Background Curvature effects on Superradiance

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Certificate of Examination

This is to certify that the dissertation titled **"Background Curvature effects on Superradiance"** submitted by **Apoorv Gaurav** (MS15133) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: May 4, 2020

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr.Kinjalk Lochan at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgment of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Apoorv Gaurav
(Candidate)

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr.Kinjalk Lochan
(Supervisor)

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Abstract

General Relativity is a theory given by Einstein as an attempt to explain gravity. This is a geometrical theory that imagines a manifold created by spacetime. On this background astrophysical object exists, which because of their masses give spacetime its curvature. And this curvature dictates the motion of particles in the spacetime. So in a sense, in general relativity, curvature takes the seat of gravitational potential in the Newtonian theory of gravity.

In this thesis, we have studied superradiance in presence of different background curvature. The project is broadly divided into three parts on the basis of the asymptote background curvature. In the first part, we have studied the superradiance in presence of Schwarzschild black hole and Kerr Balck hole, both of them are in asymptotically flat spacetime. In the second part, we have studied the superradiance in Kerr black hole in DeSitter Spacetime. And in the final part, we have studied the superradiance in presence of BTZ black hole which has AdS spacetime as its asymptote

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Chapter 1

Maximally Symmetric Spacetime

1.1 Lie Derivatives and Killing Vector

Lie derivative is used to differentiate whether the change in tensor upon transversing along a curve is due to translation or it is just due to a coordinate transformation. A change that can be carried out by co-ordinate change is not an actual change. Consider change in A^α as a result of translation along curve such that co-ordinates of Q and P is given by $x^{\alpha'}$ and x^α respectively.

$$x^{\alpha'} = x^\alpha + dx^\alpha$$

and,

$$dx^\alpha = u^\alpha d\lambda$$

Here, u^α is velocity along α direction and λ is a parameter which parameterizes the curve.
Then change in A^α due to translation on curve is

$$\begin{aligned} A^\alpha(Q) &= A^\alpha(P) + DA^\alpha \\ &= A^\alpha(P) + \partial_\beta A^\alpha dx^\beta + \Gamma^\alpha_{\mu\nu} A^\mu dx^\nu \\ &= A^\alpha(P) + \partial_\beta A^\alpha u^\beta d\lambda + \Gamma^\alpha_{\mu\nu} A^\mu u^\nu d\lambda \end{aligned}$$

Now consider change in A^α as a result of infinitesimal co-ordinate transformation $\left(x^\alpha \rightarrow x'^\alpha\right)$

$$x'^\alpha = x^\alpha + dx^\alpha$$

$$\left(\frac{\partial x'^{\alpha}}{\partial x^{\beta}}\right) = \left(\delta^{\alpha}_{\beta} + \partial_{\beta} u^{\alpha} d\lambda\right)$$

Change in A^{α} due this transformation:

$$\begin{aligned} A'^{\alpha}(x') &= \left(\frac{\partial x'^{\alpha}}{\partial x^{\beta}}\right) A^{\beta}(x) \\ &= \left(\delta^{\alpha}_{\beta} + \partial_{\beta} u^{\alpha} d\lambda\right) A^{\beta}(x) \\ &= A^{\alpha}(x) + \partial_{\beta} u^{\alpha} A^{\beta} d\lambda \end{aligned}$$

Hence we have,

$$A'^{\alpha}(Q) = A^{\alpha}(P) + \partial_{\beta} u^{\alpha} A^{\beta} d\lambda$$

Lie Derivative is defined as difference of these changes as follows:

$$L_u A^{\alpha} = \frac{A(Q) - A'(Q)}{d\lambda}$$

$$L_u A^{\alpha} = u^{\beta} \partial_{\beta} A^{\alpha} - A^{\beta} \partial_{\beta} u^{\alpha}$$

Using the definition of covariant derivative we replace the partial derivatives with covariant derivatives.

$$L_u A^{\alpha} = u^{\beta} \left(D_{\beta} A^{\alpha} - \Gamma^{\alpha}_{\beta\gamma} A^{\gamma}\right) - A^{\beta} \left(D_{\beta} u^{\alpha} - \Gamma^{\alpha}_{\beta\gamma} u^{\gamma}\right)$$

Torsionlessness of covariant derivative implies that the christoffel connection term is symmetric with respect to lower indices. So we have,

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$$

Hence, Lie derivative of A^{α} along u^{μ} is given by

$$L_u A^{\alpha} = u^{\beta} D_{\beta} A^{\alpha} - A^{\beta} D_{\beta} u^{\alpha}$$

1.1.1 Action of Lie Derivatives

Action on Scalars

Let f be the scalar function. Then Lie derivative of f along the u^{μ} will be given by

$$L_u f = \frac{f(Q) - f^{'}(Q)}{d\lambda}$$

Where Q and P are close to each other and there co-ordinates is given by $x^{\alpha'}$ and x^{α} respectively.

$$x^{\alpha'} \equiv x^{\alpha} + u^{\alpha} d\lambda \implies u^{\alpha} = \frac{dx^{\alpha}}{d\lambda}$$

$$f(Q) = f(P) + u^{\mu}(\partial_{\mu} f) d\lambda$$

Since scalars are invariant under co-ordinate transformation, so

$$f^{'}(Q) = f(P)$$

$$L_u f = \frac{f(P) + u^{\mu}(\partial_{\mu} f) d\lambda - f(P)}{d\lambda}$$

$$L_u f = u^{\mu} \partial_{\mu} f$$

Action on Tensor of any rank

Once we know the action of Lie Derivatives on scalar and vector we can compute it's action on tensor of any rank by creating scalars out of it.

$$\begin{aligned} L_u T^{a_1 a_2 \ldots a_n}_{b_1 b_2 \ldots b_m} = u^{\mu} D_{\mu} T^{a_1 a_2 \ldots a_n}_{b_1 b_2 \ldots b_m} - \big(& T^{\mu a_2 \ldots a_n}_{b_1 b_2 \ldots b_m} D_{\mu} u^{a_1} + T^{\mu a_2 \ldots a_n}_{b_1 b_2 \ldots b_m} D_{\mu} u^{a_1} + T^{a_1 \mu \ldots a_n}_{b_1 b_2 \ldots b_m} D_{\mu} u^{a_2} + \ldots \big) \\ & + \big(T^{a_1 a_2 \ldots a_n}_{\mu b_2 \ldots b_m} D_{b_1} u^{\mu} + T^{a_1 a_2 \ldots a_n}_{b_1 \mu \ldots b_m} D_{b_2} u^{\mu} + \ldots \big) \end{aligned}$$

1.1.2 Lie Derivative of metric

$$L_u g_{ab} = u^c D_c g_{ab} + g_{cb} D_a u^c + g_{ac} D_b u^c$$

Since Covariant derivative is metric compatible, we can write above equation as

$$L_u g_{ab} = D_a(g_{cb} u^c) + D_b(g_{ac} u^c)$$

$$L_u g_{ab} = D_a u_b + D_b u_a$$

If Lie deivative of metric along any direction u^{μ} is zero, then it is said to be isometry as metric remains invariant while transversing along that direction.

$$L_u g_{\mu\nu} = 0$$

$$D_\mu u_\nu + D_\nu u_\mu = 0 \tag{1.1}$$

Above equation is known as Killing's Equation. And vector u^μ satisfying above equation is said to be **Killing Vector**.

1.1.3 Conserved Charges

If K^μ is a killing vector and x^μ be geodesic. Then $K_\mu u^\mu$ is conserved quantity, where $u^\mu = \frac{dx^\mu}{d\tau}$.

$$\frac{d(K_\mu u^\mu)}{d\tau} = (u^\nu D_\nu K_\mu) u^\mu + (u^\nu D_\nu u^\mu) K_\mu$$

Second term is geodesic equation and hence it is 0.

$$\frac{d(K_\mu u^\mu)}{d\tau} = \frac{u^\mu u^\nu}{2}(D_\nu K_\mu + D_\mu K_\nu)$$

Bracketed term is killing equation and since K^μ is killing vector it is also 0.

$$\frac{d(K_\mu u^\mu)}{d\tau} = 0$$

Hence, $K_\mu u^\mu$ is conserved quantity.

1.2 Maximally Symmetric Spacetime

Maxiamlly Symmetric Spacetimes are those spacetimes which posses the maximum possible symmetries. Here symmetry is meant by invariance of the metric. Consider $(1, n-1)$ dimensional spacetime. Then possible degrees of freedom can be listed as follows:

- Translations : For n dimensions, we translate along each one of them. That gives us n degrees of freedom

- Boosts: Out of n dimensions, we have $n-1$ spacelike direction along which we can boost. That gives us $n-1$ degrees of freedom.

- Rotations: From $n-1$ spacelike directions, we can fix a plane of rotation by choosing two axes. That gives us ${}^{n-1}C_2$ degrees of freedom

Adding up all these we will get total of ${}^{n+1}C_2$ degrees of freedom. So if a spacetime admits these many vectors along which metric remains invariant,then such spacetime is said to be maximally symmetric.

1.3 Constant Curvature Spacetime

If we were to consider manifolds of constant curvature(Λ), we have three possiblities.

- $\Lambda > 0$: This is called DeSitter Spacetime. We can think of its shape to be spherical.
- $\Lambda = 0$: This is the Minkowski Spacetime.
- $\Lambda < 0$: This is called Anti DeSitter Spacetime. It's shape would resemble the horse's saddle.

In this section we will calculate killing vectors of these spacetimes and see that they are maximally symmetric.

1.3.1 Minkowski is maximally symmetric

In this section we will see that flat spacetime is maximally symmetric. For calculations, we have used $(1+2)$ dimensional flat spacetime.
Metric is given by:

$$ds^2 = -dt^2 + dx^2 + dy^2$$

We will try to look for solution of the killing equation and see how many independent parameters we get. Since we are working in $(1+2)$ dimensional spacetime, we expect it to have ${}^4C_2$ i.e. 6 independent parameters for it to be maximally symmetric. Let's say u^μ is a killing direction

$$u^\mu = \left(u^t, u^x, u^y\right)$$

Since it is a killing direction, it will satisfy the killing's equation:

$$D_\mu u_\nu + D_\nu u_\mu = 0$$

μ and ν are independent parameters, each taking 3 values. So above equation is in fact set of 9 equations. Out of these 9 equations, we will have 3 pairs of identical equation when μ and ν take same values. This leave us with set of 6 independent equations. In this co-ordinate system all the metric components are constant. As a result all the Christoffel symbols vanishes and we can use partial derivative instead of covariant derivative.

- $\partial_t u_t = 0$
- $\partial_t u_x + \partial_x u_t = 0$
- $\partial_t u_y + \partial_y u_t = 0$
- $\partial_x u_x = 0$
- $\partial_y u_y = 0$
- $\partial_x u_y + \partial_y u_x = 0$

We can manipulate above 6 equations to get set of second order equations for one component of u^μ. For instance, we can diffrentiate 2^{nd} equation with respect to x. This gives us:

$$\partial_x \partial_t u_x + \partial_x^2 u_t = 0$$

We can now use 4^{th} equation to set first term to zero. Hence we have

$$\partial_x^2 u_t = 0$$

Similarly we can diffrentiate 3^{rd} equation with respect to y, and use 5^{th} equation to eliminate some terms. This will gives us a second oreder diffrential equation for u_t in terms of y.

$$\partial_y^2 u_t = 0$$

This gives us set of second order equations for u_t. Similarly we can find equations for u_x and u_y.

These equations are as follows:

- $\partial_t^2 u_x = 0$
- $\partial_x u_x = 0$
- $\partial_y^2 u_x = 0$
- $\partial_t^2 u_y = 0$
- $\partial_x^2 u_y = 0$
- $\partial_y u_y = 0$

We can solve for 3 simulataneous diffrentional equation to get a component of u^μ and we can solve for 3 such sets to get the complete u^μ.

The solution for these equations turns out to be

$$u_t = ax + by + \lambda_1 xy$$

$$u_x = ct + dy + \lambda_2 ty$$

$$u_y = et + fx + \lambda_3 tx$$

We then substitute these solutions in equation 2 to find relation between these 9 co-efficients. It will turn out that all the λs would turn out to be 0. Hence our solution would be

$$u_t = ax + by$$

$$u_x = ct + dy$$

$$u_y = et + fx$$

So we have our 6 independent parameters which we were hoping for. Hence Minkowski Spacetime is maximally symmetric.

