Non-Commutative Space-Time and BTZ Black Hole

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Certificate of Examination

This is to certify that the dissertation titled **"Non-Commutative Space-Time and BTZ Black Hole"** submitted by **Mr. Ankur** (Reg. No. MS15078) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: June 15, 2020

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr.Sanjib Dey at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr.Sanjib Dey
(Supervisor)

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Abstract

In the thesis, we have studied BTZ black holes in non-commutative space-time. The aim is to study thermodynamics of BTZ black holes in different types of non-commutative space-time. We have used metrics obtained from two different formalisms, one via Chern-Simons theory and Moyal product approach while the other ones inspired from coherent state formalism. In coherent state formalism, we have used two types of distribution (Gaussian and Lorentzian) to study charged but non-rotating black holes in non-commutative space-time. It should be noted that static and non stationary metrics were noticed via Chern-Simons theory in non-commutative space-time. To study thermodynamics of BTZ black hole in non-commutative space-time we have used quantum tunneling formalism and Hamilton Jacobi method and GUPs are used to add quantum corrections to it. A bound on non commutative parameter θ in case of Lorentzian distribution in coherent state formalism has been proposed for a particular case of BTZ black hole. To study thermodynamics in Gravity's Rainbow, a new formalism is proposed in which Gravity's Rainbow has been studied in non-commutative space-time and intrinsic temperature has been calculated. To relate thermodynamics quantities of BTZ black hole in non-commutative space-time with thermodynamics of a CFT, Holography principle has been tested and verified.

Contents

Chapter 1

Introduction to Non-Commutative Geometry

1.1 Revisiting Quantum Mechanics

Quantum physics is quite different from our old classical physics. We start quantum mechanics by assigning an operator to every classical observable and some postulates. One of the biggest effects of this formulation is Heisenberg's uncertainty principle stated below:

$$[x, p] = i\hbar$$

Obviously, x and p in above equation are operators and not classical functions.Now physical interpretation of the above result is that, we can not know momentum and position of a particle at the same time precisely. There will be some uncertainty in determining the position and momentum of the particle simultaneously. Now this result, is not because of technological restrictions. It is more fundamental, it is how our world is. Coming from classical mechanics ,it would look bizarre, but one can interpret this notion as "fuzziness" of phase space. Also, observables in quantum physics are given by quantum expectation values. But can we match these quantum expectation values to classical observables without introducing quantum operators? This question was addressed by Wigner in his paper "On quantum correction of thermodynamic equilibrium" published in 1932 [Wigner 32].

In this paper Wigner showed that one can match quantum expectation values to the good old averages in classical statistical mechanics. He did that by introducing a probability quasi-distribution (since ,this distribution can take negative values). The form of the distribution function is given below [Gosson 17]:

$$W\psi(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}py}\psi\left(x+\frac{1}{2}y\right)\overline{\psi\left(x-\frac{1}{2}y\right)}dy$$

Also, probability amplitudes of quantum mechanics are given by [Gosson 17],

$$\int_{-\infty}^{\infty} W\psi(x,p)dp = |\psi(x)|^2$$
$$\int_{-\infty}^{\infty} W\psi(x,p)dx = |\phi(p)|^2$$

For completion, one can show that for normalized ψ, the above equations will result in:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W\psi(x,p)dpdx = 1$$

Further, in the process of quantization of phase space, a new formalism was developed by Wigner [Wigner 32] . Commonly known as Wigner-Weyl Quantization. This formalism offers insights on functioning of Quantum mechanics.In this scheme, one maps a function on phase space to an operator on Hilbert space and this is known as Weyl Quantization.

Let $f\,(q_i, p_j)$ be function on phase space and it's Fourier transformation is given by $\tilde{f}(\xi,\eta)$ and let the operators $\hat{q}_i$ and $\hat{p}_j$ follow the commutation relation $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$,then the operator associated with the function f is given by:

$$\hat{f} = \Omega(f) = \int d^n\xi d^n\eta \tilde{f}(\xi,\eta) e^{\frac{i}{\hbar}(\hat{q}\cdot\xi + \hat{p}\cdot\eta)}$$

Also, the inverse of above operation is called Wigner transform and is given by,

$$\Omega^{-1}(\hat{f}) = \int d^n\xi d^n\eta \,\mathrm{Tr}\left(\hat{f}e^{-\frac{i}{\hbar}(\hat{q}\cdot\xi+\hat{p}\cdot\eta)}\right) e^{i(q\cdot\xi+p\cdot\eta)}$$

In this context,Moyal product is defined by,

$$f \star_M g = \Omega^{-1}(\Omega(f)\Omega(g))$$

Now,for f and g to be smooth functions on $\mathbb{R}^2$, one can define canonical possion bracket given by,

$$\{f,g\}(q,p) = \partial_q f \partial_p g - \partial_p f \partial_q g, \quad (q,p) \in \mathbb{R}^2$$

which can also be rewritten as,

$$\{f,g\} = f(\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q})g = \partial_q f \partial_p g - \partial_p f \partial_q g$$

Now for such case, Moyal product is given by Groenewold's formula [Gosson 17]:

$$f *_M g = f \exp[\frac{i\hbar}{2}\left(\overleftarrow{\partial}_q \cdot \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_q\right)]g$$

In this chapter and in this thesis we will be mostly dealing with Moyal product(which is a special type of star product. So, from now we will be using only * instead of $*_M$ to denote Moyal product.One can see Moyal product as deformation of usual multiplicative product.Now, let us apply the above formula for function $f = x$ and $g = p$ on a phase space (x,p) :

$$\begin{aligned} x * p &= x \exp[\frac{i\hbar}{2}\left(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_x\right)]p \\ &= x\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{i\hbar k}{2}\right)^k\left(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_x\right)^k p \\ &= xp + \frac{i\hbar k}{2} \end{aligned}$$

Similarly, we have

$$p * x = px - \frac{i\hbar k}{2}$$

Hence,

$$[x, p]_* = x * p - p * x = \mathrm{i}\hbar$$

The above analysis shows that if we replace the usual multiplicative product of functions by the Moyal product, we can have results analogous to quantum mechanics. This offers insight to our understanding of quantum mechanics.

1.2 Example:Harmonic Oscillator

Now,in this section we will study the case of harmonic oscillator but with above described Moyal product.It is known that Hamiltonian of a quantum harmonic oscillator is given by:

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\right)^2 + \frac{mw^2}{2}x^2$$

For simplicity, let us set the value of $m = 1$ and $w = 1$ and now the above Hamiltonian would become,

$$\hat{H} = -\frac{\hbar^2}{2}\left(\frac{\partial}{\partial x}\right)^2 + \frac{1}{2}x^2$$

One can easily calculate the energy eigenvalues of the above Hamiltonian operator and they are given by:

$$E_n = \hbar\left(n + \frac{1}{2}\right), n = 0, 1, 2, \cdots$$

The classical analogous of the above mentioned harmonic oscillator has Hamiltonian given by,

$$H = \frac{1}{2}\left(p^2 + x^2\right)$$

Here p and x are functions rather than operators.As we define ladder operators in quantum mechanics, let us define "ladder functions" in the same context and they would be given by,

$$a = \frac{1}{\sqrt{2\hbar}}(x + \mathrm{i}p), \quad a^\dagger = \frac{1}{\sqrt{2\hbar}}(x - \mathrm{i}p)$$

In the above context, † just represents the complex conjugation.Now in quantum mechanics we have notion of Number operator, let us define analogous function to Number operator using star product as ($N = a^\dagger * a$) given by,

$$a^\dagger * a = \frac{1}{2\hbar}\left(p * p + \mathrm{i}[x, p]_* + x * x\right) = \frac{1}{2\hbar}(p \cdot p + \mathrm{i} \cdot \mathrm{i}\hbar + x \cdot x) = \frac{1}{2\hbar}\left(p^2 + q^2\right) - \frac{1}{2}$$

Hence,total energy or Hamiltonian and the commutator of the ladder functions in this case will be given by ,

$$H = \hbar\left(N + \frac{1}{2}\right)$$

$$\left[a, a^\dagger\right]_* = a * a^\dagger - a^\dagger * a = \frac{1}{2\hbar}2(-\mathrm{i})[x, p]_* = 1$$

Let us define a function ,$f_0 = \frac{1}{\pi\hbar}\exp\left(-2aa^\dagger\right) = \frac{1}{\pi\hbar}\exp\left(-\frac{1}{\hbar}\left(p^2 + q^2\right)\right)$
Note that for the above defined function, We have these results:

$$a * f_0 = f_0 * a^\dagger = 0$$

Moyal product is associative for polynomial functions. Now let us define a whole set of functions as,

$$f_n = \frac{1}{n!}\underbrace{a^\dagger * \cdots * a^\dagger}_{n} * f_0 * \underbrace{a * \cdots * a}_{n}$$

Observe another nice property of Number function,

$$N * a^\dagger = \left(a^\dagger * a\right) * a^\dagger = a^\dagger * \left(a * a^\dagger\right) = a^\dagger * (N + 1)$$

$$N * f_1 = N * \left(a^\dagger * f_0 * a\right) = a^\dagger * (N + 1) * f_0 * a = f_1$$

$$N * f_k = f_k * N = k f_k$$

Hence, one can define the "star eigenvalues" as,

$$H * f_n = f_n * H = \hbar\left(n + \frac{1}{2}\right)f_n = E_n f_n, \quad (n = 0, 1, 2, \cdots)$$

So, we have recovered the energy eigenvalues same as of quantum mechanics but with using Moyal product.

1.3 Non-Commutative Geometry

Motivation for non-commutative space-time started with Heisenberg when he proposed that space-time algebric properties should be modified at short length scale as a way to remove infinities appearing in quantum field theories.But this idea did not gain much attention because of success of process of renormalization.The very first paper on non-commutative space-time was written by Snyder [Snyder 47]. UV finiteness of quantum field theories on non-commutative space-time have been explored in various instances.It has been found that UV behaviour of field theories is dependent on topology of space-time [Chaichian 00]. It has been argued that quantum structure of space-time is necessary in context of quantum gravity [Doplicher 01] [Doplicher 95]. According to uncertainty principle, we can measure position with great precision on the cost of very huge uncertainty in momentum.But it is fundamentally possible to know position very well.But when we apply same principle in context of gravity.To probe very small length scale would impart a lot of energy to the small region which may result in formation of black hole and hence uncertainty in localization. The other biggest problem with quantum gravity is that it is not renormalizable and hence UV divergent. Now non-commutativity of coordinates can help in this context.

Motivation from string theory has come from the possibility of non-commutativity in space-time [Seiberg 99]. This helped the non-commutative geometry to gain attention again. Loop quantum gravity also is a type of quantization of space-time, where usual space-time is replaced with "spin-connections". It has been found that quantum analogous of some geometric entities exhibit non-commutativity [Ashtekar 98]. This shows hints of non-commutative space-time, however the relation is still not clear.

Now non-commutativity of coordinates can be looked as extension of Heisenberg's uncertainty relation. Earlier we had uncertainty in knowing position and momentum, now there will be uncertainty in knowing coordinates as well. A general and common non-commutativity can be written as,

$$\left[\hat{x}^i, \hat{x}^j\right] = \hat{c}^{ij}(\hat{x}) = i\hbar\tilde{c}^{ij}(\hat{x})$$

Now,for such kind of algebra, a "generalised" star product can be defined as [Sykora 04],

$$\begin{aligned} f \star g = & fg + \frac{1}{2}c^{ij}\partial_i f \partial_j g \\ & + \frac{1}{8}c^{mn}c^{ij}\partial_m\partial_i f \partial_n \partial_j g \\ & + \frac{1}{12}c^{ml}\partial_l c^{ij}\left(\partial_m\partial_i f\partial_j g - \partial_i f \partial_m \partial_j g\right) + \mathcal{O}(3) \end{aligned}$$

Non-commutativity arising from string theory is of the form,

$$\left[X^i, X^j\right] = i\theta^{ij}$$

and for such algebra, we can define similar star product as Moyal product,

$$f * g = fg + \frac{1}{2} i\theta^{ij} \partial_i f \partial_j g + \mathcal{O}\left(\theta^2\right)$$

Note that in above equations, i,j runs from 0 to n i.e. (0,1,2,3......n).

In **Chapter 2** we have found that just by introducing non-commutativity we can make metric function of time hence, non static and non stationary. In **Chapter 3** we have done thermodynamics of Charged and rotating BTZ black hole in non-commutative space (derived via Chern-Simons theory). In **Chapter 4** we have derived non-commutative BTZ charged black hole for Lorentzian type distribution and done thermodynamics of it. We also found bound of θ in this approach. Also for Gaussian distribution we have discussed another type of metric. In **Chapter 5** we have tried to combine non-commutative geometry and gravity's rainbow and applied to BTZ black hole and find intrinsic temperature and entropy. In **Chapter 6** we have checked the consistency of Cardy-Verlinde formula for BTZ black hole in non-commutative space.

Chapter 2

chern-simons Theory on Non-Commutative Space-Time

2.1 Gauge Theory on Non-Commutative Space-Time

Gauge theory in physics is one of the most important theories.To do gauge theory in non-commutative space-time, one of the most crucial part is Seiberg Witten Map.So first we will study Seiberg Witten map introduced in [Seiberg 99]. Another important component of Gauge theory in non-commutative geometry is Moyal product or general star products that we have discussed earlier.So using these Moyal or general star product we can find Seiberg Witten Map,which connects Gauge Fields in non-commutative space-time to Gauge Fields in commutative space-time. In this thesis,"NC" stands for Non-Commutative, "CS" stands for "chern-simons" and "SW" stands for Seiberg-Witten. In this paper canonical deformation of coordinates is considered i.e. $\left[x^i, x^j\right] = i\theta^{ij}$.

For Yang-Mills in commutative space-time, we have these results,

$$\begin{aligned}
&\delta_\lambda A_i = \partial_i \lambda + i\left[\lambda, A_i\right] \\
&F_{ij} = \partial_i A_j - \partial_j A_i - i\left[A_i, A_j\right] \\
&\delta_\lambda F_{ij} = i\left[\lambda, F_{ij}\right]
\end{aligned}$$

Corresponding to above, one can now ask for non-commutative analogous of the above mentioned equation and one can do so using Moyal product and the new equations will be :

$$\begin{aligned}
&\widehat{\delta}_{\widehat{\lambda}} \widehat{A}_i = \partial_i \widehat{\lambda} + i\widehat{\lambda} * \widehat{A}_i - i\widehat{A}_i * \widehat{\lambda} \\
&\widehat{F}_{ij} = \partial_i \widehat{A}_j - \partial_j \widehat{A}_i - i\widehat{A}_i * \widehat{A}_j + i\widehat{A}_j * \widehat{A}_i \\
&\widehat{\delta}_{\widehat{\lambda}} F_{ij} = i\widehat{\lambda} * \widehat{F}_{ij} - i\widehat{F}_{ij} * \widehat{\lambda}
\end{aligned}$$

Now one can see that,For U(1) group, the gauge invariance in commutative space-time is ,

$$\delta A_i = \partial_i \lambda$$

While, even this transformation is non-trivial even for U(1) gauge group in non-commutative space-time.

$$\delta A_i = \partial_i \lambda + i\lambda * A_i - iA_i * \lambda$$

So, mapping of gauge fields in non-commutative space-time to those gauge fields in commutative space-time has to be done very carefully. As in commutative space-time A and A' are gauge equivalent by gauge transformation $U = \exp(i\lambda)$ then it would be sensible to say that $\widehat{A}$ and $\widehat{A}'$ are also gauge equivalent by the transformation $\widehat{U} = \exp(i\widehat{\lambda})$ in non-commutative space-time. But because of non-triviality even for U(1) gauge group, it must be noticed that now $\widehat{\lambda}$ is function of both λ and A and not only λ.

Now, we are looking for a map that can make sure that gauge fields corresponding to non-commutative space-time are gauge equivalent by a corresponding gauge transformation in non-commutative space-time as of ordinary gauge transformation in commutative space-time.

