$= \sum_{i=0}^{m-1} \frac{e^{i\phi_0}}{\sqrt{M}} \left|i\right\rangle + \sum_{i=m}^{M-1} \frac{e^{i\phi_1}}{\sqrt{M}} \left|i\right\rangle ,$$

such that all the vectors $\left|i\right\rangle$ are *orthonormal* and all the coefficients have equal absolute values.

Thus, the isometry $\mathcal{S}$, transforms the 2 dim state $\left|\psi_S\right\rangle$ to a M dim state $\left|\Phi\right\rangle$, that is

$$\mathcal{S} \left|\psi_S\right\rangle = \left|\Phi\right\rangle , \tag{15}$$

and is given by

$$\mathcal{S} = \begin{array}{l} m \left\{ \begin{array}{l} \\ \\ \\ \\ \end{array} \right. \\ \\ \\ \\ \\ \end{array} \begin{bmatrix} \frac{1}{\sqrt{m}} & 0 \\ \frac{1}{\sqrt{m}} & 0 \\ . & . \\ \frac{1}{\sqrt{M}} & . \\ 0 & \frac{1}{\sqrt{M-m}} \\ 0 & \frac{1}{\sqrt{M-m}} \\ . & . \\ 0 & \frac{1}{\sqrt{M-m}} \end{bmatrix}_{M\times 2} \begin{array}{l} \\ \\ \\ \\ \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} M-m \end{array}$$

And, from equations (11), (13) and (15), we have

$$\begin{aligned} \mathcal{S} \left|\psi_S\right\rangle \left\langle\psi_S\right| \mathcal{S}^\dagger &= \operatorname{Tr}\left(U \left|\psi_{SA}\right\rangle \left\langle\psi_{SA}\right| U^\dagger\right) \\ &= \operatorname{Tr}\left(\left|\Psi_{SA}\right\rangle \left\langle\Psi_{SA}\right|\right) \\ &= \left|\Phi\right\rangle \left\langle\Phi\right| . \end{aligned}$$

The isometry $\mathcal{S}$ can thus be implemented by performing the unitary transformation, U, on the system and ancilla combined and then forgetting the redundant dimensions, to give the resultant M dim state $\left|\Phi\right\rangle$ in (14).

2.4.3 Probabilities and Born's rule

The unitary transformation,

$$U(i \leftrightarrow j) = e^{\phi_i - \phi_j} |i\rangle \langle j| + e^{\phi_j - \phi_i} |j\rangle \langle i| + H.c.$$

swaps any two basis states $|i\rangle$ and $|j\rangle$ in (14) along with their phases.

Since the absolute values of all the coefficients in (14) are equal, the transformation $U(i \leftrightarrow j)$ leaves the overall state $|\Phi\rangle$ unchanged. That is,

$$U(i \leftrightarrow j) |\Phi\rangle = |\Phi\rangle .$$

Thus, we can swap any two basis states in (14) *leaving the overall state unchanged, using unitary transformations*.

It follows that the probability of any two states that can be swapped without altering the overall state, must be equal. Since, all the states in (14) can thus be swapped, all of them must be equi-probable. Further, since there are M of them, it implies by normalization that

$$p(|0, i\rangle) = p(|1, i\rangle) = \frac{1}{M}.$$

Since all the states are mutually orthogonal, it follows that

$$p(|\sigma_0\rangle) = \sum_{i=0}^{m-1} p(|0, i\rangle) = \frac{m}{M},$$

and, $$p(|\sigma_1\rangle) = \sum_{i=m}^{M-1} p(|1, i\rangle) = \frac{M-m}{M}.$$

Thus, eq. (6) implies,

$$p(|\sigma_0\rangle) = |\alpha_0|^2, \qquad \text{and} \qquad p(|\sigma_1\rangle) = |\alpha_1|^2,$$

which is the *Born's Rule*.