1.3.2 AdS Spacetime is maximally symmetric

In this section we will see that AdS spacetime is also maximally symmetric. For calculations, we have used $(1+2)$ dimensional spacetime.
Line Element is given by[Carlip 95]:

$$ds^2 = -\left(\frac{r^2}{l^2}\right) dt^2 + \frac{dr^2}{\left(\frac{r^2}{l^2}\right)} + r^2 d\phi^2$$

Since we are working in $(1+2)$ dimensional spacetime, we expect it to have ${}^4C_2$ i.e. 6 independent parameters for it to be maximally symmetric.

Let's say u^μ is a killing direction

$$u^\mu = \left(u^t, u^x, u^y\right)$$

Since it is a killing direction, it will satisfy the killing's equation:

$$D_\mu u_\nu + D_\nu u_\mu = 0$$

$$\partial_\mu u_\nu + \partial_\nu u_\mu - 2\Gamma^\lambda_{\mu\nu} u_\lambda = 0$$

Here we have to find the Christoffel connections before writing the Killing's Equation. Non zero Christoffel symbol for the metric is given by:

- $\Gamma^t_{tr} = \frac{1}{r}$
- $\Gamma^r_{tr} = \frac{r^3}{l^4}$
- $\Gamma^r_{rr} = \frac{-1}{r}$
- $\Gamma^\phi_{r\phi} = \frac{1}{r}$
- $\Gamma^r_{\phi\phi} = \frac{-r^3}{l^2}$

Now we can use these Christoffel symbols to write killing equations. Similar to Minkowski's case we will again have 9 equations out of which 3 would be repeated. This would give us set of 6 independent equations. Those are as follows:

- $\partial_t u_t - \frac{r^3}{l^4} u_r = 0$
- $\partial_r u_r + \frac{1}{r} u_r = 0$
- $\partial_t u_r + \partial_r u_t - \frac{2}{r} u_t = 0$
- $\partial_r u_\phi + \partial_\phi u_r - \frac{2}{r} u_\phi = 0$
- $\partial_t u_\phi + \partial_\phi u_t = 0$
- $\partial_\phi u_\phi + \frac{r^3}{l^2} u_r = 0$

We can again manipulate these equations by diffrentiating with respect to relevant parameter to get set of diffrential equation for the components of killing direction and solve them.The solution to these equation is given by:

Radial Component: Set of equations for radial component turn out to be:

$$\partial_t^2 u_r = 0$$
$$\partial_r u_r + \frac{1}{r} u_r = 0$$
$$\partial_\phi^2 u_r = 0$$

The solution to these equation is

$$u_r = \frac{[(k_1\phi + k_2)\, t + (k_3\phi + k_4)]}{r}$$

Angular Component: To get to the set of differential equation for the angular part, we use our solution for radial part. The set of equation for angular part is given by:

$$\partial_t^2 u_\phi + \frac{r^2}{l^2} (k_1 t + k_2) = 0$$

$$\partial_r u_\phi - \frac{2}{r}u_\phi + \left(\frac{k_1 t + k_3}{r}\right) = 0$$

$$\partial_\phi u_\phi + \frac{r^3}{l^2}\left(\frac{(k_1\phi + k_2)\,t + (k_3\phi + k_4)}{r}\right) = 0$$

The solution to the angular part is given by:

$$u_\phi = \frac{-r^2}{l^2}\left[\frac{k_1 t^3}{6} + \frac{k_3 t^2}{2}\right] + r^2 t\left[k_5 - \frac{k_1\phi^2}{2l^2}\right] - r^2\left[\frac{k_3\phi^2}{2l^2} + \frac{(k_2 + k_4)\,\phi}{l^2}\right] + \frac{k_1 t + k_3}{2}$$

Time Component: To get to the set of differential equation for the time component, we use our solution for radial and angular part. The set of equation for angular part is given by:

$$\partial_t u_t - \frac{r^2}{l^2}\left((k_1\phi + k_2)\,t^2 + k_3\phi + k_4\right) = 0$$

$$\partial_r u_t + \frac{(k_1\phi + k_2)\,t}{r} - \frac{2}{r}u_t = 0$$

$$\partial_\phi u_t + \left(\frac{k_1}{r} - k_2 r\right) = 0$$

The solution to time component is given by:

$$u_t = \frac{r^2}{l^4}\left[(k_1\phi + k_2)\,\frac{t^2}{2} + (k_3 + k_4)\,t\right] - \frac{(k_1 t + k_3)\,\phi^2 r^2}{2l^2} - \frac{(k_2 + k_4)\,\phi r^2}{l^2} - \frac{k_1\phi}{2r^2} + \frac{k_1\phi + k_2}{2} + k_6 r^2$$

So in the complete solution of our killing direction we have the 6 independent parameters we were looking for. Hence AdS Spacetime is maximally symmetric.

1.3.3 DeSitter Spacetime is maximally symmetric

In this section we will see that DeSitter spacetime is also maximally symmetric. For calculations, we have used $(1 + 2)$ dimensional spacetime.
Line Element is given by[Akcay 11]:

$$ds^2 = -\left(1 - \frac{\Lambda r^2}{3}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{\Lambda r^2}{3}\right)} + r^2 d\phi^2 \tag{1.2}$$

Since we are working in $(1+2)$ dimensional spacetime, we expect it to have ${}^4C_2$ i.e. 6 independent parameters for it to be maximally symmetric.

Let's say u^μ is a killing direction

$$u^\mu = \left(u^t, u^x, u^y\right)$$

Since it is a killing direction, it will satisfy the killing's equation:

$$D_\mu u_\nu + D_\nu u_\mu = 0$$

$$\partial_\mu u_\nu + \partial_\nu u_\mu - 2\Gamma^\lambda_{\mu\nu} u_\lambda = 0$$

In DeSitter spacetime, we need to calculate Christoffel symbols before proceeding further. Non zero Christoffel symbol for the metric is given by:

- $\Gamma^t_{tr} = \frac{\Lambda}{\Lambda r^2 - 3}$
- $\Gamma^r_{tr} = \frac{\Lambda r(\Lambda r^2 - 3)}{9}$
- $\Gamma^r_{rr} = \frac{\Lambda}{3 - \Lambda r^2}$
- $\Gamma^\phi_{r\phi} = \frac{1}{r}$
- $\Gamma^r_{\phi\phi} = \frac{r(\Lambda r^2 - 3)}{3}$

Let's call $\sqrt{\frac{3}{\Lambda}} = r_c$.

Now we can use these Christoffel symbols to write killing equations. As before we will again have 9 equations out of which 3 would be repeated. This would give us set of 6 independent equations. Those are as follows:

- $\partial_t u_t - \frac{\Lambda^2 r(r^2 - r_c^2)}{9} u_r = 0$
- $\partial_r u_r + \frac{r}{r_c^2 - r^2} u_r = 0$
- $\partial_t u_r + \partial_r u_t - \frac{2r}{r^2 - r_c^2} = 0$
- $\partial_r u_\phi + \partial_\phi u_r - \frac{1}{r} u_\phi = 0$
- $\partial_t u_\phi + \partial_\phi u_t = 0$
- $\partial_\phi u_\phi - \frac{\Lambda r(r^2 - r_c^2)}{3} u_r = 0$

We can again manipulate these equations by differentiating with respect to relevant parameter to get set of differential equation for the components of killing direction and solve them.The solution to these equation is given by:

Radial Component: Set of equations for radial component turn out to be:

$$\partial_t^2 u_r - \frac{\Lambda}{3} u_r = 0$$

$$\partial_r u_r + \frac{r}{r^2 - r_c^2} u_r = 0$$

$$\partial_\phi^2 u_r + \frac{\Lambda r^2}{3} u_r = 0 \tag{1.3}$$

The solution to these equation is

$$u_r = \frac{\left(k_1 e^{i\frac{r}{r_c}\phi} + k_2 e^{-i\frac{r}{r_c}\phi}\right)\left(k_3 e^{-\frac{t}{r_c}} + k_4 e^{\frac{t}{r_c}}\right)}{\sqrt{\Lambda(r^2 - r_c^2)}} \tag{1.4}$$

Angular Component: To get to the set of differential equation for the angular part, we use our solution for radial part. The set of equation for angular part is given by:

$$\partial_r u_\phi + \left(\frac{ik_1}{r_c} e^{\frac{ir}{r_c}\phi} - \frac{ik_2}{r_c} e^{\frac{-ir}{r_c}\phi}\right) \frac{\left(k_3 e^{\frac{t}{r_c}} + k_4 e^{\frac{-t}{r_c}}\right)}{\sqrt{\Lambda(r^2 - r_c^2)}} - \frac{1}{r} u_\phi = 0$$

$$\partial_t^2 u_\phi + \frac{\sqrt{\Lambda^3(r^2 - r_c^2)}r}{9r_c} \left(ik_1 e^{i\frac{r}{r_c}\phi} - ik_2 e^{-i\frac{r}{r_c}\phi}\right)\left(k_3 e^{\frac{t}{r_c}} + k_4 e^{\frac{-t}{r_c}}\right) = 0$$

$$\partial_\phi u_\phi - \frac{r\sqrt{\Lambda(r^2 - r_c^2)}}{3} \left(k_1 e^{i\frac{r}{r_c}\phi} + k_2 e^{-i\frac{r}{r_c}\phi}\right)\left(k_3 e^{\frac{t}{r_c}} + k_4 e^{\frac{-t}{r_c}}\right) = 0 \tag{1.5}$$

Solving second and third equation from equation(1.5), the solution for the angular we get is:

$$u_\phi = \frac{r_c\sqrt{\Lambda(r^2 - r_c^2)}}{3}(k_3 e^{\frac{t}{r_c}} + k_2 e^{\frac{-t}{r_c}})(-k_1 e^{i\frac{r}{r_c}\phi} + k_2 e^{-i\frac{r}{r_c}\phi})\left(1 + \frac{\Lambda}{r}\right) + c_1(r, t, \phi) \tag{1.6}$$

and solution for $c_1(r, t, \phi)$ is given by first equation from equation(1.5)

$$c_1(r, t, \phi) = -\frac{r(k_3 e^{\frac{t}{r_c}} + k_2 e^{\frac{-t}{r_c}})}{r_c} \int \frac{(-k_1 e^{i\frac{r}{r\sqrt{\Lambda_c}}\phi} + k_2 e^{-i\frac{r}{r_c}\phi})}{r\sqrt{r^2 - r_c^2}} dr \tag{1.7}$$

Time Component: To get to the set of differential equation for the time component, we use our solution for radial and angular part. The set of equation for angular part is given by:

$$\partial_t u_t - \frac{r\sqrt{\Lambda^3(r^2 - r_c^2)}}{9} \left(k_1 e^{i\frac{r}{r_c}\phi} + k_2 e^{-i\frac{r}{r_c}\phi}\right)\left(k_3 e^{\frac{t}{r_c}} + k_4 e^{\frac{-t}{r_c}}\right) = 0$$

$$\partial_r u_t - \frac{2r}{r^2 - r_c^2} u_t + \frac{\left(k_1 e^{i\frac{r}{r_c}\phi} + k_2 e^{-i\frac{r}{r_c}\phi}\right)\left(k_3 e^{\frac{t}{r_c}} - k_4 e^{\frac{-t}{r_c}}\right)}{r_c\sqrt{\Lambda(r^2 - r_c^2)}} = 0$$

$$\partial_\phi^2 u_t + \frac{\Lambda r(r^2 - r_c^2)}{3r_c}\left(k_1 e^{\frac{ir}{r_c\phi}} + k_2 e^{\frac{-ir}{r_c\phi}}\right) = 0 \tag{1.8}$$

Similar to angular part, we solve first and third equation from equation(1.8). The solution is given by:

$$u_t = \frac{\sqrt{r^2 - r_c^2}}{r_c}(k_1 e^{i\frac{r}{r\sqrt{\Lambda c}}\phi} + k_2 e^{-i\frac{r}{r_c}\phi})\left(r\left(k_3 e^{\frac{t}{r_c}} - k_4 e^{\frac{-t}{r_c}}\right) + \sqrt{r^2 - r_c^2}\right) + c_2(r, t, \phi)$$

and solution for $c_2(r, t, \phi)$ is given by second equation of equation(1.8)

$$c_2 = -\frac{(r^2 - r_c^2)\left(k_3 e^{\frac{t}{r_c}} - k_4 e^{\frac{-t}{r_c}}\right)}{r_c\sqrt{\Lambda}} \int \frac{k_1 e^{i\frac{r}{r\sqrt{\Lambda c}}\phi} + k_2 e^{-i\frac{r}{r_c}\phi})}{(r^2 - r_c^2)^{3/2}} dr$$

So in the complete solution of our killing direction we have the 6 independent parameters we were looking for. Hence DeSitter Spacetime is maximally symmetric.