Mathematically, it will translate to,

$$\widehat{A}(A) + \widehat{\delta}_{\widehat{\lambda}} \widehat{A}(A) = \widehat{A}\,(A + \delta_\lambda A) \tag{2.1}$$

To solve the above equation, one can use perturbation and solve it for the first order of $\boldsymbol{\theta}$ which mathematically translates to,

$$\widehat{A} = A + A'(A)$$

$$\widehat{\lambda}(\lambda, A) = \lambda + \lambda'(\lambda, A)$$

here note that, A' and λ' are in the first order of $\boldsymbol{\theta}$ and now using Moyal product $f * g = fg + \frac{1}{2} i\theta^{ij} \partial_i f \partial_j g + \mathcal{O}\left(\theta^2\right)$, one can solve (2.1).

$$A_i'\,(A + \delta_\lambda A) - A_i'(A) - \partial_i \lambda' - i\left[\lambda', A_i\right] - i\left[\lambda, A_i'\right] = -\frac{1}{2}\theta^{kl}\left(\partial_k \lambda \partial_l A_i + \partial_l A_i \partial_k \lambda\right) + \mathcal{O}\left(\theta^2\right) \tag{2.2}$$

and (2.2) can be solved by,

$$\text{??}\;\begin{array}{l} \widehat{A}_i(A) = A_i + A_i'(A) = A_i - \frac{1}{4}\theta^{kl}\left\{A_k, \partial_l A_i + F_{li}\right\} + \mathcal{O}\left(\theta^2\right) \\ \widehat{\lambda}(\lambda, A) = \lambda + \lambda'(\lambda, A) = \lambda + \frac{1}{4}\theta^{ij}\left\{\partial_i \lambda, A_j\right\} + \mathcal{O}\left(\theta^2\right) \end{array} \tag{2.3}$$

.

2.2 BTZ Black Hole in Non-Commutative Space-Time

Now having a prescription for gauge theories in non-commutative space-time in hand. One can move forward and apply it to the chern-simons theory in commutative space-time and ask for NC chern-simons gauge fields.chern-simons theory can be related to gravity and hence this can be done to study BTZ black hole in non-commutative space-time as done in [Kawamoto 18, Chang-Young 09]. In this section that prescription done in those papers has been discussed and the possibility to get non-static and non-stationary metric by using this approach has been explored.

2.2.1 Chern-Simons in Commutative Space-Time

Now in (2+1) dimensions, we can have relation between Vielbeins fields, spin connections and two Gauge groups $SU(1,1)$ which is given by,

$$A^{(\pm)a} = \omega^a \pm \frac{1}{\ell} e^a \tag{2.4}$$

chern-simons action is given by,

$$\begin{aligned} S &= I_{\mathrm{CS}}\left[A^{(+)}\right] - I_{\mathrm{CS}}\left[A^{(-)}\right] \\ I_{\mathrm{CS}}[A] &= \frac{k}{4\pi}\int \operatorname{tr}\left[AdA + \frac{2}{3}AAA\right] \end{aligned}$$

Also in 2+1 dimensions, we have BTZ black hole (charged and rotating) whose metric is given by:

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\varphi - \frac{4GJ}{r^2}dt\right)^2 \\ &= -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2 - \frac{2\gamma}{\ell}dtd\varphi \end{aligned}$$

$$f(r) = -8GM + \frac{r^2}{\ell^2} + \frac{16G^2J^2}{r^2} - 8\pi GQ^2\ln r = \frac{1}{\ell^2}\left(-\alpha + r^2 + \frac{\gamma^2}{r^2} - \beta\ln r\right)$$

$$\alpha = 8GM\ell^2, \quad \beta = 8\pi G\ell^2Q^2, \quad \gamma = 4GJ\ell$$

and $h(r)$ is given by,

$$h(r) = f(r) - \frac{\gamma^2}{\ell^2 r^2}$$

Now for a given metric, one can form a set of valid Vielbeins field and spin connections as,

$$\begin{aligned} &e^0 = \sqrt{h(r)}dt + \tfrac{\gamma}{\ell\sqrt{h(r)}}d\varphi, \quad e^1 = \tfrac{1}{\sqrt{f(r)}}dr, \quad e^2 = r\sqrt{\tfrac{f(r)}{h(r)}}d\varphi \\ &\omega^0 = -\tfrac{\gamma h'(r)}{2\ell r\sqrt{h(r)}}dt - \sqrt{h(r)}d\varphi, \quad \omega^1 = \tfrac{\gamma h'(r)}{2\ell r h(r)\sqrt{f(r)}}dr, \quad \omega^2 = -\tfrac{h'(r)}{2}\sqrt{\tfrac{f(r)}{h(r)}}dt \end{aligned}$$

Now as described in (2.4), one can make CS gauge fields corresponding to the above set of vieblein fields and connections.

$$
\begin{aligned}
A^{(\pm)0} &= \pm\tfrac{1}{\ell}\left(\sqrt{h(r)} \mp \tfrac{h'(r)}{2r}\tfrac{\gamma}{\sqrt{h(r)}}\right)dt - \left(\sqrt{h(r)} \mp \tfrac{1}{\ell^2}\tfrac{\gamma}{\sqrt{h(r)}}\right)d\varphi \\
A^{(\pm)1} &= \tfrac{1}{\ell\sqrt{f(r)}}\left(\tfrac{\gamma h'(r)}{2rh(r)} \pm 1\right)dr \\
A^{(\pm)2} &= \tfrac{r}{\ell}\sqrt{\tfrac{f(r)}{h(r)}}\left(-\tfrac{\ell h'(r)}{2r}dt \pm d\varphi\right)
\end{aligned}
\tag{2.5}
$$

2.2.2 Chern-Simons in Non-Commutative Space-Time

In [Cacciatori 02], it was shown that for non-commutative case, Gauge group required is $U(1,1) \times U(1,1)$ rather than $SO(1,2) \times SO(1,2)$. Another reason for such a generalization can be that when ones goes from commutative case to non-commutative case SU(N) groups generally are not closed groups. While U(N) groups are closed groups even in Moyal commutator [Matsubara 00]. CS action in non-commutative space is,

$$
\hat{I}_{\text{CS}}\left[\mathcal{A}^{(\pm)}\right] = \frac{k}{4\pi}\int \text{tr}\left[\mathcal{A}^{(\pm)} \overset{\star}{\wedge} d\mathcal{A}^{(\pm)} + \frac{2}{3}\mathcal{A}^{(\pm)} \overset{\star}{\wedge} \mathcal{A}^{(\pm)} \overset{\star}{\wedge} \mathcal{A}^{(\pm)}\right]
$$

where

$$
f \overset{*}{\wedge} g = \frac{1}{i!j!} f_{\mu_1\cdots\mu_i} * g_{v_1\cdots v_j}\left(dx^{\mu_1}\cdots dx^{\mu_i}\right) \wedge \left(dx^{v_1}\cdots dx^{v_j}\right)
$$

and

$$
\mathcal{A}_\mu^{(\pm)A}\tau_A = \hat{A}_\mu^{(\pm)a}\tau_a + \hat{B}_\mu^{(\pm)}\tau_3
$$

note that here, A=0,1,2,3 while a=0,1,2 and these generators of U(1,1) follows the following properties:

$$
\tau_0 = \frac{i}{2}\sigma_3, \quad \tau_1 = \frac{1}{2}\sigma_1, \quad \tau_2 = \frac{1}{2}\sigma_2, \quad \tau_3 = \frac{i}{2}\mathbf{1}_2
$$

$$
g_{AB} = \text{tr}\left(\tau_A\tau_B\right) = \frac{1}{2}\eta_{AB}, \quad [\tau_A, \tau_B] = -\epsilon_{AB}^{C}\tau_C, \quad \epsilon_{AB}^{C} = \begin{cases} \epsilon_{ab}^{c} \\ \epsilon_{ab}^{3} = \epsilon_{3a}^{b} = 0 \end{cases}
$$

$$
\{\tau_a, \tau_b\} = \frac{1}{2}\eta_{ab}\mathbf{1}_2, \quad \{\tau_A, \tau_3\} = i\tau_A, \quad \text{tr}\left(\tau_a\tau_b\tau_c\right) = -\frac{1}{4}\epsilon_{abc}, \quad \text{tr}\left(\tau_a\tau_b\tau_3\right) = \frac{i}{4}\eta_{ab}
$$

Now, to know the metric corresponding to non-commutative case, we need chern-simons fields in non-commutative case.And for that we can use SW map to know chern-simons fields in non-commutative case in terms of ordinary chern-simons fields. But as explained above, now we have a extra field due to gauge group being U(1,1) and not SU(1,1). So to know the chern-simons fields in non-commutative case, we need information about that extra field in commutative case. From our chern-simons action above, the field equations in commutative limit will be [Kawamoto 18]:

$$F^{(\pm)a} = 0, \quad dB^{(\pm)} = 0$$

now the calculation done in [Kawamoto 18] is shown, where, B_2 is chosen to be constant B and other components of B_μ are taken to be zero. Note that this is just an assumption and we will explore more about it a bit later.

Now, Using SW map, one can find the non-commutative gauge fields: In [Kawamoto 18], the following convention is used (we will stick to this for the rest of chapter): With the gauge transformation such as :

$$\begin{aligned}\hat{\delta}_{\hat{\xi}} \hat{A}_\mu &= \partial_\mu \hat{\xi} - \hat{\xi} \star \tilde{A}_\mu + \hat{A}_\mu \star \hat{\xi} \\ &= \partial_\mu \hat{\xi} - \frac{i}{2}\theta^{\nu\rho}\left(\partial_\nu \hat{\xi} \partial_\rho \hat{A}_\mu - \partial_\nu \hat{A}_\mu \partial_\rho \hat{\xi}\right) + \mathcal{O}\left(\theta^2\right)\end{aligned}$$

one can solve for the following SW Map:

$$\hat{A}(A) + \hat{\delta}_{\hat{\xi}} \hat{A}(A) = \hat{A}\left(A + \delta_\xi A\right)$$

and the solutions upto first order correction is given by:

$$\begin{array}{l}\hat{A}_\mu(A) = A_\mu - \frac{i}{4}\theta^{v\rho}\left\{A_v, \partial_\rho A_\mu + F_{\rho\mu}\right\} + \mathcal{O}\left(\theta^2\right) \\ \hat{\xi}(\xi, A) = \xi + \frac{i}{4}\theta^{\mu v}\left\{\partial_\mu \xi, A_v\right\} + \mathcal{O}\left(\theta^2\right)\end{array} \tag{2.6}$$

Now with (2.6) in hand, one can ask solution for a particular algebra such as, $[\hat{R}, \hat{\varphi}] = 2i\theta$ where θ is a constant non-commutativity parameter and $\hat{R} = \hat{r^2}$. For such a case, (2.6) can be written explicitly as,

$$\begin{aligned}A'_\mu(A) = -\frac{i}{4}(2\theta)\Bigg[&\frac{1}{2}\eta_{ab}A^a_R\left(\partial_\varphi A^b_\mu + F^b_{\varphi\mu}\right)1 - \frac{1}{2}\eta_{ab}A^a_\varphi\left(\partial_R A^b_\mu + F^b_{R\mu}\right)1 \\ &+ i\left(A^a_R\tau_a + B_R\tau_3\right)\left(\partial_\varphi B_\mu + F^{(B)}_{\varphi\mu}\right) - i\left(A^a_\varphi\tau_a + B_\varphi\tau_3\right)\left(\partial_R B_\mu + F^{(B)}_{R\mu}\right) \\ &+ iB_R\left(\partial_\varphi A^b_\mu + F^b_{\varphi\mu}\right)\tau_b - iB_\varphi\left(\partial_R A^b_\mu + F^b_{R\mu}\right)\tau_b\Bigg]\end{aligned} \tag{2.7}$$

as mentioned above we will first discuss the case where B_ϕ =B and other components of B_μ are zero. For this case,(2.7) will be reduced to the following:

$$\begin{aligned}A^{(\pm)a'}_\mu &= -\frac{\theta B}{2}\left[\partial_R A^{(\pm)a}_\mu + F^a_{R\mu}\right] \\ B^{(\pm)\prime}_\mu &= -\frac{\theta}{2}\eta_{ab}\left[A^{(\pm)a}_R F^b_{\varphi\mu} - A^{(\pm)a}_\varphi F^b_{R\mu} - A^{(\pm)a}_\varphi \partial_R A^{(\pm)b}_\mu\right]\end{aligned}$$

Using above correction in ordinary chern-simons fields to get chern-simons fields in non-commutative space upto first order can be done as:

$$
\begin{aligned}
\hat{A}_t^{(\pm)0} &= \pm\frac{1}{\ell}\left(\sqrt{h}\mp h'\frac{\gamma}{2r\sqrt{h}}\right)-\theta B\frac{\left(2r^2-\beta\right)^2\gamma\pm 2\ell^2\left(2r^4-\beta r^2\mp 4\beta\gamma\right)h}{16\ell^5r^4h^{3/2}}\\
\hat{A}_\varphi^{(\pm)0} &= -\left(\sqrt{h}\mp\frac{1}{\ell^2}\frac{\gamma}{\sqrt{h}}\right)+\theta B\frac{\pm\gamma\left(2r^2-\beta\right)+2\ell^2\left(r^2-\beta\right)h}{8\ell^4r^2h^{3/2}}\\
\hat{A}_r^{(\pm)1} &= \frac{1}{\ell\sqrt{f}}\left(\frac{\gamma h'}{2rh(r)}\pm 1\right)+\frac{\theta B}{32\ell^7r^6h^2f^{3/2}}\left[4\gamma^3\left(2r^2-\beta\right)^2+2\ell^2h\left[3\gamma r^2\left(2r^2-\beta\right)^2-4\beta\gamma^3\right.\right.\\
&\quad\left.\left.+2\ell^2r^2h\left(\pm r^2\left(2r^2-\beta\right)+\gamma\left(2r^2-3\beta\right)\pm 2\ell^2r^2h\right)\right]\right]\\
\hat{A}_t^{(\pm)2} &= -\frac{h'}{2}\sqrt{\frac{f}{h}}+\theta B\frac{-\gamma^2\left(2r^2-\beta\right)^2+2\ell^2h\left[4\beta\gamma^2+2\ell^2r^2h\left(r^2+\beta\right)\right]}{16\ell^6r^5h^{3/2}f^{1/2}}\\
\hat{A}_\varphi^{(\pm)2} &= \pm\frac{r}{\ell}\sqrt{\frac{f}{h}}\mp\theta B\frac{2\beta\gamma\left(\pm\ell^2h+\gamma\right)+4r^2\left(\ell^4h^2-\gamma^2\right)}{16\ell^5r^3h^{3/2}f^{1/2}}\\
\hat{B}^{(\pm)} &= \left(B+\frac{\beta\theta}{4\ell^2r^2}\right)d\varphi-\beta\theta\frac{\pm r^2+2\gamma}{4\ell^3r^4}dt
\end{aligned}
$$

