Bott Periodicity and its Consequences in Complex Topological K-Theory

Ahina Nandy

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Certificate of Examination

This is to certify that the dissertation titled **"Bott Periodicity and its Consequences in Complex Topological K-Theory"** submitted by **Ms. Ahina Nandy** (Reg. No. MS15033) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Pranab Sardar   Dr. Krishnendu Gongopadhyay   Dr. Chetan Balwe

(supervisor)

Dated: April 24, 2020

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Chetan Balwe at the Indian Institute of Science Education and Research Mohali.
This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Ahina Nandy
(Candidate)

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Chetan Balwe
(Supervisor)

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I must express my profound gratitude to my parents for always encouraging me to excel in whatever I chose to and being the source of unfailing support throughout my entire life.

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Ahina Nandy

"I think that's a very powerful message — that nobody can take away from you the joy that you've had doing mathematics,"....“And people can give you grades, but that's not going to take away the freedom that you felt and the fulfillment that you felt."

Federico Ardila
'A Mathematician Who Dances to the Joys and Sorrows of Discovery'
Quanta Magazine
November 20, 2017

Notations

- $\xi_k(X)$:= collection of finite dimensional k-vector bundles over X.
- $\theta_k^n(X)$:= n-dimensional trivial k-vector bundle over X.
- $\mathbb{R}$:= field of real numbers
- $\mathbb{C}$:= field of complex numbers
- $GL_n(k^N)$:= Grassmannian maifold
- $U(n)$:= group of $n \times n$ unitary matrices
- $\pi_k(X)$:= k-th homotopy group of X
- $S^n(X)$:= n-th iterated suspension of X
- $X \wedge Y$:= smash product of the pointed space X and Y
- CX := Cone of a topological space X
- $\mathcal{L}(A)$:= the category of all finitely generated free left (or right) A modules for A, a ring with unit.
- $\mathcal{P}(A)$:= the category of all projective left (or right) A modules of finite type for A, a ring with unit.

Introduction

- The study of K-theory started with Alexander Grothendieck. In the context of the formulation of Grothendieck-Riemann-Roch theorem, he defined a group using the isomorphism classes of sheaves on an algebraic variety as the generators of that group. The name K-theory comes from the German word “Klasse”, which means class.

- Michael Atiyah and Friedrich Hirzebruch used the similar idea to study a group generated by the isomorphism classes of vector bundles over a topological space. They proved Bott periodicity theorem (3.1) and extended their formulation to construct a generalized cohomology theory. Topological K-theory is the first studied generalized cohomology theory.

- The goal of this dissertation is to learn the formulation and some marvels of classical topological K-theory. In chapter 1, the theory of vector bundles is introduced. In 1.7, an interesting approach to classify vector bundles over some compact space is sketched. Serre-Swan theorem(1.6.2) is included in 1.6. This connects topological and algebraic K-theory.

- The main formulations for K, K^{-1} and relative K functors, shown in appendix (4), are applicable for general additive categories and Banach functors between Banach categories. In chapter 2, K groups are defined for topological space (X). This is done as a special case of the abstract framework done in 4.4, 4.5, 4.6. The cohomogical properties as well as the ring structure of $K(X)$ is described.

- In chapter 3, The proof of Bott periodicity for complex K-theory, which lies in the heart of the subject, as well as some of its immediate applications (3.2) are shown. The rest of the chapter 3 is dedicated to introduce and prove Adam's theorem on Hopf invariant, along with its interesting application in finding the parallelizable spheres. Leray-Hirsch theorem and splitting principle are introduced as technical tools (3.3). Although these concepts in the context of generalized cohomology theories are worth looking at for their sheer usefulness, in this dissertation they are introduced only to facilitate the construction of Adams operations (3.5). General operations in K-theory is not discussed as well.

Contents

Chapter 1

Vector Bundles

1.1 Definition and Examples

The materials of this chapter are taken from [6], [1], [4].

Definition 1.1.1 Let, k be a field.A **k-vector bundle** of rank n is a triplet (E, X, p), where E and X are topological spaces and p is a continuous surjective map from E to X such that-

- $\forall x \in X$, $p^{-1}(x)$ is a finite dimensional k-vector space (k^n).
- $\forall x \in X$, $\exists U \subseteq X$, an open set containing x such that $p^{-1}(U)$ is homeomorphic to $U \times k^n$. (**local triviality condition**)

Remark 1.1.1 k has to be a field with some given topology. For our purpose we would be exclusively considering $\mathbb{R}$ or $\mathbb{C}$ with usual metric topology.
A similar triple without the local triviality condition is said to be a quasi vector bundle.

Example 1.1.1 Let X be any topological space and V be an $n-$dimensional k vector space. $X \times V$ is a vector bundle. This is the $n-$dimensional k-**trivial bundle** over X.

Example 1.1.2 Consider S^1 (the circle) and the **infinite cylinder** $E_1 = S^1 \times \mathbb{R}$. E_1 is a 1-dimensional $\mathbb{R}$ vector bundle over the base space S^1. This is a trivial bundle. To verify the local triviality condition, for each point $x \in S^1$, we can take the whole S^1 as the neigborhood U.

Example 1.1.3 Consider the quotient of $S^1 \times \mathbb{R}$ with respect to the equivalence relation $(x, t) \sim (\epsilon x, \epsilon t)$ where $\epsilon = \pm 1$. This is same as the quotient of $I \times \mathbb{R}$ with respect to $(0, t) \sim (1, -t)$. Let's call this **infinite mobius strip** E_2.

These two are two one dimensional real vector bundles over S^1.

Example 1.1.4 Consider **tangent bundle** TS^n over S^n. $TS^n = (E, \pi, S^n)$, where the total space E is described as the following-

$E = \{(x, v) \in S^n \times \mathbb{R}^{n+1} \mid v \perp x\}$

$\pi(x, v) = x$

Let's prove that TS^n is actually a vector bundle by showing that local triviality holds true. For $x \in S^n$, take $U = \{y \in S^n \mid \langle x, y \rangle \neq 0\}$. Here $\langle , \rangle$ is the usual scalar product of $\mathbb{R}^{n+1}$. Consider the map $\varphi : TS^n \mid_U \rightarrow U \times P_0$, where P_0 is the subspace of R^{n+1} orthogonal to x. $\varphi(x, v) = (x, w)$ where $w = v - \langle x, v \rangle x$. This w is just the orthogonal projection of v onto P_0. This φ is a homeomorphism.

Example 1.1.5 Consider the **normal bundle** NS^n over S^n. $NS^n = (E, \pi, S^n)$ where $E = \{(x, tx) \in S^n \times \mathbb{R} \mid t \in \mathbb{R}\}$. Considering the map $(x, tx) \rightarrow (x, t)$, it is a trivial bundle.

Example 1.1.6 Consider $\mathbb{R}P^n$, the space of all one dimensional subspaces of $\mathbb{R}^{n+1}$. The **canonical line bundle** over $\mathbb{R}P^n$ is defined as $(E, \pi, \mathbb{R}P^n)$, where the total space $E = (l, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid v \in l$ and $\pi(l, v) = l$. The local trivialization is given using orthogonal projection (see example 1.1.4). We are using the identification of $\mathbb{R}P^n$ as quotient space of S^n with respect to antipodal identification. Canonical line bundle over $\mathbb{C}P^n$ can be defined similarly. This particular bundle over $\mathbb{C}P^2 \cong S^2$ will play a central role in the proof of Bott Periodicity theorem.

1.2 Isomorphism of vector bundles

Definition 1.2.1 Let (E_1, π_1, X) and (E_2, π_2, X) are two k-vector bundles over X. A **homomorphism** f between vector bundles is defined as follows-

- $f : E_1 \rightarrow E_2$ is a continuous map.

- The following diagram commutes-

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & \searrow^{\pi_1} & \downarrow{\scriptstyle \pi_2} \\ & & X \end{array}$$

- $f\mid_x : E_1 \mid_x \to E_2 \mid_x$ is a linear map $\forall x \in X$.

- If f is a homeomorphism and f_x is a vector space isomorphism $\forall x \in X$, f is an **isomorphism** between the vector bundles E_1 and E_2.

Example 1.2.1 The bundles TS^1 and $S^1 \times \mathbb{R}$ are isomorphic. We can write the total space of TS^1 as $\{(e^{\imath\theta}, \imath t e^{\imath\theta}) \mid e^{\imath\theta} \in S^1, t \in \mathbb{R}\}$. Now let's consider the vector bundle isomorphism $\varphi : TS^1 \to S^1 \times \mathbb{R}$, where $\varphi(e^{\imath\theta}, \imath t e^{\imath\theta})) = (e^{\imath\theta}, t)$

Example 1.2.2 Infinite cylinder and infinite mobius bundle are not isomorphic. If there were an isomorphism between these two, there would have been a homeomorphism between these two spaces. So, following the notation of example 1.1.2 and example 1.1.3, $\{E_1 - (x, 0) \mid x \in S^1\}$ and $\{E_2 - (x, 0) \mid x \in S^1\}$ have to be homeomorphic. But the first one is disconnected and the second one is connected.

Remark 1.2.1 For a topological space X, we will denote the category consisting of all of its k- vector bundles and vector bundle homomorphisms as $\xi_k(X)$ or $\xi(X)$ when underlying field k is understood from the context.

Remark 1.2.2 Let V and V' are finite dimensional vector spaces and X is any topological space. As the following diagram commutes for any vector bundle homomorphism g-

$$\begin{array}{ccc} X \times V & \xrightarrow{g} & X \times V' \\ & \searrow & \downarrow \\ & & X \end{array}$$

$\forall x \in X$, g induces a linear map $g_x : V \to V'$. We get a map $\breve{g} : X \to \mathcal{L}(V, V')$. Here $\mathcal{L}(V, V')$ is the collection of all linear maps between V and V'.

Theorem 1.2.1 *There is a one to one correspondence between the vector bundle homomorphisms between $X \times V$ and $X \times V'$, and the continuous maps of the following form- $g : X \to \mathcal{L}(V, V')$*

PROOF First we have to show that in the notation of remark 1.2.2, $\breve{g}: X \to \mathcal{L}(V, V')$ is a continuous map with respect to the natural topology on $\mathcal{L}(V, V')$. Let's choose a basis $e_1, e_2, ..., e_n$ of V and $\epsilon_1, \epsilon_2, ..., \epsilon_p$ of V'. With respect to this basis g_x can be written as a matrix $[\alpha_{ij}(x)]$, where $\alpha_{ij}(x)$ is the i^{th} coordinate of the vector $g_x(e_j)$. Now, the function $x \to \alpha_{ij}(x)$ is the following composition-

$$X \xrightarrow{\beta_j} X \times V \xrightarrow{g} X \times V' \xrightarrow{\pi_2} V' \xrightarrow{\pi_i} k$$

Here $\beta_j(x) = (x, e_j)$, $\pi_2(x, v') = v'$ and π_i is the i^{th}projection of $V'(\approxeq k^p)$ on k. Being composition of continuous maps $\alpha_{ij}(x)$ are continuous and the map $\breve{g}$ induced by them is continuous.

Conversely, let, $h : X \to \mathcal{L}(V, V')$ is a continuous map. The map $\hat{h}$, induced by h is the following composition-

$$X \times V \xrightarrow{d_1} X \times \mathcal{L}(V, V') \times V \xrightarrow{d_2} X \times V'$$

Here $d_1(x, v) = (x, h(x), v)$ and $d_2(x, f, v) = (x, f(v))$. So, $\hat{h}$ is continuous and defines a vector bundle homomorphism.

Theorem 1.2.2 *Let E and F be two vector bundles over a space X. f is a homomorphism between them such that $f_x : E_x \to F_x$ is vector space isomorphism between E_x and F_x. Then f is a vector bundle isomorphism.*

PROOF We have to prove that f is a homeomorphism. Let $h : F \to E$ as $h(v) = f^{-1}(v)$. We just have to show that h is continuous. $\forall x \in X$, consider a neighborhood U of x and vector bundle isomorphisms $\beta_1 : E_U \to U \times V_1$ and $\beta_2 : F_U \to U \times V_2$. Let's define $f_1 = \beta_2 f_U {\beta_1}^{-1}$. We know $h_U = {\beta_1}^{-1} h_1 \beta_2$, where we get h_1 from $\breve{h_1}(x) = \breve{f_1}^{-1}(x)$ following the notation of remark 1.2.2. The map $\beta \to \beta^{-1}$ from $Iso(V_1, V_2)$ to $Iso(V_2, V_1)$ is continuous, so is h_1. So, we have shown the continuity of h on a neighborhood of every point of F, effectively proving continuity of h on the whole of F.

1.3 Clutching Construction

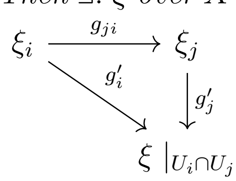
In this section we will try to construct vector bundles when we are given information about their restrictions to suitable subsets of the base space.

Theorem 1.3.1 (clutching of morphisms) *Let $\xi_1 = (E_1, \pi_1, X)$ and $\xi_2 = (E_2, \pi_2, X)$ are two vector bundles over X. Let's consider an open cover of X consisting of subsets U_i. If there exists a collection of morphisms $\alpha_i : \xi_1 |_{U_i} \to \xi_2 |_{U_i}$ such that- $\alpha_i |_{U_i \cap U_j} = \alpha_j |_{U_i \cap U_j}$, then there exists a unique morphism $\alpha : \xi_1 \to \xi_2$ such that $\alpha |_{U_i} = \alpha_i$.*

Proof

- **Uniqueness**: Let's take any point $e \in E$. As U_i is a cover of X, $e \in E_{U_i}$ for some U_i. Now α is uniquely defined by the following- $\alpha(e) = \mathfrak{J}_i(\alpha_i(e))$. Here $\mathfrak{J}_i : E'_{U_i} \hookrightarrow E'$ is the inclusion map.
- **Existence**: For $e \in E$, let $\alpha_i(e) = \alpha(e)$, where $e \in E_{U_i}$. From the uniqueness we just proved this is independent of the choice of i. The collection consisting of subsets of form $E_{U_i} = \pi^{-1}(U_i)$ gives an open cover for E. α is continuous. As $\alpha_x : E_x \to E'_x$ is linear, the map α is a morphism of vector bundles using theorem 1.2.1.

Theorem 1.3.2 (Clutching of bundles) *Let U_i be an open cover of a space X. We are given with the vector bundles $\xi_i = (E_i, \pi_i, U_i)$ on Ui's. Let $g_{ij} : \xi_i |_{U_i \cap U_j} \to \xi_j |_{U_i \cap U_j}$ are the vector bundle isomorphisms satisfying the following compatibility condition- $g_{ki} |_{U_i \cap U_j \cap U_k} = g'_{kj} \bullet g'_{ji}$, where $g'_{kj} = g_{kj} |_{U_i \cap U_j \cap U_k}$. Then $\exists!\ \xi$ over X and $g_i : \xi_i \to \xi_{U_i}$ such that the following diagram commutes-*



Proof

- **Existence**: Let's consider an equivalence relation '$\sim$' disjoint union $\sqcup E_i$ defined by the following: $e_i \sim e_j$ if $g_{ji}(e_i) = e_j$. Let $E := \sqcup E_i / \sim$. Let's define π as follows: $\pi(e) = \pi_i(e)$ if $e \in E_i$. $\forall x \in X$. π is well defined and continuous. $\pi^{-1}(x)$ are vector spaces because of $E_x \approx E_i |_x$. Let $g_i : E_i \to \pi^{-1}(U_i)$ is defined by $g_i(e) = \bar{e}$, where $\bar{e}$ is the class of e in E. By construction it is a quasi vector bundle isomorphism between (E_i, π_i, U_i) and $(E |_{U_i}, \pi |_{E_{U_i}}, U_i)$.

  Now we have to show local triviality for E. Let $\xi = (E, \pi, X)$ is the quasi vector bundle defined above. Let U_i be an open cover of X. For some $x \in U_i \subseteq X$, let $V \subseteq U_i$ be a locally trivial neighborhood of x. We just showed $\xi_i \approx \xi |_{U_i}$. So, $\xi_i |_V \approx \xi |_V$ and ξ is locally trivial.

- **Uniqueness**: Let ξ' be another bundle making the following diagram commutative:

$$\begin{array}{ccc} \xi_i & \xrightarrow{g_{ji}} & \xi_j \\ & \searrow^{\tilde{g}'_i} & \downarrow{\scriptstyle \tilde{g}'_j} \\ & & \xi' \,|_{U_i \cap U_j} \end{array}$$

  Let's consider the morphisms $\alpha_i = g'_i \cdot g_i^{-1}$, which is an isomorphism from ξ_{U_i} to ξ'_{U_i}. For $U_i \cap U_j$ we have $g_{ji} = {g'_j}^{-1} \cdot g'_i = \tilde{g'}_j^{-1} \cdot \tilde{g}'_i$. Using theorem 1.3.1, $\exists\ \alpha : \xi \to \xi'$, a vector bundle isomorphism making the following diagram commutative:

$$\begin{array}{ccc} \xi_i & & \\ \downarrow{\scriptstyle g_i} & \searrow^{g'_i} & \\ \xi \,|_{U_i} & \xrightarrow{\alpha|_{U_i}} & \xi' \,|_{U_i} \end{array}$$

Example 1.3.1 Let's consider sphere S^n. ${S^n}_+$ (respectively ${S^n}_-$) denotes the upper half (respectively lower half) sphere. ${S^n}_- \cap {S^n}_+ = S^{n-1}$. Let $f : S^{n-1} \to GL_p(k)$ be a continuous map. Now, by clutching the trivial bundles ${S^n}_+ \times k^p$ and ${S^n}_- \times k^p$ with respect to the transition map f we get a bundle over S^n. In later sections we will show that any contractible space can only have trivial bundles effectively showing that all bundles over S^n come from this kind of construction.

Now we will briefly introduce an approach of classification of vector bundles over a fixed base space and prove some interesting results in this regard.

Remark 1.3.1 Let G be a topological group and X be a space. A **G-cocycle** defined by an open cover (U_i) is a collection of continuous maps of the following form: $g_{ij} : U_i \cap U_j \to G$ with the following compatibility condition: $g_{ij}(x) \cdot g_{jk}(x) = g_{ik}(x)$, where $x \in U_i \cap U_j \cap U_k$.

Two cocycles (U_i, g_{ij}) and (V_i, h_{rs}) are considered equivalent if $\exists\ f_i^r : U_i \cap V_r \to G$ such that $g_j^s(x) \cdot g_{ji}(x) \cdot {g_i^r(x)}^{-1} = h_{sr}(x)$, where $x \in U_i \cap U_j \cap V_r \cap V_s$. This can be shown to be an equivalence relation. The collection of G cocycles over X quotiented by this equivalence relation is denoted by $H^1(X; G)$.

Lemma 1.3.1 *Let $\Phi_k^n(X)$ be the collection of all isomorphism classes of k vector bundles (finite dimensional) over X. Then there is a bijection between $\Phi_k^n(X)$ and $H^1(X; G)$*

PROOF Here we will define two set theoretic maps $h : \Phi_k^n(X) \to H^1(X; G)$, $h' : H^1(X; G) \to \Phi_k^n(X)$ and which are by construction inverse of each other. Let's consider $\xi = (E, \pi, X)$ be a given vector bundle with trivialization cover (U_i). Consider the isomorphisms $\phi_i : U_i \times k^n \to E_{U_i}$. $g_{ji}(x) = (\phi_j(x))^{-1} \cdot \phi_i(x)$ are the maps from $U_i \cap U_j$ to $GL_n(k)$.
We define h as follows: $h(\xi) = (U_i, g_{ji})$. Let (V_r, h_{sr}) is another cocycle associated to another trivialization cover (V_r) and trivialization maps $\psi_r : V_r \times k^n \to E_{V_r}$. For $x \in U_i \cap U_j \cap V_r \cap V_s$, we have-
$g_j^s(x) \cdot g_{ji}(x) \cdot (g_i^r(x))^{-1} = {(\psi_s)_x}^{-1} \cdot (\phi_j)_x \cdot ((\phi_j)_x)^{-1} \cdot (\phi_i)_x \cdot ((\phi_i)_x)^{-1} \cdot (\psi_r)_x = {(\psi_s)_x}^{-1} \cdot (\psi_r)_x = h_{sr}(x)$
So, the map h is well defined.

Conversely, let's start with a cocycle (U_i, g_{ji}). Let $\xi = (E, \pi, X)$ be the vector bundle formed by clutching $U_i \times k^n$ with the transition functions g_{ji}. Let's define $h((U_i, g_{ji})) = \xi$. If (V_r, h_{sr}) is an equivalent cocycle and $\xi' = (F, \pi', X)$ be the vector bundle we get by clutching $V_r \times k^n$ with respect to h_{sr}. Now using the ideas of theorem 1.3.2 we get a unique map $\alpha : E \to F$ which makes the following diagram commutative-

$$\begin{array}{ccc} E_i \mid_{U_i \cap V_r} & \xrightarrow{g_i^r} & F_r \mid_{U_i \cap V_r} \\ \downarrow{\scriptstyle g_i|_{U_i \cap V_r}} & & \downarrow{\scriptstyle h_r|_{U_i \cap V_r}} \\ E \mid_{U_i \cap V_r} & \xrightarrow{\alpha_i^r} & F \mid_{U_i \cap V_r} \end{array}$$

α does make sense because $\forall x \in U_i \cap U_j \cap V_r \cap V_s$, we have the following-

$$h_{sr}(x) = g_j^s(x) \cdot g_{ji}(x) \cdot (g_i^r(x))^{-1}$$

$$h_{sr}(x) \cdot g_i^r(x) \cdot g_{ji}(x) = g_j^s(x)$$

$$(h_s(x))^{-1} \cdot h_r(x) \cdot g_i^r(x) \cdot (g_i(x))^{-1} \cdot g_j(x) = g_j^s(x)$$

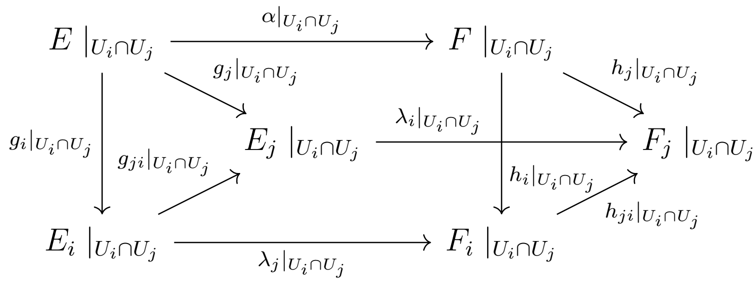
$$h_r(x) \cdot g_i^r(x) \cdot (g_i(x))^{-1} = h_s(x) \cdot g_j^s(x) \cdot (g_j(x))^{-1}$$

It effectively makes h' well defined and concudes our proof.

Now we would like to answer the following question: if we have two cocycles described with respect to the same cover, when are the two resulting vector bundles isomorphic?

Lemma 1.3.2 *Let (U_i, g_{ij}) and (U_i, h_{ij}) be two $GL_n(k)$ cocycles of X. There corresponding vector bundles E and F are isomorphic iff $\exists\ \lambda_i : U_i \to GL_n(k)$ such that $h_{ji}(x) = \lambda_j(x) \cdot g_{ji}(x) \cdot (\lambda_i(x))^{-1}$.*

Proof If we have isomorphism $\alpha : E \to F$, we obtain the following commutative diagram-



From here we get $h_{ji}(x) = \lambda_j(x) \cdot g_{ji}(x) \cdot (\lambda_i(x))^{-1}$. The converse is by definition of equivalence of cocycles.

Remark 1.3.2 Using the notation of lemma 1.3.2, if we choose $h_{ji} = Id$, then F is a trivial bundle and we get $g_{ji}(x) = (\lambda_j(x))^{-1} \cdot \lambda_j(x)$. So, for given $GL_n(k)$ cocycle (U_i, g_{ji}), the corresponding bundle is trivial iff $\exists\ \lambda_i : U_i \to GL_n(k)$ such that $g_{ji}(x) = (\lambda_j(x))^{-1} \cdot \lambda_j(x)$.

Now we want to use the previous results to look at the collection of all $p-$dimensional vector bundles over a sphere S^n.

Let's consider two maps $f_i : S^{n-1} \to GL_p(k)$, where $i \in \{0, 1\}$ and f_0, f_1 are homotopic. In section 1.7 we will prove that E_{f_0} and E_{f_1} are isomorphic. Following the notation of example 1.3.1 E_{f_i} is the vector bundle formed by clutching $S_+^{n-1} \times k^p$ and $S_-^{n-1} \times k^p$ with respect to the clutching function f_i. Now consider the base point preserving maps between S^{n-1} and $GL_p(k)$. $(f(e_1) = Id \in GL_p(k))$. So, we get a correspondence $f \to E_f$ between $\pi_{n-1}(GL_p(k))$ and $\phi_p^k(S^n)$.
On the other hand, $\pi_0(GL_p(k))$ acts on $\pi_{n-1}(GL_p(k))$ by the following map: $(b, f) = b^{-1} \cdot f \cdot b$. Now E_f and $E_{b^{-1}.f.b}$ are isomorphic using lemma 1.3.2. So, we get a well defined map from $\pi_{n-1}(GL_p(k))/\pi_0(GL_p(k))$ to $\phi_p^k(S^n)$.

Theorem 1.3.3 *The map $\pi_{n-1}(GL_p(k))/\pi_0(GL_p(k)) \to \phi_p^k(S^n)$ is injective. In section 1.7 we will show this is surjective too.*

PROOF Let's consider f, g two base point preserving maps from S^{n-1} to $GL_p(k)$ such that E_f and E_g are isomorphic. From lemma 1.3.1, we have the following diagram which commutes-

$$\begin{array}{ccc} E_1 & \xrightarrow{\alpha_1} & E_1 \\ \downarrow \hat{f} & & \downarrow \hat{g} \\ E_2 & \xrightarrow{\alpha_2} & E_2 \end{array}$$ where the vertical maps are defined only for $E_1 |_{S^{n-1}}$.

The maps $\alpha'_i = \check{\alpha'_i} |_{S^{n-1}}$ (following the notation of remark 1.2.2) are homotopic to constant maps (as they are restriction of maps defined over contractible spaces). As both f and g are basepoint preserving maps, both α'_1 and α'_2 are homotopic to the same constant map a. Again we have $g(x) = \alpha'_2(x) \cdot f(x) \cdot (\alpha'_1)^{-1}$. So, g and $a^{-1} \cdot f \cdot a$ are homotopic.
Let $h : S^{n-1} \times I \to GL_p(k)$ be the homotopy between g and $\alpha'_2(x) \cdot f(x) \cdot (\alpha'_1)^{-1}$. Then $h \cdot (h(e,t))^{-1}$ is a homotopy between g and $a(x) \cdot f(x) \cdot a^{-1}(x)$ in $\pi_{n-1}(GL_p(k))$ and we are done.

Remark 1.3.3 Let's take $k = \mathbb{C}$. $U(n)$ is a deformation retract of $GL_n(\mathbb{C})$ by Gram-Schmidt orthonormalization. $GL_n(\mathbb{C})$ is path-connected. So, $\pi_0(GL_n(\mathbb{C})) = \pi_0(U(n)) = 0$. So, $\phi_p^{\mathbb{C}}(S^n) = \pi_{n-1}(U(p))$. Let's look at the locally trivial fibration-

$$U(p-1) \to U(p) \to S^{2p-1}$$

From this we get the following exact sequence-

$$\pi_i(S^{2p-1}) \to \pi_{i-1}(U(p-1)) \to \pi_{i-1}(U(p)) \to \pi_{i-1}(S^{2p-1})$$

$\pi_j(S^r) = 0$ when $j > r$. Here for $p > i/2$ we have $\pi_i(U(p)) = \pi_i(U(p+1))$ (As the end terms of the exact sequence are zero). We get $\pi_i(U(p)) =$ inj lim $\pi_i(U(m)) = \pi_i(U)$. Where $U =$ inj lim $U(m)$. See 4.1 for detailed explanation. As an application of Bott's periodicity, in remark 3.2.3, we will prove $\pi_i(U) = \mathbb{Z}$ for i odd and 0 for i even. So, **classification of rank p complex bundles over S^r has definitive solution when $p > (r-1)/2$.**

1.4 Operations on Vector bundles

Here we will try to extend vector space operations like direct sum, tensor product, external product, dualization etc. for vector bundles.

Let T be a functor which takes finite dimensional vector spaces to finite dimensional vector spaces. We can think about the functor that takes a vector space to its dual. Now, for simplicity of calculation we are taking T to be a covariant functor of one variable.

We call T to be continuous if for all vector spaces V, W; the map $T' : Hom(V, W) \to Hom(T(V), T(W))$ is continuous.

Suppose $\xi = (E, \pi, X)$ is a given vector bundle. We define $T(\xi)$ as follows: $T(E) = \bigcup_{x \in X} T(E_x)$. For any continuous map (between vector bundles) $\phi : E \to F$, we want to topologize $T(E)$ in a way that makes the map $T(\phi) : T(E) \to T(F)$ continuous. (Giving $T(E)$ a suitable topology such that $T(\pi)$ becomes continuous is a special case of this.)

We start when $E = X \times V$ and $F = X \times W$ for some $k-$vector spaces V, W. Here $T(E) = X \times T(V)$ with **product topology**. Now $\phi : X \times V \to X \times W$ is a continuous. So, (following the notation of remark 1.2.2), $\check{\phi} : X \to Hom(V, W)$ is continuous. It is given that $T(\check{\phi}) : X \to Hom(T(V), T(W))$ is continuous. So, by theorem 1.2.1, $T(\phi) : X \times T(V) \to X \times T(W)$ is continuous (as it is a vector bundle homomorphism).

Now let's take any vector bundle $E \to X$. If $U \subseteq X$ is a trivializing open set of X, we topologize $T(E_U)$ as above. We give $T(E)$ the following topology: $V \subseteq T(E)$ is open if $V \cap T(E_U)$ is open in $T(E_U)$ $\forall U \subseteq X$ such that E_U is trivial.

Now $Y \subseteq X$, we get $T(E_Y)$ is homeomorphic to $T(E)_Y$. So, for any homomorphism $\phi : E \to F$ we get $T(\phi) : T(E) \to T(F)$ is a homomorphism, thus showing ϕ is continuous.

Remark 1.4.1 Let's consider the following diagram where (E, π, X) is a vector bundle.

$$\begin{array}{ccc} f^*(E) & \xrightarrow{\tilde{f}} & E \\ {\scriptstyle f^*(\pi)}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ Y & \xrightarrow[f]{} & X \end{array} \quad f \text{ is a continuous map}$$

$f^*(E) = \{(y, e) \in Y \times E \mid f(y) = \pi(e)\}$. This is called **pull back bundle** of E on Y with respect to the map f. Given any map $f : Y \to X$ and a vector bundle (E, π, X), this $(f^*(E), f^*(\pi), Y)$ is the unique vector bundle with a map $\tilde{g} : f^*(E) \to E$ taking the fibre of each $y \in Y$ isomorphically to the fibre of $f(y)$ in E.

So, a map $f : Y \to X$ gives rise to a map $f^* : \phi_k^n(X) \to \phi_k^n(Y)$.

Here this is mentioned to state the following: for any map $f : Y \to X$, a continuous functor T (as considered above) and (E, π, X) a given vector bundle we get-

$$T(f^*(E)) \cong f^*(T(f))$$

Similar construction can be made for contravariant functors and functors of several variable. We will show examples of the two cases which are going to be used later.

1.4.1 Direct sum

Let $\xi_1 = (E_1, \pi_1, X)$ and $\xi_2 = (E_2, \pi_2, X)$ are two vector bundles over X.Their direct sum $\xi_1 \oplus \xi_2 = (E_1 \oplus E_2, \pi_1 \oplus \pi_2, X)$. Here $(E_1 \oplus E_2)_x = E_{1x} \oplus E_{2x}$.