By now we have shown that constant curvature spacetimes are maximally symmetric. Now from next chapters onward, we will study different black holes in each of these constant curvature spacetime. Moreover we will be interested in understanding superradiance in presence of different background curvatures shaped by these black holes.

Chapter 2

Superradiance in Asymptotically Flat Spacetime

2.1 Superradiance

Superradiance is a radiation enhancement process. In General Relativity, black-hole superradiance allows for energy and momentum extraction from the vacuum, even at the classical level. Background curvature plays an important role in superradiance. It creates the potential due to which incoming field undergoes scattering. As a result of this scattering, a part of the incoming field is transmitted through the potential and falls into the black hole and a part of it is reflected back. When for a certain spacetime the reflected field has a greater energy flux than the energy flux of incoming field then we say that superradiance exists in that spacetime.

In this chapter, we will study superradiance in Asymptotically flat spacetime using massless scalar fields as our test field.

2.2 Minkowski Spacetime

Minkowski Spacetime is a solution of Einstein's Equation when spacetime is devoid of any mass throughout and the background has zero curvature everywhere. In our calculation we have considered 1+2 dimension spacetime. Metric is given by:

$$ds^2 = -dt^2 + dx^2 + dy^2$$

Determinant of is straightforward to calculate.

$$\sqrt{|g|} = 1$$

We also need to compute inverse of the metric, and too is straightforward.

$$\eta^{\mu\nu} = Diag(-1, 1, 1)$$

With these things in hand, we are ready to compute the equation of motion of our massless scalar test field.
Equation of Motion is given by (see Appendix A):

$$\frac{1}{\sqrt{|g|}}\partial_\mu \left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\psi\right) = 0$$

Plugging in the components we will have

$$-\partial_t^2\psi + \partial_x^2\psi + \partial_y^2\psi = 0$$

Solution of this equation is trivial and is given by:

$$\psi = e^{ik_\mu x^\mu}$$

This is equation of just a free particle in 1+2 dimension flat spacetime. There is no such thing of particular interest with respect to Superradiance's point of view.

2.3 Schwarzschild Spacetime

We have already seen that nothing of much interest happens in flat spacetime. Now what we can do is that we can place a non-rotating, spherically symmetric, static, and stationary mass in the flat spacetime and ask the question that how does it affect the equation of motion of our test field. Putting such a mass in flat spacetime gives rise to a new spacetime called Schwarzschild Spacetime. It is named after Karl Schwarzschild who solved Einstein's Field Equation for such a system.

Metric is given as:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$$

This metric provides the geometry of spacetime outside the mass. From the metric it may seem that $r = 2m$ is a problematic point. But once we calculate the curvature we can see that $r = 2m$ is just co-ordinate singularity and true singularity lies at $r = 0$. Kretschmann Scalar is obtained by contracting Reimann Tensor with itself and for the Schwarzschild is given by

$$K = R_{abcd}R^{abcd} = \frac{48m^2}{r^6}$$

2.3.1 Event Horizon

An event horizon is a surface beyond which no future-directed trajectory exists which will take timelike and null particles to infinity.

For Schwarzschild, it lies at $r = 2m$ and it's easier to see that in Kruskal–Szekeres coordinates. The transformations from standard Schwarzschild to Kruskal coordinates are given by:

1. $r > 2m$
   - $R = \left(\frac{r}{2m} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4m}} cosh\left(\frac{t}{4m}\right)$
   - $T = \left(\frac{r}{2m} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4m}} sinh\left(\frac{t}{4m}\right)$
2. $r < 2m$
   - $R = \left(1 - \frac{r}{2m}\right)^{\frac{1}{2}} e^{\frac{r}{4m}} sinh\left(\frac{t}{4m}\right)$
   - $T = \left(1 - \frac{r}{2m}\right)^{\frac{1}{2}} e^{\frac{r}{4m}} cosh\left(\frac{t}{4m}\right)$

After these transformations we will get our metric in Kruskal co-ordinates.

$$ds^2 = \frac{32m^3}{r} e^{-\frac{r}{2m}} \left(-dT^2 + dR^2\right) + r^2 d\Omega^2$$

Few noteworthy things are as follows:

- Radially moving null rays are given by $\frac{dT}{dR} = \pm 1$. So light cone preserves it shape throughout the spacetime.
- $T^2 - R^2 = \left(1 - \frac{r}{2m}\right) e^{\frac{r}{2m}}$ relation holds. Hence constant r curves are hyperbolic everywhere outside $r = 2m$. But at $r = 2m$ constant r curve is a pair of straight lines whose slope is 1.

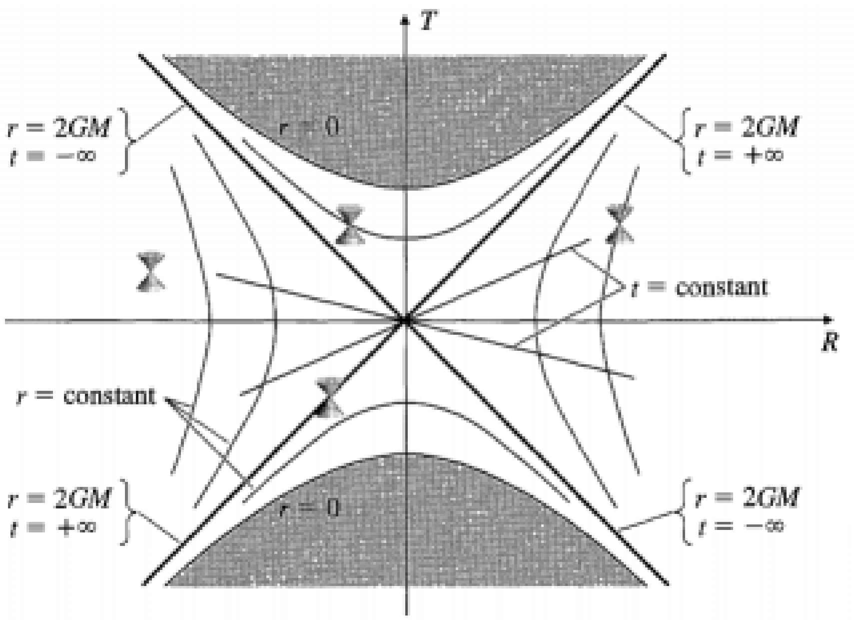


Figure 2.1: Schwarzschild Spacetime in Kruskal Co-ordinates
Image Courtsey:Spacetime Geometry, S.Carrol[Carroll.]

From figure 2.1 [Carroll.] it is clear that once a timelike particle crosses $r = 2m$ surface, there is no way it can come out of it. Because in order to come out of that surface the slope of trajectory must be less than 1 at some point. And this will lie outside the lightcone and hence it could not be causally connected to the past.

2.3.2 Massles Scalar Field in Schwarzschild Background

In order to calculate the equation of motion we go back to our standard Schwarzschild metric.

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$$

Matrix form of metric is:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2m}{r}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{\left(1-\frac{2m}{r}\right)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \end{pmatrix}$$

Since the matrix is diagonal, the determinant is the product of its element.

$$\sqrt{|g|} = r^2 sin^2\theta$$

We now need the inverse of the metric and then we can compute the equation of motion of our test field. The inverse of the metric is given by

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\left(1-\frac{2m}{r}\right)} & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2m}{r}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 sin^2\theta} \end{pmatrix}$$

Since now we have both determinant and components of inverse of metric we can calculate equation of motion.

$$\frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \psi\right) = 0$$

$$\partial_t \left(\frac{-r^2 sin^2\theta}{\left(1 - \frac{2m}{r}\right)} \partial_t \psi\right) + \partial_r \left(r^2 sin^2\theta \left(1 - \frac{2m}{r}\right) \partial_r \psi\right) + \partial_\theta \left(\sin^2\theta \partial_\theta \psi\right) + \partial_\phi^2 \psi = 0$$

Upon simplifying equation of motion turns out be

$$\partial_t^2\psi - \frac{2}{r}\left(1-\frac{m}{r}\right)\left(1-\frac{2m}{r}\right)\partial_r\psi - \left(1-\frac{2m}{r}\right)^2\partial_r^2\psi - \frac{1}{r^2sin^2\theta}\left(1-\frac{2m}{r}\right)\partial_\theta\left(\sin\theta\partial_\theta\psi\right)$$
$$- \frac{1}{r^2sin^2\theta}\left(1-\frac{2m}{r}\right)\partial_\phi^2\psi = 0$$

We try to solve this differential equation by separating the variables. Hence our solution is of form

$$\psi = T\left(t\right)R\left(r\right)\Theta\left(\theta\right)\Phi\left(\phi\right)$$

We substitute our trial solution in the differential equation, and then divide the whole equation by ψ. After introducing proper separation constants we get

- Time component:

$$\partial_t^2 T = -\omega^2 T$$
$$T = e^{-\iota\omega t}$$

- Angular component:

$$\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\Theta\right) + \frac{1}{\sin\theta^2}\partial_\phi^2\Phi = -l(l+1) \tag{2.1}$$

  The solution of the angular component are spherical harmonics.

- Radial component:

$$\left(1-\frac{2m}{r}\right)^2\partial_r^2 R + \frac{2}{r}\left(1-\frac{m}{r}\right)\left(1-\frac{2m}{r}\right)\partial_r R + \left[\omega^2 - \frac{l\left(l+1\right)}{r^2}\left(1-\frac{2m}{r}\right)\right]R = 0 \tag{2.2}$$

Radial equation

We wish to write our radial equation in Schrodinger equation like form. For this we make following transformations(See Appendix B):

- $u = rR$
- $r_* = r + 2m\ln\left(\frac{r}{2m}-1\right)$

Note that

$$r_* \rightarrow \begin{cases} -\infty & \text{when} \quad r \rightarrow 2m \\ \infty & \text{when} \quad r \rightarrow \infty \end{cases}$$

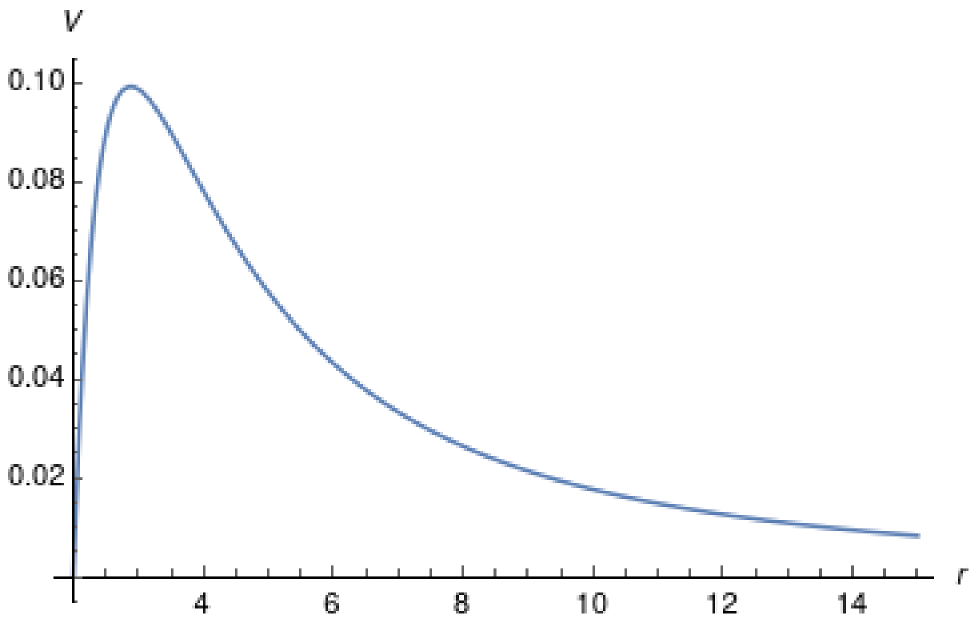
Hence the new radial co-ordinate (r_*) covers region outside the event horizon.