From these fields, one can now construct Vielbeins and quantities related to spin connection and these are given below:

$$
\hat{e}^a=\frac{\ell}{2}\left(\hat{A}^{(+)a}-\hat{A}^{(-)a}\right),\quad \hat{\omega}^a=\frac{1}{2}\left(\hat{A}^{(+)a}+\hat{A}^{(-)a}\right)
$$

$$
\begin{aligned}
\hat{e}^0 &= \left(\sqrt{h}-\theta B\frac{2r^2-\beta}{8\ell^2r^2\sqrt{h}}\right)dt+\frac{\gamma}{\ell\sqrt{h}}\left(1+\theta B\frac{2r^2-\beta}{8h\ell^2r^2}\right)d\varphi\\
\hat{e}^1 &= \left[\frac{1}{\sqrt{f}}+\theta B\frac{2\ell^2h+2r^2-\beta}{8\ell^2r^2f^{3/2}}\right]dr\\
\hat{e}^2 &= \left[r\sqrt{\frac{f}{h}}-\theta B\frac{2\ell^4r^2h^2-\left(2r^2-\beta\right)\gamma^2}{8\ell^4r^3h^{3/2}f^{1/2}}\right]d\varphi\\
\hat{\omega}^0 &= \left[-\frac{\gamma h'}{2\ell r\sqrt{h}}+\theta B\gamma\frac{8\ell^2\beta h-\left(2r^2-\beta\right)^2}{16\ell^5r^4h^{3/2}}\right]dt+\left[-\sqrt{h}+\theta B\frac{r^2-\beta}{4\ell^2r^2\sqrt{h}}\right]d\varphi\\
\hat{\omega}^1 &= \left[\frac{\gamma h'}{2\ell rh\sqrt{f}}+\theta B\gamma\right.\\
&\quad\left.\frac{2\ell^4r^2r^2\left(2r^2-3\beta\right)+2\gamma^2\left(2r^2-\beta\right)^2+\ell^2h\left(12r^6-12\beta r^4+3\beta^2r^2-4\beta\gamma^2\right)}{16\ell^7r^6h^2f^{3/2}}\right]dr,\\
\hat{\omega}^2 &= \left[-\frac{h'}{2}\sqrt{\frac{f}{h}}+\theta B\frac{4\ell^4r^2h^2\left(r^2+\beta\right)+8\beta\gamma^2\ell^2h-\left(2r^2-\beta\right)^2\gamma^2}{16\ell^6r^5h^{3/2}f^{1/2}}\right]dt-\theta B\frac{\beta\gamma}{8\ell^3r^3\sqrt{fh}}d\varphi
\end{aligned}
$$

one can easily construct the metric(upto first order of θ) corresponding to non-commutative space using these Vielbeins such as:

$$
\begin{aligned}
ds^2 =& -\left(\hat{e}^0\right)^2 + \left(\hat{e}^1\right)^2 + \left(\hat{e}^2\right)^2 \\
=& -\left[h(r) - \theta B\frac{2r^2-\beta}{4\ell^2r^2}\right]dt^2 + \left[\frac{1}{f(r)} + \theta B\frac{2h(r)\ell^2+2r^2-\beta}{4\ell^2r^2f(r)^2}\right]dr^2 + \left[r^2 - \frac{\theta B}{2}\right]d\varphi^2 - \frac{2\gamma}{\ell}dtd\varphi \\
&+ \mathcal{O}\left(\theta^2\right)
\end{aligned}
\tag{2.8}
$$

The above metric can be simplified for the case when BTZ black hole is uncharged and in that case, the above metric reduces to,

$$
\begin{aligned}
ds^2 =& -f^2dt^2 + \hat{N}^{-2}dr^2 + 2r^2N^{\phi}dtd\varphi + \left(r^2 - \frac{\theta B}{2}\right)d\varphi^2 + \mathcal{O}\left(\theta^2\right) \\
N^{\phi} =& -\frac{r_+r_-}{\ell r^2} \\
f^2 =& \frac{r^2-r_+^2-r_-^2}{\ell^2} - \frac{\theta B}{2\ell^2} \\
\hat{N}^2 =& \frac{1}{\ell^2r^2}\left[\left(r^2-r_+^2\right)\left(r^2-r_-^2\right) - \frac{\theta B}{2}\left(2r^2-r_+^2-r_-^2\right)\right]
\end{aligned}
\tag{2.9}
$$

where,

$$
\begin{aligned}
r_{\pm}^2 &= 4G\ell^2(M \pm \sqrt{M^2 - \frac{J^2}{\ell^2}}) \\
M &= \frac{r_+^2+r_-^2}{8G\ell^2}, \quad J = \frac{r_+r_-}{4G\ell}
\end{aligned}
$$

Now, we will explore more options, earlier we have taken B_φ to be a constant and now will take, $B_\varphi = B(\varphi)$ and other components of B_μ are still taken to be zero.In this case, a new correction term will also be added other than just changing the earlier constant B by $B(\varphi)$ in the metric.

From the earlier analysis, we can see that only A_μ^a are important as Vielbein fields can be constructed from them which will be used to get the metric.

Using (2.7) we can see that the new extra correction term in the A_μ^a fields will be :

$$
A_\mu^{\prime\prime a\pm} = \frac{\theta}{2}[(\partial_\varphi B_\mu)A_R^{a\pm}]
$$

Now, we know that only for $\mu = \varphi$, this correction term will not vanish since only B_φ is non-zero and function of φ.Also it is clear that for only $a = 1$ this is non-zero, as only $A_R^{1\pm}$ is non-zero. Hence,

$$
A_\varphi^{\prime\prime 1\pm} = \frac{\theta}{2}[(\partial_\varphi B_\varphi)A_R^{1\pm}]
$$

It is clear from above equation that only Vielbein field that will bear extra correction term (other than just changing B to $B(\varphi)$) is e^1. Hence the new e^1 can be calculated as,

$$\hat{e}^1 = \left[\frac{1}{\sqrt{f}} + \theta B \frac{2\ell^2 h + 2r^2 - \beta}{8\ell^2 r^2 f^{3/2}}\right] dr + \left[\frac{\theta}{2}(\partial_\varphi B)\frac{1}{\sqrt{f(r)}}\right] d\varphi$$

Now, the required metric can be computed as earlier:

$$\begin{aligned}
ds^2 &= -\left(\hat{e}^0\right)^2 + \left(\hat{e}^1\right)^2 + \left(\hat{e}^2\right)^2 \\
&= -\left[h(r) - \theta B \frac{2r^2 - \beta}{4\ell^2 r^2}\right] dt^2 + \left[\frac{1}{f(r)} + \theta B \frac{2h(r)\ell^2 + 2r^2 - \beta}{4\ell^2 r^2 f(r)^2}\right] dr^2 + \left[r^2 - \frac{\theta B}{2}\right] d\varphi^2 - \frac{2\gamma}{\ell} dt d\varphi \\
&\quad + \left[\frac{\theta}{2}(\partial_\varphi B)\frac{1}{f(r)}\right] dr d\varphi + \mathcal{O}\left(\theta^2\right)
\end{aligned} \tag{2.10}$$

One thing that can be noted here is that B is no longer a constant but a function of φ. So this is a strange thing that is happening here.We started with a metric which was just function of "r" and by going through this mechanism, we landed on a metric which is function of both (r, φ).This looks completely reasonable because we have freedom in choosing our B_μ.

Now does it mean we can make metric function of (r, φ) only?

The answer is no.We will show one more case, where metric now will be function of "t" also. This can be interesting because if we have non-static metric, it may point towards expanding or contracting BTZ black hole.

This can be done by choosing B_μ such that, $B_0 = C\varphi$, and $B_R = 0$ and $B_\varphi = Ct$, for some constant C.This is valid as it does not violate the condition on B_μ which was found earlier. With this choice of B_μ,required Vielbeins can be found by following the above machinery which can then further be used to find corresponding metric of BTZ black hole in non-commutative space. The Veilbein fields are as follow:

$$\begin{aligned}
\hat{e}^0 &= \left(\sqrt{h} - \theta C t \frac{2r^2 - \beta}{8\ell^2 r^2 \sqrt{h}}\right) dt + \frac{\gamma}{\ell\sqrt{h}}\left(1 + \theta C t \frac{2r^2 - \beta}{8h\ell^2 r^2}\right) d\varphi \\
\hat{e}^1 &= \left[\frac{1}{\sqrt{f}} + \theta C t \frac{2\ell^2 h + 2r^2 - \beta}{8\ell^2 r^2 f^{3/2}}\right] dr + \left[\frac{\theta}{2} C \frac{1}{\sqrt{f(r)}}\right] dt \\
\hat{e}^2 &= \left[r\sqrt{\frac{f}{h}} - \theta C t \frac{2\ell^4 r^2 h^2 - \left(2r^2 - \beta\right)\gamma^2}{8\ell^4 r^3 h^{3/2} f^{1/2}}\right] d\varphi
\end{aligned}$$

By using above Vielbeins one can form a metric which now will be given by(upto first order in θ):

$$
\begin{aligned}
ds^2 = &-\left(\hat{e}^0\right)^2 + \left(\hat{e}^1\right)^2 + \left(\hat{e}^2\right)^2 \\
= &-\left[h(r) - \theta Ct\frac{2r^2-\beta}{4\ell^2r^2}\right]dt^2 + \left[\frac{1}{f(r)} + \theta Ct\frac{2h(r)\ell^2+2r^2-\beta}{4\ell^2r^2f(r)^2}\right]dr^2 + \left[r^2 - \frac{\theta Ct}{2}\right]d\varphi^2 - \frac{2\gamma}{\ell}dtd\varphi \\
&+ \left[\frac{\theta}{2}C\frac{1}{f(r)}\right]drdt + \mathcal{O}\left(\theta^2\right)
\end{aligned}
\tag{2.11}
$$

Now the metric is no longer static as well as stationary. So, this is very interesting result as we started with a metric which was just function of "r" but then through this machinery to go to non-commutative space-time, we landed on a metric which is function of (t, r).we can see that now apparent horizon of this black hole is function of time as well.

There have been attempts to understand relation between non-commutative geometry and early universe [Marcolli 10]. Also in cosmology, dark matter and dark energy are still an ongoing area of research with a very little knowledge about them in hand. Understanding relation between dark energy,dark matter and non-commutative geometry has been explored in [Kuhfittig 17]. Now, since the metric has become time dependent in our BTZ black hole scenario due to non-commutativity of space-time, can this be somehow related to expansion of universe itself?

Can non-commutative structure of space-time be the cause of expansion of universe or at least it has some contribution in it? It can be very interesting to explore and since, it has been shown that by making the geometry non-commutative, we can have non-static and non-stationary black holes can be the first towards it.It can be an indication of their deeper relation.

2.3 Discussion

It would be interesting to see other types of non-commutative algebras like $\left[x^i, x^0\right] = iax^i$, where non-commutativity is explicitly between time and space coordinates and not between space-space coordinates. An attempt has been done in [Banerjee 07] to study Lie algebraic structure type non-commutativity. In that correction term to the gauge field is of the following form:

$$
\hat{A}_\mu(A) = A_\mu - \frac{1}{4}\theta^{\nu\lambda}\left\{A_\nu, \partial_\lambda A_\mu + F_{\lambda\mu}\right\} - \frac{1}{4}\theta_{\mu\nu}\theta^{\lambda\sigma}\partial_\sigma\theta^{\nu\delta}\left\{A_\lambda, A_\delta\right\}
$$

One can use this to study the algebra described above but in (2+1) dimension $\theta_{\mu\nu}$ would not be invertible and hence it would not be applicable.

Another attempt has been made in [Sykora 04] to generalize the Seiberg-Witten map for any general type of non-commutative algebra of co-ordinates. In that generalised star product mentioned in chapter 1 and formality map has been used. The correction terms in gauge fields are as follow:

$$A_X = X^n a_n + \frac{i}{4} c^{kl} X^n \{a_k, \partial_l a_n + f_{ln}\} + \frac{i}{4} c^{kl} \partial_l X^n \{a_k, a_n\}$$

In the proposed method, for algebras which are not of the form where $\theta_{\mu\nu}$ is constant there the commutative limit does not go to gauge fields on commutative flat space-time but rather curved space-time.

In previous section we have seen that introducing non-commutativity can lead to non-static metric. That affect can be more generalized if we take "space-time " non-commutativity as in that case we can have term like $B(t)$ in metric itself and with some bound on form of $B(t)$ and more freedom to pick the function. And looking at it's effect on cosmology would be very interesting and as well as to explore if that has any relation to dark energy and evaporation of black holes.

Chapter 3

Thermodynamics of BTZ Black Hole in Non-Commutative Space-Time

3.1 Black Hole Thermodynamics Laws

Bekenstein was the first one to point out that, Black hole entropy is related to area of the black hole,in fact directly proportional to it [Page 05]. This is known as Bekenstein-Hawking entropy which is given by,

$$S_{\mathrm{bh}} = S_{\mathrm{BH}} \equiv \frac{1}{4}A$$

Note that in this thesis, wherever there is "bh" in subscript, it means black hole while "BH" in subscript means Bekenstein-Hawking. Also in [Page 05] there are four laws of black hole thermodynamics which are similar to our good old known "laws of thermodynamics".They are mentioned below briefly,

Laws	bh Thermodynamics	Thermodynamics
Zeroth	κ is constant over it's event horizon for stationary bh	T is constant for a system in thermal equilibrium
1st	$\delta M = \frac{1}{8\pi}\kappa\delta A + \Omega\delta J + \Phi\delta Q$	$\Delta U = Q - W$
2nd	A can not decrease	S can not decrease for closed system
3rd	κ can not reduced to zero in finite steps	T can not reduced to zero in finite steps

Table 3.1: bh thermodynamics vs thermodynamics

3.2 Quantum Tunneling and Hawking Temperature

Particles can be emitted from Black holes was mentioned in [Hawking 75] when Hawking brought quantum physics to classical general relativity and since then black hole thermodynamics has been an interesting topic of research.
As we know that in quantum mechanics, particle tunnels through infinite potential, this picture can be applied to black holes too and particles can tunnel though black hole and one can associate temperature to such black hole.This was explored in [Banerjee 08b]. We will discuss that formalism in this section. For a metric of the form:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2$$

In this metric, there is a coordinate singularity at $r = r_h$ as $f(r_h) = g(r_h) = 0$ .Since this is just because of bad choice of coordinate system at horizon. We can get rid off it by just coordinate transformation by making the metric regular at horizon. The need to do is that, in this formalism tunneling probability of a particle through "horizon" is calculated and related to temperature. Since, "horizon" is of utmost importance here, we need metric to be regular horizon. Fortunately, it can be done easily by special transformations known as "Painleve Transformations" which is as follows,

$$dt \to dt - \sqrt{\frac{1-g(r)}{f(r)g(r)}}dr$$

Now our old metric "not regular" at horizon will change to "regular" metric at horizon of Black hole and is given by,

$$ds^2 = -f(r)dt^2 + 2f(r)\sqrt{\frac{1-g(r)}{f(r)g(r)}}dtdr + dr^2 + r^2 d\Omega^2$$

With the above metric in hand, we can ask for radial null geodesics i.e. $ds^2 = d\Omega^2 = 0$ and it can be simplified as,

$$\dot{r} \equiv \frac{dr}{dt} = \sqrt{\frac{f(r)}{g(r)}}(\pm 1 - \sqrt{1-g(r)})$$

Now, the above expression can be further simplified by using Taylor expansion of functions f and g near horizon as we are only interested in particles tunneling from near horizon.

$$f(r) = f'(r_h)(r - r_h) + \mathcal{O}\left((r - r_h)^2\right)$$
$$g(r) = g'(r_h)(r - r_h) + \mathcal{O}\left((r - r_h)^2\right)$$

and by using our above approximations, we can rewrite the radial null geodesics as,

$$\dot{r} = \frac{1}{2}\sqrt{f'(r_h)\, g'(r_h)}\,(r - r_h) + \mathcal{O}\left((r - r_h)^2\right)$$

The surface gravity for the metric which is regular at horizon can be calculated as,

$$\mathcal{K}(M) = \Gamma^0_{00}\Big|_{r=r_h} = \frac{1}{2}\left[\sqrt{\frac{1-g(r)}{f(r)g(r)}}\, g(r)\frac{df(r)}{dr}\right]\Bigg|_{r=r_h}$$

Again by using Taylor expansion we can simplify this expression for near horizon approximation

$$\mathcal{K}(M) \simeq \frac{1}{2}\sqrt{f'(r_h)\, g'(r_h)}$$

It is clear from the above calculation that there is relation between radial null geodesics and surface gravity which is in favour of quantum tunneling formalism for calculating temperature as surface gravity is directly linked to Hawking temperature.

$$\dot{r} = \mathcal{K}(M)(r - r_h) + \mathcal{O}\left((r - r_h)^2\right)$$

Imaginary part of the action for the shell crossing from r_{in} to r_{out} can be written as [Banerjee 08b, Parikh 00],

$$\operatorname{Im}\mathcal{S} = \operatorname{Im}\int_{r_{\mathsf{in}}}^{r_{\mathsf{out}}} p_r\, dr = \operatorname{Im}\int_{r_{\mathsf{in}}}^{r_{\mathsf{out}}}\int_0^{p_r} dp'_r\, dr$$