2.5 Envariance and Randomness in Quantum Physics

2.5.1 Envariance and Laplace's Principle of Indifference

Laplace's "indifference" to an observer, of the swap of two outcomes, is a consequence of observer's negligence of the underlying state of the system, which is unknown to the observer but definitely exists and is subject to change upon swap. It is the lack of knowledge of the observer about the state, that makes him indifferent to the swap, and as a consequence, the observer assigns equal probabilities to both the outcomes. Such ignorance is attributed solely to the observer, and thus is *subjective* to the observer.

In contrast to Laplace's *subjective* probabilities, Zurek claims that envariance leads to *objective* probabilities, that is, probabilities arising from an *invariance* of the underlying state of the system. Envariance cannot affect the state of the system S- when S and E are maximally entangled, the swap on states of S can be undone without acting on S, by a counterswap only on the environment E. Since, the final state after the envariant swap is same, it implies that the probabilities of the swapped states must also be same. Moreover, since the global state, of system and environment combined, is perfectly known to the observer, it implies ignorance about a part. This ignorance enters as an *objective* property of perfect entanglement between system and environment, and is not subjective to the observer. In this sense, Zurek claims that the origin of probabilities in quantum physics is objective, and calls such ignorance *objective* [5].

2.5.2 Maximally Correlated Mixed States are also Envariant

Consider the maximally correlated mixed state,

$$\rho_{AB} = \frac{1}{2} \left|\uparrow\right\rangle \left\langle\uparrow\right|_A \otimes \left|+\right\rangle \left\langle+\right|_B + \frac{1}{2} \left|\downarrow\right\rangle \left\langle\downarrow\right|_A \otimes \left|-\right\rangle \left\langle-\right|_B . \tag{16}$$

The unitary transformation,

$$u_A = \left|\downarrow\right\rangle \left\langle\uparrow\right| + \left|\uparrow\right\rangle \left\langle\downarrow\right| ,$$

acting on the system A alone, swaps the states $\left|\uparrow\right\rangle \left\langle\uparrow\right|$ and $\left|\downarrow\right\rangle \left\langle\downarrow\right|$ of system A.

This, swap can be undone by a counterswap by an unitary

$$u_B = \left|-\right\rangle \left\langle+\right| + \left|+\right\rangle \left\langle-\right| ,$$

acting on system B alone, such that the combined state, ρ_{AB}, remains same i.e.

$$\big(u_A \otimes u_B\big)\rho_{AB}\big(u_A^\dagger \otimes u_B^\dagger\big) = \rho_{AB}.$$

Thus, ρ_{AB} is envariant under swaps.

In fact, any unitary transformation on the system A,

$$u_A = \left|\alpha\right\rangle \left\langle\uparrow\right| + \left|\beta\right\rangle \left\langle\downarrow\right| \quad , \quad \langle\alpha|\beta\rangle = 0,$$

can be undone by an counter unitary transformation,

$$u_B = \left|\tilde{\alpha}\right\rangle \left\langle+\right| + |\tilde{\beta}\rangle \left\langle-\right| ,$$

on system B alone, such that

$$\begin{aligned}(u_A \otimes u_B)\,\rho_{AB}\left(u_A^\dagger \otimes u_B^\dagger\right) &= \frac{1}{2} \left|\alpha\right\rangle \left\langle\alpha\right|_A \otimes \left|\tilde{\alpha}\right\rangle \left\langle\tilde{\alpha}\right|_B + \frac{1}{2} \left|\beta\right\rangle \left\langle\beta\right|_A \otimes |\tilde{\beta}\rangle \langle\tilde{\beta}|_B , \\ &= \rho_{AB},\end{aligned}$$

where,

$$|\tilde{\alpha}\rangle = \langle\alpha|\uparrow\rangle\,|+\rangle + \langle\alpha|\downarrow\rangle\,|-\rangle\,, \qquad \text{and} \qquad |\tilde{\beta}\rangle = \langle\beta|\uparrow\rangle\,|+\rangle + \langle\beta|\downarrow\rangle\,|-\rangle\,.$$

Thus, by the definition of envariance, maximally correlated mixed states are also envariant.