These transformations gives us Regge-Wheeler equation

$$u^{**} + \omega^2 u - V(r)u = 0 \tag{2.3}$$

Here V(r) is the potential which arises due to the presence of background curvature.

$$V(r) = \left(1 - \frac{2m}{r}\right)\left[\frac{2m}{r^3} + \frac{l(l+1)}{r^2}\right] \tag{2.4}$$



[H]

Figure 2.2: Potential due to Schwarzschild background

Potential vanishes at the extremities of r_*. In these limiting region Regge Wheeler equation takes the form of classical harmonic oscillator.

$$u^{**} + \omega^2 u = 0 \tag{2.5}$$

The solution to equation is given by

$$u(r_*) \sim \begin{cases} e^{-i\omega r_*} & \text{if} \quad r_* \to -\infty \\ a_{in}e^{-i\omega r_*} + a_{out}e^{i\omega r_*} & \text{if} \quad r_* \to \infty \end{cases} \tag{2.6}$$

We can identify the solutions as part of incoming, reflected and transmitted wave.

- $a_{in}e^{-i\omega r_*}$ $\rightarrow$ This solution exists far away from the black hole and is coming towards it, hence it can be thought of as an incoming wave.
- $a_{out}e^{-i\omega r_*}$ $\rightarrow$ This solution exists far away from the black hole and is moving further away from it, hence it can be thought of as a reflected wave.
- $e^{-i\omega r_*}$ $\rightarrow$ This solution exists near the event horizon and is going towards the black hole, it can be thought of as a transmitted wave.

Once we identify these solutions, then we can define Transmittance and Reflectance as the ratio of amplitudes of these waves.

$$\text{Transmittance:} \quad T = \frac{1}{a_{in}} \quad and, \quad \text{Reflectance:} \quad R = \frac{a_{out}}{a_{in}}$$

We need to find a relation between these coefficients and for that we use Wronskian. For Regge-Wheeler equation, Wronskian is a constant of motion.(See Appendix 3) So we will calculate Wronskian at two limits where potential vanishes, and will equate them to get a relation between coefficients.

- Wronskian near Event Horizon:

$$W_1 = \begin{vmatrix} e^{-\iota\omega r_*} & e^{\iota\omega r_*} \\ -\iota^{-\iota\omega r_*} & \iota\omega e^{\iota\omega r_*} \end{vmatrix}$$

$$W_1 = 2\iota\omega$$

- Wronskian far away from black hole

$$W_2 = \begin{vmatrix} a_{in}e^{-\iota\omega r_*} + a_{out}e^{\iota\omega r_*} & \bar{a_{in}}e^{-\iota\omega r_*} + \bar{a_{out}}e^{\iota\omega r_*} \\ -\iota\omega a_{in}e^{-\iota\omega r_*} + \iota\omega a_{out}e^{\iota\omega r_*} & -\iota\omega \bar{a_{in}}e^{-\iota\omega r_*} + \iota\omega \bar{a_{out}}e^{\iota\omega r_*} \end{vmatrix}$$

$$W_2 = 2\iota\omega \left(|a_{in}|^2 - |a_{out}|^2\right)$$

Equating them we will get

$$1 + |a_{out}|^2 = |a_{in}|^2$$

$$T^2 + R^2 = 1 \tag{2.7}$$

This equation states that whatever energy comes in either gets reflected back by the potential or is transmitted through it. It is a sense is a statement of energy conservation. So there is no radiation enhancement in this case and hence we conclude that there is no superradiance in Schwarzschild spacetime.

2.4 Kerr Spacetime

2.4.1 Brief Introduction to Kerr Spacetime

We have seen in the previous section that there is no superradiance in the case of Schwarzschild spacetime. However, we saw that massless scalar field scatters in Schwarzschild spacetime whereas no such scattering was there in Minkowski Spacetime. That was due to the fact that the presence of mass-generated a background curvature due to which potential was created which lead to scattering.

Now we step up a bit further and ask what would happen if we put a rotating spherically symmetric mass instead of non-rotating mass. This would lead to different set of Einstein's equations whose solution was given by Roy Kerr in 1963, and hence the solution is called Kerr Spacetime. Metric is[Derek Raine]:

$$ds^2 = -\left(\frac{\Delta - a^2 sin^2\theta}{\rho^2}\right) dt^2 - \frac{2marsin^2\theta}{\rho^2}\left(dtd\phi + d\phi dt\right) + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{Asin^2\theta}{\rho^2} d\phi^2 \tag{2.8}$$

- $\Delta = r^2 - 2mr + a^2$
- $\rho^2 = r^2 + a^2 \cos^2\theta$
- $A = (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta$

Here a is Angular momentum per unit mass and if we set $a = 0$, we will recover the Schwarzschild spacetime is its standard form. This is the metric for Kerr spacetime in Boyer-Lindquist coordinates.

Event Horizon in Kerr Spacetime

Consider a surface $s \equiv r = constant$ surface. Normal to the surface is given by:

$$n_\mu = \partial_\mu s$$

Norm of the normal would be

$$n_\mu n^\mu = g^{\mu\nu} \partial_\mu s \partial_\nu s$$

Since we are considering $r = constant$ surface

$$n_\mu n^\mu = g^{rr} \partial_r s \partial_r s$$

For null hypersurface:

$$|n| = n_\mu n^\mu = 0 \implies g^{rr} = 0 \tag{2.9}$$

To see why $g^{rr} = 0$ would give event horizon, we check the sign for $|n|$. As we move closer to the black hole, we will find that $|n|$ becomes positive. That is to say that only spacelike particles can be on $r = constant$ surface. In other words, timelike and null particles cannot stay at a fixed r and hence would inevitably in the black hole.

For Kerr black hole we see that $g^{rr} = 0$ would give that event horizon for Kerr spacetime lies at

$$\Delta = 0$$

$$r_+ = r + \sqrt{m^2 - a^2}$$

Note that even though we don't have complete spherical symmetry because we can distinguish between polar angles by taking direction on angular momentum as a reference, the event horizon is spherically symmetric.

Ergosphere

Apart from the event horizon, another very interesting feature of rotating black holes is the existence of the Ergosphere. The word Ergosphere has its root in the Latin word erg, which means energy. So the ergosphere is basically a region where the energy of the black hole resides. For the case of rotating black holes, it lies outside the event

horizon and hence is accessible. So we can withdraw energy from the rotating black hole. To see this, consider a particle which is at a fixed given (r, θ, ϕ).
Then 4 velocity of such a particle would be

$$u^{\mu} = \left(\frac{dt}{d\tau}, 0, 0, 0 \right)$$

Using $u^{\mu}u_{\mu} = -1$ we have

$$g_{00} \left(\frac{dt}{d\tau} \right)^2 = -1$$

This statement will hold true only when $g_{00} < 0$. This gives us the static limit surface

$$r_{static} = r + \sqrt{m^2 - a^2 \cos^2 \theta}$$

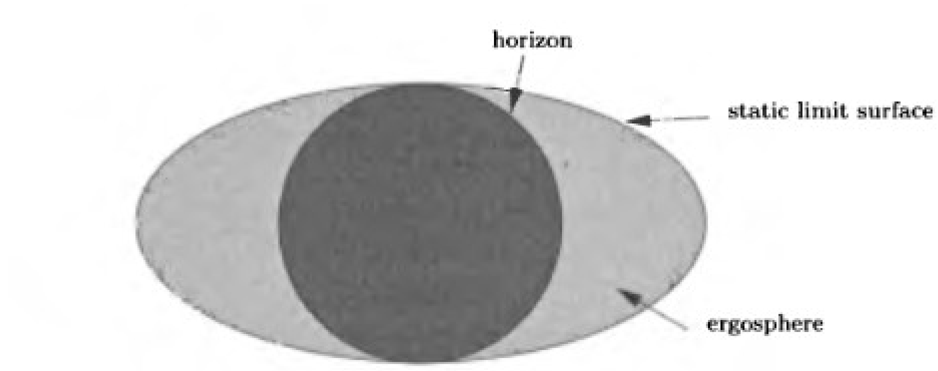


Figure 2.3: Event Horizon and Ergosphere in Kerr Spacetime
Image Courtesy:Black Holes:An Introduction, Derek Raine Edwin Thomas[Derek Raine]

Since $\cos^2 \theta \geq 0$, static limit surface lies outside the event horizon, except at poles($\theta = 0, \pi$).From the figure(2.3)[Derek Raine] it is clear that at ($\theta = 0, \pi$) static limit surface coincides with event horizon at these points.

2.4.2 Massless Scalar Field in Kerr Background

We use the Kerr metric in Boyer-Lindquist co-ordinate to calculate the equation of motion of massless scalar field in Kerr geometry.

$$ds^2 = -\left(\frac{\Delta - a^2 sin^2\theta}{\rho^2} \right) dt^2 - \frac{2mar sin^2\theta}{\rho^2} (dtd\phi + d\phi dt) + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{A sin^2\theta}{\rho^2} d\phi^2$$

Matrix form of the metric is:

$$g_{\mu\nu} = \begin{pmatrix} -\left(\frac{\Delta - a^2 sin^2\theta}{\rho^2}\right) & 0 & 0 & -\frac{2marsin^2\theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{2marsin^2\theta}{\rho^2} & 0 & 0 & \frac{Asin^2\theta}{\rho^2} \end{pmatrix}$$

Determinant of the matrix is

$$g = -\sin^2\theta \left(r^2 + a^2\cos^2\theta\right)^2$$

$$\sqrt{|g|} = \sin\theta \left(r^2 + a^2\cos^2\theta\right) \tag{2.10}$$

Inverse of the metric is given by:

$$g^{\mu\nu} = \begin{pmatrix} \frac{A\rho^2}{(-A\Delta + a^2(A\theta - 4m^2r^{22}\theta))} & 0 & 0 & \frac{-2amr\rho^2}{A\Delta + a^2(-A\theta + 4m^2r^{22}\theta)} \\ 0 & \frac{\Delta}{\rho^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ \frac{-2amr\rho^2}{A\Delta + a^2(-A\sin\theta + 4m^2r^2\sin^2\theta)} & 0 & 0 & \frac{\left(a^2\sin\theta - \Delta\right)\rho^2}{\sin^2\theta\left(A\Delta + a^2\left(-A\sin\theta + 4m^2r^2\sin^2\theta\right)\right)} \end{pmatrix}$$

Now that we have determinant and inverse of the metric we can calculate the equation of motion.

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\psi\right) = 0$$

$$\frac{A}{\Delta}\partial_t^2\psi + \frac{4mar}{\Delta}\partial_{t,\phi}\psi + \left(\frac{a^2}{\Delta} - \frac{1}{sin^2\theta}\right)\partial_\phi^2\psi - \partial_r\left(\Delta\partial_r\psi\right) - \frac{1}{\sin\theta}\partial_\theta\left(sin\theta\partial_\theta\psi\right) = 0 \tag{2.11}$$

This diffrential equation can be separated in variables using the following trial solution:

$$\psi = R(r)_{\omega kl}S(\theta)e^{ik\phi}e^{-i\omega t}$$

We substitute our trial solution in the diffrential equation, and then divide the whole equation by ψ. After introducing proper separation constants we get:

- Angular Part:

$$\frac{1}{S\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta S\right)+\omega^2a^2\sin^2\theta-\frac{k^2}{\sin^2\theta}-l\left(l+1\right)=0 \tag{2.12}$$

  The solution for angular part is given by spheroidal harmonics.