One can rewrite this expression by using hamiltonian equation $\dot{r} = \frac{dH}{dp_r}\Big|_r$,

$$\operatorname{Im}\mathcal{S} = \operatorname{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^H \frac{dH'}{\dot{r}}\, dr$$

This integral can be evaluated as in [Banerjee 08b, Parikh 00] to give expression of temperature as entropy and this integral is related by quantum tunneling probability.
As we know tunneling probability can be related to entropy as entropy is realted to total number of microstates by, $S_{\mathrm{bh}} = \log\Omega$ we have entropy related to tunneling probability by the following relation,

$$\Gamma = \frac{P_{\mathsf{emission}}}{P_{\mathsf{absorption}}} = e^{S_f - S_i} = e^{\Delta S_{\mathsf{bh}}}$$

and thus,

$$\Gamma \sim e^{-2\,\mathrm{lm}\,S} = e^{\Delta S_{\mathrm{bh}}}$$

Which gives relation between entropy and the action for energy shell crossing through horizon.

$$\Delta S_{\mathrm{bh}} = -2\,\mathrm{Im}\,\mathcal{S}$$

And thus the Hawking temperature can be calculated as in [Banerjee 08b],

$$T_h = \frac{\mathcal{K}}{2\pi} = \frac{1}{4\pi}\sqrt{f'\left(r_h\right) g'\left(r_h\right)} \tag{3.1}$$

3.3 Chargeless BTZ Black Hole in Non-Commutative Space-Time

3.3.1 Semi-Classical Treatment

In the previous chapter, various metrics were obtained which describes a BTZ black hole in non-commutative space.In this section we will explore the thermodynamics of one special metric which is chargeless and this work has been described in [Anacleto 18]. In the units 8G=1=c, (2.9) metric mentioned in the previous chapter can be rewritten by doing the following transformation,

$$d\phi = d\varphi - \frac{J}{2\left(1 - \theta B/2r^2\right) r^2} dt$$

Now after performing the above transformation, our metric will come in the form in which we can apply tunneling formalism result to calculate temperature of this BTZ black hole as in [Anacleto 18],

$$ds^2 = -\mathcal{F}dt^2 + \mathcal{Q}^{-1}dr^2 + \left(1 - \frac{\theta B}{2r^2}\right) r^2 d\phi^2$$

$$\mathcal{F} = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} + \frac{\theta B J^2}{8r^4} - \frac{\theta B}{2l^2}$$
$$\mathcal{Q} = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\theta B}{2}\left(\frac{2}{l^2} - \frac{M}{r^2}\right)$$

Hence now temperature of this black hole can be calculated via the formula found in previous section (3.1) and by using simply first law of black hole thermodynamics one can find entropy too for the above mentioned metric [Anacleto 18]. Note that here, $\hat{r}_+$ denotes the event horizon of the black hole with non-commutative correction while r_+ simply denotes horizon of black hole in commutative space-time.

$$
\begin{aligned}
\mathcal{T}_H &= \frac{\bar{\kappa}}{4\pi} = \frac{\sqrt{\mathcal{F}'\left(\hat{r}_+\right)\mathcal{Q}'\left(\hat{r}_+\right)}}{4\pi} \\
&= \frac{2\hat{r}_+}{4\pi l^2}\left(1-\frac{l^2J^2}{4\hat{r}_+^4}\right)\sqrt{1-\left[\frac{\theta B l^2 M}{2\hat{r}_+^4}+\frac{\theta B l^2 J^2}{4\hat{r}_+^6}\left(1-\frac{l^2M}{2\hat{r}_+^2}-\frac{l^2J^2}{4\hat{r}_+^4}\right)\right]\left(1-\frac{l^2J^2}{4\hat{r}_+^4}\right)^{-2}}+\mathcal{O}\left(\theta^2\right)
\end{aligned} \tag{3.2}
$$

To calculate entropy we can use 1st law of black hole thermodynamics as for a rotating black hole as,

$$
dM = \mathcal{T}_H dS + \Omega dJ
$$

but we also know that M is function of $\hat{r}_+$ and J which means that dM can also be written as the following ($\Omega = \frac{\partial M}{\partial J}$),

$$
dM = \frac{\partial M}{\partial \hat{r}_+} d\hat{r}_+ + \Omega dJ
$$

Now, it is very clear from the above two equations that entropy can be written as,

$$
dS = \frac{1}{T_H}\frac{\partial M}{\partial \hat{r}_+} d\hat{r}_+ \tag{3.3}
$$

We can find M by putting $Q = 0$ for $r = \hat{r}_+$

$$
\begin{aligned}
M &= \left(1-\frac{\theta B}{2\hat{r}_+^2}\right)^{-1}\left[\frac{\hat{r}_+^2}{l^2}+\frac{J^2}{4\hat{r}_+^2}-\frac{\theta B}{l^2}\right] \\
&= \frac{\hat{r}_+^2}{l^2}+\frac{J^2}{4\hat{r}_+^2}+\theta B\left(\frac{J^2}{8\hat{r}_+^4}-\frac{1}{2l^2}\right)+\mathcal{O}\left(\theta^2\right)
\end{aligned}
$$

and thus,

$$
\frac{\partial M}{\partial \hat{r}_+} = \frac{2\hat{r}_+}{l^2}-\frac{J^2}{2\hat{r}_+^3}\left(1+\frac{\theta B}{\hat{r}_+^2}\right)+\mathcal{O}\left(\theta^2\right) = \frac{2\hat{r}_+}{l^2}\left(1-\frac{l^2J^2}{4\hat{r}_+^4}\right)-\frac{\theta B J^2}{\hat{r}_+^5}
$$

Now by using above equation in (3.3), we can finally get to semi-classical entropy which can be found as,

$$
\begin{aligned}
\hat{S} = 4\pi \int \Bigg\{ & 1+\left[\frac{\theta B}{4\hat{r}_+^2}+\frac{\theta B l^2 J^2}{8\hat{r}_+^6}\left(1-\frac{3l^2J^2}{8\hat{r}_+^4}\right)\right]\left(1-\frac{l^2J^2}{4\hat{r}_+^4}\right)^{-2} \\
& -\frac{\theta B J^2}{\hat{r}_+^5}\left[\frac{2\hat{r}_+}{l^2}\left(1-\frac{l^2J^2}{4\hat{r}_+^4}\right)\right]^{-1}+\mathcal{O}\left(\theta^2\right)\Bigg\} d\hat{r}_+
\end{aligned}
$$

Note that above expression gets back to usual entropy if θ goes to zero.
In terms of the horizon of commutative space-time, the above expression can be rewritten

as,

$$\hat{S} = 4\pi \int \left(1 - \frac{\theta B}{4r_+^2}\right) \left\{1 + \left[\frac{\theta B}{4r_+^2} + \frac{\theta B l^2 J^2}{8r_+^6}\left(1 - \frac{3l^2J^2}{8r_+^4}\right)\right]\left(1 + \frac{l^2J^2}{2r_+^4}\right) - \frac{\theta B l^2 J^2}{2r_+^6}\left(1 + \frac{l^2J^2}{4r_+^4}\right) + \mathcal{O}\left(\theta^2\right)\right\} dr_+$$
$$= 4\pi r_+ + \frac{\pi B^2\theta^2}{12r_+^3} + \frac{\pi B\theta l^2 J^2}{5r_+^5} + \frac{7\pi B\theta l^4 J^4}{144r_+^9} + \frac{3\pi B\theta l^6 J^6}{416r_+^{13}} + \cdots$$

Hence, θ correction to entropy of BTZ black hole in non-commutative space-time will start from 2nd order if $J = 0$.

3.3.2 Quantum Correction

Generalized Uncertainty Principle

In [Anacleto 18], generalized uncertainty principle is used to find quantum correction to the entropy and temperature that were found in the previous section.We know that uncertainty principle is give by,

$$\Delta x \Delta p \geq \hbar$$

There has been generalized version of the above formula to incorporate minimum length. For a long time, a minimum fundamental length is considered to be in theory of quantum gravity. Like in string theory we have fundamental length of string. Such notion is carried out in other versions of quantum gravity too.
The one possible extension of the above uncertainty principle is discussed in [Hai-Xia 07] and it has been used to calculate corrections to the Bekenstein-Hawking temperature. The generalized uncertainty principle used in [Hai-Xia 07] is,

$$\Delta x_i \geq \frac{\hbar}{\Delta p_i} + \alpha^2 l_{pl}^2 \frac{\Delta p_i}{\hbar} \tag{3.4}$$

Also, in [Anacleto 18] a more general version of the above uncertainty principle is discussed which is given by (α is the deformation parameter),

$$\Delta x \Delta p \geq \hbar \left(1 - \frac{\alpha l_p}{\hbar}\Delta p + \frac{\alpha^2 l_p^2}{\hbar^2}(\Delta p)^2\right)$$

Now, this can be further written as,

$$\Delta p \geq \frac{\hbar\left(\Delta x + \alpha l_p\right)}{2\alpha^2 l_p^2}(1 - \sqrt{1 - \frac{4\alpha^2 l_p^2}{(\Delta x + \alpha l_p)^2}})$$

For such a relation there exist a minimum bound on the uncertainty in position and hence a maximum bound on the uncertainty in momentum which will be given by,

$$\Delta x \geq (\Delta x)_{\min} \approx \alpha l_p$$

and,

$$\Delta p \leq (\Delta p)_{\max} \approx \hbar/\left(\alpha l_p\right)$$

Using Taylor expansion, it can be further simplified as$(G = c = k_B = \hbar = l_p = 1)$,

$$\Delta p \geq \frac{1}{2\Delta x}\left[1 - \frac{\alpha}{2\Delta x} + \frac{\alpha^2}{2(\Delta x)^2} + \cdots\right]$$

for $\alpha = 0$, one can get the usual relation (absorbing 2 in Δx),

$$\Delta x \Delta p \geq 1$$

Now, we know that for massless particle the energy momentum relation can be given by,

$$E^2 = p^2$$

if we assume, $p \sim \Delta p \geq 1/\Delta x$, then we will have,

$$E\Delta x \geq 1$$

And hence, we will have energy corresponding to the generalized uncertainty principle as,

$$E_{GUP} \geq E\left[1 - \frac{\alpha}{2(\Delta x)} + \frac{\alpha^2}{2(\Delta x)^2} + \cdots\right]$$

Now, we can replace the E by E_{GUP} in tunneling formalism to get temperature which now will be given by,

$$T \leq \mathcal{T}_H\left[1 - \frac{\alpha}{2(\Delta x)} + \frac{\alpha^2}{2(\Delta x)^2} + \cdots\right]^{-1} \tag{3.5}$$

Note that here $\mathcal{T}_H$ is same as calculated in (3.2)

Now if we use the generalised uncertainty principle of the form (3.4) then, by following the same procedure, the temperature will be given by,

$$T \leq \mathcal{T}_H\left[1 + \left(\frac{\alpha^2 l_{pl}^2}{(\Delta x)^2}\right) + 2\left(\frac{\alpha^2 l_{pl}^2}{(\Delta x)^2}\right)^2 + \cdots\right]^{-1}$$

In $l_{pl} = 1$, this expression would be written as,

$$T \leq \mathcal{T}_H \left[1 + \left(\frac{\alpha^2}{(\Delta x)^2}\right) + 2\left(\frac{\alpha^2}{(\Delta x)^2}\right)^2 + \cdots\right]^{-1} \quad (3.6)$$

Now we can compare expression (3.5) and (3.6). It is clear that all the terms that are contained in expression (3.6), are also contained in expression (3.5) which should have been the case. But which terms should be there? and which terms should be absent? can we modify generalized uncertainty principle that it gives only particular favourable terms in quantum correction to temperature? If so, then what terms should be in favour and hence which form of generalized principle should be correct?

One reasonable guess for choice of the Δx could be the radius of horizon itself i.e. $\Delta x = \hat{r}_+$. Once this is done, we can use the expression (3.3) to calculate quantum correction to the entropy. In [Anacleto 18] entropy has been calculated for a special case for which the black hole is not rotating and in that $\Delta x = 2\hat{r}_+$ is set in (3.5) and (3.3) is used.

$$S_{GUP} = \int \frac{1}{T_{GUP}} \frac{\partial M}{\partial \hat{r}_+} d\hat{r}_+ = 4\pi \int \left(1 - \frac{\theta B}{4r_+^2}\right) \left\{\left(1 + \frac{\theta B}{4r_+^2}\right)\left[1 - \frac{\alpha}{4r_+}\left(1 - \frac{\theta B}{4r_+^2}\right) + \frac{\alpha^2}{8r_+^2}\left(1 - \frac{\theta B}{2r_+^2}\right)\right]\right\} dr_+$$
$$= 4\pi r_+ + \frac{\pi\theta^2 B^2}{12 r_+^3} - \pi\alpha \ln(r_+) - \frac{1}{8}\frac{\pi\alpha\theta B}{r_+^2} - \frac{1}{2}\frac{\pi\alpha^2}{r_+} + \frac{1}{12}\frac{\pi\alpha^2\theta B}{r_+^3} + \cdots$$

It must be noted that in semi-classical entropy there was no correction term of 1st order in θ but here we can clearly see that because of deformation of uncertainty principle, there has been contribution of 1st order in θ to the quantum correction of entropy. Also, it is important that logarithmic correction to entropy has been observed which was also observed in various other approaches [Bargueño 15, Das 02].

Hamilton-Jacobi Method

In [Modak 09, Banerjee 08a], quantum correction were added to Hamilton-Jacobi method for a BTZ black hole in commutative space-time. That method can also be applied to a general metric of the form:

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{G(r)} + r^2 d\Omega^2$$

For a massless particle in the space-time described by above metric, we can write K-G equation as follows,

$$-\frac{\hbar^2}{\sqrt{-g}} \partial_\mu \left[g^{\mu\nu}\sqrt{-g}\partial_\nu\right] \phi = 0$$

Like earlier, we are only concerned with the radial trajectories and hence K-G equation will be simplified to the following,

$$\frac{\partial^2\phi}{\partial t^2} - \frac{1}{2}\frac{\partial(FG)}{\partial r}\frac{\partial\phi}{\partial r} - FG\frac{\partial^2\phi}{\partial r^2} = 0$$

One can start by assuming the solution of the above equation as,

$$\phi(r,t) = \exp\left[-\frac{i}{\hbar}\mathcal{S}(r,t)\right]$$

Putting the above solution in K-G equation, we will have following partial differential equation,

$$\left(\frac{\partial S}{\partial t}\right)^2 - FG\left(\frac{\partial S}{\partial r}\right)^2 + i\hbar\left[\frac{\partial^2 S}{\partial t^2} - \frac{1}{2}\frac{\partial(FG)}{\partial r}\frac{\partial S}{\partial r} - FG\frac{\partial^2 S}{\partial r^2}\right] = 0$$

To find the quantum correction we can write $\mathcal{S}(r,t)$ as,

$$\mathcal{S}(r,t) = \mathcal{S}_0(r,t) + \sum_i \hbar^i \mathcal{S}_i(r,t)$$

For each order of $\hbar$, the partial differential equations will be of the form:

$$\hbar^0 : \left(\frac{\partial S}{\partial t}\right)^2 - AB\left(\frac{\partial S}{\partial r}\right)^2 = 0$$

$$\hbar^1 : \frac{\partial S_0}{\partial t}\frac{\partial S_1}{\partial t} - FG\frac{\partial S_0}{\partial r}\frac{\partial S_1}{\partial r} + \frac{i}{2}\left[\frac{\partial^2 S_0}{\partial t^2} - \frac{1}{2}\frac{\partial(FG)}{\partial r}\frac{\partial S_0}{\partial r} - FG\frac{\partial^2 S_0}{\partial r^2}\right] = 0$$

$$\hbar^2 : \left(\frac{\partial S_1}{\partial t}\right)^2 + 2\frac{\partial S_0}{\partial t}\frac{\partial S_2}{\partial t} - FG\left(\frac{\partial S_1}{\partial r}\right)^2 - 2FG\frac{\partial S_0}{\partial r}\frac{\partial S_2}{\partial r} + i\left[\frac{\partial^2 S_1}{\partial t^2} - \frac{1}{2}\frac{\partial(FG)}{\partial r}\frac{\partial S_1}{\partial r} - FG\frac{\partial^2 S_1}{\partial r^2}\right] = 0$$