Example 1.4.1 The tangent bundle TS^n over S^n may not be trivial. But its direct sum with the normal bundle NS^n is trivial. $TS^n \oplus NS^n \cong S^n \times \mathbb{R}^{n+1}$ considering the following map: $\phi : TS^n \oplus NS^n \to S^n \times \mathbb{R}^{n+1}$. We know $TS^n \oplus NS^n = \{(x, v, tx) \in S^n \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid v \perp x\}$ Here $\phi(x, v, tx) = (x, v + tx)$.

1.4.2 Tensor product

In a similar fashion we define tensor product of vector bundles. Let $\xi_1 = (E_1, \pi_1, X)$ and $\xi_2 = (E_2, \pi_2, X)$ are two vector bundles over X. Their tensor product $\xi_1 \otimes \xi_2 = (E_1 \otimes E_2, \pi_1 \otimes \pi_2, X)$. Here $(E_1 \otimes E_2)_x = E_{1x} \otimes E_{2x}$.

Example 1.4.2 Consider the set of all **line bundles** over a space X. This collection forms an abelian group **Picard group** with respect to tensor product and the identity element is the trivial line bundle over X. For each line bundle we get its inverse by clutching the trivial line bundles over the trivializing subsets of the given vector bundle with respect to the clutching functions which are inverses of the clutching functions for the given line bundles.

Here because the multiplication in $\mathbb{R} - \{0\}$ or $\mathbb{C} - \{0\}$ is commutative, compatibility conditions for a cocycle (remark 1.3.1) are satisfied. As we do not have commutativity in matrix multiplication for higher dimension, we do not have such simple group structure there.

1.5 Sections

Definition 1.5.1 Let (E, π, X) be a vector bundle over X. A **section** is a map $s : X \to E$ that assigns each element $x \in X$, a vector $s(x)$ in $\pi^{-1}(x)$. Or we can say a section is a map $s : X \to E$ such that $\pi \circ s = Id$.

Here we will only think about continuous sections (the map $s : X \to E$ has to be continuous.)

Example 1.5.1 The most trivial example is the zero section, taking each element x of X to the 0 of the vector space V_x.

Remark 1.5.1 Let's consider a vector bundle E with m continuous sections $s_1, s_2, ..., s_m$. If the vectors $s_i(x)$'s are linearly independent $\forall x \in X$, the sections s_i's are called linearly independent sections.

Consider the map: $\psi : X \times k^m \to E$ defined as follows:
$\psi(x, a_1, a_2, ..., a_m) = (x, \sum_{i=1}^{m} a_i s_i(x))$. If E is an m dimensional vector bundle this is a vector space isomorphism in each fibre of E. As it is continuous using theorem 1.2.2 it is a vector bundle isomorphism.

So, in a nutshell, **an m-dimensional vector bundle E is trivial iff it has exactly m linearly independent continuous sections.**

Remark 1.5.2 Let's consider the ccollection of all continuous sections of (E, π, X) and denote it by $\Gamma(X, E)$. It has a k-vector space structure as follows: $(s_1 + s_2)(x) = s_1(x) + s_2(x)$ and $(k_i s)(x) = k_i(s(x))$, where $s_j \in \Gamma(X, E)$ and $k_i \in k$. Now let's consider $Y \subseteq X$ or $Y \xhookrightarrow{i} X$. Following the notion of pullback in remark 1.4.1, for $s \in \Gamma(X, E)$, $si \in \Gamma(Y, i^*(E)) = \Gamma(Y, E_Y)$

Now we are going to prove some technical results which will be used in section 1.7. For this purpose we need to use the existence of partitions of unity subordinate to any trivializing cover (for a given vector bundle E) of the base space X. So, we will be considering **Hausdorff paracompact** spaces unless otherwise mentioned.

Lemma 1.5.1 *If Y is a closed subset of X, every element of $\Gamma(Y, E_Y)$ is the restriction of some element of $\Gamma(X, E)$.*

Proof At first we will prove it for trivial bundles and then using that we will show it for the general case.

a. Let's take a bundle $E = X \times V$ where V is a finite dimensional k-vector space. Now using remark 1.5.1, we can identify $\Gamma(X, E)$ with $F(X, E)$, the set of continuous functions from X to E with the special property that any $x \in X$ will go to an element of E_x. So, instead of talking about $\Gamma(X, E)$ and $\Gamma(Y, E_Y)$, we can consider $F(X, E)$ and $F(Y, E_Y)$ respectively. But any element of $F(Y, E_Y)$ can be extended to some element of $F(X, E)$ using Tietze extension theorem. (As any paracompact Hausdorff space is normal, we can use this). So we have effectively shown that **the restriction map from $\Gamma(X, E)$ to $\Gamma(Y, E_Y)$ is surjective.**

b. Now let's take any vector bundle E on X. Consider a trivializing open cover $(U_i \mid i \in I)$ of X which is locally finite. Let (V_i) be an open cover of X such that $\bar{V_i} \subseteq U_i$. Let $W_i := \bar{V_i} \cap Y$. For $t \in \Gamma(Y, E_Y)$, let's denote $t \mid W_i$ by t_i. Now using the previous argument, $\exists s_i \in \Gamma(V_i, E_{V_i})$ such that $s_i \mid_{W_i} = t_i$. Let's consider α_i be a partition of unity subordinate to the cover (V_i). Now set $s(x) := \sum_{i \in I} \alpha_i(x) s_i(x)$. For $y \in Y$, $s(y) = \sum_{i \in I} \alpha_i(x) t(x) = (\sum_{i \in I} \alpha_i(x)) t(x) = t(x)$. So we have got an $s \in \Gamma(X, E)$ such that $s_Y = t$ for any arbitrary $t \in \Gamma(Y, E_Y)$.

Let's consider two vector bundles E, F on X. We get a functor from the category $\xi_k(X) \times \xi_k(X)$ to the category of $k-$vector spaces as follows: $(E, F) \to Hom_k(E, F)$. We have come another functor from $\xi_k(X) \times \xi_k(X)$ to the category of $k-$vector spaces given by $(E, F) \to \Gamma(X, Hom(E, F))$

Lemma 1.5.2 *The above mentioned two functors are equivalent.*

Proof Consider $\alpha \in Hom(E, F)$. The map $x \to \alpha \mid_{E_x}$ gives a section of $Hom(E, F)$. We have to show it is continuous. Let's consider the following isomorphisms: $E_U \cong U \times V_1$ and $F_U \cong U \times V_2$. Here U is a trivializing neighborhood of X. Now we get the following commutative diagram:

$$\begin{array}{ccc} U \times V_1 & \xrightarrow{\check{\alpha}} & U \times V_1 \\ \uparrow & & \uparrow \\ E_U & \xrightarrow{\alpha_U} & F_U \end{array}$$

Here the vertical maps are isomorphisms.

Being composition of continuous maps the map $x \to \alpha \mid_{E_x}$ has to be continuous.

On the other hand $\forall s \in \Gamma(X, Hom(E,F))$, let's consider the following correspondence: $s \to \alpha$, where $\alpha \in Hom(E,F)$. $\alpha_x = s(x)$. Similarly this correspondence is continuous.

Remark 1.5.3 Using the identification of $\Gamma(X, Hom(E,F))$ and $Hom(E,F)$ and lemma 1.5.1 we can conclude that $\forall \alpha \in Hom(E_Y, F_Y)$, $\exists \beta \in Hom(E,F)$ such that $\beta \mid_Y = \alpha$.

.

Theorem 1.5.1 *Following the notations of remark 1.5.3, for any isomorphism $\alpha : E_Y \to F_Y$, $\exists U \subseteq X$, a neighborhood of Y and an* ***isomorphism*** *$\beta : E_U \to F_U$ such that $\beta_Y = \alpha$.*

Proof Let $\hat{\alpha} : E \to F$ be the extension of α. Let $V = \{x \in X \mid \hat{\alpha}_x : E_x \to F_x$ is an isomorphism$\}$. We want to show V is open. $\forall x \in V$, we have open set $W_x \subseteq X$ such that $E \mid_{W_x} \cong F \mid_{W_x} \cong W_x \times k^n$. $\gamma_x := \hat{\alpha} \mid_{W_x}$. Using the notation of remark 1.2.2, $\check{\gamma_x} : X \to Hom(k^n, k^n)$ is a continuous map. We can rephrase $V \cap W_x$ as the set of points $x \in W_x$ such that $x \in (\check{\gamma_x})^{-1}(Iso(k^n, k^n))$. As $(Iso(k^n, k^n)) \subset Hom(k^n, k^n)$ is open, $V \cap W_x$ is open in W_x. As $W_x \subseteq X$ is open so is $V \cap W_x \subseteq X$. $U = \cup_{x \in V}(W_x \cap V)$ is an open neighborhood containing V, which does our job.

Theorem 1.5.2 *Let, E and F be vector bundles over X and $\alpha : E \to F$ is a vector bundle morphism such that $\alpha_x : E_x \to F_x$ is surjective $\forall x \in X$. Then $\exists$ a morphism $\psi : F \to E$ such that $\alpha\psi = Id_F$.*

Proof Let's choose any arbitrary point $x \in X$. $\exists\, U$ *subseteq* X, a neighborhood containing x such that $E_U \cong U \times V_1$ and $F_U \cong U \times V_2$. Again, using the notations of remark 1.2.2, $\check{\alpha_U} : U \to Hom(V_1, V_2)$ is a continuous map. Using the fact that V_1 and V_2 are vector spaces, we can write $V_1 = V_2 \oplus Ker(\check{\alpha_{U}}_x)$. Denote $\check{\alpha_{U}}_x$ by ϕ. Now with respect to appropriate basis the map ϕ has the following form:
$\phi(y) : V_2 \oplus Ker(\phi(x)) = (\phi_1(y), \phi_2(y))$
Here ϕ_i's are continuous functions such that $\phi_1(x) = 1$ and $\phi_2(x) = 0$. Now again using the fact that $Aut(V_2)$ is open in $End(V_2)$, $\exists U_x \subseteq X$, an open set containing x, where $\phi_1(y)$ is an isomorphism.
We can construct $\phi' : U_x \to Hom(V_2, V_1)$, which can be written in the following form: $\begin{pmatrix} \phi_1(y)^{-1} \\ 0 \end{pmatrix}$. So, we get an induced morphism $\hat{\phi}'_x : F \mid_{U_x} \to E \mid_{U_x}$

such that $\alpha_{U_x} \cdot \hat{\phi}'_x = Id$. We can vary the point x and construct a locally finite open cover (U_i) of X and morphisms $\psi_i : F_{U_i} \to E_{U_i}$, where $\alpha_{V_i}\psi_i = Id$. We have a partition of unity (η_i) subordinate to the open cover U_i. Now we define $\psi : F \to E$ as follows: $\psi(f) = \sum_i \eta_i(x)\psi_i(f)$, where $f \in F$ and $f \in F_x$.
$(\alpha\psi)(e) = \sum_i \eta_i(x)(\alpha \cdot \psi_i)(e) = (\sum_i \eta_i(x))(\alpha \cdot \psi_i(e)) = e$

1.6 Algebraic counterpart

In this section we are going to provide a connection between topological K-theory and algebraic K-theory. In the later, the interesting objects of study are collection of finitely generated projective modules over a given ring; just like the collection of finite dimensional vector bundles over a given space is of central interest in the previous one. For this we have thoroughly followed [6, pp. 26–32].

Definition 1.6.1 We call a category C to be **pre-additive** if $\forall\ M, N \in \mathcal{Ob}(C)$, $Hom_C(M, N)$ has an abelian group structure and the composition of morphisms is distributive over group addition (we are referring to the addition defined in $Hom_C(M, N)$).

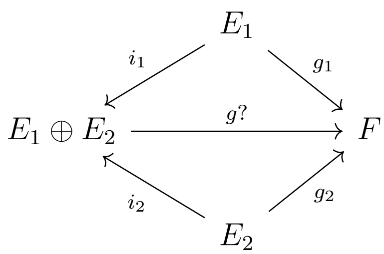
If in a pre-additive category all finitary products are coproduct, it is said to be an **additive category**.

Lemma 1.6.1 *The category $\xi_k(X)$ is an additive category.*

Proof For vector bundles E, F over X, we have seen $Hom(E, F)$ is equivqlent to $\Gamma(X, Hom(E, F))$ (following the notation of remark 1.5.2), which is a vector space. So, obviously it has an abelian group structure.

The composition $Hom(E, F) \times Hom(F, G) \longrightarrow Hom(E, G)$ is bilinear. So, $\xi_k(X)$ is pre-additive.

Now we want to show that direct sum of vector bundles is actually the coproduct in the category $\xi_k(X)$. Let's consider the following diagram-



For $j \in \{1,2\}$, $i_j : E_j \to E_1 \oplus E_2$ is the canonical morphism. g_j's are arbitrarily chosen morphisms between E_j and F. We have to prove the existence and uniqueness of a map g.

uniqueness: As we want the diagram commute $\forall x \in X$, we want $g_{j_x}(e_j) = g_x \circ i_x(e_j)$. So $g_x(e_1, e_2) = g_{1x}(e_1) + g_{2x}(e_2)$. We have essentially shown a unique construction for g.

existence: We just have to show the above defined construction for g is continuous. Let's consider an open set $U \subseteq X$, such that $E_j \cong U \times V_j$ and $F_U \cong U \times W$. Using the notation of remark 1.2.2, $\check{g}_j \mid_U : U \to Hom(V_j, W)$ is a continuous map (see theorem 1.2.1). Similarly $g \mid_U : U \times (V_1 \oplus V_2) \to U \times W$ is defined like $g_U(x, (v_1, v_2)) = (x, g_{1x}(v_1) + g_{2x}(v_2)$. Again by connstruction this is continuous. So, we have shown the g we constructed is continuous.

Consider a vector bundle E on X.An element $p \in End(E)$ is called a **projector** if $p^2 = p$. Kernel of any projector has a quasi vector bundle structure but it is not obvious whether it should satisfy local triviality or not. $ker(p)$ is of special interest for us.

Lemma 1.6.2 *Let E be a vector bundle over X. For any projector $p \in End(E)$, $Ker(p) \in \mathcal{Ob}(\xi(X))$.*

Proof Here we have to show that $Ker(p) = \bigsqcup_{x \in X} Ker(p_x)$ is locally trivial. As we are considering how $Ker(p)$ looks like locally, Without loss of generality we can assume $E = X \times V$ for some finite dimensional vector space V. Let's fix some arbitrarily chosen $x_0 \in X$. Set $g : X \to \xi(X)(E, E)$ as follows: $g := 1 - (p_x + p_{x_0})^2 = 1 - p_x - p_{x_0} - 2p_x p_{x_0}$. By straightforward calculation $f(x) \cdot p_x = p_{x_0} \cdot f(x)$. We have the following diagram:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & Ker(p) & \longrightarrow & X \times V & \xrightarrow{p} & X \times V \\
 & & \Big\downarrow ? & & \Big\downarrow \hat{g} & & \Big\downarrow \hat{g} \\
0 & \longrightarrow & X \times Ker(p_{x_0}) & \longrightarrow & X \times V & \xrightarrow[Id_X \times p_{x_0}]{} & X \times M
\end{array}
$$

Again as $g(x_0) = 1$, $\exists U_{x_0}$, a neighborhood of x_0 such that f is an automorphism $\forall x \in X$. (As $Aut(V,V) \subset End(V)$ is open and f is continuous). Now we can define $(\hat{g}_{U_{x_0}})^{-1} = \widehat{g \mid_{U_{x_0}}}$. So we have shown $Ker(p) \cong X \times Ker(p_{x_0})$, which is a vector bundle.

For the rest of this section we will take the base space X to be **compact** unless otherwise stated.

Theorem 1.6.1 *$\forall\ E \in \xi(X)$, $\exists\ E' \in \xi(X)$ such that $E \oplus E' \cong X \times k^n$ for some $n \in \mathbb{N}$.*

Proof Using the fact that X is compact, at first let's choose a finite open cover $(U_i), i \in \mathbb{N} \cap [1,j]$ such that $E_{U_i} \cong U_i \times k^n$. Let (η_i) be a partition of unity subordinate to (U_i). Using remark 1.5.1, we can find n linear independent sections $s_i^1, s_i^2, ..., s_i^n$ of E_{U_i}. Now the sections $\sigma_i^j := \eta_i s_i^j$ can be extended outside U_i as 0. They give us n linearly independent continuous section for E_{V_i}, where $V_i = \eta_i^{-1}((0,1])$. $\sigma_i^j(x)$ generates the vector space E_x. We have got the following morphism:

$$\psi : X \times k^n \to E$$

. Here $\psi(x, \lambda_1, ..., \lambda_n) = \sum_j \lambda_j \sigma_i^j(x)$. ψ_x is surjective for all x. Now we use theorem 1.5.2 to get a morphism $\phi : X \times k^n \to E$ such that $\psi \cdot \phi = Id_E$. $(\phi \cdot \psi)^2 = \phi \cdot (\psi \cdot \phi) \cdot \psi = \phi \cdot \psi$ So, $p := \phi \cdot \psi$ is a projector of $X \times k^n$.

$E \cong Ker(1-p)$ and using lemma 1.6.1, $\xi(X)$ is an additive category. So, $E \oplus Ker(p) \cong X \times k^n$. Using lemma 1.6.2, $Ker(p) = E' \in \xi(X)$ and we are done.

Let's consider an abelian category $\mathcal{C}$. if $\forall E \in \mathcal{Ob}(\mathcal{C})$ and for all $p : E \to E$, where p is projector of E; $Ker(p) \in \mathcal{Ob}(\mathcal{C})$, then $\mathcal{C}$ is called a **pseudo abelian category**.

We have just shown that $\xi(E)$ is a pseudo abelian category.

For any arbitrary ring R with unit, $\mathcal{P}(R)$, the category consisting of all finitely generated left (or right) projective R modules and morphisms between them is a pseudo abelian category.

A non-example would be $\mathcal{L}(R)$ the category of all finitely generated left free R modules and morphisms between them.

Let's consider a pseudo abelian category $\mathcal{C}$. $E \in \mathcal{Ob}(\mathcal{C})$. If $p : E \to E$ be a projector so is $(1-p)$. Now by definition $Ker(p) \in \mathcal{Ob}(\mathcal{C})$ and $Ker(1-p) \in \mathcal{Ob}(\mathcal{C})$. Now using Mitchell's embedding theorem, (remark 4.2.1) we can consider abstract abelian categories as rather concrete category of modules. So, we can write $E = Ker(p) \oplus Ker(1-p) \cong Im(p) \oplus Im(1-p)$. (For more details see 4.2).

Lemma 1.6.3 *Let $\mathcal{C}$ be an abelian category. Then $\exists\ \tilde{\mathcal{C}}$, a pseudo abelian category and and additive fuctor $\phi : \mathcal{C} \to \tilde{\mathcal{C}}$ with the following universal property. If $\mathcal{D}$ is another pseudo abelian category with an abelian functor $\psi : \mathcal{C} \to \mathcal{D}$ then $\exists\ \psi' : \tilde{\mathcal{C}} \to \mathcal{D}$ such that the following diagram commutes-*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\phi} & \tilde{\mathcal{C}} \\ \downarrow{\scriptstyle\psi} & \swarrow{\scriptstyle\psi'} & \\ \mathcal{D} & & \end{array}$$

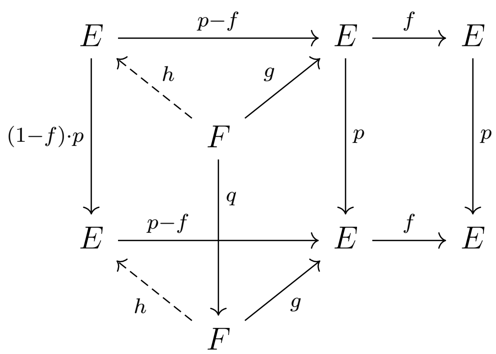
As a solution of a universal property problem the pair $(\phi, \tilde{\mathcal{C}})$ is unique upto equivalence of categories.

Proof Let's construct a pseudo abelian category we require. Consider the collection of pairs (E, p). $E \in \mathcal{Ob}(\mathcal{C})$ and $p : E \to E$ is a projection. The pair (E, p) is thought of as the image of p. A map between (E, p) and (F, q) are $\mathcal{C}$ morphisms $f : E \to F$ such that $fp = qf$.

We digress a bit to make this formulation palatable. Now p and q are projections of E. With respect to the decomposition $E \cong Im(p) \oplus Im(1-p)$ the map f has the form $\begin{pmatrix} f_1 & 0 \\ 0 & 0 \end{pmatrix}$. So it takes $Im(p)$ to $Im(q)$. We are considering the decomposition $E \cong Im(q) \oplus Im(1-q)$.

Now the composition of morphism in $\tilde{\mathcal{C}}$ is induced by the composition of morphism in $\mathcal{C}$. $(E, p) \oplus (F, q) = (E \oplus F, p \oplus q)$. The identity morphism for (E, p) is p. As $\mathcal{C}$ is **additive** so is $\tilde{\mathcal{C}}$.

Let, f be a projector of (E, p). We want to show that $Ker(p) \in \mathcal{Ob}(\tilde{\mathcal{C}})$. Let's consider the following diagram-



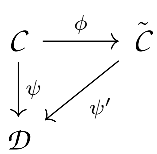
$(1-f)\cdot p$ is a projector of E. $(E,(1-f)\cdot p)\in \mathcal{O}b(\tilde{\mathcal{C}})$. $p-f$ is a morphism from this object to (E,p). Now our claim is that the pair consisting of the object the image of E under $(1-f)\cdot p$ (that is the object $(E,(1-f)\cdot p)$) and the morphism $p-f$ is the kernel of f.

Let's consider an object (F,q) and a morphism $g:(F,q)\to(E,p)$ such that $f\cdot g=0$. If we take $h:(F,q)\to(E,(1-f)\cdot p)$ that makes the diagram commutative, we have $h=(1-f)\cdot ph=p(1-f)h=g$, so h is unique. If we take $h=g$, the diagram anyway commutes.

Now we will construct the functor $\phi:\mathcal{C}\to\tilde{\mathcal{C}}$. $\phi(E):=(E,Id)$ and $\phi(f)=f$.

Using the above argument $(E,p)=Ker(E,1-p)$. So, $\phi(E)\cong(E,p)\oplus(E,1-p)$.

Now we want to construct the functor ψ'. If $\psi:\mathcal{C}\to\mathcal{D}$ is an additive functor such the following diagram commutes-



We have for all projector f, $\psi'(Ker(f))\cong Ker(\psi'(f))$. So, $\psi'(E,p)=Ker(\psi(1-p))$ and $\psi'(f)=\psi(f)_{Ker\psi(1-p)}$. These formulas define ψ' uniquely.

$\tilde{\mathcal{C}}$ is called the pseudo abelian category attached to the additive category $\mathcal{C}$. A functor is called full (respectively faithful) if it is surjective (respectively injective) when restricted to each set of morphisms between given source and target objects. A functor which is both is called **fully faithful**.

Lemma 1.6.4 *Let $\mathcal{C}$ be an additive category. $\mathcal{D}$ is a pseudo abelian category and $\psi : \mathcal{C} \to \mathcal{D}$ is an additive fully faithful functor such that every object of $\mathcal{D}$ is a direct functor of an object in the image of ψ. Then the functor ψ' we defined in lemma 1.6.3 is a categorical equivalence between $\tilde{\mathcal{C}}$ and $\mathcal{D}$.*

Proof We want to show that ψ' is fully faithful and dense.

Dense: Let $G \in \mathcal{O}b(\mathcal{D})$. By hypothesis $\exists$ $E \in \mathcal{O}b(\mathcal{C})$ and using the formulation of 4.2, there exists a projector $q : \psi(E) \to \psi(E)$ such that $G \cong Ker(q)$. As ψ is fully faithful, we can write q as $\psi(p)$, where $p : E \to E$ is a projector. Then $G \cong \psi'(E, 1-p)$.

Fully faithful: Let's consider $H, H' \in \mathcal{O}b(\tilde{\mathcal{C}})$, which are direct functors of $\phi(E)$ and $\phi(E')$. Then we have the following diagram-

$$\begin{array}{ccccc}
\tilde{\mathcal{C}}(\phi(E),\phi(E')) & \xrightarrow{\simeq} & \mathcal{C}(E,E') & \longleftrightarrow & \tilde{\mathcal{C}}(H,H') \\
 & \searrow^{\psi'_{\phi(E),\phi(E')}} & \downarrow \psi_{E,E'} & & \downarrow \psi'_{H,H'} \\
 & & \mathcal{D}(\psi(E),\psi(E')) & \longleftrightarrow & \mathcal{D}(\psi'(E),\psi'(E'))
\end{array}$$

We get the horizontal maps due to the decompositions $\phi(E) = H \oplus H_1$ and $\phi(E') = H' \oplus H'_1$. As $\psi_{E,E'}$ is an isomorphism, so is $\psi'_{H,H'}$. So, ψ' is fully faithful.

Remark 1.6.1 Let $\mathcal{C} = \xi_T(X)$, the subcategory of finite dimensional trivial bundles over X and $\mathcal{D} = \xi(X)$. Let $\psi : \mathcal{C} \to \mathcal{D}$ be the inclusion functor. Using theorem 1.6.1 every object of $\mathcal{D}$ is a direct functor of $\psi(E)$ for some $E \in \mathcal{O}b(\mathcal{C})$. Using lemma 1.6.4, $\mathcal{D}$ is equivalent to the pseudo abelian category attached to $\mathcal{C}$.

This is valid only when X is compact. Similarly for any arbitrary ring R with unit, using the notation of 1.6, $\mathcal{P}(R)$ can be considered as the pseudo abelian category attached to $\mathcal{L}(R)$.

Remark 1.6.2 Denote $C_k(X)$, the ring of continuous functions from X to k by A. Let E be a k-vector bundle over X. Using the notation of remark 1.5.2, $\Gamma(X, E)$ can be identified with A^n if E is the trivial bundle of dimension n.

If E is an arbitrary bundle, *exists* E' such that $E \oplus E' \cong A^n$. $\Gamma(X, E) \oplus \Gamma(X, E') \cong A^n$ (isomorphic as A-modules). So, $\Gamma(X, E) \in \mathcal{P}(A)$. We have actually got an additive functor $\Gamma : \xi(X) \to \mathcal{P}(A)$.

Theorem 1.6.2 (Serre-Swan) *The functor defined in remark 1.6.2 induces a categorical equivalence between $\xi(X)$ and $\mathcal{P}(A)$.*

Proof At first we want to show that the restriction of Γ on $\xi_T(X)$ gives an equivalene: $\Gamma_T : \xi_T(X) \to \mathcal{L}(A)$.

As $A^n \cong \Gamma(X, X \times k^n)$, Γ is dense. Consider $f : X \times K^n \to X \times K^p$ be any morphism. Using the notation of remark 1.2.2, the map $\Gamma_T(f)_x$ coincides with $\check{f}_x$, where $\check{f} : X \to \xi(k^n, k^p)$ for any arbitrarily fixed $x \in X$. Now using theorem 1.2.1, Γ_T is fully faithful.

Let ψ be the composition of Γ_T followed by the inclusion of $\mathcal{L}(A)$ into $\mathcal{P}(A)$. Now, let's consider the following commutative diagram-

$$\begin{array}{ccc} \xi_T(X) & \longrightarrow & \xi(X) \\ \downarrow \psi & \swarrow \Gamma & \\ \mathcal{P}(A) & & \end{array}$$

Using lemma 1.6.4, Γ can be identified with ψ' we constructed in lemma 1.6.3. We have shown Γ is a categorical equivalence.

We will use this identification in the proof of Bott's periodicity theorem in section 3.1.

1.7 Homotopy of vector bundles

Here we will take the base-space X to be compact unless otherwise stated. I denotes the closed interval $[0, 1]$.

Lemma 1.7.1 *Let E be a vector bundle over $X \times I$. $\beta_t : X \to X \times I$ and $\pi : X \times I \to X$ be defined as follows: $\beta_t(x) = (x, t)$ and $\pi(x, t') = x$. Then the vector bundles $E_0 = \beta_0^*(E)$ and $E_1 = \beta_1^*(E)$ are isomorphic.*

Proof Let's denote β_t^* as E_t. By construction E and $\pi^*(E_t)$ are isomorphic over the closed subset $X \times \{t\}$ of $X \times I$. Now using theorem 1.5.1, there exists a neighborhood U of $X \times \{t\}$ in $X \times I$ such that $\pi^*(E_t) |_U$ and E_U are isomorphic. Now as X is compact, using tube lemma U contains a $X \times V$, where V is an open neighborhood of t in I. Now if we fix t, we get a neighborhood V of t such that $E'_t \cong E_t$, $\forall t' \in U$. As I is connected, $E_0 \cong E_1$.

Theorem 1.7.1 *Let $f_0, f_1 : X \to Y$ continuous maps which are homotopic. If E is any vector bundle over Y, the pull-back bundles $f_0^*(E)$ and $f_1^*(E)$ are isomorphic.*

Proof Let $H : X \times I \to Y$ be a homotopy between f_0 and f_1. We are using the following convention: $f_t(x) = H(x,t) = H \cdot \beta_t(x)$ (β_t is defined in lemma 1.7.1). So, $f_0^*(E) = \beta_0^*(H^*(E) = (H^*(E))_0$ and $f_1^*(E) = \beta_1^*(H^*(E) = (H^*(E))_1$. Now we just have to use lemma 1.7.1 on the bundle $H^*(E)$ over $X \times I$.

Corollary 1.7.1 *If we follow the notation of lemma 1.7.1, π^*E and E are isomorphic.*

Proof Let $q : X \times I \times I \to X \times I$ defined as follows: $q(x, t_1, t_2) = (x, t_1 t_2)$. Now $\tilde{E} = q^*(E)$ is a vector bundle on $X \times I \times I$. Here $\tilde{E}_0 \cong \pi^* E_0$ and $\tilde{E}_1 \cong E$. Now we apply lemma 1.7.1 on $q^*(E)$.

Corollary 1.7.2 *If X is a compact contractible space every bundle over X is trivial.*

Proof Let x be any point of X. Let $i : \{x\} \to X$ be the inclusion and $j : X \to x$ be the projection. As per assumption, $i \cdot j$ is homotopic to Id_X. Let E be any vector bundle on X. Using theorem 1.7.1, $E \cong (i \cdot j)^*(E) \cong i^*(j^*(E))$. As $j^*(E)$ is trivial so is $i^*(j^*(E))$. So, E has to be trivial.

Theorem 1.7.2 *The map $\pi_{n-1}(GL_p(k))/\pi_0(GL_p(k)) \to \phi_p^k(S^n)$, we defined in theorem 1.3.3, is bijective.*

Proof For this we are following the notation developed to prove theorem 1.3.3. We want to show that any bundle over S^n is of form E_f, where $f : S^{n-1} \to GL_p(k)$ is a continuous map such that $f(e) = Id$. We have fixed a base point e in S^{n-1}. If E is a bundle over S^n, its restriction over S^n_+ and S^n_- are trivial (using corollary 1.7.2). Let, $E_1 := S^n_+ \times k^p$ and $E_2 := S^n_- \times k^p$. Let $g_2 : E_2 \to E_{S^n_-}$ and $g_1 : E_1 \to E_{S^n_+}$ are isomorphisms. Using theorem 1.3.2, E is isomorphic to the bundle we obtain by clutching E_1 and E_2 with respect to the clutching function $f(x) = ((g_1 |_{S^{n-1}})^{-1} g_2 |_{S^{n-1}})(x) \cdot ((g_1 |_{S^{n-1}})^{-1} g_2 |_{S^{n-1}})^{-1}(e)$.

The following rather technical result will be used later.