- Radial Part:

$$\partial_r\left(\Delta\partial_r R\right)+\left[\frac{\omega^2\left[\left(r^2+a^2\right)+a^2k^2+4markω\right.}{\Delta}-l\left(l+1\right)\right]R=0 \tag{2.13}$$

Radial Equation

Similar to the Schwarzschild's case we wish to write our radial equation in Schrodinger equation like form. For this we make following transformations:

- $u=\left(\sqrt{r^2+a^2}\right)R\left(r\right)$
- $r_*=r+m\ln\left(\Delta\right)+\frac{m}{m^2-a^2}\ln\left(\frac{r-r_+}{r-r_-}\right)$

  Note that

$$r_*\rightarrow\begin{cases}-\infty & \text{when}\quad \Delta\rightarrow 0\\ \infty & \text{when}\quad r\rightarrow\infty\end{cases}$$

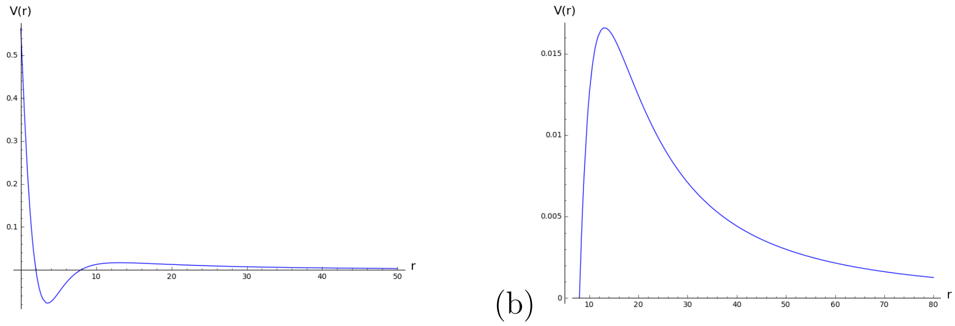
Hence the new radial co-ordinate (r_*), similar to the Schwarzschild's case, covers region outside the event horizon.

These transformations gives us Tuekolsky Equation.

$$u^{**}+\frac{\left[\left(r^2+a^2\right)\omega-ak\right]^2}{\left(r^2+a^2\right)^2}u-V\left(r\right)u=0$$

Here $V\left(r\right)$ is the potential due to the background kerr curvature.

$$V\left(r\right)=\frac{\Delta\left(l\left(l+1\right)+a^2\omega^2-2ak\omega\right)}{\left(r^2+a^2\right)^2}+\frac{r^2\Delta}{\left(r^2+a^2\right)^4}+d_{r*}\left(\frac{r\Delta}{\left(r^2+a^2\right)^2}\right)$$



(a) (b)

Figure 2.4: Potential due to Kerr background

Figure(a) depicts potential for entire radial co-ordinate. Two points where potential vanishes correspond to the inner and outer horizon. Figure(b) shows the same potential outside the outer event horizon. Note its similarity with the Schwarzschild's potential

As we saw in Schwarzschild, here too potential vanishes at the extremities of r_*. This reduces the Tuekolsky equation to the equations of a harmonic oscillator with different frequencies at each extremity. The equation takes the form:

$$u^{**} \sim \begin{cases} -\omega^2 u & \text{if} \quad r_* \to \infty \\ -\left(\omega - k\omega_H\right)^2 u & \text{if} \quad r_* \to -\infty \end{cases}$$

The solution to these equations is given by:

$$u(r_*) \sim \begin{cases} a_{in} e^{-i\omega r_*} + a_{out} e^{i\omega r_*} & \text{if} \quad r_* \to \infty \\ e^{-i(\omega - k\omega_H) r_*} & \text{if} \quad r_* \to -\infty \end{cases}$$

Again we can identify the solutions as part of incoming, reflected, and transmitted waves.

- $a_{in} e^{-i\omega r_*}$ $\to$ This solution exists far away from the black hole and is coming towards it, hence it can be thought of as an incoming wave.
- $a_{out} e^{-i\omega r_*}$ $\to$ This solution exists far away from the black hole and is moving further away from it, hence it can be thought of as a reflected wave.
- $e^{-i(\omega - k\omega_H) r_*}$ $\to$ This solution exist near the event horizon and is going towards the black hole, it can be thought of as Transmitted wave.

Similar to Schwarzschild's case we define reflection and transmission coefficients as :

$$\text{Transmittance:} \quad T = \frac{1}{a_{in}} \quad and, \quad \text{Reflectance:} \quad R = \frac{a_{out}}{a_{in}}$$

Having identified these solutions, we can find Wronskian at the extremities where potential vanishes and equate them.

- Wronskian near Event Horizon:

$$W_1 = \begin{vmatrix} e^{-i(\omega - k\omega_H)r_*} & e^{i(\omega-\omega_H)r_*} \\ -i(\omega - k\omega_H)\, e^{-i(\omega-k\omega_H)r_*} & i(\omega - k\omega_H)\, e^{i(\omega-k\omega_H)r_*} \end{vmatrix}$$

$$W_1 = -2\iota(\omega - k\omega_+) \qquad \text{here} \quad \omega_+ = \frac{a}{2mr_+}$$

- Wronskian far away from black hole

$$W_2 = \begin{vmatrix} a_{in}e^{-\iota\omega r_*} + a_{out}e^{\iota\omega r_*} & \bar{a_{in}}e^{-\iota\omega r_*} + \bar{a_{out}}e^{\iota\omega r_*} \\ -\iota\omega a_{in}e^{-\iota\omega r_*} + \iota\omega a_{out}e^{\iota\omega r_*} & -\iota\omega \bar{a_{in}}e^{-\iota\omega r_*} + \iota\omega \bar{a_{out}}e^{\iota\omega r_*} \end{vmatrix}$$

$$W_2 = 2\iota\omega\left(|a_{in}|^2 - |a_{out}|^2\right)$$

Equating these we will get:

$$-(\omega - k\omega_+) = \omega\left(|a_{in}|^2 - |a_{out}|^2\right)$$

Hence the relation between transmittance and reflectance is given by :

$$|R|^2 = 1 + \frac{(k\omega_H - \omega)}{\omega}|T|^2$$

It is clear from the above expression that for certain modes which satisfy the relation $k\omega_H > \omega$, we would have $|R|^2 > 1$. This would mean that whatever energy flux is reflected back after scattering due to potential which was created by background curvature, is greater than that of incoming flux. So Kerr Spacetime allows us to draw energy from it. It is due to the fact that ergoregion for Kerr black hole lies outside the event horizon and hence is accessible to draw energy.

Chapter 3

Superradiance in Kerr DeSitter Spacetime

3.1 Introduction

After the asymptotically flat spacetime, we wish to see whether we can get superradiant modes in DeSiiter Spacetime. The DeSitter Spacetime is a maximally symmetric spacetime with constant positive curvature throughout. The metric for DeSitter spacetime is given by[Akcay 11]:

$$ds^2 = -\left(1 - \frac{\Lambda r^2}{3}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{\Lambda r^2}{3}\right)} + r^2 d\Omega^2 \tag{3.1}$$

The curvature(K) of this spacetime turns out to be:

$$K = 4\Lambda$$

DeSitter spacetime has a horizon associated with it called cosmological horizon. It is given by

$$g^{rr} = 0$$

This gives that

$$r_{ch} = \sqrt{\frac{3}{\Lambda}} \tag{3.2}$$

If we put a rotating black hole in the DeSitter spacetime, then Kerr DeSitter spacetime is generated.

Metric used for Kerr DeSitter spacetime is given by is given by[Zhang 14]:

$$
\begin{aligned}
ds^2 =& \frac{-1}{(1+\alpha)^2 \rho^2} \left(\Delta_r - a^2 \Delta_\theta \sin^2\theta\right) dt^2 \\
& - \frac{a \sin^2\theta}{(1+\alpha)^2 \rho^2} \left[\Delta_\theta \left(r^2 + a^2\right) + \Delta_r\right] (dt d\phi + d\phi dt) + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 \\
& + \frac{\sin^2\theta}{(1+\alpha)^2 \rho^2} \left(\Delta_\theta \left(r^2 + a^2\right)^2 - a^2 \sin^2\theta \Delta_r\right) d\phi^2
\end{aligned}
$$

Here a is angular momentum per unit mass as we had in Kerr spacetime and,

- $\alpha = \frac{a^2 \Lambda}{3}$
- $\Delta_\theta = 1 + \alpha \cos^2\theta$
- $\rho^2 = r^2 + a^2 \cos^2\theta$
- $\Delta_r = (r^2 + a^2)\left(1 - \frac{\alpha r^2}{a^2}\right) - 2mr$

If we set the curvature to zero, we recover the Kerr metric in asymptotically flat spacetime.

3.1.1 Horizons of Kerr DeSitter

We wish to look at the horizons for kerr de-sitter spacetime. It is determined by

$$\Delta_r = 0 \tag{3.3}$$

This will give a biquadratic equation.

$$\frac{\Lambda}{3} r^4 - \left(1 - \frac{\Lambda a^2}{3}\right) r^2 + 2mr - a^2 = 0 \tag{3.4}$$

Note that this equation will transform to previous equation if you remove the black hole from the spacetime, i.e. set $m = 0, a = 0$. And if we set curvature ($\Lambda = 0$) then we recover the equation which determines event horizons for kerr spacetime. Let the root of this equation be r_1, r_2, r_3, r_4 such that $r_1 < r_2 < r_3 < r_4$.

From the equation we have product of roots

$$r_1 r_2 r_3 r_4 = \frac{-3a^2}{\Lambda} < 0 \tag{3.5}$$

So either one or three of these roots are negative. But the spacetime should have at least 3 horizon, two due to the presence of Kerr black hole and another cosmological horizon of DeSitter background. This leaves us with one option i.e. $r_1 < 0$ and $r_2, r_3, r_4 > 0$. So the solution corresponds to:

- $r_2 \rightarrow$ Inner Event Horizon(r_-)
- $r_3 \rightarrow$ Outer Event Horizon(r_+)
- $r_4 \rightarrow$ Cosmological Horizon(r_c)

3.1.2 Angular velocity at Horizons

Consider far away from the black hole, a freely falling particle with zero angular momentum. A rotating black hole drags spacetime along with it and as a result of this, the freely falling particle will acquire some angular velocity. Angular Velocity is given by

$$\begin{aligned} \omega &= \frac{d\phi}{dt} \\ &= \frac{g^{\phi t} u_t + g^{\phi\phi} u_\phi}{g^{tt} u_t + g^{t\phi} u_\phi} \end{aligned} \tag{3.6}$$

Since ϕ and t directions are killing directions, so u_t and u_ϕ are conserved quantities, i.e. energy and z-component of angular momentum respectively. As our particle has zero angular momentum, we have $u_\phi = 0$ This gives us:

$$\omega = \frac{g^{\phi t}}{g^{tt}} \tag{3.7}$$

Upon solving this we get,

$$\omega = \frac{a\left(-\Delta_r + \left(r^2 + a^2\right)\Delta_\theta\right)}{\left(\left(r^2 + a^2\right)^2 \Delta_\theta - a^2 \Delta_r \sin^2\theta\right)} \tag{3.8}$$

Since we wish to look for angular velocities at horizon we put $\Delta_r = 0$ in above equation. This gives us

$$\omega_i = \frac{a}{r_i^2 + a^2} \tag{3.9}$$

Here r_i denotes different horizons for $i = (2, 3, 4)$. So angular velocity at outer horizon and DeSitter horizon is given by:

$$\omega_+ = \frac{a}{r_+^2 + a^2} \qquad \text{and} \qquad \omega_c = \frac{a}{r_c^2 + a^2}$$

3.2 Equation of Motion of Test Field

Now we proceed to calculate the equation of motion of massless scalar field in the Kerr DeSitter Spacetime. Matrix form of the metric is:

$$g_{\mu\nu} = \begin{pmatrix} -\frac{\Delta_r - a^2\Delta_\theta \sin^2(\theta)}{(\alpha+1)^2\rho^2} & 0 & 0 & -\frac{a\Delta_{theta}\left(a^2+r^2\right)\sin^2(\theta) - a\Delta_r \sin^2(\theta)}{(\alpha+1)^2\rho^2} \\ 0 & \frac{\rho^2}{\Delta_r} & 0 & 0 \\ 0 & 0 & \frac{\rho^2}{\Delta_\theta} & 0 \\ -\frac{a\Delta_\theta\left(a^2+r^2\right)\sin^2(\theta) - a\Delta_r \sin^2(\theta)}{(\alpha+1)^2\rho^2} & 0 & 0 & \frac{\Delta_\theta\left(a^2+r^2\right)^2 \sin^2(\theta) - a^2\Delta_r \sin^4(\theta)}{(\alpha+1)^2\rho^2} \end{pmatrix}$$