In general we will have the solution of the above set of equation as,

$$\hbar^k : \frac{\partial S_k}{\partial t} = \pm\sqrt{F(r)G(r)}\frac{\partial S_k}{\partial r}$$

As, S_k are proportional to S_0, we can write the solution as(γ_i 's have the dimension $\hbar^{-i}$),

$$\mathcal{S}(r,t) = \left(1 + \sum_i \gamma_i \hbar^i\right)\mathcal{S}_0(r,t)$$

We can write γ_i 's to be of the form, $\frac{\beta_i}{r_+}$ as r_+ have dimensions of $\hbar$ and β_i 's are dimensionless constants.

$$\mathcal{S}(r,t) = \left(1 + \sum_i \frac{\beta_i \hbar^i}{r_+^i}\right) \mathcal{S}_0(r,t)$$

Because of symmetry of the metric, $\mathcal{S}_0(r,t)$ can be the written as (ω is the energy of the particle,

$$\mathcal{S}_0(r,t) = \omega t + \tilde{\mathcal{S}}_0(r)$$

solving the differential equation for the $\mathcal{S}_0(r,t)$ will result in($\pm$ denotes ingoing and outgoing particle),

$$\tilde{\mathcal{S}}_0(r) = \pm\omega \int_C \frac{dr}{\sqrt{F(r)G(r)}}$$

Probabilities for the ingoing and outgoing particles will be,

$$P_{\mathsf{in}} = |\phi_{\mathsf{in}}|^2 = \exp\left[\frac{2}{\hbar}\left(1 + \sum_i \beta_i \frac{\hbar^i}{r_+^i}\right)\left(\omega \operatorname{Im} t + \omega \operatorname{Im} \int_C \frac{dr}{\sqrt{F(r)G(r)}}\right)\right]$$

$$P_{\mathsf{out}} = |\phi_{\mathsf{out}}|^2 = \exp\left[\frac{2}{\hbar}\left(1 + \sum_i \beta_i \frac{\hbar^i}{r_+^i}\right)\left(\omega \operatorname{Im} t - \omega \operatorname{Im} \int_C \frac{dr}{\sqrt{F(r)G(r)}}\right)\right]$$

we know that classically there should be no tunneling outside, hence $P_{\mathsf{in}} = 1$ which implies,

$$\operatorname{Im} t = -\operatorname{Im} \int_C \frac{dr}{\sqrt{F(r)G(r)}}$$

Using the above relation in P_{out} we will have,

$$P_{\mathsf{out}} = \exp\left[-\frac{4}{\hbar}\omega\left(1 + \sum_i \beta_i \frac{\hbar^i}{r_+^i}\right) \operatorname{Im} \int_C \frac{dr}{\sqrt{F(r)G(r)}}\right]$$

for $P_{\mathsf{in}} = 1$, P_{out} can be written as,

$$P_{\mathsf{out}} = \exp\left(-\frac{\omega}{T_h}\right)$$

and hence, quantum correction to semi-classical Hawking temperature (T) can be calculated as,

$$T_h = T\left(1 + \sum_i \beta_i \frac{\hbar^i}{r_+^i}\right)^{-1} \tag{3.7}$$

Now, if we apply (3.7) to the (3.2) we will have,

$$T_h = \mathcal{T}_H \left(1 + \sum_i \beta_i \frac{\hbar^i}{r_+^i}\right)^{-1} \tag{3.8}$$

One thing is clear that the expression (3.8) can be cast into (3.5) or (3.6) for particular choices of β_i's. Also it has been found that generalized uncertainty principle is consistent with corpuscular gravity [Buoninfante 19].
Consistency of generalized uncertainty principle with every result from different approaches shows that generalized uncertainty principle or minimum length models have a deeper connection with quantum gravity.

3.4 Charged BTZ Black Hole in Non-Commutative Space-Time

In (2.8) a metric is mentioned for a charged Black hole in non-commutative space. In this section, thermodynamics of that metric will be studied.

If one performs the following transformation that was also performed in the previous section, we can get rid of $dtd\varphi$ term in the metric and thus we can perform tunneling approach to calculate temperature and hence the entropy of the Black Hole.

$$d\phi = d\varphi - \frac{J}{2\left(1 - \theta B/2r^2\right) r^2} dt$$

By doing the above transformation on (2.8), the metric will be of the form ($8G = c = 1$):

$$ds^2 = -\mathcal{F} dt^2 + \mathcal{Q}^{-1} dr^2 + \left(1 - \frac{\theta B}{2r^2}\right) r^2 d\phi^2$$

$$\begin{aligned} \mathcal{F} &= -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \pi q^2 \ln r + \frac{\theta B J^2}{8r^4} - \frac{\theta B}{2l^2} + \frac{\theta B \pi q^2}{4r^2} \\ \mathcal{Q} &= -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \pi q^2 \ln r - \frac{\theta B}{2}\left(\frac{2}{l^2} - \frac{M}{r^2} - \frac{\pi q^2}{2r^2}\right) \end{aligned}$$

Now using the temperature calculated from semi-classical treatment as (3.1), we can calculate the temperature of the metric as,

$$\mathcal{T}_H = \frac{\bar{\kappa}}{4\pi} = \frac{\sqrt{\mathcal{F}'(\hat{r}_+)\,\mathcal{Q}'(\hat{r}_+)}}{4\pi}$$
$$= \frac{2\hat{r}_+ x}{4\pi l^2}\sqrt{1-\theta B\left[\frac{l^2M}{2\hat{r}_+^4}+\frac{\pi q^2 l^2}{2\hat{r}_+^4}\left(1-\frac{\pi q^2l^2}{2\hat{r}_+^2}-\frac{Ml^2}{2\hat{r}_+^2}\right)+\frac{l^2J^2}{4\hat{r}_+^6}\left(1-\frac{l^2M}{2\hat{r}_+^2}-\frac{l^2J^2}{4\hat{r}_+^4}-\frac{\pi q^2l^2}{\hat{r}_+^2}\right)\right]x^{-2}+\mathcal{O}\left(\theta^2\right)} \tag{3.9}$$

here,

$$x = \left(1-\frac{l^2J^2}{4\hat{r}_+^4}-\frac{\pi q^2l^2}{2\hat{r}_+^2}\right)$$

We can find M by putting $Q = 0$ for $r = \hat{r}_+$

$$M = \left(1-\frac{\theta B}{2\hat{r}_+^2}\right)^{-1}\left[\frac{\hat{r}_+^2}{l^2}+\frac{J^2}{4\hat{r}_+^2}-\pi q^2\ln r-\theta B\left(\frac{1}{l^2}-\frac{\pi q^2}{4\hat{r}_+^2}\right)\right]$$
$$= \frac{\hat{r}_+^2}{l^2}+\frac{J^2}{4\hat{r}_+^2}-\pi q^2\ln r+\theta B\left(\frac{J^2}{8\hat{r}_+^4}-\frac{1}{2l^2}-\frac{\pi q^2\ln r}{2\hat{r}_+^2}+\frac{\pi q^2}{4\hat{r}_+^2}\right)+\mathcal{O}\left(\theta^2\right)$$

and thus,

$$\frac{\partial M}{\partial \hat{r}_+} = \frac{2\hat{r}_+}{l^2}\left(1-\frac{l^2J^2}{4\hat{r}_+^4}-\frac{\pi q^2l^2}{2\hat{r}_+^2}\right)+\theta B\left(-\frac{J^2}{2\hat{r}_+^5}-\frac{\pi q^2}{\hat{r}_+^3}+\frac{\pi q^2\ln r}{\hat{r}_+^3}\right)$$

Now expression to calculate entropy for charged black hole will be same as of (3.3) as,

$$dM = \frac{\partial M}{\partial \hat{r}_+}d\hat{r}_+ + \Omega dJ + \Phi dq$$

Also $(\Omega = \frac{\partial M}{\partial J}, \Phi = \frac{\partial M}{\partial q})$,

$$dM = \mathcal{T}_H dS + \Omega dJ + +\Phi dq$$

$$dS = \frac{1}{T_H}\frac{\partial M}{\partial \hat{r}_+}d\hat{r}_+ \tag{3.10}$$

$$\hat{S} = 4\pi\int\left\{1+\left[\frac{\theta B}{4\hat{r}_+^2}+\frac{\theta B l^2J^2}{8\hat{r}_+^6}\left(1-\frac{3l^2J^2}{8\hat{r}_+^4}\right)\right]\left(1-\frac{l^2J^2}{4\hat{r}_+^4}-\frac{\pi q^2l^2}{2\hat{r}_+^2}\right)^{-2}\right.$$
$$+\left[\theta B\frac{\pi q^2l^4}{8\hat{r}_+^4}\left(\frac{1}{l^2}-\frac{5J^2}{4\hat{r}_+^4}-\frac{\pi q^2}{\hat{r}_+^2}\right)\right]\left(1-\frac{l^2J^2}{4\hat{r}_+^4}-\frac{\pi q^2l^2}{2\hat{r}_+^2}\right)^{-2}$$
$$+\left[\theta B\frac{\pi q^2l^4\ln r}{8\hat{r}_+^4}\left(\frac{-2}{l^2}+\frac{J^2}{2\hat{r}_+^4}+\frac{\pi q^2}{\hat{r}_+^2}\right)\right]\left(1-\frac{l^2J^2}{4\hat{r}_+^4}-\frac{\pi q^2l^2}{2\hat{r}_+^2}\right)^{-2}$$
$$\left.+\theta B\left(-\frac{J^2}{2\hat{r}_+^5}-\frac{\pi q^2}{\hat{r}_+^3}+\frac{\pi q^2\ln r}{\hat{r}_+^3}\right)\left[\frac{2\hat{r}_+}{l^2}\left(1-\frac{l^2J^2}{4\hat{r}_+^4}-\frac{\pi q^2l^2}{2\hat{r}_+^2}\right)\right]^{-1}+\mathcal{O}\left(\theta^2\right)\right\}d\hat{r}_+$$

By solving the above integral we can have expression for semi-classical entropy.

One can add quantum correction to temperature (3.9) by the methods describe in previous section and then again use (3.3) to calculate quantum correction to entropy.

Quantum corrections to temperature can be:

$$T \leq \mathcal{T}_H \left[1 - \frac{\alpha}{2(2\hat{r}_+)} + \frac{\alpha^2}{2(2\hat{r}_+)^2} + \cdots\right]^{-1} \tag{3.11}$$

$$T \leq \mathcal{T}_H \left[1 + \left(\frac{\alpha^2}{(2\hat{r}_+)^2}\right) + 2\left(\frac{\alpha^2}{(2\hat{r}_+)^2}\right)^2 + \cdots\right]^{-1} \tag{3.12}$$

$$T_h = \mathcal{T}_H \left(1 + \sum_i \beta_i \frac{\hbar^i}{r_+^i}\right)^{-1} \tag{3.13}$$

In chapter 2, other two metric (2.10),(2.11) that were derived are also functions of φ and t respectively.It would be interesting to study thermodynamics of those. In [Chakraborty 14] it was shown that quantum tunneling formalism both in terms of semi-classical and quantum description can be generalised to non-static black holes. It would be interesting to figure out something for non-stationary as well as non-static metric like (2.11).

Plot of (3.9) is drawn for the comparison.

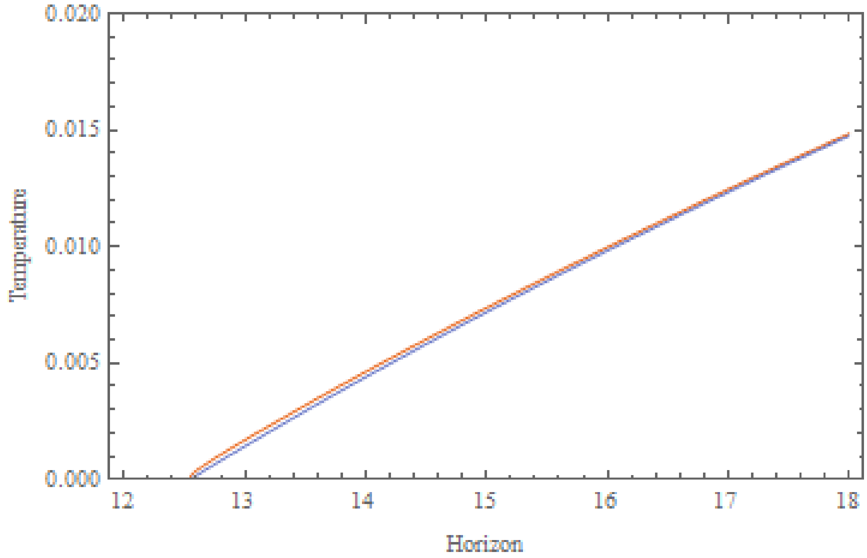

Figure 3.1: T vs $\hat{r_+}$ graph with Non-Commutative Chern-Simons correction.Orange line represents Non-Commutative Space-Time with θB=5, blue line represents Commutative Space-Time. Here, $J = q = 1$ and $l = 10$.

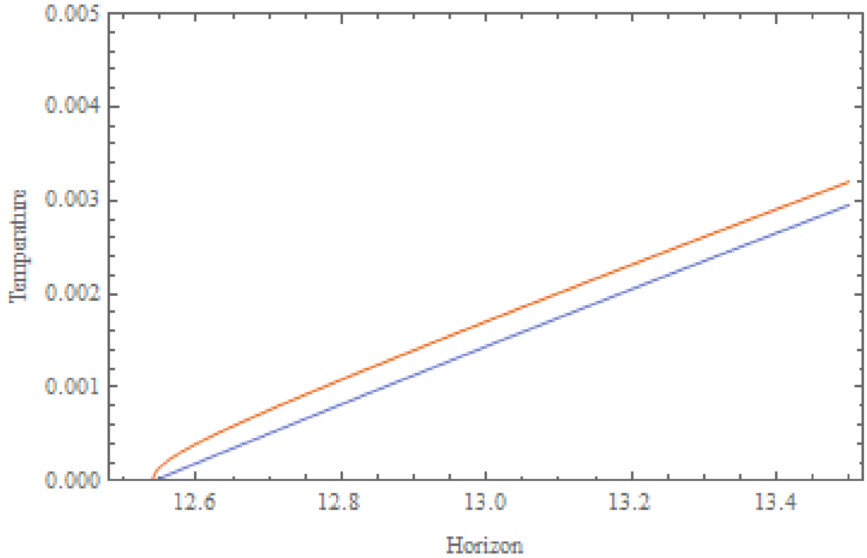

Figure 3.2: Zoom in for 3.1

Chapter 4

Essence of Non-Commutative Space-Time

In previous chapters we have studied BTZ Black hole in non-commutative space and studied thermodynamics of it using quantum tunneling and generalized uncertainty principle. The physical picture in previous chapters is not much clear. It is based on "good mathematics" but questions like "what is the physical meaning of B?" in the metric (2.8) are still unclear. We know that $\mathcal{A}_\mu^{(\pm)a}$ will give us the gravity in commutative space. Mathematical reason was stated in chapter 2 for the existence of "B-field" in non-commutative space but why they have to be physically present in non-commutative space with gravity? In commutative space-time, they are just decoupled U(1) gauge fields. Non-commutative corrections are supposed to be effective on short length scales or high energy scales. Does it mean that in low energy limit in (2+1)D space-time, we would have experienced $\mathcal{A}_\mu^{(\pm)a}$ in form of gravity? But then where would be those "B-field" in low energy limit and in what way we would have experienced them? Would they be present in (3+1)d universe in low energy limit? if yes, are we experiencing them?
In one way, we can try to give meaning to these "B field" is that effectively we can see that in the metric (2.8), constant B is always paired up with θ, so we can think of θB as an effective non-commutative parameter. Which would make the constant B just a scaling factor. In (2.9) we can treat B a scaling factor that depends on the angle (φ). Similarly, in (2.10), "Ct"can be treated as time dependent scaling factor.