Lemma 1.7.2 *Let E be a vector bundle over X and p, p' are two projectors of E such that $Im(p) \cong Im(p')$. Let's define $\tilde{p} := p + 0$, and $\tilde{p'} := p' + 0$, two projectors of $E \oplus E$. Then $\exists\ \delta \in Aut(E \oplus E)$, isotopic to identity such that $\tilde{p'} = \delta \cdot \tilde{p} \cdot \delta^{-1}$.*

PROOF $E_1 := Im(p)$, $E_2 := Im(1-p)$, $E_1' := Im(p')$, $E_2' := Im(1-p')$. Using 4.2, we can write $E \oplus E$ as $E_1 \oplus E_2 \oplus E_1' \oplus E_2'$. If we take an isomorphism $\alpha : E_1 \to E_1'$, from this we can construct an automorphism δ of $E \oplus E$ as the following:

$$\delta = \begin{pmatrix} 0 & 0 & -\alpha^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With respect to the previous decomposition we get the following:

$$\tilde{p} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tilde{p'} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We have got $\tilde{p'} = \delta \cdot \tilde{p} \cdot \delta^{-1}$.
Now all we have to show is that δ is isotopic to Id.

$$\delta = \begin{pmatrix} 0 & -\alpha^{-1} \\ \alpha & 0 \end{pmatrix} \oplus Id_{T_2 \oplus T_2'}$$

$$\begin{pmatrix} 0 & -\alpha^{-1} \\ \alpha & 0 \end{pmatrix} = \begin{pmatrix} 1 & -\alpha^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & -\alpha^{-1} \\ 0 & 1 \end{pmatrix}$$

Now for $t \in I$, consider the following path in $Aut(E)$-

$$H(t) = \begin{pmatrix} 1 & -t\alpha^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t\alpha & 1 \end{pmatrix} \begin{pmatrix} 1 & -t\alpha^{-1} \\ 0 & 1 \end{pmatrix}$$

So δ is isotopic to identity.

Now we will state an interesting standard approach taken in the context of the problem of classifying vector bundles of a given rank for a given base space. (For this we have thoroughly followed [6, pp. 35–37] and [4, pp. 27–31].)

Let's look at the topological space of all projection operators q on k^N such that the dimension of $Im(q)$ is n. Let's denote it as $Proj_n(k^N)$.

Let Y to be a compact space. Consider $f : Y \to Proj_n(k^N)$, a continuous map. Using the notation of remark 1.2.2, it gives us a projection $\hat{f} : Y \times k^n \to Y \times k^n$. So, it defines a bundle $Im(f)$ of rank n over Y.

If we have another continuous map $g : X \to Y$, we have $Im(g \cdot f) = f^*(Im(g))$.

If we take the map $f : X \to Proj_n(k^N)$ and $Im : Proj_n(k^N) \to Proj_n(k^N)$, the bundle $Im(Id)$ is called the canonical line bundle over $Proj_n(k^N)$ and denoted as ξ_n, N. Let's denote the vector bundle $f^*(\xi_n, N)$ as ξ_f. The isomorphism class of the bundle ξ_f depends only on the homotopy class of the map f when X is compact (Using theorem 1.7.1). So, we have a well defined map $C_{n,N} : [X, Proj_n(k^N)] \to \phi_n^k(X)$, where $C_{n,N}(f) = \xi_f$.

We have a directed system of the collection of all $Proj_n(k^N)$ for N variable with respect to the following inclusion-like map. For $N' > N$, we represent an element $q \in Proj_n(k^N)$ as $\tilde{q} = q \oplus 0 \in Proj_n(k^{N'})$.

Taking the direct limit of this directed system, we get an induced map $C_n : inj\ limit[X, Proj_n(k^N)] \to \phi_n^k(X)$.

Lemma 1.7.3 *The map $C_n : inj\ limit[X, Proj_n(k^N)] \to \phi_n^k(X)$, defined above is bijective when X is compact.*

PROOF **a.**surjectivity: If X is compact, using the results in theorem 1.6.1 and 4.2, any vector bundle $\xi \in \phi_n^k(X)$ is isomorphic to $Im(p)$ for some projection $p : X \times k^{N'} \to X \times k^{N'}$. Now for any given bundle ξ over X, we have a projector (to be more precise, the image of $p \in limit[X, Proj_n(k^N)]$ in $inj\ limit[X, Proj_n(k^N)]$) $p \in inj\ limit[X, Proj_n(k^N)]$ such that $\xi \cong \xi_p$.

b.injectivity: We will be done if we manage to show that when we have two continuous maps $f_0, f_1 : X \to Proj_n(k^N)$ such that $\xi_{f_0} \cong \xi_{f_1}$, then (denoting the inclusion map $[X, Proj_n(k^{N_1})] \to [X, Proj_n(k^{N_2})]$ as i_{N_1,N_2} for $N_2 > N_2$) the maps $i_{N,2N}(f_0)$ and $i_{N,2N}(f_1)$ are homotopic.

Again using the notation of remark 1.2.2, consider the projectors $p_i = \hat{f}_i$ for $i \in \{0, 1\}$ of $X \times k^N$. Now we can use lemma 1.7.2, $\exists\ \delta \in Aut(k^{2N})$ isotopic to identity, such that $\overline{p_1} = \delta \cdot \overline{p_0} \cdot \delta^{-1}$. Here $\overline{p_i} = p_i \oplus 0$. So, p_0 and p_1 are homotopic and so are $i_{N,2N}(f_0) = \check{\overline{p_0}}$ and $i_{N,2N}(f_1) = \check{\overline{p_1}}$.

Corollary 1.7.3 *Let's denote $BGL_n(k) := inj\ limit[X, Proj_n(k^N)]$. For X compact we get a functor isomorphism induced from C_n, between $BGL_n(k)$ and $\phi_n^k(X)$.*

PROOF $Proj_n(k^{N_1})$ is closed in $Proj_n(k^{N_2})$ for $N_1 < N_2$. Now as X is compact we can use 4.1 to conclude the following:

$$injlimit[X, Proj_n(k^N)] = [X, injlimit(Proj_n(k^N))] = [X, BGL_n(k)]$$

Remark 1.7.1 Let's consider the usual bilinear form on k^n (we are taking $k = \mathbb{R}$ or $\mathbb{C}$). An n-dimensional subspace V of k^N gives a self adjoint projector p of k^N. Here $V = Im(p)$ and $V^{\perp} = Im(1-p)$. That's how we get a set-theoretic bijection between the collection of n-dimensional subspaces of k^N ($GL_n(k^N)$) and the subspace of self-adjoint operators of $Proj_n(k^N)$. We give $GL_n(k^N)$ subspace topology of $Proj_n(k^N)$. In 4.3, it is shown that the Grassmannian $GL_n(k^N)$ is a deformation retract of $Proj_n(k^N)$.

Theorem 1.7.3 *Combining lemma 1.7.3 and remark 1.7.1, the maps C_n induce functor isomorphisms*
$[X, BO(n)] \sim \phi_n^{\mathbb{R}}(X)$ *and* $[X, BU(n)] \sim \phi_n^{\mathbb{C}}(X)$.
Here $BO(n) = inj\ limG_n(\mathbb{R}^N)$ *and* $BU(n) = inj\ limG_n(\mathbb{C}^N)$.

1.8 Forms on vector bundles

An inner product on a vector bundle E is a positive definite symmetric bilinear from $<,>: E \oplus E \to k$, where $k = \mathbb{R}$ or $\mathbb{C}$. In this section we just want to show that if the base space X is Hausdorff paracompact, we can induce an inner product on any vector bundle using the existing inner product structure of k^n.

Theorem 1.8.1 *An inner product exist for any vector bundle over a Hausdorff paracompact base space X.*

PROOF Let's consider a vector bundle $P : E \to X$. Let (U_α) be a trivialization cover of X and we have the following homeomorphisms- $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times k^n$. We can take the exactly same inner product of k^n for $p^{-1}(U_\alpha)$. Now set $< e_1, e_2 >= \sum_\alpha \phi_\alpha p(e_1) < e_1, e_2 >_{U_\alpha}$. Where (ϕ_α) is a partition of unity subordinate to the cover (U_α) and $<,>_{U_\alpha}$ is the pullback of the inner product of k^n for U_α.

Remark 1.8.1 Using theorem 1.8.1, we can get a similar result like theorem 1.6.1 for any paracompact base space X. So, for any vector bundle E, if E_0 is a sub bundle of E, then $\exists\ E_0^{\perp}$, a vector sub bundle of E; such that $E_0 \oplus E_0^{\perp} \cong E$.

Chapter 2

Introduction to K-theory

2.1 First K group

For this section the treatment given in [6] is followed. For the basic construction of Grothendieck group of an additive category and its elementary properties please see 4.4.

Remark 2.1.1 We have seen in lemma 1.6.1, that the category $\xi_k(X)$ is an additive category. We already consider the isomorphism class of vector bundles as objects of $\xi_k(X)$. So, using the notation in remark 4.4.4, $\Phi(\xi_k(X)) = \xi_k(X)$. Symmetrization of the monoid $\xi_k(X) = K(\xi_k(X))$ (which is the Grothendieck group of the abelian category $\xi_k(X)$) is called the first K- group of the space X and denoted as $K(X)$.

Using theorem 1.6.2, $\xi_k(X)$ and $\mathcal{P}(\mathcal{A})$ are categorically equivalent if X is compact, Hausdorff. Here $\mathcal{A}$ is the ring of continuous maps from X to k and $\mathcal{P}(\mathcal{A})$ is the category of all finitely generated left (or right) projective $\mathcal{A}$ module. So, $K(\mathcal{P}(\mathcal{A}))$ and $K(X)$ are isomorphic in this scenario.

For the rest of this dissertation the focus would be to study $K(X)$ for different base spaces (X) and we will mostly consider $k = \mathbb{C}$ and occassionally $k = \mathbb{R}$ while talking about the abelian monoid $\xi_k(X)$.

Just to add, in algebraic K-theory the focus is on studying $K(\mathcal{P}(\mathcal{A}))$ for different rings $\mathcal{A}$.

So far we have defined the first K-group for any compact, Hausdorff space X. We will be considering this condition unless otherwise mentioned.

Proposition 2.1.1 *We denote the n-dimensional trivial vector bundle over X by θ^n. In $K(X)$, any element can be written as $[A] - [\theta^m]$ for some $m \in \mathbb{N}$.*

Here $[A_1] - [\theta^{m_1}] = [A_2] - [\theta^{m_2}]$ *if and only if* $\exists\, m \in \mathbb{N} \cup \{0\}$*, such that* $A_1 \oplus \theta^{m_2+m} \approx A_2 \oplus \theta^{m_1+m}$.

Proof We use the proposition 4.4.1 to get that every element x of $K(X)$ can be written as $[E_1] - [E_2]$ for some $E_1, E_2 \in \xi(X)$. Now using theorem 1.6.1, $\exists\, F_2 \in \xi(X)$ such that $E_2 \oplus F_2 = \theta^m$ for some $m \in \mathbb{N} \cup \{0\}$. Now we can write $[E_1] - [E_2] = [E_1] + [F_2] - [E_2] - [F_2] = [E] - [\theta^m]$.

Again using proposition 4.4.1, $[A_1] - [\theta^{m_1}] = [A_2] - [\theta^{m_2}]$ if and only if $\exists$ $B \in \xi(X)$ such that $A_1 \oplus \theta^{m_2} \oplus B \approx A_2 \oplus \theta^{m_1} \oplus B$. Again using theorem 1.6.1, $\exists\, B' \in \xi(X)$ such that $B \oplus B' \approx \theta^m$. Now $A_1 \oplus \theta^{m_2} \oplus B \oplus B' \approx A_2 \oplus \theta^{m_1} \oplus B \oplus B' \Rightarrow A_1 \oplus \theta^{m_2+m} \approx A_2 \oplus \theta^{m_1+m}$.

Corollary 2.1.1 *Let* $E_1, E_2 \in \xi(X)$*. In* $K(X)$*,* $[E_1] = [E_2]$ *iff* $\exists\, n \in \mathbb{N} \cup 0$ *such that* $E_1 \oplus \theta^n \approx E_2 \oplus \theta^n$.

Proof Again we get this by following corollary 4.4.1 for $\mathcal{A} = \xi(X)$.

Remark 2.1.2 This construction of first K-group actually gives us a **contravariant functor** from the category of compact, Hausdorff spaces to the category of abelian groups.

Let $f : Y \to X$ be a continuous map between two compact spaces. Now using the construction of **pull-back** bundle (remark 1.4.1), for any object $E \in \xi(X)$, we get an object $f^*(E) \in \xi(Y)$. If we consider $Id : X \to X$, $Id^*(E) = E$ and $(f \cdot g)^*(E) = f^* \cdot g^*(E)$. So, this construction of pull-back gives a contravariant functor from the category of compact spaces and continuous maps to the additive category of the monoid of collection of finite dimensional vector bundles (upto isomomorphism) and monoid homomorphisms.

Now $K(\mathcal{A})$ gives a covariant functor (remark 4.4.3) from any additive category $\mathcal{A}$ to the category of abelian groups and group homomorphism. So, combiing all this we get the claimed contravariant functor.

Remark 2.1.3 Using theorem 1.7.1, for two homotopic maps $f_0, f_1 : Y \to X$, $[f_0^*(E)] = [f_1^*(E)]$ in $K(X)$, for any $E \in \xi(X)$.

If X, Y are homotopically equivalent, $K(X) = k(Y)$. So, K-group is a **topological invariant** of the base space X.

Example 2.1.1 For any contractible space X, using corollary 1.7.1, $\xi(X) \approx \mathbb{N} \cup \{0\}$. So, using example 4.4.2, $K(X) \approx \mathbb{Z}$.

Now we will define **reduced K-groups** and show a very interesting formulation of its **representability** following [6, pp. 57–59].

Remark 2.1.4 Let, $x \in X$ be a point of the topological space X. Using remark 2.1.2, the projection $\pi : X \to \{x\}$ gives rise to a homomorphism $i' : K(x) \to K(X)$. As $K(x) \approx \mathbb{Z}$, we get a homomorphism $i : \mathbb{Z} \to K(X)$. The cokernel of i is the reduced K-group of X and is denoted by $\widetilde{K}(X)$.

If $X \neq \phi$, we have the following short exact sequence-

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\pi^*} K(X) \xrightarrow{j} \widetilde{K}(X) \longrightarrow 0$$

with a canonical splitting $K(X) \approx \widetilde{K}(X) \oplus \mathbb{Z}$, once we fix a point $x \in X$.Here we consider the induced map (i^*) by the following incusion- $i : x \hookrightarrow X$ and $i^* \cdot \pi^* = Id_{K(x)}$.

Now by $\xi(X)$, we have considered the isomorphism classes of finite dimensional vector bundles (over a fixed field) over X, following the notation of remark 4.4.4, $\Phi(\xi(X)) = \xi(X)$. Now consider the following sequence of maps- $\xi(X) \xrightarrow{s} K(X) \xrightarrow{j} \widetilde{K}(X)$

The map $j \cdot s : \xi(X) \to \widetilde{K}(X)$ is surjective. As $j(\theta^n) = 0$ and using proposition 2.1.1, any element of $K(X)$ is of form $[E] - [\theta^m]$, where $E \in \xi(X)$ and $m \in \mathbb{N} \cup \{0\}$. So, $j([E] - [\theta^m]) = j([E]) = j \cdot s(E)$.

Remark 2.1.5 Let $\alpha := j \cdot s$. In the group $\widetilde{K}(X)$, $\alpha(E_1) = \alpha(E_2)$ if and only if $\exists\ m_1, m_2 \in \mathbb{N} \cup \{0\}$ such that $E_1 \oplus \theta^{m_1} \approx E_2 \oplus \theta^{m_2}$.

In $\widetilde{K}(X)$, $\alpha(E_1) = \alpha(E_2) \Rightarrow [E_1] - [E_2] = [\theta^m] = [\theta^{n_2}] - [\theta^{n_1}] \Rightarrow E_1 \oplus \theta^{n_1} \oplus \theta^n \approx E_2 \oplus \theta^{n_2} \oplus \theta^n \Rightarrow E_1 \oplus \theta^{m_1} \approx E_2 \oplus \theta^{m_2}$, where $n_i + n = m_i$.

Proposition 2.1.2 *Let $f_0, f_1 : Y \to X$ be homotopic maps between two compact spaces Y, X. The induced maps from $\widetilde{K}(X) \to \widetilde{K}(Y)$ are same.*

Proof Using remark 2.1.3, f_0, f_1 induce same homomorphism between $K(X)$ and $K(Y)$. Condider the following diagram-

$$\begin{array}{ccc} K(Y) & \xrightarrow{f_i^*} & K(X) \\ \uparrow & \nearrow & \\ K(x) & & \end{array}$$

Here $i \in \{0, 1\}$.

As the diagram commutes the induced map between $\widetilde{K}(X)$ and $\widetilde{K}(Y)$ are same.

We will show a property of K-groups, peculiar to a cohomology theory.

Proposition 2.1.3 *Let $X = \bigsqcup_{i=1}^{i=n} X_i$ and X_i's are open in X. The inclusion $i_j : X_j \hookrightarrow X$ induces the following decomposition of $K(X)$: $K(X) = K(X_1) \times K(X_2) \times ... \times K(X_n)$.*

Proof Any element of $\xi(X)$ is completely characterised by its restrictions on each X_j. So, at the vector bundle level we have, $\xi(X) = \xi(X_1) \times \xi(X_2) \times ... \times \xi(X_n)$. So, doing symmetrization of the monoid we get $K(X) = K(X_1) \times K(X_2) \times ... \times K(X_n)$.

This is **not true** for $\widetilde{K}(X)$. For example, let's consider $X = \{x_1\} \cup \{x_2\}$ Now $K(X) \approx \mathbb{Z} \oplus \mathbb{Z}$ and $\widetilde{K}(X) \approx \mathbb{Z}$. But $\widetilde{K}(\{x_i\}) = 0$ as $K(\{x_i\}) = \mathbb{Z}$.

Remark 2.1.6 By $\xi(X)$, let's consider the collection of **isomorphism class** of finite dimensional vector bundles over X. Considering direct sum by trivial bundles, we get the following directed system-
$\xi^0(X) \to \xi^1(X) \to .... \to \xi^n(X)$. Let's denote the direct limit by $\xi(X)$. It has a monoid structure induced by direct sum of vector bundles:
$\xi^{n_1}(X) \times \xi^{n_2}(X) \to \xi^{n_1+n_2}(X)$

So far, we have considered vector bundles (E) whose dimension (dimension of E_x where $x \in X$) is constant for the whole of X. We can do away with this condition when X is not connected. In that case we consider $H^0(X, \mathbb{Z})$:= the abelian group of locally constant functions on X with value in $\mathbb{Z}$ or the first *Čech* cohomology group of X. Let's define $K'(X) := Ker(K(X) \to H^0(X, \mathbb{Z}))$. In case X is connected this is $Ker(K(X) \to \mathbb{Z}) = \widetilde{K}(X)$.

Similarly we have the following short exact sequence: $0 \to H^0(X, \mathbb{Z}) \to K(X) \to K'(X) \to 0$.

In the reminder of the chapter we will be considering bundles with globally constant dimensions and show an interesting result for $\widetilde{K}(X)$. Exactly same thing can be done for $K'(X)$.

Considering the map $E \to [E] - [\theta^n]$, where $E \in \xi(X)$, we have a monoid homomorphism between $\xi(X)$ and $\widetilde{K}(X)$ (It makes sense because remark 2.1.5).

This homomorphism is actually an isomorphism. Surjectivity is shown before. In $\widetilde{K}(X)$, $[\theta^n]$ denotes the class of $[0]$. So, $[E_1] - [\theta^{n_1}] = [E_2] - [\theta^{n_2}]$ if and only if $[E_1] = [E_2]$.

Remark 2.1.7 We denote $BO(n) := inj\ lim\ G_n(\mathbb{R}^{\mathbb{N}})$. Let's denote the directed system $BO(1) \to BO(2) \to ... \to BO(n) \to ...$ The map between $BO(n) \to BO(n+1)$ is induced by the map $j_n : G_n(\mathbb{R}^{\mathbb{N}}) \to G_{n+1}(\mathbb{R}^{\mathbb{N}+1})$, where the the subspace generated by $e_{\mathbb{N}+1} = (0, 0, ..., 1)$ is added to the previous one.

Theorem 2.1.1 *we have the following functorial equivalence:*
$\widetilde{K}_{\mathbb{R}}(X) \approx [X, BO]$

Proof $BO(n)$ is closed in $BO(n+1)$. As X compact, using 4.1, we have $[X, BO] \approx injlim[X, BO(n)]$. Using theorem 1.7.3, $\xi^n_{\mathbb{R}}(X) \approx [X, BO(n)]$. Now we have $\widetilde{K}_{\mathbb{R}}(X) \approx \xi_{\mathbb{R}}(X) \approx injlim\xi^n_{\mathbb{R}}(X) \approx injlim[X, BO(n)] \approx [X, BO]$.

Remark 2.1.8 Similarly we have $\widetilde{K}_{\mathbb{C}}(X) \approx [X, BU]$. Here BU is the direct limit of the following directed system-
$BU(1) \to BU(2) \to ... \to BU(n) \to ...$

2.2 Relative K group

Our goal is to construct a cohomology theory starting from $\xi(X)$. The $K^0(X)$ or $K(X)$ is defined for a compact space X. In this section the goal to define $K(X, Y)$ for any compact pair (X, Y). Excision for $K(X, Y)$ is shown and an exact sequence concerning $K(X)$ and $K(X, Y)$ is introduced.
For the basic construction and properties of Grothendieck group of a Banach functor please see 4.5.

Remark 2.2.1 For a **compact** space X, $\xi(X)$ can be given a **Banach category** structure. Let's denote the ring of all continuous functions from X to k ($k = \mathbb{R}$ or $\mathbb{C}$) by A. A is a Banach space. For any two vector bundles $E_1, E_2 \in \xi(X)$, there are M_1, M_2, projective A modules; such E_i corresponds to M_i, using the categorical equivalence constructed in remark 1.6.2.So, it is enough to show that $Hom_A(M_1, M_2)$ has a Banach space structure. Now we have surjections $u_i : A^{n_i} \to M_i$. So, M_i can be provided with a norm induced from the usual sup norm used for A. As M_1, M_2 are projective A modules, we get decompositions of the following form: $A^{n_1} \approx M_1 \oplus M_1'$, $A^{n_2} \approx M_2 \oplus M_2'$. So $Hom_A(M_1, M_2)$ is a closed subspace of $Hom_A(A^{n_1}, A^{n_2}) = A^{n_1 n_2}$. So, we have given $\mathcal{P}(A)$ (equivalently $\xi(X)$) a Banach category structure.

Remark 2.2.2 Let's consider a compact pair (X, Y). As, $Y \subseteq X$ is closed, Y is compact and $\xi(Y)$ is a Banach category. Let's consider the functor $\phi : \xi(X) \to \xi(Y)$. Here $\phi(E) = E_Y$.

ϕ is **full**. As using remark 1.5.3, $\alpha \in Hom(E_Y, E_Y')$, $\exists\ \alpha' \in Hom(E, E')$ such that $\alpha' \mid_Y = \alpha$.

ϕ is **quasi-surjective**. For any object $E' \in \xi(Y)$, $F' \in \xi(Y)$ such that $E' \oplus F' \approx Y \times k^m$ for some non-negative integer m. Now $\phi(X \times k^m) = Y \times k^m \approx E' \oplus F'$.

ϕ is a **Banach functor**. Using lemma 1.5.2, the following maps are identical: $Hom_{\xi(X)}(E, E') \to Hom_{\xi(Y)}(E_Y, E_Y')$ and $\Gamma(X, Hom(E, E')) \to \Gamma(Y, Hom(E_Y, E_Y'))$. The second correspondence is already shown to be continuous and linear.

Remark 2.2.3 Now, let's consider the functor ϕ defined above. $\phi : \xi(X) \to \xi(Y)$ is a quasi-surjective, full Banach functor. So, we can construct $K(\phi)$ (remark 4.5.2). Here we write $K(\phi)$ as $K(X, Y)$ (when the basic field is not clear from the context, $K_{\mathbb{R}}(X, Y)$ or $K_{\mathbb{C}}(X, Y)$ is specified.)

As ϕ is full, using remark 4.5.2, proposition 4.5.4 and corollary 4.5.1, every element of $K(X, Y)$ is of form $[E, E', \beta]$, where E, E' are vector bundles over X. $\beta : E_Y \to E'_Y$ is an isomorphism. Two triples $[E, E', \beta_1]$, $[F, F', \beta_2]$ are equivalent in the collection of triples if and only if $\exists\ [G, G, id_{G_Y}], [H, H, id_{H_Y}]$ and isomorphisms $g : E \oplus G \to F \oplus H$, $h : E' \oplus G \to F' \oplus H$, which make the following diagram commute.

$$\begin{array}{ccc}
(E \oplus G)_Y & \xrightarrow{\beta_1 \oplus Id_{G_Y}} & (E' \oplus G)_Y \\
{\scriptstyle g_Y}\downarrow & & \downarrow{\scriptstyle h_Y} \\
(F \oplus H)_Y & \xrightarrow[\beta_2 \oplus Id_{H_Y}]{} & (F' \oplus H)_Y
\end{array}$$

Remark 2.2.4 For any compact pair (X, Y), directly from theorem 4.5.1, we have the following exact sequence:

$$K(X,Y) \xrightarrow{i} K(X) \xrightarrow{j} K(Y)$$

Here $i([E,E',\alpha] = [E] - [E']$ and $j([E]-[F]) = [E_Y] - [F_Y]$.

If the inclusion $i : Y \to X$ has a left inverse, or in other words, Y is a retract of X and we have the following:

$$Y \xhookrightarrow{i} X \xrightarrow{r} Y$$

$ri = Id_Y$.

Directly from theorem 4.5.2, we have the following short exact sequence:

$$0 \longrightarrow K(X,Y) \xrightarrow{i} K(X) \xrightarrow{j} K(Y) \longrightarrow 0$$

Remark 2.2.5 Using proposition 4.5.3, we get **normalization** property for $K(X,Y)$ as $K(X,\phi) = K(X)$.

Example 2.2.1 Let's consider the pair $(X,Y) = (B^2, S^1)$. The element $[E,F,\alpha]$, we are going to define now will be extremely important in 3.1. $E = F = B^2 \times \mathbb{C}$ and $\alpha(s,u) = su$, where $s \in S^1$ and $u \in \mathbb{C}$. In 3.2, we will see that $K(B^2,S^1) \approx \mathbb{Z}$ and this $[E,F,\alpha]$ is a generator.

Proposition 2.2.1 *The correspondence* $(X,Y) \to K(X,Y)$ *is functorial.*

Proof Let, $f : (X,Y) \to (X_1,Y_1)$ is a continuous map between pairs (ie. $f(Y) \subseteq Y_1$). We get the following diagram that commutes:

$$\begin{array}{ccc} \xi(X) & \longrightarrow & \xi(Y) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f_Y^*} \\ \xi(X_1) & \longrightarrow & \xi(Y_1) \end{array}$$

So, the following correspondence makes sense:
$f^*([E,F,\beta]) = [f^*(E), f^*(F), f^*(\beta)]$.

Now, we will try to show excision for K-theory. Let's Consider a compact pair (X,Y). X/Y is the compact space we get by identifying Y to a single point. In case $Y = \phi$, X/Y is the disjoint union of X and a single point (Let's denote it by $\{\infty\}$). Here $X-Y$ is a locally compact space and $\widehat{X \dot{-} Y} = X/Y$ ($\widehat{X \dot{-} Y}$ is the one point compactification of $X-Y$).

Proposition 2.2.2 *Let's consider the projection $\pi : X \to X/Y$. Using proposition 2.2.1, we get the following induced map: $\pi^* : K(X/Y, \infty) \to K(X,Y)$. This map π^* is an isomorphism.*

Proof In case $Y = \phi$, using proposition 2.2.5, $K(X,\phi) = K(X/\phi, \{\infty\})$. For $Y \neq \phi$, we will prove the homomorphism π^* is bilinear.

a. π^* is **injective**. Let, $[E,F,\beta] \in K(X/Y,\infty)$ such that $\pi^*([E,F,\beta]) = 0$. Using proposition 2.2.1, $[\pi^*(E), \pi^*(F), \pi^*(\beta)] = 0$. Using corollary 4.5.1, $\exists T \in \xi(X)$ such that for $\pi^*(\beta) \oplus Id_{T_Y}$, we get an isomorphism $\alpha : \pi^*(E) \oplus T \to \pi^*(F) \oplus T$, such that β is the restriction of α. As in the proof of corollary 4.5.1, let's assume T to be trivial. Let, $\tilde{T}$ be the trivial bundle over X/Y of the same rank as of T. Let's consider the isomorphism $\alpha' : E \oplus \tilde{T} \to F \oplus \tilde{T}$, whose restriction on $X-Y$ is is α and on $\{\infty\}$ is β. As, this α' is an extension of $\beta \oplus Id_{T'_Y}$ on X/Y. Using corollary 4.5.1, $[E,F,\alpha] = [E \oplus \tilde{T}, F \oplus \tilde{T}, \alpha \oplus Id_{\tilde{T}}]$.

b. π^* is **surjective**. Let's consider $[E,F,\beta] \in K(X,Y)$. As, X is compact and $\exists\ F' \in K(X)$ such that $F \oplus F' \approx \theta^m$. So, wlog we can take the triple to be of form $[E, \theta^m, \alpha]$. We have to find the existence of a triple $[E_1, F_2, \alpha'] \in K(X/Y, \{\infty\})$ such that $[\pi^*(E_1), \pi^*(F_1), \pi^*(\alpha')] = [E, \theta^m, \alpha]$. Now, we use theorem 1.5.1 to find $V \subseteq X$ (a close neighborhood of Y) and $\alpha : E_V \to F_V$, an isomorphism such that $\alpha_Y = \beta$. Now, let's clutch E_{X-Y} and $\theta^m_{V/Y}$ with respect to α_{V-Y}. Let's denote the resultant bundle by E_1. Now because of the commutativity of the following diagram, we are done.

$$\begin{array}{ccc} E_Y & \xrightarrow{\beta} & \theta^m \\ \big\downarrow{\scriptstyle g_Y} & & \big\| \\ \pi^*(E_1)_Y & \xrightarrow{\pi^*(\beta')} & \pi^*(\theta^m))_Y \end{array}$$

Here $f : E \to \pi^*(E_1)$ is the isomorphism such that $f_{X-Y} = Id$ (using $\pi^*(E_{1X-Y}) = E_{1X-Y} = E_{X-Y}$) and $f_V = Id$ (Using $\pi^*(E_{1V}) = \theta^m_V$). $\beta' : E_{1\infty} \to F_{1\infty}$ is the isomorphism we get due to clutching.

Theorem 2.2.1 (Excision) *Let, $X = X_1 \cup X_2$ and X_1, X_2 are closed subsets of X. Now consider the inclusion $i : (X_1, X_1 \cap X_2) \to (X, X_2)$. The induced map $i^* : K(X_1 \cup X_2, X_2) \to K(X_1, X_1 \cap X_2)$ is an isomorphhism.*

Proof This directly follows because of the commutativity of the following diagram.

$$\begin{array}{ccccc}
(X_1, X_1 \cap X_2) & \longrightarrow & (X, X_2) & = & (X_1 \cup X_2, X_2) \\
\downarrow & & \downarrow & & \\
(X_1/(X_1 \cap X_2), \{\infty\}) & \longrightarrow & ((X_1 \cup X_2)/X_2, \{\infty\}) & &
\end{array}$$

Applying the fuctor $K(,)$ on the elements of the above mentioned diagram we get:

$$\begin{array}{ccc}
K(X_1 \cup X_2, X_2) & \longrightarrow & K(X_1, X_1 \cap X_2) \\
\uparrow & & \uparrow \\
K((X_1 \cup X_2)/X_2, \{\infty\}) & \longrightarrow & K(X_1/(X_1 \cap X_2), \{\infty\})
\end{array}$$

Here using theorem 2.2.2, the vertical maps are isomorphism, so is the map between $K((X_1 \cup X_2)/X_2, \{\infty\})$ and $K(X_1/(X_1 \cap X_2), \{\infty\})$ as $X_1 - (X_1 \cap X_2)$ and $(X_1 \cup X_2) - X_2$ are homeomorphic and so are their one point compactifications.

