Hyperrigidity conjecture and recent developments

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MS15129

A dissertation submitted for the partial fulfilment of BS-MS dual degree in Science

Indian Institute of Science Education and Research Mohali

May 2020

Certificate of Examination

This is to certify that the dissertation titled **"Hyperrigidity conjecture and recent developments"** submitted by **Mr. Manujith K. Michel** (Reg. No. MS15129) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Lingaraj Sahu at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Manujith K. Michel
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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Lingaraj Sahu
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Acknowledgement

I am thankful to everyone who helped me in completing this project. I extend my gratitude to all of them, especially Dr. Lingaraj Sahu who supervised the project and advised me through regular discussions.
I sincerely thank Dr. S.P. Murugan Paramasivam, who being a part of the discussions helped me whenever necessary.
I would like to thank my parents and friends for their support and encouragement.
I acknowledge the facilities and the pleasant environment provided by IISER Mohali for carrying out the project.
I also thank Department of Science and Technology, GoI, for the financial support through INSPIRE fellowship for past 5 years.

Contents

Abstract

This work presents the recent developments in hyperrigidity conjecture in the theory of non commutative Choquet boundary. The pioneer work in the theory was done by Arveson who proposed the conjecture in 1969 among others. Arveson generalised the idea of boundary and Korovkin set in commutative C*-algebra through unique extension property (UEP) of representations in a non commutative C*-algebra. The counterpart of Korovkin set in commutative theory is called a hyperrigid set. The conjecture when proposed by Arveson claimed that a set is hyperrigid if and only if all the irreducible representations of the C*-algebra generated has UEP relative to the set. The forward implication was solved by Arveson himself in 2011 and the other implication is still open. For the sake of completion, this thesis surveys all the necessary results in the theory of non commutative Choquet boundary, most of them being very recent.

Chapter 1

Preliminaries

This chapter is the prerequisite knowledge for discussions on operator systems and hyperrigidity in the later chapters which is the main part of this thesis. Every result required for later chapters is presented in this chapter to make this thesis self contained so that this work is accessible to anyone with a basic knowldege in functional analysis. This chapter focuses on nice subalgebras of $B(\mathcal{H})$, the set of bounded linear operators on Hilbert space $\mathcal{H}$, called C*-algebras and von Neumann algebras. For more generality we begin with a wider class called Banach algebra.

1.1 Banach algebra

Definition 1.1.1 *A normed algebra over* $\mathbb{C}, (\mathfrak{A}, \|.\|), \mathfrak{A} \neq \{0\}$ *is called a Banach algebra if it is complete under the norm and* $\|AB\| \leq \|A\|\|B\|$*. If there is a unit(denoted by* $\mathbf{1}$*) in the multiplication inside a Banach algebra, we call it a unital Banach algebra.*

It is easy to see that multiplication is jointly continuous, i.e,
if $A_n \to A$ and $B_n \to B$ then $A_nB_n \to AB$, because

$$\begin{aligned} \|A_nB_n - AB\| &= \|A_nB_n - A_nB + A_nB - AB\| \\ &\leq \|A_nB_n - A_nB\| + \|A_nB - AB\| \\ &\leq \|A_n\|\|B_n - B\| + \|A_n - A\|\|B\| \end{aligned}$$

and the continuity follows.

In Banach algebras, we see how the algebraic structure and analytic structure interplay. The algebraic structure can be used to approximate, and in some cases find exactly norm of elements. To this end, we will define spectrum of an element. Before that we will have to have some important results. The first theorem will also show how invertibility, an algebraic property, is related to norm and convergence in norm topology.

Theorem 1.1.2 *If $\|A\|$ <1, $A \in \mathfrak{A}$, then $\mathbf{1} - A$ is invertble, and*

$$(\mathbf{1} - A)^{-1} = \sum_{n=1}^{\infty} A^n.$$

Proof The infinite sum in the theorem is absolutely convergent, therefore convergent in the Banach space. Call the limit B. Let S_n be the sequence of partial sum of the series. Then $S_n(\mathbf{1} - A) = (\mathbf{1} - A)S_n = \mathbf{1} - A^{n+1}$.

$$\begin{aligned} B(\mathbf{1} - A) &= (limS_n)(\mathbf{1} - A) \\ &= lim[(S_n(\mathbf{1} - A)] \\ &= lim(\mathbf{1} - A^{n+1}) = \mathbf{1} \end{aligned}$$

Continuity of multiplication used in the second equality and the fact that A^{n+1} is the general term of convergent series is used in the last equality to get $limA^{n+1} = 0$. B is a left sided inverse for **1**-A. Right sided inverse is shown similarly.

From $\|A + A'\| \leq \|A\| + \|A'\|$ and $\|AA'\| \leq \|A\|\|A'\|$, we can estimate the norm of B=$(\mathbf{1} - A)^{-1}$ in the theorem. $\|B\| \leq \sum_{n=1}^{\infty} \|A\|^n = 1/(1 - \|A\|)$.

Corollary 1.1.3 *The open unit ball around* **1** *contains only invertible elements.*

Proof Let $A \in \mathfrak{A}$ be such that $\|\mathbf{1} - A\| < 1$. Then by previous theorem $\mathbf{1} - (\mathbf{1} - A)$ $=A$ is invertible.

Corollary 1.1.4 *Let $A \in \mathfrak{A}$ be invertible. $B \in \mathfrak{A}$ be such that $r := \|(A-B)A^{-1}\| < 1$. Then B is invertible. $\|B^{-1}\| \leq \|A^{-1}\|[1/(1-r)]$.*

Corollary 1.1.5 *The set of invertible elements is open in a Banach algebra.*

Proof Given an invertible element A, we know that $\|A^{-1}\| \neq 0$. Take the open ball around A with radius $1/\|A^{-1}\|$. Then for any B in the ball we have $\|A - B\| < 1/\|A^{-1}\|$. Then $\|(A-B)A^{-1}\| < 1$. By previous corollary BA^{-1} is invertible. Therefore B is invertible (B^{-1} is given by $A^{-1}[(BA^{-1})^{-1}]$.

Theorem 1.1.6 *The map $A \mapsto A^{-1}$ is continuous map from the set of invertible elements onto itself.*

Proof Let A_n be a sequence of invertible elements converging to A which is also invertible. Then we have $\|(A-A_n)A^{-1}\| \leq \|A-A_n\|\|A^{-1}\|$ converges to 0. Therefore $r_n = \|(A-A_n)A^{-1}\| < 1$ for sufficiently large n.

$$\begin{aligned}\|A_n^{-1} - A^{-1}\| &= \|A_n^{-1}(A-A_n)A^{-1}\| \leq \|A_n^{-1}\|\|(A-A_n)A^{-1}\| \\ &\leq \|A^{-1}\|\frac{r_n}{1-r_n} \text{ converges to } 0\ .\end{aligned}$$

1.1.1 Spectrum

Assume that $\mathfrak{A}$ is a unital Banach algebra with unit $\mathbf{1}$ in this discussion on spectrum. Spectrum of an element $A \in \mathfrak{A}$, denoted $\sigma(A)$ is $\{\lambda \in \mathbb{C} \mid A - \lambda\mathbf{1} \text{ is not invertible}\}$.

Theorem 1.1.7 *In a Banach algebra, Spectrum of an element A, $\sigma(A)$ is a compact subset of $\mathbb{C}$.*

Proof We first show that $\sigma(A)$ is bounded by $\|A\|$. If $|k| > \|A\|$, (note $|k| > 0$) then $\|A/k\| < 1$. Therefore $\mathbf{1} - (A/k)$ is invertible. $A - k\mathbf{1} = k[A/k - \mathbf{1}]$. Since k is non zero, k is invertible and therefore, $A - k\mathbf{1}$ is invertible, i.e, $k \notin \sigma(A)$. Next task is to show that spectrum is closed. Let k_n be a sequence in spectrum such that k_n converges to k, then $A - k_n\mathbf{1}$ is a sequence of non- invertible elements in the Banach algebra which converges to $A - k\mathbf{1}$ in the algebra. Since the set of non-invertible elements are closed in the algebra we have $A - k\mathbf{1}$ is not invertible, i.e, $k \in \sigma(A)$. Since any closed and bounded subset of $\mathbb{C}$ is compact, the theorem follows.

Theorem 1.1.8 *Spectrum of an element A, $\sigma(A)$ is non-empty.*

Proof Let f be a bounded linear functional on Banach space $\mathfrak{A}$. For $z \notin \sigma(A)$ define $w_f(z) = f[(A - z\mathbf{1})^{-1}]$. We will show that w_f is analytic on the domain of definition called the resolvent of A and is open subset of $\mathbb{C}$.

$$(A - z\mathbf{1})^{-1} - (A - z_0\mathbf{1})^{-1} = (z - z_0)(A - z\mathbf{1})^{-1}(A - z_0\mathbf{1})^{-1}.$$

$$\lim_{z \to z_0} \frac{w_f(z) - w_f(z_0)}{z - z_0} = \lim_{z \to z_0} f[(A - z\mathbf{1})^{-1}(A - z_0\mathbf{1})^{-1}] = f(A - z_0\mathbf{1})^{-2}.$$

The last equality is due to continuity of inversion and we have, w_f is analytic on the resolvent. If the spectrum is empty then w_f is an entire function, when $|z| > \|A\|$, we have

$$\|(A - z\mathbf{1})^{-1}\| = \|1/z(A/z - \mathbf{1})^{-1}\| \leq \frac{1}{|z| - \|A\|}.$$

Therefore $\lim_{z\to\infty} f(A - z\mathbf{1})^{-1}$ converges to 0. w_f is a bounded entire function. By Liouville, w_f is constant throughout and equals 0. $w_f(0) = 0$ in particular, i.e, $f(A) = 0$. This is true for all bounded linear functionals f. By Hahn-Banach extension theorem we have $A = 0$. But A is invertible as 0 is in the resolvent. This is a contradiction since 0 is not invertible.

For simplicity $z\mathbf{1}$ will be denoted by z.

Lemma 1.1.9 *If $p(z)$ is a polynomial and $A \in \mathfrak{A}$ then $\sigma(p(A)) = p(\sigma(A))$. This is called spectral mapping property.*

Proof Given $\lambda \in \mathbb{C}$, we can factorize $p(z) - \lambda$ as

$$p(z) - \lambda = c\prod_{k=1}^{n}(z - \alpha_k).$$

Then

$$p(A) - \lambda = c\prod_{k=1}^{n}(A - \alpha_k).$$

Suppose $p(A)-\lambda$ is not invertible. Then there exist α_i in the $\sigma(A)$ such that $p(\alpha_i) = \lambda$. Now for the other implication suppose $\lambda = p(k)$ for some $k \in \sigma(A)$. Then $k = \alpha_i$ in the factorization. Then $A - k$ appears in the factorization of $p(A) - \lambda$. Since $A - k$ is not invertible, so is $p(A) - \lambda$.

Definition 1.1.10 *Spectral radius of $A \in \mathfrak{A}$, spr(A), is the sup*$\{|\lambda| : \lambda \in \sigma(A)\}$.

Remark 1.1.11 *By theorem* 1.1.7 *spr(A) is finite* $\forall A \in \mathfrak{A}$.

Theorem 1.1.12 *Spectral radius of A,* $spr(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$.

Proof For $\lambda \in$ resolvent of A, we have

$$(\lambda - A)^{-1} = \sum_{n=1}^{\infty} \frac{A^n}{\lambda^{n+1}}.$$

The given series converges absolutely and uniformly for $|\lambda| > r > spr(A)$. Therefore for $r > spr(A)$, the Taylor series coefficients $r^{-n-1}\|A^n\|$ converge to 0. Hence

$$\limsup_{n\to\infty} \|A^n\|^{1/n} \leq spr(A).$$

By compactness of spectrum, there is an $\alpha \in \sigma(A)$ such that $|\alpha| = spr(A)$. By spectral mapping property α^n is in $\sigma(A^n)$. Then $|\alpha^n| \leq \|A^n\|$. Hence

$$spr(A) = |\alpha^n|^{1/n} \leq \|A^n\|^{1/n}.$$

Thus

$$\limsup_{n\to\infty} \|A^n\|^{1/n} \leq spr(A) \leq inf\|A^n\|^{1/n}.$$

The limit exists and equality is established.

1.1.2 Abelian Banach algebra

This section is extremely useful in classification of abelian C*- algebra. In fact the Gelfand transform which is introduced here will give the isometric $*$-isomorphism in the classification. One example which is extensively used for our purpose is $C(X)$, the collection of continuous complex valued functions on compact Hausdorff space X with sup norm and pointwise addition and multiplication. It is important to note down some of its properties.

- The Banach algebra $C(X)$ is unital with the unit being the function taking value 1 everywhere, denoted by $\mathbf{1}$.
- A function f in $C(X)$ is invertible iff $f(x) \neq 0$ for all $x \in X$ and inverse is given by $f^{-1}(x) = \frac{1}{f(x)}$.
- Spectrum of f in $C(X)$ is the range of f.

Theorem 1.1.13 *The only simple unital abelian Banach Algebra is* $\mathbb{C}$.

Proof Let A be a non scalar in a unital abelian Banach algebra $\mathfrak{A}$. We will produce a proper closed ideal of $\mathfrak{A}$. Choose λ from the spectrum of A. Since $A - \lambda \neq 0$, the closure $\overline{(A-\lambda)\mathfrak{A}}$ of the ideal $(A-\lambda)\mathfrak{A}$ is a nonzero closed ideal. We only need to verify that it is proper. $(A-\lambda)B$ is not invertible for any $B \in \mathfrak{A}$. As open unit ball around $\mathbf{1}$ contains only invertible elements, $\|(A-\lambda)B - \mathbf{1}\| \geq 1$. Therefore, its closure doesn't contain $\mathbf{1}$, hence proper.

Definition 1.1.14 *Maximal ideal space of* $\mathfrak{A}$, $\mathfrak{M}_{\mathfrak{A}}$ *is the collection of non-zero algebra homomorphism from* $\mathfrak{A}$ *to* $\mathbb{C}$.

Theorem 1.1.15 *Let* $\mathfrak{A}$ *be unital abelian Banach algebra. Then for any* $\phi \in \mathfrak{M}_{\mathfrak{A}}$, ϕ *is bounded with* $\|\phi\| = 1$.

Proof Suppose there is $A \in \mathfrak{A}$ such that $|\phi(A)| > \|A\|$. By rescaling we may assume, $\|A\| < 1$ and $\phi(A) = 1$. Let $B := \sum_{n \geq 1} A^n$. Then $B = A + AB$ and

$$\phi(B) = \phi(A + AB) = \phi(A) + \phi(A)(B) \Rightarrow \phi(B) = 1 + \phi(B)$$

which is absurd. Therefore $|\phi(A)| \leq \|A\|$. Hence ϕ is bounded and $\|\phi\| \leq 1$. Since ϕ is non zero $\phi(\mathbf{1}) = 1$ and we have that $\|\phi\| = 1$.

Lemma 1.1.16 *Maximal ideals in a unital Banach algebra are closed.*

Proof Let M be a maximal ideal, it doesn't contain any invertible elements. Therefore $\|B - \mathbf{1}\| \geq 1$ for any $B \in M$. Its closure $\bar{M}$ is a larger proper ideal as it doesn't contain $\mathbf{1}$. By maximality of M, $\bar{M} = M$.

The next theorem will justify the name maximal ideal space.

