# Dimension Subgroups and Augmentation Quotients

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# Certificate of Examination

This is to certify that the dissertation titled "**Dimension Subgroups and Augmentation Quotients**" submitted by **Ms. Rajni Ranjan** (Reg. No. MS07018) for the partial fullment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Amit Kulshrestha    Dr. Chanchal Kumar    Prof. Inder Bir Singh Passi
(Supervisor)

Dated: May 08, 2012

# Declaration

The work presented in this dissertation has been carried out by me under the guidance of Prof. I.B.S. Passi at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Ms. Rajni Ranjan
(Candidate)

Dated: May 08, 2012

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Prof. Inder Bir Singh Passi
(Supervisor)

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# Introduction

The aim of this report is to highlight the major developments in the topics of dimension subgroups and augmentation quotients.

The identification of certain normal subgroups determined by the powers of the augmentation ideal of a group ring $R(G)$, known as the dimension subgroups, is one of the most challenging problems in group rings. The study of these subgroups is supposed to be originated by W. Magnus [Mag35] when he conjectured that for any group $G$, the lower central and the integral dimension series coincide. The works of Magnus [Mag37] and E. Witt [Wit37] implied that the conjecture is true for finitely generated free groups. The conjecture remained undecided until more than three decades later, it was proved to be false by E. Rips [Rip72]. As of now, it is known that the first three terms of the integral dimension series coincide with the lower central series but the same cannot be said for the subsequent terms. The first chapter of this report gives a broad overview of the results concerning dimension subgroups. We begin with some basic definitions and as an example, compute the second dimension subgroup of the group ring $R(T)$, where $T = \mathbb{Q}/\mathbb{Z}$. We then state and prove the theorem of Magnus regarding dimension subgroups of free groups using the theory of Lie algebras. We then list some special cases in which we can compute these subgroups. The results concerning integral dimension subgroups in low dimensions are stated next. In this section, we define polynomial maps and give their connection with dimension subgroups, which forms a motivation for the next chapter.

The canonical filtration of the augmentation ideal by its powers gives us two sequences of Abelian groups, known as the polynomial groups. These were first computed by I.B.S. Passi for cyclic and elementary Abelian groups [Pas68b]. One of these, known as the sequence of augmentation quotients has been extensively studied by many authors. The second chapter begins with a brief description of some early results. One of the important results regarding augmentation quotients due to F. Bachmann and L. Grünenfelder [BG74] states that for all finite groups, the sequence of augmentation quotients is periodic. In particular, for finite Abelian groups, it is stationary. This stationary structure was given by A. Hales in terms of generators and relations [Hal85]. We study this result in some detail. The structure of augmentation quotients for all finite Abelian groups was described by S. Chang and G. Tang [CT11]. Description of their proof forms the next part of this chapter. Finally, we conclude by summarising some results regarding non-Abelian groups, in particular, the case of symmetric groups [ZT08]. We also list some developments for the non-Abelian case [LL79, ZY09, ZY10].

# Chapter 1

# Dimension Subgroups

To every group $G$, we can associate a series of normal subgroups determined by commutators known as the lower central series, which is denoted by $\{\gamma_n(G)\}_{n\geq 1}$. On considering the group ring $R(G)$, we obtain another series of normal subgroups, determined by the powers of the augmentation ideals known as the dimension series, denoted by $\{D_{n,R}(G)\}_{n\geq 1}$. One of the oldest and the most challenging problems in group rings is the identification of these subgroups. Of special interest is the case when $R = \mathbb{Z}$, the ring of integers.

Dimension subgroups were first studied by Magnus [Mag37] who proved that for free groups the integral dimension series and the lower central series coincide. This led to a conjecture known as the dimension conjecture that these series are the same for all groups [Coh52]. This was proved to be false by Rips [Rip72] who gave an example of a group $G$ for which $D_4(G) \neq \gamma_4(G)$. It is now known that for all groups $D_n(G)$ ,i.e., the integral dimension subgroups for $n = 1, 2, 3$ are the same as the corresponding lower central terms (see [Pas79]), while it is not necessarily true for the subsequent terms.

This chapter focuses on some main results concerning dimension subgroups, in particular, when the ring under consideration is $\mathbb{Z}$.

## 1.1 Motivation

We begin with some definitions and notations which will be followed subsequently.

**Definition 1.1.1** Let $G$ be a group and $R$ be a ring with identity. The set of all formal $R$-linear combinations of elements of $G$ forms a ring known as the **group ring**

of G with respect to the ring $R$ and is denoted by $R(G)$.

**Definition 1.1.2** The trivial map from $G$ to $R$ which sends every element $x \in G$ to $1_R$ gives rise to a unique ring homomorphism $\epsilon : R(G) \to R$, known as the **augmentation map**. The kernel of the augmentation map defined above is an ideal of $R(G)$, known as the **augmentation ideal** and is denoted by $\Delta_R(G)$.

It can be easily seen that the augmentation ideal is a 2 sided ideal of $R(G)$. For the sake of simplicity of notation, we shall drop the subscript $R$ when $R = \mathbb{Z}$.

As an Abelian group, $\Delta_R(G)$ is free on the set

$$W = \{g - 1 | \, 1 \neq g \in G\}. \tag{1.1}$$

Also, if $S$ is a generating set for $G$, then as a $G$-module, $\Delta_R(G)$ is generated by the set

$$S - 1 = \{s - 1 | \, s \in S\}. \tag{1.2}$$

Let $\mathcal{N}$ denote the set of all normal subgroups of $G$ and $\mathcal{I}$ denote the set of all 2-sided ideals of $R(G)$. Let $N \trianglelefteq G$ and let $\Delta_R(G, N)$ denote the kernel of the epimorphism $R(G) \to R(G/N)$ induced by the natural map $G \to G/N$. Then, $\Delta_R(G, N)$ is a 2-sided ideal of $R(G)$. Hence, every normal subgroup of $G$ defines a 2-sided ideal of the group ring $R(G)$.

In view of the above, we have a map

$$\phi : \mathcal{N} \to \mathcal{I} \tag{1.3}$$

defined by $\phi(N) = \Delta_R(G, N)$.

On the other hand, to every $I \in \mathcal{I}$, we can associate the normal subgroup $G \cap (1+I)$. This gives a map

$$\psi : \mathcal{I} \to \mathcal{N} \tag{1.4}$$

defined by $\psi(I) = G \cap (1 + I)$.

For $g \in G$, if $g - 1 \in \Delta_R(G, N)$, then $g \in N$. Hence, $\psi \circ \phi = 1$. But this may not be true for $\phi \circ \psi$. For example, when we take $I = R(G)$, then $\phi \circ \psi(R(G)) = \phi(G) = \Delta_R(G, G) = \Delta_R(G)$.

The above correspondence motivates us to study certain normal subgroups of the group $G$ which reflect the properties of the ring $R$ as well. One such type, namely the dimension subgroups are obtained by considering the powers of the augmentation ideal as given below.

**Definition 1.1.3** The $n^{th}$ **dimension subgroup** of a group $G$ with respect to a ring $R$ is defined as

$$D_{n,R}(G) = \psi({\Delta^n}_R(G)) = G \cap (1 + \Delta_R^n(G)). \tag{1.5}$$

Since ${\Delta^{n+1}}_R(G) \subseteq {\Delta^n}_R(G)$, we have $D_{n+1,R}(G) \subseteq D_{n,R}(G)$. Hence, we have the following series of normal subgroups of $G$

$$G = D_{1,R}(G) \supseteq D_{2,R}(G) \supseteq \ldots \supset D_{n,R}(G) \supseteq \ldots \tag{1.6}$$

known as the **dimension series** of the group $G$ with respect to the ring $R$. When $R = \mathbb{Z}$, we denote $D_{n,\mathbb{Z}}(G)$ by $D_n(G)$.

In this chapter, we shall denote the commutator $x^{-1}y^{-1}xy$ of two elements $x$ and $y$ by $(x, y)$ to avoid confusion with the Lie bracket notation introduced later. We now define the lower central series of a group $G$.

**Definition 1.1.4** We define the subgroups $\gamma_n(G)$ inductively as

$$\gamma_1(G) = G, \gamma_n(G) = (G, \gamma_{n-1}(G)). \tag{1.7}$$

Then, the normal series

$$G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \ldots \supseteq \gamma_n(G) \supseteq \gamma_{n+1}(G) \supseteq \ldots \tag{1.8}$$

is known as the **lower central series** of the group $G$.

We shall denote the $n^{th}$ lower central terms of a group $G$ by $G_n$, $n \geq 1$ whenever there is no scope of confusion. Using some results from commutator calculus, we shall now see that for any group $G$, the lower central series is contained in the dimension series.

Let $G$ be a group and $a, b \in G$. Then, we define $a^b = b^{-1}ab$. The following theorem can be proved using simple computations.

**Theorem 1.1.5.** *(see [MKS75, p. 290]) For $a, b, c \in G$, we have*

1. $(a, b)^{-1} = (b, a)$

2. $(a, bc) = (a, c)(a, b)((a, b), c)$

3. $(ab, c) = (a, c)((a, c), b)(b, c)$

4. $((a, b), c^a)((c, a), b^c)((b, c), a^b) = 1$

5. $((a, b), c)((b, c), a)((c, a), b) = (b, a)(c, a)(c, b)^a(a, b)(a, c)^b(b, c)^a(a, c)(c, a)^b$.

The identity (5) is known as the Hall-Witt identity.

Let $A$ and $B$ be subgroups of $G$. Then, we denote by $(A, B)$, the subgroup generated by all elements of the type $(a, b)$, where $a \in A$ and $b \in B$.

It is clear that if $A$ and $B$ are normal subgroups, then $(A, B)$ is a normal subgroup of $G$ and $(A, B) \subseteq A \cap B$.

The following result can be easily proved using Theorem 1.1.5.

**Theorem 1.1.6.** *Let $A$, $B$ and $C$ be any normal subgroups of a group $G$. Then, each of the three normal subgroups*

$$((A, B), C), ((B, C), A), ((C, A), B)$$

*is contained in the product of the other two.*

**Definition 1.1.7** A series $G = H_1 \supseteq H_2 \supseteq \ldots \supseteq H_i \supseteq \ldots$ of subgroups of a group $G$ is called an $N$**-series** if $(H_i, H_j) \subseteq H_{i+j}$ for all $i, j \geq 1$.

**Definition 1.1.8** An $N$-series is called a **restricted $N$-series relative to a prime** $p$, or an $N_p$-series, if $H_i^p \subseteq H_{ip}$ for all $i \geq 1$.

**Remark 1.1.9** The condition $(H_i, H_j) \subseteq H_{i+j}$ implies that the terms of the $N$-series are normal subgroups of $G$ (take $j = 1$).

The next result can be easily obtained using induction.

**Theorem 1.1.10.** *The lower central series $\{G_i\}_{i \geq 1}$ of a group $G$ is an $N$-series.*

**Definition 1.1.11** A decreasing series

$$\Delta_R(G) = A_1 \supseteq \ldots \supseteq A_n \supseteq \ldots \tag{1.9}$$

of two-sided ideals of $R(G)$ is called a **filtration** of the augmentation ideal $\Delta_R(G)$.

One way to arrive at $N$-series is by filtrations of augmentation ideals by using the correspondence stated in Proposition 1.1.12. In fact, it was proved by Lazard [Laz54] that every $N_p$-series is obtained by some filtraion of the augmentation ideal.

**Proposition 1.1.12.** *Let $G$ be a group and $R$ be a ring with identity. Let*

$$\Delta_R(G) = A_1 \supseteq \ldots \supseteq A_n \supseteq \ldots \tag{1.10}$$

*be a filtration of the augmentation ideal such that $A_iA_j \subseteq A_{i+j}$ for all $i, j \geq 1$. If $H_i = G \cap (1 + A_i)$, then $\{H_i\}_{i\geq 1}$ is an $N$-series. Further, if the characteristic of $R$ is a prime $p$, then $\{H_i\}_{i\geq 1}$ is an $N_p$-series.*

*Proof.* Let $a \in H_i$ and $b \in H_j$. Then, $a - 1 \in A_i$ and $b - 1 \in A_j$. Therefore, both $(a - 1)(b - 1)$ and $(b - 1)(a - 1)$ belong to $A_{i+j}$. Hence,

$$(a, b) - 1 = a^{-1}b^{-1}((a - 1)(b - 1) - (b - 1)(a - 1)) \in A_{i+j}. \tag{1.11}$$

Thus, $\{H_i\}_{i\geq 1}$ is an $N$-series.

If the characteristic of $R$ is $p$, then for $x \in H_i$, using the identity

$$(x - 1)^p = x^p - 1, \tag{1.12}$$

we get $x^p - 1 \in A_p$, i.e., $H_i^p \subseteq H_{ip}$ which proves that $\{H_i\}_{i\geq 1}$ is an $N_p$-series. □

Clearly, since ${\Delta_R}^i(G){\Delta_R}^j(G) \subseteq {\Delta_R}^{i+j}(G)$ for all $i, j \geq 1$, we have the following

**Corollary 1.1.13.** *If $G$ is a group and $R$ is a ring with identity, then the dimension series of $G$ with respect to $R$ is an $N$-series. Further, if $R$ is of characteristic $p$, then it is an $N_p$-series.*

The next result shows that the lower central series is the 'smallest' $N$-series of the group $G$.

**Theorem 1.1.14.** *Let $\{\gamma_i(G)\}_{i\geq 1}$ denote the lower central series of a group $G$. If $\{H_i\}_{i\geq 1}$ is any $N$-series of $G$, then $\gamma_i(G) \subseteq H_i$ for all $i \geq 1$.*

In view of Theorem 1.1.14 and Corollary 1.1.13, we get that $\gamma_n(G) \subseteq D_n(G)$. It is thus a natural question whether $\gamma_n(G) = D_n(G)$ and if not, what is the structure of $D_n(G)/\gamma_n(G)$. The next result is a reduction which says that to study the above problem, it suffices to study the case of finite $p$-groups.

**Theorem 1.1.15.** *(see [Pas68b]) If $D_n(G) \neq G_n$, for some groups $G$, then there is a finite $p$-group for which the same holds.*

We also have the following result which states that for computation of dimension subgroups over an arbitrary ring $R$, it is sufficient to consider the cases when $R$ is either $\mathbb{Z}$ or $\mathbb{Z}/r\mathbb{Z}$ for some integer $r$.

**Theorem 1.1.16.** *[PPS73, San72] Let $G$ be a group and $R$ be a commutative ring with unity.*

1. *If characteristic of $R$ is zero, then*

$$D_{n,R}(G) = \prod_{p\in\sigma(R)} \{\tau_p(G \ mod \ D_{n,\mathbb{Z}}(G)) \cap D_{n,\mathbb{Z}/p^e\mathbb{Z}}(G)\}$$

*where $\sigma(R) = \{p|p$ is a prime and $p^nR = p^{n+1}R$ for some $n \geq 0\}$ and for $p \in \sigma(R)$, $p^e$ is the smallest power of $p$ for which $p^e = p^{e+1}R$. When $\sigma(R)$ is empty, the right hand side is interpreted as $D_{n,\mathbb{Z}}(G)$.*

2. *If characteristic of $R$ is $r > 0$, then for all $n \geq 1$,*

$$D_{n,R}(G) = D_{n,\mathbb{Z}/r\mathbb{Z}}(G) = \cap_i D_{n,\mathbb{Z}/{p_i}^{e_i}\mathbb{Z}}(G)$$

*where $r = \prod_i {p_i}^{e_i}$ is the prime factorization of $r$.*

**Example 1.1.17** We calculate the $2^{nd}$ dimension subgroup of the group ring $R(T)$, where $T = \mathbb{Q}/\mathbb{Z}$ and $R$ is a commutative ring with identity.