Another method which can be used to get a metric, corresponding to non-commutative space. And this approach is more intuitive and it is easy to understand the physical meaning of it. This method was first introduced in [Nicolini 06] where non-commutativity is introduced in stress energy tensor of Einstein's field equations for a Schwarzchild black hole. This approach is inspired by the results obtained in [Smailagic 03a, Smailagic 03b, Smailagic 02a, Smailagic 02b] where it was shown that non-commutativity of coordinates affects distribution as well as propagation of energy and momentum. In these papers it has been shown that commutative space-time is replaced by so called "smeared" objects which mathematically translates to changing the Dirac delta distribution to Gaussian ones. It should be noted that in these papers, another approach known as coherent state formalism of non-commutative space was followed where star $*$ or Moyal product was not used. In [Nicolini 06, Ansoldi 07, Spallucci 09] different 4D and higher dimensional black hole geometries are discussed in non-commutative space by using this intuitive approach where they have mentioned removal of singularities and Hawking process. In [Larranaga 10, Liang 12] BTZ black hole geometries are discussed. We will explore this approach for charged BTZ black hole geometries in this chapter.

4.1 Einstein-Maxwell Equation for Non-Commutative Space-Time

For a (2+1)D geometry, Einstein Maxwell equations are written and non-commutativity is introduced in stress energy tensor for both matter and charge via their distribution function. It makes sense also physically, since we know that matter and energy affects the geometry of space, changing matter-energy distribution to introduce non-commutativity will help to get correction term to geometry of space-time casued by non-commutativity of space.

Physically one can say that particle's mass and energy is diffused in some region (of the Planck order) instead of ideally localized at one point and this can be attributed to the fuzziness of space due to non-commutative coordinates.

Particle's mass and energy density will now be replaced by some distribution function ρ other than Dirac delta distribution $\delta(\vec{x})$.

Set of Einstein-Maxwell equations for AdS_3 space (cosmological constant,$\Lambda = -\frac{1}{\ell^2}$) are:

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu R = 8\pi G\left(T^\mu_\nu|_{\text{matt.}} + T^\mu_\nu|_{el.}\right) + \frac{1}{\ell^2}g_{\mu\nu}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) = J^\nu$$

For a static charge distribution, current density is of the following form(vanishes for space directions):

$$J^\mu(x) = \rho_{el}(r)\delta^\mu_0$$

Earlier this would have a Dirac delta charge density,

$$e\delta(\vec{x}) \longrightarrow \rho_{el.}(\vec{x})$$

and field strength can be taken of the form:

$$F^{\mu\nu} = \delta^{0[\mu l}\delta^{r|\nu]}E(r)$$

for an electric field $E(r)$.

Now to compute Einstein-Maxwell equations, we need the form of stress energy tensor for both matter and charge distribution.

For matter part, the stress energy tensor can be considered of an ideal anisotropic fluid as,

$$T_{\mu\nu}|_{\text{matt.}} = (\rho + p_t)\, u_\mu u_\nu + p_t g_{\mu\nu} + (p_r - p_t)\, \chi_\mu \chi_\nu$$

here ρ is energy density, p_t and p_r are tangential and radial pressure respectively and u^i and χ^i are (2+1)D velocity and unit vector in radial direction respectively.

For Electromagnetic stress energy tensor, we have,

$$T_{\mu\nu}|_{el.} = -\frac{1}{4\pi}\left(F_{\mu\alpha}g^{\alpha\beta}F_{\beta\nu} - \frac{1}{4}g_{\mu\nu}F_{\sigma\alpha}g^{\alpha\beta}F_{\beta\rho}g^{\rho\sigma}\right)$$

For a spherically symmetric and static solution, the general form of metric can be assumed to be of the form:

$$ds^2 = -f(r)dt^2 + [g(r)]^{-1}dr^2 + r^2 d\phi^2$$

Let us assume, $f(r) = e^{2\alpha(r)}$ and $[g(r)]^{-1} = e^{2\beta(r)}$. Now, if we put this form of metric into Einstein's field equation with the above mentioned stress-energy tensors, the three Einstein-field equations will take the following form (assuming 8G=1=c and $\chi^\mu = e^{-\beta(r)}\delta^\mu_r$, $k = \frac{1}{8\pi}$) :

$$\begin{aligned}
\frac{\beta' e^{-2\beta}}{r} &= \pi\rho - \frac{1}{l^2} + \pi k e^{2(\alpha+\beta)}E^2 \\
\frac{\alpha' e^{-2\beta}}{r} &= \pi p_r + \frac{1}{l^2} - \pi k e^{2(\alpha+\beta)}E^2 \\
e^{-2\beta}\left(\alpha'^2 + \alpha'' - \alpha'\beta'\right) &= \pi p_t + \frac{1}{l^2} + \pi k e^{2(\alpha+\beta)}E^2
\end{aligned} \tag{4.1}$$

Along with these equations, We will have Maxwell equation as well,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) = J^\nu \tag{4.2}$$

One equation will also come from conservation of stress energy tensor which will take the following form:

$$e^{-2\beta}\left[(\rho+p_r)\,\alpha'+p_r'+\frac{1}{r}\,(p_r-p_t)\right]+2ke^{2\alpha}\left(-EE'-\alpha'E^2-\beta'E^2-\frac{E^2}{r}\right)=0$$

But the above equation is not independent from the set of three Einstein's field equations.

In this chapter two types of distribution will be studied, Gaussian and Lorentzian distribution for both mass and charge.

4.2 Gaussian Distribution

In this section first we will discuss that, if mass and charge densities are of Gaussian form then what will be the form of the metric? The motivation behind choosing the Gaussian one is that it has been shown that in non-commutative space-time, Dirac delta distribution is replaced by Gaussian in [Smailagic 03a, Smailagic 03b, Smailagic 02a, Smailagic 02b]. In [Nicolini 06, Ansoldi 07, Spallucci 09] this approach has been applied to Schwarzchild and higher dimensional black holes with Gaussian distribution where they have successfully removed singularities and also discussed Hawking process.

In the previous section it was noted that there are four equations and five variables and hence, it will not be possible to solve the system without putting restriction on the system. We will discuss two cases with meaningful assumption/restriction to solve for this Gaussian distribution:

$$\rho_{el}(r)=\frac{q}{4\pi\theta}e^{-r^2/4\theta}$$

$$\rho(r)=\frac{M}{4\pi\theta}e^{-r^2/4\theta}$$

4.2.1 $\alpha+\beta=0$

Now, this assumption seems valid, as in commutative space this condition holds true, we can expect it to hold true in non-commutative space too. This case is discussed in [Larranaga 10]. Now we can easily solve, (4.2) with the assumption $\alpha+\beta=0$. This will lead to expression for $E(r)$:

$$E(r)=\frac{1}{r}\int_0^r r'\rho_{el}\left(r'\right)dr'=\frac{q}{2\pi r}\left(1-e^{-r^2/4\theta}\right)$$

Once we get $E(r)$, we can use this expression to solve the 1st equation in (4.1) with the above mentioned assumption that $\alpha+\beta=0$. This will lead to $e^{-2\beta}$ expression:

$$e^{-2\beta} = g(r) = f(r) = -M\left(1 - e^{-r^2/4\theta}\right) + \frac{r^2}{\ell^2} - \frac{q^2 k}{2\pi}\left[\ln|r| + \Gamma\left(o, \frac{r^2}{4\theta}\right) - \frac{1}{2}\Gamma\left(o, \frac{r^2}{2\theta}\right)\right]$$

where, $\Gamma(o,x)$ is upper incomplete gamma function.

Now, it is easy to calculate Hawking temperature for such a case. Using (3.1) we have,

$$T_H = \frac{r_+}{2\pi\ell^2}\left[1 - \frac{M\ell^2}{4\theta}e^{-r_+^2/4\theta} - \frac{q^2 k\ell^2}{4\pi r_+^2}\left(1 + e^{-r_+^2/2\theta} - 2e^{-r_+^2/4\theta}\right)\right] \tag{4.3}$$

We can write M in terms of r_+, by making $f(r_+) = g(r_+) = 0$

$$M = \frac{1}{\left(1 - e^{-r_+^2/4\theta}\right)}\left[\frac{r^2}{\ell^2} - \frac{q^2 k}{2\pi}\left[\ln|r| + \Gamma\left(o, \frac{r^2}{4\theta}\right) - \frac{1}{2}\Gamma\left(o, \frac{r^2}{2\theta}\right)\right]\right] \tag{4.4}$$

Now it should be noted that as r_+ keeps getting smaller and smaller, T_H will approach $-\infty$ and since T_H can not be negative. It means that T_H will go to zero for a finite value of r_+. This means that, as the black hole keeps contracting because of the Hawking process, it will eventually stop contracting and a cold frozen remnant of black hole will be left.

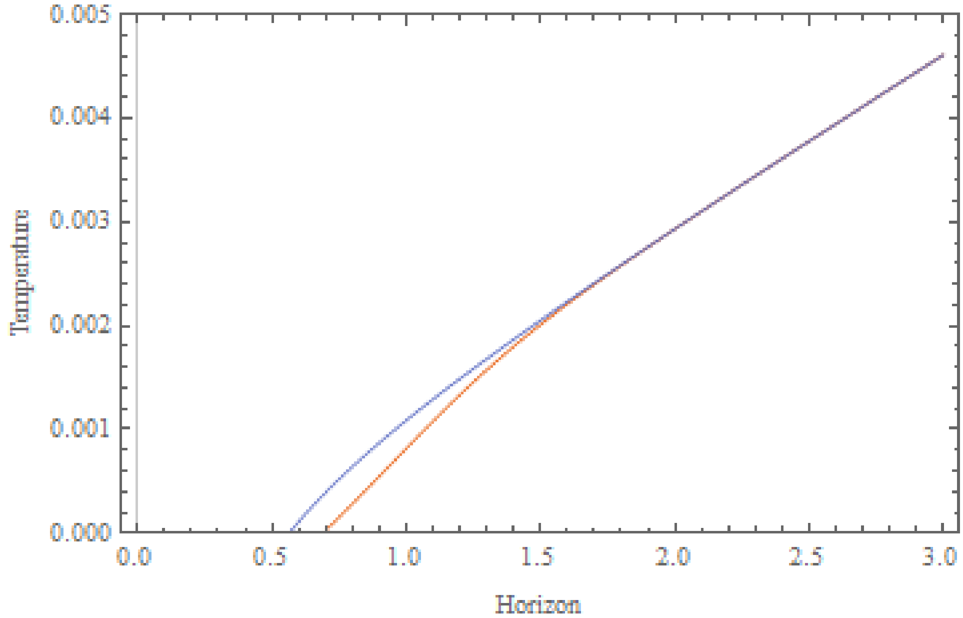

Figure 4.1: T vs r_+ for Gaussian with Orange line represents Non-Commutative Space-Time with $\theta = .1$, Blue line represents Commutative Space-Time. Here,$l = 10$ and $q = 1$

4.2.2 $\alpha \neq -\beta$

We have seen in (2.8), that though in commutative limit we have, $f(r) = g(r)$, it is not necessary that non-commutative correction also has to be equal in both terms as the case

in (2.8).
So, we need a relation in $f(r)$ and $g(r)$ such that in commutative limit they are equal. One valid assumption can be:

$$e^{2(\alpha+\beta)} = (1 - e^{-r^2/4\theta})$$

If this relation holds true, $E(r)$ can be calculated from (4.2):

$$E(r) = \frac{1}{r\sqrt{1 - e^{-r^2/4\theta}}} \int_0^r r' \sqrt{1 - e^{-r'^2/4\theta}} \rho_{el}(r')\, dr' = \frac{q}{3\pi r}\left(1 - e^{-r^2/4\theta}\right)$$

With this value of $E(r)$ and the assumption that $e^{2(\alpha+\beta)} = (1 - e^{-r^2/4\theta})$, we can solve for the first equation in (4.1) which will give:

$$e^{-2\beta} = g(r) = -M\left(1 - e^{-r^2/4\theta}\right) + \frac{r^2}{\ell^2} - \frac{2q^2 k}{9\pi}\left[\ln(|r|) + \frac{\Gamma\left(0, \frac{3r^2}{4\theta}\right)}{2} - \frac{3\Gamma\left(0, \frac{r^2}{2\theta}\right)}{2} + \frac{3\Gamma\left(0, \frac{r^2}{4\theta}\right)}{2}\right]$$

And,

$$f(r) = e^{2\alpha} = e^{-2\beta}(1 - e^{-r^2/4\theta})$$

Mass can be calculated from the relation that $f(r_+) = g(r_+) = 0$

$$M = \frac{1}{\left(1 - e^{-r_+^2/4\theta}\right)}\left[\frac{r^2}{\ell^2} - \frac{2q^2 k}{9\pi}\left[\ln(|r|) + \frac{\Gamma\left(0, \frac{3r^2}{4\theta}\right)}{2} - \frac{3\Gamma\left(0, \frac{r^2}{2\theta}\right)}{2} + \frac{3\Gamma\left(0, \frac{r^2}{4\theta}\right)}{2}\right]\right]$$

Hawking temperature can be calculated using (3.1)

4.3 Lorentzian Distribution

The motivation behind choosing Lorentzian distribution is that there has been found interesting relation between black-holes,warmholes and gravstars in [Kuhfittig 20] that they seem to be indistinguishable in non-commutative space. Also [Kuhfittig 17, Kuhfittig 20], a relation between dark energy and non-commutativity of space-time has been explored.
Geometry for BTZ black hole has been discussed in [Liang 12] for chargeless case. In this section, charge will be included for BTZ black hole (with Lorentzian distribution).

$$\rho(r) = \frac{M\sqrt{\theta}}{2\pi\left(r^2 + \theta\right)^{\frac{3}{2}}}$$

$$\rho_{el}(r) = \frac{q\sqrt{\theta}}{2\pi\left(r^2+\theta\right)^{\frac{3}{2}}}$$

For such a case, we will assume $\alpha = -\beta$. $E(r)$ can be calculated from (4.2) as,

$$E(r) = \frac{1}{r}\int_0^r r'\rho_{el}\left(r'\right)dr' = \frac{q}{2\pi r}\left(1 - \frac{1}{\sqrt{\frac{r^2}{\theta}+1}}\right)$$

Now, with the above $E(r)$ and the assumption that $\alpha = -\beta$, we can solve first equation in (4.1) to get expression for $e^{-2\beta}$:

$$e^{-2\beta} = g(r) = f(r) = -M\left(1 - \frac{1}{\sqrt{\frac{r^2}{\theta}+1}}\right) + \frac{r^2}{\ell^2} - \frac{q^2k}{2\pi}\left[2\ln(\sqrt{r^2+\theta}+\sqrt{\theta}) - \frac{\ln\left(\left|r^2+\theta\right|\right)}{2}\right]$$

We can plot f(r) vs r to analyse the horizon of the black-hole.