Example 2.2.2 We can identify B^n/S^{n-1} with S^n. Using remark 2.2.3 and proposition 2.2.2, We have the following isomorphisms:
$K(B^n, S^{n-1}) \approx K(S^n, \{\infty\}) \approx \widetilde{K}(S^n) = \pi_{n-1}(GL(k))$. In case $k = \mathbb{R}$, $\widetilde{K}(S^n) = \pi_{n-1}(O)$ and if $k = \mathbb{C}$, $\widetilde{K}(S^n) = \pi_{n-1}(U)$.
So, $\widetilde{K}_{\mathbb{C}}(S^1) = \pi_1(U) = \pi_1(U(1)) = \mathbb{Z}$. In general, in the following exact sequence:

$$K(X \times B^n, X \times S^{n-1}) \xrightarrow{u_1} K(X \times S^n, X) \xrightarrow{u_2} \widetilde{K}((X \times S^n)/X)$$

u_1 is an isomorphism. using proposition 2.2.2, u_2 is isomorphism too. So is the composition $u_2 u_1$.

Remark 2.2.6 In a similar fashion to what was done for $K()$ functor, here two homotopic maps $f_0, f_1 : (X_1, Y_1) \to (X_2, Y_2)$ give rise to the same map between $K(X_2, Y_2)$ and $K(X_1, Y_1)$.

We get the following exact sequence:
$\widetilde{K}(X/Y) \to \widetilde{K}(X) \to \widetilde{K}(Y)$
Using remark 2.2.3 and proposition 2.2.2, we have that the following sequence

is exact.
$\widetilde{K}(X/Y) \to K(X) \to K(Y)$
Now as the following diagram is commutative and the vertical sequences are exact, we have exactness of $\widetilde{K}(X/Y) \to \widetilde{K}(X) \to \widetilde{K}(Y)$.

$$
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & \mathbb{Z} & & \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
K(X,Y) & \longrightarrow & K(X) & \longrightarrow & K(Y) \\
\approx\downarrow & & \downarrow & & \downarrow \\
\widetilde{K}(X/Y) & \longrightarrow & \widetilde{K}(X) & \longrightarrow & \widetilde{K}(Y) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

2.3 K^{-1} group

In this section our target is to construct the group $K^{-1}(X)$, for any compact space X. It gives us a picture concerning the automorphisms of vector bundles. Following what is done for $K(X)$ and $K(X,Y)$, $K^{-1}(X)$ would be constructed using arguments applicable for Banach categories in general. The main objective is to construct a connecting homomorphism $\delta : K^{-1}(Y) \to K(X,Y)$, for a compact pair (X,Y). For the basic construction and properties of K^{-1} of a Banach category, please see 4.6.

Remark 2.3.1 For any compact space X, using remark 2.2.1, $\xi(X)$ is a Banach category. Using proposition 4.6.1, K^{-1} of $\xi(X)$, which is denoted by $K^{-1}(X)$ is collection of pairs (E,β), where $E \in \xi(X)$ and $\beta \in Aut(E)$; upto the following equivalence relation: $(E,\beta) \sim (E',\beta')$ if and only if $\exists$ $F \in \xi(X)$ such that $\beta \oplus Id_{E'} \oplus Id_F$ and $Id_E \oplus \beta' \oplus Id_F$ are homotopic within $Aut(E \oplus E' \oplus F)$.

Using remark 4.6.1, $K^{-1}(X)$ is an abelian group. If we denote the class of an element (E,β) in $K^{-1}(X)$ by $[E,\beta]$, $[E,\beta] = [0]$, when β is homotopic to Id_E within $Aut(E)$.

Remark 2.3.2 Following remark 2.2.1, remark 2.2.2 and remark 2.2.3, For a compact pair (X, Y), we consider the quasi-surjective Banach functor $\phi : \xi(X) \to \xi(Y)$, where $\phi(E) = E_Y$, for any vector bundle E over X. In 2.2, $K(X, Y) = K(\phi)$ is constructed. Using remark 4.6.4, the connecting homomorphism $\delta : K^{-1}(Y) \to K(X, Y)$, defined as follows is well-defined and natural. $\delta(E', \alpha') = [E, E, \alpha]$. Here $\exists\ F' \in \xi(Y)$ and an isomomorphism $h : E_Y \to E' \oplus F'$. $\alpha = h \cdot (\alpha' \oplus Id_{F'}) \cdot h^{-1}$.

Following the notation of theorem 4.6.2, if we take $\mathcal{A} = \xi(X)$ and $\mathcal{A}' = \xi(Y)$, we get the following exact sequence:

$$K^{-1}(X) \xrightarrow{j'} K^{-1}(Y) \xrightarrow{\delta} K(X, Y) \xrightarrow{i} K(X) \xrightarrow{j} K(Y)$$

Remark 2.3.3 Following the notation of remark 4.6.3, Let's consider $\mathcal{A} = \xi_T(X)$ and $\mathcal{A}' = \xi(X)$ (Here $\mathcal{A}'$ is the pseudo-abelian category attached to the additive category $\mathcal{A}$). $\xi_T(X)$ is the category of finite dimensional trivial bundles over X. Now, $K^{-1}(\xi_T(X)) = K^{-1}(\xi(X)) = K^{-1}(X)$.

Let, A be the Banach algebra of all continuous maps from X to k (the base field $\mathbb{R}$ or $\mathbb{C}$). Using theorem 1.6.2, $\xi_T(X) \sim \mathcal{L}(A)$ and $\xi(X) \sim \mathcal{P}(A)$ ($\sim$ implies categorical equivalence).

Using remark 4.6.2, $K^{-1}(\mathcal{L}(A)) = \pi_0(GL(A))$. So, $K^{-1}(X) \approx K^{-1}(\mathcal{P}(A)) \approx K^{-1}(\mathcal{L}(A)) \approx \pi_0(GL(A))$.

Remark 2.3.4 Let's consider a compact space X. k denotes $\mathbb{R}$ or $\mathbb{C}$. Because of compactness of X, using 4.1, $[X, GL(k)] = [X, injlimGL_n(k)] \approx injlim[X, GL_n(k)]$. Here in $[X, GL(k)]$, we consider the group structure induced from matrix multiplication.

Now, we consider the map $\gamma : injlim[X, GL(k)] \to K^{-1}(\xi_T(X))$ given by $\gamma(\alpha_n) = [X \times k^n, \check{\alpha_n}]$. This γ is shown to be an isomorphism.

Let's consider A, the ring of all continuous maps from X to k. Now $[X, GL_n(k)] \approx \pi_0(GL_n(A))$. Now we have the map γ, which is factored as follows:

$$\begin{array}{ccccc} injlim[X, GL_n(k)] & \longrightarrow & K^{-1}(\xi_T(X)) & \xrightarrow{\approx} & K^{-1}(\xi(X)) \\ \downarrow\approx & & \downarrow\approx & & \\ injlim(\pi_0(GL_n(A))) & \xrightarrow{\approx} & K^{-1}(\mathcal{L}(A)) & & \end{array}$$

So, γ is indeed an isomorphism.

Example 2.3.1 We are thoroughly using the notation of remark 2.3.3. Let's consider $X = \{x\}$, a single point. Then, $A \approx k$. So, $K_{\mathbb{C}}^{-1}\{x\}) \approx \pi_0(GL(\mathbb{C})) = 0$ and $K_{\mathbb{R}}^{-1}\{x\}) \approx \pi_0(GL(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$.

2.4 Extending to a cohomology theory

In this section, at first, K-functor will be defined for locally compact Hausdorff spaces. Then, taking inspiration from a property of $K^{-1}(X)$, the higher K-functors (K^{-n}) will be defined. The goal here is to introduce the long exact sequence concerning these higher K-groups. The approach taken in [6] is more or less followed.

Remark 2.4.1 Let, X be a locally compact Hausdorff space. We denote its one point compactification by $\dot{X}$. $K(X)$ is defined as $Ker[K(\dot{X}) \to K(\{\infty\})]$ and similarly $K^{-1}(X) := Ker[K^{-1}(\dot{X}) \to K^{-1}(\{\infty\})]$.

In case X itself is compact, $\dot{X} = X \cup \{\infty\}$ and $K(X) = Ker[K(\dot{X}) \to K(\{\infty\})] = \widetilde{K}(X, \{\infty\})$. It matches with the definition of $K(X)$, using proposition 2.1.3.

Similarly using remark 2.3.4 and example 2.3.1, for compact space X, the definition of $K^{-1}(X)$ matches with this.

Now we want to show that this construction of K (or K^{-1}) groups actually defines a functor from the category of locally compact spaces (and continuous maps) to the category of abelian groups. Let, X and Y are two locally compact spaces. A continuous map $f : \dot{X} \to \dot{Y}$ is such that $f(\infty_X) = \infty_Y$. Because of the commutativity of the following diagrams, the functors are well defined.

$$\begin{array}{ccc} K(\dot{Y}) & \xrightarrow{f^*} & K(\dot{X}) \\ \downarrow & & \downarrow \\ K(\infty_Y) & \xrightarrow{=} & K(\infty_X) \end{array} \qquad \begin{array}{ccc} K^{-1}(\dot{Y}) & \xrightarrow{f^*} & K^{-1}(\dot{X}) \\ \downarrow & & \downarrow \\ K^{-1}(\infty_Y) & \xrightarrow{=} & K^{-1}(\infty_X) \end{array}$$

For a proper map $f' : X \to Y$, it can be extended into a continuous map $f : \dot{X} \to \dot{Y}$, such that $f(\infty_X) = \infty_Y$. But all such maps between $\dot{X}$ and $\dot{Y}$ does not come from these proper maps.

Remark 2.4.2 Let's consider a compact pair (X, Y). Now $X - Y \approx X/Y - \{\infty\}$. So $\widehat{\dot{X - Y}} = X/Y$. By definition, $K(X - Y) = K(\widehat{\dot{X - Y}}, \{\infty\}) =$

$K(X/Y, \{\infty\}) = \widetilde{K}(X/Y)$. Using proposition 2.2.2, $\widetilde{K}(X/Y) \approx K(X,Y)$. So, using remark 2.3.2, we have the following exact sequence:

$$K^{-1}(X) \xrightarrow{j'} K^{-1}(Y) \xrightarrow{\delta} K(X-Y) \xrightarrow{i} K(X) \xrightarrow{j} K(Y)$$

Even for X locally compact and $Y \subseteq X$, closed; we get the same exact sequence using commutativity of the following diagram:

$$\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
K^{-1}(\{\infty_X\}) & \longrightarrow & K^{-1}(\{\infty_Y\}) & \longrightarrow & 0 & \longrightarrow & K(\{\infty_X\}) & \longrightarrow & K(\{\infty_X\}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
K^{-1}(\dot{X}) & \longrightarrow & K^{-1}(\dot{Y}) & \longrightarrow & K^{-1}(\dot{X}-\dot{Y}) & \longrightarrow & K(\dot{X}) & \longrightarrow & K(\dot{Y}) \\
\uparrow & & \uparrow & & \approx\uparrow & & \uparrow & & \uparrow \\
K^{-1}(X) & \longrightarrow & K^{-1}(Y) & \longrightarrow & K^{-1}(X-Y) & \longrightarrow & K(X) & \longrightarrow & K(Y) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

All the vertical sequences are exact (actually split-exact). As, the first two horizontal sequences are exact, so is the third (which is the required) one.

Theorem 2.4.1 *Let, X be a locally compact space. Then we have: $K^{-1}(X) \approx K(X \times \mathbb{R})$.*

PROOF For any two locally compact space X_1, X_2, $\dot{X}_1 \vee \dot{X}_2 := \{(\infty_{X_1}) \times x_2 \mid x_2 \in X_2\} \cup \{x_2 \times (\infty_{X_2}) \mid x_1 \in X_1\}$. Let's consider the space $X' := X \times [0,1)$. So, X' and $\dot{X} \times I - \dot{X} \vee I$. We consider the base-point of $\dot{[0,1)} = [0,1]$ to be 1. So, $\dot{X'} = \dot{X} \times I/\dot{X} \vee I$. Now, we have a homotopy $h : \dot{X'} \times I \to \dot{X'}$, where $h(x, t_1, t_2) = (x, 1 + (1-t_1)t_2)$.

Now, $K(X \times [0,1)) = \widetilde{K}(\dot{X'}) = 0$. Similarly, $K^{-1}(X \times [0,1)) = 0$. Now, as $X \subseteq X \times [0,1)$ is closed, for the pair $(X \times [0,1), X)$, we have the exact sequence:

$$K^{-1}(X \times [0,1)) \longrightarrow K^{-1}(X) \longrightarrow K(Y \times (0,1)) \longrightarrow K(X \times [0,1))$$

$Y \times (0,1) \approx Y \times \mathbb{R}$ (homeomorphic). So, $K^{-1}(X) \approx K(Y \times \mathbb{R})$.

Remark 2.4.3 Using remark 2.4.2 and theorem 2.4.1, for a locally compact space X and its closed subset Y, we have the following exact sequence:

$$K(X \times \mathbb{R}) \xrightarrow{j'} K(Y \times \mathbb{R}) \xrightarrow{\delta} K(X - Y) \xrightarrow{i} K(X) \xrightarrow{j} K(Y)$$

Definition 2.4.1 Let, X be a locally compact space and $Y \subseteq X$ is closed. We define $K^{-n}(X, Y) := K((X - Y) \times \mathbb{R}^n)$.

For $Y = \phi$, $K^{-1}(X) = K^{-1}(X, \phi)$. Previously we have defined, $K^{-1}(X, Y) := Ker[K^{-1}(X/Y) \to K^{-1}(\{\infty\})]$. This agrees with definition 2.4.1.

Remark 2.4.4 For a pair (X, Y) (with the conditions for (X, Y), used in remark 2.4.3) we have the long exact sequence:

$$K^{-n-1}(X) \longrightarrow K^{-n-1}(Y) \longrightarrow K^{-n}(X, Y) \longrightarrow K^{-n}(X) \longrightarrow K^{-n}(Y)$$

For $n = 0$, we have already shown this. For higher n, this directly follows by using remark 2.4.3, for pair $(X \times \mathbb{R}^n, Y \times \mathbb{R}^n)$. Similarly, we use remark 2.4.3 for the pair $(X_1 - X_3, X_2 - X_3)$ to get the following long exact sequence:

$$K^{-n-1}(X_1, X_3) \longrightarrow K^{-n-1}(X_2, X_3) \longrightarrow K^{-n}(X_1, X_2) \longrightarrow K^{-n}(X_1, X_3) \longrightarrow K^{-n}(X_2, X_3)$$

Here X_1 is a locally compact space. X_2 is closed in X_1 and X_3 is closed in X_2.

Remark 2.4.5 Again we will consider a locally compact space X and its closed subset Y. $S^n(X/Y) \approx (B^n/S^{n-1} \times X/Y)/(B^n/S^{n-1} \vee X/Y) \approx X \times B^n/(X \times S^{n-1} \cup Y \times B^n)$
Now we know, $X \times B^n - X \times S^{n-1} \cup Y \times B^n \approx (X - Y) \times (B^n - S^{n-1}) \approx (X - Y) \times \mathbb{R}^n$.
So, $K^{-n}(X, Y) := K((X-Y)\times\mathbb{R}^n) \approx K(X\times B^n - X\times S^{n-1} \cup Y \times B^n)\widetilde{K}(X \times B^n/(X \times S^{n-1} \cup Y \times B^n)) \approx \widetilde{K}(S^n(X/Y))$.
So, we have $K^{-n}(X, Y) \approx \widetilde{K}(S^n(X/Y))$. This is indeed expected from a cohomology theory. In [4] and [7], $\widetilde{K}(S^n(X))$ is given as definition of $K^{-n}(X)$, for a compact space X.

Remark 2.4.6 K^{-n} is a **homotopy invariant functor** of the category of locally compact spaces. Let, $f_0, f_1 : Y \to X$ are homotopic maps. We want to show that $f_0^* : K^{-n}(X) \to K^{-n}(Y)$ is exactly same as f_1^*. Enough to show it for $f_i^* : K(X) \to K(Y)$, the rest follows by applying this result for $X \times \mathbb{R}^n$ and $Y \times \mathbb{R}^n$. We have already shown it for compact space X.
At first, we want to show that $\dot{f}_0 : \dot{Y} \to \dot{X}$ and $\dot{f}_1 : \dot{Y} \to \dot{X}$ are homotopic. $\exists F : Y \times I \to X$, a homotopy between f_0 and f_1 such that $f_j = F \cdot i_j$, where $i_j : X \to X \times I$ such that $i_j(x) = (x, j)$, for $j \in \{1, 2\}$. Now using the following commutative diagram, $\dot{f}_0$ and $\dot{f}_1$ are homotopic.

$$\begin{array}{ccccc} \dot{Y} \times I & \longrightarrow & \dot{\widehat{Y \times I}} & \longrightarrow & \dot{X} \\ & \nwarrow^{\dot{i}_j} & \uparrow & \nearrow^{\dot{f}_j} & \\ & & \dot{Y} & & \end{array}$$

So, $\dot{f}_0$ and $\dot{f}_1$ induce the same homomorphism between $K(\dot{X})$ and $K(\dot{Y})$.

$$\begin{array}{ccc} K(\dot{X}) & \longrightarrow & K(\dot{Y}) \\ \downarrow & & \downarrow \\ K(\infty_Y) & \longrightarrow & K(\infty_X) \end{array}$$

As this diagram is commutative, we are done.

2.5 Mayer-Vietoris Sequence

For this section, the formulation given by Karoubi in [6] is followed.

Remark 2.5.1 Let, X be a locally compact space and T is a closed subset of X. For notational purpose, the sequence $T \to X \to X - T$ and any other sequence $X_1 \to X \to X_2$ of locally compact spaces, which is isomorphic to the previous one is called exact.
Using remark 2.4.4, we have the following exact sequence:

$$K^{-n-1}(X) \longrightarrow K^{-n-1}(X_1) \longrightarrow K^{-n}(X_2) \longrightarrow K^{-n}(X) \longrightarrow K^{-n}(X_1)$$

Theorem 2.5.1 *Let, X be a locally compact space. X_1, X_2 are two closed subsets of X and $X = X_1 \cup X_2$. Then the following sequence is exact.*

$$\begin{array}{ccccccc} K^{-n-1}(X_1) \oplus K^{-n-1}(X_2) & \xrightarrow{u_1} & K^{-n-1}(X_1 \cap X_2) & \xrightarrow{\Delta} & K^{-n}(X_1 \cup X_2) & \xrightarrow{u_2} & K^{-n}(X_1) \oplus K^{-n}(X_2) \\ & & & & & & \downarrow{\scriptstyle u_1} \\ & & & & & & K^{-n}(X_1 \cap X_2) \end{array}$$

Here, $u_1(e_1, e_2) = e_1 \mid_{X_1 \cap X_2} - e_2 \mid_{X_1 \cap X_2}$ and $u_2(e) = (e \mid_{X_1}, e \mid_{X_2})$.

Proof We haave the following commutative diagram, where the vertical sequences are exact.

$$\begin{array}{ccccc} X_1 & \longrightarrow & X_1 \cup X_2 & \longrightarrow & X - X_1 \\ \uparrow & & \uparrow & & \| \\ X_1 \cap X_2 & \longrightarrow & X_2 & \longrightarrow & X_2 - X_1 \cap X_2 \end{array}$$

Applying remark 2.5.1, we get the following-

$$\begin{array}{ccccccccc} K^{-n-1}(X_1 \cup X_2) & \longrightarrow & K^{-n-1}(X_1) & \longrightarrow & K^{-n}(X_1 \cup X_2 - X_1) & \to & K^{-n}(X_1 \cup X_2) & \longrightarrow & K^{-n}(X_1) \\ \downarrow & & \downarrow & & \| & & \downarrow & & \downarrow \\ K^{-n-1}(X_2) & \longrightarrow & K^{-n-1}(X_1 \cap X_2) & \to & K^{-n}(X_2 - X_1 \cap X_2) & \longrightarrow & K^{-n}(X_2) & \longrightarrow & K^{-n}(X_1 \cap X_2) \end{array}$$

Δ is the zig-zag composition of maps $K^{-n-1}(X_1 \cap X_2) \to K^{-n}(X_1 \cup X_2)$.
So, for $n \geq 0$, we are done for-

$$K^{-n-1}(X_1) \oplus K^{-n-1}(X_2) \xrightarrow{u_1} K^{-n-1}(X_1 \cap X_2) \xrightarrow{\Delta} K^{-n}(X_1 \cup X_2)$$

We have to show exactness of-

$$K(X_1 \cup X_2) \xrightarrow{u_2} K(X_1) \oplus K(X_2) \xrightarrow{u_1} K(X_1 \cap X_2)$$

If we can show it for X_1, X_2 compact, we can easily extend the result for X_1, X_2 locally compact using the following commutative diagram. As all the vertical sequences are split-exact and first two horizontal sequences are exact, so is the third (the required) one.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K(\infty_{\dot{X}}) & \longrightarrow & K(\infty_{\dot{X}_1}) \oplus K(\infty_{\dot{X}_2}) & \longrightarrow & K(\infty_{\dot{X}_1 \cap \dot{X}_2}) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & K(\dot{X}_1 \cup \dot{X}_2) & \longrightarrow & K(\dot{X}_1) \oplus K(\dot{X}_2) & \longrightarrow & K(\dot{X}_1 \cap \dot{X}_2) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & K(X_1 \cup X_2) & \longrightarrow & K(X_1) \oplus K(X_2) & \longrightarrow & K(X_1 \cap X_2) & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Now, it remains to show it when X_1, X_2 compact. By definition, $u_1 u_2 = 0$. Let, $e_0 = E_0 - T_0$ and $e_1 = E_1 - T_1$, where $e_i \in K(X_i)$, $T_i \approx X_i \times k^n$ and $e_0 \mid_{X_1 \cap X_2} = e_1 \mid_{X_1 \cap X_2}$. Using 2.1.1, $E_0 \mid_{X_1 \cap X_2} \approx E_1 \mid_{X_1 \cap X_2}$. We obtain an isomorphism $h : E_0 \mid_{X_1 \cap X_2} \to E_1 \mid_{X_1 \cap X_2}$. By clutching E_1 and E_2 with h as the clutching function, we get a bundle E on X. $[E] - \theta^n \in K(X_1 \cup X_2)$ is the required element. We have shown $Ker(u_1) \subseteq Im(u_2)$.

Remark 2.5.2 Exactly same sequence is obtained when X_1, X_2 are open subsets of X. We need to change the previous argument a bit to get exact sequences of locally compact spaces. We consider-

$$
\begin{array}{ccccc}
X' = X_1 - X_1 \cap X_2 & \longrightarrow & X_1 & \longrightarrow & X_1 \cap X_2 \\
\| & & \uparrow & & \uparrow \\
X' = X_1 \cup X_2 - U_2 & \to & X_1 \cup X_2 & \longrightarrow & X_2
\end{array}
$$

Applying remark 2.5.1, we get-

$$
\begin{array}{ccccccccc}
K^{-n-1}(X_1) & \longrightarrow & K^{-n-1}(X_1 - X_1 \cap X_2) & \to & K^{-n}(X_1 \cup X_2) & \longrightarrow & K^{-n}(X_1) & \longrightarrow & K^{-n}(X') \\
\downarrow & & \| & & \downarrow & & \downarrow & & \| \\
K^{-n-1}(X_1 \cup X_2) & \to & K^{-n-1}(X_1 \cup X_2 - X_2) & \longrightarrow & K^{-n}(X_2) & \longrightarrow & K^{-n}(X_1 \cup X_2) & \to & K^{-n}(X')
\end{array}
$$

Now, exactly by the previous argument used in the proof of theorem 2.5.1, we are done.

2.6 Ring structure

In any cohomology theory, multiplicative ring structure is found due to contravariance. For K-theory as well, we find ring structure in $K(X)$.

Remark 2.6.1 External tensor product: Let, E_1, E_2 are vector bundles respectively over X_1, X_2, two compact spaces. Let's consider projections $\pi_1 : X_1 \times X_2 \to X_1$ are $\pi_2 : X_1 \times X_2 \to X_2$. For notational simplicity, let's denote the pull-back of E_i with respect to π_i by $\pi^*(E_1)$. Now on $X \times Y$, we have got a bundle $\pi^*(E_1) \otimes \pi^*(E_2)$.
So, we have got a map $\xi(X_1) \times \xi(X_2) \to \xi(X_1 \times X_2)$ given by $(E_1, E_2) \to \pi^*(E_1) \otimes \pi^*(E_2)$. We denote this external product as $E_1 \boxtimes E_2$. This is actually a bilinear functor. Let's denote it by ϕ. $\phi(E_1, E_2) = E_1 \boxtimes E_2$.
Similarly, we get a map $K(X_1) \times K(X_2) \to K(X_1 \times X_2)$ as follows:
$\phi(([E_1] - [F_1]), ([E_2] - [F_2])) = [E_1 \boxtimes E_2] + [F_1 \boxtimes F_2] - [E_1 \boxtimes F_2] - [E_2 \boxtimes F_1]$.
This is called **cup-product** and denoted by $x_1 \cup x_2$ for $x_1 \in K(X_i)$.

As tensor product is associative, for compact spaces X, Y, Z the following diagram is commutative.

$$\begin{array}{ccc} K(X) \times K(Y) \times K(Z) & \longrightarrow & K(X \times Y) \times K(Z) \\ \downarrow & & \downarrow \\ K(X) \times K(Y \times Z) & \longrightarrow & K(X \times Y \times Z) \end{array}$$

Because of commutativity of tensor product, the following diagram commutes:

$$\begin{array}{ccc} K(X) \times K(Y) & \longrightarrow & K(X \times Y) \\ \downarrow P & & \downarrow P^* \\ K(Y) \times K(X) & \longrightarrow & K(Y \times X) \end{array}$$

P is the following map. For $(x_1, x_2) \in K(X) \times K(Y)$, $P(x, y) = (y, x)$.

Remark 2.6.2 Ring structure in $K(X)$: Let's consider the diagonal map $\Delta : X \to X \times X$, $\Delta(x) = (x, x)$. Now we get the induced map $\Delta^* : K(X \times X) \to K(X)$. Now, let's consider the following composition:

$$K(X) \times K(X) \xrightarrow{\cup} K(X \times X) \xrightarrow{\Delta^*} K(X)$$

We have got a multiplicative structure for $K(X)$. Using the last commutative diagram of remark 2.6.1, K(X) is a commutative ring with this multiplication. For $k = \mathbb{R}$ or $\mathbb{C}$, $[\theta^1]$, the class of rank one trivial bundle is the multiplicative identity.

Remark 2.6.3 Now we want to extend this cup-product for $K(X), K(Y)$, where X, Y are locally compact Hausdorff space. At first we will try to construct a "cup product" in this context which happens to be commutative and associative. In case of compact spaces, the two construction match as well. Then, we will show this the unique way to extend the cup product construction for locally compact spaces.
Let, $x \in K(X) = K(X, \infty)$ and $y \in K(Y) = K(Y, \infty)$. $x \cup y \in K(\dot{X} \times \dot{Y})$ and the restriction of $x \cup y$ on $K(\dot{X} \vee \dot{Y})$ has to be zero. (Here, we are considering the inclusion of the element x of $K(X)$ in $K(\dot{X})$).
We already have $\theta' : K(\dot{X}) \times K(\dot{Y}) \to K(\dot{X} \times \dot{Y})$. Its restriction $\theta : K(X, \infty) \times K(Y, \infty) \to K(\dot{X} \times \dot{Y}, \dot{X} \vee \dot{Y})$ gives us what we want.
Now, we have the following exact sequence:
$K^{-1}(\dot{X} \times \dot{Y}) \to K^{-1}(\dot{X} \vee \dot{Y}) \to K(X \times Y) \to K(\dot{X} \times \dot{Y}) \to K(\dot{X} \vee \dot{Y})$
If we can show that the sequence $0 \to K(X \times Y) \to K(\dot{X} \times \dot{Y}) \to K(\dot{X} \vee \dot{Y})$ is exact, due to commutativity of the following diagram, we will be done in proving uniqueness of θ.

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ K(X) \times K(Y) & \xrightarrow{\theta} & K(X \times Y) \\ \downarrow & & \downarrow \\ K(\dot{X}) \times K(\dot{Y}) & \longrightarrow & K(\dot{X} \times \dot{Y}) \\ & & \downarrow \\ & & K(\dot{X} \vee \dot{Y}) \end{array}$$

Now, if we can show the map $K^{-1}(\dot{X} \times \dot{Y}) \to K^{-1}(\dot{X} \vee \dot{Y})$, is surjective, we are done in showing exactness of $0 \to K(X \times Y) \to K(\dot{X} \times \dot{Y}) \to K(\dot{X} \vee \dot{Y})$.

Following the identification showed in remark 2.3.4, for any $[E, \beta] \in K^{-1}(\dot{X} \vee \dot{Y})$, $\beta : \dot{X} \vee \dot{Y} \to GL(k)$. Now β is just restriction of $\beta' : \dot{X} \times \dot{Y} \to GL(k)$, where $\beta'(x_1, x_2) = \beta_{\dot{X}}(x_1)\beta_{\dot{X}}^{-1}(\infty)\beta_{\dot{Y}}(x_2)$.

Remark 2.6.4 Now, we want to extend the idea of inner product for compact pairs. For compact pairs (X, X_1) and (Y, Y_1), we want a bilinear functor $K(X, X_1) \times K(Y, Y_1) \to K(X \times Y, X \times Y_1 \cup X_1 \times Y)$.
As $X \times Y - X \times Y_1 \cup X_1 \times Y = (X - X_1) \times (Y - Y_1)$, using the construction done in remark 2.6.3 for locally compact spaces $(X - X_1), (Y - Y_1)$, we get a product.
In case $X = Y$, we have the product : $K(X, X_1) \times K(X, Y_1) \to K(X \times X, X_1 \cup Y_1) \to K(X, X_1 \cup Y_1)$. The last map is induced by the diagonal map.

This section will be used in 3.2, to calculate the ring structure of $K(X)$, for certain ubiquitous spaces X.

Chapter 3

Some computation regarding K groups

3.1 Bott's Periodicity

In this chapter, we are exclusively considering complex K-theory. For most of these chapter, the focus will be on clarifying and summarizing the approaches and ideas rather than meticulous technicalities. Bott's periodicity lies in the heart of classical K-theory. We have constructed infinitely many K-functors from the category of locally compact spaces to the category of abelian groups. Here we are going to show that upto group isomorphism $K^0_{\mathbb{C}}$ and $K^{-1}_{\mathbb{C}}$ are the only ones. The approach taken in [6], following the ideas developed in [8] is thoroughly followed.

Theorem 3.1.1 *Let,X be a locally compact space and Y is a closed subspace of X. Then, $K^{-n}(X,Y)$ is isomorphic to $K^{-n-2}(X,Y)$, for $n \in \mathbb{N} \cup \{0\}$.*

Remark 3.1.1 As, $K^{-n-2}(X,Y) \approx K^{-n}(X \times B^2, X \times S^{-1} \cup Y \times B^2)$, we have to prove the existence of an isomorphism $\alpha : K^{-n}(X,Y) \to K^{-n}(X \times B^2, X \times S^1 \cup Y \times B^2)$.
If we can prove it for $K(X,Y)$ and $K(X \times B^2, X \times S^1 \cup Y \times B^2)$, the rest is done just by replacing (X,Y) by $(X \times B^n, X \times S^{n-1} \cup Y \times B^n)$.
Now, we have the following diagram that commutes:

$$\begin{array}{ccc} K(X,Y) & \xrightarrow{\quad\alpha\quad} & K(X\times B^2, X\times S^1\cup Y\times B^2) \\ \approx\uparrow & & \approx\uparrow \\ K(X/Y,\{\infty\}) & \xrightarrow{\alpha} & K(X/Y\times B^2, X/Y\times S^1\cup\{\infty\}\times B^2) \end{array}$$

So, it is enough to show for $Y=\{x\}$, a single point and X, a compact space.