We know that $T = \sum_p Z(p^\infty)$, where the sum is over all primes $p$. We shall write $T$ multiplicatively.

**Claim 1.1.18**

$$D_{2,R}(T) = \sum_{p\in\sigma(R)} \mathbb{Z}(p^\infty) \tag{1.13}$$

where $\sigma(R) = \{p \,|\, p$ is a prime and $p^nR = p^{n+1}R$ for some$n \geq 0\}$.

For $p \in \sigma(R)$, let $t \in \mathbb{Z}(p^\infty)$. Then, since $\mathbb{Z}(p^\infty)$ is a divisible Abelian group, $\exists\, n \geq 0$ and $x \in \mathbb{Z}(p^\infty)$ such that $p^nR = p^{n+1}R$ and $t = x^{p^n}$. Thus,

$$\begin{aligned} t-1 &= x^{p^n}-1 \\ &= p^n(x-1)+\binom{p^n}{2}(x-1)^2+\ldots+(x-1)^{p^n} \\ &\equiv p^n(x-1) \bmod \Delta_R^2(\mathbb{Z}(p^\infty)) \end{aligned} \tag{1.14}$$

Let $p^m$ be the order of $x$ in $T$. By the choice of $p$, there exists $r \in R$ such that $p^n = p^{n+m}r$. Also, the equation

$$0 = x^{p^m}-1 = p^m(x-1)+\binom{p^m}{2}(x-1)^2+\ldots+(x-1)^{p^m} \tag{1.15}$$

shows that $p^m(x-1) \in {\Delta_R}^2(\mathbb{Z}(p^\infty))$.

Hence, $t-1 \equiv p^n(x-1) \equiv rp^np^m(x-1) \equiv 0 \bmod {\Delta_R}^2(\mathbb{Z}(p^\infty))$. Thus,

$$\sum_{p\in\sigma(R)} \mathbb{Z}(p^\infty) \subseteq D_{2,R}(T). \tag{1.16}$$

Conversely, suppose $t \in D_{2,R}(T)$. Then, for any prime $p$, on considering the projection of $T$ on its direct summand $\mathbb{Z}(p^\infty)$, we get that the $p$-primary component of $t$, say $t_p$ is in $D_{2,R}(\mathbb{Z}(p^\infty))$.

Let $H$ be the subgroup generated by the elements of $\mathbb{Z}(p^\infty)$ which appear in an expression of $t_p - 1$ as an element of $D_{2,R}(\mathbb{Z}(p^\infty))$. Then, since $\mathbb{Z}(p^\infty)$ is locally cyclic, $H$ is a cyclic group of order $p^r$, say, and $t_p \in D_{2,R}(H)$. Now, suppose $p \notin \sigma(R)$. Then, the rings $R/p^nR$ for $n \geq 1$ have increasing characteristics and hence $D_{2,R}(H) \subseteq D_{2,R/p^nR}(H)$. Thus, $t_p - 1 \in D_{2,R/p^nR}(H)$ for all $n \geq 1$.

Now, if $t_p = h^{p^s}$, where $h$ is a generator of $H$, then $t_p - 1 \in {\Delta^2}_{R/p^nR}(H)$ and the equation

$$t_p - 1 = h^{p^s} - 1 \equiv p^s(h-1) \bmod {\Delta^2}_{R/p^nR}(H) \tag{1.17}$$

imply that $p^s(h-1) \in {\Delta^2}_{R/p^nR}(H)$. Thus, since ${\Delta^2}_{R/p^nR}(H)$ is generated by $(h-1)^2$, $\exists\, u \in R/p^nR(H)$ such that

$$p^s(h-1) = (h-1)^2u. \tag{1.18}$$

Let $\alpha = p^s - (h-1)u$. Then, $h\alpha = \alpha$ and on comparing coefficients on both sides, $\alpha$ can be written as

$$\alpha = \beta(1 + h + \dots h^{p^r - 1}) \tag{1.19}$$

where $\beta \in R/p^nR$.

Now, on applying the augmentation map $\epsilon : R/p^nR(H) \to R/p^nR$ to both sides of the equation

$$p^s - (h-1)u = \beta(1 + h + \dots h^{p^r - 1}), \tag{1.20}$$

we get that $p^s = 0$ in $R/p^nR$. Thus, $s \geq r$ and hence $t_p = 1$. This proves that if $t \in D_{2,R}(T)$, then for every $p \notin \sigma(R)$, $t_p = 1$. Thus,

$$t \in \sum_{p \in \sigma(R)} \mathbb{Z}(p^\infty) \tag{1.21}$$

and Claim 1.1.18 is established.

## 1.2 Dimension Subgroups of Free Groups

For a finitely generated free group $F$, the truth of the dimension conjecture was established by Magnus [Mag37] using an identity of Witt [Wit37]. This result is proved in this section using techniques from the theory of Lie algebras. The part on Lie algebras have been derived from the books by J-P. Serre [Ser06] and N. Jacobson [Jac79] and that on commutator calculus from [MKS75]. We begin with a few definitions.

Let $R$ be a commutative ring with identity.

**Definition 1.2.1** An associative algebra $A$ over $R$ is a module along with a bilinear map

$$\theta : A \times A \to A \tag{1.22}$$

where $\theta$ is an $R$-homomorphism satisfying the condition

$$\theta(\theta(x, y), z) = \theta(x, \theta(y, z)). \tag{1.23}$$

**Definition 1.2.2** A Lie algebra $A$ over $R$ is an algebra equipped with a bilinear map $[.,.]$ called the Lie bracket which satisfies the following properties:

1. $[x, x] = 0 \,\forall\, x \in A$,

2. $[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

By the first condition, we have

$$[x, y] = -[y, x]. \tag{1.24}$$

**Remark** Any associative algebra can be converted to a Lie algebra by defining the Lie bracket as:

$$[x, y] = xy - yx. \tag{1.25}$$

**Definition 1.2.3** A universal enveloping algebra of a Lie algebra $L$ over $R$ is an associative algebra with identity, $UL$ along with a map

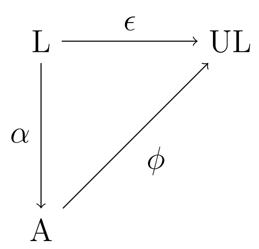
$$\epsilon : L \to UL \tag{1.26}$$

satisfying the following properties:

1. $\epsilon$ is a Lie algebra homomorphism i.e. it is $R$ linear and

$$\epsilon[x, y] = \epsilon x \epsilon y - \epsilon y \epsilon x$$

2. If $A$ is any associative algebra with identity and $\alpha : L \to A$ is any Lie algebra homomorphism, then there is a unique homomorphism of associative algebras $\phi : UL \to A$ such that the diagram




is commutative.

This $UL$ is unique up to isomorphism.

**Definition 1.2.4** A free associative algebra on a set $X$ is an associative algebra $\mathcal{A}$ along with a map $\theta : X \to \mathcal{A}$ such that if $\beta : X \to \mathcal{B}$ is any map, where $\mathcal{B}$ is an associative algebra, then there exists a unique associative algebra homomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ such that $\beta = \theta \circ \alpha$.

The notion of a free Lie algebra can be similarly defined. We denote the free associative algebra on a set $X$ by $\mathcal{A}_X$ and the free Lie algebra on $X$ by $L_X$.

Let $X$ be a set and $F$ be the free group on $X$. Let us consider the lower central series of $F$.

We have the associated graded ring $\bigoplus_{i=1}^{\infty} F_i/F_{i+1}$. This is isomorphic to the free associative algebra $\mathcal{A}_X$. Also, since every associative algebra can be viewed as a Lie algebra by defining the Lie bracket as $[x, y] = xy - yx$, the above is a Lie algebra.

Let $UL_X$ be the universal enveloping algebra of $L_X$. It is isomorphic to $\mathcal{A}_X$ in view of the following:

**Theorem 1.2.5.** *(see [Jac79, p.168]) The universal enveloping algebra of the free Lie algebra over a set $X$ is the free associative algebra $\mathcal{A}_X$.*

Poincaré-Birkhoff-Witt( [Poi00, Bir37, Wit37]) Theorem implies that the universal map $\epsilon : L_X \to UL_X \cong \mathcal{A}_X$, where $L_X$ and $\mathcal{A}_X$ are as defined above, is an embedding.

Let $F_n$ denote the terms of the lower central series of $F$. For $x_m \in F_m \setminus F_{m+1}$ and $y_n \in F_n \setminus F_{n+1}$, we define a multiplication in $\bigoplus_{n=1}^{\infty} F_n/F_{n+1}$ as

$$[xF_{m+1}, yF_{n+1}] = (x, y)F_{m+n+1}. \tag{1.27}$$

This can be extended linearly to all elements of $\bigoplus_{n=1}^{\infty} F_n/F_{n+1}$ and converts it into a Lie algebra over $\mathbb{Z}$.

The following theorem shows that this is the free Lie algebra on the set $X$.

**Theorem 1.2.6.** *(see [MKS75, p.337]) The canonical map $X \to \bigoplus_{n=1}^{\infty} F_n/F_{n+1}$ induces an isomorphism of Lie algebras*

$$\phi : L_X \to \bigoplus_{n=1}^{\infty} F_n/F_{n+1} \tag{1.28}$$

*where $L_X$ denotes the free Lie algebra generated by the set $X$.*

Now, let us consider the dimension subgroups of $F$. Since $F_n \subseteq D_n$, we have the canonical injection

$$F_i/F_{i+1} \hookrightarrow D_i/D_{i+1}. \tag{1.29}$$

Similar to the above construction, $\bigoplus_{i=1}^{\infty} D_i/D_{i+1}$ can be viewed as a Lie algebra.

There is a canonical map from $\bigoplus_{i=1}^{\infty} F_i/F_{i+1}$ to $\bigoplus_{i=1}^{\infty} D_i/D_{i+1}$.

Next, let $\mathbb{Z}F$ be the integral group ring of $F$. Then, we have the filtration

$$\mathbb{Z}F \supseteq \Delta(F) \supseteq \Delta^2(F) \supseteq \ldots \supseteq \Delta^n(F) \supseteq \ldots \tag{1.30}$$

In view of the relations

$$(xy-1) \equiv (x-1)+(y-1) \bmod \Delta^2(F) \tag{1.31}$$

and

$$(x^{-1}-1) \equiv (x-1) \bmod \Delta^2(F), \tag{1.32}$$

we get that

$$\Delta(F) \equiv \sum_i \alpha_i(x_i-1) \bmod \Delta^2(F). \tag{1.33}$$

Similarly, we have with us

$$\Delta^n(F)/\Delta^{n+1}(F) \equiv \langle \sum (x_{i_1}-1)\ldots(x_{i_n}-1) + \Delta^{n+1}(F)\rangle \tag{1.34}$$

We define the homogeneous elements of degree $n$ as the elements of $\Delta^n(F)/\Delta^{n+1}(F)$.

For two elements $\alpha_i \in \Delta^i(F)$ and $\alpha_j \in \Delta^j(F)$, we define a multiplication in $\bigoplus_{i=1}^{\infty} \Delta^i(F)/\Delta^{i+1}(F)$ as follows

$$(\alpha_i + \Delta^{i+1}(F)).(\alpha_j + \Delta^{j+1}(F)) = \alpha_i\alpha_j + \Delta^{i+j+1}(F). \tag{1.35}$$

We extend this product by linearity to $\bigoplus_{i=1}^{\infty} \Delta^i(F)/\Delta^{i+1}(F)$. This converts it into an associative graded algebra, denoted as $Gr(F)$.

It follows from a result of Quillen [Qui68] that for a free group $F$, $Gr(F)$ is isomorphic to the free associative algebra over $X$, $\mathcal{A}_X$.

Now, we state the main result of this section and give its proof.

**Theorem 1.2.7.** *Let $F$ be the free group on the set $X = \{x_1, x_2, \ldots, x_r\}$. Then, $D_n(F) = F_n$ for all $n \geq 1$.*

*Proof.* Since $F_n \subseteq D_n(F)$, there is a canonical map $\theta_n : F_n/F_{n+1} \to D_n/D_{n+1}$ defined by $\theta_n(\bar{x_n}) = x_n + D_{n+1}$.

This map can be extended to the Lie algebra $\bigoplus_{n=1}^{\infty} F_n/F_{n+1}$ by linearity. We thus have a Lie algebra homomorphism

$$\theta : \bigoplus_{n=1}^{\infty} F_n/F_{n+1} \to \bigoplus_{n=1}^{\infty} D_n/D_{n+1}. \tag{1.36}$$

We also have a map

$$\phi_n : D_n/D_{n+1} \to \Delta^n(F)/\Delta^{n+1}(F) \tag{1.37}$$

given by

$$\phi_n(x + D_{n+1}) = (x - 1) + \Delta^{n+1}(F). \tag{1.38}$$

As above, we extend this map linearly to $\bigoplus_{n=1}^{\infty} D_n/D_{n+1}$ to get an associative algebra homomorphism

$$\phi : \bigoplus_{n=1}^{\infty} D_n/D_{n+1} \to \bigoplus_{n=0}^{\infty} \Delta^n(F)/\Delta^{n+1}(F). \tag{1.39}$$

Combining the above, we have the following sequence of maps:

$$\bigoplus_{n=1}^{\infty} F_n/F_{n+1} \xrightarrow{\theta} \bigoplus_{n=1}^{\infty} D_n/D_{n+1} \xrightarrow{\phi} \bigoplus_{n=0}^{\infty} \Delta^n(F)/\Delta^{n+1}(F). \tag{1.40}$$

The universal envelope of the free Lie algebra is the free associative algebra, $\mathcal{A}_X$. From the previous discussion, $\bigoplus_{n=0}^{\infty} \Delta^n(F)/\Delta^{n+1}(F) \cong \mathcal{A}_X$.

Also, by Poincare-Berkhoff-Witt theorem, the map $\alpha : L_X \to UL_X \cong \mathcal{A}_X$ is an embedding. Thus, the map $\phi \circ \theta$ is one-to-one.

Now, the proof proceeds by induction. For $n = 1$, $D_n(F) = F = F_n$. Suppose the result holds for $k \leq n$. Then, consider an element $x \in D_{n+1} \setminus F_{n+1}$. Then, $x \in D_{n+1} \subseteq D_n = F_n$. Thus, $x + F_{n+1}$ is a non-zero element of $F_n/F_{n+1}$. Then, $\phi \circ \theta(x + F_{n+1}) = 0$ and hence $x \in F_{n+1}$.
This completes the induction. □

## 1.3 Dimension Subgroups over Fields

In this section, we study the results regarding dimension subgroups when the ring $R$ is a field. These subgroups have been studied extensively and a complete description of them has been given, in terms of known subgroups of a group $G$.

In view of Theorem 1.1.16 for a field $k$, we have for any integer $n \geq 1$,

$$D_{n,k}(G) = \begin{cases} D_{n,\mathbb{Q}}(G) & \text{if characteristic of } k \text{ is } 0, \\ D_{n,\mathbb{Z}/p\mathbb{Z}}(G) & \text{if characteristic of } k \text{ is } p > 0. \end{cases}$$

To state the main result, we need a few definitions.

**Definition 1.3.1** For a subgroup $H$ of a group $G$, we define a subgroup $\sqrt{H}$ as the set of all elements $x \in G$ such that $x^m \in H$ for some $m \geq 0$.