Theorem 1.1.17 *If $\mathfrak{A}$ is unital abelian Banach algebra then $\mathfrak{M}_{\mathfrak{A}}$ is in bijection with the collection of maximal ideals of $\mathfrak{A}$.*

Proof Let $\mathfrak{I}$ be the collection of maximal ideals. Then define map from $\mathfrak{M}_{\mathfrak{A}}$ to $\mathfrak{I}$ by setting $\phi \mapsto ker\phi$. The map is well defined as kernel is a closed ideal of co-dimension 1. The map is injective as if two non zero algebra homomorphism to $\mathbb{C}$ have same kernel, the fact that $\phi(\mathbf{1}) = 1$ implies that both homorphisms are equal, since a linear functional is determined by kernel and value of one element outside kernel. For surjectivity, given any maximal ideal I, since it is closed, consider the quotient Banach Algebra $\mathfrak{A}/I$. Since I is maximal, the quotient is simple abelian unital algebra and by theorem 2.1.8 it is isomorphic to $\mathbb{C}$. Compose the canonical projection onto quotient with the isomorphism to get a non zero algebra homomorphism to $\mathbb{C}$. This precisely has kernel I.

For unital abelian Banach algebra $\mathfrak{A}$, since $\mathfrak{M}_{\mathfrak{A}}$ is a subset of closed unit ball of $\mathfrak{A}^*$ (collection of bounded linear functionals), we can topologise $\mathfrak{M}_{\mathfrak{A}}$ with the subspace topology inherited from weak* topology on $\mathfrak{A}^*$. It is easy to see that $\mathfrak{M}_{\mathfrak{A}}$ is weak* closed. The closed unit ball of $\mathfrak{A}^*$ is compact under weak* topology by well-known Banach-Alaoglu theorem [1]. It is Hausdorff as well. As closed subspace of compact space is compact, we have that $\mathfrak{M}_{\mathfrak{A}}$ is compact Hausdorff with the inherited weak* topology. Thus from now on wards $\mathfrak{M}_{\mathfrak{A}}$ will denote not merely the set, but the topological space. For fixed $A \in \mathfrak{A}$ define ev_A (read evaluation at A), $ev_A : \mathfrak{A}^* \to \mathbb{C}$ by $ev_A(\phi) = \phi(A)$. By definition of weak* topology on dual, ev_A is continuous if we equip the domain with weak* topology. Thus ev_A belongs to $C(\mathfrak{M}_{\mathfrak{A}})$, the set of all complex valued continuous functions on $\mathfrak{M}_{\mathfrak{A}}$.

The next definition of Gelfand transform and its properties will play a crucial role in the next chapter.

Definition 1.1.18 *Gelgand transform, Γ is the map from unital abelian Banach algebra $\mathfrak{A}$ to $C(\mathfrak{M}_{\mathfrak{A}})$ taking $A \in \mathfrak{A}$ to $ev_A \in C(\mathfrak{M}_{\mathfrak{A}})$. ev_A will be denoted by $\hat{A}$.*

Considering $C(\mathfrak{M}_{\mathfrak{A}})$ as a Banach algebra in the sup norm, we have the following immediate observations.

1. The Gelfand transform is an algebra homomorphism because product in the range is pointwise and functionals in domain are linear and multiplicative.

2. The Gelfand transform is contractive as functionals in the domain are contractive and the norm in range is the sup norm.

3. The range separates the points of $\mathfrak{M}_{\mathfrak{A}}$ as points in $\mathfrak{M}_{\mathfrak{A}}$ are distinct algebra homomorphisms.

Corollary 1.1.19 *A in a unital abelian Banach algebra is invertible if and only if $\hat{A}$ is invertible. Thus*

$$\sigma(A) = \sigma(\hat{A}) \text{ and } \|\hat{A}\| = spr(A).$$

Proof Since Γ is a non-trivial algebra homomorphism, inverse of $\Gamma(A)$ is given by $\Gamma(A^{-1})$. Now suppose, A is not invertible. Then $A\mathfrak{A}$ is a proper ideal, thus contained in a maximal ideal, M. Let $\phi \in \mathfrak{M}_{\mathfrak{A}}$ be such that $\ker \phi = M$. Then $\hat{A}(\phi) = ev_A(\phi) = 0$, and therefore not invertible. Now the equality on spectrum is immediate. Since the norm on the range is sup norm, the second equality follows.

$C(X)$ has more structure to it which makes it a C*-algebra (defined in the next section). As said earlier the theory on Gelfand transform is an important tool in studying C* algebras which will be our main goal from next section.

1.2 C*-algebra

Definition 1.2.1 *A map* $* : \mathfrak{A} \to \mathfrak{A}$ *is called an involution if the following holds* $\forall A, B \in \mathfrak{A}$ *and* $\lambda \in \mathbb{C}$ *where* A^* *denotes* $*(A)$.

1. $(A^*)^* = A$.
2. $(\lambda A)^* = \bar{\lambda} A$.
3. $(A+B)^* = A^* + B^*$.
4. $(AB)^* = B^* A^*$.

Definition 1.2.2 *A Banach* $*$*-algebra is a Banach algebra with an involution. A Banach* $*$*-algebra is called a C*-algebra if* $\|A^*A\| = \|A\|^2$.

It follows that, in a C*-algebra, $\|A\| = \|A^*\|$ because

$$\|A\|^2 = \|A^*A\| \leq \|A^*\|\|A\| \Rightarrow \|A\| \leq \|A^*\|$$

and since (A*)*=A we also have $\|A^*\| \leq \|(A^*)^*\| = \|A\|$. Note that the map $*$ is uniformly continuous as $\|A^* - B^*\| = \|(A-B)^*\| = \|A - B\|$.

Example 1.2.3 *The set of* $n \times n$ *matrices,* $\mathcal{M}_n(\mathbb{C})$ *with usual matrix multiplication and addition.*

Example 1.2.4 *The set of all complex valued continuous functions on compact Hausdorff space* X*, C(X) is a C*-algebra if we equip it with sup norm. Addition is given by pointwise addition and multiplication by pointwise multiplication.* $f^*(x) = \overline{f(x)}$ *gives the involution.*

Example 1.2.5 *The set of all complex valued continuous functions on locally compact Hausdorff space* X *vanishing at infinity,* $C_o(X)$ *is a C*-algebra, with same norm and structure as above example.*

Example 1.2.6 *The set of all bounded linear operators on Hilbert space $\mathcal{H}$, $B(\mathcal{H})$, equipped with operator norm, pointwise addition and composition and $*$ being adjoint operator. Any norm closed self-adjoint subalgebra of $B(\mathcal{H})$ is also an example, like the subalgebra of compact operators.*

Remark 1.2.7 *Examples 1.2.4 and 1.2.5 are commutative whereas others are non commutative. In fact there is a classification theorem by Gelfand which says that any commutative C*-algebra is one of these two kinds upto isometric $*$-isomorphism.*

If the C*-algebra contains a unit (identity in the ring multiplication of algebra often denoted by $\mathbf{1}$) then it is called unital C*-algebra. $C(X)$ is unital whereas $C_0(X)$ is not when X is not compact. Whenever $\mathcal{H}$ is infinite dimensional, algebra of compact operators is a non unital subalgebra of $B(\mathcal{H})$ which is unital.

From the axioms and uniqueness of unit we have $\mathbf{1}^* = \mathbf{1}$ as $\mathbf{1}^*A = (A^*\mathbf{1})^* = (A^*)^* = A$. $A\mathbf{1}^* = A$ follows similarly. Moreover, since $\|\mathbf{1}\|^2 = \|\mathbf{1}^*\mathbf{1}\| = \|\mathbf{1}\|$ we have $\|\mathbf{1}\| = 1$.

Definition 1.2.8

1. *If $A = A^*$, then A is called self-adjoint.*

2. *If $AA^* = A^*A$, then A is called normal.*

3. *In a unital C*-algebra, A is called unitary if $AA^* = A^*A = \mathbf{1}$.*

It is imporant to note the following whose proofs are immediate.

1. A is invertible if and only if A^* is invertible. In that case $(A^*)^{-1} = (A^{-1})^*$.

2. λ is in the spectrum of A if and only if $\bar{\lambda}$ is in the spectrum of A^*.

3. Norm of a unitary element A is 1 as $\|A\|^2 = \|A^*A\| = \|\mathbf{1}\| = 1$.

Theorem 1.2.9 *If A is self-adjoint, $\|A\| = spr(A)$.*

Proof This follows from spectral radius formula (1.1.12) in the following fashion. From the sequence $a_n = \|A^n\|^{1/n}$, take the subsequence a_{2^n}. Since $\|A\|^2 = \|AA^*\| = \|A^2\|$, using induction $a_{2^n} = \|A\|$ and the equality in the theorem follows.

Proposition 1.2.10 $\|A\| = sup\{\|AX\| \mid X \in \mathfrak{A}, \|X\| \leq 1\}$.

Proof Let r be the supremum in the proposition. Clearly, $r \leq \|A\|$. If $A = 0$, then $r = 0 = \|A\|$. Suppose A $\neq 0$ and set $X = A^*/\|A\|$. Then $\|X\| = 1$ and therefore $\|AX\| = \|A\|$ belongs to the set over which supremum is taken. Therefore $r \geq \|A\|$ and hence $r = \|A\|$.

Theorem 1.2.11 *If $\mathfrak{A}$ is a non unital C*-algebra there exist a unital C*-algebra $\tilde{\mathfrak{A}}$ such that we have an injective algebra homomorphism, $\pi : \mathfrak{A} \to \tilde{\mathfrak{A}}$ which is also an isometry such that $\pi(\mathfrak{A})$ is an ideal of co-dimension 1 in $\tilde{\mathfrak{A}}$.*

Proof Let $B(\mathfrak{A})$ be the set of all bounded linear operators on $\mathfrak{A}$. Consider the map $\pi : \mathfrak{A} \to B(\mathfrak{A})$ defined by $\pi(A)B = AB$ for all A, B in $\mathfrak{A}$. Clearly, π is a homomorphism. By previous proposition π is an isometry. Let $\mathbf{1}$ be the identity operator on $\mathfrak{A}$. Let $\tilde{\mathfrak{A}}$ be the algebra of operators of form $\pi(A) + \lambda\mathbf{1}$. Since $\pi(A)$ is complete and $\tilde{\mathfrak{A}}/\pi(A)$ is $\mathbb{C}$, $\tilde{\mathfrak{A}}$ is also complete. Define adjoint by $(\pi(A) + \lambda\mathbf{1})^* = \pi(A^*) + \bar{\lambda}\mathbf{1}$. For each ϵ >0, there is $B \in \mathfrak{A}$ with $\|B\| = 1$ such that

$$\begin{aligned}\|\pi(A) + \lambda\mathbf{1}\|^2 \leq \epsilon + \|(A + \lambda\mathbf{1})B\|^2 &= \epsilon + \|B^*(A^* + \bar{\lambda})(A + \lambda)B\| \\ &\leq \epsilon + \|(A^* + \bar{\lambda})(A + \lambda)B\| \\ &\leq \epsilon + \|(\pi(A^*) + \bar{\lambda})(\pi(A) + \lambda)\|.\end{aligned}$$

Thus $\tilde{\mathfrak{A}}$ satisfies C*-algebra norm condition.

Remark 1.2.12

When $\mathfrak{A}$ is non unital C-algebra, for $A \in \mathfrak{A}$, by $\sigma(A)$, we mean the spectrum of $\pi(A)$ in $\tilde{\mathfrak{A}}$. When $\mathfrak{A}$ is abelian $\tilde{\mathfrak{A}}$ is also abelian.*

1.2.1 Abelian C*-algebra

Now we will study unital abelian C*-algebra using Gelfand theory developed in previous section. Recall that Gelfand transform is the algebra homomorphism from unital abelian Banach algebra $\mathfrak{A}$ to $C(\mathfrak{M}_{\mathfrak{A}})$, taking A to ev_A. Non unital case can be dealt with easily using the unitization $\tilde{\mathfrak{A}}$.

Theorem 1.2.13 *When $\mathfrak{A}$ is a unital abelian C*-algebra, the Gelfand transform Γ is isometric $*$-isomorphism.*

Proof First we will show that Γ preserves involution and norm on self-adjoint elements. Let A be a self-adjoint element. For $t \in \mathbb{R}$ define

$$U_t = e^{itA} = \sum_{n\geq 0} \frac{(itA)^n}{n!}.$$

Note that $U_t^{-1} = U_{-t}$. Then

$$U_t^* = \sum_{n\geq 0} \frac{(\bar{i}tA)^n}{n!} = \sum_{n\geq 0} \frac{(-itA)^n}{n!} = U_t^{-1}.$$

This shows that U_t is unitary for all $t \in \mathbb{R}$. Hence $\|U_t\| = 1$. So for any $\phi \in \mathfrak{M}_{\mathfrak{A}}$

$$1 \geq |\phi(U_t)| = |\sum_{n\geq 0} \frac{(it\phi(A))^n}{n!})| = |e^{it\phi(A)}| = e^{-t\Im(\phi(A))},$$

where $\Im$ denotes the imaginary part of a complex number. Since this is true for all $t \in \mathbb{R}$, we conclude that $\Im(\phi(A)) = 0$. Since involution in $C(\mathfrak{M}_{\mathfrak{A}})$ is pointwise complex conjugation involution is preserved under Gelfand transform on self-adjoint elements.

Norm of self-adjoint element A is $spr(A)$. Since $\|\hat{A}\| = spr(A)$, norm is preserved on self-adjoint elements. For an arbitrary element X, define

$$A = \frac{X + X^*}{2} \text{ and } B = \frac{X - X^*}{2i}.$$

Then $X = A + iB$. A and B are self-adjoint and $X^* = A - iB$.

$$\Gamma(X^*) = \Gamma(A - iB) = \Gamma(A) - i\Gamma(B) = (\Gamma(A) + i\Gamma(B))^* = (\Gamma(A + iB))^* = (\Gamma(X))^*.$$

For an arbitrary element X, X^*X is self-adjoint.

$$\|X\|^2 = \|X^*X\| = \|\Gamma(X^*X)\| = \|(\Gamma(X))^*\Gamma(X)\| = \|\Gamma(X)\|^2.$$

Injectivity follows from isometry. Surjectivity follows from Stone-Weierstrass approximation as the range is a norm closed subalgebra of $C(X)$ which separates every point.

Now we will turn to non unital abelian C*-algebra $\mathfrak{A}$ with the unitization $\tilde{\mathfrak{A}}$. We know that $\tilde{\mathfrak{A}}$ is isometrically $*$- isomorphic to $C(\mathfrak{M}_{\tilde{\mathfrak{A}}})$. So we only need to find what is the image of $\mathfrak{A} = \{(A,0) \mid A \in \mathfrak{A}\}$ under the Gelfand transform of $\tilde{\mathfrak{A}}$. Consider $\phi_0 \in \mathfrak{M}_{\tilde{\mathfrak{A}}}, \phi_0(A,\lambda) = \lambda$. Then $\Gamma(A,0)(\phi_0) = 0$. Conversely, $f \in C(\mathfrak{M}_{\tilde{\mathfrak{A}}})$ be such that $f(\phi_0) = 0$. Then $f = \Gamma(A,\lambda)$ for some (A,λ). Necessarily $\lambda = 0$. Therefore under Gelfand transform $\mathfrak{A}$ is surjectively mapped to $\Gamma(\mathfrak{A}) = \{f \in C(\mathfrak{M}_{\tilde{\mathfrak{A}}}) \mid f(\phi_0) = 0\}$. However we know that a continuous scalar valued function g on a locally compact Hausdorff space X, can be extended to its one point compactification by setting $g(\infty) = c$ if and only if $g - c$ belongs to $C_0(X)$. Thus $\Gamma(\mathfrak{A})$ is identified with $C_0(\mathfrak{M}_{\tilde{\mathfrak{A}}} \setminus \phi_0)$. Because any element in the maximal ideal space of unitization other than ϕ_0, when restricted to $\mathfrak{A}$ is a non zero algebra homomorphism, and also any nonzero $\phi \in \mathfrak{M}_{\mathfrak{A}}$ is extended uniquely to unitization by setting $\tilde{\phi}(A,\lambda) = \phi(A) + \lambda$, $\mathfrak{M}_{\tilde{\mathfrak{A}}} \setminus \phi_0$ is identified with $\mathfrak{M}_{\mathfrak{A}}$ through restriction.