So, we want to show the following:

Theorem 3.1.2 *Let, X be a compact space. Then, we have a group isomorphism between $K(X)$ and $K(X\times B^2, X\times S^1)$.*

Now, as we have done before, let's denote the ring of all continupus maps from X to $\mathbb{C}$ by A. Now, we want to find a connection between $K(X\times B^2, X\times S^1)$ and $GL(A)$, the direct limit of the inductive system: $GL_1(A)\to GL_2(A)\to \ldots.\to GL_n(A)\to \ldots.$

Proposition 3.1.1 *Every element of $K(X\times B^2, X\times S^1)$ is of form $[T,T,\beta]$,where T is a trivial bundle and $\beta(x,e)=Id$.*
$[T,T,\beta_1]=[T,T,\beta_2]$ iff $\exists T'$(a trivial bundle) such that $\beta_1\oplus Id_{T'|_{X\times S^1}}$ is homotopic to $\beta\oplus Id_{\tilde{T}|_{X\times S^1}}$ within normalized automorphisms of $T\oplus T'$.

Proof Let's consider any element $[E_1', E_2', \alpha]$ of $K(X\times B^2, X\times S^1)$. As, B^2 is contractible, $\pi: X\times B^2\to X$ is a homotopy equivalence. So, we can consider E_i' to be of form $\pi^*(E_i)$, for $i\in\{1,2\}$. If we restrict the isomorphism α on $X\times\{e\}$ (e is the arbitrarily fixed base point of S^1), we get an isomorphism $\pi^*(\alpha_e):\pi^*(E_1)\to\pi^*(E_2)$. Now, we have-
$(\Pi^*(E_1'),\Pi^*(E_2'),\alpha)=(\Pi^*(E_1'),\Pi^*(E_2'),\alpha)+[\Pi^*(E_2'),\Pi^*(E_1'),\Pi^*(\alpha_e^{-1}\mid_{X\times S^1})]=[\Pi^*(E_1'),\Pi^*(E_1'),\gamma]$ (Using proposition 4.5.3). Where $\gamma=\Pi^*(\alpha_e^{-1}\mid_{X\times S^1})\cdot\alpha$. So, γ is normalized.
If $F_1'\in\xi(X)$, such that $E_1'\oplus F_1'\approx T'$, a trivial bundle. $[\pi^*(E_1'),\pi^*(E_2'),\gamma]=[\pi^*(E_1'),\pi^*(E_2'),\gamma]+[\pi^*(F_1'),\pi^*(F_2'),Id]=[T,T,\beta]$. Here, $T=\pi^*(T')$, β is normalized.

If an element $[T,T,\beta]=0\in K(X\times B^2, X\times S^1)$, using proposition 4.5.1, $\exists\ \tilde{T}\in\xi(X\times B^2)$, another trivial bundle such that we have an automorphism $\tilde{\beta}: T\oplus\tilde{T}\to T\oplus\tilde{T}$. Here, $\tilde{\beta}\mid_{X\times S^1}=\beta\oplus Id$.

Let, $dim(T \oplus \tilde{T}) = n$. Now, we want to construct a map $h : X \times S^1 \times I \to GL_n(\mathbb{C})$. Using the notations of remark 1.2.2, $h(x,s,t) = \check{\tilde{\beta}}(x,st)(\check{\tilde{\beta}}(x,et))^{-1}$. This h actually gives a normalized homotopy between $\beta \oplus Id_{\tilde{T}|_{X\times S^1}}$ and $Id_{T\oplus\tilde{T}|_{X\times S^1}}$.

Let's take two triples $[T,T,\alpha_1] = [T,T,\alpha_2] \Rightarrow [T,T,\alpha_1(\alpha_2)^{-1}] = 0$. Using the formulation we just did, $\alpha_1(\alpha_2) \oplus Id_{\tilde{T}|_{X\times S^1}}$ is homotopic to $Id_{T\oplus\tilde{T}|_{X\times S^1}}$. Just by right-multiplying by $\alpha_2 \oplus Id_{\tilde{T}|_{X\times S^1}}$, we get $\alpha_1 \oplus Id_{\tilde{T}|_{X\times S^1}}$ and $\alpha_2 \oplus Id_{\tilde{T}|_{X\times S^1}}$ are homotopic within $Aut((T \oplus \tilde{T}) \mid_{X\times S^1})$ through normalized homotopies.

Remark 3.1.2 Let's consider a triple $[E,E,\beta] \in K(X \times B^2, X \times S^1)$. So, E can be considered a trivial bundle and β, a normalized automorphism. Again using notations of remark 1.2.2, we get the following continuous map: $\check{\beta} : X \times S^1 \to GL_n(\mathbb{C})$ and $\check{\beta}(x,e) = Id$, for any $x \in X$. Let's denote the space of all continuous maps from X to $GL_n(\mathbb{C})$ by $H(X, GL_n(\mathbb{C}))$. So, we get a map $\sigma : S^1 \to H(X, GL_n(\mathbb{C}))$. Thus we get $\sigma : S^1 \to GL_n(A)$.

So, we get a correspondence from $inj\ lim\ \pi_1(GL_n(A)) = \pi_1(GL(A))$ to $K(X \times B^2, X \times S^1)$ given by $\beta \to [E,E,\beta]$, where E is any trivial bundle over $X \times B^2$.

Using proposition 3.1.1, this is actually an isomorphism.

Now, instead of looking directly at $K(X \times B^2, X \times S^1)$, **we will try to find an isomomorphism between $K(X)$ and $\pi_1(GL(A))$.**

Remark 3.1.3 Using theorem 1.6.2, we have a categorical equivalence between $K(X)$ and $\mathcal{P}(A)$, for compact X. Let, $E \in \mathcal{P}(A)$. So, we get a projector q of A^m, for some $m \in \mathbb{N}$ such that $Im(q) \approx E$. Now, we get a map $\gamma : S^1 \to GL_n(A)$ given by $\gamma(u) = qu + 1 - q$. So, thus we get a map from $\mathcal{P}(A)$ to $\pi_1(GL_n(A))$. Passing through direct limit, we get a correspondence from $\mathcal{P}(A)$ to $\pi_1(GL(A))$.

Proposition 3.1.2 *The correspondence γ is well-defined or the element $\gamma(E) \in \pi_1(GL(A))$ does not depend on the choice of projector q.*

Proof When $E \approx F$, we want to show that $\gamma(E,p_1) = \gamma(F,p_2)$ in $\pi_1(GL_n(A))$.Here, $E \approx p_1(A^{n_1})$ and $F \approx p_1(A^{n_2})$ for some $n_1, n_2 \in \mathbb{N}$.
We can represent p_1 and p_2 as follows:

$$p_1 = \begin{pmatrix} p_{1(n_1\times n_1)} & 0 \\ 0 & 0_{(n_2\times n_2)} \end{pmatrix} \quad p_2 = \begin{pmatrix} 0_{(n_1\times n_1)} & 0 \\ 0 & p_{1(n_2\times n_2)} \end{pmatrix} \text{ set } n_1 + n_2 = n$$

Now using, proposition 1.7.2, In $GL_{2n}(A)$, $\exists\delta$ homotopic to Id such that $p_2 = \delta p_1 \delta^{-1}$. Let, H be a homotopy between δ and Id within $GL_{2n}(A)$. Now, we get the following homotopy (θ) between the loops associated to p_1, p_2. $\theta : S^1 \times I \to GL_{2n}(A)$ such that $\theta(z,t) = H(t)(pz + 1 - p)H^{-1}(t)$. So, we are done.

Proposition 3.1.3 *The correspondence γ is actually a group homomorphism.*

Proof In $\pi_1(GL(A))$, we consider the group structure we get from matrix multiplication in $GL(A)$ as the group structure gotten from concatenation of loops is equivalent to the group structure induced from the existing group operation in topological group are equivalent in fundamental group of any topological group. Let, $E_1 \approx p_1(A^{n_1})$ and $E_2 \approx p_2(A^{n_2})$ in $\mathcal{P}(A)$. $E_1 \oplus E_2 \approx Im(p_1 \oplus p_2)$. We represent p_1 as $p_1 \oplus 0$ and p_2 as $0 \oplus p_2$. Now, $(p_1 z + 1 - p_1)(p_2 z + 1 - p_2) = ((p_1 \oplus p_2)z + 1 - (p_1 \oplus p_2))$.

Remark 3.1.4 Let's consider a special element u in $K(B^2, S^1)$. $u = [B^2 \times \mathbb{C}, B^2 \times \mathbb{C}, \beta]$, where $\beta(s,x) = (s, sx))$ for $s \in S^1$. The map $\gamma : K(A) \to \pi_1(GL(A))$ is actually cup-product by the element u. We essentially want to show the following diagram is commutative.

$$\begin{array}{ccc} K(X) & \xrightarrow{\psi} & K(X \times B^2, X \times S^1) \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} \\ K(A) & \xrightarrow{\gamma} & \pi_1(GL(A)) \end{array}$$

Here ϕ_1 and ϕ_2 are isomorphisms and ψ is the cup product with u.
Let, $E, E' \in \xi(X)$ such that $E \oplus E' \approx A^n$. Now, $\psi([E]) = [E \times B^2, E \times B^2, \beta]$. $\beta : E \times S^1 \to E \times S^1$ and $\beta(e,s) = (se, s)$. Now, $\psi([E]) = [(E \oplus E') \times B^2, (E \oplus E') \times B^2, \tilde{\beta}]$, where $\tilde{\beta} = \beta \oplus Id_{E' \times S^1}$. Let, $p : A^n \to A^n$, a projector such that $Im(p) \approx E$. Again, using notations of remark 1.2.2, $\check{\beta}(s,t) = t\check{p}(s) + 1 - \check{p}(s)$. So, $\phi_2\psi([E]) = \gamma\phi_1([E])$.

Now we want to show that the map $\gamma : K(A) \to \pi_1(GL(A))$ is an isomorphism. At first we will show injectivity of γ by constructing its left inverse $\tilde{\gamma}$.

At first, we will focus on $\pi_1(GL(A))$.
A loop σ based at Id in $\pi_1(GL(A))$ is called a **Laurential** if it is of the form $\sum_{k=-N}^{k=N} a_k z^k . a_k \in M_n(A), z \in S^1$.

Now, we consider only the Laurential loops and Laurential homotopy (at each time t, the loop we get has to be Laurential) and call this collection $\pi_1^L(GL(A)$.

Now, we want to show that $\pi_1^L(GL(A)$ is actually $\pi_1(GL(A)$. For every loop in $\pi_1(GL(A)$, there is a Laurentian loop which is homotopic to it and If we have any two homotopic Laurential loops, there has to be a Laurential loop between them.

Proposition 3.1.4 *$\pi_1^L(GL(A)$ and $\pi_1(GL(A)$ are equivalent.*

Proof At first we will show that $\forall \sigma \in \pi_1(GL(A)$, $\exists \sigma^L \in \pi_1^L(GL(A)$ such that they are homotopic in $\pi_1(GL(A)$.

Let, $\sigma \in \pi_1(GL_n(A)$. Let's define $a_k := \frac{1}{2\pi}\int_0^{2\pi} \sigma(e^{i\theta})e^{-ik\theta}d\theta$ and $s_k := \sum_{l=-k}^{l=k} a_k z^k$. $\sigma'_k := \frac{s_0+s_1+\ldots+s_k}{k+1}$ is the k-th **Cesaro mean** of σ. Now, $\sigma : S^1 \to GL(A)$, a continuous map. As, S^1 is compact, using 4.1, σ has to factor through a finite stage. So, we can consider the map $\sigma : S^1 \to GL_n(A)$, for some $n \in \mathbb{N}$. As, $M_n(A)$ is a Banach space, we can use **Fejer's** theorem to say σ'_k converges to σ uniformly. For a proof, please see [10, pp. 86–88].
Now, $\sigma(e) = Id$. So, for k large enough, $\sigma'_k(e)$ lies inside the open ball of radius 1 in $M_n(A)$, which is contained inside $GL_n(k)$. Now, as the norm in $M_n(A)$ follows triangle inequality, we have local convexity of $GL_n(k)$ in $M_n(A)$. So, the linear homotopy between σ'_k (for large enough k) and σ' $u\sigma'_k(s) + (1-u)\sigma(s)$ lies in $GL_n(k)$, $\forall u \in [0,1]$.
Now, we defne $\sigma_k := \sigma'_k \bullet (\sigma'_k(e))^{-1}$. Considering the homotopy $(s,t) \to (t\sigma'_k(s) + (1-t)\sigma(s))(t\sigma'_k(s) + (1-t)\sigma(s))^{-1}$ between σ and σ_k, we are done.

Let, $s : S^1 \times I \to GL_n(A)$, be a homotopy between Laurentian loops σ, τ. Now, consider the Bnach algebra $A(I)$ of all continuous maps from I to A. Now, $s : S^1 \to GL_n(A(I))$ is a continuous map. Using the previous argument, there exists a linear homotopy in $GL_n(A(I))$ between s_k and s. for k large. s_k is a Laurentian loop in $GL_n(A(I))$. Now, we combine all the data we have to get the following map $r : S^1 \times I \to GL_n(A)$.

- $r(z,t) = 3ts(z,0) + (1-3t)s_k(z,0)$ for $0 \leq t \leq \frac{1}{3}$
- $r(z,t) = s_k(z, 3t-1)$ for $\frac{1}{3} \leq t \leq \frac{2}{3}$
- $r(z,t) = (3t-2)s(z,1) + (3-3t)s_k(z,1)$ for $\leq \frac{1}{3} \leq t \leq 1$

This r is a Laurential homotopy between σ and τ. So, we are done.

Remark 3.1.5 Let's consider the Banach algebra $A = C(X)$, the ring of continuous maps from X to $\mathbb{C}$. Let's consider an $A(I)$ module E. Let's denote the restriction of E on $\{0\}$ and $\{1\}$ by E_0 and E_1 respectively. Applying proposition 1.7.1, we get that E_0 and E_1 have to be isomorphic as A modules.

Remark 3.1.6 Let, $\sigma_z \in \pi_1^L(GL_n(A))$ be any Laurential loop. We get its inverse loop $\tau := (\sigma)^{-1}$, which is C^∞. By construction of a Laurential loop $\sigma_z = \sum_{l=-k}^{l=k} a_l z^l$. Let's consider the Fourier series of τ. Let's denote it by $\tau_z = \sum_{l=-\infty}^{l=\infty} b_l z^l$. Now, we have $\sigma_z^k := z^k \sum_{l=-k}^{l=\infty} a_l z^l$, $\tau_z^k := z^k \sum_{l=-k}^{l=\infty} b_l z^l$. So, $\sigma_z^k \tau_z^k = z^{2k}(1 + \epsilon_k(z))$. Here,$\epsilon_k(z) \to 0$ as $k \to 0$.

Let's consider the Banach algebras
$A < z >$, $A < z, z^{-1} >$. Where $A < z >$ consists of formal power series $\sum_{k=0}^{\infty} a_k z^k$, where $\sum_{k=0}^{\infty} |a_k| < \infty$ and
$A < z, z^{-1} >$ consists of formal Laurent series $\sum_{k=-\infty}^{\infty} a_k z^k$, where
$\sum_{k=-\infty}^{\infty} |a_k| < \infty$.

Remark 3.1.7 If we choose k large enough such that $\sigma_z^k \in End(A < z >) = M_n(A < z >)$ and $(1 + \epsilon_k(z))$ is invertible.
We define $M_n(\sigma, k) :=$ cokernel(σ_z^k). Under the assumptions of the previous line we want to show that $M_n(\sigma, k)$ is a finitely generated projective A module.

We have the following sequence:

$$A < z > \xhookrightarrow{i} A < z, z^{-1} > \xmapsto{P} A < z >$$

Here i is inclusion and $P(\sum_{k=-\infty}^{\infty} a_k z^k) = \sum_{k=0}^{\infty} a_k z^k$.
We have the exact sequence:

$$(A < z >)^n \xrightarrow{\sigma_z^k} (A < z >)^n \longmapsto M_n(\sigma, k)$$

Let's define $\theta_z^k := P z^{-k} \tau_z i$.As, $\theta_z^k \sigma_z^k = Id$, we get a splitting for the previous sequence:

$$(A < z >)^n \underset{\theta_z^k}{\overset{\sigma_z^k}{\rightleftarrows}} (A < z >)^n \longmapsto M_n(\sigma, k)$$

So, $M_n(\sigma, k)$ is a direct factor of $(A < z >)^n$ as an A-module. Now, we want to show that it is a finitely generated projective A module. We get the following commutative diagram.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (A < z >)^n & \xhookrightarrow{\sigma_z^k} & (A < z >)^n & \xmapsto{\pi_1} & M_n(\sigma, k) & \longrightarrow & 0 \\
 & & \Big\uparrow {\scriptstyle \tau_z^k(1+\epsilon_k(z))^{-1}} & & \Big\| & \xleftarrow{s} & \Big\uparrow {\scriptstyle p} & & \\
0 & \longrightarrow & (A < z >)^n & \xhookrightarrow{z^{2k}} & (A < z >)^n & \xmapsto{\pi_2} & (A^n)^{2k} & \longrightarrow & 0
\end{array}
$$

Now, the map p is surjective. So, we are done.

Remark 3.1.8 We have $z^l \sigma_z^k = \sigma_z^{k+l}$. Let's consider the following sequence of A-modules:

$$(A < z >)^n \xrightarrow{\sigma_z^k} (A < z >)^n \xrightarrow{z^l} (A < z >)^n$$

Now, we have the short exact sequence of cokernels-

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & cok(\sigma_z^k) & \longrightarrow & cok(\sigma_z^{k+l}) & \longrightarrow & cok(z^l) & \longrightarrow & 0 \\
 & & \Big\| & & \Big\| & & \Big\| & & \\
0 & \longrightarrow & M_n(\sigma, k) & \longrightarrow & M_n(\sigma, k+l) & \longrightarrow & A^{nl} & \longrightarrow & 0
\end{array}
$$

As, A^{nl} is free (thus projective) A module, the sequence splits. So, we get $M_n(\sigma, k+l) = M_n(\sigma, k) \oplus A^{ln}$.

Now we are ready to construct a map $\gamma' : \pi_1^L(GL(A)) \to K(A)$, which will be proven to be the left inverse of γ.

Remark 3.1.9 Construction: Let, $\sigma \in \pi_1^L(GL(A))$. $\gamma'(\sigma) = M_n(\sigma, k) - A^{nk} \in K(A)$, k, n st $z^k \sigma_z \in M_n(A < z >)$. We choose k large enough such that $(1 + \epsilon_z^k)$ is invertible. Now, at first, we want to show that this construction does make sense.

Proposition 3.1.5 γ' *is a well-defined homomorphism.*

Proof Using remark 3.1.8, whenever $(1 + \epsilon_k(z))$ is invertible the map γ' does not depend on the choice of k.

Now, suppose we have two homotopic loops σ_0 and σ_1. We want to show that $\gamma'(\sigma_0) = \gamma'(\sigma_1)$. Now, here we have a laurential homotopy $\sigma : S^1 \times I \to \pi_1(GL_n(A))$ between σ_0 and σ_1. Basically, we have a loop σ :

$S^1 \to \pi_1(GL_n(A(I)))$ and the restrictions of σ on $\{0\}$ and $\{1\}$ are σ_0 and σ_1 respectively. Similarly, the restriction of $M_n(\sigma, k)$ on $\{0\}$ and $\{1\}$ are $M_n(\sigma_0, k)$ and $M_n(\sigma_1, k)$ respectively. Now, using remark 3.1.5, $M_n(\sigma_0, k)$ and $M_n(\sigma_1, k)$ are isomorphic. So, the map γ does make sense.

Now, we want to show that this is a homomorphism. Let's consider two Laurential loops σ, τ. We are again considering the group structure induced from matrix multiplication in $\pi_1(GL(A))$. For large enough k_1, k_2, we want to show that $[M_n(\sigma\tau, k_1+k_2)] - [A^{n(k_1+k_2)}] = [M_n(\sigma, k_1)] - [A^{n(k_1)}] + [M_n(\tau, k_2)] - [A^{n(k_2)}]$. It's enough to show that $[M_n(\sigma\tau, k_1+k_2)] \approx [M_n(\sigma, k_1)] + [M_n(\tau, k_2)]$. We have $(\sigma\tau)^{k_1+k_2} = \sigma^{k_1} \cdot \tau^{k_2}$. So, we have the following short exact sequence of A-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & cok(\sigma_z^{k_1}) & \longrightarrow & cok(\sigma_z^{k_1}\tau_z^{k_2}) & \longrightarrow & cok(\tau_z^{k_2})) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & M_n(\sigma, k_1) & \longrightarrow & M_n(\sigma\tau, k_1+k_2) & \longrightarrow & M_n(\tau, k_2) & \longrightarrow & 0
\end{array}$$

As, $M_n(\tau, k_2)$ is a projective A-module, the sequence splits and we are done.

Proposition 3.1.6 *γ' is left inverse of γ.*

PROOF Let, $E \in \mathcal{P}(A)$. As, E is a finitely generated projective A-module, $\exists\, p$, a projector of A^n, for some $n \in \mathbb{N}$, such that $p(A^n) = E$. Let's denote $(1-p)(A^n)$ by E'. As defined before, $\gamma(E) = pz + 1 - p$. Let's consider the following short exact sequence:

$$0 \longrightarrow (A < z >)^n \xrightarrow{pz+1-p} (A < z >)^n \longmapsto E \longrightarrow 0$$

$(A < z >)^n$ can be identified with $A^n < z > \approx E < z > \oplus E' < z >$. Now, On E, $pz + 1 - p$ is multiplication by z and on E', it is Id. So, $\gamma'\gamma(E) = E$. As, $\gamma'\gamma$ is a homomorphism, $\gamma'\gamma([E]-[F]) = \gamma'\gamma([E]) - \gamma'\gamma([F]) = [E] - [F]$. So, we have shown that $\gamma'\gamma = Id$ on $K(A)$.

Now, we have to show that the map γ is surjective. The approach is rather technical and a sketch is given.

Proposition 3.1.7 *Each element of $\pi_1^L(GL(A))$ can be written as difference of polynomial loops $[\tau_1] - [\tau_2]$.*

PROOF Let's denote the class of a loop $\tau \in \pi_1^L(GL_n(A))$ by $[\tau]$. $[z^k\tau] = [z^k] + [\tau] \Rightarrow [\tau] = [z^k\tau] - [z^k]$. As for k large enough $z^k\sigma$ is polynomial, we are done.

Proposition 3.1.8 *Any polynomial loop $[\tau] \in \pi_1^L(GL_n(A))$ is in class of a loop of form $a_0 + a_1 z$ (affine loop).*

PROOF Let, $\tau(z) = a_0 + a_1 z + ... + a_m z^m \in GL_n(A < z >)$. We want to show that it is homotopic to an affine loop. As, $m \in \mathbb{N}$ is finite, if we can find a homotopy between τ and any polynomial of degree less than $m-1$, we will be done.

$\tau(z,t) = \begin{bmatrix} 1 & -tz^{m-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tau(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ ta_m z & 1 \end{bmatrix}$

Now, $\tau(z,t)\tau(1,t)^{-1}$ is a normalized homotopy between $\tau(z)$ and a loop of degree $< m-1$.

Remark 3.1.10 From straightforward calculation, we get the following. Let, $\sigma(z) = a_0 + a_1 z \in GL_n(A)$ is any Laurential loop and $\tau(z) = \sum_{k=-\infty}^{k=\infty} b_k z^k$ in $GL_n(A < z, z^{-1} >)$ is its inverse loop.Then-

- $a_0 b_k + a_1 b_{k-1} = b_k a_0 + b_{k-1} a_1 = 1$ if $k = 1$
- $a_0 b_k + a_1 b_{k-1} = b_k a_0 + b_{k-1} a_1 = 0$ if $k \neq 1$
- $k < 0, l \geq 0$ or $l < 0, k \geq 0$
  - $b_k a_j b_l = 0$
  - $b_k b_l = 0$
- $a_j b_i = b_i a_j$ for $j = 0, 1$

Proposition 3.1.9 *$q = a_0 b_0$ is a projector. σ can be written as $\sigma(z) = \sigma^+(z)\sigma^-(z^{-1})(pz+q)$ where $p = 1-q$ and $\sigma^+(z) \in GL_n(A < z >), \sigma^-(z^{-1}) \in GL_n(A < z^{-1} >)$*

PROOF $q^2 = a_0 b_0 a_0 b_0 = a_0 b_0 (1 - a_1 b_{-1}) = a_0 b_0 - a_0(b_0 a_1 b_{-1}) = a_0 b_0$. So, q is a projector.

We can take $\sigma^+(z) = p + \sigma(z) q$ and $\sigma^-(z^{-1}) = q + \sigma(z) p z^{-1}$ and we are done.

Proposition 3.1.10 *The map γ is surjective.*

PROOF Following the previous propositions, it will be enough to show that any affine loop $[\tau]$ lies in $Im(\gamma)$.
Let, $[\tau(z)] = [b_0+b_1z]$. We can write $\tau(z)$ as $\tau^+(z)\tau^-(z^{-1})(pz+q)$. Let's consider the map $H(z,t) : S^1\times I \to GL(A)$ is defined as $H(z,t) = \sigma^+(zt)\sigma^-(z^{-1}t)(pz+q)$. We have found a normalized homotopy $H(z,t)H(1,t)^{-1}$ between τ and $[\gamma([Im(p)])]$.

So, we have proved Bott's periodicity theorem for complex K-groups.

Remark 3.1.11 This proof does not work for real K-theory. One reason is that we can not use Fejer's theorem in that case.

3.2 First application of periodicity

Here we will explicitly compute complex K-groups of some ubiquitous base space. In some cases effort is made to compute the ring structure of $K_{\mathbb{C}}(X)$ as well. Some immediate consequences of periodicity is also shown. Mostly [10] and [4] are followed.

3.2.1 K-theory of S^n and $\mathbb{R}^n$

$\mathbb{R}^n \cong \mathbb{R}^n \times \{p\}$. $\{p\}$ is a single point.
So, $K(\mathbb{R}^n) \cong K(\mathbb{R}^n \times \{p\}) \cong K(\{p\} \times \mathbb{R}^n) := K^{-n}(\{p\})$.
Now,

$$K^{-n}(\{p\}) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

So,

$$K(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

In general, for $n \in \mathbb{N}$ and $p \in \mathbb{Z}$,

$$K^p(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & \text{if } p+n \text{ is even} \\ 0 & \text{if } p+n \text{ is odd} \end{cases}$$

Now, S^n is the one point compactification of $\mathbb{R}^n$. So, $K(S^n, \{p\}) = K(\mathbb{R}^n)$ and $K(S^n) = K(\mathbb{R}^n) \oplus K(\{p\})$.

So,

$$K^0(S^n) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n \text{ is even} \\ \mathbb{Z} & \text{if } n \text{ is odd} \end{cases}$$

Again, As, $\tilde{K}^{-1}(S^n) = K^{-1}(\mathbb{R}^n)$-

$$K^{-1}(S^n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \mathbb{Z} & \text{if } n \text{ is odd} \end{cases}$$

Remark 3.2.1 In remark 3.1.4, we have shown a generator u of $\tilde{K}_{\mathbb{C}}(S^2)$. So, $\tilde{K}_{\mathbb{C}}(S^{2n}) \cong \mathbb{Z}$, with generator u^n.

This generator u is actually $H - 1$, where H is the canonical line bundle over $\mathbb{C}P^1 = S^2$. (defined in example 1.1.6).

Now, the composition of the following maps $\tilde{K}(X) \to \tilde{K}(S^2) \otimes \tilde{K}(X) \to \tilde{K}(S^2 X)$ is an isomorphism. Here, the first map is external product with $(H - 1)$, which is an isomorphism by periodicity theorem. As an iterate of this the external product $\tilde{K}(S^{2k}) \otimes \tilde{K}(X) \to \tilde{K}(S^{2k} \wedge X)$ is an isomorphism. Here $S^{2k} \wedge X$ is the smash product, which is equivalent to 2-fold reduced suspension of X.

Remark 3.2.2 Now we will look the ring structure of $K(S^2)$ explicitly. Let, H be the canonical line bundle over $\mathbb{C}P^1 = S^2$. Now, as shown in 4.7, $(H \otimes H) \oplus 1 = H \oplus H \Rightarrow H^2 + 1 = 2H \Rightarrow (H - 1)^2 = 0$. As, $K(\mathbb{R}^2)$ is generated by $H - 1$. As, the group $K(S^2)$ is generated by $H - 1$ and 1, which is the generator of $K(\{p\})$. Now let's consider the polynomial ring $\frac{\mathbb{Z}[H]}{(H-1)^2}$, which is generated by $\{1, H\}$. We have a ring isomorphism between $\frac{\mathbb{Z}[H]}{(H-1)^2}$ and $K(S^2)$.

Now, using periodicity, $K(S^{2m}) \cong \frac{\mathbb{Z}[H]}{(H-1)^2}$ (this is a ring isomorphism).

$K(S^{2m+1}) \cong \mathbb{Z}$ is generated by 1, the trivial generator of $K(\{p\})$.

Remark 3.2.3 Now, we have shown that $\xi_p^{\mathbb{C}}(S^n) \approx \pi_{n-1}(GL_p(\mathbb{C}))$. Using the fact that $U(n) \hookrightarrow GL_n(\mathbb{C})$ is a homotopy equivalence, $\xi_p^{\mathbb{C}}(S^n) \approx \pi_{n-1}(U(p))$. So, $\tilde{K}_{\mathbb{C}}(S^n) \approx \text{inj lim } \pi_{n-1}(U(p)) \cong \pi_{n-1}(U)$.

So,

$$\pi_n(U) \cong \pi_n(GL(\mathbb{C})) = \begin{cases} 0, \text{ when } n \text{ is even} \\ \mathbb{Z}, \text{ when } n \text{ is odd} \end{cases}$$

3.2.2 K-theory of torus ($S^1 \times S^1$)

Let's consider the sequence of inclusion and projection maps respectively:
$S^1 \times \{e\} \xhookrightarrow{i} S^1 \times S^1 \xrightarrow{\pi} S^1 \times \{e\}$

Here, $\pi i = Id$ and $((S^1 \times S^1) - (S^1 \times \{e\})) \cong S^1 \times \mathbb{R}$. So, we have-

$$K(S^1 \times S^1) \cong K(S^1 \times \{e\}) \oplus K(S^1 \times \mathbb{R}) \cong K^0(S^1) \oplus K^{-1}(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$$
$$K^{-1}(S^1 \times S^1) \cong K^{-1}(S^1 \times \{e\}) \oplus K^{-1}(S^1 \times \mathbb{R}) \cong K^{-1}(S^1) \oplus K^{-2}(S^1) \cong K^{-1}(S^1) \oplus K^0(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$$

3.2.3 K-theory of figure eight ($S^1 \vee S^1$)

Let, $i : S^1 \hookrightarrow S^1 \vee S^1$ is the inclusion of one of the copies of S^1 into $S^1 \vee S^1$ and $\pi : S^1 \vee S^1 \to S^1$ identifies both of the copies of S^1 to a single cirle.
So, $\pi i = Id_{S^1}$. $(S^1 \vee S^1/S^1) \cong \mathbb{R}$. So, we get the following:

$$K^0(S^1 \vee S^1) \cong K^0(S^1) \oplus K^0(\mathbb{R}) \cong \mathbb{Z}$$

$$K^{-1}(S^1 \vee S^1) \cong K^{-1}(S^1) \oplus K^{-1}(\mathbb{R}) \cong \mathbb{Z} \oplus \mathbb{Z}$$

3.2.4 K-theory of complex projective space ($\mathbb{C}P^n$)

Here, we will focus on deriving the additive as well as multiplicative structure of $K(\mathbb{C}P^n)$. The treatment given in [4] and [2] is more or less followed.
At first, we want to show a result for the groups $K^i(X)$,where $i \in \{0, -1\}$ and X is a finite CW complex.
$K^*(X) := K^{-1}(X) \oplus K^0(X)$, is a $\mathbb{Z}_2$-graded ring. Similarly, $\tilde{K^*}(X) := \tilde{K^{-1}}(X) \oplus \tilde{K^0}(X)$
. Because of Bott's periodicity, instead of working with the full $\mathbb{Z}$-graded ring $\oplus_{r=0}^{-\infty} K^r(X)$, we can equivalently work with the $\mathbb{Z}_2$-graded ring $K^*(X)$.
The following proposition focuses on the additive structure of $K^*(X)$.