The following result gives us the dimension subgroups for fields of characteristic zero.

**Theorem 1.3.2.** *[Jen55, Hal70] For all $n \geq 1$,*

$$D_{n,\mathbb{Q}}(G) = \sqrt{\gamma_n(G)}. \tag{1.41}$$

Next, we proceed to the case where $k$ is of characteristic a prime $p$.

**Definition 1.3.3** The Brauer-Jennings-Zassenhaus $M$-series $\{M_{n,p}(G)\}_{n\geq 1}$ of a group $G$ is defined inductively as:

$$M_{1,p}(G) = G, M_{n,p}(G) = (G, M_{n-1,p}(G))M^p_{(\frac{n}{p})}(G) \quad \text{for } n \geq 2. \tag{1.42}$$

where $(\frac{n}{p})$ denotes the least integer $\geq \frac{n}{p}$. This series is the minimal with the property $M^p_{n,p}(G) \subseteq M_{np,p}(G)$.

By Corollary 1.1.13, the series $D_{n,\mathbb{Z}/p\mathbb{Z}}(G)$ is an $N_p$-series and hence $D^p_{n,\mathbb{Z}/p\mathbb{Z}}(G) \subseteq D_{np,\mathbb{Z}/p\mathbb{Z}}(G)$. This implies that

$$M_{n,p}(G) \subseteq D_{n,\mathbb{Z}/p\mathbb{Z}}(G), \quad \forall\, n \geq 1. \tag{1.43}$$

**Definition 1.3.4** We define a series $\{G_{n,p}\}_{n\geq 1}$ of normal subgroups by setting

$$G_{n,p} = \prod_{ip^j \geq n} \gamma_i(G)^{p^j} \tag{1.44}$$

Since $\{M_{n,p}(G)\}_{n\geq 1}$ is an $N_p$-series and contains the lower central series, we have

$$G_{n,p} \subseteq M_{n,p}(G), \quad \forall\, n \geq 1. \tag{1.45}$$

The next theorem states that the inclusions (1.43) and (1.45) are equalities and hence gives us the dimension subgroups for fields of characteristic $p$.

**Theorem 1.3.5.** *(see [Pas79]) For every group $G$ and prime $p$,*

$$G_{n,p} = M_{n,p}(G) = D_{n,\mathbb{Z}/p\mathbb{Z}}(G), \quad \forall\, n \geq 1. \tag{1.46}$$

## 1.4 Integral Dimension Subgroups in Low Dimensions

When the ring of coefficients is $\mathbb{Z}$, the ring of integers, a lot of interesting results regarding dimension subgroups have been proved. This section lists some of them. We first study polynomial groups, which are an important tool in the computation of the former and exhibit the connection between the two.

**Definition 1.4.1** Let $M$ be a monoid and $G$ be an additive Abelian group. A map $f : M \to G$ is called a polynomial map of degree $\leq n$ if the linear extension of $f$ to $\mathbb{Z}M$ vanishes on $\Delta^{n+1}(M)$.

We denote the set of all polynomial maps of degree $\leq n$ from a monoid $M$ to an Abelian group $G$ is denoted by $P_n(M, G)$. If $f : M \to G$ is given by $f(x) = f_1(x) + f_2(x)$, $x \in M$, then the map $f$ is in $P_n(M, G)$.

For every $n \geq 1$, the map $\lambda_n : M \to \mathbb{Z}M/\Delta^{n+1}(M)$ defined by

$$\lambda_n(x) = x + \Delta^{n+1}(M)\ x \in M$$

is a polynomial map of degree $\leq n$. For every polynomial map $f : M \to G$ of degree $\leq n$, there exists a unique homomorphism $\phi : \mathbb{Z}M/\Delta^{n+1}(M) \to G$ such that $f = \phi \circ \lambda_n$. Thus, the map $\lambda_n$ is the universal polynomial map in this sense. This gives us that

$$P_n(M, G) \cong Hom(\mathbb{Z}M/\Delta^{n+1}(M), G).$$

The relationship between dimension subgroups and polynomial groups is given in the form of the next theorem.

**Theorem 1.4.2.** *[Pas68a] For every group $G$ and integer $n \geq 0$,*

$$D_{n+1}(G) = \{x \in G | \phi(x) = \phi(1) \forall \phi \in P_n(G, \mathbb{Q}/\mathbb{Z})\}. \tag{1.47}$$

*Proof.* Let $\delta_{n+1}(G) = \{x \in G | \phi(x) = \phi(1) \forall \phi \in P_n(G, \mathbb{Q}/\mathbb{Z})\}$. Let $y \in D_{n+1}(G)$. Then, $y - 1 \in \Delta^{n+1}(G)$ and hence, since $\phi$ is a polynomial map of degree $\leq n$, $\phi(y-1) = 0$, i.e. $\phi(y) = \phi(1)$. Thus, $D_{n+1}(G) \subseteq \delta_{n+1}(G)$.

Conversely, suppose $y \in \delta_{n+1}(G)$. If $y \notin D_{n+1}(G)$, then $y - 1 + \Delta^{n+1}(G)$ is a non-zero element of $\mathbb{Z}(G)/\Delta^{n+1}(G)$. Thus, there exists a homomorphism $\phi : \mathbb{Z}(G)/\Delta^{n+1}(G) \to T$ such that $\phi(y - 1 + \Delta^{n+1}(G)) \neq 0$. Let us consider the map $\eta : G \to T$ defined as $\eta(g) = \phi(g - 1 + \Delta^{n+1})$. Then, clearly, $\eta$ is a polynomial map of degree $\leq n$. By the choice of $\phi$, we have $\eta(y) \neq \eta(1)$. But, since $y \in \delta_{n+1}(G)$, we have $\eta(y) = \eta(1)$, a contradiction!

Hence $D_{n+1}(G) = \delta_{n+1}(G)$. □

The following theorem states that to compute the integral dimension subgroups by induction on the class of a group $G$, we are faced with extending homomorphisms from the last non-identity term in the lower central series to polynomial maps on the whole group.

**Theorem 1.4.3.** *([Pas68a]) Let $G$ be a nilpotent group of class $n$. then, $\gamma_n(G) \cap D_{n+1}(G) = 1$ if and only if every homomorphism $f : \gamma_n(G) \to \mathbb{Q}/\mathbb{Z}$ can be extended to a polynomial map $\theta : G \to \mathbb{Q}/\mathbb{Z}$ of degree $\leq n$.*

Thus polynomial maps have been effective in computation of integral dimension subgroups. The following results can be obtained using polynomial maps.

**Theorem 1.4.4.** *[Pas68a] For every group $G$, $D_2(G) = \gamma_2(G)$.*

*Proof.* It suffices to prove that for an Abelian group $G$, $D_2(G) = 1$. Let $1 \neq x \in D_2(G)$. let $H = \langle x \rangle$, the subgroup generated by $x$. Then, we can find a non-trivial homomorphism $f : H \to \mathbb{Q}/\mathbb{Z}$. now, since $T = \mathbb{Q}/\mathbb{Z}$ is a divisible Abelian group, $f$ can be extended to a homomorphism $f' :\to T$. Since $G$ is Abelian, it can be easily seen that $f'$ vanishes on $\Delta^2(G)$. Thus, $f'(x-1) = 0$. But, $f'(x-1) = f'(x) - f'(1) = f(x) \neq 0$, which is a contradiction. Hence $D_2(G) = 1$. □

In view of the above results, we note that polynomial maps play an important role in the computation of dimension subgroups. This connection is a motivation for the next chapter where we shall study the groups associated with these maps, known as polynomial groups.

The following result gives us the third integral dimension subgroup and is independently due to G. Higman and D. Rees independently.

**Theorem 1.4.5.** *(see [Pas79]) For every group $G$, $D_3(G) = \gamma_3(G)$.*

The next result states that the dimension conjecture holds for all $p$-groups, where $p$ is an odd prime.

**Theorem 1.4.6.** *[Pas68a] If $G$ is a $p$-group, where $p$ is a prime other than 2, then*

$$D_4(G) = \gamma_4(G). \tag{1.48}$$

In fact, for all groups $G$, we have the following result which shows that there is an upper bound to the exponent of the group $D_4(G)/\gamma_4(G)$. It was proved by G. Losey, J. Sjögren and K. Tahara.

**Theorem 1.4.7.** *(see [Pas79]) For every group $G$, $D_4(G)/\gamma_4(G)$ has exponent at most 2.*

This result shows that though the dimension subgroups $D_n(G)$ are the same as the lower central terms $\gamma_n(G)$ for $n = 1, 2, 3$, it may fail to be the same for $n = 4$ when $G$ is a 2-group. The dimension conjecture was ultimately refuted by Rips [Rip72] who constructed a 2-group $G$ of order $2^{38}$ such that $\gamma_4(G) = 1$ but $D_4(G) \neq 1$.

The fourth and fifth dimension subgroups were computed by Tahara for all finite groups. Though the dimension subgroups may differ from the lower central terms, there is an upper bound to the exponent of the groups $D_n(G)/\gamma_n(G)$ for all groups $G$. This was proved by Sjögren in form of the next result.

**Theorem 1.4.8.** *[Sjö79] Let $b(m) = lcm\{1, 2, \ldots, m\}$, $c(1) = c(2) = 1$, $c(n) = \prod_{k=1}^{n-2} b(k)^{\binom{n-2}{k}}$, $n \geq 3$. Then, for every group $G$,*

$$D_n(G)^{c(n)} \subseteq \gamma_n(G), \quad n \geq 1. \tag{1.49}$$

Several examples of groups without the dimension property, i.e., groups where the lower central series does not coincide with the dimension series, are presented in [MP09].

# Chapter 2

# Augmentation Quotients

In the last section of the previous chapter, we saw a connection between polynomial groups and integral dimension subgroups. The group $Q_n(G)$ is termed as the $n^{\text{th}}$ augmentation quotient for the group ring $\mathbb{Z}(G)$, and has been studied extensively. $Q_n$, for $n \geq 1$ can also be considered as functors from the category of abelian groups, $\mathcal{A}$ to itself. Some of these functorial properties have been studied in [Pas69]. The groups $Q_n(G)$ for cyclic and elementary Abelian groups were computed by Passi [Pas68b]. In [Kar83], Karpilovsky raised the problem of computing $Q_n(G)$ for $n \geq 1$. Bachmann and Grünenfelder proved that for all finite groups, the sequence $\{Q_n(G)\}_{n\geq 1}$ of augmentation quotients becomes periodic [BG74]. In particular, for finite Abelian groups, this sequence is stationary. This eventual isomorphism type was given by Hales [Hal85] in terms of generators and relations. Motivated by his result, Chang and Tang [CT11] solved Karpilovsky's problem for finite Abelian Groups. They completely described the groups $Q_n(G)$ and gave an explicit basis for them for all $n \geq 1$.

The main aim of this chapter is an exposition of the results given in [Hal85] and [CT11]. We end with some results concerning augmentation quotients for non-Abelian groups.

## 2.1 Polynomial Groups

For an integral group ring $\mathbb{Z}G$, we have the following filtration of the augmentation ideal

$$\Delta(G) \supseteq \Delta^2(G) \supseteq \ldots \supseteq \Delta^n(G) \supseteq \ldots$$

Using the above sequence, we can define the following two sequences of abelian groups

$$P_n(G) = \mathbb{Z}G/\Delta^{n+1}(G) \tag{2.1}$$

$$Q_n(G) = \Delta^n(G)/\Delta^{n+1}(G). \tag{2.2}$$

These are known as polynomial groups. The group $Q_n(G)$ is called the $n^{th}$ augmentation quotient. They are described in some detail in the subsequent chapters. The polynomial groups of cyclic and elementary Abelian groups were computed by Passi using their correspondence with certain polynomial rings [Pas68b]. It was evident from his results that the computation of the group $P_n(G)$ is in general a more difficult problem than that of $Q_n(G)$. In this section, we briefly study these results.

We first give some simple identities in $\mathbb{Z}G$ which will be used throughout the report. Let $g, h \in G$. Then,

$$(gh - 1) = (g - 1) + (h - 1) + (g - 1)(h - 1) \tag{2.3}$$

Also, if an element $g \in G$ is of exponent $r$, then using binomial theorem, we get

$$r(g - 1) \in \Delta^2(G). \tag{2.4}$$

**Remark 2.1.1** (2.4) shows that an exponent of $G$ is an exponent of $Q_n(G)$.

Now, we demonstrate the computation of $Q_n(G)$ for a cyclic group $G$. First, let $G$ be a finite cyclic group of order $m$ and let $a$ be a genrator of $G$. Then, under the correspondence $a \leftrightarrow X$, the group ring $\mathbb{Z}G \cong \mathbb{Z}[X]/\langle X^m - 1\rangle$, where $\langle \alpha \rangle$ denotes the ideal generated by $\alpha$ in $\mathbb{Z}G$. Let $A = \langle X^m - 1\rangle$. Then, we have

$$\Delta(G) = \langle X - 1\rangle/A, \quad \Delta^n(G) = \langle (X - 1)^n\rangle + A/A. \tag{2.5}$$

If $G$ is the infinite cyclic group, then we have $\mathbb{Z}G = \mathbb{Z}[X, Y]/B$ where $B = \langle XY - 1\rangle$. Clearly, $\Delta G = \langle X - 1, Y - 1\rangle/B$. In fact, the equation

$$XY - 1 = (X - 1) + (Y - 1) + (X - 1)(Y - 1) \tag{2.6}$$

in $\mathbb{Z}[X,Y]$ shows that

$$\Delta(G) = \langle X-1\rangle + B/B, \quad \Delta^n(G) = \langle (X-1)^n\rangle + B/B. \tag{2.7}$$

These correspondences are used in proving the following result.

**Theorem 2.1.2.** *[Pas68b] If $G$ is a cyclic group, then $Q_n(G) = G \,\forall\, n \geq 1$.*

*Proof.* For the integral group ring $\mathbb{Z}G$, it is well known that $\Delta G/\Delta^2(G) = G/G'$ (see [HS71, p. 192]), where $G'$ denotes the commutator subgroup of $G$.

Now, if $G$ is cyclic, clearly $G/G' = G$. Hence, in our case, $Q_1(G) = G$. The result will hence be proved if we establish an isomorphism between $Q_n(G)$ and $Q_1(G)$ for all $n \geq 2$.

Firstly, let $G$ be a finite cyclic group, say of order $m$. In view of (2.5), for any $n \geq 2$, we have the homomorphism

$$\alpha : \Delta(G) \to \Delta^n(G) \tag{2.8}$$

given by

$$\alpha((X-1)f(X) + A) = (X-1)^n f(X) + A \tag{2.9}$$

where $f(X) \in \mathbb{Z}[X]$. Now, $\alpha((X-1)^n f(X) + A) \in \Delta^{n+1}(G)$ if and only if there exist polynomials $g(X)$ and $h(X)$ in $\mathbb{Z}[X]$ such that

$$(X-1)^n f(X) = (X-1)^{n+1} g(X) + (X^m - 1)h(X) \tag{2.10}$$

that is possible if and only if

$$(X-1)f(X) = (X-1)^2 g(X) + (X^m - 1)h'(X) \tag{2.11}$$

for some polynomial $h'(X) \in \mathbb{Z}[X]$. In view of 2.5, this happens if and only if

$$(X-1)f(X) + A \in \Delta^2(G). \tag{2.12}$$

Thus, $\alpha$ induces a monomorphism

$$\alpha' : \Delta(G)/\Delta^2(G) \to \Delta^n(G)/\Delta^{n+1}(G). \tag{2.13}$$

Since $\alpha$ is an epimorphism, so is $\alpha'$. Hence, $\alpha'$ is an isomorphism.