To summarize this section, we have classified abelian C*-algebra as $C_0(X)$ upto isometric $*$- isomorphism. X is the maximal ideal space of the algebra which is compact Hausdorff when the algebra is unital and locally compact Hausdorff otherwise. From this section, we develope what is called the continuous functional calculus on normal elements.

1.2.2 Continuous Functional Calculus on Normal element

The motivation for this section comes from the question that how can we make sense of $f(A)$ where A is a bounded linear operator and f is a function on some subset of $\mathbb{C}$. In the theory of finite dimensional vector space we have spectral resolution. That is any normal operator N can be written as $N = \sum_{i=1}^{n} c_i E_i$ where c_i are eigenvalues and E_i are orthogonal projections onto eigenspace of c_i. Thus for any f we define $f(A) = \sum_{i=1}^{n} f(c_i)E_i$. The restriction on f being that f be defined on $\sigma(N)$ (set of eigenvalues here). Note that since $\sigma(N)$ is discrete any function on it is continuous.

Now, we will turn to the general question. When $\mathfrak{A}$ is a unital C*-algebra, and N a normal element of it, how can we make sense of $f(N)$. From earlier, we demand that f is defined on $\sigma(N)$. The idea is to use the Gelfand theory. Note that $C^*(N)$ defined as closure of polynomials in N, N^* and $\mathbf{1}$ is a unital abelian C*-algebra hence isomorphic to $C(X)$. Thus what remains is to identify the maximal ideal space.

Lemma 1.2.14 *The map $\phi \mapsto \phi(N)$ is a homeomorphism from the maximal ideal space of $C^*(N)$ onto $\sigma(N)$.*

Proof Since ϕ is multiplicative and linear, $\phi(N)$ determines the value of ϕ on the polynomials and hence on its closure. Hence the map is injective. From Gelfand theory, we know that the range is $\sigma(N)$. Hence onto. By definition of weak* topology, we have that the map is a homeomorphism.

Theorem 1.2.15 *The C*-algebra generated by N, C*(N) is isomorphic to $C(\sigma(N))$ such that under the isomorphism N is mapped to f with $f(z) = z$ (denoted by z).*

Proof Isomorphism follows from the previous lemma and Gelfand theory. Under Gelfand transform, we know that N is mapped to $\hat{N}$ where $\hat{N}(\phi) = \phi(N)$. Since ϕ is identified with $\phi(N)$, we have the theorem.

Definition 1.2.16 *For a normal element N and $f \in C(\sigma(N))$, we define f(N)=A where $\hat{A} = f$.*

Corollary 1.2.17 $\sigma(f(N)) = \sigma(f) = f(\sigma(N))$.

Corollary 1.2.18 *If g is continuous on $f(\sigma(N))$, then $(g \circ f)(N) = g(f(N))$.*

Proof Clearly, if p is a polynomial, $p(f(N)) = (p \circ f)(N)$. General case follows from approximating continuous functions by polynomials.

Corollary 1.2.19

1. *$\|N\| = spr(N)$, when N is normal. This is because N is mapped to z on the spectrum under Gelfand transform and the norm is sup norm and Gelfand transform is isometric.*

2. *For a self-adjoint element A, $\sigma(A)$ is real since A is mapped to z on the spectrum under Gelfand transform and the Gelfand transform preserves involution.*

3. *If U is unitary $\sigma(U)$ is contained in the unit circle. Since under Gelfand transform $\hat{U}(z) = z$, we have $z\overline{z} = 1$.*

1.3 Positive elements and Approximate Identity

Positive elements and their ordering is used to prove existence of an approximate identity. We know that a Banach space quotiented by closed linear subspace is Banach. Also a Banach algebra quotiented by closed ideal is a Banach algebra with obvious vector space structure and quotient norm. Similar result holds true for C*-algebra and shall be proved using approximate identity. We will make use of the continuous functional calculus and work with unital C*-algebras.

Definition 1.3.1 *A self-adjoint element A in a unital C*-algebra $\mathfrak{A}$ is called positive (denoted $A \geq 0$) if spectrum of A is contained in non-negative real line. We also say $A \leq B$ if $B - A$ is positive.*

Definition 1.3.2 *A bounded net E_λ in $\mathfrak{A}$, $E_\lambda \geq 0$, $\|E_\lambda\| \leq 1$, $E_\lambda \leq E_\mu$ when $\lambda \leq \mu$ is called approximate identity for $\mathfrak{A}$ if $\forall A \in \mathfrak{A}$*

$$\lim_{\lambda \in \Lambda} E_\lambda A = \lim_{\lambda \in \Lambda} A E_\lambda = A.$$

The next theorem is an important tool in characterizing positive elements.

Theorem 1.3.3 *A positive element A has a unique positive square root, element B such that $B^2 = A$.*

Proof Consider $f(x) = \sqrt{x}$, a well defined continuous function on non-negative real line, and thus on $\sigma(A)$. Let $B := f(A)$. We will show that B is the required element. B is self-adjoint as f is real valued. B is positive as $\sigma(f(A)) = f(\sigma(A))$. Clearly $\hat{B}^2 = z$. Therefore $B^2 = A$. Thus B is a positive square root. For uniqueness, let C be another positive square root. Then $C = f(C^2) = f(A) = B$.

Theorem 1.3.4 *Any self-adjoint element A can be written as $A = A_+ - A_-$ where A_+ and A_- are positive and $A_+A_- = 0$.*

Proof In order to use the functional calculus on A, we define two continuous functions f, g on real line and thus on spectrum of A by restriction.

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases}$$

Now f and g are positive and $(f - g)(x) = x$ and $(fg)(x) = 0$. Set $A_+ = f(A)$ and $A_- = f(A)$.

The next theorem characterizes the positive elements.

Lemma 1.3.5 *For a self-adjoint element A, the following are equivalent:*

1. *A is positive*
2. *$A = B^2$ for some $B = B^*$*
3. *$\|c - A\| \le c$ for all $c \ge \|A\|$*
4. *$\|c - A\| \le c$ for some $c \ge \|A\|$.*

Proof Statement 1 implies 2 by the existence of square root .

Statement 2 implies 3: Suppose 2 holds and let c be such that $c \ge \|A\|$. From 2, we have $A = f(B)$ where $f(x) = x^2$ is defined on $\sigma(B)$. So, $\|f\|_{\sigma(B)} = \|A\|$. Thus $0 \le f \le \|A\| \le c$. Therefore $0 \le c - f \le c$ and

$$\|c - A\| = \|(c - f)(B)\| = \|c - f\|_{\sigma(B)} \le c.$$

Statement 3 implies 4 is obvious.

Statement 4 implies 1: Let $c \geq \|A\|$ be fixed and also assume $\|c - A\| \leq c$.Let z denote the function $f(z) = z$ on the spectrum of A.

$$\|c - A\| = \|(c - z)A\| = \|c - z\|_{\sigma(A)}.$$

By hypothesis $\|c-A\| \leq c$. Therefore $\|c-z\|_{\sigma(A)} \leq c$. Thus for all $z \in \sigma(A), c-z \leq c$. Hence $z \geq 0$ which is what was to be shown.

Corollary 1.3.6 *If A and B are positive, so is $A + B$.*

Proof Choose $c_1 \geq \|A\|$ and $c_2 \geq \|B\|$.Then $c_1 + c_2 \geq \|A\| + \|B\| \geq \|A + B\|$. Now

$$\|c_1 + c_2 - (A + B)\| \leq \|c_1 - A\| + \|c_2 - B\| \leq c_1 + c_2.$$

By previous lemma, $A + B$ is positive.

Theorem 1.3.7 *A^*A is positive for all A.*

Proof Let $B = A^*A$. B is self-adjoint. Use theorem 2.2.4 to write $B = B_+ - B_-$ with B_+ and B_- positive and $B_+B_- = 0$. Let C be the unique positive square root of B_-. Then C can be approximated by polynomials in B_- with zero as constant co-efficients. We have $CB_+ = 0$. Define $T = AC$. Then

$$-T^*T = -CA^*AC = -C(B_+ - B_-)C = CB_-C = B_-^2.$$

Thus $-T^*T$ is positive. Write $T = X + iY$ where X and Y are the real and imaginary parts of T. Then

$$T^*T + TT^* = 2(X^2 + Y^2)$$

is a sum of positive elements and hence positive by previous corollary. But

$$TT^* = T^*T + TT^* - T^*T = T^*T + TT^* + B_-^2$$

also turns out to be positive. Both $\pm T^*T$ are positive. Therefore $\sigma(T^*T) = \{0\}$.

Since T^*T is self-adjoint $T^*T = 0$. $\|B_-\|^2 = \|T^*T\| = 0$. We have $B_- = 0$.

Corollary 1.3.8 *If $A \leq B$ and X is arbitrary, then $X^*AX \leq X^*BX$.*

Proof Let C be the positive square root of $B - A$. Then

$$X^*BX - X^*AX = X^*(B - A)X = (CX)^*(CX).$$

is a positive element by previous theorem.

Corollary 1.3.9 *If $0 \leq A \leq B$ with A and B invertible, then $B^{-1} \leq A^{-1}$.*

Theorem 1.3.10 *The set $\Lambda = \{A \in \mathfrak{A} : A \geq 0, \|A\| \leq 1\}$ with the ordering '$\leq$' is an approximate identity for C*-algebra $\mathfrak{A}$ [1].*

1.4 Ideals and Quotients

The main theorem of this section will show that a C*-algebra quotiented by any closed ideal is a C*-algebra.

Lemma 1.4.1 *For any closed ideal $\mathfrak{J}$, $\mathfrak{J}^* := \{J^* : J \in \mathfrak{J}\} = \mathfrak{J}$ and we say that $\mathfrak{J}$ is self-adjoint.*

Proof Consider the C*-algebra $\mathfrak{B} = \mathfrak{J} \cap \mathfrak{J}^*$. Let E_λ be the approximate identitiy for $\mathfrak{B}$. Then for any $J \in \mathfrak{J}$

$$\lim_{\lambda\in\Lambda} \|J^* - J^*E_\lambda\|^2 = \lim_{\lambda\in\Lambda} \|JJ^* - JJ^*E_\lambda - E_\lambda(JJ^* - JJ^*E_\lambda)\| = 0.$$

Since J^*E_λ belongs to the ideal, we have J^* belongs to the ideal $\mathfrak{J}$. Hence $\mathfrak{J}^*$ is contained in $\mathfrak{J}$. Since $(J^*)^* = J$, we have $\mathfrak{J}^* = \mathfrak{J}$.

Corollary 1.4.2 *Any closed ideal of $\mathfrak{A}$ is a C*- subalgebra of $\mathfrak{A}$.*

Theorem 1.4.3 *For any closed ideal $\mathfrak{J}$ of $\mathfrak{A}$, the quotient Banach algebra $\mathfrak{A}/\mathfrak{J}$ is a C*-algebra.*

Proof We only need to verify that $\|\dot{A}^*\dot{A}\|^2 = \|\dot{A}\|^2$ where $\|\cdot\|$ is the quotient norm and $\dot{A}$ is the coset $A + \mathfrak{J}$. If E_λ is an approximate identity for $\mathfrak{J}$, we claim that for any $A \in \mathfrak{A}$

$$\|\dot{A}\| = \lim \|A - AE_\lambda\|.$$

$\|\dot{A}\| \leq \|A - AE_\lambda\|$ as AE_λ belongs to ideal. On the other hand, for $\epsilon > 0$ there is an element such that $\|A - J\| < \|\dot{A}\| + \epsilon$. So

$$lim\|A - AE_\lambda\| \leq lim\|(A - J)(\mathbf{1} - E_\lambda)\| + \|J - JE_\lambda\| \leq \|A - J\| < \|\dot{A}\| + \epsilon.$$

Now

$$\begin{aligned}\|\dot{A}^*\dot{A}\| &= lim\|A^*A(I - E_\lambda)\| \geq lim\|(\mathbf{1} - E_\lambda)A^*A(I - E_\lambda)\| \\ &= lim\|A(\mathbf{1} - E_\lambda)\|^2 = \|\dot{A}\|^2 = \|\dot{A}^*\|\|\dot{A}\| \geq \|\dot{A}^*\dot{A}\|.\end{aligned}$$

Therefore this is an equality and hence the C*-norm condition is satisfied.

1.5 Representations of a C*-algebra

A representation of a C*-algebra $\mathfrak{A}$ is a pair $(\pi, \mathcal{H})$, where π is a $*$- homomorphism form $\mathfrak{A}$ to $B(\mathcal{H})$, which is the C*-algebra of bounded linear operators on the Hilbert space $\mathcal{H}$. If $\mathfrak{A}$ is unital, we also demand that $\pi(\mathbf{1}) = I$, the identity operator. When there is no confusion we suppress $\mathcal{H}$. In fact in next section we will prove that any C*-algebra can be realized as a concrete C*-algebra, i.e, any C*-algebra is isometrically $*$-isomorphic to a subalgebra of $B(\mathcal{H})$.

If $\{(\pi_i, \mathcal{H}_i)\}$ is a collection of representations, then, we define $\oplus\pi_i = \pi$, where $\pi : \mathfrak{A} \to B(\oplus\mathcal{H}_i)$ is the representation such that $\pi(A) = \oplus\pi_i(A)$. First we start with a theorem on $*-$homomorphisms.

Theorem 1.5.1 *If $\pi : \mathfrak{A} \to \mathfrak{B}$, is a $* -$ homomorphism then π is continuous. If π is injective, then it is isometric.*

Proof It is clear that $\sigma(\pi(A))$ is contained in $\sigma(A)$. Suppose $A \in \mathfrak{A}$ is self-adjoint. Then

$$\|\pi(A)\| = spr(\pi(A)) \leq spr(A) = \|A\|.$$

For arbitrary A

$$\|\pi(A)\|^2 = \|\pi(A^*A)\| \leq \|A^*A\| = \|A\|^2.$$

Thus π is continuous with $\|\pi\| \leq 1$. Now suppose π is injective, but not isometric. Then there exists A such that $\|\pi(A)\| < \|A\|$. Let $r := \|\pi(A^*A)\|$ and $s := \|A^*A\|$. Clearly, $r < s$. Choose a continuous function f, from $C[0, s]$ such that f is identically 0 on $[0, r]$ contained in $[0, s]$ and $f(s) = 1$. (Such a function exists. Also [0,s] contains the spectrum of both $\pi(A^*A)$ and A^*A.) So, by functional calculus,

$$0 = f(\pi(A^*A)) = \pi(f(A^*A)).$$

But $f(A^*A) \neq 0$. This contradicts injectivity of π.

Corollary 1.5.2 *For any $*-$homomorphism π from a C*-algebra $\mathfrak{A}$ into C*-algebra $\mathfrak{B}$, $\|\pi\| = 1$.*

Proof $Ker\pi$ is a closed ideal and the induced homomorphism $\dot{\pi}$ from the C*-algebra $\mathfrak{A}/ker\pi$ is injective and therefore an isometry by previous theorem.

Definition 1.5.3 *An injective representation is called faithful.*

By previous theorem, we have that a faithful representation is automatically isometric.