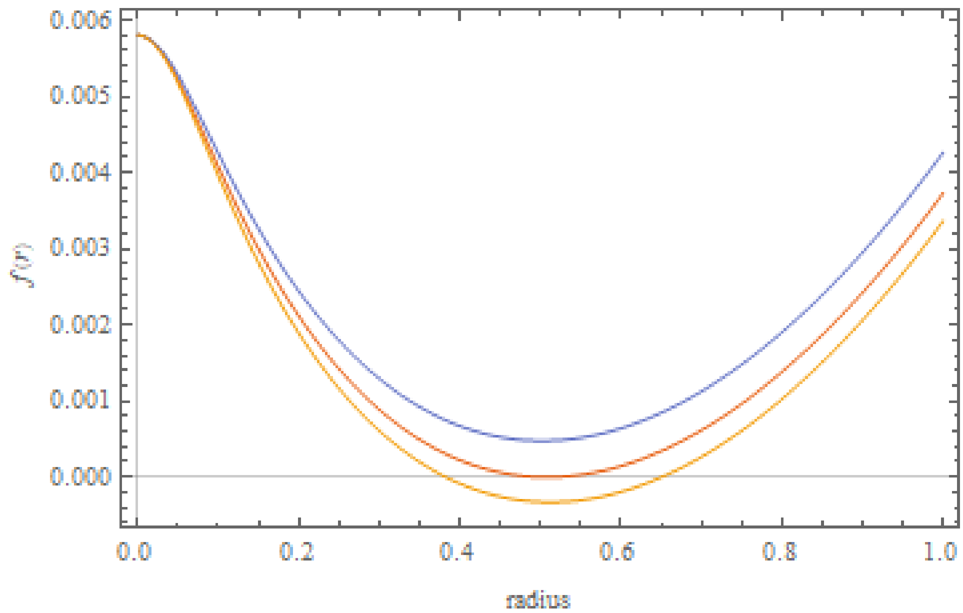

Figure 4.2: f(r) vs r for Lorentzian with $q = 1, l = 10$ and $\theta = .01$. Blue for $M = .005000$ red for $M = .005593$ and orange for $M = .006000$

M can be calculated from the relation $f(r_+) = g(r_+) = 0$,

$$M = \left(1 - \frac{1}{\sqrt{\frac{r_+^2}{\theta}+1}}\right)^{-1}\left[\frac{r_+^2}{\ell^2} - \frac{q^2k}{2\pi}\left[2\ln(\sqrt{r_+^2+\theta}+\sqrt{\theta}) - \frac{\ln\left(\left|r_+^2+\theta\right|\right)}{2}\right]\right] \tag{4.5}$$

Hawking temperature for this case can also be calculated using (3.1),

$$T_H = \frac{1}{4\pi}\left[\frac{2r_+}{l^2} - \frac{Mr_+}{\theta(r_+^2/\theta+1)^{\frac{3}{2}}} - \frac{q^2k}{2\pi r_+}\left(1 - \frac{1}{\sqrt{r_+^2/\theta+1}}\right)^2\right] \tag{4.6}$$

Now if r_+ keeps decreasing, T_H will give two cases, for $r_+ \to 0$

It will explode to ∞ if $4\sqrt{\theta} > 1$. Such divergences are not expected in non-commutative space-time, since non-commutativity is supposed to remove divergences at small distances. But if $4\sqrt{\theta} < 1$ then, T_H will reach to $-\infty$ and since temperature can not be negative, in this case T_H will go to zero for a finite non-zero value of r_+. In this case, after hawking process, a frozen remnant of black hole will be left. Hence, a bound is achieved on θ, $4\sqrt{\theta} < 1$. This can be seen in figure given below:

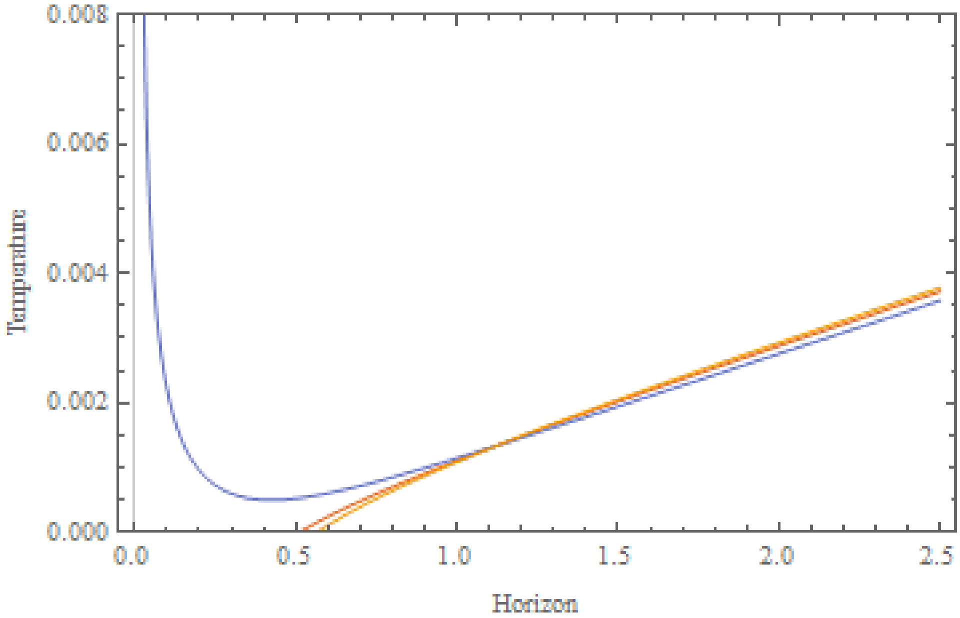

Figure 4.3: T vs r_+ for Lorentzian with $q = 1$ and $l = 10$. Red for $\theta = .01$, blue for $\theta = .1$,orange for commutative space

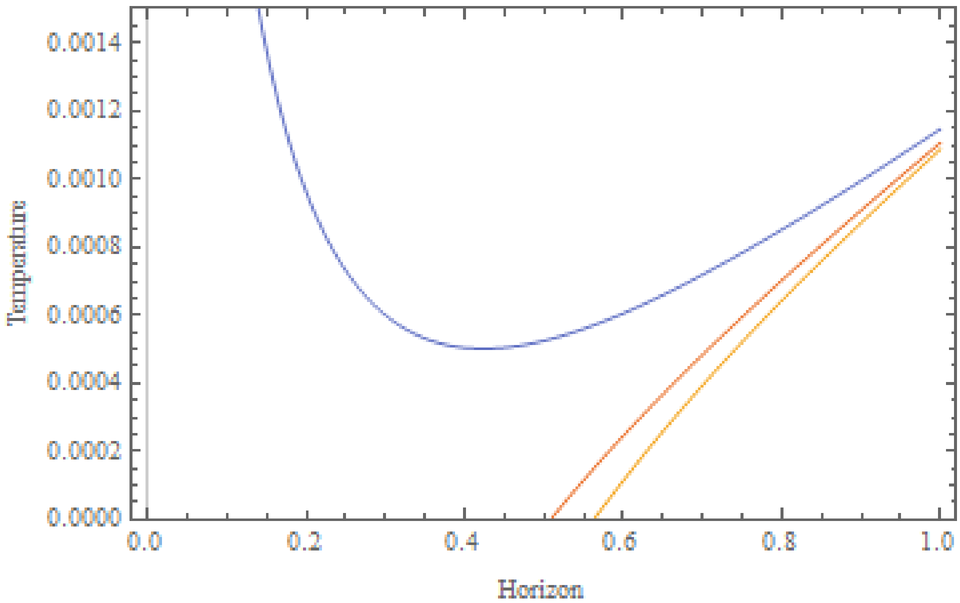

Figure 4.4: Zoom in for 4.3

Chapter 5

Gravity's Rainbow and Non-Commutative Space-Time

In past years, there have been various approaches to quantize gravity and almost all of them predicts that usual dispersion relation that we use in physics should be modified. Like in quantum discreteness [Hooft 96], in non-commutative geometry [Carroll 01, Amelino-Camelia 97b, Ye 18, Alexander 01], in spin networks [Gambini 99] and in string theory [Kostelecký 89]. Generalizing the concept of modified dispersion relation falls under the name of "gravity's rainbow" where the metric itself get modified in such a way a way that it becomes function of E of the probe. A very good review of gravity's rainbow and quantum space-time phenomenology is given in [Magueijo 04, Amelino-Camelia 13]. There are various papers in which various black holes and their thermodynamics are studied in gravity's rainbow [Ling 07, Gim 19, Gim 14, Ali 14, Amelino-Camelia 06]. Not only black holes but this concept had also been applied to cosmology as well as non-singular universe has been found in gravity's rainbow [Awad 13]. BTZ black holes in gravity's rainbow has also been studied [Alsaleh 17, Panah 19]. There have been some speculations that these effects can be tested in gamma ray bursts observations [Amelino-Camelia 97a]. In this chapter we will explore combine effect of non-commutative geometry and gravity's rainbow. Term "Flat non-commutative space" used in this chapter means non-commutative geometry whose commutative limit is flat minkowski space.

5.1 Gravity's Rainbow

The general modification in usual dispersion relation can be given by,

$$E^2 f^2 \left(E/E_{Pl}\right) - p \cdot p g^2 \left(E/E_{Pl}\right) = m^2$$

Note that E is not the energy of space-time itself, but it is the energy to which space-time is probed. One can have a system of particles with energy E that can be used to probe the geometry, then we can assign the metric $g_{ab}(E)$. Since, geometry is energy dependent, Vielbein field's be too. Now for system of particles with energy E, the geometry is given by,

$$g(E) = \eta^{ab} e_a \otimes e_b$$

Here, the Vielbein fields (energy dependent) can be written in form of energy independent Vielbein fields (which will be realised by particles with low energy) as,

$$e_0 = f^{-1}(E/E_{Pl})\,\tilde{e}_0, \quad e_i = g^{-1}(E/E_{Pl})\,\tilde{e}_i$$

These functions are chosen such that for low energy we have,

$$\lim_{E\to 0} g_{ab}(E/E_{Pl}) = g_{ab}^{\mathsf{classical}}$$

Therefore,

$$\lim_{E/E_P\to 0} f(E/E_P) = 1, \quad \lim_{E/E_P\to 0} g(E/E_P) = 1$$

5.2 Consequences of Gravity's Rainbow

In terms of renormalizable group flow, one can say metric "runs". Also, in absences of gravity or free falling frames, there is not just one flat space-time geometry but a class of flat space-time geometries given by $g_{ab}(E)$. If an observer observes two particles with different energies, then he/she will attribute different geometries/metrics as realised by them even though they are moving in same space-time region. Not only different particles(with different energies) observed by the observer will realise different metrics but also if one particle is observed by two different observers, and they assign different energy to the same particle then they will attribute different geometry/metric realised by the particle. Only when two particles are moving with very low energy($E_1/E_P \to 0$,$E_2/E_P \to 0$), geometries/metrics as experienced by them will be effectively same and equal to the usual classical geometry.

Inspired by non-commutative geometry and loop quantum gravity, one popular choice of these energy dependent functions are [Amelino-Camelia 13],

$$f(E) := 1 \quad g(E) := \sqrt{1 - \eta\,(E/E_p)^{\nu}} \tag{5.1}$$

5.3 Non-Commutative Space-Time and Gravity's Rainbow

Now what if instead of taking Vielbein fields(independent of energy) corresponding to the geometry of commutative space-time , we include the non-commutative correction terms in them i.e. $\tilde{e}_a \rightarrow \tilde{e}_{\theta a}$. In this way we include non-commutative geometry and gravity's rainbow at the same time.
Motivated by the previous section, we can make energy dependent functions to be of the form:

$$\lim_{E/E_\theta \rightarrow 0} f\left(E/E_\theta\right) = 1, \quad \lim_{E/E_\theta \rightarrow 0} g\left(E/E_\theta\right) = 1$$

For a non-commutative parameter θ, since θ is generally of $(Plancklength)^2$, we can take E_θ to be energy corresponding to $(\sqrt{\theta})^{-1}$.
Now Vielbein fields are:

$$e_0 = f^{-1}\left(E/E_\theta\right)\tilde{e}_{\theta 0}, \quad e_i = g^{-1}\left(E/E_\theta\right)\tilde{e}_{\theta i}$$

and thus now the metric will of the form:

$$g(E) = \eta^{ab} e_a \otimes e_b$$

Double special relativity supports the idea of minimum length and the same is case with non-commutative geometry. So, it would be best to incorporate features of both into one. non-commutative geometry has been successful in removing divergences in many cases, so it would be interesting to study "running" metric in non-commutative space.

5.4 Consequences of Non-Commutative Space-Time and Gravity's Rainbow

We will recover commutative space-time picture in the limit $\theta \rightarrow 0$, and in that case not only space will become commutative but also energy dependence will now be removed as $E_\theta \rightarrow \infty$ and hence $E/E_\theta \rightarrow 0$. This picture makes sense, as now if there is no minimum length then, both non-commutative geometry and gravity's rainbow (because of doubly special relativity) will be ineffective. The main difference here is that even if $E/E_\theta \rightarrow 0$ that does not necessarily mean that non-commutative geometry effects are excluded now. In this case, there will be non-commutative geometry effects in the metric. But now there will be single metric instead of a wide class of metric. Similarly, in absence of gravity, there will be wide class of flat but non-commutative geometries. If an observer observes two particles with different energies then geometries realised by them will be not only be different but also non-commutative.

From now we will call the combine description of non-commutative geometry and gravity's rainbow as "Non-Commutative Gravity's Rainbow"

5.5 Thermodynamics in Non-Commutative Gravity's Rainbow

5.5.1 Intrinsic Temperature

In this section we will discuss temperature and entropy for few non-commutative gravity's rainbow pictures of BTZ black hole. A general form of metric for BTZ black hole can be represented by,

$$ds^2 = -\frac{\mathcal{F}}{f^2(E)}dt^2 + \mathcal{Q}^{-1}g^{-2}(E)dr^2 + \frac{r^2}{g^2(E)}d\phi^2$$

Here, $\mathcal{F}$ and $\mathcal{Q}$ are with non-commutative correction terms.
By using same formalism as of quantum tunneling and using (3.1), we can calculated temperature as,

$$T = T_\theta \frac{g(E)}{f(E)} \tag{5.2}$$

here T_θ is the temperature in non-commutative background for $E/E_\theta \to 0$.
In this chapter we will use similar relation as of (5.1)

$$f(E) := 1 \quad g(E) := \sqrt{1 - \eta\,(E/E_\theta)^\nu} \tag{5.3}$$

and $E_\theta = (\sqrt{\theta})^{-1}$. We will be using $\eta = 1, \nu = 2$ in this chapter. We can see that there is whole class of temperature now. We can assign the intrinsic temperature in gravity's rainbow by focusing on particles near horizon which are assumed to be emitted during Hawking radiation. Since they will be near horizon and using the relation $E \geq 1/\Delta x \approx 1/r_+$, we can find intrinsic temperature in non-commutative gravity's rainbow. The reason we are focusing on particles near horizon is that their energy must be same as of the Hawking temperature of the black hole (in $k_b = 1$).

and thus,

$$T = T_\theta \sqrt{1 - \eta \left(\frac{\sqrt{\theta}}{r_+}\right)^\nu} \tag{5.4}$$

Now, for $\eta = 1, \nu = 2$ in (5.4), we will have,

$$T = T_\theta \sqrt{1 - \left(\frac{\sqrt{\theta}}{r_+}\right)^2} \tag{5.5}$$

5.5.2 Examples

In this section we will use non-rotating but charged BTZ black hole.

Gaussian type non-commutative gravity's rainbow

By using (4.4), and putting $q = 0$, we have:

$$M = \frac{1}{\left(1 - e^{-r_+^2/4\theta}\right)} \left(\frac{r^2}{\ell^2}\right)$$

And for such a case, we can put $q = 0$ in (4.3) to get T_θ as,

$$T_\theta = \frac{r_+}{2\pi\ell^2} \left(1 - \frac{M\ell^2}{4\theta} e^{-r_+^2/4\theta}\right)$$

By using the above expression in (5.5) we have,

$$T = \frac{r_+}{2\pi\ell^2} \left(1 - \frac{M\ell^2}{4\theta} e^{-r_+^2/4\theta}\right) \sqrt{1 - \left(\frac{\sqrt{\theta}}{r_+}\right)^2} \tag{5.6}$$

Using (5.6) in (3.3) entropy will be given by,

$$S = 4\pi \int_{r_o}^{r_+} \frac{1}{\left(1 - e^{-\xi^2/4\theta}\right) \sqrt{1 - \left(\frac{\sqrt{\theta}}{\xi}\right)^2}} d\xi$$

We will now plot (5.6) for comparison with commutative case.