Now, suppose $G$ is the infinite cyclic group. Using (2.7) and proceeding similar to the previous case, we define a homomorphism $\alpha : \Delta(G) \to \Delta^n(G)$ and we get that

$$\Delta(G)/\Delta^2(G) \cong \Delta^n(G)/\Delta^{n+1}(G) \quad n \geq 2. \tag{2.14}$$

Thus, $Q_n(G) = G$ for all cyclic groups $G$. □

The groups $P_n(G)$ are more complicated to compute and require more machinery. We first state the following result which reduces the problem from arbitrary finite Abelian groups to finite Abelian $p$-groups.

**Theorem 2.1.3.** *[Pas68b] If $G$ and $H$ are groups, $s$ and $t$ integers, $(s,t) = 1$ such that $x^s = 1$ and $y^t = 1$ for all $x \in G$ and $y \in H$, then*

1. *$P_n(G \oplus H) \cong P_n(G) \oplus P_n(H)$,*

2. *$Q_n(G \oplus H) \cong Q_n(G) \oplus Q_n(H)$.*

*Proof.* We consider the canonical projections $f_1 : G \oplus H \to G$ and $f_2 : G \oplus H \to H$ and extend them naturally first to the integral group rings and then to the corresponding polynomial groups, $P_n$ and $Q_n$. We shall use the same notation for the induced maps.

We can combine both maps to get a homomorphism $\theta : P_n(G \oplus H) \to P_n(G) \oplus P_n(H)$ given by $\theta(z) = f_1(z) + f_2(z)$. It is clear that $\theta$ is an epimorphism.

A general element $z$ of $P_n(G \oplus H)$ is of the form

$$\begin{aligned} z &= \sum_{x \in G, y \in H, \alpha \in \mathbb{Z}} \alpha(xy - 1) + \Delta^{n+1}(G \oplus H) \\ &= \sum \alpha((x-1) + (y-1) + (x-1)(y-1)) + \Delta^{n+1}(G \oplus H). \end{aligned} \tag{2.15}$$

Now, $\theta(z) = 0$ implies that $\sum \alpha(x-1) \in \Delta^{n+1}(G)$ and $\sum \alpha(y-1) \in \Delta^{n+1}(H)$. Also, $s(x-1) \in \Delta^2(G)$ implies that $s^{n-1}\Delta(G) \subseteq \Delta^n(G)$. Similarly, $t^{n-1}\Delta(H) \subseteq \Delta^n(H)$. Since $s$ and $t$ are coprime, $\exists\, a, b \in \mathbb{Z}$ such that $s^{n-1}a + t^{n-1}b = 1$. Thus, we can write

$$(x-1)(y-1) = as^{n-1}(x-1)(y-1) + b(x-1)t^{n-1}(y-1). \tag{2.16}$$

Thus $(x-1)(y-1) \in \Delta^{n+1}(G \oplus H)$ and hence the element $z$ is 0, proving that $\theta$ is an isomorphism.

On considering the restriction to $Q_n(G \oplus H)$ of the map $\theta$ and combining them as earlier, we get a homomorphism

$$\phi : Q_n(G \oplus H) \to Q_n(G) \oplus Q_n(H) \tag{2.17}$$

which as above can be proved to be an isomorphism. □

To compute the groups $P_n(G)$, for a finite cyclic group $G$, we again establish a relation between the integral group ring $\mathbb{Z}G$ and the polynomial ring $\mathbb{Z}[X]$. In view of Theorem 2.1.3, we only need to consider the case of a cyclic group $G$ of prime power order, say $p^m$, for $p$ prime.

The elements of $\mathbb{Z}G$ can be written as sums of the form

$$\sum_{i=0}^{p^m-1} \alpha_i x^i \tag{2.18}$$

where $\alpha_i \in \mathbb{Z}$ for $0 \le i < p^m$ and $x$ is a generator of $G$.

Let $C$ denote the ideal $\langle \sum_{i=1}^{p^m} \binom{p^m}{i} X^i \rangle$. The map

$$\theta : \mathbb{Z}G \to \mathbb{Z}[X]/C \tag{2.19}$$

defined by

$$\theta\left(\sum_{i=0}^{p^m-1} \alpha_i x^i\right) = \sum_{i=0}^{p^m-1} \alpha_i (1+X)^i + C \tag{2.20}$$

is a ring isomorphism. It maps the ideal $\Delta(G)$ onto the ideal $\langle X \rangle / C$.

Thus, $\theta$ maps the ideal $\Delta^{n+1}(G)$ onto $\langle X^{n+1}, \sum_{i=1}^{p^m} \binom{p^m}{i} X^i \rangle / C$. So, we have an isomorphism

$$\Delta(G)/\Delta^{n+1}(G) \cong \langle X \rangle / \langle X^{n+1}, \sum_{i=1}^{p^m} \binom{p^m}{i} X^i \rangle. \tag{2.21}$$

For $r, t \in \mathbb{N}$, Let ${\mathbb{Z}_r}^{(t)}$ denote the direct sum of $t$ copies of the cyclic group $\mathbb{Z}_r$. Using the above correspondence and a series of computations, the following result was proved by Passi.

**Theorem 2.1.4.** *[Pas68b] If $m$ and $n$ are integers $\ge 1$ and $p$ is a prime, then*

$$P_n(\mathbb{Z}_{p^m}) \cong \mathbb{Z}^{(r_1)}_{p^{m+q_1}} \oplus \mathbb{Z}^{(p-1-r_1)}_{p^{m+q_1-1}} \oplus \ldots \oplus \mathbb{Z}^{(r_m)}_{p^{q_m+1}} \oplus \mathbb{Z}^{(p^m-p^{m-1}-r_m)}_{p^{q_m}} \tag{2.22}$$

*for* $n > p^{m-1} - 1$, *and*

$$\begin{aligned} P_n(\mathbb{Z}_{p^m}) \cong \mathbb{Z}_{p^{m+q_1}}^{(r_1)} \oplus \mathbb{Z}_{p^{m+q_1-1}}^{(p-1-r_1)} \oplus \ldots \oplus \mathbb{Z}_{p^{m-(s-1)+q_{s-1}+1}}^{(r_{s-1})} \\ \oplus \mathbb{Z}_{p^{m-(s-1)+q_{s-1}}}^{(p^{s-1}-p^{s-2}-r_{s-1})} \oplus \mathbb{Z}_{p^{m-s+1}}^{(n-p^{s-1}+1)} \end{aligned} \tag{2.23}$$

*where* $p^{s-1} - 1 < n \leq p^s - 1$, $1 \leq s \leq m-1$, $q_i$ *and* $r_i$, $1 \leq i \leq m$ *are integers satisfying*

$$n - p^{i-1} + 1 = (p^i - p^{i-1})q_i + r_i,\ 0 \leq r_i < p^i - p^{i-1}. \tag{2.24}$$

The groups $P_n(\mathbb{Z}_r)$ for an arbitrary $r$ can now be obtained by using Theorem 2.1.3 and Theorem 2.1.4.

Next, for elementary Abelian $p$-groups ${\mathbb{Z}_p}^{(t)}$, we have the following theorem which gives an inductive formula for $P_n({\mathbb{Z}_p}^{(m)})$, for any $m \geq 1$.

**Theorem 2.1.5.** *[Pas68b] For any* $m \geq 1$,

$$P_n(\mathbb{Z}_p^{(m)}) \cong P_n(G) \oplus \ldots \oplus P_{n-p+1}(G) \oplus P_n(\mathbb{Z}_p) \tag{2.25}$$

*where* $G$ *is the group* $\mathbb{Z}_p^{(m-1)}$.

Once the structure of $P_n(\mathbb{Z}_p^{(m)})$ is known, the structure of the groups $Q_n(\mathbb{Z}_p^{(m)})$ can be easily determined. By Remark 2.1.1, $Q_n(\mathbb{Z}_p^{(m)})$ are of exponent $p$ and hence it suffices to know the order of these groups. Using the exact seqence

$$0 \to Q_n(\mathbb{Z}_p^{(m)}) \to P_n(\mathbb{Z}_p^{(m)}) \to P_{n-1}(\mathbb{Z}_p^{(m)}) \to 0, \tag{2.26}$$

we get

$$|Q_n(\mathbb{Z}_p^{(m)})| = \frac{|P_n(\mathbb{Z}_p^{(m)})|}{|P_{n-1}(\mathbb{Z}_p^{(m)})|} \tag{2.27}$$

and hence the structure of $Q_n(\mathbb{Z}_p^{(m)})$ is determined.

From the above theory, it is easily observed that for $G = \mathbb{Z}_p^{(m)}$, if $n \geq (m-1)(p-1)+1$,

$$Q_n(G) \cong {\mathbb{Z}_p}^{\frac{(p^m-1)}{(p-1)}}. \tag{2.28}$$

## 2.2 Stable Structure of Augmentation Quotients

We observed in the previous section that for an elementary Abelian $p$-group of rank $m$, the groups $Q_n(G)$'s are all isomorphic for $n \geq (m-1)(p-1)+1$. Thus, a natural question arises whether such a result is true for a larger class of groups, say all finite groups. This question was answered by Bachmann and Grünenfelder [BG74] in the form of the following result:

**Theorem 2.2.1.** *[BG74] Let $G$ be a finite group and $c$ be the least integer such that $G_{c+1} = G_{c+2}$, where $G_j$ denotes the $j^{th}$ lower central term of $G$. Let $\bar{c} = lcm\{1, 2, \ldots, c\}$. Then there exist positive integers $n_0 = n_0(G)$ and $\pi = \pi(G)$ such that $\pi \mid \bar{c}$ and $Q_{n+\pi}(G) \cong Q_n(G)\ \forall\, n \geq n_0$.*

For finite nilpotent groups, $c$ is the class of the group. Theorem 2.2.1 says that for a finite Abelian group $G$, the sequence $\{Q_n(G)\}_{n\geq 1}$ becomes stationary for $n \geq n_0$. The type of this eventual isomorphism and the number $n_0$ was determined by Hales [Hal85]. He gave the structure of this group, in terms of generators and relations. In this section, we shall study his result in some detail.

In view of Theorem 2.1.3, it is enough to consider the case of finite Abelian $p$-groups. For a finite Abelian $p$-group $G$, we first define an Abelian group $Q_G$ additively. Let $P$ denote the set of all cyclic subgroups of $G$ partially ordered by inclusion. We say that a subgroup $H$ of $G$ is *over* $K$ if

$$K < H' \leq H \quad \implies H' = H. \tag{2.29}$$

For each cyclic subgroup $H \in P$, we define a generator $x_H$ of $Q_G$ and impose the following relation on the generators:

$$px_H = x_K, \quad K \leq H \tag{2.30}$$

whenever $H$ is over $K$. Set $x_{\{1\}} = 0$. It can be easily seen that if $c = |P|$, then any element $x$ of $Q_G$ can be uniquely written as

$$x = m_1 x_{H_1} + \ldots + m_{c-1} H_{c-1} \quad 0 \leq m_i < p\, \forall\, 1 \leq i \leq c-1. \tag{2.31}$$

Thus, the order of $Q_G$ is $p^{c-1}$. It is also evident from the definition that the minimum number of elements required to generate $Q_G$ is the number of maximal cyclic subgroups of $G$.

**Claim 2.2.2** The number of elements required to generate $p^k Q_G$ is the number of maximal cyclic subgroups of $G^{p^k}$.

*Proof.* The result is clear for $k = 0$. We proceed by induction. Suppose every element $x$ of $p^k Q_G$ can be written as

$$x = a_1 x_{H_1} + \ldots + a_c x_{H_c}, \quad a_i \in \mathbb{Z},\ 1 \leq i \leq c, \tag{2.32}$$

where $c$ is the number of maximal cyclic subgroups of $G^{p^k}$. We observe that if $H$ is a cyclic subgroup of $G$, then it is over $H_p$. Now, using (2.32), we can write every element $y$ of $p^{k+1}Q_G$ as a linear combination

$$\begin{aligned} y &= a_1 p x_{H_1} + \ldots + a_c p x_{H_c} \\ &= a_1 x_{{H_1}^p} + \ldots + a_c x_{{H_c}^p} \end{aligned} \tag{2.33}$$

Now, since $H_j$ is a maximal cyclic subgroup of $G^{p^k}$, $x_{{H_j}^p} \neq 0$ if and only if ${H_j}^p$ is a non-trivial maximal subgroup of $G^{p^{k+1}}$. Thus, by induction hypothesis, we are done. □

The following examples will help clarify the above definition.

**Example 2.2.3** $G = C_{p^2} = \langle a \rangle$. In this case,

$$P = \{H_0 = 1, H_1 = \langle a^p \rangle, H_2 = \langle a \rangle\}. \tag{2.34}$$

$$Q_G = \langle x_{H_1}, x_{H_2} \,:\, p x_{H_1} = 0, p x_{H_2} = x_{H_1} \rangle. \tag{2.35}$$

Thus, any element of $Q_G$ can be written as

$$m_1 x_{H_1} + m_2 x_{H_2}, \quad 0 \leq m_1, m_2, < p. \tag{2.36}$$

Thus, $|Q_G| = p^2 = p^{3-1}$.

**Example 2.2.4** $G = C_{p^n} \times C_p = \langle a \rangle \times \langle b \rangle$. In this case, the cyclic subgroups are generated by the elements $(a^{p^i}, 1)$, for $0 \leq i \leq n$, $(1, b)$ and $(a^{p^{n-1}}, b)$. Thus, the number of cyclic subgroups is $n + 1 + 1 + 1 = n + 3$. Thus, the order of $|Q_G|$ is $p^{n+2}$.

Let $Q_\infty(G)$ denote the type of eventual isomorphism type of the groups $Q_n(G)$, $n \geq n_0$. The following theorem is the main aim of this section:

**Theorem 2.2.5.** *[Hal85] For a finite Abelian p-group G, $Q_\infty(G) \cong Q_G$.*

We first state the following two lemmas proved by Hales [Hal85] which will be used subsequently.

**Lemma 2.2.6.** *Let $G = \langle g \rangle \times \langle h \rangle$, where $g$ and $h$ have orders $p^a$ and $p^b$ respectively, where $a \geq b$. Let $x = g - 1$ and $y = h - 1$. Then, for each $0 \leq k < b$,*

$$p^k y^{(a-b)(p^{b-k}-p^{b-k-1})}(x^{p^{b-k}} y^{p^{b-k-1}} - x^{p^{b-k-1}} y^{p^{b-k}}) \in \Delta^l(G), \tag{2.37}$$

*where*

$$l = (a-b)(p^{b-k} - p^{b-k-1}) + p^{b-k} + p^{b-k-1} + 1. \tag{2.38}$$

Let $G = \langle g \rangle \times \langle h \rangle \times \langle a_1 \rangle \ldots \times \langle a_r \rangle$, with $g$ and $h$ as in Lemma 2.2.6 and $a_i$ having order $p^{e_i}$ for each $i$. Let $x = g - 1$, $y = h - 1$ and $z_i = a_i - 1$ for $1 \leq i \leq r$. Let

$$X_{m,n,k} = y^{(a-b)(p^{b-k}-p^{b-k-1})}(x^{p^{b-k}} y^{p^{b-k-1}} - x^{p^{b-k-1}} y^{p^{b-k}}). \tag{2.39}$$

**Lemma 2.2.7.** *Suppose $b = e_0 > e_1 > \ldots e_r$ and for each $1 \leq i \leq r$, $k_i$ is a positive integer such that $e_i + k_i \leq e_{i-1}$. Suppose $k \geq 0$ and define $d_i = e_i - \sum_{j=i+1}^{r} k_j - k$ and $s_i = p^{d_i - 1} + k_i(p^{d_i} - p^{d_i - 1})$. Let $q = \sum_{i=1}^{r} k_i + k$. Then,*

$$p^k X_{m,n,k} {z_1}^{s_1} \ldots {z_r}^{s_r} \in \Delta^l(G), \tag{2.40}$$

*where*

$$l = (a-b)(p^{b-q} - p^{b-q-1}) + p^{b-q} + p^{b-q-1} + \sum_{i=1}^{r} s_i + 1. \tag{2.41}$$

Now, we move to describe the proof of Theorem 2.2.5.