Definition 1.5.4

1. *A representation π is called cyclic if there exists $e \in \mathcal{H}$ such that $\{\pi(A)e : A \in \mathfrak{A}\}$ is dense in $\mathcal{H}$. Such an e is called a cyclic vector for the representation.*

2. *A representation π is called non-degenerate if $\{\pi(A)h : A \in \mathfrak{A}, h \in \mathcal{H}\}$ is dense in $\mathcal{H}$.*

3. *Two representations of $\mathfrak{A}$, $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are called equivalent if there is a unitary (isometric and surjective) linear transformation $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi_2(A) = U\pi_1(A)U^{-1}$.*

Proposition 1.5.5 *$(\pi, \mathcal{H})$ is a non-degenerate representation of C*-algebra $\mathfrak{A}$ if and only if for any $h \neq 0$ in $\mathcal{H}$, there exists A in $\mathfrak{A}$ such that $\pi(A)h \neq 0$.*

Proof Suppose $(\pi, \mathcal{H})$ is non-degenerate and $\pi(A)h = 0$ for all A. Then, for arbitrary $v \in \mathcal{H}$,

$$\langle \pi(A)v, h \rangle = \langle v, \pi(A^*)h \rangle = 0.$$

Therefore orthogonal complement of $span(h)$ is $\mathcal{H}$ which implies $h = 0$. Conversely, suppose h is orthogonal to $\overline{\pi(\mathfrak{A})\mathcal{H}}$, then

$$0 = \langle h, \pi(A^*Ah) \rangle = \langle \pi(A)h, \pi(A)h \rangle$$

so that $\|\pi(A)h\|^2 = 0$ for all A. By hypothesis we have $h = 0$ and therefore $\overline{\pi(\mathfrak{A})\mathcal{H}} = \mathcal{H}$

Theorem 1.5.6 *Every representation of a C*-algebra is a direct sum of cyclic representations [1].*

1.6 Positive linear functionals and the GNS construction

The GNS construction stands for Gelfand-Naimark-Segal construction which realizes any C*-algebra as concrete C*-algebra, not necessarily isometric to, but $*-$homomorphic to the realization. The construction is done assuming the existence of a positive linear functional of norm 1. Isometry will be shown by showing existence of many positive linear functionals which preserves norm in some sense.

Definition 1.6.1 *A linear functional is called positive if it takes positive elements to positive real numbers.*

Definition 1.6.2 *A positive linear functional is called a state if it has norm* 1.

If $\pi : \mathfrak{A} \to B(\mathcal{H})$ is a representation, then $\phi(A) := \langle \pi(A)e, e \rangle$ is a state for a fixed vector e of norm 1 in $\mathcal{H}$.

The main theorem which is the above mentioned GNS construction shall be a converse for this. The proposition below is a routine check along with Cauchy-Schwarz.

Proposition 1.6.3 *If ϕ is a state, then $f(A,B) = \phi(B^*A)$ is a semi inner product on $\mathfrak{A}$ and hence $|\phi(B^*A)|^2 \leq \phi(B^*B)\phi(A^*A)$.*

From Cauchy Bunyakovsky Schwarz inequality it follows that for any positive linear functional ϕ on a unital C*-algebra, ϕ is bounded and $\|\phi\| = \phi(1)$. The next proposition which is a converse uses the well known representation theorem for bounded linear functionals on $C(X)$.

Proposition 1.6.4 *If $\mathfrak{A}$ is a unital C*-algebra and $\phi : \mathfrak{A} \to \mathbb{C}$ is a bounded linear functional such that $\|\phi\| = \phi(\mathbf{1})$, then ϕ is positive.*

Proof First, suppose $\mathfrak{A} = C(X)$, then ϕ corresponds to a measure μ. By hypothesis $\mu(X) = \|\mu\|$ and $\mu \geq 0$. For arbitrary C*-algebra $\mathfrak{A}$, take a positive element $A \in \mathfrak{A}$ and consider $\mathcal{C} = C^*(A)$ which is isomorphic to $C(\sigma(A))$. Letting $\phi_0 = \phi|_{\mathcal{C}}$, we get

$$\phi_0(\mathbf{1}) \leq \|\phi_0\| \leq \|\phi\| = \phi(\mathbf{1}) = \phi_0(\mathbf{1}).$$

Corollary 1.6.5 *If $\mathfrak{B}$ is a C^* subalgebra of $\mathfrak{A}$, any state ϕ on $\mathfrak{B}$ can be extended to a state on $\mathfrak{A}$.*

Proof If $\mathfrak{B}$ is unital with same unit as that of $\mathfrak{A}$, consider the norm preserving extension to $\mathfrak{A}$ which exists by Hahn-Banach extension theorem. Now apply previous proposition so that the extension is positive. If $\mathcal{B}$ is non-unital, let $\mathcal{B}_1 = \mathcal{B} \oplus \mathbb{C}\mathbf{1}$, which is unital. On $\mathcal{B}_1$, define the positive linear functional $\tilde{\phi}(B+\lambda) = \phi(B) + \lambda$. Now use the first part on unital case.

Theorem 1.6.6 *The GNS construction: If ϕ is a state on unital C*-algebra $\mathfrak{A}$, then there is a cyclic representation $(\pi, \mathcal{H})$ with unit cyclic vector e such that $\phi(A) = \langle \pi(A)e, e \rangle$.*

Proof Define $\mathcal{L} = \{A \in \mathfrak{A} : \phi(A^*A) = 0)\}$. Clearly $\mathcal{L}$ is closed. For any $A \in \mathfrak{A}$ and $L \in \mathcal{L}$, we have

$$\phi((AL)^*(AL)) = \phi(L^*A^*AL) \leq \phi(L^*\|A\|^2L) = \|A\|^2\phi(L^*L) = 0$$

which shows that $\mathcal{L}$ is an ideal too. Consider the vector space $\mathfrak{A}/\mathcal{L}$. Let $\bar{X}$ denote the coset $X + \mathcal{L}$. Define an inner product on $\mathfrak{A}/\mathcal{L}$ by setting $\langle \bar{X}, \bar{Y} \rangle = \phi(Y^*X)$. Let $\mathcal{H}$ be the Hibert space formed by completing the inner product space $\mathfrak{A}/\mathcal{L}$. For $A \in \mathfrak{A}$, define $\pi(A) : \mathfrak{A}/\mathcal{L} \to \mathfrak{A}/\mathcal{L}$ by setting $\pi(A)\bar{X} = \overline{AX}$. We will prove that this is a bounded linear operator on $\mathfrak{A}/\mathcal{L}$ for all A and thus extends to one on its completion. For the proof let A be fixed and X be arbitrary, then

$$\langle \overline{AX}, \overline{AX} \rangle = \phi(X^*A^*AX) \leq \phi(X^*\|A\|^2X) = \|A\|^2\langle \bar{X}, \bar{X} \rangle.$$

That is $\|\pi(A)\bar{X}\|^2 \leq \|A\|^2_{\mathfrak{A}}\|\bar{X}\|^2$ where $\|\cdot\|_{\mathfrak{A}}$ is the algebra norm and $\|\cdot\|$ without subscript is the inner product space norm. We have proved our claim. Now it is easy to see that $\pi : \mathfrak{A} \to B(\mathcal{H})$ is a representation. We also claim that the representation is also cyclic with cyclic vector $e = \bar{\mathbf{1}}$ because

$$\{\pi(A)e : A \in \mathfrak{A}\} = \{\bar{A} : A \in \mathfrak{A}\}$$

is by definition dense in $\mathcal{H}$.

We have not achieved our goal as the representation through GNS construction is not necessarily an isometry. To get a faithful representation, we are away from only a definition and a proposition which guarantees existence of enough states to approximate norm.

Definition 1.6.7 *If π_ϕ denotes the representation through GNS construction using state ϕ, then $\pi = \oplus\pi_\phi$ where ϕ runs over all states is called the universal representation of $\mathfrak{A}$.*

Proposition 1.6.8 *For self ajoint A in $\mathfrak{A}$, $\{\phi(A) : \phi \text{ is a state}\} = [\alpha, \beta]$ where $\alpha = min\, \sigma(A)$ and $\beta = max\, \sigma(A)$.*

Proof Consider $\mathcal{C} := C^*(A) = \{f(A) : f \in C(\sigma(A))\}$. If ϕ is a state its restriction to $\mathcal{C}$ corresponds to a probability measure μ so that $\phi(f) = \int f d\mu$. Since A corresponds to function $f(t) = t$ on the spectrum, we have $\phi(A) = \int t d\mu(t)$ which belongs to $[\alpha, \beta]$ since the integral is over the spectrum of A.

For the other inclusion, given t such that $\alpha \leq t \leq \beta$, define the functional ϕ on $C(\sigma(A))$ by setting

$$\phi(f) = \frac{t-\alpha}{\beta-\alpha} f(\beta) + \frac{\beta-t}{\beta-\alpha} f(\alpha).$$

We have under the identification $\phi(A) = t$ and ϕ is a state. Extend it to a sate on $\mathfrak{A}$ by corollary 1.6.5 and we are done.

Corollary 1.6.9 *For any $A \in \mathfrak{A}$, $\|A\|^2 = sup\{\phi(A^*A) : \phi$ is a state$\}$.*

Now it is easy to read off from the corollary that the universal representation is isometric.

To summarize, even though we started from axiomatic definition for a C*-algebra, we have shown that essentially it is a norm closed self-adjoint algebra of operators on a Hilbert space. Loosely speaking a C* -algebra is algebra of operators on a Hilbert space without mentioning the Hilbert space on which it acts.

1.6.1 Positive linear functionals on operator systems

Definition 1.6.10 *Let $\mathfrak{A}$ be a unital C*-algebra. A subspace of $\mathfrak{A}$ containing the unit is called an operator space of $\mathfrak{A}$. An operator system of $\mathfrak{A}$ is a self-adjoint operator space of $\mathfrak{A}$.*

Proposition 1.6.11 *Let S be an operator system of $\mathfrak{A}$ and $f : S \to \mathbb{C}$, a linear functional.*

1. *If f is positive, then f is bounded and $\|f\| = f(\mathbf{1})$.*
2. *If f is bounded and $\|f\| = f(\mathbf{1})$, then f is positive.*
3. *If f is positive, then f extends to a positive linear functional on $\mathfrak{A}$.*

The proofs are identical to the proofs of similar statements for positive linear functionals on C-algebra.*

1.7 von Neumann algebra

1.7.1 Topologies on $B(\mathcal{H})$

$B(\mathcal{H})$ has a natural topology which is the norm topology induced by the operator norm. But there are various other interesting topologies. We will see three topologies in this section, all of them being locally convex.

1. Strong operator topology(SOT): The topology defined by the semi-norms $\phi_x : B(\mathcal{H}) \to \mathbb{C}$ where $T \mapsto \|Tx\|$.

2. Weak operator topology(WOT): The topology defined by the semi-norms $\phi_{x,y} : B(\mathcal{H}) \to \mathbb{C}$ where $T \mapsto \langle Tx, y \rangle$.

3. σ- weak topology: The weak* topology inherited after identification with the dual of trace class operators.

Even though there are vast treatments of several other topologies on $B(\mathcal{H})$, we will be interested in the first two. One can study these topologies in detail and see that on many nice subsets these topologies coincide, we will be interested in exploring the properties of WOT closed self-adjoint unital subalgebras of $B(\mathcal{H})$.

Definition 1.7.1 *A unital self-adjoint WOT closed subalgebra of $B(\mathcal{H})$ is called von Neumann algebra.*

From definitions we know that norm topology is finer than SOT which in turn is finer than WOT. Therefore a von Neumann algebra is a C*-algebra.

Definition 1.7.2 *Let $\mathfrak{M} \subseteq B(\mathcal{H})$ be nonempty, then the commutant of $\mathfrak{M}, \mathfrak{M}'$ is defined as the set of operators in $B(\mathcal{H})$ that commutes with every element in $\mathfrak{M}$. We also denote $(\mathfrak{M}')'$ by $\mathfrak{M}''$ called double commutant.*

Definition 1.7.3 *A factor is a von Neumann algebra, $\mathfrak{M}$ whose centre, $\mathfrak{M} \cap \mathfrak{M}'$ consists of only scalars..*

Regardless of the structure of $\mathfrak{M}$, $\mathfrak{M}'$ is always a unital algebra. If $\mathfrak{M}$ is self-adjoint then $\mathfrak{M}'$ is a $*$-algebra. It also follows easily that

$$\mathfrak{M} \subseteq \mathfrak{M}'' = (\mathfrak{M}'')'' = \cdots\cdots$$

and also

$$\mathfrak{M}' = (\mathfrak{M}')'' = \cdots\cdots.$$

Let $x \in B(\mathcal{H})$. A closed subspace $\mathcal{M} \subseteq \mathcal{H}$ is called invariant under x if $x\mathcal{M} \subseteq M$. An subspace invariant under x is called reducing if it is also invariant under x^*. We say $\mathcal{M}$ is an invariant (resp. reducing) subspace for a family of operators $\mathfrak{M}$, it it is invariant (resp. reducing) for all $x \in \mathfrak{M}$.

Lemma 1.7.4 *Let $\mathfrak{M}$ be a $*$- subalgebra of $B(\mathcal{H})$, then a closed subspace $\mathcal{M}$ of $\mathcal{H}$ is reducing if and only if the projection onto $\mathcal{M}$ is in the commutant of $\mathfrak{M}$.*

Proof Let $x \in \mathfrak{M}$ and $p \in \mathfrak{M}'$. $x\mathcal{M} = xp\mathcal{H} = px\mathcal{H} \subseteq \mathcal{M}$. Since $\mathfrak{M}$ is closed under adjoints $\mathcal{M}$ is reducing. Now suppose $\mathcal{M}$ is reducing. Then for $\zeta \in \mathcal{M}$ $xp(\zeta) = x\zeta = px(\zeta)$. For η in the orthogonal complement of $\mathcal{M}$, we know that orthogonal complement is also invariant under x, therefore $xp = px$, i.e, $p \in \mathfrak{M}'$.

Now we will prove the famous double commutant theorem which shows how algebraically defined commutant is analytically important.

Theorem 1.7.5 *Let $\mathfrak{M} \subseteq B(\mathcal{H})$, be a $*$-subalgebra, then $\bar{\mathfrak{M}}^{SOT} = \bar{\mathfrak{M}}^{WOT} = \mathfrak{M}''$.*

Proof We will prove in the following sequence $\bar{\mathfrak{M}}^{SOT} \subseteq \bar{\mathfrak{M}}^{WOT} \subseteq \mathfrak{M}'' \subseteq \bar{\mathfrak{M}}^{SOT}$. The first inclusion follows because SOT is finer. For the second, let $x \in \bar{\mathfrak{M}}^{WOT}$ and $x' \in \mathfrak{M}'$. Let x_α be a net in $\mathfrak{M}$ converging in WOT to x. Now

$$\langle xx'\zeta, \eta\rangle = lim\langle x_\alpha x'\zeta, \eta\rangle = lim\langle x'x_\alpha\zeta, \eta\rangle = \langle x'x\zeta, \eta\rangle.$$

Therefore $x \in \mathfrak{M}''$. For the last let $x'' \in \mathfrak{M}''$. We have to show for given $\zeta_1, \zeta_2, ..., \zeta_n$ and $\epsilon > 0$, there exists $x_0 \in \mathfrak{M}$ such that $\|(x''-x_0)\zeta_k\| < \epsilon$ for $k = 1, 2, ..., n$. Consider $v = (\zeta_1, \zeta_2, ..., \zeta_n) \in \mathcal{H}^n$ and $\mathfrak{M} \otimes I_n$ as a family of operators in $B(\mathcal{H}^n)$. Let $\mathfrak{S}$ be the closure of the set $\{x \otimes I_n(v) : x \in \mathfrak{M}\}$. Clearly $\mathfrak{S}$ is a reducing subspace for $\mathfrak{M} \otimes I_n$. By 1.7.4 we know that the corresponding projection $p \in (\mathfrak{M} \otimes I_n)' = \mathfrak{M}' \otimes M_n(\mathbb{C})$ and $x'' \otimes I_n \in (\mathfrak{M}' \otimes I_n)'$. Therefore $p(x'' \otimes I_n) = (x'' \otimes I_n)p$. $x'' \otimes I_n v \in \mathfrak{S}$ and we are done.