Now, suppose Alice has an ensemble of 100 spin-$\frac{1}{2}$ particles, with 50 states prepared in $|\uparrow\rangle$ and 50 states prepared in $|\downarrow\rangle$. She draws out a state at random and prepares another spin-$\frac{1}{2}$ system B in $|+\rangle$ if she gets $|\uparrow\rangle$, and in $|-\rangle$ if she gets $|\downarrow\rangle$; and sends it to Bob.

In this case, the state of each particle exists and is definite, irrespective of the knowledge of Alice. There is no role of entanglement, since the correlation between Bob and Alice is purely classical. What I want to emphasize is that the only kind of ignorance present here is purely *subjective*, due to lack of knowledge of Alice about which particle she is going to pick. In such a scenario, Bob must assign the mixed state (16), to the state he receives from Alice.

Thus, envariance is not a characteristic property of maximally entangled systems, whose global state is perfectly known and is thus described by a pure state. That is, envariance is not a symmetry of maximally entangled pure states alone and is thus, not a consequence of quantum entanglement (not correlation).

2.6 Conclusion

Zurek claimed that environment-assisted invariance *or* envariance is a quantum symmetry originating from perfect entanglement between the system and the environment. We showed that this symmetry is also exhibited by maximally correlated mixed states which implies that envariance is not a consequence of entanglement, but instead a consequence of correlations which can even be classical; and thus is not a distinction between subjective ignorance, due to lack of knowledge of the observer, and objective ignorance, due to perfect entanglement. Since, *only* maximally correlated states show envariance, this symmetry is rather a property of rotationally invariant states. In other words, environment-assisted invariance is just another way to portray what is already known as *rotational invariance*, since unitary transformations are just rotations from one basis to another and maximally correlated mixed or pure states are inherently rotationally invariant, that is, for each rotation in one basis (unitary transformation on one subsystem) there exists a rotation in another basis (unitary transformation on another subsystem) such that the joint state remains invariant.

3 Summary and Future Perspective

3.1 Summary

In the first part of the thesis, we discussed the article "Quantum Theory cannot consistently describe the use of itself" [2], which shows that Quantum Theory is inconsistent. We scrutinized the inferences of all the observers. The analysis led us to the understanding that, the inconsistency is a manifestation of inferences about other's measurement outcome, based on unique correlations between entangled systems. We showed some other inconsistencies associated with such inferences, and how they manifested themselves in [2]. We proposed a thought experiment in which an observer draws a prediction, based on the other observer's inference, inconsistent with the prediction from his own measurement outcome.

In the second part of the thesis, we discussed Zurek's work on environment-assisted invariance, or *envariance*[5]. Firstly, we presented Zurek's derivation of Born's Rule based on envariance[4]. We showed the problems with his derivation, and how Born's Rule can be derived using isometries. We explored further the implications of envariance as a symmetry of perfectly entangled quantum states, as claimed by Zurek. We found that this symmetry is also exhibited by maximally correlated mixed states which implies that envariance is not a consequence of entanglement, but instead a consequence of correlations which can even be classical. We further show that envariance is only a property of maximally correlated mixed or pure states, and thus is rather a property known as rotational invariance, rebranded as the term envariance. We conclude, that envariance does not quantify entanglement or quantumness as such, as claimed by Zurek, and does not answer the long withstanding question of the origin of randomness in quantum world.

3.2 Future Perspective

We would like to investigate further the implications of correlations and why exactly the inferences based on correlations lead to inconsistencies. Further, we would like to answer whether such inconsistencies are inherent in the structure of quantum mechanics and superposition, or are resolvable within quantum mechanics. If these inconsistencies are inherent, then something beyond the present structure of quantum, must be needed to resolve them or may be a more fundamental law like Heisenberg Uncertainty principle, which justifies their origin. We would like to visit Hardy's Paradox, as well, to see if such inconsistencies manifest there or not.

We would also like to explore further implications of Wigner's Friend paradox, particularly, the possibility of existence of subjective realities in quantum mechanics. It would answer a very fundamental question, whether the collapse, is absolute or just relative to the observer. And it might be a profound step towards understanding the measurement problem.

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