It was proved by Singer [Sin77] that the order of the group $Q_\infty(G)$ is $p^{c-1}$, where $c$ is the number of cyclic subgroups of $G$. Thus, $|Q_\infty(G)| = |Q_G|$. Since both of these are finite Abelian $p$-groups, to prove Theorem 2.2.5, it is enough to show that their Ulm invariants are the same [Ulm33]. This is done once we show that for large $n$, $p^k Q_n(G)$ can be generated by the number of elements required to generate $p^k Q_G$. In view of Claim 2.2.2, we are done if we prove that there is a set of elements generating $p^k Q_n(G)$ having cardinality equal to the number of maximal cyclic subgroups of $G^{p^k}$. We shall do this by finding a partition of $\mathcal{C}(G^{p^k})$, the set of maximal cyclic subgroups of $G^{p^k}$. We shall then find a collection of subsets of $p^k Q_n(G)$ with the same cardinalities as the corresponding subsets of $\mathcal{C}(G^{p^k})$ respectively. Finally, we shall show that the elements of these sets generate $p^k Q_n(G)$. Let

$$G = \langle g_1 \rangle \times \langle g_2 \rangle \times \ldots \times \langle g_m \rangle \tag{2.42}$$

where $g_i$ has order $p^{e_i}$ and

$$e_1 \geq e_2 \geq \ldots \geq e_m. \tag{2.43}$$

For each $0 \leq i \leq e_1 - e_m$, define $m_i$ such that $m_i < m$ is maximal with the property

$$e_{m_i} \geq e_m + i. \tag{2.44}$$

For instance, $m_0 = m - 1$. Set

$$G_i = \langle g_1 \rangle \times \langle g_2 \rangle \times \ldots \times \langle g_{m_i} \rangle. \tag{2.45}$$

Define $x_i = g_i - 1$ for $1 \leq i \leq m$. The proof of Theorem 2.2.5 is presented step-wise.

**Step 1:** Partition of $\mathcal{C}(G)$.
Let $\langle ({g_1}^{\beta_1} {g_2}^{\beta_2} \ldots {g_m}^{\beta_m}) \rangle$ denote the cyclic subgroup generated by the element $({g_1}^{\beta_1} {g_2}^{\beta_2} \ldots {g_m}^{\beta_m})$ of $G$. Let $o(h)$ denote the order of an element $h$. Let $\beta_1, \beta_2, \ldots, \beta_m$ be such that $\langle ({g_1}^{\beta_1} {g_2}^{\beta_2} \ldots {g_m}^{\beta_m}) \rangle$ is a maximal cyclic subgroup of $G$. The set $\mathcal{C}(G)$ can be partitioned into $e_1 - e_m + 2$ classes as

$$\mathcal{C}(G) = S \cup T_0 \cup T_1 \cup \ldots \cup T_{e_1 - e_m}, \tag{2.46}$$

where $S$ and $T_i$ are defined inductively as follows:

$$S = \{ \langle ({g_1}^{\beta_1} {g_2}^{\beta_2} \ldots {g_m}^{\beta_m}) \rangle \mid o({g_j}^{\beta_j}) < p^{e_m} \; \forall j < m \}, \tag{2.47}$$

and, for $0 \leq i \leq e_1 - e_m$,

$$\begin{aligned} T_i = \{ \langle ({g_1}^{\beta_1} {g_2}^{\beta_2} \ldots {g_m}^{\beta_m}) \rangle \mid o({g_j}^{\beta_j}) < p^{e_m} \, \forall \, m_i < j < m, \\ \langle ({g_1}^{\beta_1} {g_2}^{\beta_2} \ldots {g_{m_i}}^{\beta_{m_i}}) \rangle \in \mathcal{C}({G_i}^{p^i}) \}. \end{aligned} \tag{2.48}$$

It is clear that (2.46) is a partition of $\mathcal{C}(G)$.

**Step 2:** Computation of $|T_i|$ and $|S|$.
It is clear from the previous step that

$$|T_0| = p^{e_m} |\mathcal{C}(G_0)|. \tag{2.49}$$

For $1 \le i \le e_1 - e_m$,

$$|T_i| = (p^{e_m} - p^{e_m-1}) p^{(e_m-1)(m-m_i-1)} |\mathcal{C}(G_i{}^{p^i})|. \tag{2.50}$$

$$|S| = (p^{e_m-1})^{m-1}. \tag{2.51}$$

**Step 3:** Construction of subsets of $Q_n(G)$.
Let

$$T_G(n) = \{\alpha = (\alpha_1, \ldots, \alpha_m) | \sum_j \alpha_j = n\}. \tag{2.52}$$

We set

$$x^\alpha = x_1{}^{\alpha_1} \ldots x_m{}^{\alpha_m}. \tag{2.53}$$

We shall denote the element $x^\alpha + \Delta^{n+1}(G)$ of $Q_n(G)$ by its representative $x^\alpha$. Then, $Q_n(G)$ is generated by all monomials of the form $x^\alpha$, where $\alpha \in T_G(n)$. Let $\mathcal{S}(n, G)$ denote a minimal set of generators for $Q_n(G)$. Let $\alpha \in T_G(n)$. We define the sets $C_i$ for $0 \le i \le e_1 - e_m$ and $D(n)$ inductively, induction being on $m$.

$$C_0(n) = \{x^\alpha \,:\, 0 < \alpha_m < p^{e_m}, \prod_{j=1}^{m-1} x_j{}^{\alpha_j} \in \mathcal{S}(n - \alpha_m, G_0)\}. \tag{2.54}$$

For $1 \le i \le e_1 - e_m$,

$$\begin{aligned} C_i(n) = \{x^\alpha \,|\, p^{e_m-1} + i(p^{e_m} - p^{e_m-1}) \le \alpha_m < p^{e_m-1} + (i+1)(p^{e_m} - p^{e_m-1}), \\ \alpha_j < p^{e_m-1} \,\forall\, m_i < j < m, \prod_{j=1}^{m_i} x_j{}^{\alpha_j} \in \mathcal{S}(n - \sum_{j=m_i+1}^{m} \alpha_j, G_i^{p^i})\}. \end{aligned} \tag{2.55}$$

$$D(n) = \{x^\alpha \,:\, \alpha_j < p^{e_m-1} \,\forall\, j < m\}. \tag{2.56}$$

**Step 4:** Computation of $|D(n)|$ and $|C_i(n)|$, for $0 \le i \le e_1 - e_m$.
Since $\alpha \in T_G(n)$, $\alpha_m$ is fixed once we fix $\alpha_1, \ldots, \alpha_{m-1}$. Hence, by the definition of $D(n)$, we get that

$$|D(n)| = (p^{e_m-1})^{m-1}. \tag{2.57}$$

Similarly,

$$|C_0(n)| = p^{e_m} |\mathcal{S}(n - \alpha_m, G_0)|. \tag{2.58}$$

Now, consider the sets $C_i$ for $1 \leq i \leq e_1 - e_m$. We have

$$|C_i(n)| = (p^{e_m} - p^{e_m - 1}) p^{(e_m - 1)(m - m_i - 1)} |\mathcal{S}(n - \sum_{j=m_i+1}^{m} \alpha_j, G_i^{p^i})|. \tag{2.59}$$

**Step 5:** The sets $C_i, 0 \leq i \leq e_1 - e_m$ and $D$ are disjoint for large $n$.
For $m = 1$, the statement is trivial. The proof proceeds by induction. Suppose $n'$ is such that the result is true for $G_0$. In view of (2.54), we only need to choose $n$ such that $n - \alpha_m \geq n'$. Since $\alpha_m \leq p^{e_m} - 1$, any $n \geq n' + p^{e_m} - 1$ suffices for $G$.

**Step 6:** $|\mathcal{S}(n, G)| = |\mathcal{C}(G)|$.
By induction hypothesis, we have $|\mathcal{S}(n - \alpha_m, G_0)| = |\mathcal{C}(G_0)|$ and $|\mathcal{S}(n - \sum_{j=m_i+1}^{m}, G_i^{p^i})| = |\mathcal{C}(G_i^{p^i})|$. Thus, from steps 2, 4 and 5, we get

$$|T_i| = |C_i|, \quad 0 \leq i \leq e_1 - e_m, \tag{2.60}$$

$$|S| = |D|. \tag{2.61}$$

**Step 7:** $\cup_{i=0}^{e_1 - e_m} C_i(n) \cup D$ generates $Q_n(G)$.
We shall show that every element $x^\alpha = {x_1}^{\alpha_1} \dots {x_m}^{\alpha_m}$ can be written as a $\mathbb{Z}$-linear combination of elements of $C_i$'s and $D$.
Let $x^\alpha$ be an element of $Q_n(G)$ for large $n$ which cannot be written as a linear combination of elements of $C_i$ and $D$. We pick $\alpha = (\alpha_1, \dots, \alpha_m)$ such that it is lexicographically greatest with this property. Then, we have three cases:

**Case I:** $\alpha_m < p^{e_m}$.
By induction, $\prod_{j=1}^{m-1} {x_j}^{\alpha_j}$ is a linear combination of elements of $\mathcal{S}(n - \alpha_m, G_0)$. Let

$$\prod_{j=1}^{m-1} {x_j}^{\alpha_j} = \sum_k r_k x(k) \tag{2.62}$$

where $x(k) \in \mathcal{S}(n - \alpha_m, G_0)$. On multiplying both sides by ${x_m}^{\alpha_m}$, we get that $x^\alpha$ can be written as

$$x^\alpha = \prod_{j=1}^{m} {x_j}^{\alpha_j} = \sum_k r_k x(k) {x_m}^{\alpha_m}. \tag{2.63}$$

By definition of $C_0$, each $x(k){x_m}^{\alpha_m}$ is in $C_0$ and we are done in this case.

**Case II:** $p^{e_m-1} + i(p^{e_m} - p^{e_m-1}) \leq \alpha_m < p^{e_m-1} + (i+1)(p^{e_m} - p^{e_m-1})$.
If for any $m_i < j < m$, $\alpha_j$ satisfies $\alpha_j \geq p^{e_m-1}$, then by Lemma 2.2.6, we can replace $\alpha_j$ by $\alpha_j + (p^{e_m} - p^{e_m-1})$ and $\alpha_m$ by $\alpha_m - (p^{e_m} - p^{e_m-1})$ without changing its class modulo $\Delta^{n+1}(G)$. This contradicts the maximality of $\alpha$ and hence $\alpha_j < p^{e_m-1}$ for $m_i < j < m$. Also, by induction, the element $\prod_{j=1}^{m_i} {x_j}^{\alpha_j}$ of $Q_{n-\sum_{j=m_i+1}^{m}\alpha_j}(G_i^{p^i})$ can be written as a linear combination of elements of $\mathcal{S}(n - \sum_{j=m_i+1}^{m} \alpha_j, G_i^{p^i})$. This shows that the element $x_1^{\alpha_1} \ldots x_m^{\alpha_m}$ is a linear combination of elements of $C_i$.

**Case III:** $\alpha_m \geq p^{e_m-1} + (i+1)(p^{e_m} - p^{e_m-1})$.
We proceed similar to the second case and use Lemma 2.2.6 to conclude that $\alpha_j < p^{e_m-1}$ for all $j < m$ and hence $x^\alpha \in D$.

In view of Steps 1-7, we get that for large $n$, $Q_n(G)$ can be generated by a set of cardinality equal to $|\mathcal{C}(G)|$. Proceeding in a similar fashion as above and using Lemma 2.2.7 for that $k$, we can show that the number of elements required to generate $p^k Q_n(G)$ is equal to $|\mathcal{C}(G^{p^k})|$. Now, in view of Claim 2.2.2, the proof of Theorem 2.2.5 is complete.

**Corollary 2.2.8.** *$Q_n(G)$ is isomorphic to $Q_\infty(G)$ if and only if*

$$n \geq (e_1 - e_2)(p^{e_2} - p^{e_2-1}) + p^{e_2-1} + \sum_{j=2}^{m}(p^{e_j} - 1). \tag{2.64}$$

*Proof.* Let

$$U = U(n) = \cup_{i=0}^{e_1-e_m} C_i \cup D. \tag{2.65}$$

By the above discussion, for $n \geq n_0 = (e_1 - e_2)(p^{e_2} - p^{e_2-1}) + p^{e_2-1} + \sum_{j=2}^{m}(p^{e_j} - 1)$, the sets $C_i$ and $D$ are disjoint. If possible, let

$$\sum_\alpha r_\alpha x^\alpha = 0 \tag{2.66}$$

be a linear relation not implied by Lemma 2.2.6 in the set $U(n)$. Then, by multiplying this relation with $x_j^m$ for a suitable value of $j$, we can convert (2.66) into a linear relation in the elements of the set $U(n+m)$ which form a generating set for $Q_{n+m}(G)$. If $m$ is large, $Q_{n+m}(G) \cong Q_\infty(G)$ and hence we get a relation in $Q_\infty(G)$. But no relation not implied by Lemma 2.2.6 can hold in $Q_\infty(G)$, and hence in $Q_n(G)$ for any $n$.

For $k \geq 1$, we use Lemma 2.2.7 for the corresponding $k$ and a similar argument as above which concludes the proof of the result. □

## 2.3 Finite Abelian Groups

The augmentation quotients for finite Abelian groups were computed by Chang and Tang [CT11]. They also gave an explicit basis for the same. In this section, we present their result in some detail.

Using the ideas of the previous section, it is clear that we only need to find a minimal generating set for $p^k Q_n(G)$ for $1 \leq k < e_1$. We shall do this by considering all generators of $Q_n(G)$ and then defining an equivalence relation on them for each $k$ such that any two equivalent elements represent the same element of $p^k Q_n(G)$. A set defined using at most one element from each equivalence class will give us a minimal generating set for $p^k Q_n(G)$.

We set $G$ as in (2.42) i.e.,

$$G = \langle g_1 \rangle \times \langle g_2 \rangle \times \ldots \times \langle g_m \rangle \tag{2.67}$$

where $g_i$ has order $p^{e_i}$ and

$$e_1 \geq e_2 \geq \ldots \geq e_m. \tag{2.68}$$

Let $T_G(n)$ denote the set of $m$-tuples as in previous section. To find a minimal generating set for $p^k Q_n(G)$, we shall find subsets of $T_G(n)$, denoted by $T_G(k, n)$ such that every generator $p^k x^\alpha$ of $p^k Q_n(G)$ is such that $\alpha \in T_G(k, n)$.