The determinant of the matrix is given by:

$$g = -\frac{\left((r^2+a^2) + a^2\sin^2\theta\right)}{(1+\alpha)^4}$$

$$\sqrt{|g|} = \frac{\sqrt{\left((r^2+a^2) + a^2\sin^2\theta\right)}}{(1+\alpha)^2}$$

Inverse of the metric is:

$$g^{\mu\nu} = \begin{pmatrix} -\frac{4(\alpha+1)^2\rho^2\csc^2(\theta)\left(\Delta_\theta\left(a^2+r^2\right)^2\sin^2(\theta) - a^2\Delta_r\sin^4(\theta)\right)}{\Delta_r\Delta_\theta(a^2\cos(2\theta)+a^2+2r^2)^2} & 0 & 0 & \frac{4a(\alpha+1)^2\rho^2\left(\Delta_\theta\left(a^2+r^2\right) - \Delta_r\right)}{\Delta_r\Delta_\theta(a^2\cos(2\theta)+a^2+2r^2)^2} \\ 0 & \frac{\Delta_r}{\rho^2} & 0 & 0 \\ 0 & 0 & \frac{\Delta_\theta}{\rho^2} & 0 \\ \frac{4a(\alpha+1)^2\rho^2\left(\Delta_\theta\left(a^2+r^2\right) - \Delta_r\right)}{\Delta_r\Delta_\theta(a^2\cos(2\theta)+a^2+2r^2)^2} & 0 & 0 & \frac{4(\alpha+1)^2\rho^2\csc^2(\theta)\left(\Delta_r - a^2\Delta_\theta\sin^2(\theta)\right)}{\Delta_r\Delta_{theta}(a^2\cos(2\theta)+a^2+2r^2)^2} \end{pmatrix}$$

Since we have the determinant and metric inverse, we can now calculate our equation of motion.

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\psi\right) = 0$$

Upon substituting relevant terms, we get:

$$\frac{4\rho^2}{\left(a^2+r^2+a^2\cos^2\theta\right)^2}\left(-\partial_t^2\psi+2a\left(\left(r^2+a^2\right)\Delta_\theta-\Delta_r\right)\partial_t\partial_\phi\psi+\partial_\phi^2\psi\right)$$
$$+\frac{1}{\left(1+\alpha\right)^2\rho^2}\left(\frac{1}{\Delta_\theta}\left(\partial_r^2\psi+2r\partial_r\left[\frac{1}{r^2+a^2+a^2\sin^2\theta}-\frac{1}{\rho^2}+\frac{1}{\Delta_r}\left(1-\alpha-\frac{2\alpha r^2}{a^2}-\frac{m}{r}\right)\right]\psi\right)\right.$$
$$\left.+\frac{1}{\Delta_r}\left(\partial_\theta^2\psi+\sin 2\theta\partial_\theta\left[\frac{a^2}{2\left(r^2+a^2+a^2\sin^2\theta\right)}-\frac{\alpha}{\Delta_\theta}+\frac{2a^2}{\rho^2}\right]\psi\right)\right)=0$$

The equation can be separated by following trial solution:

$$\psi=e^{-\iota\omega t}e^{\iota k\phi}S\left(\theta\right)R\left(r\right)$$

We substitute our trial solution in the differential equation, and then divide the whole equation by ψ. After introducing proper separation constants we get:

- Angular Part:

$$\partial_\theta\left(\sin\theta\Delta_\theta\partial_\theta S\right)+\left[\lambda-\frac{\left(1+\alpha\right)^2}{\Delta_\theta}\left(a\omega\sin\theta-\frac{k}{\sin\theta}\right)^2-2\alpha\cos^2\theta\right]S=0 \quad (3.10)$$

- Radial Part:

$$\Delta_r\partial_r\left(\Delta_r\partial_r R\right)+\left[\left(1+\alpha\right)^2\left(\omega\left(r^2+a^2\right)-ak\right)^2-\frac{2\alpha\Delta_r}{a^2}r^2+\Delta_r\lambda\right]R=0 \quad (3.11)$$

3.2.1 Radial Equation

Similar to the Kerr case we wish to write our radial equation in Schrodinger equation like form. For this we make the following transformations:

- $u=\sqrt{r^2+a^2}R\left(r\right)$
- $dr_*=\frac{r^2+a^2}{\Delta_r}dr$

$$u^{**}+\left(1+\alpha\right)^2\left(\omega-\frac{ak}{r^2+a^2}\right)u-V\left(r\right)u=0 \quad (3.12)$$

Here $V\left(r\right)$ is the potential due to the background Kerr curvature.

$$V\left(r\right)=-\frac{\Delta_r}{(r^2+a^2)^3}\left[r\partial_r\Delta_r+\frac{a^2-2r^2}{r^2+a^2}\Delta_r+(r^2+a^2)\left(\lambda+\frac{2\alpha r^2}{a^2}\right)\right] \quad (3.13)$$

Potential vanishes at the horizons and we write equation at $r = r_H$ and $r = r_C$. Equation takes the form:

$$u^{**} \sim \begin{cases} -(1+\alpha)^2 \left(\omega - k\omega_+\right)^2 u & \text{if} \quad r \to r_+ \\ -(1+\alpha)^2 \left(\omega - k\omega_c\right)^2 u & \text{if} \quad r \to r_c \end{cases} \tag{3.14}$$

The solution to these equations is given by:

$$u(r_*) \sim \begin{cases} e^{-\iota(1+\alpha)(\omega - k\omega_+)r_*} & \text{if} \quad r \to r_+ \\ a_{in} e^{-\iota(1+\alpha)(\omega - k\omega_c)r_*} + a_{out} e^{\iota(1+\alpha)(\omega - k\omega_c)r_*} & \text{if} \quad r \to r_c \end{cases} \tag{3.15}$$

Now we can identify the solutions as part of incoming, reflected and transmitted wave.

- $a_{in} e^{-\iota(1+\alpha)(\omega - k\omega_c)r_*} \to$ This solution exists at the cosmological horizon of de-sitter spacetime and is coming towards the black hole, hence it can be thought of as an incoming wave.

- $a_{out} e^{\iota(1+\alpha)(\omega - k\omega_c)r_*} \to$ This solution exist at the cosmological horizon of de-sitter spacetime and is moving further away from it, hence it can be thought of as a reflected wave.

- $e^{-\iota(1+\alpha)(\omega - k\omega_+)r_*} \to$ This solution exists near the event horizon and is going towards the black hole, it can be thought of as Transmitted wave.

As we did in Kerr's spacetime, we can again define reflection and transmission coefficients as :

$$\text{Transmittance:} \quad T = \frac{1}{a_{in}} \quad and, \quad \text{Reflectance:} \quad R = \frac{a_{out}}{a_{in}}$$

Having identified these solutions, we can find Wronskian at the extremities where potential vanishes and equate them.

- Wronskian near Event Horizon:

$$W_1 = \begin{vmatrix} e^{-\iota(1+\alpha)(\omega - k\omega_+)r_*} & e^{\iota(1+\alpha)(\omega - k\omega_+)r_*} \\ -\iota(1+\alpha)(\omega - k\omega_+)e^{-\iota(1+\alpha)(\omega - k\omega_+)r_*} & \iota(1+\alpha)(\omega - k\omega_+)e^{\iota(1+\alpha)(\omega - k\omega_+)r_*} \end{vmatrix}$$

$$W_1 = -2\iota \left(1+\alpha\right)\left(\omega - k\omega_+\right) \tag{3.16}$$

- Wronskian near cosmological horizon

$$W_2 = \left(a_{in}e^{-\iota(1+\alpha)(\omega-k\omega_c)r_*} + a_{out}e^{\iota(1+\alpha)(\omega-k\omega_c)r_*}\right)$$
$$\left(\bar{a_{in}}\iota(1+\alpha)(\omega-k\omega_c)e^{\iota(1+\alpha)(\omega-k\omega_c)r_*} - \bar{a_{out}}\iota(1+\alpha)(\omega-k\omega_c)e^{-\iota(1+\alpha)(\omega-k\omega_c)r_*}\right)$$
$$-\left(\bar{a_{in}}e^{-\iota(1+\alpha)(\omega-k\omega_c)r_*} + \bar{a_{out}}e^{\iota(1+\alpha)(\omega-k\omega_c)r_*}\right)\left(\bar{a_{in}}e^{-\iota(1+\alpha)(\omega-k\omega_c)r_*} + \bar{a_{out}}e^{\iota(1+\alpha)(\omega-k\omega_c)r_*}\right)$$

$$W_2 = 2\iota(1+\alpha)(\omega-k\omega_c)\left(|a_{in}|^2 - |a_{out}|^2\right) \tag{3.17}$$

Equating these we will get:

$$(\omega-k\omega_c)[|a_{in}|^2 - |a_{out}|^2]) = (\omega-k\omega_+)$$
$$(\omega-k\omega_c)[1-|R|^2]) = (\omega-k\omega_+)|T|^2$$

Hence the relation between transmittance and reflectance is given by :

$$|R|^2 = 1 - \frac{(\omega-k\omega_+)}{(\omega-k\omega_c)}|T|^2 \tag{3.18}$$

Hence condition for superradiace is:

$$\frac{(\omega-k\omega_+)}{(\omega-k\omega_c)} < 0$$
$$\implies k\omega_c < \omega < k\omega_+ \tag{3.19}$$

Modes whose frequency is lower than the angular velocity of the black hole but is greater than the angular velocity of DeSitter horizon will have reflectance greater than 1. Hence only these modes will be superradiant modes and will be able to extract the energy from the rotating black hole and will exhibit superradiance.

Chapter 4

Superradiance in AdS Spacetime

4.1 Introduction to BTZ Spacetime

After studying superradiance in asymptotically flat and de sitter spacetimes, we wish to study it in the third possible constant curvature spacetime, i.e. AdS Spacetime. It is a constant negative curvature ($\Lambda = \frac{-6}{l^2}$) spacetime. We will study it for the case of BTZ black hole.

Metric used for BTZ(1+2) black hole is given by[Carlip 95]:

$$ds^2 = -\left(-M + \frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)} - \frac{J}{2}\left(dtd\phi + d\phi dt\right) + r^2 d\phi^2 \tag{4.1}$$

Here

- M: mass of the black hole
- J: angular momentum of the black hole

4.1.1 Horizons of BTZ

The horizon for the BTZ spacetime is given by:

$$g^{rr} = 0$$

This gives us a biquadratic equation in r

$$\frac{r^4}{l^2} - Mr^2 + \frac{J}{4} = 0 \tag{4.2}$$

Let the root of this equation be r_1, r_2, r_3, r_4 such that $r_1 < r_2 < r_3 < r_4$.

From the equation we have product of roots

$$r_1 r_2 r_3 r_4 = \frac{Jl^2}{4} > 0 \tag{4.3}$$

So either all roots are positive, or all roots are negative or two of them is positive. But we also note that the sum of roots vanishes.

$$r_1 + r_2 + r_3 + r_4 = 0 \tag{4.4}$$

Hence not all roots can be positive or negative. This leaves us with only one choice, $r_1, r_2 < 0$ and $r_3, r_4 > 0$. So we have r_3 as inner horizon(r_-) and r_4 as outer horizon(r_+).