Corollary 1.7.6 *A $*$-subalgebra is a von Neumann algebra if and only if it equals its double commutant. Commutant of a $*$-algebra is always a von Neumann algebra. A maximal abelian $*$-subalgebra is von Neumann because it equals its commutant.*

1.7.2 Types of von Neumann algebra

A von Neumann algebra is said to be of

- type I if it has an abelian projection with central support **1**.
- typeI_n, if **1** is the sum of n equivalent abelian projections (for some cardinal n).
- type II, if it has no non zero abelian projections, but has a finite projection with central support **1**.
- type III if it has no non zero finite projections.

It is well known that a von Neumann algebra and its commutant are of same type.

Example 1.7.7 *Let n be cardinal and S be any set with cardinality n ($n \in \mathbb{N}$ or $n = \aleph_0$, cardinality of $\mathbb{N}$). Consider $l^\infty(S)$ acting on $l^2(S)$ by multiplication. The algebra generated is denoted by $\mathfrak{M}_n$. The projections correspond to characteristic functions and the ordering corresponds to ordering under inclusion on subsets of S. Therefore minimal projections correspond to singleton sets and therefore are exactly n in number.*

Example 1.7.8 *Consider $L^\infty[0, 1]$ acting on $L^2[0, 1]$ by multiplication. It is well known that it is a von Neumann algebra. Let us denote it by $\mathfrak{M}_c$.*

Taking direct sum gives more example. These examples are very important in the theory of von Neumann algebra because their direct sum exhausts all maximal abelian von Neumann algebra acting on separable Hilbert space as we will see. We will prove it in following steps.

Definition 1.7.9 *Let $\mathfrak{M}$ be a von Neumann algebra acting on $\mathcal{H}$. A vector $\zeta \in \mathcal{H}$ is called separating vector for $\mathfrak{M}$ if $A\zeta \neq 0$ whenever $A \neq 0$ and $A \in \mathfrak{M}$.*

Proposition 1.7.10 *A vector is cyclic for a von Neumann algebra $\mathfrak{M}$ if and only if it is separating for $\mathfrak{M}'$.*

Proof Suppose ζ is cyclic for $\mathfrak{M}$ and $A\zeta = 0$, $A \in \mathfrak{M}'$. Let p be the projection onto kernel A. Then closed linear span of ζ, $[\zeta] \subseteq p\mathcal{H}$. Now $p \in \mathfrak{M}'$ and therefore $\mathcal{H} = [\mathfrak{M}\zeta] \subseteq p\mathcal{H}$. Therefore $p = \mathbf{1}$ and $A = 0$. For the other implication Suppose ζ is separating for $\mathfrak{M}'$. Let p be the projection onto $[\mathfrak{M}\zeta]$. Now $p \in \mathfrak{M}'$ and $(\mathbf{1} - p)\zeta = 0$. By assumption $\mathbf{1} - p = 0$. $p = \mathbf{1}$ and we are done.

Corollary 1.7.11 *A maximal abelian von Neumann algebra acting on a a separable Hilbert space has a cyclic separating vector.*

Proof A C*-algebra acting on a separable Hilbert space has a cyclic vector and since $\mathfrak{M} = \mathfrak{M}'$, the vector is separating.

Proposition 1.7.12 *A maximal abelian von Neumann algebra $\mathfrak{M}$ acting on a separable Hilbert space with no minimal projection is unitary equivalent to $\mathfrak{M}_c$.*

Before actually beginning the proof let us see some properties of $\mathfrak{M}_c$. Denote the characteristic function on $[0, t]$ by χ_t. Let p_t denote the corresponding projection. Note that χ_1 is a cyclic and separating vector. The collection p_t forms a totally ordered subset of projections which is order isomorphic to $[0, 1]$. Also, $p_t\chi_1 = \chi_t$. Now the span of $\{p_t\chi_1 : t \in [0, 1]\}$ are the step functions which is dense in $L^2[0, 1]$. also, we have the formula:

$$\langle p_t\chi_1, p_s\chi_s\rangle = \langle \chi_t, \chi_s\rangle = min\{s, t\}.$$

In the poof we will show that $\mathfrak{M}$ is generated as a von Neumann algebra by a sequence of projections.

Proof First we will show that $\mathfrak{M}$ is generated by a sequence of projections. Let $\mathcal{P}$ be the set of projections with usual order $\leq$ and ζ be the cyclic vector. Consider the set $\{P\zeta : P \in \mathcal{P}\}$ has a countable dense subset by separability, say $\{P_1\zeta, P_2\zeta, ...\}$. If $P \in \mathcal{P}$, P_{n_k} be the subsequence with $P_{n_k}(\zeta) \to P\zeta$. Now for any $A \in \mathfrak{M}$, we have by commutativity,

$$PA\zeta = AP\zeta = limAP_{n_k}\zeta = limP_{n_k}A\zeta.$$

Since $\mathfrak{M}\zeta$ is dense in $\mathcal{H}$. P_{n_k} converges to P in SOT. Hence the von Neumann algebra generated by $\{P_1, P_2, ...\}$ contains $\mathcal{P}$ and hence $\mathfrak{M}$. Define inductively the totally ordered chain of finite subsets of $\mathcal{P}$, $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq .....$ as follows. $\mathcal{F}_1 = \{E_0 = 0, E_1 = P_1, E_2 = 0\}$. Assuming we have defined $\mathcal{F}_n = \{E_0, E_1, ..., E_m\}$, define $\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{E_r + (E_{r+1} - E_r)P_{n+1} : 0 \leq r < m\}$. Notice that P_n is in the span of $\mathcal{F}_n$.

Now let $\mathcal{F}_\infty = \cup\mathcal{F}_n$ which is a totally ordered subset of $\mathcal{P}$ containing every P_n. Call the family of all totally ordered subsets of $\mathcal{P}$ containing $\mathcal{F}_\infty$, $\mathfrak{J}$ which is ordered by inclusion. By standard argument of Zorn's lemma, it has a maximal element, say $\mathcal{J}$. We show that this corresponds to $\{p_t : t \in [0,1]\}$ in $\mathfrak{M}_c$.

Define a map $\phi : \mathcal{J} \to [0,1]$ by $\phi(P) = \langle P\zeta, \zeta\rangle$ where ζ is the cyclic separating vector. ϕ is clearly SOT continuous. Since ζ is separating $\phi(P_1 - P_2) = \langle P_1 - P_2\zeta, \zeta\rangle = ||P_1 - P_2\zeta||^2 > 0$ whenever $P_1 \neq P_2$ and since $0, \mathbf{1}$ are in $\mathcal{J}$ and $\mathfrak{M}$ contains no minimal projections we have ϕ is surjective. The inverse of ϕ, $t \mapsto P_t$ is strictly increasing. Hence we have

$$\langle P_s\zeta, P_t\zeta\rangle = \langle P_tP_s\zeta, \zeta\rangle = \langle P_{min\{s,t\}}\zeta, \zeta\rangle = min\{s,t\} = \langle p_s\chi_1, p_t\chi_1\rangle.$$

Now since linear span of $\mathcal{J}$ is SOT dense in $\mathfrak{M}$, linear span of vectors of form $P_t\zeta$ is dense in $\mathfrak{M}\zeta$ and therefore in $\mathcal{H}$ too. We have already seen that $p_t\chi_1$ is dense in $L^2[0,1]$. Consider the map $U : H \to L^2[0,1]$ defined by the formula $U(P_t\zeta) = p_t\chi_1$.

A simple computation shows that U is a unitary and the equivalence follows from the following fact since $P_t\zeta$ generates $\mathcal{H}$.

$$UP_sP_t\zeta = UP_{min\{s,t\}}\zeta = p_{min\{s,t\}}\chi_1 = p_sp_t\chi_1 = p_sUP_t\zeta.$$

Theorem 1.7.13 *A maximal abelian von Neumann algebra, $\mathfrak{M}$ acting on a separable Hilbert space $\mathcal{H}$ is unitary equivalent to exactly one of the algebras $\mathfrak{M}_c, \mathfrak{M}_n, \mathfrak{M}_n \oplus \mathfrak{M}_c$.*

Proof The algebras listed are non unitary equivalent as we have already seen. We are done if it contains no minimal projections by previous proposition. So without loss of generality we may assume $\mathfrak{M}$ contains minimal projections whose collection we call, $\mathcal{P}$. Let ζ be a separating and cyclic vector. We show that any $P \in \mathcal{P}$ has one dimensional range and any two distinct elements $P_1, P_2 \in \mathcal{P}$ are orthogonal.

For the first part since P is a minimal projection $P\mathfrak{M} = \mathbb{C}P$. $PA\zeta = aP\zeta$ and since ζ is cyclic we have range of P is spanned by $P\zeta$. Now if $P_1, P_2 \in \mathcal{P}$ are distinct P_1P_2 is a projection majorised by both P_1 and P_2. By minimality of projections in $\mathcal{P}$ we have $P_1P_2 = 0$ as we have claimed. Since $\mathcal{H}$ is separable, we have $\mathcal{P}$ is countable. Let n denotes its cardinality. For each $P_k \in \mathcal{P}$, choose unit vector e_k in its range. The collection $\{e_k\}$ forms an orthonormal basis for $Q(\mathcal{H})$ where Q is the sum of all minimal projections. This determines a unitary from $l^2(S)$ to $Q(\mathcal{H})$, where S is a set containing n elements, given by $Ux = \sum x(k)e_k$ where sum runs over all k. Fix $A \in \mathfrak{M}$ and define $f(k)$ by the formula $P_kA = f(k)P_k$. Note that $\|f(k)\| \leq \|A\|$. Therefore $f \in l^\infty(S)$. Now

$$AQ = \sum AP_k = \sum f(k)P_k.$$

Therefore $\mathfrak{M}Q$ consists of all operators of the form $\sum f(k)P_k$.

For $x \in l^2(S)$ and $f \in l^\infty(S)$, defining U by $UM_fx = \sum(M_fx)(k)e_k$ we have

$$UM_fx = \sum(M_fx)(k)e_k = \sum f(k)x(k)P_ke_k = \sum f(k)P_k(x(k)e_k) = \sum f(k)P_k(Uy)$$

and thus $UM_fU^{-1} = \mathfrak{M}Q$. Therefore $\mathfrak{M}Q$ is unitary equivalent to $\mathfrak{M}_n$. If $Q = \mathbf{1}$, $\mathfrak{M}$ is unitary equivalent to $\mathfrak{M}_n$. Otherwise $\mathfrak{M}(\mathbf{1}-Q)$ is a maximal abelian von Neumann algebra acting on a separable Hilbert space and has no minimal projection. Now applying the previous proposition, $\mathfrak{M}(\mathbf{1}-Q)$ is unitary equivalent to $\mathfrak{M}_c$. Therefore $\mathfrak{M}$ is unitary equivalent to $\mathfrak{M}Q \oplus \mathfrak{M}(\mathbf{1}-Q)$ which is unitary equivalent to $\mathfrak{M}_n \oplus \mathfrak{M}_c$.

Definition 1.7.14 *Let $\mathfrak{M}$ be a von Neumann algebra acting on $\mathcal{H}$. Let n be a cardinal. The n^{th} inflation of $\mathfrak{M}$ is the algebra of operators in $B(\mathcal{H}^n)$, whose matrix representation has diagonal entries as same operator in $\mathfrak{M}$ and 0 elsewhere.*

Theorem 1.7.15 *Let $\mathfrak{A}$ be an abelian von Neumann algebra acting on separable Hilbert space, with type I_n commutant then $\mathfrak{A}$ is unitary equivalent to n^{th} inflation of a maximal abelian von Neumann algebra.*

Even though a von Neumann algebra is isomorphic to its n^{th} inflation always, we do not always have unitary equivalence.

Proof We have $\mathfrak{A}'$ is type I. Let it be type I_n for some cardinal. $\{P_j\}$ be the set of n equivalent abelian projections which sum to $\mathbf{1}$. Suppose P_j has range $\mathcal{K}_j$. Since P_j is abelian in $\mathfrak{A}'$, $\mathfrak{M} = P\mathfrak{A}'P$ is abelian. Also $\mathfrak{M}' = P\mathfrak{A}''P = P\mathfrak{A}P$ is a subalgebra of abelian algebra $\mathfrak{A}$ and therefore abelian. Therefore $\mathfrak{M} = \mathfrak{M}'$ is maximal abelian. Let $V : \mathcal{H} \to \oplus\mathcal{K}_j$ be unitary defined by $V\zeta = \oplus(V_j\zeta)$ where V_j is the partial isometry with initial projection P_j and final projection P where P is a fixed projection in $\{P_j\}$. Since $V_j, P \in \mathfrak{A}'$, for $T \in \mathfrak{A}$, we have $VTV^{-1} = \oplus PTP$ and $V\mathfrak{A}V^{-1} = \mathfrak{M}^n$ and $\mathfrak{M}$ is a maximal abelian von Neumann algebra.

Now we know all abelian von Neumann algebras acting on separable Hilbert space by this theorem and 1.7.13. We will use this in the disintegration theory in following section.

1.7.3 Disintegration theory

This section can be seen as generalisation of direct sums of Hilbert spaces. We use the disintegration approach rather than integration approach., i.e, we will say conditions when given Hilbert space can be written as direct integral of components rather than trying to make sense of conditions to be imposed on a family of Hilbert spaces to speak about their direct integral. First we have to make some new definitions which are to be read keeping in mind about direct sum decomposition of vector spaces.

Definition 1.7.16 *Let $\mathcal{H}$ be a Hilbert space and (X, μ) be a standard Borel space. $\{\mathcal{H}_x\}$ be a family of Hilbert spaces indexed with X. We say that $\mathcal{H}$ is the direct integrall of $\{H_x\}$ over X if the following holds.*

1. *For each $\zeta \in \mathcal{H}$, we have a map $x \in X \mapsto \zeta(x) \in \mathcal{H}_x$.*
2. *For $\zeta, \eta \in \mathcal{H}$ the map $x \mapsto \langle \zeta(x), \eta(x) \rangle$ is measurable and integrable and the inner product on $\mathcal{H}$ is given by*

$$\langle \zeta, \eta \rangle = \int_X \langle \zeta(x), \eta(x) \rangle d\mu.$$

3. *If $\theta \in \prod \mathcal{H}_x$ is such that for all $\zeta \in \mathcal{H}$ $x \mapsto \langle \theta(x), \zeta(x) \rangle$ is measurable and integrable, then there exists a vector $\Theta \in \mathcal{H}$ such that $\Theta(x) = \theta(x)$.*

In this case we write

$$\mathcal{H} = \int_X^{\oplus} \mathcal{H}_x d\mu(x) \text{ and } \zeta = \int_X^{\oplus} \zeta(x) d\mu(x).$$

Definition 1.7.17 *Let $\mathcal{H}$ be a direct integral of $\mathcal{H}_x$ over (X, μ). We say an operator $A \in B(\mathcal{H})$ is decomposable, if A defines a map, $x \in X \mapsto A(x) \in B(\mathcal{H}_x)$ such that $(A\zeta)(x) = A(x)\zeta(x)$ for all $\zeta \in \mathcal{H}$. If in addition $A(x)$ is a scalar operator in $B(\mathcal{H}_x)$ for almost all x, then we say that the operator A is diagonalisable with respect to the direct integral decomposition. We write*

$$A = \int_X^{\oplus} A(x) d\mu(x).$$

We denote the von Neumann algebra of decomposable operators by $\mathfrak{D}$ and its commutant, the algebra of diagonalisable operators by $\mathfrak{C}$. Now we list some basic equalities which follows from definitions directly. If $\mathcal{H}$ is a direct integral of $\mathcal{H}_x$ over (X, μ), then the following holds.