Any element of $Q_n(G)$ can be written as a linear combination of elements of the form

$$x^\alpha = {x_1}^{\alpha_1} \ldots {x_m}^{\alpha_m}. \tag{2.69}$$

Instead of partitioning the set of monomials, we shall work with the set

$$T_G(n) = \{\alpha = (\alpha_1, \ldots, \alpha_m) | \sum_{j=1}^{m} \alpha_j = n\} \tag{2.70}$$

of $m$-tuples of integers, as for any $m$-tuple $\alpha$, we can associate the element

$$x^\alpha = {x_1}^{\alpha_1} \ldots {x_m}^{\alpha_m} \tag{2.71}$$

of $\Delta^n(G)$. We shall write $x^\alpha$ for the element $x^\alpha + \Delta^{n+1}(G)$ of $Q_n(G)$, whenever there is no scope of confusion.

We define the set

$$\mathcal{T}_G(n) = \{x^\alpha + \Delta^{n+1} | \alpha \in T_G(n)\}. \tag{2.72}$$

**Remark 2.3.1** It is clear that $Q_n(G)$ is generated by $\mathcal{T}_G(n)$.

We shall define a subset $T_G(k, n)$ of $T_G(n)$ for each $0 \leq k < e_1$ such that the set $p^k \mathcal{T}_G(k, n)$, where

$$\mathcal{T}_G(k, n) = \{x^\alpha + \Delta^{n+1}(G) | \alpha \in T_G(k, n)\}, \tag{2.73}$$

is a minimal generating set of $p^k Q_n(G)$.

For the same, we now define a series of equivalence relations $\sim_{G,k}$ for each $0 \leq k < e_1$ on the set $T_G(n)$. These are motivated by Lemma 2.2.6 and Lemma 2.2.7.

**Definition 2.3.2** Let $m \geq 2$, $\alpha = (\alpha_1, \ldots, \alpha_m) \in T_G(n)$. Suppose $1 \leq i < j \leq m$ and $0 \leq k < e_i$. If $\alpha$ satisfies

- $\alpha_i \geq p^{e_i - k - 1}$, and
- $\alpha_j \geq (e_i - e_j)(p^{e_j - k} - p^{e_j - k - 1}) + p^{e_j - k}$,

then we set

$$\beta = (\alpha_1, \ldots, \alpha_i + (p^{e_j - k} - p^{e_j - k - 1}), \ldots, \alpha_j - (p^{e_j - k} - p^{e_j - k - 1}), \ldots, \alpha_m) \tag{2.74}$$

and say that $\beta \in T_G(n)$ is the image of $\alpha$ under the map $(G, k, i, j)$ and is denoted as

$$\beta = (G, k, i, j)(\alpha). \tag{2.75}$$

**Remark 2.3.3** The element $\beta$ of $T_G(n)$ is greater than the element $\alpha$ with respect to the lexicographic ordering.

Motivated by the above definition, we now define a map $(G, k, i, j_0, j_1, \ldots, j_r, k_1, \ldots, k_r)$ where $1 \leq i < j_0 < j_1 < \ldots < j_r \leq m$, $0 \leq k < e_{i_r}$ and $0 < k_t \leq e_{j_{t-1}} - e_{j_t}$ where $1 \leq t \leq r$.

**Definition 2.3.4** Let $q = \sum_{t=1}^{r} k_t + k$ and $d_t = e_{j_t} - \sum_{s=t+1}^{r} k_s - k$ for $1 \leq t \leq r$. If $\alpha \in T_G(n)$ satisfies

- $\alpha_i \geq p^{e_i - k - 1}$,

- $\alpha_{j_0} \geq (e_i - e_{j_0})(p^{e_{j_0}-k} - p^{e_{j_0}-k-1}) + p^{e_{j_0}-k}$,
- $\alpha_{i_t} \geq p^{d_t-1} + k_t(p^{d_t} - p^{d_{t-1}})$ for each $1 \leq t \leq r$,

then we set

$$\beta = (G, k, i, j_0, j_1, \ldots, j_r, k_1, \ldots, k_r)(\alpha). \tag{2.76}$$

If $\beta$ is of the form (2.76), we write

$$\alpha = (G, k, i, -j_0, j_1, \ldots, j_r, k_1, \ldots, k_r)(\beta). \tag{2.77}$$

**Definition 2.3.5** If $\alpha$ and $\beta$ are elements of $T_G(n)$, we define a relation $\sim_{G,k}$ as:

$$\alpha \sim_{G,k} \beta \iff \begin{cases} \beta = \sigma_s \circ \ldots \sigma_1(\alpha) \text{ or } \alpha & \text{if } m \geq 2,\ k < e_2, \\ \alpha = \beta & \text{otherwise.} \end{cases} \tag{2.78}$$

where $\sigma_l =$ for $1 \leq l \leq s$ are maps defined in 2.3.4.

**Remark 2.3.6** It follows trivially from the definition of $\sim_{G,k}$ that it is an equivalence relation.

The next lemma states that a minimal set of generators for $p^k Q_n(G)$ contains at most one element from each equivalence class of $T_G(n)$ for the corresponding $k$.

**Lemma 2.3.7.** *Let $\alpha, \beta \in T_G(n)$ be such that $\alpha \sim_{G,k} \beta$. Then, $p^k x^\alpha - p^k x^\beta \in \Delta^{n+1}(G)$.*

*Proof.* Since $G$ is Abelian, we only need to consider the case for $m = 2$. The result is clear if $\beta = \alpha$. Otherwise, $\beta = (G, k, 1, 2)(\alpha)$ and in this case, the result follows from Lemma 2.2.6. □

Let

$$M_k = \max\{1 \leq j \leq m | e_j > k\}, 0 \leq k < e_1. \tag{2.79}$$

Clearly, if $\exists j > M_k$ such that $\alpha_j > 0$, then $p^k {x_j}^{\alpha_j} \in \Delta^{\alpha_j+1}$ and hence $x^\alpha + \Delta^{n+1}(G)$ is zero in $Q_n(G)$. In view of this fact and Lemma 2.3.7, we define the sets $T_G(k, n)$ as:

**Definition 2.3.8** For each fixed integer $0 \leq k < e_1$,

$$T_G(k, n) = \{\alpha = (\alpha_1, \ldots, \alpha_{M_k}, 0, \ldots, 0) \in T_G(n) \,|\, \tau_k(\alpha) = \alpha\} \tag{2.80}$$

where

$$\tau_k(\alpha) = \max\{\gamma \in T_G(n) \mid \gamma \sim_{G,k} \alpha\} \tag{2.81}$$

deontes the lexicographically greatest element of the equivalence class containing $\alpha$.

**Remark 2.3.9** For $0 \leq k < e_2$, $M_k = 1$ and hence $T_G(k,n) = \{(n, 0, \dots, 0)\}$.

It is clear that $m = M_0 \geq M_1 \geq \dots \geq M_{e_1-1}$. Thus,

$$T_G(0,n) \supseteq T_G(1,n) \supseteq \dots \supseteq T_G(e_1-1,n). \tag{2.82}$$

We define $T_G(e_1, n) = \emptyset$.

**Lemma 2.3.10.** *The set $p^k\mathcal{T}_G(k,n)$ generates $p^kQ_n(G)$ for each $0 \leq k < e_1$.*

*Proof.* In view of Remark 2.3.1, we only need to show that $p^k\mathcal{T}_G(k,n)$ generates $p^k\mathcal{T}_G(n)$. Clearly, for $\alpha \in T_G(n)$, if $\exists\, j > M_k$, with $\alpha_j > 0$, then $k > e_j$ and hence $p^k x_j^{\alpha_j}$ belongs to a higher power of the augmentation ideal, i.e, $p^k x^\alpha \in \Delta^{n+1}(G)$. Hence the set

$$\{x^\alpha + \Delta^{n+1}(G) \mid \alpha = (\alpha_1, \dots, \alpha_{M_k}, 0, \dots, 0) \in T_G(n)\}. \tag{2.83}$$

generates $p^k\mathcal{T}_G(n)$. □

**Remark 2.3.11** For each $0 \leq k < e_1$, it follows from the definition of $T_G(k,n)$ that

$$T_G(k,n) = \{\alpha \sqcup (0, \dots, 0) \in T_G(n) | \alpha \in T_{G^{p^k}}(0,n)\}, \tag{2.84}$$

where $\alpha \sqcup \beta$ denotes the $m+n$-tuple obtained by juxtaposing an $m$-tuple $\alpha$ and an $n$-tuple $\beta$. It remains to show that the set $p^k\mathcal{T}_G(k,n)$ is a minimal generating set for $p^kQ_n(G)$ for each $0 \leq k < e_1$. Proceeding in this direction, we state the next result which gives a partition of $T_G(0,n)$ for large $n$. The proof is clear from the equivalence of the sets $T_G(0,n)$ and $\mathcal{S}(n,G)$ in Step 6 of the Theorem 2.2.5.

**Lemma 2.3.12.** *Let $m \geq 2$ and $\alpha = (\alpha_1, \dots \alpha_m) \in T_G(0,n)$. Then, $\alpha$ satisfies at least one of the following conditions and these conditions are mutually exclusive if $n \geq (m-1)(p^{e_m-1}-1) + p^{e_m-1} + (e_1 - e_m + 1)(p^{e_m} - p^{e_m-1})$:*

1. *$\alpha_m < p^{e_m}, (\alpha_1, \dots, \alpha_{m-1}) \in T_{G_0}(0, n-\alpha_m)$.*
2. *$p^{e_m-1} + if(e_m) \leq \alpha_m < p^{e_m-1} + (i+1)f(e_m)$ for some fixed integer $i$ with $1 \leq i \leq e_1 - e_m$, and*

- $\alpha_j < p^{e_m - 1}$ *for* $m_i < j < m$,
- $(\alpha_1, \ldots, \alpha_{m_i}) \in T_{G_i{}^{p^i}}(0, n - \sum_{j>m_i} \alpha_j)$.

*3.* $\alpha_j < p^{e_m - 1}$ *for all* $j < m$.

It is clear that the conditions 1 and 2 are mutually exclusive for all $n$. The following result also follows from the previous section.

**Theorem 2.3.13.** *There exists a positive integer* $n_0$ *such that for any* $n \geq n_0$, $Q_n(G)$ *is isomorphic to the group* $Q_G$ *and* $p^k \mathcal{T}_G(k, n)$ *is a minimal generating set of* $p^k Q_n(G)$ *for each* $0 \leq k < e_1$.

In view of Lemma 2.3.12, the set $T_G(0, n)$ can be partitioned as follows:

$$T_G(0, n) = B \cup_{i=0}^{e_1 - e_m} A_i, \tag{2.85}$$

where

$$B = \{\alpha \,|\, \alpha \text{ satisfies condition 3}\}, \tag{2.86}$$

$$A_0 = \{\alpha \,|\, \alpha \text{ satisfies condition 1 but not 3}\}, \tag{2.87}$$

$$A_i = \{\alpha \,|\, \alpha \text{ satisfies condition 2 but not 3}\}. \tag{2.88}$$

Now, for smaller $n$, we shall prove that this set is minimal by constructing maps from $T_G(k, n)$ to $T_G(k, n+1)$. For the same, we shall use the following result:

**Lemma 2.3.14.** *Let* $m \geq 2, \alpha = (\alpha_1, \ldots, \alpha_m) \in T_G(n)$, *and* $0 \leq i \leq e_1 - e_m$ *be a fixed integer. If* $\alpha$ *satisfies the following conditions*

1. $\alpha_m < p^{e_m - 1} + (i+1) f(e_m)$,
2. $\alpha_j < p^{e_m - 1}$ *for* $m_i < j < m$,
3. $(\alpha_1, \ldots, \alpha_{m_i}) \in T_{G_i^{p^i}}(0, n - \sum_{j>m_i} \alpha_j)$,

*then* $\alpha \in T_G(0, n)$.

Its proof is complicated and has been avoided here. We refer the reader to [CT11] for the same.

Let us first consider the case for $k = 0$. We define this map inductively, induction being on $m$. For $m = 1$, set

$$\phi_G(\alpha) = \alpha + 1. \tag{2.89}$$

For $k \geq 1$, we define

$$\phi_G(\alpha) = \begin{cases} (\alpha_1, \ldots, \alpha_m + 1) & \text{if } \alpha \in B, \\ \phi_{G_i p^i}((\alpha_1, \ldots, \alpha_{m_i})) \sqcup (\alpha_{m_i+1}, \ldots, \alpha_m) & \text{if } \alpha \in A_i. \end{cases} \tag{2.90}$$

By Remark 2.3.11, this map is well defined. Also, it is clear by definition that $\phi_G(\alpha) = \alpha + \epsilon_j$ for some $1 \leq j \leq m$.

**Claim 2.3.15** $\phi_G(T_G(0, n)) \subseteq T_G(0, n+1)$.

Clearly, $\phi_G(T_G(0, n)) \subseteq T_G(n+1)$. The claim is trivial for $m = 1$. We proceed by induction. Suppose the claim holds for groups with minimal number of generators less than $m$. Let $\alpha \in T_G(0, n)$. We have the following two cases depending upon $\alpha$.

**Case I:** $\alpha \in B$. In this case, the first $m-1$ entries of $\phi_G(\alpha)$ are less than $p^{e_m - 1}$. If $\exists\, \beta \sim_{G,k} \alpha$, then by the conditions on $\alpha$ and $\beta$ in Definition 2.3.4, we arrive at a contradiction. Hence the only element in the equivalence class of $\phi_G(\alpha)$ is itself. Thus, $\phi_G(\alpha) \in T_G(0, n+1)$.

**Case II:** $\alpha \in A_i$ for some $0 \leq i \leq e_1 - e_m$. In this case, $(\alpha_1, \ldots, \alpha_{m_i}) \in T_{G_i p^i}(0, n - \sum_{j > m_i} \alpha_j)$. By induction, we have

$$\phi_{G_i p^i}((\alpha_1, \ldots, \alpha_{m_i})) \in T_{G_i p^i}(0, n + 1 - \sum_{j > m_i} \alpha_j) \tag{2.91}$$

and hence in view of Lemma 2.3.14,

$$\phi_G(\alpha) = \phi_{G_i p^i}((\alpha_1, \ldots, \alpha_{m_i})) \sqcup (\alpha_{m_i+1}, \ldots, \alpha_m) \in T_G(0, n+1) \tag{2.92}$$

and we are done.

Now, we define maps $\phi_{G,k}$ for each $k$ as follows:

$$\phi_{G,k}(\alpha) = \phi_{G p^k}((\alpha_1, \ldots, \alpha_{M_k})) \sqcup (0, \ldots, 0). \tag{2.93}$$

We shall denote $\phi_G$ by $\phi_{G,0}$. Thus, in view of Remark 2.3.11 and Claim 2.3.15, we have

$$\phi_{G,k}(T_G(k, n)) \subseteq T_G(k, n+1). \tag{2.94}$$

We can now prove the following:

**Theorem 2.3.16.** *$p^k\mathcal{T}_G(k,n)$ is a minimal generating set of $p^kQ_n(G)$.*

*Proof.* When $n \geq n_0$, the result is clear from Theorem 2.3.13.

For $n \geq n_0$, let $l = n - n_0$. We shall prove the result by induction on $l$. The case $l = 0$ is clear. We assume that the result is true for $l-1$ , i.e., $p^k\mathcal{T}_G(k,n+1)$ is a minimal generating set for $p^kQ_{n+1}(G)$.