Proposition 3.2.1 *Let, X be a finite CW complex which has n cells. Then the group $K^i(X)$ is finitely generated by at most n generators. Here, $i \in \{0, -1\}$.*

PROOF The proof is done using induction on the number of cells. Let's assume the statement works for a subcomplex A.

Here, X is obtained by attaching a k-cell to A. Now, considering the exact sequence (explained in proposition 2.2.6) for the pair (X, A), we have $\tilde{K^i}(X/A) \to \tilde{K^i}(X) \to \tilde{K^i}(A)$. Now, we know $X/A \cong S^k$. Now, $\tilde{K^i}(X/A) = \mathbb{Z}$ or 0. As the sequence is exact, $K^i(X)$ requires at most one more generator than $K^i(A)$.

Proposition 3.2.2 *If X has cells of even dimensions only, $K^{-1}(X) = 0$ and $K^0(X)$ is a free abelian group with one generator for each cell.*

Proof Let's consider the exact sequence: $K^{-1}(X/A) \to K^{-1}(X) \to K^{-1}(A)$. Now, if all cells are of even dimension, $X/A \cong S^{2m}$, for some $m \in \mathbb{N}$. So, the first term $K^{-1}(X/A) = 0$. Now, $K^{-1}(\{p\}) = 0$. So, by induction on number of cells $K^{-1}(X) = 0$.

In this situation, we get a short exact sequence: $0 \to \tilde{K^0}(X/A) \to \tilde{K^0}(X) \to \tilde{K^0}(A) \to 0$ and $\tilde{K^0}(X/A) = \tilde{K^0}(S^{2m}) \cong \mathbb{Z}$. Again, we are assuming that the statement is true for $\tilde{K^0}(A)$. As, $\tilde{K^0}(A)$ is free, the sequence splits. So, we get $\tilde{K^0}(X) \cong \mathbb{Z} \oplus \tilde{K^0}(A)$ and we are done.

Corollary 3.2.1 *$\mathbb{C}P^n$, as a CW complex has one cell of each dimension $0, 2, 4, ..., 2n$. So, using proposition 3.2.2, $K^{-1}(\mathbb{C}P^n) = 0$ and $K^0(\mathbb{C}P^n) = \oplus_{i=1}^{i=n+1} \mathbb{Z}_i$ and for all i, $\mathbb{Z}_i = \mathbb{Z}$.*

Now we will focus on computing the multiplicative structure of $K^*(\mathbb{C}P^n)$.

Theorem 3.2.1 *Let, L be the canonical line bundle over $\mathbb{C}P^n$. $K(\mathbb{C}P^n)$ is isomorphic (as a ring) to the quotient ring $\mathbb{Z}[L]/(L-1)^{n+1}$.*

Step 1: Let's consider the short exact sequence for the pair $(\mathbb{C}P^n, \mathbb{C}P^{n-1})$, we get-

$$0 \to K(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \to K(\mathbb{C}P^n) \to K(\mathbb{C}P^{n-1}) \to 0$$

We proceed by induction. We assume the statement to be true for $K(\mathbb{C}P^{n-1})$. Let's denote the map $K(\mathbb{C}P^n) \to K(\mathbb{C}P^{n-1})$ by ρ. If we can prove that $Ker(\rho)$ is generated by $(L-1)^n$,we will be done. We have assumed that $K(\mathbb{C}P^{n-1}) = \mathbb{Z}[L]/(L-1)^n$. Due to exactness, we get that $\{1, L-1, .., (L-1)^n\}$ will be an additive basis of $K(\mathbb{C}P^n)$. Now, $(L-1)^{n+1} = 0$ in $K(\mathbb{C}P^n)$. So, $K(\mathbb{C}P^n)$ has to be the ring $\mathbb{Z}[L]/(L-1)^{n+1}$.

Step 2: One intuitive idea for what $Ker(\rho)$ should be generated by $(L-1)^n$ is the following. Now, $Ker(\rho)$ is to be identified with a copy of

$K(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \cong \tilde{K}(S2n)$. $\tilde{K}(S2n)$ is generated by n-fold reduced external product $(L-1) * (L-1) * .... * (L-1)$.

Step 3: $\mathbb{C}P^n$ is the quotient of $S^{2n+1} \subset \mathbb{C}^n$ with respect to scalar multiplication by $S^1 \subset \mathbb{C}$. Let's consider D_i^2, the unit disk in the ith coordinate of $\mathbb{C}^{n+1}$, for $i \in \{0, 1, ..., n\}$. Instead of S^{2n+1}, we can consider the boundary $\partial(D_0^2 \times D_1^2 \times ... \times D_n^2) = \cup_i (D_0^2 \times D_1^2 \times ... \times \partial D_i^2 \times ... \times D_n^2)$. The S^1 action (scalar multiplication) preserves this decomposition. The orbit of $D_0^2 \times D_1^2 \times ... \times \partial D_i^2 \times ... \times D_n^2$ under the scalar multiplication of S^1 is $D_0^2 \times .... \times \hat{D_i^2} \times ... \times D_n^2$. Let's denote $C_i := D_0^2 \times .... \times \hat{D_i^2} \times ... \times D_n^2$.
So, we get $\mathbb{C}P^n = \cup_i C_i$. $C_i \cong D^{2n}$ and $C_i \cap C_{i'} = \partial C_i \cap \partial C_{i'}$.
Let's consider $C_0 = D_1^2 \times ... \times D_n^2$ and $\partial C_0 = \cup_i (D_1^2 \times .... \times \partial D_i^2 \times ... \times D_n^2)$.
Let's denote $\partial_i C_0 := D_1^2 \times .... \times \partial D_i^2 \times ... \times D_n^2$ Considering the inclusion $(D_i^2, \partial D_i^2) \subset (C_0, \partial C_0) \subset (\mathbb{C}P^n, C_i)$, we get the following commutative diagram.

$$
\begin{array}{ccccc}
K(D_1^2, \partial D_1^2) \otimes ... \otimes K(D_n^2, \partial D_n^2) & \overset{\approx}{\searrow} & & & \\
\uparrow \approx & & & & \\
K(C_0, \partial_1 C_0) \otimes ... \otimes K(C_0, \partial_n C_0) & \xrightarrow{\approx} & K(C_0, \partial C_0) & & \\
\uparrow \approx & & \uparrow & & \\
K(\mathbb{C}P^n, C_1) \otimes ... \otimes K(\mathbb{C}P^n, C_n) & \longrightarrow & K(\mathbb{C}P^n, C_1 \cup .... \cup C_n) & \xrightarrow{\approx} & K(\mathbb{C}P^n, \mathbb{C}P^{n+1}) \\
\downarrow & & \downarrow & \swarrow & \\
K(\mathbb{C}P^n) \otimes ... \otimes K(\mathbb{C}P^n) & \longrightarrow & K(\mathbb{C}P^n) & &
\end{array}
$$

Here, all the maps between first and second columns are n-fold external products.
The map $K(\mathbb{C}P^n, C_1 \cup .... \cup C_n) \to K(C_0, \partial C_0)$ is isomorphism as from the inclusion $C_0 \hookrightarrow \mathbb{C}P^n$, the following homeomorphism is induced. $C_0/\partial C_0 \cong \mathbb{C}P^n/(C_1 \cup ... \cup C_n)$.
$\mathbb{C}P^n/\mathbb{C}P^{n-1} \to \mathbb{C}P^n/(C_1 \cup ... \cup C_n)$ is a homotopy equivalence if we identify $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$, in the last coordinates of $\mathbb{C}^{n+1}$.

step 4: $y_i \in K(\mathbb{C}P^n, C_i)$ maps to $L - 1 \in K(\mathbb{C}P^n)$ and y_i maps to a generator of $K(C_0, \partial_i C_0)$. As a result of commutativity, $y_1 * .... * y_n$ generates $K(\mathbb{C}P^n, C_1 \cup .... \cup C_n)$. So, $(L-1)^n$ generates $Im(K(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \to K(\mathbb{C}P^n)) = Ker(\rho)$. So, we are done.

3.3 Splitting principle

Splitting principle is a technical tool that allows us to reduce the questions about general vector bundles into questions regarding line bundles. this will be used to prove Adam's theorem in 3.5.

We give the statement first. We are considering complex vector bundles exclusively. The proof will be done in several steps.

Theorem 3.3.1 (Splitting principle) *Let, X be a compact Hausdorff space and E is a vector bundle over X. Then we can get a compact Hausdorff space $F(E)$ and a map $p : F(E) \to X$, such that the pull-back $p^* : K^*(X) \to K^*(F(E))$ is injective.*

$p^*(E)$ can be decomposed (or 'split') into Whitney sum of line bundles.

This is a general consequence of **Leray-Hirsch** theorem for K-theory. For certain fiber bundles $E \to X$, it allows us to consider $K^*(E)$ as a finitely generated free $K^*(X)$ module. We will provide the statement and show a sketch of its proof following [4].

Theorem 3.3.2 (Leray-Hirsch theorem) *Let, E, X are compact Hausdorff spaces and (E, p, X) be a fiber bundle with fibre F such that $\tilde{K}^*(F)$ is free. Suppose there exists elements $c_1, ..., c_k \in K^*(E)$, restrictions of which give a basis for $K^*(F)$, for each fiber F_x. Then, $K^*(E)$ can be written as a free $K^*(X)$ module, with $\{c_1, c_2, ..., c_k\}$ as a basis if either*

- *X is a finite CW complex.*
- *F is a finite CW complex with cells of even dimension only.*

Before getting into the proof, some clarifications are required.

$K^*(E)$ is given a $K^*(X)$ module structure as follows. For $\gamma \in K^*(X)$ and $\beta \in K^*(E)$, $\gamma\dot{\beta} := p^*(\gamma)\beta$.

So, we can also state the theorem a bit differently. The map $\Phi : K^*(X) \otimes K^*(F) \to K^*(E)$, where $\Phi(\sum_j b_j \otimes i^*(c_j)) = \sum_j p^*(b_j)c_j$ is an isomorphism. Here $i : F \hookrightarrow E$ is the inclusion.

Proof We will prove the theorem for the two conditions differently.

a. Suppose, X is a finite CW complex. Consider any subspace $X' \subset X$, let's denote $p^{-1}(X')$ as E'. Now, we have the following diagram.

$$\begin{array}{ccccc} K^*(X,X')\otimes K^*(F) & \longrightarrow & K^*(X)\otimes K^*(F) & \longrightarrow & K^*(X')\otimes K^*(F) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ K^*(E,E') & \longrightarrow & K^*(E) & \longrightarrow & K^*(E') \end{array}$$

The left most vertical map Φ is given by the same form $\Phi(\sum_j b_j \otimes i^*(c_j)) = \sum_j p^*(b_j)c_j$, but $p^*(b_j)c_j$ corresponds to the relative product $K^*(E,E') \otimes K^*(E) \to K^*(E)$, defined in remark 2.6.4.
We define the right most vertical map (Φ) by taking into account the restrictions of c_i's on E'.
Now we have to show that the diagram commutes.
Now, we factor Φ as
$\sum_i b_i \otimes i^*(c_i) \xrightarrow{\Phi_1} \sum_i p^*(b_i) \otimes i^*(c_i) \xrightarrow{\Phi_2} \sum_i p^*(b_i)c_i$

We get the following enlarged diagram:

$$\begin{array}{ccccc} K^*(X,X')\otimes K^*(F) & \longrightarrow & K^*(X)\otimes K^*(F) & \longrightarrow & K^*(X')\otimes K^*(F) \\ \downarrow{\scriptstyle \Phi_1} & & \downarrow{\scriptstyle \Phi_1} & & \downarrow{\scriptstyle \Phi_1} \\ K^*(X,X')\otimes K^*(F) & \longrightarrow & K^*(X)\otimes K^*(F) & \longrightarrow & K^*(X')\otimes K^*(F) \\ \downarrow{\scriptstyle \Phi_2} & & \downarrow{\scriptstyle \Phi_2} & & \downarrow{\scriptstyle \Phi_2} \\ K^*(E,E') & \longrightarrow & K^*(E) & \longrightarrow & K^*(E') \end{array}$$

The upper squares are commutative by construction. The lower squares are commutative using the argument given in 4.8.
The lower is exact. As, $K^*(F)$ is assumed to be free and the upper row is got by tensoring the exact sequence $K^*(X,X') \to K^*(X) \to K^*(X')$ by the free abelian group $K^*(F)$. So, the upper row is exact as well.
For the first condition, X is a finite CW complex, we use induction twice. First, for dimension of X and then for a fixed dimension, for the number of cells in X.
If $Dim(X) = 0$, X has to be a finite discrete set.
Let's assume that X' is a subcomplex of X and to get X, we attach a cell e^n with X'. Let's assume, for this the rightmost vertical map Φ in the first diagram is an isomorphism. Here, if the left most Φ is an isomorphism, the

middle one has to be (As in the diagram, a part of the long exact sequence is shown, similar vertical maps can be considered and we can use five lemma). Let, $\Psi : (D^n, S^{n-1}) \to (X, X')$ be the characteristic map for the attached cell e^n, which is an n-cell. Using the fact that D^n is contractible, $\Psi^*(E)$, the pullback of E has to be trivial. Now, we have the following diagram, that commutes.

$$\begin{array}{ccccc} K^*(X,X') \otimes K^*(F) & \xrightarrow{\cong} & K^*(D^n, S^{n-1}) \otimes K^*(F) & & \\ \downarrow \Phi & & \downarrow \Phi & \searrow^{\Phi} & \\ K^*(E,E') & \xrightarrow{\cong} & K^*(\Psi^*(E), \Psi^*(E')) & \xrightarrow{\cong} & K^*(D^n \times F, S^{n-1} \times F) \end{array}$$

Here, the restriction of Ψ on the interior of D^n is a homeomorphism. So, we get the homeomorphisms $X/X' \cong D^n/S^{n-1}$ and $E/E' \cong \Psi^*(E)/\Psi^*(E')$, which explains the horizontal isomorphisms. So, now, it is enough to show that the left most Φ is an isomorphism.
Let's consider the first diagram replacing (X, X') by (D^n, S^{n-1}). As, S^{n-1} has lower dimension than X, we can assume the right most Φ to be an isomorphism. As, D^n is contractible, the middle Φ is isomorphism using discrete case. Using, five-lemma again, the left Φ has to be an isomorphism. We are done.

b. Now, we assume that F is a finite CW complex with cells of even dimension only. Here, we will prove in two steps. At first, we will do it for trivial bundles $E = X \times F$. Then we will do it for general fibre bundles.

Step 1: Let's take $E = X \times F$. Here, Φ is just an external product, which gives us the freedom to interchange F and X. Again, we consider the first diagram of the previous argument taking F to be any compact Hausdorff space and X to be a finite CW complex with only even dimensional cells. Again we consider $X' \subset X$, a subcomplex, to which a cell e^n has to be attached in order to get X. The upper row is obtained by tensoring the following split exact sequence: $0 \to K^*(X, X') \to K^*(X) \to K^*(X') \to 0$ by $K^*(F)$ (a fixed group). So, it is exact. If, somehow we manage to show that the left Φ is an isomorphism, by induction (on cell number), we can assume the right most one to be an isomorphism. So, thanks to five-lemma again, the middle one will be an isomorphism.
As, $X/X' \cong S^n$, without loss of generality, we can take (D^n, S^{n-1}) instead of arbitrary (X, X'). Here the middle Φ is an isomorphism. The left one will be a isomorphism if and only if the right one is. If S^{n-1} is even dimensional, Φ is

an isomorphism. For odd dimensional sphere also this works by interchanging K^0 and K^{-1}.

Step 2: Now, we consider any arbitrary fiber bundle E. As, X is not a CW comlex and just a compact Hausdorff space, we have to change the induction. Consider any compact subspace $X' \subset X$. If for all compact subspaces X'' of X', we have $\Phi : K^*(X'') \otimes K^*(F) \to K^*(p^{-1}(X''))$ to be an isomorphism, we call X' to be "good". Using the previous step and local triviality of fiber bundle, every point x of X has such a "good" neighborhood. As, finite number of such neighborhoods cover X, by induction all we have to show is if X'_1 and X'_2 are "good", so is $X'_1 \cup X'_2$.
Let, $U \subset X'_1 \cup X'_2$ is compact. Now, U is union of $U_1 := U \cap X'_1$ and $U_2 := U \cap X'_2$. Again we consider the very first commutative diagram used in the argument of (a.) for the pair (U, U_2). The upper row is exact as $K^*(F)$ is free. As, X'_2 is good, Φ is an isomorphism for U_2. To show Φ to be an isomorphism of U, we have to show that it is an isomorphism for (U, U_2). $U/U_2 \cong U_1/(U_1 \cap U_2)$. We will be done if we can prove Φ to be an isomorphism for $(U_1, U_1 \cap U_2)$. Now, we consider the same diagram again for $(U_1, U_1 \cap U_2)$. As, X'_1 is "good", Φ is isomorphism for U_1 and $U_1 \cap U_2$. So, it is an isomorphism for $(U_1, U_1 \cap U_2)$. We are done.

Example 3.3.1 Let, E be a vector bundle over a compact space X with fibre $\mathbb{C}^n$. From this we get a fibre bundle $p : P(E) \to X$ with fiber $\mathbb{C}P^{n-1}$. $P(E)$ is the space of one dimensional linear subspaces of fibers of E.Let's consider the canonical line bundle L over $P(E)$. It is the bundle over X, whose fiber over x is the canonical line bundle over $P(E_x)$. For each fiber $(\mathbb{C}P^{n-1})$ of $P(E)$, the classes of the elements $1, L, L^2, ..., L^{n-1} \in K^*(P(E))$ restricts to a basis for $K^*(\mathbb{C}P^{n-1})$. Using the previous theorem, $K^*(P(E))$ is a free $K^*(X)$ module with $\{1, L, L^2, ..., L^{n-1}\}$ as basis. As, 1 is among the basis set of $K^*(P(E))$, the map $p^* : K^*(X) \to K^*(P(E))$ is injective.

Now, we are ready to prove **splitting principle**.

PROOF Let's consider the pullback bundle $p^*(E)$ over $P(E)$, it contains the line bundle L, which sits as a subbundle. So, we get another subbundle E' on $P(E)$, such that $L \oplus E' \cong p^*(E)$. $E' \perp L$, for some choice of inner product on $p^*(E)$. Now, we do the same splitting where the pullback of E' on $P(E')$ splits off another line bundle. As, $P(E')$ is consists of pairs of orthogonal line in fibers of E, repeating finite number of time, we get the flag bundle $F(E)$ on X (described in 4.9). Elements of $F(E)$ are n tuples of orthonal lines in

the fibers of E. Here, E has dimension n.
The pullback of E on $F(E)$ does split as a line bundle and induces an injection, using the argument used in the previous example.

3.4 Parallelizability of spheres

One of the most interesting application of classical K-theory is showing the non existence of a division algebra structure on $\mathbb{R}^n$, if $n \notin \{1, 2, 4, 8\}$. In this section, we will focus on the question of finding values of n, for which $\mathbb{R}^n$ is a division algebra or S^{n-1} is parallelizable. The celebrated "Adam's theorem on the Hopf invariant" is the most crucial part of this quest. It will be proved in 3.5.

Definition 3.4.1 A **division algebra** is a ring where each non-zero element has a multiplicative inverse.

Here, the multiplication is not assumed to be commutative.

At, the same time, the existence of unit in the underlying ring R is not assumed. All we want is the following. For any $r \in R$, the maps $R \to R$, given by $x \to rx$ and $x \to xr$ are linear (we are emphasizing distributive property of ring multiplication over addition in a different way!) and for all non-zero r, this map has to be **invertible**.
Examples

- Any field is a commutative division algebra. Most obvious non commutative division algebra is the quaternions, $\mathbb{H}$.

- On $\mathbb{R}^8$, we have a well-known multiplication structure called Cayley's octonions ($\mathbb{O}$). Although the multiplication is non associative here.

As, the quaternions can be thought of ordered pairs of complex numbers, similarly octonions can be represented as ordered pairs of quaternions. The multiplication is defined as follow:

$$(p_1, q_1)(p_2, q_2) = (p_1p_2 - \bar{q_2}q_1, q_2p_1 + q_1\bar{p_2})$$

If we consider usual Euclidean norm on $\mathbb{R}^8$, we see this multiplication is "norm-preserving". In $\mathbb{R}$, $\mathbb{R}^2$ (with the multiplication structure of $\mathbb{C}$), $\mathbb{R}^4$ (with the multiplication structure of $\mathbb{H}$) and $\mathbb{R}^8$ (with the multiplication structure of $\mathbb{O}$), the multiplications are "norm-preserving".
By the phrase "norm-preserving", we mean $\|c_1c_2\| = \|c_1\|\|c_2\|$

Remark 3.4.1 Division algebra structure on $\mathbb{R}^n$

To give R^n a division algebra structure, we consider a multiplication map $m' : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$.
For an arbitrarily fixed element a, the maps- $x \to m'(a, x)$ and $x \to m'(x, a)$ are linear and for $a \neq 0$, it is invertible.

As, we are considering linear maps in $\mathbb{R}^n$, we want these linear maps to have trivial kernel. So, the multiplication does not have any zero divisors. Here, existence of unit is not assumed

Although we can modify a bit to set a unit. Let's fix a unit vector $e \in \mathbb{R}^n$. Let's consider an invertible linear map Ψ in $\mathbb{R}^n$ that takes $e^2 = m'(e, e)$ to e. Now if we compose m' with Ψ. So, $\Psi m'(e) = e$.
Let,
$y \xrightarrow{\alpha} ye$

and,
$y \xrightarrow{\alpha} ey$

The map $(x, y) \to \alpha^{-1}(x)\beta^{-1}(y)$ takes (y, e) to $\alpha^{-1}(y)\beta^{-1}(e) = \alpha^{-1}(y)e = y$. Similarly, (e, y) is mapped to y. The maps $x \to ax$ and $x \to xa$ are surjective (they are invertible linear maps in $GL_n(\mathbb{R})$!). So, $ax = e$ and $xa = e$ have solution for all nonzero a.
So, we have a division algebra structure with a unit, where every nonzero element has left as well as inverse.

Remark 3.4.2 If $\mathbb{R}^n$ has a division algebra structure with a "norm-preserving" multiplication, the same restricted on S^{n-1} gives us a multiplicative map-

$$\phi : S^{n-1} \times S^{n-1} \to S^{n-1}$$

If we have a multiplicative structure on $\mathbb{R}^n$ with no non zero structure, we get a multiplication for S^{n-1} as shown in 3.4.1.

Definition 3.4.2 An H-space is a space X with with a continuous multiplication map $\phi : X \times X \to X$, with a both sided identity element. So, $\exists$ $e \in X$ such that $\phi(x, e) = \phi(e, x) = x$.

Any topological group is an H-space. Although, being an H-space is a much weaker condition that being a topological group. In the former, the existence of inverse is not assumed.

Proposition 3.4.1 *S^{n-1} is an H-space if-*

- *$\mathbb{R}^n$ is a division algebra.*
- *S^{n-1} is parallelizable.*

PROOF

- It is given that we have a division algebra structure on $\mathbb{R}^n$ with two-sided identity. Now, on S^{n-1}, we get the following map $(x, y) \to xy/|xy|$. It is well-defined as the multiplication on $\mathbb{R}^n$ does not have any zero-divisor and $0 \notin S^{n-1}$.
- As S^{n-1} has trivial tangent bundle, at each point s of S^{n-1}, we have linearly independent tangent vectors $v_1, ..., v_{n-1}$. We use Gram-Schmidt orthonormalization to get n orthonormal vectors $s, s(v_1), ..., s(v_{n-1})$. We can take the vectors $e_1, e_1(v_1), ..., e_1(v_{n-1})$ to be the standard orthonormal basis $e_1, e_2, ..., e_n$. Now, consider the element $\beta_s \in SO(n)$ sending $s, s(v_1), ..., s(v_{n-1})$ to $e_1, e_1(v_1), ..., e_1(v_{n-1})$. We may need to change sign of $s(v_i)$ to get the same orientation, in order to find an element in $SO(n)$. Now consider the map $(s_1, s_2) \to \beta_{s_1}(s_2)$. It gives an H-space structure with e_1 as identity.

In 3.2, we have shown that $K(S^{2m}) \cong \frac{\mathbb{Z}[H]}{(H-1)^2}$ (this is a ring isomorphism). So, by a change of variable, we get $K(S^{2m}) \cong \frac{\mathbb{Z}[t]}{t^2}$. Now, we will focus on finding n, for which S^n is an H-space.

Proposition 3.4.2 *For $n > 0$, S^{2n} is NOT an H-space.*

PROOF Let, we have the following H-space multiplication.

$$\phi : S^{2n} \times S^{2n} \to S^{2n}$$

From this we get the induced homomorphism ϕ^* between K-rings has the following form.

$$\phi^* : \mathbb{Z}[t]/(t^2) \to \mathbb{Z}[\alpha, \gamma]/(\alpha^2, \gamma^2)$$

Suppose, $\phi^*(t) = c_1\alpha + c_2\gamma + k\alpha\gamma$, where $k, c_1, c_2 \in \mathbb{Z}$.
Now, we consider the composition, which is identity

$S^{2n} \xhookrightarrow{i} S^{2n} \times S^{2n} \xrightarrow{\phi} S^{2n}$

i is the inclusion of $S^{2n} \times \{e\}$ (or {e} $\times$S^{2n}). e is the identity of the H-space map ϕ.
Considering i to be the inclusion of the first factor, we get

$\alpha \longrightarrow t$

$\beta \longrightarrow 0$

As $\phi^* \circ \imath^* = Id$, the coefficient of α in $\phi^*(t)$ has to be 1. Similarly, the coefficient of β in $\phi^*(t)$ is 1.
Now,

$$\phi^*(t^2) = (\alpha + \gamma + k\alpha\gamma)^2 = 2\alpha\gamma \neq 0$$

But $t^2 = 0$. That is the contradiction.

Now, we want to find $k \in \mathbb{Z}$, for which S^{2k+1} is a H-space. It is not a trivial problem. We approach the problem in several steps.

- Let, S^{k-1} is an H-space. Let's consider the H-space map

  $$\phi : S^{k-1} \times S^{k-1} \to S^{k-1}$$

  We want to associate another map $\hat{\phi} : S^{2k-1} \to S^k$ to the given ϕ. We can write S^{2k-1} as $\partial(D^k \times D^k) = \partial D^k \times D^k \cup D^k \times \partial D^k$.
  Now we consider S^k to be the union of the disks D^k_+ and D^k_- (the upper and lower hemispheres respectively) with their boundaries identified.
  When none of x and y is 0, we define $\hat{\phi}$ as follows:

  $$\hat{\phi}(x, y) = \begin{cases} |y|\phi(x, \frac{y}{|y|}) \in D^k_+, \text{ where } (x, y) \in \partial D^k \times D^k \\ |x|\phi(\frac{x}{|x|}, y) \in D^k_-, \text{ where } (x, y) \in D^k \times \partial D^k \end{cases}$$

  In case any of the x or y is 0, just define $\hat{\phi}(x, y) = 0$. $\hat{\phi}$ is well-defined and continuous and if we restrict it on $S^{k-1} \times S^{k-1}$, it agrees with ϕ. This formulation is applicable on any map $f : S^{k-1} \times S^{k-1} \to S^{k-1}$.

- Now, we replace k by $2k$ and consider any continuous map

$$f : S^{4k-1} \to S^{2k}$$

Let's define

$$C_f := e^{4k} \cup_f S^{2k}$$

. e^{4n} is a closed $4k$ cell. For $a \in \partial(e^{4K}) = S^{4k-1}$, $f(a) \in S^{2n}$.
C_f is a CW complex with one cell each in dimensions $0, 2n, 4n$. So

$$\tilde{K}^i(C_f) = \begin{cases} 0, \text{ i = -1} \\ \mathbb{Z} \oplus \mathbb{Z}, \text{ i = 0} \end{cases}$$

$C_f/S^{2k} \cong S^{4k}$. $\tilde{K}^{-1}(S^{2k}) \cong \tilde{K}^{-1}(S^{4k}) \cong 0$. We consider the exact sequence for the pair (C_f, S^{2k}), which becomes the following short exact sequence

$$0 \longrightarrow \tilde{K}(S^{4k}) \xrightarrow{h_1} \tilde{K}(C_f) \xrightarrow{h_2} \tilde{K}(S^{2k}) \longrightarrow 0$$

Let, $\alpha \in \tilde{K}(C_f)$ is the image of $2k$ fold product of $(H-1)$, which is the generator of $\tilde{K}(S^{4k})$, under h_1.
Let, $\beta \in \tilde{K}(C_f)$, such that $h_2(\beta)$ is k fold product of $(H-1)$, which is the generator of $\tilde{K}(S^{2k})$. $\beta^2 = 0 \in \tilde{K}(S^{2k})$. Using exactness, $\beta^2 = h\alpha$, where $h \in \mathbb{Z}$.
This integer h is called the **Hopf invariant** of f.

- Now, we want to show that h does not depend on the choice of β.
First of all β has to be unique upto adding integer multiple of α. As $\alpha^2 = 0$, $(\beta + m\alpha)^2 = \beta^2 + 2m\alpha\beta$. We will be done if we manage to show $\alpha\beta = 0$.
$h_2 h_1(\alpha) = 0 \in \tilde{K}(S^{2k})$. So, $\alpha\beta = n\alpha$, for some integer n. $n\alpha = \alpha\beta \Rightarrow n\alpha\beta = \alpha(\beta)^2 = h(\alpha)^2 = 0$. $n\alpha\beta = 0 \Rightarrow \alpha\beta \in h_1(\tilde{K}(S^{4k})) \cong \mathbb{Z}$. So, $n\alpha\beta = 0$ implies $\alpha\beta = 0$.

Theorem 3.4.1 *Let, S^{2n-1} is an H-space and we have the H-space multiplication*

$$\phi : S^{2n-1} \times S^{2n-1} \to S^{2n-1}$$

Then the map

$$\hat{\phi} : S^{4n-1} \to S^{2n}$$

is of Hopf invariant ± 1.