1. $\zeta, \eta \in \mathcal{H}$ are equal if and only if $\zeta(x) = \eta(x)$ almost everywhere(a.e).

2. $(\alpha\zeta + \eta)(x) = \alpha\zeta(x) + \eta(x)$ a.e.($\alpha \in \mathbb{C}, \zeta, \eta \in \mathcal{H}$).

3. For $A, B \in \mathfrak{D}$, $A = B$, if and only if $A(x) = B(x)$ a.e.

4. For $A, B \in \mathfrak{D}$, $\alpha A + B \in \mathfrak{D}$ and $(\alpha A + B)(x) = \alpha A(x) + B(x)$ a.e. ($\alpha \in \mathbb{C}$).

5. For $A \in \mathfrak{D}$, $A^* \in \mathfrak{D}$ and $A^*(x) = A(x)^*$ a.e.

6. $I_{\mathcal{H}} \in \mathfrak{C}$ and $I_{\mathcal{H}}(x) = I_{\mathcal{H}_x}$ a.e.

Definition 1.7.18 *Let $\mathfrak{A}$ be a C^*-algebra. Suppose $\mathcal{H}$ is a direct integral of $\mathcal{H}_x$ and $(\pi, \mathcal{H})$ is a representation of $\mathfrak{A}$ such that $\pi(\mathfrak{A}) \subseteq \mathfrak{D}$. Then $(\pi, \mathcal{H})$ is called decomposable if it determines a map $x \in X \mapsto \pi_x$ such that π_x is a representation of $\mathfrak{A}$ on $\mathcal{H}_x$ and $\pi_x(a) = \pi(a)(x)$. We write*

$$\pi = \int_X^{\oplus} \pi_x d\mu(x).$$

Similar results as before holds here too, when $\mathfrak{A}$ is separable.

Definition 1.7.19 *Let $\mathfrak{M}$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. Let $\mathfrak{A}$ be a norm separable SOT dense C^* subalgebra (which exists by Kaplansky's density theorem) of $\mathfrak{M}$. Suppose $\mathcal{H}$ is a direct integral of $\mathcal{H}_x$, then we call $\mathfrak{M}$ decomposable with decomposition $x \mapsto \mathfrak{M}_x$ if the identity representation i of $\mathfrak{A}$ is decomposable and $\mathfrak{A}_x := i_x(\mathfrak{A})$ is strong operator dense in $\mathfrak{M}_x$.*

We will now give the result which is interesting on its own right and is also crucial in proving our main result on hyperrigidity. This will use the classification of von Neumann algebras acting on separable Hilbert space which was presented in 1.7.15.

Theorem 1.7.20 *If $\mathfrak{M}$ is an abelian von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, then there is a standard Borel space (X, μ) such that $\mathcal{H}$ is a direct integral of $\mathcal{H}_x$ over X and $\mathfrak{M}$ coincides with the von Neumann algebra of diagonalisable operators, $\mathfrak{C}$ with respect to the decomposition.*

Proof We know that $\mathfrak{M}$ is of type I and so is $\mathfrak{M}'$. We prove the theorem in cases where $\mathfrak{M}'$ is of type I_n where $n = 1$ or n is a natural number or $n = \aleph_0$.

Case 1: $\mathfrak{M}'$ is of type I_1. Then $\mathfrak{M}'$ is abelian. $\mathfrak{M}' = \mathfrak{M}$ and therefore a maximal abelian von Neumann algebra. By 1.7.13, we know that $\mathfrak{M}$ is unitary equivalent to $\mathfrak{M}_c, \mathfrak{M}_n$ or $\mathfrak{M}_n \oplus \mathfrak{M}_c$. If it is $\mathfrak{M}_c$, take X to be $[0, 1]$, $\mathcal{H}$ is isomorphic to $L^2[0, 1]$ which is a direct integral of $\mathbb{C}$ over X and $\mathfrak{M}$ is unitary equivalent to $L^\infty[0, 1]$ which is exactly the diagonalisable operators w.r.t the decomposition. The case $\mathfrak{M}_n$ is similarly dealt with. If it is a direct sum, we can take the direct sum of corresponding measure spaces.

Case 2: $\mathfrak{M}'$ is type I_n, n is natural number. By 1.7.15, $\mathfrak{M}$ is n^{th} inflation of maximal abelian von Neumann algebra. Therefore there is a K isomorphic to $L^2(X, \mu)$ and von Neumann algebra $\mathfrak{A}$ unitary equivalent to $L^\infty(X, \mu)$ such that $\mathfrak{M}$ is unitary equivalent to $\mathfrak{A}^n$. So $\mathcal{H}$ is the direct integral of $\mathbb{C}^n$ over X and as in case 1 $\mathfrak{M}$ coincides with diagonalisable operators.

Case 3: If $n = \aleph_0$, the same proof in case 2 works with a slight modification. Using same notation we have $(f_1, f_2, ...) \in \mathcal{H}$ if and only if $\sum \|f_i\|^2$ is finite and therefore $\sum |f_n(p)|^2$ is finite almost everywhere. Therefore $\mathcal{H}$ is the direct integral of l^2.

Case 4: In general $\mathfrak{M}'$ is the direct sum of type I_n where $n = 1, 2, ...\aleph_0$. So $\mathfrak{M}' = \oplus \mathfrak{M}' P_n$ where P_n is a central projection in $\mathfrak{M}'$. So, We have $P_n(\mathcal{H})$ is unitary equivalent to sum of n copies $L^2(X_n, \mu_n)$ denoted $\mathcal{K}_n$ where (X_n, μ_n) is the measure space associated to $\mathfrak{M} P_n$ by the above approach. Write (X, μ) for the direct sum of (X_n, μ_n). Then writing

$$\mathcal{K}_n = \oplus \int_{X_n}^{\oplus} \mathcal{H}_x d\mu_n$$

we have

$$\mathcal{K} = \oplus \mathcal{K}_n = \int_X^{\oplus} \mathcal{H}_x d\mu$$

using convention when $p \notin X_n$ $p \mapsto x_n(p) = 0$. Direct sum of unitaries mapping $P_n\mathcal{H}$ to $\mathcal{K}_n$ is a unitary from $\mathcal{H}$ to $\mathcal{K}$ and the operators coincide as required and we are done.

Theorem 1.7.21 *Let $\mathfrak{M}$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$ with center $\mathcal{Z}$. Then there is a decomposition $x \in X \mapsto \mathfrak{M}_x$ such that $\mathfrak{M}_x$ is a factor almost everywhere.*

Proof By previous theorem we have a decomposition of $\mathcal{H}$ relative to $\mathcal{Z}$ and $\mathcal{Z}$ corresponds to algebra of diagonalisable operators, $\mathfrak{C}$. Since $\mathfrak{M} \subseteq \mathcal{Z}'$ and $\mathfrak{C}' = \mathfrak{D}$, the algebra of decomposable operators $\mathfrak{M}$ is also decomposable.

Let $\mathfrak{A}_1, \mathfrak{A}_2$ be SOT dense norm separable C* algebras inside $\mathfrak{M}$ and $\mathfrak{M}'$ respectively. Call the C*-algebra generated by $\mathfrak{A}_1$ and $\mathfrak{A}_2$, $\mathfrak{A}$. Now $\mathcal{Z}'$ is generated by $\mathfrak{M}$ and $\mathfrak{M}'$, $\mathfrak{B}$ is SOT dense in $\mathcal{Z}'$. $(\mathfrak{B}_1)_x, (\mathfrak{B}_2)_x$ and $\mathfrak{B}_x$ are SOT dense in $\mathfrak{M}_x$, $(\mathfrak{M}')_x$ and $(\mathcal{Z}')_x$ respectively a.e. .$(\mathfrak{B}_1)_x$ and $(\mathfrak{B}_2)_x$ generate the commutant of centre of $\mathfrak{M}_x$ a.e. $(\mathfrak{B}_1)_x$ and $(\mathfrak{B}_2)_x$ generate $\mathfrak{B}_x$. $\mathfrak{B}_x$ is SOT dense in $\mathcal{Z}_x'$ a.e. Hence $\mathcal{Z}_x'$ is the commutant of the centre of $\mathfrak{M}_x$ a.e. Hence $\mathcal{Z}_x$ is the centre of $\mathfrak{M}_x$ a.e. Now since $\mathcal{Z}$ consists of diagonalisable operators, we have $\mathcal{Z}_x$ are all scalars and therefore $\mathfrak{M}_x$ is a factor a.e.

Chapter 2

Boundary representations and C*-envelope

2.1 Completely positive maps

Here we are going to generalize the notion of positive linear functionals. One way to do this is by allowing range to be any C*-algebra rather than $\mathbb{C}$ and demanding positive elements map to positive elements. For more generality, we may replace the domain with operator systems. But in this setting, we will find through the next example how these maps are not well behaved like positive linear functionals that statements similar to proposition 1.6.11 cannot be generalized readily if the range is higher dimensional and what we have to demand more than mere positivity of maps so that these maps have nice properties.

Example 2.1.1 *Let $\mathbb{T}$ denote the unit circle in the complex plane. Consider the operator system S of $C(\mathbb{T})$ spanned by $1, z, \overline{z}$ and the map $\phi : S \to \mathcal{M}_2$(algebra of 2× 2 matrices) defined as follows:*

$$\phi(a + bz + c\overline{z}) = \begin{bmatrix} a & 2b \\ 2c & a \end{bmatrix}.$$

$a + bz + c\overline{z}$ is positive if and only if $c = \overline{b}$ and $a \geq 2|b|$ and a self-adjoint matrix is positive if and only if its diagonal entries and determinant are nonnegeative real numbers. Now ϕ is positive readily follows. Note that

$$2\|\phi(\mathbf{1})\| = 2 = \|\phi(z)\| \leq \|\phi\|.$$

What is important is to note that, even though a positive map is automatically bounded, $\|\phi\| = \|\phi(\mathbf{1})\|$ does not hold for arbitrary positive maps. But if the domain and range are unital C*-algebras, it is true. For similar reasons the map in the example cannot be extended to $C(\mathbb{T})$ positively. Therefore property of extension of positive linear functionals on operator system to C*-algebras cannot be generalized to positive maps. In order to overcome this difficulty, completely positive maps, a subclass of postive maps for which these nice properties hold is defined.

Let S be an operator system of $\mathfrak{A}$. Denote by $\mathcal{M}_n(\mathfrak{A})$(respectively $\mathcal{M}_n(S)$), the set of $n \times n$ matrices with entries in $\mathfrak{A}$ (respectively S). If ϕ is a linear map from S to C*-algebra $\mathfrak{B}$, define ϕ_n from $\mathcal{M}_n(S)$ to $\mathcal{M}_n(\mathfrak{B})$ as follows:

$$\phi_n[a_{ij}] = [\phi(a_{ij})]$$

for $1 \leq i, j \leq n$ and $a_{ij} \in S$. Call ϕ n-positive, if ϕ_n is positive. Call ϕ completely positive (CP) if ϕ_n is postive for all n. Call ϕ completely bounded (CB) if $\sup_n \|\phi_n\|$ is finite and in that case define, the CB norm as $\|\phi\|_{cb} = \sup_n \|\phi_n\|$. Call ϕ completely isometric (respectively contractive) if ϕ_n is isometric (respectively contractive) for all n.

Our aim is to show that completely postive maps are generalizations of positive linear functionals, in the sense that they can be extended and their norm can be given by the norm of the image of the unit. Before that we have to discuss about C*-algebra structure and positive elements in $\mathcal{M}_n(\mathfrak{A})$ and mention the bounded weak topology.

Proposition 2.1.2 *Consider $\mathcal{M}_n(\mathfrak{A})$ where $\mathfrak{A}$ is a unital C*-algebra under entry-wise addition and conjugation and usual matrix multiplication. Then there is a unique norm on $\mathcal{M}_n(\mathfrak{A})$ so that it is a C*-algebra.*

Proof Identify $\mathfrak{A}$ with subalgebra of some $B(\mathcal{H})$ through GNS. Then $\mathcal{M}_n(\mathfrak{A})$ acts on $\mathcal{H}^n$ and therefore can be identified with a C* subalgebra of $B(\mathcal{H}^n)$. Lift this norm after identification. Uniqueness part is easy as for any Banach-$*$-algebra norm of self-adjoint element is determined by spectral radius which is algebraic and then using C*-algebra norm condition.

Lemma 2.1.3 *Let $\mathfrak{A}$ be a C*-algebra with unit. For a and b in $\mathfrak{A}$ we have,*

$\|a\| \leq 1$ *if and only if*

$$\begin{bmatrix} \mathbf{1} & a \\ a^* & \mathbf{1} \end{bmatrix}$$

is positive in $\mathcal{M}_2(\mathfrak{A})$.

Proof Without loss of generality assume $a \in B(\mathcal{H})$ by identifying through GNS. Now argue for the matrix with identity operator and a as entries using the inner product.

Similarly we have

$$\begin{bmatrix} \mathbf{1} & a \\ a^* & b \end{bmatrix}$$

is positive in $\mathcal{M}_2(\mathfrak{A})$ if and only if $a^*a \leq b$.

Proposition 2.1.4 *Let S be an operator system and $\mathfrak{B}$ be a unital C*-algebra. Assume $\phi : S \to \mathfrak{B}$ is a unital 2-positive map. Then ϕ is contractive.*

Proof Let $a \in S$ be such that $\|a\| \leq 1$. Consider the matrix

$$A = \begin{bmatrix} \mathbf{1} & a \\ a^* & \mathbf{1} \end{bmatrix}$$

in $\mathcal{M}_2(S)$ which is positive by previous lemma.

$$\phi_2(A) = \begin{bmatrix} \mathbf{1} & \phi(a) \\ (\phi(a))^* & \mathbf{1} \end{bmatrix}$$

is positive. By previous lemma $\|\phi(a)\| \leq 1$.

Theorem 2.1.5 *Let S be an operator system, $\mathfrak{B}$ a C*-algebra. $\phi : S \to \mathfrak{B}$ be a CP map. Then ϕ is CB and $\|\phi(\mathbf{1})\| = \|\phi\| = \|\phi\|_{cb}$.*

Proof $\|\phi(\mathbf{1})\| \leq \|\phi\| \leq \|\phi\|_{cb}$ is clear. For $\|\phi\|_{cb} \leq \|\phi(\mathbf{1})\|$, let $I_n \in \mathcal{M}_n(S)$ be the matrix whose diagonal entries are $\mathbf{1}$ and 0 elsewhere. Let $A \in \mathcal{M}_n(S)$ be such that $\|A\| \leq 1$. Cosider the matrix $\mathcal{A} \in \mathcal{M}_{2n}(S)$,

$$\mathcal{A} = \begin{bmatrix} I_n & A \\ A^* & I_n. \end{bmatrix}$$

We know that $\mathcal{A}$ is positive. Therefore $\phi_{2n}(\mathcal{A})$ is positive. That is,

$$\begin{bmatrix} \phi_n(I_n) & \phi_n(A) \\ \phi_n(A)^* & \phi_n(I_n) \end{bmatrix}$$

is positive. So

$$\phi_n(A)^*\phi_n(A) \leq \|\phi_n(I_n)\|\phi_n(I_n) \implies \|\phi_n(A)\| \leq \|\phi_n(I_n)\| = \|\phi(\mathbf{1})\|.$$

2.1.1 Bounded weak topology

We now know now that norm of a CP map is determined by the norm of image of unit like positive linear functionals. What remains is to show that CP maps extend to the C*-algebra from the operator system. To do this we need compactness of some set in bounded weak topology.