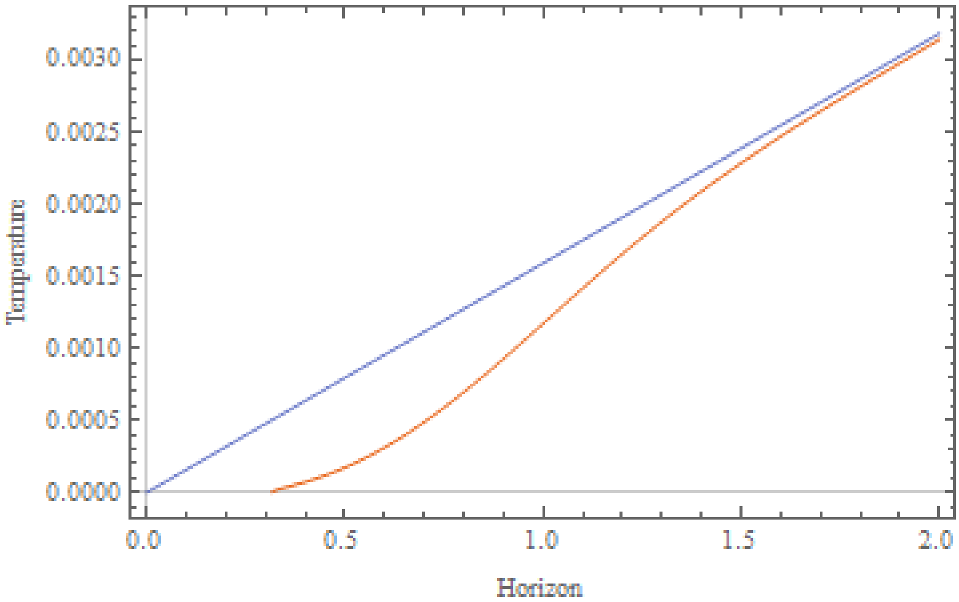

Figure 5.1: T vs r_+ for Gaussian Non-commutative Gravity's rainbow with$l = 10$, Blue line represents commutative case, Orange represents Non-Commutative Gravity's Rainbow case with $\theta = .1$

Lorentzian Non-Commutative Gravity's rainbow

Using (4.5) and (4.6) for $q = 0$, we have mass and T_θ given by:

$$M = \left(1 - \frac{1}{\sqrt{\frac{r_+^2}{\theta} + 1}}\right)^{-1} \left(\frac{r_+^2}{\ell^2}\right)$$

$$T_\theta = \frac{1}{4\pi}\left(\frac{2r_+}{l^2} - \frac{Mr_+}{\theta\left(r_+^2/\theta + 1\right)^{\frac{3}{2}}}\right)$$

Using above expression in (5.5) temperature in this non-commutative gravity's rainbow will be given by,

$$T = \frac{1}{4\pi}\left(\frac{2r_+}{l^2} - \frac{Mr_+}{\theta\left(r_+^2/\theta + 1\right)^{\frac{3}{2}}}\right)\sqrt{1 - \left(\frac{\sqrt{\theta}}{r_+}\right)^2} \qquad (5.7)$$

Using above expression in (3.3) entropy will be given by,

$$S = 4\pi \int_{r_o}^{r_+} \frac{1}{\left(1 - \frac{1}{\sqrt{\frac{\xi^2}{\theta}+1}}\right)\sqrt{1 - \left(\frac{\sqrt{\theta}}{\xi}\right)^2}} d\xi$$

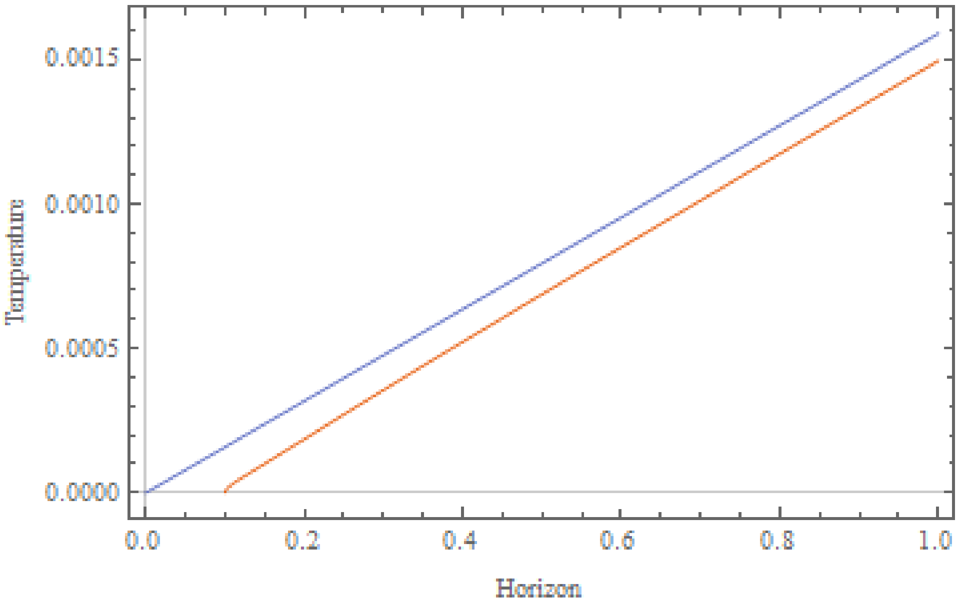

Figure 5.2: T vs r_+ for Lorentzian Non-Commutative Gravity's Rainbow with $l = 10$. Blue line for Commutative space, Orange line for Non-Commutative Space-Time with $\theta = .01$

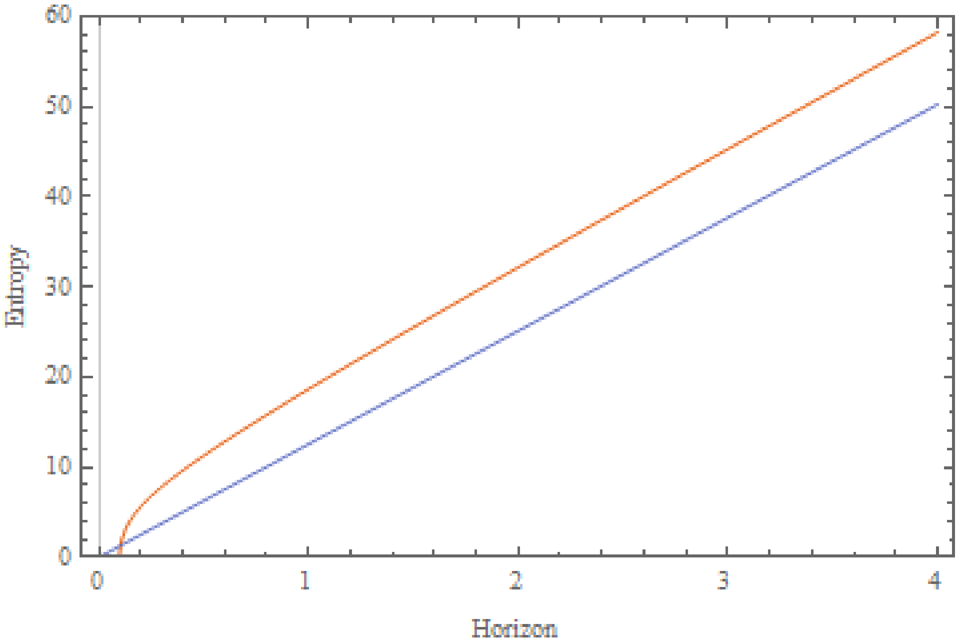

Figure 5.3: S vs r_+ for Lorentzian Non-Commutative Gravity's Rainbow with $l = 10$. 0range for Commutative Space-Time, Blue for Non-Commutative Space-Time with $\theta = .01$

Graph of temperature and entropy are plotted for comparison.

5.6 Discussion

Another attempt has been made in [Faizal 18] to unite gravity's rainbow and non-commutative geometry. In that approach θ itself is made function of E, such that when $\frac{E}{E_{pl}} \to 0$, then θ itself vanishes. In that approach, low energy particles will not realise non-commutative space-time structure. In that case for low energy particles there might be singularities in metric realised by them.For example, schwarzchild has singularity at $r = 0$, for low energy particles in gravity's rainbow formalism developed in [Faizal 18].
In formalism developed in this chapter, even for low energy particles, metric realised by them will be non-commutative and hence there will be no divergences at $r = 0$ in schwarzchild case.

Chapter 6

Holography for BTZ in Non-Commutative Space-Time

6.1 Cardy Verlinde Formula

There have been a lot of progress in AdS-CFT correspondence in the recent decades. Another formulation known as "Cardy" formula relates the entropy of a (1+1)D conformal field theory to it's Casimir energy and total energy [Cardy 86]. That formula was later generalized to arbitrary dimension in [Verlinde 00] where the entropy of CFT in arbitrary dimension is related to it's Casimir energy and the total energy. Cardy Verlinde formula has been shown to be valid for a numerous types of black holes [Cai 01, Jing 02, Halyo 02, Lee 08, Setare 04, Cai 02]. But this is not a trivial thing as Cardy Verlinde formula was shown not to hold for many AdS black holes [Gibbons 05, Cai 01]. The consistency of Cardy Verlinde formula for a charged and rotating BTZ black hole is checked in [Setare 09]. A Schwarzchild case in non-commutative case is discussed in [Abbas 14]. In this chapter, we will discuss the case of BTZ black hole in non-commutative case.

Casimir energy is defined by the violation of Euler's formula.

$$E_C = n\left(E + PV - TS - \Phi Q - \Omega_+ J\right)$$

In terms of Casimir energy, total energy is defined as,

$$E = E_E + \frac{1}{2}E_C$$

These extensive part of energy and Casimir energy can also be written in radius of the sphere R and entropy S as,

$$E_E = \tfrac{a}{4\pi R}S^{1+\frac{1}{n}}$$
$$E_C = \tfrac{b}{2\pi R}S^{1-\frac{1}{n}}$$

and hence,the generalised Cardy Verlinde can be written as,

$$S_{\mathrm{CFT}} = \frac{2\pi R}{\sqrt{ab}} \sqrt{E_C\left(2E - E_C\right)} \tag{6.1}$$

6.2 BTZ Black Hole in Non-Commutative Space-Time

For a BTZ (chargeless and non-rotating in non-commutative space) case, n can be taken to be 1 and hence, Casimir energy can be defined as,

$$E_C = (E + PV - TS)$$

and,

$$PV = E$$

$$E_C = (2E - TS)$$

The metric that is considered in this case is,

$$ds^2 = -f(r)dt^2 + [g(r)]^{-1}dr^2 + r^2 d\phi^2$$

$$g(r) = f(r) = -M\left(1 - e^{-r^2/4\theta}\right) + \frac{r^2}{\ell^2}$$

Horizon can be calculated from $f(r_+ = 0)$,

$$M = \frac{r_+^2}{\ell^2\left(1 - \exp\left(\frac{r_+^2}{4\theta}\right)\right)}$$

This equation can not be solved for a closed solution but an approximate solution can be found as,

$$r_+ = \sqrt{M}\ell\left(1 - \frac{1}{2}\exp\left(-\frac{M\ell^2}{4\theta}\right)\right)$$

By using the above equation, we can find entropy as well as temperature as,

$$S = 4\pi r_+ = 4\pi\sqrt{M}\ell\left(1 - \frac{1}{2}\exp\left(-\frac{M\ell^2}{4\theta}\right)\right)$$

$$T = \frac{1}{4\pi}\left[\sqrt{M}\ell\left\{1 - \frac{1}{2}\exp\left(-\frac{M\ell^2}{4\theta}\right)\right\}\right]\left[\frac{2}{\ell^2} - \frac{M}{2\theta}\exp\left\{-\frac{M\ell^2}{4\theta}\left[1 - \exp\left(-\frac{M\ell^2}{4\theta}\right)\right]\right\}\right]$$

Casimir energy can now be written as,

$$E_C = 2E - TS$$

we can write $TS = 2M(1-p)$ and hence,

$$E_C = 2Mp \tag{6.2}$$

in the above expression p is given by,

p=$\left(\frac{M\ell^2 \exp(q)}{4\theta} - \frac{M\ell^2}{4\theta}\exp(q)\exp\left(-\frac{M\ell^2}{4\theta}\right)\left[1-\frac{1}{4}\exp\left(-\frac{M\ell^2}{4\theta}\right)\right] + \exp\left(-\frac{M\ell^2}{4\theta}\right)\left[1-\frac{1}{4}\exp\left(-\frac{M\ell^2}{4\theta}\right)\right]\right)$

where q is,

$$q = -\frac{M\ell^2}{4\theta}\left(1-\exp\left(-\frac{M\ell^2}{4\theta}\right)\right)$$

Extensive part of energy can be written as,

$$E_E = E - \frac{1}{2}E_C = E - \frac{1}{2}(2E - TS) = \frac{TS}{2} = M(1-p) \tag{6.3}$$

or

$$2E - E_C = 2M(1-p) \tag{6.4}$$

We can also write, extensive and Casimir part of energy in form of R and S as,

$$E_C = \frac{b}{2\pi R}$$

From which we can say that,

$$R = \frac{b}{2\pi(2Mp)} \tag{6.5}$$

also,

$$E_E = \frac{a}{4\pi R}S^2$$

which implies,

$$R = \frac{aS^2}{4\pi E_E} = \frac{4\pi a r_+^2}{M(1-p)} \tag{6.6}$$

From (6.5) and (6.6) we can write R as,

$$R = \frac{\sqrt{ab}\, r_+}{M\sqrt{p(1-p)}} \tag{6.7}$$

Using (6.7),(6.2) and (6.4) in (6.1), we can say that,

$$S_{CFT} = S$$

Hence, Cardy Verlinde formula is valid for BTZ black holes which are chargeless and non-rotating in the non-commutative space.
Here we have used distribution picture in which distribution function that will set non-commutativity is taken to be Gaussian. Similar results may be found if we use Lorentzian distribution instead of Gaussain.

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Chapter 7

Conclusion

Non-Commutative Space-Time has been used as an approach to remove UV divergences in quantum field theories. It has also significant applications in case of gravity where it is believed that at quantum scale, the structure of space-time is not smooth but non-commutative. Such a picture helps to remove divergences at the centre of Schwarzchild black hole. Other motivations has also been discussed in chapter 1 for non-commutative structure of space-time. With these things in mind BTZ black hole is studied in non-commutative space-time in this thesis. In **chapter 2** we have used Chern-Simons theory to get the corresponding metric of BTZ in non-commutative case. Moyal product and Seiberg-Witten map has been used to find Chern-Simons gauge fields in non-commutative space-time and they are further used to get the metric of BTZ black hole in non-commutative space-time. In this approach, we have noticed that it is possible to make metric of BTZ black hole non-static as well as non-stationary in non-commutative space-time. Thermodynamics of black-holes has been an interesting area of research in past some decades. Various approaches to study thermodynamics of black holes has been explored by various research groups. In this thesis, we have used the formalism of quantum tunneling to study thermodynamics of BTZ black hole in non-commutative space-time and quantum corrections are added via GUPs and Hamilton-Jacobi method. In **chapter 3**, quantum tunneling has been used to study thermodynamics of the BTZ black hole obtained from Chern-Simons and Moyal product approach. Another formalism has been developed for non-commutative space-time known as coherent state formalism where Moyal product is not used to get the metric in non-commutative space-time but Dirac delta function for distribution of charge and mass of a test particle is replaced by some other general distribution. Gaussian distribution has been already study for a charged BTZ black hole. In **chapter 4**, we have studied Lorentzian distribution and explored thermodynamics of this type of BTZ black hole in non-commutative space-time and found an upper bound on non-commutative parameter θ. Gravity's rainbow has gained some attention in last two decades and it would be interesting to see combine effects of Gravity's Rainbow and Non-Commutative space-time in case of BTZ black hole. A formalism has been developed and thermodynamics of BTZ black hole in Non commutative gravity's rainbow has been discussed for both the Gaussian and Lorentzian type distribution in **chapter 5**. There has been a great progress regarding AdS-CFT correspondence in recent decades and hence it would be interesting to see if we could map thermodynamics quantities of BTZ black hole in non-commutative case to some CFT and to check that we have used Cardy-Verlinde formula in **chapter 6**.

It would be interesting in future to see the relation between non-commutative geometry and dark energy as metrics can be made time dependent just by introducing non-commutative structure of space-time. Also the thermodynamics of non-stationary and non-static black holes would be interesting to explore in non-commutative space-time. Different types of non-commutative geometries can also be explored in future work in context of black holes. In gravity's rainbow one can try to make θ itself dependent on energy but in such a way that it would not vanish even for low energy probes.

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