Proof Let $e \in S^{2n-1}$ be the identity element of ϕ and let's denote $\hat{\phi} : S^{4n-1} \to S^{2n}$ as g. The characteristic map Φ of e^{4n} can be thought of as map $(D^{2n} \times D^{2n}, \partial(D^{2n} \times D^{2n})) \to (C_g, S^{2n})$. Now, we have the following commutative diagram.

$$\begin{array}{ccc}
\tilde{K}(C_g) \otimes \tilde{K}(C_g) & \longrightarrow & \tilde{K}(C_g) \\
\approx \uparrow & & \uparrow \\
\tilde{K}(C_g, D^{2n}_{-}) \otimes \tilde{K}(C_g, D^{2n}_{+}) & \longrightarrow & \tilde{K}(C_g, S^{2n}) \\
\downarrow \Phi^* \otimes \Phi^* & & \Phi^* \downarrow \approx \\
\tilde{K}(D^{2n} \times D^{2n}), \partial D^{2n} \times D^{2n}) \otimes \tilde{K}(D^{2n} \times D^{2n}), D^{2n} \times \partial D^{2n}) & \longrightarrow & \tilde{K}((D^{2n} \times D^{2n}, \partial(D^{2n} \times D^{2n}))) \\
\downarrow \approx & \nearrow \approx & \\
\tilde{K}(D^{2n} \times \{e\}, \partial D^{2n} \times \{e\}) \otimes \tilde{K}(\{e\} \times D^{2n}, \{e\} \times \partial D^{2n}) & &
\end{array}$$

Here, the horizontal maps are internal product maps. The diagonal map is an external product, which is an isomorphism, as it is equivalent to the iterated Bott's periodicity isomorphism: $\tilde{K}(S^{2n}) \otimes \tilde{K}(S^{2n}) \to \tilde{K}(S^{4n})$.
If we restrict Φ on $D^{2n} \times \{e\}$, we get a homeomorphism onto D^{2n}_{-}. Similarly $\{e\} \times D^{2n} \to D^{2n}_{+}$ is a homeomorphism. As β goes to a generator of $\tilde{K}(S^{2n})$, the element $\beta \otimes \beta$ goes to a generator of $\tilde{K}(D^{2n} \times D^{2n}, \partial D^{2n} \times \{e\}) \otimes \tilde{K}(D^{2n} \times D^{2n}, \{e\} \times \partial D^{2n})$. Thanks to commutativity, $\beta \otimes \beta$ is sent to $\pm\alpha$. So, we have got $\beta^2 = \pm\alpha$. We are done.

Remark 3.4.3 So, we have reduced the question of finding n, for which S^{2n-1} is parallelizable into finding integers n such that the map $\hat{\phi} : S^{4n-1} \to S^{2n}$ has Hopf invariant ± 1. Here, $\phi : S^{2n-1} \times S^{2n-1} \to S^{2n-1}$ is the H-space multiplication.
The following theorem completes the solution of the question described in remark 3.4.3.

Theorem 3.4.2 *There exists a map* $f : S^{4n-1} \to S^{2n}$ *with Hopf invariant* ± 1*, if and only if* $n \in \{1, 2, 4\}$.

If $n \in \{1, 2, 4\}$, it is obvious because of existence of division algebra structure on $\mathbb{R}^2, \mathbb{R}^4, \mathbb{R}^8$ and our previous argument.
The next section is all about proving the other way statement of the theorem.

3.5 Adam's theorem on Hopf invariant

An operation is a natural transformation from K^0 to K^0. It may not respect the additive or multiplicative structure of K^0. There are many operations for generalized cohomology theories. We will not go into the details of operations in K-theory. Here, for the sake of completing the quest described in 3.4, with a goal to prove theorem 3.4.2, we will construct certain ring homomorphisms (Adam's operations) $\Psi^k : K(X) \to K(X)$ and show some of their properties quite mechanically.

Theorem 3.5.1 *Let, X be a compact Hausdorff space and $k \in \mathbb{N} \cup \{0\}$. There exists ring homomorphisms $\Psi^k : K(X) \to K(X)$, which satisfies:*
(1) Ψ^k is natural. For any map $f : X \to Y$, $\Psi^k f^* = f^* \Psi^k$.
(2) $\Psi^k(L) = L^k$, when L is a line bundle.
(3) $\Psi^k \circ \Psi^k = \Psi^{kl}$.
(4) $\Psi^p(\alpha) - \alpha^p = p\beta$, for some $\beta \in K(X)$.

Remark 3.5.1 Let's consider the case $\Psi^k(L_1 \oplus L_2 \oplus ... \oplus L_n)$, where L_i's are line bundles. Using property (2) and the fact that Ψ^k is a ring homomorphism, we get $\Psi^k(L_1 \oplus L_2 \oplus ... \oplus L_n) = L_1^k \oplus L_2^k \oplus ... \oplus L_n^k$.
Now, we want a formula of Ψ^k for general vector bundles E. In case of direct sum of line bundles it has to give us what we expect.
We want to use exterior powers $\Lambda^k(E)$. We list some properties of exterior power for our reference(for details please see [10, pp. 144–147]).
(a) $\Lambda^k(E_1 \oplus E_2) \approx \bigoplus_i (\Lambda^i(E_1) \otimes \Lambda^{k-i}(E_1))$.
(b) $\Lambda^0(E) = \theta^1$, the trivial line bundle.
(c) $\Lambda^1(E) = E$
(d) $\Lambda^k(E) = 0$, if $k > dim(E)$.
If we can find polynomials s_k with integer coefficients such that $s_k(\Lambda^1(E), \Lambda^2(E), ..., \Lambda^n(E)) = L_1^k \oplus L_2^k \oplus ... \oplus L_n^k$, when $E = \oplus_{i=1}^{i=n} L_i$ and L_i's are line bundles. We can define $\Psi^k(E) = s_k(\Lambda^1(E), \Lambda^2(E), ..., \Lambda^n(E))$.

Construction of Ψ^k:
We want to concretely find out the polynomial s_k.

- Let's define $\Lambda_t(E) = \sum_i \Lambda^i(E)t^i \in K(X)[t]$. This sum makes sense because using property (b), it is a finite sum. Using (a), $\Lambda_t(E_1 \oplus E_2) = \Lambda_t(E_1)\Lambda_t(E_2)$.
  When $E = \oplus_{i=1}^{i=n} L_i$, we get $\Lambda_t(E) = \prod_i \Lambda_t(L_i) = \prod_i (1 + L_i t)$ (Using (d) and the fact that $dim(L_i) = 1$). Λ^j, the coefficient of t^j is σ_j, the jth elementary symmetric polynomial in L_i's. So,

$$\Lambda^j(\oplus_{i=1}^{i=n} L_i) = \sigma_j(L_1, .., L_n)$$

- We have $(1+L_1)(1+L_2)...(1+L_n) = 1+\sigma_1+...+\sigma_n$. σ_i is the ith elementary symmetric polynomial in the L_j's. Now, we use the fundamental theorem of symmetric polynomials ([3]). Every degree k symmetric polynomial in $L_1, L_2, ..., L_n$ can be written uniquely as a polynomial in $\sigma_1, \sigma_2, .., \sigma_k$. $L_1^k + L_2^k + ... + L_n^k$ is a polynomial $s_k(\sigma_1, \sigma_2, ..., \sigma_k)$.
  s_k is independent of n. If we substitute L_i's by variables $t_1, t_2, .., t_n$, setting $t_n = 0$, we can pass from n to $n - 1$.

- Now, we want a recursive formula s_k. Let's take $n = k$. In the identity $(x + t_1)(x + t_2)...(x + t_k) = x^k + \sigma_1 t^{k-1} + .. + \sigma_k$, we replace x by $-t_i$ and sum over i to get-

$$s_k = \sigma_1 s_{k-1} - \sigma_2 s_{k-2} + .. + (-1)^{k-2}\sigma_{k-1}s_1 + (-1)^{k-1}\sigma_k$$

$$s_1 = \sigma_1$$

- In a nutshell, we define $\Psi^k(E) = s_k(\Lambda^1(E), \Lambda^2(E), ..., \Lambda^k(E))$.

- For $E = \oplus_{i=1}^{i=n} L_i$,
  $\Psi^k(E) = s_k(\Lambda^1(E), ..., \Lambda^k(E))$
  $= s_k(\sigma_1(L_1, L_2, .., L_n), .., \sigma_k(L_1, L_2, .., L_n)) = L_1^k + L_2^k + .. + L_n^k$.

Now, we want to show that this construction of Ψ^k indeed follows the properties described in theorem 3.5.1. In this we will heavily use **splitting principle**.

Theorem 3.5.2 *A ring homomorphism $K(X) \to K(X)$, extended from $\Psi^k(E) := s_k(\Lambda^1(E), \Lambda^2(E), ..., \Lambda^k(E))$ follows the required properties listed in theorem 3.5.1.*

Proof At first, the definition of Ψ^k will be extended for $K(X)$. It will be shown that it indeed preserves the additive as well as the multiplicative structures.
Then the properties will be proved.
Splitting principle allows us to essentially prove the theorem for Whitney sum of line bundles, in order to prove this for general vector bundles.

- (1) naturality follows because $f^*(\Lambda^i(E)) = \Lambda^i(f^*(E))$.

- (2) To show: $\Psi^k(E_1 \oplus E_2) = \Psi^k(E_1) + \Psi^k(E_2)$. Using splitting principle, we first pullback E_1 and split. Then another pullback to split E_2 is done. Using naturality and the fact $\Psi^k(L_1 \oplus ... \oplus L_n) = L_1^k + ... + L_n^k$, we are done. As it preserves the additive structure at the vector bundle level, in $K(X)$, the map defined as $\Psi^k(E_1 - E_2) = \Psi^k(E_1) - \Psi^k(E_2)$ preserves additive structure.

- For this definition in $K(X)$, property (1) and property (2) are satisfied.

- We want to show that it prserves the multiplicative structure. We will use splitting principle again to prove this. If E_1, E_2 are sum of line bundles L_i's and L'_i's respectively, $E_1 \otimes E_2 = \sum_{i,j}(L_i \otimes L'_j)$. So, $\Psi^k(E_1E_2) = \Psi^k(E_1 \otimes E_2) = \Psi^k(\sum_{i,j}(L_i \otimes L'_j)) = \sum_{i,j}(L_i \otimes L'_j)^k = \sum_{i,j} L_i^k \otimes {L'_j}^k = \sum_i L_i^k \sum_j {L'_j}^k$. We can do this interchanging as this is a finite sum. So, we have got $\Psi^k(E_1E_2) = \Psi^k(E_1)\Psi^k(E_2)$. From additivity, it follows that multiplication is preserved for elements $[E_1]-[E_2]$ of $K(X)$ as well.

- For property (3), using splitting principle and additivity, it is enough to show for line bundles. $\Psi^l \circ \Psi^k(L) = \Psi^l(L^k) = (L^k)^l = L^{kl} = \Psi^{kl}(L)$.

- Using same argument, it is enough to show property (4), for sum of line bundles. Let, $L = L_1 \oplus L_2 \oplus ... \oplus L_n$.

Then $\Psi^p(E) = L_1^p + ... + L_n^p = (E^p) \bmod p$. As, tensor product is commutative, we can use binomial theorem.

Restricting Ψ^k on $\tilde{K}(X)$:

$\tilde{K}(X) := Ker[K(X) \to K(\{x\})]$. So, using naturality, Ψ^k can be restricted on $\tilde{K}(X)$.
Now, we want to look at $\Psi^k(\alpha * \beta)$ for $\alpha \in \tilde{K}(X)$ and $\beta \in \tilde{K}(Y)$. The homomorphism $\tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \wedge Y)$ is defined as $(\alpha, \beta) \to p_1^*(\alpha)p_2^*(\beta)$. Where, p_1, p_2 are projections of $X \times Y$ to X and Y respectively.

$$\begin{aligned}\Psi^k(\alpha * \beta) &= \Psi^k(p_1^*(\alpha)p_2^*(\beta)) \\ &= \Psi^k(p_1^*(\alpha))\Psi^k(p_2^*(\beta)) \\ &= p_1^*(\Psi^k(\alpha))p_2^*(\Psi^k(\beta)) \\ &= \Psi^k(\alpha) * \Psi^k(\beta)\end{aligned}$$

Our motive is to compute Ψ^k on $\tilde{K}(S^{2n}) \cong \mathbb{Z}$. As, it is an additive homomorphism in $\mathbb{Z}$, it has to be multiplication by some element in $\mathbb{Z}$.

Proposition 3.5.1 *The ring homomorphism $\Psi^k : \tilde{K}(S^{2n}) \to \tilde{K}(S^{2n})$ is multiplication by k^n.*

Proof We will prove it using induction.

- For $n = 1$, using additivity, we will be done if we manage to show $\Psi^k(\beta) = k\beta$. β is a generator of $\tilde{K}(S^2) \cong \mathbb{Z}$. Without loss of generality, we can take $\beta = H - 1$. H is the canonical line bundle over $\mathbb{C}P^1 \cong S^2$.

$$\begin{aligned}\Psi^k(\beta) &= H^k - 1 \\ &= (1+\beta)^k - 1 \\ &= 1 + k\beta - 1 \\ &= k\beta\end{aligned}$$

We are simply using the fact that multiplication in $\tilde{K}(S^2)$ is trivial. $\beta^2 = 0 \Rightarrow \alpha^i = 0$, for $i \geq 2$.

- $n > 1$. Now, we use the following isomorphism (Bott's periodicity) involving external product:

  $$\tilde{K}(S^2) \otimes \tilde{K}(S^{2n-2}) \to \tilde{K}(S^{2n})$$

  By induction hypothesis, the theorem holds true for $\tilde{K}(S^{2n-2})$. So, for $\alpha \in \tilde{K}(S^2)$ and $\beta \in \tilde{K}(S^{2n-2})$, $\Psi^k(\alpha * \beta) = \Psi^k(\alpha) * \Psi^k(\beta) = k\alpha * k^{n-1}\beta = k^n(\alpha * \beta)$.

Now, we are ready to prove our holy grail **Adam's theorem on Hopf invariant** (theorem 3.4.2).

PROOF Let $f : S^{4n-1} \to S^{2n}$ be a map with Hopf invariant ± 1. We are using the notations, we used to define Hopf invariant in the previous section. $\alpha, \beta \in \tilde{K}(C_f)$, such that α is the image of the generator of $\tilde{K}(S^{4n})$ and β maps to a generator of $\tilde{K}(S^{2n})$.
So, $\Psi^k(\alpha) = k^{2n}\alpha$ and $\Psi^k(\beta) = k^n\beta + \phi_k\alpha$. $\phi_k \in \mathbb{Z}$. So

$$\Psi^k\Psi^l(\beta) = \Psi^k(l^n\beta + \phi_l\alpha) = k^n l^n \beta + (k^{2n}\phi_l + l^n\phi_k)\alpha$$

$\Psi^k\Psi^l = \Psi^{kl} = \Psi^{lk} = \Psi^l\Psi^k$. If we interchange l and k, the coefficient of α is unchanged in $\Psi^k\Psi^l$. So,

$$k^{2n}\phi_l + l^n\phi_k = l^{2n}\phi_k + k^n\phi_l$$
$$\Rightarrow (k^{2n} - k^n)\phi_l = (l^{2n} - l^n)\phi_k$$

Now, $\Psi^2(\beta) = \beta^2 \bmod 2$. Here, $\beta^2 = \pm\alpha$. By the formula for Ψ^2,

$$\Psi^2(\beta) = 2^n\beta + \phi_2\alpha$$

So, ϕ_2 has to be an odd number.Putting $k = 2$, $l = 3$, we get-

$$(2^{2n} - 2^n)\phi_3 = (3^{2n} - 3^n)\phi_2$$
$$\Rightarrow 2^n(2^n - 1)\phi_3 = 3^n(3^n - 1)\phi_2$$

So, 2^n divides $3^n(3^n - 1)\phi_2$.
As 3^n, ϕ_2 are odd, 2^n divides $3^n - 1$.
If 2^n divides $3^n - 1$ then $n \in \{1, 2, 4\}$.For the proof of this fact, please see [4, p. 65].

So, we have shown that only for $n = 1, 2$ or 4, S^{2n-1} is a Hopf space. So, S^1, S^3 and S^7 are the only parallelizable spheres. Consequently, $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^4$ and $\mathbb{R}^8$ are the only $\mathbb{R}^n$ to have a division algebra structure.

Chapter 4

Appendix

4.1 On 'finiteness' of compact space

The idea of the proof is taken from [5, pp. 49–50].

Theorem 4.1.1 *Let, X be a compact space. Let's consider a directed system $A_1 \to A_2 \to ... \to A_n \to ...$, where $A_i \to A_{i+1}$ is closed inclusion and A_i is a T_1 space for all i. Let's denote the direct limit of the directed system by A. Then, for any map $f : X \to A$, $\exists\ m \in \mathbb{N}$, such that $f(X) \subseteq A_m$.*

PROOF If for an arbitrarily fixed $N \in \mathbb{N}$, let's assume, $f(X) \not\subset A_n$, for any $n \leq N$. Then, we can get a sequence of points $\{a_i\}_{i=1}^{\infty}$, such that $a_i \in A_{i+1} \setminus A_i$. Let's denote this collection of points by S. Now, for any subset $S' \subseteq S$, $S' \cap A_n$ is finite, so closed in A_n. $A_n \to A$ is closed inclusion, so any subset is closed in A or with respect to subspace topology, S has discrete topology. $S \subseteq f(X)$, a compact space and S is infinite. A compact set can not have an infinite, discrete subset. So, that is the contradiction and we are done.

Remark 4.1.1 Let's consider a directed system $A_1 \to A_2 \to ... \to A_n \to ...$, with the conditions above mentioned. The direct limit of the directed system is denoted by A. $[X, A] = [X, injlimA_n]$ denotes the collection of continuous maps from X to A upto homotopy. Now if X is compact, $[X, A] = [X, injlimA_n] \approx injlim[X, A_n]$. For any class of element $[f] \in [X, injlimA_n]$, f has to factor through some A_n.
The reverse inclusion is by definition true.
In [5], the proof is done for any directed system using transfinite induction.

4.2 On pseudo-abelian categories

In theorem 1.6.1, we have explicitly shown that for a compact base space X, any finite dimensional vector bundle E over X is direct summand of a trivial bundle over X. Here, similar result will be shown for any pseudo-abelian category. So, necessarily theorem 1.6.1 is a very special case of the following.

Remark 4.2.1 Mitchell embedding theorem [9] says that for any small abelian category $\mathcal{C}$, we can find a ring R with unit, such that there exists a fully faithful exact functor $\mathcal{C} \to R$-mod. R-mod is the category of all left R-modules. So, while working with abelian categories, we can actually use diagram-chasing techniques.

While working with pseudo-abelian categories and only projection morphisms, as all the kernels and co-kernels are objects of the category; we are allowed to use diagram-chasing.

Theorem 4.2.1 *Let, $\mathcal{A}$ be a pseudo-abelian category and $E \in \mathcal{O}b(\mathcal{C})$. p is a projection of E. Then E has the following decomposition: $E \approx Ker(p) \oplus Ker(1-p)$.*

Proof Let's consider $j_1 : Ker(p) \hookrightarrow E$ and $j_2 : Ker(1-p) \hookrightarrow E$, the inclusions. We will be done if we can show the existence of $j_1' : E \to Ker(p)$ and $j_2' : E \to Ker(1-p)$ such that $j_1'j_1 = Id_{Ker(p)}$ and $j_2'j_2 = Id_{Ker(1-p)}$. $j_1'j_2 = 0$ and $j_2'j_1 = 0$. $j_1j_1' + j_2j_2' = Id_E$.
Let's define j_1', j_2' as the unique homomorphisms to make the following diagrams commute:

$$\begin{array}{ccccc} Ker(p) & \xrightarrow{j_1} & E & \xrightarrow{p} & E \\ {\scriptstyle Id}\uparrow & \nwarrow{\scriptstyle j_1'} & \uparrow{\scriptstyle 1-p} & & \\ Ker(p) & \xrightarrow[j_1]{} & E & & \end{array} \qquad \begin{array}{ccccc} Ker(1-p) & \xrightarrow{j_2} & E & \xrightarrow{1-p} & E \\ {\scriptstyle Id}\uparrow & \nwarrow{\scriptstyle j_2'} & \uparrow{\scriptstyle p} & & \\ Ker(1-p) & \xrightarrow[j_2]{} & E & & \end{array}$$

$j_1j_1'j_1 = j_1$ and $j_2j_2'j_2 = j_2$. So, $j_1'j_1 = Id_{Ker(p)}$ and $j_2'j_2 = Id_{Ker(1-p)}$.
Again we use universal property of kernel to get:

$$\begin{array}{ccccc} Ker(p) & \xrightarrow{j_1} & E & \xrightarrow{p} & E \\ {\scriptstyle 0}\uparrow & \nwarrow{\scriptstyle j_1'} & \uparrow{\scriptstyle 1-p} & & \\ Ker(1-p) & \xrightarrow[j_2]{} & E & & \end{array} \qquad \begin{array}{ccccc} Ker(1-p) & \xrightarrow{j_2} & E & \xrightarrow{1-p} & E \\ {\scriptstyle 0}\uparrow & \nwarrow{\scriptstyle j_2'} & \uparrow{\scriptstyle p} & & \\ Ker(p) & \xrightarrow[j_1]{} & E & & \end{array}$$

$j'_1 j_2 = 0$ and $j'_2 j_1 = 0$. As $j_2 j'_2 = p$, $j_1 j'_1 + j_2 j'_2 = p + (1-p) = Id_E$.

4.3 Grassmannian manifold

In 1.7, the space of all projection operators p of k^N ($Proj_n(k^N)$) such that $dim(Imp) = n$ and the collections of all n-dimensional subspaces of k^N ($G_n(k^N)$) or Grassmannian are described. There is a bijective correspondence between $G_n(k^N)$ and the subspace of all self-adjoint projectors of $Proj_n(k^N)$.

Theorem 4.3.1 *$G_n(k^N)$ is a deformation retract of $Proj_n(k^N)$.*

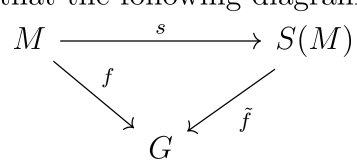
Proof Any self adjoint positive operator p has a self-adjoint positive square root $\sqrt{p}$. Let's consider the map $H : Proj_n(k^N) \times I \to Proj_n(k^N)$, given by $H(q,u) = \sqrt{1 + u(2q-1)^*(2q-1)}\,q(\sqrt{1 + u(2q-1)^*(2q-1)})^{-1}$.
If $q^*q = Id$, $H(q,u) \in G_n(k^N)$, $H(q,1) \in G_n(k^N)$ and $H(q,0) = q$.

4.4 Symmetrization of an abelian monoid

Definition 4.4.1 An abelian **monoid** is a set provided with a composition law satisfying all properties of an abelian group except for the existence of an inverse. The existence of inverse is not essential here.

Example 4.4.1 One example is $\mathbb{N} \cup \{0\}$ (the set of all natural numbers including 0.) with usual addition.

Remark 4.4.1 For each monoid M, we can attach an abelian group $S(M)$ and a monoid homomorphism $s : M \to S(M)$ with it which satisfies the following universal property: For any abelian group G and group homomorphism $f : M \to G$, $\exists!\ \tilde{f} : S(M) \to G$, a unique group homomorphism such that the following diagram commutes:



Remark 4.4.2 Constructions of $S(M)$

There are equivalent (upto group isomorphism) construction for $S(M)$. Let's consider the free abelian group $\mathcal{F}(M)$ generated by the elements $[m]$ of M. $S(M)$ can be taken as the quotient of $\mathcal{F}(M)$ by the subgroup generated by the elements of form: $[m_1 + m_2] - [m_1] - [m_2]$. For any $m \in M$, s takes m to the class of $[m]$ in the quotient.

Another formulation is the following. Consider the quotient of $M \times M$ with respect to the following equivalence relation- $(m_1, m_2) \sim (m_1', m_2')$ if $\exists$ $m \in M$ such that $m_1 + m_2' + m = m_2 + m_1' + m$. $s(m) = [(m, 0)]$

The following is equivalent as well. Consider the quotient of $M \times M$ with respect to the following equivalence relation- $(m_1, m_2) \sim (m_1', m_2')$ if $\exists\ m, n \in M$ such that $(m_1, m_2) + (m, m) = (m_1', m_2') + (n, n)$. Here also $s(m) = [(m, 0)]$. Now for any element (m, n), its inverse is (n, m) in $S(M)$.

Any element (m, n) can be written as $(m, 0) + (0, n) = s(m) - s(n)$.

Example 4.4.2 If we take $M = \mathbb{N} \cup 0$ with addition, $S(M) = \mathbb{Z}$.

Taking $M = \mathbb{Z} - \{0\}$, $S(M) = \mathbb{Q} - \{0\}$.

Example 4.4.3 Let M is a monoid where $\exists\ \infty$, an element of M such that $m \cdot \infty = \infty$, $\forall m \in M$. In this case, $S(M) = 0$. We can write each element of $S(M)$ as $s(m) - s(n)$ for some $m, n \in M$. $s(m) - s(n) = s(m) + s(\infty) - s(n) - s(\infty)$. As s is a monoid homomorphism, $s(m + \infty) = s(\infty) = s(m) + s(\infty)$. So, $s(m) - s(n) = s(\infty) - s(\infty) = 0$.
If we take $M = \mathbb{Z}$ with respect to multiplication, as $0.m = 0$, $\forall m \in M$, using the previous argument $S(M) = 0$.
Here the map s is definitely not injective.

Remark 4.4.3 For any abelian monoid M, the construction of $S(M)$ gives us a covariant functor from the category of abelian monoids and monoid homomorphism to the category of abelian groups and group homomorphisms.

Let's consider $f : M \to N$, a monoid homomorphism. As $s_N \cdot f : M \to S(N)$ is a group homomorphism, using the universal property of $S(M)$, we get a unique map (let's call it $S(f)$) between $S(M)$ and $S(N)$ making the following diagram commutative-

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle s_M} & & \downarrow{\scriptstyle s_N} \\ S(M) & \xrightarrow{S(f)} & S(N) \end{array} \quad S(g \cdot f) = S(g) \cdot S(f) \text{ and } S(Id_M) = Id_{S(M)}$$

Remark 4.4.4 (Grothendieck group of an additive category) Let's consider any additive category $\mathcal{A}$. For any $E \in \mathcal{Ob}(\mathcal{A})$, let's denote its isomorphism class by $\dot{E}$. Defining, $\dot{E}_1 + \dot{E}_2 = \widehat{E_1 \oplus E_2}$ does make sense as $\widehat{E_1 \oplus E_2}$ (the isomorphism class of $E_1 \oplus E_2$ depends only on $\dot{E}_1$ and $\dot{E}_2$. $E_1 \oplus (E_2 \oplus E_3) \approx (E_1 \oplus E_2) \oplus E_3$. $E_1 \oplus E_2 \approx E_2 \oplus E_1$. So, considering the isomorphism classes of objects of $\mathcal{A}$, we have got an abelian monoid out of the abelian category. Let's denote it by $\Phi(\mathcal{A})$.
Following the construction in remark 4.4.2, we can attach a unique abelian group with the abelian monoid just defined above. This is called the **Grothendieck group** of the additive category $\mathcal{A}$ and is denoted as $K(\mathcal{A})$.

Let, $\psi : \mathcal{A} \to \mathcal{A}'$ be an additive functor, we get a monoid homomorphism $\Phi(\psi) : \Phi(\mathcal{A}) \to \Phi(\mathcal{A}')$. Now using remark 4.4.3, we get a group homomorphism $K(\Phi(\psi)) : K(\mathcal{A}) \to K(\mathcal{A}')$.

Again if we have another abelian category and additive functor $\psi' : \mathcal{A}' \to \mathcal{A}''$, $K(\Phi(\psi' \cdot \psi)) = K(\Phi(\Psi) \cdot \Phi(\Psi')) = K(\Phi(\Psi)) \cdot K(\Phi(\Psi'))$ and $K(Id_{\mathcal{A}}) = Id_{K(\mathcal{A})}$.

Proposition 4.4.1 *Let's consider an additive category $\mathcal{A}$. We denote the the class of an object A of $\mathcal{A}$ as $[A]$ in $K(\mathcal{A})$.Then $[A]$ can be written as $[A_1] - [A_2]$ for some objects A_1, A_2 in $\mathcal{A}$.*

Two objects $[A_1]-[A_2] = [B_1]-[B_2]$, iff $\exists\, C \in \mathcal{A}$, such that $A_1 \oplus B_2 \oplus C \approx A_2 \oplus B_1 \oplus C$.

Proof Using remark 4.4.2, as the formulation of $K(\mathcal{A})$ for $\mathcal{A}$ is exactly what we do for any monoid, any element can be represented as difference of the classes of two other objects of the category $\mathcal{A}$.

Again using the second formulation of remark 4.4.2, if $s(\dot{A}_1) - s(\dot{A}_2) = s(\dot{B}_1) - s(\dot{B}_2)$, $\exists\, C \in \mathcal{A}$ such that $\dot{A}_1 + \dot{B}_2 + \dot{C} = \dot{A}_2 + \dot{B}_1 + \dot{C} \Rightarrow A_1 \oplus B_2 \oplus C \approx A_2 \oplus B_1 \oplus C$.

Corollary 4.4.1 *Suppose $A_1,\ A_2 \in \mathcal{Ob}(\mathcal{A})$. In $K(\mathcal{A})$, $[A_1] = [A_2]$ iff $\exists\, B \in \mathcal{Ob}(\mathcal{A})$ such that $A_1 \oplus B \approx A_2 \oplus B$.*

Proof We know for any object A of $\mathcal{A}$, in $K(\mathcal{A})$, $[A] = [A] - [0]$. Here 0 is the identity element of the monoid $\Phi(\mathcal{A})$ (using the notation of remark 4.4.4). Now we just use proposition 4.4.1 to get the required result.

4.5 Grothendieck group of a Banach functor

Remark 4.5.1 A **Banach category** $\mathcal{A}$ is an additive category where $\forall$ $E_1, E_2 \in \mathcal{Ob}(\mathcal{A})$, $Hom(E_1, E_2)$ can be provided with a Banach space structure over some fixed field $\mathbb{R}$ or $\mathbb{C}$.

$\xi(X)$ is a Banach category for X compact (described in 2.2).

An additive functor $\phi : \mathcal{A} \to \mathcal{A}'$ is said to be **quasi-surjectve** if for every object E of $\mathcal{A}'$, $\exists$ E', another object of $\mathcal{A}'$ such that $E \oplus E' = \phi(F)$, for some $F \in \mathcal{Ob}(\mathcal{A})$.

An additive functor $\phi : \mathcal{A} \to \mathcal{A}'$ between two Banach categories $\mathcal{A}$ and $\mathcal{A}'$ is said to be a **Banach functor** if $\forall$ $E, E' \in \mathcal{Ob}(\mathcal{A})$, the map $\mathcal{A}(E, E') \to \mathcal{A}'(\phi(E), \phi(E'))$ is linear and continuous.

For a compact pair (X, Y) (X is a compact space and $Y \subseteq X$ is closed), the functor $\phi : \xi(X) \to \xi(Y)$ taking any vector bundle over X to its restriction over Y is quasi-surjective, Banach functor. (described in 2.2)

Now we will be defining the Grothendieck group of a quasi surjective, Banach functor. For the definition quasi surjectivity is not necessary but for the interesting (and useful to describe relative K-groups) properties regarding exact sequences, it will be useful.

Remark 4.5.2 Let $\phi : \mathcal{A} \to \mathcal{A}'$ be a quasi-surjective Banach functor. Let's consider triples (E_1, E_2, β), where $E_1, E_2 \in \mathcal{Ob}(\mathcal{A})$ and $\beta : \phi(E_1) \to \phi(E_2)$ is an isomorphism.

Two triples (E_1, F_1, β_1) and (E_2, F_2, β_2) are isomorphic if there exists isomorphism $f_1 : E_1 \to E_2$ and $f_2 : F_1 \to F_2$ such that the following diagram commutes-

$$\begin{array}{ccc} \phi(E_1) & \xrightarrow{\beta_1} & \phi(F_1) \\ \Big\downarrow{\scriptstyle \phi(f_1)} & & \Big\downarrow{\scriptstyle \phi(f_2)} \\ \phi(E_2) & \xrightarrow{\beta_2} & \phi(F_2) \end{array}$$

Consider the collection of all such triples upto isomorphism and denote it by $\Gamma(X)$.