Let X and Y be Banach spaces, Y^*, the dual of Y. $B(X, Y^*)$ denote the set of bounded linear operators from X to Y^*. The fact we use here to define the bounded weak topology on this set is that $B(X, Y^*)$ can be identified with dual of some Banach space Z. Assuming this is true we can lift the weak-$*$-topology on Z^* to $B(X, Y^*)$. The identification is as follows.

Given $x \in X$ and $y \in Y$ $x \otimes y$ denote the linear functional on $B(X, Y^*)$ defined by $x \otimes y(L) = (L(x))y$ $\forall L \in B(X, Y^*)$. Let Z denote the closed linear span of $\{x \otimes y | x \in X, y \in Y\}$ in $(B(X, Y^*))^*$. Let $\phi_L \in Z^*$ be such that $\phi_L(x \otimes y) = x \otimes y(L)$. Now the map $\phi : B(X, Y^*) \to Z^*$ given by $\phi(L) = \phi_L$ is isometric $*$-isomorphism. Now, as said earlier topologize $B(X, Y^*)$ with weak* topology of Z^* to obtain bounded weak (BW) topology.

We are going to only use the fact about convergence of nets in BW topology along with Banach- Alaoglu for weak* topology. A net $L_\lambda \in B(X, Y^*)$ converges to L in BW topology iff $L_\lambda(x)$ weakly converges to $L(x)$ for all $x \in X$.

Proposition 2.1.6 *S be an operator system of C*-algebra $\mathfrak{A}$. If $\phi : S \to \mathcal{M}_n$ is CP, then ϕ can be extended to $\mathfrak{A}$ as a CP map.*

Proof Use the one-one correspondence of CP maps from S to $\mathcal{M}_n$ and positive linear functionals on $\mathcal{M}_n(S)$. For a given CP map, construct the corresponding positive linear functional, extend it positively to the algebra, now find the corresponding CP map of the extended positive linear functional.

If S is a closed operator system $B(S, B(\mathcal{H}))$ can be given BW topology by identifying $B(H)$ as the dual of trace class operators.

Proposition 2.1.7 *Let S be a closed operator system of C*-algebra $\mathfrak{A}$. Then*

1. *$CP_r(S, H) := \{L \in B(S, B(\mathcal{H})) : L \text{ is } CP, \|L\| \leq r\}$ is compact in the BW topology.*
2. *$CP(S, H; P) := \{L \in B(S, B(\mathcal{H})) : L \text{ is } CP, L(\mathbf{1}) = P\}$ is compact in the BW topology.*

Proof Use Banach- Alaoglu and the fact that these subsets are closed which easily follows from the criteria for convergence.

Theorem 2.1.8 Arveson's Extension theorem: CP maps on an operator system can be extended to the C*-algebra [5]

Proof Let S be an operator system of $\mathfrak{A}$, $\phi : S \to B(\mathcal{H})$ be CP. For a finite dimensional subspace, $\mathcal{F}$ of $\mathcal{H}$, let $\phi_{\mathcal{F}}$ be the compression of ϕ to $\mathcal{F}$. Since $B(\mathcal{F})$ is isomorphic to $\mathcal{M}_n$ for some n, $\phi_{\mathcal{F}}$ extends to $\psi_{\mathcal{F}} : \mathfrak{A} \to B(\mathcal{F})$. Define $\psi_{\mathcal{F}} : S \to B(\mathcal{H})$ by setting $\psi'_{\mathcal{F}}(a)h = \psi_{\mathcal{F}}(a)h$, for any $h \in \mathcal{F}$ and 0 for any $h \in \mathcal{F}^{\perp}$. Now the collection $\{\psi'_{\mathcal{F}} : \mathcal{F} \text{ is a finite dimensional subspace of } \mathcal{H}\}$ is a net in $CP_{\|\phi\|}(\mathfrak{A}, \mathcal{H})$. By compactness choose a subnet which converges to $\psi \in CP_r(\mathfrak{A}, \mathcal{H})$. This ψ is the desired extension.

2.2 Compression and Dilation

Definition 2.2.1 *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces such that $\mathcal{H} \subseteq \mathcal{K}$. $A \in B(\mathcal{H})$ is called a compression to $\mathcal{H}$ of the operator $B \in B(\mathcal{K})$ if $A = PB|_{\mathcal{H}}$ where $P \in B(\mathcal{K})$ is the projection onto $\mathcal{H}$. In that case, we say B is a dilation of A to $\mathcal{K}$ and we denote it $A \preceq B$.*

Similarly given 2 unital completely positive maps (UCP), $\phi_{1,2} : S \to B(\mathcal{H}_{1,2})$, with $\mathcal{H}_1 \subseteq \mathcal{H}_2$, we say ϕ_1 is a compression of ϕ_2 if $\phi_1(s)$ is a compression of $\phi_2(s)$ to $\mathcal{H}_1$ (in the earlier sense) for all $s \in S$. In that case we write, $\phi_1 \preceq \phi_2$. Context will make it clear whether compression is in operator sense or UCP map sense.

Definition 2.2.2 *A UCP map $\phi : S \to B(\mathcal{H})$ is said to be maximal at $F \subseteq S \times \mathcal{H}$ if for every dilation ψ of ϕ, $\psi(s)x = \phi(s)x$ for all $(s, x) \in F$.*

Definition 2.2.3 *Let $\phi_1 : S \to B(\mathcal{H}_1)$ be UCP and $\phi_2 : S \to B(\mathcal{H}_2)$, a UCP dilation of ϕ_1. We say that dilation is minimal if $\mathcal{H}_2 = [C^*(\phi_2(S))\mathcal{H}_1]$.*

Lemma 2.2.4 *For a UCP map $\phi : S \to B(\mathcal{H})$, the following are equivalent.*

1. *The only minimal dilation of ϕ is ϕ itself.*
2. *ϕ is maximal at $S \times \mathcal{H}$.*
3. *Any dilation ψ of ϕ keeps $\mathcal{H}$ invariant and $\psi = \phi \oplus \psi'$ for some UCP map ψ'.*

Call ϕ maximal if any of the equivalent conditions is satisfied, in which case all the three occur.

Proposition 2.2.5 *Let $\phi : S \to B(\mathcal{H})$ be UCP. For given $(s, x) \in S \times \mathcal{H}$, there exists a UCP dilation ψ of ϕ such that ψ is maximal at (s, x).*

Proof The set $\{\|\psi(s)x\| : \psi \text{ is a UCP dilation of } \phi\}$ is bounded with sup, say r. Then for each n we can find a dilation, $\phi_{n+1} : S \to B(\mathcal{H}_{n+1})$ such that $\|\phi_{n+1}x\| < r - 1/n + 1$. Taking the limit on closure of $\cup\mathcal{H}_n$, we have the required dilation.

Corollary 2.2.6 *Any UCP map can be dilated to a maximal UCP map.*

Proof By lemma 2.2.4, we only need to find a map which is maximal at $S \times \mathcal{H}$. Previous proposition followed by a trasfinite induction on $S \times \mathcal{H}$ produces the required dilation.

2.3 Unique Extension Property

Definition 2.3.1

1. *A UCP map $\phi : S \to B(\mathcal{H})$ is said to have unique extension property (UEP) if ϕ has a unique completely positive extension to $C^*(S)$ and this extension is a representation.*

2. *A boundary ideal for S is a closed 2 sided ideal $\mathfrak{J}$ of $C^*(S)$ such that the natural projection of $C^*(S)$ to $C^*(S)/\mathfrak{J}$ is complete isometry when restricted to S.*

3. *Silov Boundary of S is a boundary ideal for S which contains all other boundary ideals.*

Lemma 2.3.2 *A UCP map has UEP if and only if every CP extension of the map is multiplicative.*

Theorem 2.3.3 *A UCP map has UEP if and only if it is maximal.*

Proof Let $\phi : S \to B(\mathcal{H})$ be a maximal map. Let $\tilde{\phi}$ be a CP extension to $C^*(S)$. Construct the minimal Stinespring dilation of $\tilde{\phi}$. By maximality of ϕ, we have that the extension itself is the representation. For the converse, for a UCP $\phi : S \to B(\mathcal{H})$ with UEP, and $\pi : S \to B(\mathcal{K})$ a minimal dilation, extend π to $C^*(S)$ as $\tilde{\pi}$. Compress

$\tilde{\pi}$ to $\mathcal{H}$. The compression when restricted to S equals ϕ. By UEP, the compression of $\tilde{\pi}$ to $\mathcal{H}$ is a representation. From $\tilde{\pi}(x^*x) \geq \tilde{\pi}(x)^*\tilde{\pi}(x)$, it follows that

$$P_{\mathcal{H}}\tilde{\pi}(x)^* P_{\mathcal{H}}\tilde{\pi}(x)P_{\mathcal{H}} = P_{\mathcal{H}}\tilde{\pi}(x^*x)P_{\mathcal{H}} \geq P_{\mathcal{H}}\tilde{\pi}(x)^*\tilde{\pi}(x)P_{\mathcal{H}}.$$

Therefore $|(I - P_{\mathcal{H}})\tilde{\pi}(x)P_{\mathcal{H}}|^2 \leq 0$. Thus $\mathcal{H}$ is invariant under $C^*(\pi(S))$. By minimality of dilation π, we have $\mathcal{K} = \mathcal{H}$ and $\pi = \phi$.

Corollary 2.3.4 *Any UCP map can be dilated to a UCP map with UEP.*

Theorem 2.3.5 *Let S be an operator system of $\mathfrak{A}$. Silov boundary exists for S [6].*

Proof Begin with complete isometric UCP map $\phi : S \to B(\mathfrak{H})$. Now dilate it to a maximal map. Let it be $\tilde{\phi}$. Extend it uniquely to a representation of $C^*(S)$. Call it π. Kernel of this map is the desired Silov boundary.

Corollary 2.3.6 *Let S be an operator system and $\mathfrak{J}$ be its Silov boundary, the natural projection of S to $C^*(S)/\mathfrak{J}$ identifies $C^*(S)/\mathfrak{J}$ as the C^*-envelope of S [6].*

2.4 Boundary representations

Definition 2.4.1 *Let S be an operator system. An irreducible representation of $C^*(S)$, $(\pi, \mathcal{H})$ is called a boundary representation relative to S if $\pi|_S$ has UEP.*

Lemma 2.4.2 *If $\phi : S \to B(\mathcal{H})$ is a pure and maximal UCP, then ϕ extends to a representation of $C^*(S)$, boundary to S.*

Proof We only need to check that the extension is irreducible as maximality accounts for UEP. Let π be the extension and suppose it is not irreducible. Now choose a proper projection P in the commutant of the image of π. Define ψ by setting $\psi(s) = P\phi(s)$. Now ψ is an intermediate map violating purity of ϕ.

Proposition 2.4.3 *Any pure UCP map can be dilated to a pure and maximal UCP.*

Proof By corollary 2.2.6 and previous lemma.

Theorem 2.4.4 *Let S be an operator system. Sufficiently many boundary representations exist relative to S. That is given $A \in \mathcal{M}_n(S)$, there exists boundary representation π of $C^*(S)$ such that $\|A\| = \|\pi_n(A)\|$. In other words boundary representations completely norm S [8].*

Proof Suppose $A \in \mathcal{M}_n(S)$. We only need to prove the existence of π satisfying the properties in theorem for $T = A^*A$. We know that a state f exists on S such that $f(T) = \|T\|$. Extend it by Hahn Banach extension theorem to $C^*(\mathcal{M}_n(S))$. Call it $\tilde{f}$. Consider the collection of states,

$$\{g : \text{g is a state on } C^*(\mathcal{M}_n(S)), g(T) = \|T\|\}.$$

This is non empty and convex. Choose an extreme point by Krein-Millman, say f_0. This is a pure state as the above collection is a face to the collection of states. Now dilate it to pure and maximal UCP and extend to an irreducible representation π. By Hopenwasser [9], we know that π is unitarily equivalent to ϕ_n, for some irreducible representation ϕ of $C^*(S)$. Now ϕ is the required boundary representation.

Corollary 2.4.5 *Direct sum taken over all boundary representations is a complete isometry. And also the image algebra is a C^*-envelope.*

Chapter 3

Hyperrigidity conjecture

3.1 Boundary representations and hyperrigidity

The classical Korovkin's theorem says that for a sequence of UCP maps $\phi_n : C[0,1] \to C[0,1]$ if $\|\phi_n(f_i) - f_i\| \to 0$ for $f_1(x) = x$ and $f_2(x) = x^2$, then $\|\phi_n(f) - f\| \to 0$ for all f in $C[0,1]$. The set $\{f_1, f_2\}$ is called Korovkin set. Arveson generalized this idea to non-commutative C*-algebra and defined hyperrigid set and hyperrigid operator systems in separable cases. We assume any operator system, C*-algebra and Hilbert space in ths chapter to be separable and also any representation to be non-degenerate. The precise definition of hyperrigidity is as follows.

Definition 3.1.1 *A finite or countable subset $\mathcal{G}$ of a separable operator system generating a C*-algebra $\mathfrak{A}$ is called hyperrigid if for any faithful representation $\mathfrak{A} \subseteq B(\mathcal{H})$ and any sequence of UCP maps $\phi_n : B(\mathcal{H}) \to B(\mathcal{H})$*

$$\|\phi_n(g) - g\| \to 0 \implies \|\phi_n(a) - a\| \to 0$$

for all $a \in \mathfrak{A}$.

If we closely look at the definition we can see that if $\mathcal{G}$ is hyperrigid, so is linear span of $\mathcal{G} \cup \mathcal{G}^*$. We can also adjoin identity if we please. Therefore it makes perfect sense to speak about hyperrigid operator system. The question Arveson asked is how hyperrigid operator systems and its boundary representations are related. Arveson himself proved in a recent paper if an operator system S generating a C*-algebra $\mathfrak{A}$ is hyperrigid in $\mathfrak{A}$, then every irreducible representation of $\mathfrak{A}$ is a boundary representation relative to S and therefore $\mathfrak{A}$ is the C*-envelope of S. The hyperrigidity conjecture is the converse and is still open.

Hyperrigidity conjecture: Let S be an operator system generating a C*-algebra $\mathfrak{A}$. If every irreducible representation of $\mathfrak{A}$ is boundary relative to S, then S is hyperrigid.

In this chapter we will go through the proof of converse of the conjecture and also the improvements made so far in solving the conjecture. What follows is the main theorem of this thesis.

Theorem 3.1.2 *For a separable operator system S generating a C*-algebra $\mathfrak{A}$ following statements are equivalent.*

1. *S is hyperrigid.*
2. *For any nondegenerate representation $\pi : \mathfrak{A} \to B(\mathcal{H})$ and any sequence of UCP maps $\phi_n : \mathfrak{A} \to B(\mathcal{H})$*

$$\|\phi_n(s) - \pi(s)\| \to 0 \implies \|\phi_n(a) - \pi(a)\| \to 0$$

   for all $a \in \mathfrak{A}$.

3. *Any nondegenerate representation of $\mathfrak{A}$ has UEP when restricted to S.*
4. *For any unital C*-algebra $\mathfrak{B}$ and any $*$-homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$, and any UCP map $\Phi : \mathfrak{B} \to \mathfrak{B}$*

$$\Phi(\pi(s)) = \pi(s) \implies \Phi(\pi(a)) = \pi(a)$$

   for all $a \in \mathfrak{A}$.