We can now write mass and angular momentum in terms of these inner and outer horizon. Above biquadratic equation in r is also a quadratic in r^2. So we can write: from sum of roots,

$$r_+^2 + r_-^2 = Ml^2 \implies M = \frac{r_+^2 + r_-^2}{l^2} \tag{4.5}$$

and from product of roots,

$$r_+^2 r_-^2 = \frac{Jl^2}{4} \implies J = \frac{2r_+ r_-}{l} \tag{4.6}$$

4.1.2 Angular velocity of horizon

We have already seen in previous chapter that angular velocity is given by $\omega = \frac{g^{\phi t}}{g^{tt}}$. For the BTZ black hole, angular speed is given by:

$$\omega = \frac{J}{2r^2}. \tag{4.7}$$

Substituting for J from equation(4.6) we have,

$$\omega = \frac{r_+ r_+}{r^2 l} \tag{4.8}$$

Hence angular speed at outer event horizon is :

$$\omega_+ = \frac{r_-}{r_+ l} \tag{4.9}$$

4.2 Equation of motion of massive scalar field

Matrix form of metric is:

$$g_{\mu\nu} = \begin{pmatrix} M - \frac{r^2}{l^2} & 0 & -\frac{J}{2} \\ 0 & \frac{1}{\frac{J^2}{4r^2}+\frac{r^2}{l^2}-M} & 0 \\ -\frac{J}{2} & 0 & r^2 \end{pmatrix}$$

Determinant of the matrix is $g = -r^2$

$$\sqrt{|g|} = r$$

We now need the inverse of the metric and then we can compute the equation of motion of our test field. The inverse of the is metric given by

$$g^{\mu\nu} = \begin{pmatrix} \frac{r^2}{-\frac{J^2}{4}-\frac{r^4}{l^2}+Mr^2} & 0 & \frac{J}{2\left(-\frac{J^2}{4}-\frac{r^4}{l^2}+Mr^2\right)} \\ 0 & \frac{J^2}{4r^2}+\frac{r^2}{l^2}-M & 0 \\ \frac{J}{2\left(-\frac{J^2}{4}-\frac{r^4}{l^2}+Mr^2\right)} & 0 & \frac{M-\frac{r^2}{l^2}}{-\frac{J^2}{4}-\frac{r^4}{l^2}+Mr^2} \end{pmatrix}$$

Since we are dealing with a massive scalar field, we modify the equation of motion a bit to accommodate the mass term as well. The equation of motion is given by:

$$\left(D_\mu D^\mu - \frac{\mu}{l^2}\right)\psi = 0 \tag{4.10}$$

μ here is the mass parameter for the scalar field.

$$\frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\psi\right) - \frac{\mu}{l^2}\psi = 0 \tag{4.11}$$

Substituting the relevant terms, we will get our equation of motion as:

$$\left(-M+\frac{r^2}{l^2}+\frac{J^2}{4r^2}\right)\partial_r^2\psi + \left(-M+\frac{3r^2}{l^2}-\frac{J^2}{4r^2}\right)\partial_r\psi + \frac{r}{\left(-M+\frac{r^2}{l^2}+\frac{J^2}{4r^2}\right)}\left(-\partial_t^2\psi - \frac{J}{r^2}\partial_t\partial_\phi\psi \right.$$
$$\left. + \frac{1}{r^2}\left(-M+\frac{r^2}{l^2}\right)\partial_\phi^2\psi - \frac{r\mu}{l^2}\psi = 0\right.$$

This equation can be separated using the trial solution $\psi = e^{-\iota\omega t}e^{\iota k\phi}R$.

This gives us the the radial equation.

$$R^{''}+\frac{\left(-M+\frac{3r^2}{l^2}-\frac{J^2}{4r^2}\right)}{\left(-M+\frac{r^2}{l^2}+\frac{J^2}{4r^2}\right)}R^{'}+\frac{1}{r\left(-M+\frac{r^2}{l^2}+\frac{J^2}{4r^2}\right)^2}\left(-\omega^2r^2-J\omega k-k^2\left(-M+\frac{r^2}{l^2}\right)\right)R-\frac{r\mu}{l^2}R=0$$

Here, $R^{'}=\frac{dR}{dr}$. and $R^{''}=\frac{d^2R}{dr^2}$

We replace the Mass and Angular momentum in terms of outer and inner horizon.

$$R^{''}+\frac{3r^2-\left(r_+^2+r_-^2+\frac{r_+^2r_-^2}{r^2}\right)}{r^2-\left(r_+^2+r_-^2-\frac{r_+^2r_-^2}{r^2}\right)}R^{'}+\frac{l^2}{r\left(r_+^2+r_-^2-\frac{r_+^2r_-^2}{r^2}\right)}\left(\omega^2r^2+\frac{2r_+r_-}{l}\omega k-\frac{k}{l^2}\left(r_+^2+r_-^2-r^2\right)R\right.$$
$$-\frac{\mu r}{l^2}R=0$$

We make a change in variable as follows[Dappiaggi 18]:

$$z=\frac{r^2-r_+^2}{r^2-r_-^2} \tag{4.12}$$

Inner and outer horizon lies at $z=-\infty$ and $z=0$ respectively. Radial infinity lies at $z=1$. So in the interval $0\leq z\leq 1$, z covers the complete region outside the outer horizon.

This changes the radial differential equation and it takes the following form:

$$z(1-z)\frac{d^2R}{dz^2}+(1-z)\frac{dR}{dz}+\left(\frac{A}{z}+B+\frac{C}{(1-z)}\right)R=0 \tag{4.13}$$

Here,

$$A=\frac{l^4}{4(r_+^2-r_-^2)^2\left(\omega r_+-\frac{m}{l}r_-\right)^2}$$

$$B=-\frac{l^4}{4(r_+^2-r_-^2)^2\left(\omega r_--\frac{m}{l}r_+\right)^2}$$

$$C=-\frac{\mu}{4}$$

We redefine R in order to get out differential equation in Euler hyper geometric form as:

$$R(z)=z^\alpha(1-z)^\beta F(z)$$

This gives us :

$$z(1-z)\frac{d^2F}{dz^2}+(c-(a+b+1)\,z)\,\frac{dF}{dz}-abF=0 \tag{4.14}$$

where,

$$\begin{aligned} c &= 1+2\alpha \\ a+b &= \frac{\alpha+\beta}{2} \\ ab &= (\alpha+\beta)^2-B \end{aligned} \tag{4.15}$$

and,

$$\begin{aligned} \alpha^2=-A \implies \alpha &= -\iota\frac{l^2}{2(r_+^2-r_-^2)\left(\omega r_+-\frac{m}{l}r_-\right)} \\ \beta &= \frac{1}{2}(1-\sqrt{1+\mu}) \end{aligned} \tag{4.16}$$

The two solution for the equation is given by $F(a,b,c,z)$ and $z^{1-c}F(a-c+1,b-c+1,2-c,z)$.

4.3 Boundary Conditions

We wish to see whether BTZ exhibits superradiance or not. For that, we impose conditions on horizon and at infinity.

4.3.1 Condition at Outer Horizon

We impose the condition on the solution that near outer horizon modes should be ingoing. The general solution for the radial equation can be written as:

$$R(z)=c_1z^{\alpha}(1-z)^{\beta}F(a,b,c,z)+c_2z^{\alpha}(1-z)^{\beta}z^{1-c}F(a-c+1,b-c+1,2-c,z) \tag{4.17}$$

In order to see which of these is ingoing mode we need to make a trasnformation.

$$x=p\ln z \implies z=e^{\frac{x}{p}} \tag{4.18}$$

Here, $p=p(a,b,c)$ is some scaling function which facilitates this transformation. We also rewrite α from equation(4.16) as

$$\alpha=-i\lambda \tag{4.19}$$

where

$$\lambda = \frac{l^2}{2(r_+^2 - r_-^2)\left(\omega r_+ - \frac{m}{l} r_-\right)}$$

Now using equations (4.18),(4.19) and (4.15) we can rewrite (4.17) as:

$$R(x) = c_1 e^{-i\frac{\lambda x}{p}}(1-e^{\frac{x}{p}})^{\beta} F(a,b,c,z) + c_2 e^{i\frac{\lambda x}{p}}(1-e^{\frac{x}{p}})^{\beta} F(a-c+1, b-c+1, 2-c, z) \quad (4.20)$$

From equation(4.20) we can see that the first term is ingoing while second term is outgoing. So we set c_2 to zero. And we have,

$$R(z) = c_1 F(a,b,c,z) \quad (4.21)$$

4.3.2 Condition at infinity

At infinity, we impose two different boundary conditions

- Robinson Boundary Condition
- Vanishing Boundary Condition

4.4 Superradiance

We will obtain conditions on the modes which will exhibit superradiance in BTZ spacetime with different boundary conditions at infinity

4.4.1 Robinson Boundary Condition

Robinson Boundary condition is a linear combination of Neumann and Dirichlet boundary conditions.

We use a linear transformation result for hypergeometric functions and rewrite $F(a,b,c,z)$ as:

$$F(a,b,c,z) = (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a, c-b, c-a-b+1, 1-z) + \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b, a+b-c+1, 1-z) \text{[Birmingham 01]}$$

So we have,

$$R(z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} R^{(D)} + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} R^{(N)} \tag{4.22}$$

where,

$$R^{(N)} = z^{\alpha}(1-z)^{\beta}(1-z)^{c-a-b} F(c-a, c-b, c-a-b+1, 1-z)$$

and

$$R^{(D)}(z) = z^{\alpha}(1-z)^{\beta} F(a, b, a+b-c+1, 1-z) \tag{4.23}$$

are Nuemann and Dirichlet Boundary condition.

We can parametrise the radial solution with ξ

$$R(z) \sim \sin(\xi) R^{(N)} + \cos(\xi) R^{(D)} \tag{4.24}$$

ξ here is Robinson Boundary condition's parameter. $\xi = 0$ will give us Dirichlet boundary condition and $\xi = \pi/2$ will give Neumann boundary condition.

Since the equation(4.14) which we are dealing with has a non zero coefficient of the first derivative, the usual way to find the condition of superradiance by equating Wronskian at extremities will not work. Hence we need to find a new way to look at superradiance. We do so by defining Superradiant modes as those modes whose energy flux is towards the exterior region of the black hole at the horizon.

For the sake of convenience, we transform our co-ordinate system to Eddington Finkelstein co-ordinates. Transformations are given by:

$$dv = dt + dr_*$$

$$dr_* = \frac{dr}{(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2})}$$

and

$$d\phi' = d\phi + \frac{J}{r^2(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2})} \tag{4.25}$$

This transforms our radial solution as

$$R(v, r_+, \phi') = \hat{C} e^{-\iota \omega v} e^{\iota k \phi'},$$

here, $\hat{C} = c_1 \left(\frac{4r_+^2}{r_+^2 - r_-^2}\right)^\alpha$

The metric takes the form[Carlip 95]:

$$ds^2 = -(-M + \frac{r^2}{l^2})dv^2 + (dvdr + drdv) + \frac{-J}{2}(d\phi dv + dvd\phi) + r^2 d\phi^2 \tag{4.26}$$

Inverse of the metric is given by:

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -(-M + \frac{r^2}{l^2} - \frac{J^2}{2r^2}) & \frac{J}{2r^2} \\ 0 & \frac{J}{2r^2} & \frac{1}{r^2} \end{pmatrix}$$

We then use the definition for stress energy for scalar field[Dappiaggi 18]:

$$T_{\mu\nu} = \partial_{(\mu} R \partial_{\nu)} R - \frac{1}{2} g_{\mu\nu} \left(g^{\rho\lambda} \partial_{(\rho} R \partial_{\lambda)} R + \frac{\mu}{2l^2} |R|^2 \right) \tag{4.27}$$

This gives us:

$$T_{\mu\nu} = |\hat{C}|^2 \begin{pmatrix} \left[-2\omega^2 e^{-2\iota(\omega v + k\phi)} - \left(-\frac{M}{2} + \frac{r^2}{2l^2}\right)\left(\frac{-2k}{r^2} + \frac{\mu}{2l^2}\right)\right] & \left(\frac{-k}{r^2} + \frac{\mu}{4l^2}\right) & \left[-2k^{-2\iota(\omega v + k\phi)} - \frac{Jk}{2r^2} + \frac{J\mu}{8l^2}\right] \\ 0 & 0 & 0 \\ \left[-2k^{-2\iota(\omega v + k\phi)} + \frac{J}{4}\left(\frac{-2k}{r^2} + \frac{\mu}{2l^2}\right)\right] & 0 & \left[-2k^2 e^{-2\iota(\omega v + k\phi)} + k - \frac{\mu}{2l^2}\right] \end{pmatrix}$$

Once we have $T_{\mu\nu}$ we can calculate the Energy Flux at horizon:

$$F_E = \int_0^{2\pi} d\phi' r_+ \chi_\mu T^\mu_\nu k^\nu \tag{4.28}$$

Here,

- r_+ is outer event horizon
- $\chi_\mu = \partial_\mu + \omega_+ \partial_{\phi'}$
- $T_{\mu\nu}$ is Stress Energy Tensor
- $k^\mu = \partial_\mu$

Plugging in the $T_{\mu\nu}$ we will have:[Dappiaggi 18]

$$F_E = 2\pi r_+ \hat{C}^2 (Im[\omega]^2 + Re[\omega](Re[\omega] - k\omega_+))e^{2Im[\omega]v} \tag{4.29}$$

In order to have energy flux towards the exterior region at horizon, it should be negative. This gives us the condition that

$$Im[\omega]^2 + Re[\omega](Re[\omega] - k\omega_+) < 0$$

It will hold only if

$$k\omega_+ > Re[\omega]$$

So condition for superradiance becomes:

$$0 < Re[\omega] < k\omega_+ \tag{4.30}$$

Modes which satisfy the above relation have outgoing flux at outer horizon so such modes are superradiant modes and they exhibit superradiance in BTZ spacetime.