Now, if $p^k\mathcal{T}_G(k,n)$ is not a minimal generating set for $Q_n(G)$, then there exists an element $\beta \in T_G(k,n)$ such that

$$p^k x^\beta \equiv \sum_{\alpha \in T_G(k,n)\setminus\{\beta\}} c_\alpha p^k x^\alpha \pmod{\Delta^{n+1}(G)} \tag{2.95}$$

and $\beta$ is the smallest with this property. If for some $\alpha < \beta$, $p \nmid c_\alpha$, then $p^kx^\alpha$ can be written as a linear combination which contradicts the minimality of $\beta$. Thus, $p \mid c_\alpha$ for all $\alpha < \beta$. Let $\phi_{G,k}(\beta) = \beta + \epsilon_j$ for some $1 \leq j \leq M_k$. On multiplying both sides of (2.95) by $x_j$, we get

$$p^k x^{\beta+\epsilon_j} = p^k x^{\phi_{G,k}(\beta)} = \sum_{\alpha \in T_G(k,n)\setminus\{\beta\}} c_\alpha p^k x^{\alpha+\epsilon_j} \pmod{\Delta^{n+1}(G)} \tag{2.96}$$

By Lemma 2.3.7 and (2.84), we get that $p^kx^{\alpha+\epsilon_j} - p^kx^{\tau_k(\alpha+\epsilon_j)} \in \Delta^{n+2}(G)$. Thus, we can replace $p^kx^{\alpha+\epsilon_j}$ by $p^kx^{\tau_k(\alpha+\epsilon_j)}$ in (2.96).

We partition the set $A = T_G(k,n) \setminus \{\beta\}$ as follows:

$$A = A_1 \sqcup A_2, \tag{2.97}$$

where

$$A_1 = \{\alpha \in A \mid \tau_k(\alpha+\epsilon_j) = \phi_{G,k}(\beta)\}, \tag{2.98}$$

$$A_2 = \{\alpha \in A \mid \tau_k(\alpha+\epsilon_j) \neq \phi_{G,k}(\beta)\}. \tag{2.99}$$

In view of the above, we can write

$$\begin{aligned} p^k x^{\phi_{G,k}(\beta)} &= \sum_{\alpha \in T_G(k,n)\setminus\{\beta\}} c_\alpha p^k x^{\alpha+\epsilon_j} \\ &= \sum_{\alpha \in T_G(k,n)\setminus\{\beta\}} c_\alpha p^k x^{\tau_k(\alpha+\epsilon_j)} \\ &= \sum_{\alpha \in A_1} c_\alpha p^k x^{\phi_{G,k}(\beta)} + \sum_{\alpha \in A_2} c_\alpha p^k x^{\tau_k(\alpha+\epsilon_j)}. \end{aligned} \tag{2.100}$$

For any $\alpha \in A_1$, we have

$$\alpha + \epsilon_j \leq \tau_k(\alpha + \epsilon_j) = \phi_{G,k}(\beta) = \beta + \epsilon_j, \tag{2.101}$$

and hence $\alpha < \beta$ which implies $c_\alpha$ is a multiple of $p$. Thus, $1 - \sum_{\alpha \in A_1} c_\alpha$ is coprime to $p$, i.e., there exists an integer $b$ such that

$$b\left(1 - \sum_{\alpha \in A_1} c_\alpha\right) \equiv 1 \pmod{p^{e_1}}. \tag{2.102}$$

Multiplying both sides of the congruence (2.100) by $b$ and using (2.102), we get

$$p^k x^{\phi_{G,k}(\beta)} \equiv \sum_{\alpha \in A_2} b c_\alpha p^k x^{\tau_k(\alpha+\epsilon_j)} \pmod{\Delta^{n+2}(G)}. \tag{2.103}$$

For $\alpha \in A_2$, $\tau_k(\alpha + \epsilon_j)$ is in $T_G(k, n+1)$. Thus, in view of (2.103), $p^k x^{\phi_{G,k}(\beta)}$ can be written as a $\mathbb{Z}$-linear combination of elements of $p^k \mathcal{T}_G(k, n+1)$ contradicting the induction assumption. Hence proved. □

Theorem 2.3.16 gives us the Ulm invariants of the groups $Q_n(G)$.

**Theorem 2.3.17.** *Let $F_G(k,n) = |T_G(k,n)| - |T_G(k+1,n)|, 0 \leq e_1$. Then, the Ulm invariants of $Q_n(G)$ are*

$$F_G(0,n), F_G(1,n), \ldots, F_G(e_1 - 1, n). \tag{2.104}$$

*Consequently,*

$$Q_n(G) \cong (\mathbb{Z}/p\mathbb{Z})^{F_G(0,n)} \oplus (\mathbb{Z}/p^2\mathbb{Z})^{F_G(1,n)} \oplus \ldots \oplus (\mathbb{Z}/p_1^e\mathbb{Z})^{F_G(e_1-1,n)}. \tag{2.105}$$

Having found the structure of the augmentation quotients, we now proceed to find an explicit basis for the same.

Let

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in T_G(k,n) \setminus T_G(k+1,n). \tag{2.106}$$

We choose elements $X_k^\alpha \in Q_n(G)$ as follows

$$X_k^\alpha = \begin{cases} x^\alpha + \Delta^{n+1}(G) & k = e_1 - 1, \\ x^\alpha + \Delta^{n+1}(G) & k < e_1 - 1, \sum_{j > M_{k+1}} \alpha_j > 0, \\ x^\alpha - x^{\tau_{k+1}(\alpha)} + \Delta^{n+1}(G) & k < e_1 - 1, \sum_{j > M_{k+1}} \alpha_j = 0. \end{cases}$$

**Remark 2.3.18** From the fact that $p^{e_1}Q_n(G) = 0$, Lemma 2.3.7 and the argument in the proof of Lemma 2.3.10, it is clear that $X_k^\alpha$ is $p^{k+1}$ torsion.

When $k < e_1 - 1$ and $\sum_{j>M_{k+1}} \alpha_j = 0$, then from Lemma 2.3.7, we have

$$p^{k+1}X_k^\alpha \in \Delta^{n+1}(G). \tag{2.107}$$

Thus, for any $\alpha \in T_G(k,n) \setminus T_G(k+1,n)$, $p^{k+1}X_k^\alpha = 0$.

**Remark 2.3.19** We observe that if $k < e_1 - 1$ and $\sum_{j>M_{k+1}} \alpha_j = 0$, then $\alpha_j = 0$ for all $j > M_{k+1}$. Hence, $\tau_{k+1}(\alpha) \in T_G(k+1,n)$. Thus, $\cup_{k=0}^{e_1-1} X_k$ generates $\mathcal{T}_G(0,n)$.

**Theorem 2.3.20.** *Let $\langle X_k \rangle$ denote the subgroup of $Q_n(G)$ generated by $X_k$. Then, $\langle X_k \rangle$ is a free $\mathbb{Z}/p^{k+1}\mathbb{Z}$ module and $Q_n(G) = \bigoplus_{k=0}^{e_1-1} \langle X_k \rangle$.*

*Proof.* By Remark 2.3.19, it is clear that $\cup_{k=0}^{e_1-1} X_k$ generates $Q_n(G)$. Now, each element $X \in Q_n(G)$ can be written as

$$X = \sum_{0 \le k \le e_1} \sum_{\alpha \in T_G(k,n)\setminus T_G(k+1,n)} c_{k,\alpha} X_k^\alpha \tag{2.108}$$

where $0 \le c_{k,\alpha} < p^{k+1}$ in view of Remark 2.3.18. This expression gives at most $\prod_{k=0}^{e_1-1} p^{(k+1)|X_k|}$ elements.

Also,

$$\prod_{k=0}^{e_1-1} p^{(k+1)|X_k|} \le \prod_{k=0}^{e_1-1} p^{(k+1)F_G(k,n)} = |Q_n(G)|. \tag{2.109}$$

Thus, the expression (2.108) is unique and this proves the theorem. □

We assign as positive integer $T(G)$ to each $G$ as follows:

$$T(G) = \begin{cases} 1, & \text{if } m = 1, \\ (e_1 - e_2)(p^{e_2} - p^{e_2-1}) + p^{e_2-1} + \sum_{j=2}^{m}(p^{e_j} - 1), & \text{if } m \ge 2. \end{cases} \tag{2.110}$$

It has been proved by Hales that $Q_n(G) \cong Q_\infty(G)$ if and only if $n \ge T(G)$. Having found a basis for $Q_n(G)$ for all $n \ge 1$, we shall now see that the groups $Q_n(G)$ are non-isomorphic for all $n \leq T(G)$. We have by the definition of $T_G(k,n)$,

$$\begin{aligned} F_G(k,n) &= |T_G(k,n)| - |T_G(k+1,n)| \\ &= |T_{G^{p^k}}(0,n)| - |T_{G^{p^{k+1}}}(0,n)| \\ &= |T_{G^{p^k}}(0,n)| - |T_{G^{p^k}}(1,n)| = |F_{G^{p^k}}(0,n)|. \end{aligned} \tag{2.111}$$

The above equations show that we can focus on $F_G(0,n)$ instead $F_G(k,n)$.

**Lemma 2.3.21.** *The map $\phi_G : T_G(0,n) \to T_G(0,n+1)$ is injective for each $n \geq 1$.*

*Proof.* The proof is by induction on $m$. For $m = 1$, the statement is trivial. Suppose the result holds for all finite Abelian groups with at most $m-1$ summands in their cyclic decompositions. Let $\alpha, \beta \in T_G(0,n)$ such that $\phi_G(\alpha) = \phi_G(\beta)$.

Suppose one of $\alpha, \beta$, say $\alpha \in B$, then $\phi_G(\alpha) = (\alpha_1, \ldots, \alpha_{m+1})$. Thus, $\phi_G(\beta) = \beta + \epsilon_j$ for some $1 \leq j \leq m$ implies that $\alpha + \epsilon_m = \beta + \epsilon_j$ for some $1 \leq j \leq m$. Thus,

$$(\alpha_1, \ldots, \alpha_m + 1) = (\beta_1, \ldots, \beta_j + \epsilon_j, \ldots, \beta_m). \tag{2.112}$$

Since $\alpha_j < p^{e_m - 1}$, we get $\beta_j \leq \alpha_j < p^{e_m-1}$ for all $1 \leq j < m$. Thus, $\beta \in B$ and hence $\alpha = \beta$.

Now, suppose neither $\alpha$ nor $\beta$ belongs to $B$. Then, $\phi_G(\alpha) = \phi_G(\beta)$ implies that $\alpha_m = \beta_m$. Hence both $\alpha$ and $\beta$ have type $A_i$ for some $0 \leq i \leq e_1 - e_m$. Thus, by definition of $\phi_G$, we get

$$\phi_{G_i^{p^i}}(\alpha_1, \ldots, \alpha_{m_i}) = \phi_{G_i^{p^i}}(\beta_1, \ldots, \beta_{m_i}) \tag{2.113}$$

and

$$(\alpha_{m_i+1}, \ldots, \alpha_m) = (\beta_{m_i+1}, \ldots, \beta_m). \tag{2.114}$$

The result follows from above and the induction hypothesis. □

Since $\phi_G$ is injective, we have

$$\begin{aligned} F_G(0,n) &= |T_G(0,n) \setminus T_G(1,n)| \\ &\leq |\phi_G(T_G(0,n) \setminus T_G(1,n))| \\ &\leq |T_G(0,n+1) \setminus T_G(1,n+1)| \\ &\leq F_G(0,n+1). \end{aligned} \tag{2.115}$$

**Theorem 2.3.22.** *The groups $Q_1(G), Q_2(G), \ldots, Q_{T(G)}(G)$ are pairwise nonisomorphic and*

$$|Q_1(G)| < |Q_2(G)| < \ldots < |Q_{T(G)}|. \tag{2.116}$$

*Moreover,*

$$Q_{T(G)}(G) \cong Q_{T(G)+1} \cong \ldots \tag{2.117}$$

*Proof.* The second part follows from Theorem 2.2.5.

Now, we prove (2.116). For each $0 \leq k < e_2$, the following is obtained by the injectivity of the maps $\phi_{G,k}$

$$F_G(k,1) \leq \ldots \leq F_G(k, T(G^{p^{k+1}})) < \ldots < F_G(k, T(G^{p^k})). \tag{2.118}$$

Suppose there are two positive integers $n_1$ and $n_2$ such that $n_1 < n_2 < T(G)$ and $Q_{n_1}(G) \cong Q_{n_2}(G)$. Then, their Ulm invariants are same, i.e., $F_G(k, n_1) = F_G(k, n_2)$ for each $0 \leq k < e_1$. The equation (2.118) with $k = 0$ yields that $n_2 \leq T(G^p)$. On repeating this process, we get that $n_2 \leq T(G^{p^{e_2}}) = 1$. This implies that $n_1 < 1$, which is a contradiction. This proves the theorem. □

Thus, in view of Theorem 2.1.3, the problem of Karpilovsky is now fully solved for all finite Abelian groups. In the next subsection, we shall consider a simple example and compute the augmentation quotients and an explicit basis for the same.

**Example 2.3.23** Let

$$G = C_9 \times C_3 \times C_3 = \langle g_1 \rangle \times \langle g_2 \rangle \times \langle g_3 \rangle, \tag{2.119}$$

where $C_r$ denotes the cyclic group of order $r$.

Here $p = 3, m = 3$, $e_1 = 2$, $e_2 = e_3 = 1$. From earlier, we get

$$T(G) = (2-1)(3^1 - 3^0) + 3^{1-1} + (3^1 - 1) + (3^1 - 1) = 7 \tag{2.120}$$

We also have $m_0 = 2$, $m_1 = 1$. Thus,

$$G_0 = C_9 \times C_3, \quad G_1 = C_9. \tag{2.121}$$

In this case, $k = 0, 1$. Now, we determine the sets $T_G(0, n)$ and $T_G(1, n)$.