Proof $1 \implies 2$: Following the notation in the theorem we have to prove that

$$\|\phi_n(s) - \pi(s)\| \to 0 \implies \|\phi_n(a) - \pi(a)\| \to 0$$

for all $a \in \mathfrak{A}$. Fix any faithful representation of $\mathfrak{A}$, say $(\rho, \mathcal{K})$. Now $(\rho \oplus \pi, \mathcal{H} \oplus \mathcal{K})$ is a faithful representation. Consider the sequence of maps $\omega_n : \rho \oplus \pi(A) \to B(\mathcal{H} \oplus \mathcal{K})$ defined by

$$\rho \oplus \pi(a) \mapsto \rho(a) \oplus \phi_n(a).$$

This is UCP as ρ is a homomorphism and ϕ_n are UCP. Now, by Arveson extension theorem, extend the map ω_n to $B(\mathcal{H} \oplus \mathcal{K})$, call it $\tilde{\omega_n}$. $\omega_n(s) \to s$ in norm because of the assumption $\phi_n(s) \to s$ in norm. Now use hyperrigidity of S for the representation $(\rho \oplus \pi, \mathcal{H} \oplus \mathcal{K})$ and sequence of maps $\tilde{\omega_n}$. And the theorem follows from an easy approximation.

$2 \implies 3$ is trivial.

$3 \implies 4$: Following the notation define $B_0 = \pi(\mathfrak{A})$ and inductively define

$$B_{n+1} = B_n \cup \phi(B_n) \cup \phi^2(B_n) \cdots .$$

The closure of the union of B_n, say B_∞ is a separable C*-algebra inside $\mathfrak{B}$ such that $\phi(B_\infty) \subseteq B_\infty$. Represent B_∞ in a separable Hilbert space $\mathcal{H}$. Now under the obvious identification we have a representation of $\mathfrak{A}$ on $\mathcal{H}$=, which is clearly nondegenerate. By hypothesis, it has UEP when restricted to S and the theorem follows.

$4 \implies 1$. Following the notation of the theorem, write $B = B(\mathcal{H})$. Let $l^\infty(B)$ be the C*-algebra of bounded sequences in B and c_0 be the ideal of sequences converging to 0 in norm. Consider the UCP map $\Phi : l^\infty(B) \to l^\infty(B)$ defined as follows.

$$\Phi(b_1, b_2, \cdots) = (\phi_1(b_1), \phi_2(b_2), \cdots).$$

This carries $c_0(B)$ to $c_0(B)$. Therefore we have a natural map from, $\tilde{\Phi} : l^\infty(B)/c_0(B) \to l^\infty(B)/c_0(B)$ defined by

$$x + c_0(B) \mapsto \Phi(x) + c_0(B).$$

Now consider the homomorphism $\pi : (A) \to l^\infty(B)/c_0(B)$ defined by

$$a \mapsto (a, a, a, ...) + c_0(B).$$

$$\Phi(\pi(s)) = (\phi_1(S), \phi_2(S), \cdots) + c_0(B) = (s, s, s, \cdots) + c_0(B) = \pi(s)$$

Now by hypothesis, $\Phi(\pi(a)) = \pi(a)$ for all $a \in \mathfrak{A}$ and we are done.

Now it is easy to read off the converse of hyperrigidity conjecture from the above theorem which is the next corollary.

Corollary 3.1.3 *For a separable operator system generating a C^*-algebra $\mathfrak{A}$, every irreducible representation of $\mathfrak{A}$ is boundary relative to S and therefore Silov boundary is the zero ideal which implies $\mathfrak{A}$ is the C^*-envelope of S.*

Proof Statement 3 of the above theorem.

3.2 Examples of hyperrigid sets

Theorem 3.2.1 *Let $x \in B(\mathcal{H})$ be self-adjoint and $\mathfrak{A}$ be the C^*-algebra generated by x and $\mathbf{1}$. Then the set(equivalently the operator system generated by) $\{\mathbf{1}, x, x^2\}$ is hyperrigid in $\mathfrak{A}$.*

Proof By the many characterisations we have proved about hyperrigidity, we only need to prove that given any representation $\pi : \mathfrak{A} \to B(\mathcal{K})$, π restricted to S has UEP. Suppose $\phi : \mathfrak{A} \to B(\mathcal{K})$ be UCP such that $\phi(x) = \pi(x)$ and $\phi(x^2) = \pi(x^2)$. Now by Stinespring dilation we have a Hilbert space $\mathcal{L}$ containing $\mathcal{K}$ and a representation $\sigma : \mathfrak{A} \to B(\mathcal{L})$ such that $\phi(a) = P\sigma(a) \restriction_{\mathcal{K}}$ where $P \in B(\mathcal{L})$ is the projection onto $\mathcal{K}$. Note that

$$\begin{aligned} P\sigma(x)(\mathbf{1} - P)\sigma(x)P &= P\sigma(x^2)P - P\sigma(x)P\sigma(x)P \\ &= \phi(x^2)P - \phi(x)^2P = \pi(x^2)P - \pi(x)^2P = 0. \end{aligned}$$

This shows that $|(\mathbf{1} - P)\sigma(x)P|^2 = 0$. Therefore $(\mathbf{1} - P)\sigma(x)P = 0$. Hence $\sigma(x)$ leaves $\mathcal{K}$ invariant. Since $\mathfrak{A}$ is generated by x, $\sigma(A)$ leaves $\mathcal{K}$ invariant. Hence ϕ is multiplicative and we are done.

Theorem 3.2.2 *Let x and $\mathfrak{A}$ be as in above theorem with spectrum of x containing at least 3 points. Then, the set $\{\mathbf{1}, x\}$ is not hyperrigid.*

Proof Let $\lambda_1 < \lambda_2 < \lambda_3$ be distinct points in the spectrum. For $k = 1, 2, 3$ define states ρ_k as follows.

$$\rho_k(f(x)) = f(\lambda_k) f \in C(\sigma(x)).$$

These are irreducible representations of $\mathfrak{A}$. Write λ_2 as convex combination of λ_1 and λ_3. Let $S = span\{\mathbf{1}, x\}$. Now $\rho_2 \restriction_S$ is a convex combination of ρ_1 and ρ_3. Since $\rho_1 \neq \rho_3$, $\rho_2 \restriction_S$ fails to have UEP. Therefore $\{\mathbf{1}, x\}$ is not hyperrigid in $\mathfrak{A}$.

Theorem 3.2.3 *Let $u_1, u_2, ..., u_k$ be isometries generating a C^*-algebra $\mathfrak{A}$, then the set $\{u_1, u_2, ..., u_k, u_1u_1^* + u_2u_2^* + ... + u_ku_k^*\}$ is hyperrigid in $\mathfrak{A}$.*

Proof Let $(\pi, \mathcal{H})$ be any representation. Set $v_k = \pi(u_k)$, clearly isometries and $\phi : \mathfrak{A} \to B(\mathcal{H})$ be UCP such that $\phi(u_k) = v_k$ and $\phi(u_1u_1^* + u_2u_2^* + ... + u_ku_k^*) = v_1v_1^* + v_2v_2^* + ... + v_kv_k^*$. Let $(\sigma, V, \mathcal{K})$ be the minimal Stinespring dilation. We proceed in 3 steps.

Step 1: To show that $\sigma(u_k)V = Vv_k$.

Step 2: $V\mathcal{H}$ is invariant under $\sigma(\mathfrak{A})$.

Step 3: $\phi = \pi$

Proof for step 1:

$$V^*\sigma(u_k)^*VV^*\sigma(u_k)V = \phi(u_k)^*\phi(u_k) = v_k^*v_k = \mathbf{1}_{\mathcal{H}}.$$

Hence $\sigma(u_k)$ leaves $V\mathcal{H}$ invariant and therefore

$$\sigma(u_k)V = VV^*\sigma(u_k)V = V\phi(u_k) = Vv_k.$$

Proof for step 2:

$$\begin{aligned}
\sum_{i=1}^{k} \sigma(u_k)VV^*\sigma(u_k)^* &= \sum_{i=1}^{k} Vv_kv_k^*V^* = V\phi(\sum_{i=1}^{k} u_ku_k^*) \\
&= VV^* \sum_{i=1}^{k} \sigma(u_ku_K^*)VV^* \\
&= \sum_{i=1}^{k} VV^*\sigma(u_k)\sigma(u_k^*)VV^*
\end{aligned}$$

and since $\sigma(u_k)V = VV^*\sigma(u_k)V$, subtracting left from right and rewriting we get $\sigma(v_k)^*$ also leaves $V\mathcal{H}$ invariant. Since the C*-algebra $\sigma(\mathfrak{A})$ is generated by $\sigma(x)$ and $\sigma(x)^*$, $V\mathcal{H}$ is invariant under $\sigma(\mathfrak{A})$.

Proof for step 3: By minimality of $\mathcal{K}$ and step 2, we have $V\mathcal{H} = \mathcal{K}$, V is unitary and $\phi(x) = V^{-1}\sigma(x)V$ is a representation. Since ϕ agrees on a generating set with π, we have $\phi = \pi$.

3.3 Hyperrigidity and type I C*-algebra

This section deals with the very recent developments in hyperrigidity conjectures. It was proven by Arveson that if C*- algebra generated by operator system has countable spectrum then hyperrigidity conjecture is true. When C*-algebra has countable spectrum it is type I, i.e, its double commutant is a type I von Neumann algebra. Now we know that conjecture is true even if we only assume that C*-algera is type I rather than having countable spectrum. The proof given by Kleski is detailed here. In this section S is a separable operator system acting on separable Hilbert space $\mathcal{H}$ and the C*-algebra it generates, $\mathfrak{A}$ is type I unless stated otherwise.

First we weaken the hypothesis of type I to nuclearity.

Theorem 3.3.1 *Let S generate a nuclear C*-algebra in $B(\mathcal{H})$. Suppose every factor representation of $\mathfrak{A}$ has UEP relative to S. Let $(\rho, \mathcal{K})$ be a faithful representation of $\mathfrak{A}$ and $\gamma : \rho(\mathfrak{A}) \to B(\mathcal{K})$ be UCP extending Identity on $\rho(S)$. For any conditional expectation $E : B(\mathcal{K}) \to \rho(A)''$, then $E\gamma$ is identity on $\rho(\mathfrak{A})$.*

Proof Consider the abelian von Neumann algebra $\mathfrak{M}$, the centre of $\rho(\mathfrak{A})''$. By theorem 1.7.21 of disintegration theory $\mathcal{K}$ decomposes as $\int_X^{\oplus} \mathcal{K}_x d\mu(x)$ where (X, μ) is a measure space where each $\mathcal{K}_x$ is a factor. The identity representation of $\rho(\mathfrak{A})''$ decomposes into $\int_X^{\oplus} \pi_x(b)d\mu(x)$, each π_x is a factor representation of $\rho(\mathfrak{A})$. Since $E\gamma\rho(\mathfrak{A}) \subseteq \rho(\mathfrak{A})''$, we have that

$$E\gamma\rho(a) = \int_X^{\oplus} \pi_x(E\gamma\rho(a))d\mu(x).$$

Now $\gamma\rho = \rho$ on S. Therefore $\pi_x E\rho = \pi_x\rho$ on S. By hypothesis $\pi_x\rho$ has UEP relative to S and therefore $\pi_x E\gamma\rho = \pi_x\rho$ from which we conclude $E\gamma\rho = \rho$.

Corollary 3.3.2 *Let S be an operator system generating $\mathfrak{A}$. If every irreducible representation of $\mathfrak{A}$ is boundary relative to S, then for any representation $(\pi, \mathcal{K})$ and any UCP $\psi : \pi(\mathfrak{A}) \to B(\mathcal{K})$ extending identity on $\pi(S)$ and any conditional expectation $E : B(\mathcal{K}) \to \pi(\mathfrak{A})''$, $E\psi\pi = \pi$.*

Proof Let ρ be a faithful representation and $F : B(\mathcal{K}) \to \rho(\mathfrak{A})''$, a conditional expectation. By previous theorem applied to the faithful representation $\rho \oplus \pi$ and conditional expectation $F \oplus E$, we have

$$(F \oplus E)(\rho \oplus \psi\pi(a)) = (\rho \oplus \pi)(a).$$

Therefore $E\psi\pi = \pi$.

Corollary 3.3.3 *Let S generate $\mathfrak{A}$. Suppose every irreducible representation of $\mathfrak{A}$ is boundary relative to S. For any UCP $\psi : \mathfrak{A} \to \mathfrak{A}''$ extending identity on S, we have that ψ is identity on $\mathfrak{A}$.*

Proof Use previous theorem by taking ρ as the identity representation. When $\mathfrak{A}$ is type I, every factor representation is a multiple of an irreducible representation. The hypothesis of the theorem is satisfied and since $\psi(\mathfrak{A}) \subseteq A''$, we have $E\psi = \psi$ and $\psi(a) = a$. The following theorem asserts that hyperrigidity conjecture is true in the case when C*-algebra generated by the operator system is type I.

Theorem 3.3.4 *Let S be an operator system such that $\mathfrak{A}$=$C^*(S)$ is type I. If every irreducible representation of $\mathfrak{A}$ is boundary relative to S, then S is hyperrigid.*

Proof Let $\mathfrak{B}$ be arbitrary unital C*-algebra and $\pi : \mathfrak{A} \to \mathfrak{B}$ be a $*-$homomorphism. After concretely realising $\mathfrak{B}$ as algebra of operators, we have that $\pi(\mathfrak{A})$ is a type I C* algebra. By 3.3.3 and statement 4 of theorem 3.1.2, we only need to verify that any UCP map from $\mathfrak{B}$ to $\mathfrak{B}$ fixing $\pi(S)$ fixes $\pi(\mathfrak{A})$. By 3.3.3, we know that statement holds for UCP maps from $\pi(\mathfrak{A})$ to $\pi(A)''$. For any arbitrary UCP map from $\mathfrak{B}$ to $\mathfrak{B}$ fixing $\pi(S)$, restricting it to $\pi(\mathfrak{A})$, we have the result.

3.4 Conclusion

In summary this work presents important results in the theory of non commutative Choquet boundary which was started by Arveson by generalising the idea of boundary in commutative C*-algebra to non commutative C*-algebra. The tools used for generalisation are completely positive maps and their unique extension property. Hamana proved that Silov boundary exists and K R Davidson together with Mathew Kennedy proved that boundary representations completely norm an operator system. The recent studies on the area is about investigating the hyperrigidity conjecture. This conjecture was proposed by Arveson. The critical observation is that $\{\mathbf{1}, x, x^2\}$ has Korovkin property and also C*-algebra generated, $C(X)$ is the C*-envelope. Arveson himself proved that this is true in general for a hyperrigid operator system. Now what is still open in the conjecture is to study whether UEP of restriction of every irreducible representation to an operator system amounts to its hyperrigidity. There are several cases which exhibits the truth of the conjecture, among which the case where C*-algebra generated is type I, proved by C. Kleski is discussed in detail. For another important case where the C*-algebra generated has countable spectrum, one can refer to [7]. Like classical Korovkin theorem became an important tool and resulted in advancement of approximation theory, the investigations on hyperrigidity conjecture will be extremely helpful in non commutative approximation theory.

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Subject Index

Notation Index

Abbreviations

a.e	almost everywhere
BW	Bounded Weak
CB	Completely Bounded
CC	Completely Contractive
CP	Completely Positive
SOT	Strong Operator Topology
UCP	Unital Completely Postive
UEP	Unique Extension Property
WOT	Weak Operator Topology