4.4.2 Vanishing Boundary Condition

For Field to vanish at infinity it is sufficient to obtain a condition for $F(a, b, c, z) = 0$ at $z = 1$. We have,

$$F(a,b,c,z) = (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a, c-b, c-a-b+1, 1-z)$$
$$+\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1,1-z)\text{[Birmingham 01]}$$

First term is 0 at $z = 1$ and for the second term to vanish, we impose conditions on Gamma function so that the denominator shoots to infinity. We know that the Gamma functions diverge to infinity if the input of the function is a negative integer. This gives us the condition:

$$c - a = -n \qquad \text{or} \qquad c - b = -n \tag{4.31}$$

Modes which satisfy these conditons is given by[Birmingham 01]

$$\omega = \frac{k}{l} - 2i\left(\frac{r_+ - r_-}{l^2}\right)\left(n + \frac{1}{2}\left(1 + \sqrt{1+\mu}\right)\right) \tag{4.32}$$

Real part of the frequency is $Re[\omega] = \frac{k}{l}$.
Since $r_- < r_+$ we have,

$$\frac{kr_-}{r_+} < \frac{k}{l}$$

Since the frequency of vanishing modes does not satisfy the relation given by equation(4.26), it does not have an outgoing flux at the outer horizon. Hence such modes will not show superradiance.

So we have seen that Superradiance is governed not only by the background curvature of the spacetime, but it also depends on what are the boundary conditions imposed on the field.

Chapter 5

Conclusion and Discussion

The Thesis starts with the discussion on the maximally symmetric spacetimes in Chapter 1. We showed that for a $(1, n-1)$ dimension spacetime, ${}^nC_2$ independent killing vectors ensure that the given spacetime is maximally symmetric. We then considered all possible types of constant curvature spacetimes and showed that they all are maximally symmetric. In subsequent chapters, we studied superradiance in all of these spacetimes as asymptotic backgrounds.

In Chapter 2, we discussed superradiance and studied it for the case of asymptotically flat spacetime. In the process of doing so, we discussed Schwarzschild and Kerr spacetimes. We discussed the event horizon for both the spacetimes and learned about the existence of the ergosphere in the Kerr spacetime. Using the Klein Gordon equation for the curved spacetimes, we calculated the equation of motion for massless scalar fields in Schwarzschild spacetime and saw that no modes exhibit superradiance. But the more important thing which we learned was about how to approach the calculations regarding superradiance, something which we followed for the rest of the thesis. Further, in the second chapter, we calculated the equation of motion of massless scalar fields in the Kerr spacetime and obtained a condition for superradiance. We learned that low frequency modes compared to the angular velocity of the event horizon exhibit superradiance. We also noted that the presence of the ergosphere is necessary from which field can draw energy.

In Chapter 3 and Chapter 4, we discussed superradiance with asymptotically DeSitter and Anti DeSitter spacetime respectively. We discussed the existence of the cosmological horizon in the DeSitter spacetime. In DeSitter spacetime, we studied a rotating black hole which leads to Kerr DeSitter. We calculated the equation of motion of massless scalar field in Kerr DeSitter and obtained a condition on the modes which will show superradiance. While in AdS spacetime, we considered the BTZ black hole to study superradiance. We worked with different boundary conditions at infinity for BTZ and learned that superradiance depends not only on the black hole which generates the background curvature but also on the boundary conditions which are imposed on the scalar field.

We can extend this work in the future by exploring the Robin condition for the Kerr DeSitter spacetime. Throughout the thesis, the scalar field was considered. It would be interesting to explore whether the spin-1/2,spin-1 patricles would exhibit superradiance and what would be conditions for superradiance for different spacetimes and boundary conditions. We can also for sake of completeness, consider including the change in curvature of spacetime due to the presence of field itself and treat it as a perturbation to the background curvature of spacetime and study how it affects the superradiance.

Appendix A

Equation of motion of massless scalar field

Consider the test field to be ψ. Since it is a scalar field, its equation of motion will be governed by the Klein-Gordon Equation. Klein Gordon Equation is given by:

$$(\partial_\mu \partial^\mu - m^2)\psi = 0$$

Since our field is massless, the equation of motion simplifies to

$$\partial_\mu \partial^\mu \psi = 0$$

This relation holds only for flat spacetime. To accommodate various possible backgrounds, we use Covariant Derivative instead of partial derivatives.

$$D_\mu D^\mu \psi = 0$$

Here, D is the Covariant Derivative operator.

We intent to simplify this relation using actions of covariant derivative on scalars and covectors and get a new relation in entirely in terms of metric components so that given the metric we can straightaway write down its equation of motion. First, let us recall how covariant derivative operates on scalars and covectors.

- Action on Scalar(f): $D_\mu f = \partial_\mu f$
- Action on Covector(A_ν): $D_\mu A_\nu = \partial_\mu A_\nu - \Gamma^a_{\mu\nu} A_a$

Using these two relations we can write our equation of motion of scalar test field in terms of metric components.

$$D_\mu D^\mu \psi = 0$$

$$g^{\mu\nu} D_\mu D_\nu \psi = 0$$

We operate covariant derivative on ψ and get

$$g^{\mu\nu} D_\mu(\partial_\nu \psi) = 0$$

Now Covariant derivative will operate on $\partial_\nu \psi$ which is a covariant vector. Since we know it's action on covariant vector we can write it as:

$$D_\mu D^\mu \psi = g^{\mu\nu} \left(\partial_\mu \partial_\nu \psi - \Gamma^a_{\mu\nu} \partial_a \psi\right) = 0 \tag{A.1}$$

We just now need to deal with the Christoffel symbol and we will have our equation of motion in terms of the metric component. For that, we use the fact that covariant derivative is metric compatible, i.e. Covariant derivative of metric vanishes.

$$D_\mu(g^{\mu\nu}) = \partial_\mu g^{\mu\nu} + \Gamma^\mu_{\mu\lambda} g^{\nu\lambda} + \Gamma^\mu_{\nu\lambda} g^{\mu\lambda} = 0$$

This gives us

$$-\Gamma^\nu_{\mu\lambda} g^{\mu\lambda} \partial_\nu \psi = \left(\partial_\mu g^{\mu\nu} + \Gamma^\mu_{\mu\lambda} g^{\nu\lambda}\right) \partial_\nu \psi \tag{A.2}$$

We can expand Christoffel symbol in terms of metric and can further simplify our expression

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu a} \left(\partial_\mu g_a + \partial_\lambda g_{\mu a} - \partial_a g_\mu\right)$$

Since both μ and a are dummy variables, first and third terms will out each other. This gives us

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu a} \partial_\lambda g_{\mu a}$$

Using relation between trace and determinant we get

$$\frac{1}{2} g^{\mu a} \partial_\lambda g_{\mu a} = \frac{1}{2} Tr(g^{-1} \partial_\lambda g) = \frac{1}{2} \partial_\lambda ln(|det(g)|) \tag{A.3}$$

Using the result of the second and third equation in the first equation we get our equation of motion in terms of inverse of metric component and determinant of metric as follows:

$$D_\mu D^\mu \psi = \frac{1}{\sqrt{|g|}} \left[g^{\mu\nu} (\partial_\mu (\partial_\nu \psi)) \sqrt{|g|} + g^{\nu\mu} \partial_\nu \psi (\partial_\mu \sqrt{|g|}) + (\partial_\mu g^{\mu\nu}) \partial_\nu \psi \sqrt{|g|} \right]$$

$$D_\mu D^\mu \psi = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\nu \psi \right) = 0 \tag{A.4}$$

Appendix B

Transformation of an equation to Schrodinger like form

Consider a second order differential equation which we wish to transform into Schrodinger like form:

$$R^{''}(r) + a(\omega, k, r) R^{'}(r) + \left(b^2(r)\omega^2 + c(r)\right) R(r) = 0 \tag{B.1}$$

Here, $R^{'} = \frac{dR}{dr}$ After transformation, we want the equation in form:

$$-\frac{d^2u}{d_{r_*^2}} + V(r) u = -\omega^2 u \tag{B.2}$$

We wish to find the transformation which will give us the desired result.

- $u(r_*) = f(r) R(r)$
- $r_* = g(r)$

We need to find the functions f and g in terms of known coefficient of equation B.1

$$R = \frac{u}{f}$$

$$R^{'} = -\frac{f^{'}}{f^2}u + \frac{1}{f}u^* g^{'}$$

Here, $u^* = \frac{du}{dr_*}$

$$R^{''} = \frac{-f^{''}}{f^2}u + \frac{2f^{'2}}{f^3}u - \frac{2f^{'}}{f^2}u^* g^{'} + \frac{u^{**}}{f}g^{'2} + \frac{u^*}{f}g^{''}$$

Substituting R and $R^{''}$ in equation B.1 we get the equation in terms of function f and g and their derivatives.

$$u^{**} + \left(\frac{g^{''}}{f} - \frac{2f^{'}g^{'}}{f^2} + \frac{ag^{'}}{f}\right)u^{*} + \frac{b^2\omega^2}{g^{'2}}u - \left[\frac{f^{''}}{f} - \frac{2f^{'}}{f} + af^{'} - cf\right]u = 0$$

Comparing above equation with equation B.2 we can impose two conditions:

- Set coefficient of u^{*} to 0.
- Set coefficient of ω^2 to 1.

First condition gives us:

$$\frac{g^{''}}{f} - \frac{2f^{'}g^{'}}{f^2} + \frac{ag^{'}}{f} = 0 \implies \frac{g^{''}}{g^{'}} = \frac{2f^{'}}{f} - a$$

And second condition gives us:

$$\frac{b^2}{g^{'2}} = 1 \implies g^{'} = \pm b$$

Using these two relations, we can solve for f and g. These functions will give us the desired transformation.

Appendix C

Wronskian as constant of motion

In this appendix we will show that for Schrodinger like differential equations, Wronskian is a constant. Consider the following differential equation:

$$y^{''} + ay^{'} + by = 0 \quad \text{here} \quad y^{'} = \frac{dy}{dx} \tag{C.1}$$

Let $u(x)$ and $v(x)$ be two solutions for this second order differential equation. Then Wronskian would be:

$$W = \begin{vmatrix} u & v \\ u^{'} & v^{'} \end{vmatrix}$$

$$W = uv^{'} - vu^{'}$$

Differentiate the above equation with respect to x:

$$W^{'} = uv^{''} - vu^{''} \tag{C.2}$$

Since both u and v are solutions of differential equation, we can replace double derivative as:

$$u^{''} = -au^{'} - bu$$

and

$$v^{''} = -av^{'} - bv$$

Using these we can rewrite equation $(C.2)$ as:

$$W^{'} = u\left(-av^{'} - bv\right) - v\left(-au^{'} - bu\right)$$

$$W' = -aW \tag{C.3}$$

Note that a here is the coefficient of the first derivative. If our differential equation is of the form of Schrodinger's equation then it would not have first derivative term, i.e. $a = 0$ So for such differential equations, from $(C.3)$ we have,

$$W' = 0$$

$$W = constant$$

Hence Wronskian would be constant for such differential equation.

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