In view of Lemma 2.3.12 and Lemma 2.3.14, we get

$$T_G(0,n) = A_n \cup B_n \cup C_n \tag{2.122}$$

where

$$A_n = \{(0,0,n)\}, \tag{2.123}$$

$$B_n = \{(n-3,0,3),(n-4,0,4)\}, \tag{2.124}$$

$$\begin{aligned} C_n = \{&(n,0,0),(n-1,1,0),(n-2,2,0),(n-3,3,0),(n-4,4,0),(0,n,0),\\ &(n-1,0,1),(n-2,1,1),(n-3,2,1),(n-4,3,1),(n-5,4,1),(0,n-1,1),\\ &(n-2,0,2),(n-3,1,2),(n-4,2,2),(n-5,3,2),(n-6,4,2),(0,n-2,2)\}. \end{aligned} \tag{2.125}$$

On computation, we get the following table for $T_G(0,n)$:

Table 2.1: Elements of $T_G(0,n)$

| $n$ | $A_n$ | $B_n$ | $C_n$ |
|---|---|---|---|
| 1 | $(0,0,1)$ | $\emptyset$ | $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ |
| 2 | $(0,0,2)$ | $\emptyset$ | $(2,0,0)$, $(1,1,0)$, $(0,2,0)$, $(1,0,1)$, $(0,1,1)$, $(0,0,2)$ |
| 3 | $(0,0,3)$ | $(0,0,3)$ | $(3,0,0)$, $(2,1,0)$, $(1,2,0)$, $(0,3,0)$, $(2,0,1)$, $(1,1,1)$, $(0,2,1)$, $(1,0,2)$, $(0,1,2)$ |
| 4 | $(0,0,4)$ | $(1,0,3)$, $(0,0,4)$ | $(4,0,0)$, $(3,1,0)$, $(2,2,0)$, $(1,3,0)$, $(0,4,0)$, $(3,0,1)$, $(2,1,1)$, $(1,2,1)$, $(0,3,1)$, $(2,0,2)$, $(1,1,2)$, $(0,2,2)$ |
| 5 | $(0,0,5)$ | $(2,0,3)$, $(1,0,4)$ | $(5,0,0)$, $(4,1,0)$, $(3,2,0)$, $(2,3,0)$, $(1,4,0)$, $(0,5,0)$, $(4,0,1)$, $(3,1,1)$, $(2,2,1)$, $(1,3,1)$, $(0,4,1)$, $(3,0,2)$, $(2,1,2)$, $(1,2,2)$, $(0,3,2)$ |
| 6 | $(0,0,6)$ | $(3,0,3)$, $(2,0,4)$ | $(6,0,0)$, $(5,1,0)$, $(4,2,0)$, $(3,3,0)$, $(2,4,0)$, $(0,6,0)$, $(5,0,1)$, $(4,1,1)$, $(3,2,1)$, $(2,3,1)$, $(1,4,1)$, $(0,5,1)$, $(4,0,2)$, $(3,1,2)$, $(2,2,2)$, $(1,3,2)$, $(0,4,2)$ |
| $\geq 7$ | $(0,0,n)$ | $(n-3,0,3)$, $(n-4,0,4)$ | $(n,0,0)$, $(n-1,1,0)$, $(n-2,2,0)$, $(n-3,3,0)$, $(n-4,4,0)$, $(0,n,0)$, $(n-1,0,1)$, $(n-2,1,1)$, $(n-3,2,1)$, $(n-4,3,1)$, $(n-5,4,1)$, $(0,n-1,1)$, $(n-2,0,2)$, $(n-3,1,2)$, $(n-4,2,2)$, $(n-5,3,2)$, $(n-6,4,2)$, $(0,n-2,2)$ |

Also, $T_G(1, n) = \{(n, 0, 0)\}$ for all $n \geq 1$.

Using the above, we get the following table for Ulm invariants.

Table 2.2: Ulm Invariants of $Q_n(G)$

| $n$ | $F_G(0, n)$ | $F_G(1, n)$ | $Q_n(G)$ |
|---|---|---|---|
| 1 | 3-1 = 2 | 1 | ${C_3}^2 \times C_9$ |
| 2 | 6-1 = 5 | 1 | ${C_3}^5 \times C_9$ |
| 3 | 10-1 = 9 | 1 | ${C_3}^9 \times C_9$ |
| 4 | 14-1 = 13 | 1 | ${C_3}^{13} \times C_9$ |
| 5 | 18-1 = 17 | 1 | ${C_3}^{17} \times C_9$ |
| 6 | 20-1 = 18 | 1 | ${C_3}^{18} \times C_9$ |
| $\geq 7$ | 21-1 = 20 | 1 | ${C_3}^{20} \times C_9$ |

For finding a basis of $Q_n(G)$, we use Theorem 2.3.20. Let $x = g_1 - 1$, $y = g_2 - 1$ and $z = g_3 - 1$. Then, the following table lists the representatives of the basis elements of $Q_n(G)$.

Table 2.3: Basis of $Q_n(G)$

| $n$ | $X_1$ | $X_0$ |
|---|---|---|
| 1 | $x$ | $z, y$ |
| 2 | $x^2$ | $z^2$, $xy$, $y^2$, $xz$, $yz$ |
| 3 | $x^3$ | $z^3$, $x^2y$, $xy^2$, $y^3$, $x^2z$, $xyz$, $y^2z$, $xz^2$, $yz^2$ |
| 4 | $x^4$ | $z^4$, $xz^3$, $x^3y$, $x^2y^2$, $xy^3$, $y^4$, $x^3z$, $x^2yz$ |
| 5 | $x^5$ | $z^5$, $x^2z^3$, $xz^4$, $x^4y$, $x^3y^2$, $x^2y^3$, $xy^4$, $y^5$, $x^4z$, $x^3yz$, $x^2y^2z$, $xy^3z$, $y^4z$, $x^3z^2$, $x^2yz^2$, $xy^2z^2$, $y^3z^2$ |
| 6 | $x^6$ | $z^6$, $x^3z^3$, $x^2z^4$, $x^5y$, $x^4y^2$, $x^3y^3$, $x^2y^4$, $y^6$, $x^5z$, $x^4yz$, $x^3y^2z$, $x^2y^3z$, $xy^4z$, $y^5z$, $x^4z^2$, $x^3yz^2$, $x^2y^2z^2$, $xy^3z^2$, $y^4z^2$ |
| $\geq 7$ | $x^n$ | $z^n$, $x^{n-3}z^3$, $x^{n-4}z^4$, $x^{n-1}y$, $x^{n-2}y^2$, $x^{n-3}y^3$, $x^{n-4}y^4$, $y^n$, $x^{n-1}z$, $x^{n-2}yz$, $x^{n-3}y^2z$, $x^{n-4}y^3z$, $x^{n-5}y^4z$, $y^{n-1}z$, $x^{n-2}z^2$, $x^{n-3}yz^2$, $x^{n-4}y^2z^2$, $x^{n-5}y^3z^2$, $x^{n-6}y^4z^2$, $y^{n-2}z^2$ |

## 2.4 Non-Abelian Groups

Since we do not have a structure theorem for non-Abelian groups, the computation of augmentation quotients in this case as compared to that for Abelian groups, is in general, a much more difficult problem. In this section, we shall summarise the main results obtained for some non-Abelian groups.

The Augmentation Quotients for symmetric and dihedral groups were computed by Zhao and Tang[ZT08]. We describe the case of symmetric groups in some detail. We begin with a few definitions.

**Definition 2.4.1** A group $G$ is said to be perfect if it equals its own commutator subgroup.

It is clear that for a perfect group, $D_n(G) = G$ and hence $\Delta^n(G) = \Delta(G)$ for all $n \geq 1$. Thus, the set

$$B_n = \{g - 1|\, g \in G \setminus 1\} \tag{2.126}$$

is a $\mathbb{Z}$-basis for $\Delta^n(G)$.

**Lemma 2.4.2.** *Let $G$ be a group and $H$ be a perfect subgroup of $G$. Then, for any $n \in \mathbb{Z}$,*

$$Q_n(G) \cong Q_n(G/H).$$

*Proof.* The canonical epimorphism from $G$ to $G/H$ will give us an epimorphism

$$\phi : Q_n(G) \to Q_n(G/H).$$

We are done if we show that ker $(\phi) = 0$.

We first find a set of generators for $Q_n(G)$ and $Q_n(G/H)$.

Let $X = \{x_i | i \in I\}$ denote a representative set of preimages of $G/H \setminus \{eH\}$ in $G$. Then, $G = H \cup (\cup_{x_i \in X} x_i H)$ and $G/H = \{\bar{1}, \bar{x_i} | i \in I\}$.

Thus, for any $g \in G$, $\exists x_i \in X$, $h, h_i \in H$ such that $x = h$ or $x = x_i h_i$. Thus, either

$$x - 1 = h - 1 \in \Delta^{n+1}(G) \tag{2.127}$$

or

$$x - 1 = x_i h_i - 1 = (x_i - 1)(h_i - 1) + (x_i - 1) + (h_i - 1) \equiv x_i - 1 (mod \Delta^{n+1}(G)). \tag{2.128}$$

In view of the above, $Q_n(G)$ is generated by the set

$$\{(x_{i_1} - 1) \ldots (x_{i_n} - 1) + \Delta^{n+1}(G) | x_{i_j} \in X\}. \tag{2.129}$$

and $Q_n(G/H)$ is generated by the set

$$\{(\bar{x}_{i_1} - 1) \ldots (\bar{x}_{i_n} - 1) + \Delta^{n+1}(G) | x_{i_j} \in X\}. \tag{2.130}$$

Now, we consider the natural homomorphism

$$\varphi : \mathbb{Z}G \to \mathbb{Z}(G/H). \tag{2.131}$$

Since $G = H \cup (\cup_{x_i \in X} x_i H)$, every element of $\mathbb{Z}$ can be written as

$$\alpha = \sum_j b_j(h_j - 1) + \sum_{i,j} a_{ij}(x_i h_j - 1). \tag{2.132}$$

Thus, $\phi(\alpha) = \sum_{i,j} a_{ij}(\bar{x}_i - 1)$. So, if $\phi(\alpha) = 0$, then for any fixed $i$, $\sum_{i,j} a_{ij} = 0$. Hence,

$$\ker(\varphi) = \{\sum_j b_j(h_j - 1) + \sum_{i,j} a_{ij}(x_i h_j - 1) | \sum_j a_{ij} = 0\}. \tag{2.133}$$

For any $\alpha$ in $ker(\varphi)$, we have

$$\alpha = \sum_j b_j(h_j - 1) + \sum_{i,j} a_{ij}(x_i - 1)(h_j - 1) + \sum_j \sum_i a_{ij}(h_{ij} - 1) \in \Delta^{n+1}(G). \tag{2.134}$$

So, $ker(\varphi) \subseteq \Delta^{n+1}(G)$.

Now, we consider the epimorphism

$$\phi : Q_n(G) \to Q_n(G/H). \tag{2.135}$$

Suppose an element $\beta = \sum a_{i_{1\ldots n}}(x_{i_1} - 1)\ldots(x_{i_n} - 1) + \Delta^{n+1}(G) \in Q_n(G)$ satisfies

$$\phi(\beta) = 0. \tag{2.136}$$

That is $\sum a_{i_{1\ldots n}}(\bar{x}_{i_1} - 1)\ldots(\bar{x}_{i_n} - 1) \in \Delta^{n+1}(G)$. Now, since $\Delta^{n+1}(G)$ has generators $(\bar{x}_{j_1} - 1)\ldots(\bar{x}_{j_{n+1}} - 1)$, where $x_{j_i} \in X$, there exists $b_{j_{1\ldots n+1}} \in \mathbb{Z}$ such that

$$\sum a_{i_{1\ldots n}}(\bar{x}_{i_1} - 1)\ldots(\bar{x}_{i_n} - 1) - \sum b_{j_{1\ldots n+1}}(\bar{x}_{j_1} - 1)\ldots(\bar{x}_{j_{n+1}} - 1) = 0, \tag{2.137}$$

and hence

$$\sum a_{i_{1\ldots n}}(x_{i_1} - 1)\ldots(x_{i_n} - 1) - \sum b_{j_{1\ldots n+1}}(x_{j_1} - 1)\ldots(x_{j_{n+1}} - 1) \in \ker(\varphi). \tag{2.138}$$

Now, since $\sum b_{j_{1\dots n+1}}(x_{j_1}-1)\dots(x_{j_{n+1}}-1),\ \ker(\varphi)\in\Delta^{n+1}(G)$, so

$$\sum a_{i_{1\dots n}}(x_{i_1}-1)\dots(x_{i_n}-1)\in\Delta^{n+1}(G) \tag{2.139}$$

and hence $ker(\phi)=0$. Thus $\phi$ is an epimorphism and we are done. □

The derived group of $S_m$ is $A_m$, which is the subgroup of $S_m$ consisting of all even permutations. Clearly, for $m\geq 5$, $A_m$ is perfect. Also, $S_m/A_m$ is cyclic of order 2. Hence, on applying the previous result, we get the following:

**Theorem 2.4.3.** *Let $m\geq 5$, $S_m$ is the $m^{th}$ symmetric group and $A_m$ is the $m^{th}$ alternating group. Then,*

$$Q_n(S_m)\cong\mathbb{Z}_2. \tag{2.140}$$

**Alternative Method** Now, we provide a method to obtain the augmentation quotients of $S_m$ for all $m\geq 1$. Unlike the previous approach, this does not provide a basis for $\Delta^n(S_m)$.

We first note that if $g\in G'$, then it can be easily seen that $g-1\in\Delta^2(G)$. Any element of $S_m$ can be written as either $\tau$ or $t\tau$ where $\tau$ is an element of $A_m$ and $t$ is a transposition.

Let $n$ be a positive integer. Then, the Abelian group $\Delta^n(S_m)/\Delta^{n+1}(S_m)$ is generated by elements of the type

$$(\sigma_1-1)(\sigma_2-1)\dots(\sigma_n-1)+\Delta^{n+1}(S_m). \tag{2.141}$$

When $\sigma=\tau$, where $\tau\in A_m$, we have

$$\sigma-1\cong 0 \mod \Delta^2(S_m) \tag{2.142}$$

When $\sigma=t\tau$, then

$$\begin{aligned}\sigma-1&=t\tau-1\\&=(t-1)(\tau-1)+(t-1)+(\tau-1)\\&\equiv(t-1)\mod \Delta^2(S_m).\end{aligned} \tag{2.143}$$

Thus, $Q_n(S_m)$ is generated by elements of the type $t^n+\Delta^{n+1}(S_m)$.

Since $t$ is a transposition, which is of order 2, we have $t^2=1$. The equation

$$0 = t^2 - 1 = (t-1)^2 + 2(t-1) \tag{2.144}$$

shows that $2(t-1) \in \Delta^2(S_m)$.

In view of the above, any element of $Q_n(S_m)$ is of the type $\alpha t^n + \Delta^{n+1}(S_m)$, where $\alpha = 0, 1$. Thus,

$$Q_n(S_m) \cong \mathbb{Z}_2. \tag{2.145}$$

We now consider the case of dihedral groups, $D_n$, $n \geq 1$. The group $D_k$ has a representation

$$D_k = \langle r, s | r^k = s^2 = 1, rsr = s\rangle. \tag{2.146}$$

The following theorem gives the augmentation quotients of $D_k$ for odd $k$.

**Theorem 2.4.4.** *Let $D_k$ be a dihedral group, with $k$ odd. Then, for any $n \geq 1$, $\Delta^n(D_k)$ has a $\mathbb{Z}$-basis*

$$B_n = \{(s-1)^n, r^i - 1, (r^i - 1)(s-1) \,|\, 1 \leq i \leq k-1\}. \tag{2.147}$$

*Consequently, $Q_n(D_k) \cong \mathbb{Z}_2$.*

*Proof.* Since every element of $D_k$ can be written as $r^i s^j$, $0 \leq i \leq k-1, j = 0, 1$, we get that every generator of $\Delta^n(D_k)$ can be written as a $\mathbb{Z}$-linear combination of elements of $B_n$. Also, since $|B_n| = 2k-1 = |D_k| - 1$, it suffices to prove that $B_n \subseteq \Delta^n(D_k)$. In view of the relations (2.3) and (2.4), this is done once we prove that $r - 1 \in \Delta^n(D_k)$. Since $k$ is odd, the number $k - 1/2 \in \mathbb{N}$. Also, $sr = r^{-1}s$. The equation

$$r - 1 = (s-1)(r^{\frac{k-1}{2}} - 1) - (r^{\frac{k+1}{2}} - 1)(s-1) - (r^{\frac{k-1}{2}} - 1)(r-1) \tag{2.148}$$

implies that $r - 1 \in \Delta^2(D_k)$. On proceeding inductively and using 2.148, we get that $r - 1 \in \Delta^n(D_k)$ for any $n \geq 1$. Thus, $B_n \subseteq \Delta^n(D_k)$ for all $n \geq 1$. □

The augmentation quotients for non-Abelian finite groups in which the lower central series is an $N$-series were determined by G. Losey and N. Losey [LL79]. For groups of order $p^5$, Q. Zhou and H. You [ZY09, ZY10] computed the augmentation quotients using the group presentations. We have a complete description of the case for finite Abelian groups, but in the case of non-Abelian groups, only little is known.

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