We call a triple (E, F, β) to be **elementary** if $E = F$ and β is homotopic to $Id_{\phi(E)}$ within $Aut(\phi(E))$. We define sum of triples like the following: $(E_1, F_1, \beta_1) \oplus (E_2, F_2, \beta_2) = (E_1 \oplus E_2, F_1 \oplus F_2, \beta_1 \oplus \beta_2)$

Consider an equivalence relation $(\sim)$ in $\Gamma(X)$. Two triples σ_1, σ_2 are equivalent if there exists elementary triples τ_1, τ_2 such that $\sigma_1 \oplus \tau_1 \approx \sigma_2 \oplus \tau_2$.

Now consider $K(\phi) := \Gamma(X)/\sim$. $K(\phi)$ has a monoid structure with respect to sum of triples.

Let's denote the class of a triple (E, F, β) in the monoid $K(\phi)$ by $[E, F, \beta]$. An element $[E, F, \beta] = 0$ in $K(\phi)$ if there exists $G, H \in \mathcal{O}b(\mathcal{A})$ and isomorphisms $u_1 : E \oplus G \to H$, $u_2 : F \oplus G \to H$ and $\phi(u_2) \cdot (\beta \oplus Id_{\phi(G)}) \cdot (\phi(u_1))^{-1}$ is homotopic to $Id_{\phi(H)}$ within the automorphisms of $\phi(H)$.

Proposition 4.5.1 *The abelian monoid $K(\phi)$ is actually a group.*

PROOF Let $[E, F, \beta] \in K(\phi)$. $[E, F, \beta] + [F, E, \beta^{-1}] = [E \oplus F, F \oplus E, \beta \oplus \beta^{-1}] \approx [E \oplus F, E \oplus F, \alpha]$. Here α is represented by: $\begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix}$
In $Aut(\phi(E) \oplus \phi(F))$, we have-

$$\begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix} = \begin{pmatrix} 1 & -\beta^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \beta^{-1} \\ 0 & -1 \end{pmatrix}$$

So, within $Aut(\phi(E) \oplus \phi(F))$, we have the following path between Id and α. $\gamma : I \to Aut(\phi(E) \oplus \phi(F))$, where-

$$\gamma(t) = \begin{pmatrix} 1 & -t\beta^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ t\beta & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & t\beta^{-1} \\ 0 & -1 \end{pmatrix}$$

So, the triple $[E \oplus F, F \oplus E, \beta \oplus \beta^{-1}]$ is elementary and for any element $[E, F, \beta]$, we have got its inverse $[F, E, \beta^{-1}]$.

Remark 4.5.3 Consider $\phi : \mathcal{A} \to \mathcal{A}'$ and the following homomorphism $j : K(\phi) \to K(\mathcal{A})$ given by $j([E, F, \beta]) = [E] - [F]$. If $\mathcal{A}' = 0$, $[E, F, \beta]$ is essentially determined by E and F and j is an isomorphism.

So, $K(\mathcal{A}) = K(\phi)$ if $\mathcal{A}' = 0$.

Now we will determine easier to implement conditions of equivalence of class of triples and some other technical results which is used in 2.2.

Proposition 4.5.2 *Let $[E, F, \beta_1]$ and $[E, F, \beta_2]$ are elements of $K(\phi)$, where β_1 and β_2 are homotopic within $Iso_{\mathcal{A}'}(\phi(E), \phi(F))$. Then $[E, F, \beta_1] = [E, F, \beta_2]$.*

Proof We will show $[E,F,\beta_1]-[E,F,\beta_2]=0$ in $K(\phi)$ (Here the class of elementary triples is the 0 of the monoid). $[E,F,\beta_1]-[E,F,\beta_2]=[E,F,\beta_1]+[F,E,\beta_2^{-1}]=[E\oplus F,E\oplus F,\alpha]$. $\alpha\in Aut(\phi(E)\oplus\phi(F))$ is represented by $\begin{pmatrix}0 & -\beta_2^{-1}\\ \beta_1 & 0\end{pmatrix}$. As β_2 is homotopic to β_1 within $Iso_{\mathcal{A}'}(\phi(E),\phi(F))$, α is homotopic to $\begin{pmatrix}0 & -\beta_1^{-1}\\ \beta_1 & 0\end{pmatrix}$, which is homotopic to $Id_{\phi(E)\oplus\phi(F)}$, using the proof of proposition 4.5.1.

Proposition 4.5.3 *Let,* $[E,F,\beta_1],[F,G,\beta_2]\in K(\phi)$. *Then* $[E,F,\beta_1]+[F,G,\beta_2]=[E,G,\beta_2\beta_1]$.

Proof $[E,F,\beta_1]+[F,G,\beta_2]=[E\oplus F,F\oplus G,\beta_1\oplus\beta_2]=[E\oplus F,G\oplus F,\gamma_1]$. Here $\gamma_1=\begin{pmatrix}0 & -\beta_2\\ \beta_1 & 0\end{pmatrix}$. Now $[E,G,\beta_2\beta_1]=[E\oplus F,G\oplus F,\beta_2\beta_1\oplus Id_{\phi(F)}]=[E\oplus F,G\oplus F,\gamma_2]$. Now $\gamma_1\gamma_2=\begin{pmatrix}0 & -\beta_2\\ \beta_2^{-1} & 0\end{pmatrix}$. So, $\gamma_1\gamma_2$ is homotopic to $Id_{\phi(E)\oplus\phi(F)}$ within $Aut(\phi(E)\oplus\phi(F))$. So, γ_1 is homotopic to γ_2 within the $Aut(\phi(E)\oplus\phi(F))$.

Example 4.5.1 Consider the functor $\phi:\xi_{\mathbb{R}}\to\xi_{\mathbb{R}}$. Here ξ_R is the category of finite dimensional real vector spaces and $\phi(V)=V\oplus V$. Now $GL_n(\mathbb{R})$, for any $n\in\mathbb{N}$, has two path connected components. The matrices of positive determinant and the matrices of negative determinant form these two components. Let, $beta: E\oplus E\to F\oplus F$ be an isomorphism. For any $v\in Iso(F,E)$, the sign of the determinant of the composite $(v\oplus v)\cdot\beta$ is independent of v. We are considering

$$E\oplus E\xrightarrow{\beta}F\oplus F\xrightarrow{v\oplus v}E\oplus E.$$

Now the only possible triples are of form $[E,E',\alpha]$, where E and E' are isomorphic and it is totally characterized by the sign of determinant of α. So, $K(\phi)\approx\mathbb{Z}_2$. For the similar functor ϕ in case of $\xi_{\mathbb{C}}$, $K(\phi)=0$.

Now we will introduce the exact sequence for which all this formulations are done.

Theorem 4.5.1 *Let,* $\phi:\mathcal{A}\to\mathcal{A}'$ *be a quasi-surjective, Banach functor. Let* $i:K(\phi)\to K(\mathcal{A})$ *be defined by* $i([E,F,\alpha])=[E]-[F]$ *and* $j:K(\mathcal{A})\to K(\mathcal{A}')$

is defined as follows. $j([E]-[F]) = [\phi(E)] - [\phi(F)]$. *Then we have the following exact sequence-*

$$K(\phi) \xrightarrow{i} K(\mathcal{A}) \xrightarrow{j} K(\mathcal{A}')$$

Proof **a.** $Im(i) \subseteq Ker(j)$

Let $[E, F, \beta] \in K(\phi)$. $j \cdot i([E, F, \beta]) = j([E] - [F]) = [\phi(E)] - [\phi(F)] = 0$, as $\beta : \phi(E) \to \phi(F)$ is an isomorphism.

b. $Ker(j) \subseteq Im(i)$

Let $[E] - [F] \in K(\mathcal{A})$ such that $[\phi(E)] - [\phi(F)] = 0$, then using corollary 4.4.1, $\exists E' \in \mathcal{A}'$ such that $\phi(E) \oplus E' \approx \phi(F) \oplus E'$. As the functor ϕ is **quasi-surjective**, $\exists\ T \in \mathcal{A}$ and $\exists\ T' \in \mathcal{A}'$ such that $\phi(T) \approx E' \oplus T'$. So, $\phi(E \oplus T) \approx \phi(E) \oplus E' \oplus T'$ is isomorphic to $\phi(F \oplus T) \approx \phi(F) \oplus E' \oplus T'$. So, $[E] - [F] = [E \oplus T] - [F \oplus T] = i([E \oplus T, F \oplus T, \gamma])$. Here $\gamma : \phi(E \oplus T) \to \phi(F \oplus T)$ is an isomorphism.

Theorem 4.5.2 *Considering the premise of theorem 4.5.1, if there exists a functor* $\Psi : \mathcal{A}' \to \mathcal{A}'$ *such that* $\phi\Psi = Id_{\mathcal{A}'}$, *we get the following split exact sequece-*

$$0 \longrightarrow K(\phi) \xrightarrow{i} K(\mathcal{A}) \xrightarrow{j} K(\mathcal{A}') \longrightarrow 0$$

Proof **a.** i injective.

Let $i([E, F, \beta]) = 0$. We want to show that $[E, F, \beta]$ is an elementary triple. So, $\exists\ T \in \mathcal{O}b(\mathcal{A})$ such that $E \oplus T \approx F \oplus T$. Now, $[E, F, \beta] = [E \oplus T, F \oplus T, \beta \oplus Id] := [H, H, \alpha]$. We denote $H := E \oplus T \approx F \oplus T$. We get the following diagram that commutes:

$$\begin{array}{ccc} \phi(H) & \xrightarrow{\alpha} & \phi(H) \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\ \phi(\phi\Psi(H)) & \xrightarrow{\phi\Psi(\alpha)} & \phi(\phi\Psi(H)) \end{array}$$

γ is an isomorphism taken to make the diagram commute.Using proposition 4.5.3, we get $[H, H, \alpha] = [H, \phi\Psi(H), \gamma] + [\Psi\phi(H), \Psi\phi(H), \phi\Psi(\beta)] + [\Psi\phi(H), H, \gamma^{-1}]$. Using proposition 4.5.1, $[H, \Psi\phi(H), \gamma] + [\Psi\phi(H), H, \gamma^{-1}] =$

0 Because of the commutativity of the following diagram $[\Psi\phi(H), \Psi\phi(H), \phi\Psi(\beta)] = 0$ in $K(\phi)$, being isomorphic to $[\Psi\phi(H), \Psi\phi(H), Id]$.

$$\begin{array}{ccc} \phi(\Psi\phi(H)) & \xrightarrow{\phi\Psi(\alpha)} & (\phi\Psi)(\phi(H)) \\ \big\downarrow{\scriptstyle \phi\Psi(\alpha)} & & \big\downarrow{\scriptstyle Id} \\ \phi(\Psi\phi(H)) & \xrightarrow{Id} & (\phi\Psi)(\phi(H)) \end{array}$$

b. j surjective
Even without the existence of the functor Ψ, j is anyway surjective because of quasi-surjectivity of ϕ. For any $[E_1'] - [E_2'] \in K(\mathcal{A}')$; $\exists\ T_1', T_2' \in \mathcal{Ob}(\mathcal{A}')$ and $T_1, T_2 \in \mathcal{Ob}(\mathcal{A})$ such that $\phi(T_1) = E_1' \oplus T_1'$ and $\phi(T_2) = E_2' \oplus T_2'$. So, $\phi([T_1] - [T_2]) = [E_1' \oplus T_1'] - [E_2' \oplus T_2'] = [E_1'] - [E_2']$.

Now some technical results will be introduced to give a simpler description of the elementary triples in $K(\phi)$, when ϕ is a full Banach functor.

Remark 4.5.4 Let C, C' be two Banach algebras with unit and $f : C \to C'$ is a continuous map, which is surjective. Let, $\sigma : I \to C'$ be another continuous map such that $Im(\sigma) \subseteq C'^*$ (The group of invertibles in C (respectively C' is denoted by C^* (or C'^*). $\exists\ a \in C^*$ such that $f(a) = \sigma(0)$. Then $\exists\ a' \in C^*$ such that $f(a') = \sigma(1)$.

Consider X, a compact space and $C(X)$ denotes the Banach algebra of all continuous functions from X to C. In that case if if $f : C \to C'$ (with properties same as proposition 4.5.4) gives rise to a surjective homomorphism $f^* : C(I) \to C'(I)$. We get this just by using the previous result for $f^* : C(I) \to C'(I)$.

Proposition 4.5.4 *Let $\phi : \mathcal{A} \to \mathcal{A}'$ is a* ***full*** *Banach functor and $[E, E, \beta] = 0$ in $K(\phi)$. Then $[E, E, \beta] \approx [E, E, Id]$.*

Proof Consider the Banach algebras (E) and $End(\phi(E))$. As ϕ is full, ϕ induces a surjective ring homomorphism between $End(E)$ and $End(\phi(E))$. As there is a continuous path within $Aut(\phi(E))$ between Id and β, we have got a $\sigma : I \to End(\phi(E))^*$, where $\sigma(0) = Id, \sigma(1) = \beta$. Using remark 4.5.4, $\exists\ \alpha \in Aut(E)$ and $\phi(\alpha) = \beta$.

$$\begin{array}{ccc} \phi(E) & \xrightarrow{\beta} & \phi(E) \\ \downarrow{\scriptstyle \phi(\alpha)} & & \downarrow{\scriptstyle \phi(Id)} \\ \phi(E) & \xrightarrow{Id} & \phi(E) \end{array}$$

Because of the commutativity of the following diagram, $[E, E, \beta]$ is isomorphic to $[E, E, Id]$.

So, we get a simpler definition of $K(\phi)$, when ϕ is full, quasi-surjective Banach functor. In that we can replace the definition of elementary triples by the triples of form $[E, E, Id]$ or $[E_1, E_2, \alpha]$ of the form defined in corollary 4.5.1.

Corollary 4.5.1 *Let, $\phi : \mathcal{A} \to \mathcal{A}'$ be a full, quasi-surjective Banach functor. An element $[E, F, \beta] = 0$ in $K(\phi)$ if and only if $\exists\, H \in \mathcal{O}b(\mathcal{A})$ and $\alpha : E \oplus G \to F \oplus G$, an isomorphism, where $\phi(\alpha) = \beta \oplus Id_{\phi(H)}$.*

Proof If $[E, E, \beta]$ is an elementary triple, then using proposition 4.5.4, there exists elementary triples $[G, G, Id], [H, H, Id] \in K(\phi)$ such that $[E \oplus H, F \oplus H, \beta \oplus Id_{\phi(H)}] \approx [G, G, Id_{\phi(G)}]$. Let's consider $f_1 : E \oplus H \to G$ and $f_2 : F \oplus H \to G$, the corresponding isomorphisms. The following diagram commutes.

$$\begin{array}{ccc} \phi(E \oplus H) & \xrightarrow{\beta \oplus Id_{\phi(H)}} & \phi(F \oplus G) \\ \downarrow{\scriptstyle \phi(f_1)} & & \downarrow{\scriptstyle \phi(f_2)} \\ \phi(G) & \xrightarrow{Id_{\phi(G)}} & \phi(G) \end{array}$$

4.6 K^{-1} of a Banach category

Here we will try to define the functor $K^{-1}(\mathcal{A})$, for a Banach category $\mathcal{A}$. The main aim is to construct the connecting homomorphism, for a Banach functor $\phi : \mathcal{A} \to \mathcal{A}'$, $\delta : K^{-1}(\mathcal{A}') \to K(\phi)$, which makes the following sequence exact:

$$K^{-1}(\mathcal{A}) \longrightarrow K^{-1}(\mathcal{A}') \xrightarrow{\delta} K(\phi) \longrightarrow K(\mathcal{A}) \longrightarrow K(\mathcal{A}')$$

Remark 4.6.1 Let $\mathscr{A}$ be a Banach category (although for the definition additive category would suffice). Let's consider pairs of the following forms: (E, β). Here $E \in \mathcal{Ob}(\mathscr{A})$ and $\beta \in Aut(E)$. Two pairs (E_1, β_1) and (E_2, β_2) are said to be isomorphic if $\exists\ h : E_1 \to E_2$, an isomorphism that makes the following diagram commute.

$$\begin{array}{ccc} E_1 & \xrightarrow{h} & E_2 \\ \Big\downarrow{\scriptstyle\beta_1} & & \Big\downarrow{\scriptstyle\beta_2} \\ E_1 & \xrightarrow{h} & E_2 \end{array}$$

A pair (E, β) is considered **elementary** if β is homotopic to Id_E within $Aut(E)$. We define $(E_1, \beta_1) + (E_2, \beta_2) = (E_1 \oplus E_2, \beta_1 \oplus \beta_2)$.

Consider the collection of all such pairs with respect to the following equivalence relation: $(E_1, \beta_1) \sim (E_2, \beta_2)$ if $\exists\ \tau_1, \tau_2$, elementary triples such that $(E_1, \beta_1) + \tau_1 \approx (E_2, \beta_2) + \tau_2$. This collection (quotiented out by the above mentioned equivalence relation) is a **group** with respect to the addition of pairs defined and is said to be $K^{-1}(\mathscr{A})$. In $K^{-1}(\mathscr{A})$ the class of a pair (E, β) is denoted by $[E, \beta]$

To show $K^{-1}(\mathscr{A})$ indeed is a group, we will show that for any class of pair $[E, \beta]$, $[E, \beta^{-1}]$ is its inverse (ie. $[E, \beta] + [E, \beta^{-1}] = [0]$ in $K^{-1}(\mathscr{A})$.)

Now, $[E, \beta] + [E, \beta^{-1}] = [E \oplus E, \beta \oplus \beta^{-1}]$. $\beta \oplus \beta^{-1}$ can be represented by the following matrix-

$$\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} = \begin{pmatrix} 0 & -\beta \\ \beta^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

So, $\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}$ is homotopic to $Id_{E \oplus E}$ within $Aut(E \oplus E)$ and $(E \oplus E, \beta \oplus \beta^{-1})$ is an elementary pair and we are done.

Now we will show some technical results in order to provide a simpler description of $K^{-1}(\mathscr{A})$.

Proposition 4.6.1 *If $\beta_1, \beta_2 \in Aut(E)$ such that β_1 is homotopic to β_2 within $Aut(E)$, then in $K^{-1}(\mathscr{A})$, $[E, \beta_1] = [E, \beta_2]$.*

Proof $[E, \beta_1] - [E, \beta_2] = [E, \beta_1] + [E, \beta_2^{-1}] = [E \oplus E, \beta_1 \oplus \beta_2^{-1}]$. By assumption $\begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2^{-1} \end{pmatrix}$ is homotopic to $\begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_1^{-1} \end{pmatrix}$ within $Aut(E \oplus E)$, which again using remark 4.6.1 is homotopic to $Id_{E \oplus E}$, within $Aut(E \oplus E)$.

Proposition 4.6.2 *In $K^{-1}(\mathcal{A})$, $[E,\beta_1]+[E,\beta_2]=[E,\beta_1\beta_2]=[E,\beta_2\beta_1]$*

Proof $[E,\beta_1]+[E,\beta_2]=[E\oplus E,\beta_1\oplus\beta_2]$ and $[E,\beta_1\beta_2]=[E\oplus E,\beta_1\beta_2\oplus Id_E]$. So, we want to show that $\beta_1\oplus\beta_2$ is homotopic to $\beta_1\beta_2\oplus Id_E$ within $Aut(E\oplus E)$. Now

$$\begin{pmatrix}\beta_1 & 0\\ 0 & \beta_2\end{pmatrix}^{-1}\begin{pmatrix}\beta_1\beta_2 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\beta_2 & 0\\ 0 & \beta_2^{-1}\end{pmatrix}$$

$\begin{pmatrix}\beta_2 & 0\\ 0 & \beta_2^{-1}\end{pmatrix}$ is homotopic to $Id_{E\oplus E}$ within $Aut(E\oplus E)$. Again $[E,\beta_2\beta_1]=[E,\beta_2]+[E,\beta_1]=[E,\beta_1]+[E,\beta_2]$.

Proposition 4.6.3 *In $K^{-1}(\mathcal{A})$, $[E,\beta]=0$ if and only if exists $F\in\mathcal{O}b(\mathcal{A})$, such that $\beta\oplus Id_F$ and $Id_{E\oplus F}$ are homotopic within $Aut(E\oplus F)$.*

Proof Let's consider $[E,\beta]=[0]$ (ie.(E,β) is an elementary pair). By definition $\exists\ (F,\alpha),(F',\alpha')\in K^{-1}(\mathcal{A})$ (elementary pairs) and $g:E\oplus F\to F'$, an isomorphism, which makes the following diagram commute.

$$\begin{array}{ccc} E\oplus F & \xrightarrow{g} & F' \\ {\scriptstyle\beta\oplus\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ E\oplus F & \xrightarrow{g} & F' \end{array}$$

So, $\beta\oplus\alpha=g^{-1}\cdot\alpha'\cdot g$, which is homotopic to $g^{-1}\cdot Id_{F'}\cdot g=Id_{E\oplus F}$. As, $\beta\oplus Id_F$ and $\beta\oplus\alpha$ are homotopic, we are done.
The converse is true by definition.

With the following theorem, the group $K^{-1}(\mathcal{A})$ can be defined in a much easier way equivalently.

Theorem 4.6.1 *In $K^{-1}(\mathcal{A})$, two elements $[E,\beta_1]=[F,\beta_2]$ iff $\exists\ H\in\mathcal{O}b(\mathcal{A})$, such that $\beta_1\oplus Id_F\oplus Id_H$ and $Id_E\oplus\beta_2\oplus Id_H$ are homotopic within $Aut(E\oplus F\oplus H)$.*

Proof Let, $[E,\beta_1]=[F,\beta_2]\Rightarrow[E,\beta_1]+[F,{\beta_2}^{-1}]=0\Rightarrow[E\oplus F,\beta_1\oplus\beta_2^{-1}]=0$. Using proposition 4.6.3, $\exists H\in\mathcal{O}b(\mathcal{A})$, such that $\beta_1\oplus\beta_2^{-1}\oplus Id_H$ is homotopic to $Id_{E\oplus F\oplus H}$. Multiplying by $Id_E\oplus\beta_2\oplus Id_H$, we get $\beta_1\oplus Id_F\oplus Id_H$ is homotopic to $Id_F\oplus\beta_2\oplus Id_H$.

Now a very interesting formulation of K^{-1} is given for a particular case.

Remark 4.6.2 Let's cosider a Banach algebra A and $\mathcal{L}(A)$, the category of finitely generated free (left or right) A modules. As A is a Banach algebra, $\mathcal{L}(A)$ is a Banach category. We want to describe $K^{-1}(\mathcal{L}(A))$. Let's consider the directed system: $GL_1(A) \to GL_2(A) \to .... \to GL_n(A) \to ....$ Let's denote the direct limit of this system by $GL(A)$. Now we will show $K^{-1}(\mathcal{L}(A))$ and $\pi_0(GL(A))$ are isomorphic as groups. (In $\pi_0(GL(A))$, we consider the group structure induced from matrix multiplication). Consider the map $\gamma : K^{-1}(\mathcal{L}(A)) \to \pi_0(GL(A))$, given by γ takes $([A^m, \beta])$ to the class of β in $\pi_0(GL(A))$. This is well defined because according to computation in remark 4.6.1, $\begin{pmatrix} \beta & 0 \\ 0 & Id \end{pmatrix}$ and $\begin{pmatrix} Id & 0 \\ 0 & \beta \end{pmatrix}$ are in the same path-connected component in $GL_{2n}(A)$, using proposition 4.6.1, it makes sense. Using proposition 4.6.2, γ is a group homomorphism, which is injective because of proposition 4.6.3 and surjective by construction.

Remark 4.6.3 For any additive category $\mathcal{A}$, in lemma 1.6.3, we have constructed its pseudo-abelian category $\mathcal{A}'$. We want to focus on $K^{-1}(\mathcal{A}')$. A functor $\phi : \mathcal{A} \to \mathcal{A}'$ has been constructed. Now the group homomorphism $K(\phi)$ is definitely not bijective. (For example we can think about, for any Banach algebra with unit A, the the category of finitely generated (left) free A-modules ($\mathcal{L}(A)$) and its associated pseudo-abelian category consisting of all finitely generated (left) projective A modules ($\mathcal{P}(A)$). $K(\mathcal{L}(A)) \approx \mathbb{Z}$ but $K(\mathcal{P}(A))$ is usually hard to compute.)

But the homomorphism $K^{-1}(\phi) : K^{-1}(A) \to K^{-1}(A')$ is bijective.

Let, $[E, \beta] \in K^{-1}(\mathcal{A}')$. Let, $E_1 \in \mathcal{A}$ and $F \in \mathcal{A}'$, such that $E \oplus F \approx \phi(E_1)$. Let's denote this isomorphism by f. Now $[E, \beta] = [E \oplus F, \beta \oplus Id_F] = [E_1, \alpha] \in K^{-1}(\mathcal{A})$, where α is such that $\phi(\alpha) = f^{-1} \cdot (\beta \oplus Id_F) \cdot f$. So $K^{-1}(\phi)$ is **surjective**.

If $[E, \beta] \in K^{-1}(\mathcal{A})$ such that $K^{-1}(\phi)([E, \beta]) = 0 \in K^{-1}(\mathcal{A}')$. Using proposition 4.6.3, $\exists\ F \in \mathcal{A}'$(wlog we can take it of form $\phi(F')$ using quasi-surjectivity of ϕ) such that $\phi(\beta) \oplus Id_{\phi(F')}$ and $Id_{\phi(E) \oplus \phi(F')}$ are homotopic within $Aut(\phi(E) \oplus \phi(F'))$. So, $[E, \beta] = [E \oplus F', \beta \oplus Id_{F'}] = 0 \in K^{-1}(\mathcal{A})$. So, $K^{-1}(\phi)$ is **injective**.

Now we will be considering a quasi-surjective Banach functor $\phi : \mathcal{A} \to \mathcal{A}'$ and trying to construct the connecting homomorphism $\delta : K^{-1}(\mathcal{A}') \to K(\phi)$.

Remark 4.6.4 Let's consider an element $[E', \beta'] \in K^{-1}(\mathcal{A}')$. Because of quasi-surjectivity of ϕ, $\exists F' \in \mathcal{Ob}(\mathcal{A}')$ and $\exists F \in \mathcal{Ob}(\mathcal{A})$ such that we get an isomorphism $g : E' \oplus F' \to \phi(F)$. Let's denote the isomorphism from $\phi(F)$ to $\phi(F)$ by β, that makes the following diagram commute:

$$\begin{array}{ccc} E' \oplus F' & \xrightarrow{g^{-1}} & \phi(F) \\ {\scriptstyle \beta' \oplus Id'_F}\downarrow & & \downarrow{\scriptstyle \beta} \\ E' \oplus F' & \xrightarrow{g^{-1}} & \phi(F) \end{array}$$

Now we define $\delta([E', \beta']) := [F, F, \beta]$.

At first, we have to show δ **makes sense**, that is it does not depend on the choice of F' and g'. Let's consider another choices H', H, g', for which we get a new element $[H, H, \overline{\beta}] = \delta([E', \beta'])$.

As $[F, F, \beta] = [F \oplus H, F \oplus H, \beta \oplus 1]$. Now we get the following commutative diagram:

$$\begin{array}{ccc} (E' \oplus F') \oplus (E' \oplus H') & \xrightarrow{g^{-1} \oplus g'^{-1}} & \phi(F) \oplus \phi(H) \\ {\scriptstyle (\beta' \oplus Id_{F'}) \oplus (Id_{E'} \oplus Id_{H'})}\downarrow & & \downarrow{\scriptstyle \beta \oplus Id_{\phi(H)}} \\ (E' \oplus F') \oplus (E' \oplus H') & \xrightarrow[g^{-1} \oplus g'^{-1}]{} & \phi(F) \oplus \phi(H) \end{array}$$

Now, $(\beta' \oplus Id_{F'}) \oplus (Id_{E'} \oplus Id_{H'})$ is homotopic to $(Id_{E'} \oplus Id_{H'}) \oplus (\beta' \oplus Id_{F'})$ within $Aut(E' \oplus F' \oplus E' \oplus H')$.

$$\begin{array}{ccc} (E' \oplus F') \oplus (E' \oplus H') & \xrightarrow{g^{-1} \oplus g'^{-1}} & \phi(F) \oplus \phi(H) \\ {\scriptstyle (Id_{E'} \oplus Id_{H'}) \oplus (\beta' \oplus Id_{F'})}\downarrow & & \downarrow{\scriptstyle Id_{\phi(E)} \oplus \beta} \\ (E' \oplus F') \oplus (E' \oplus H') & \xrightarrow[g^{-1} \oplus g'^{-1}]{} & \phi(F) \oplus \phi(H) \end{array}$$

As the last diagram commutes and $\beta \oplus Id_{\phi(H')}$ and $Id_{\phi(E)} \oplus \beta$ are homotopic, we have $[E \oplus H, E \oplus H, \beta \oplus Id_{\phi(H)}] = [E \oplus H, E \oplus H, Id_{\phi(E)} \oplus \beta'] = [H, H, \beta']$. So, δ is well-defined.

At the same time we have shown that δ is **natural**.

Theorem 4.6.2 *Using the notation used in theorem 4.5.1, we get the following exact sequence:*

$$K^{-1}(\mathcal{A}) \xrightarrow{j'} K^{-1}(\mathcal{A}') \xrightarrow{\delta} K(\phi) \xrightarrow{i} K(\mathcal{A}) \xrightarrow{j} K(\mathcal{A}')$$

Proof **a.** exactness at $K(\phi)$: Because ϕ is quasi-surjective, for any element in $K^{-1}(\mathcal{A}')$, without loss of generality, we can assume E' to be of form $\phi(E)$. Now, $\delta([E', \beta']) = [E, E, \beta']$. $i \cdot \delta([E', \beta']) = i([E, E, \beta']) = 0$.

Let $[E, F, \beta] \in K(\phi)$ and $i([E, F, \beta]) = 0 = [E] - [F]$. Using proposition 4.4.1, $\exists\ H \in \mathcal{O}b(\mathcal{A})$ and there is an isomorphism $g : E \oplus H \to F \oplus H$. So, $[E, F, \beta] = [E \oplus H, F \oplus H, \beta \oplus Id_{\phi(H)}] = [E \oplus H, F \oplus H, \phi(g^{-1}) \cdot (\beta \oplus Id_{\phi(H)})]$. As $[E \oplus H, F \oplus H, \phi(g^{-1}) \cdot (\beta \oplus Id_{\phi(H)})]$ and $[E \oplus H, F \oplus H, \beta \oplus Id_{\phi(H)}]$ are isomorphic, $\delta([\phi(E) \oplus \phi(T), \beta]) = [E, F, \beta]$

b. exactness at $K^{-1}(\mathcal{A}')$: Let, $[E, \beta] \in K^{-1}(\mathcal{A})$. $\delta \cdot j'([E, \beta]) = [E, E, \beta] = 0$ as $[E, E, \beta]$ and $[E, E, Id_{\phi(E)}]$ are isomorphic.

Again, using quasi-surjectivity of ϕ, we can choose an arbitrary element $[E', \beta'] \in K^{-1}(\mathcal{A}')$ such that $E' = \phi(E)$, for some $E \in \mathcal{O}b(\mathcal{A})$. If $\delta([E']) = [E, E, \alpha'] = 0$, there are $[H_1, H_1, \eta_1]$ and $[H_2, H_2, \eta_2]$, two elementary triples and isomorphisms $u_1 : E \oplus H_1 \to H_2$ and $u_2 : E \oplus H_1 \to H_2$, such that the following diagram is commutative:

$$\begin{array}{ccc} \phi(E) \oplus \phi(H_1) & \xrightarrow{\beta' \oplus \eta_1} & \phi(E) \oplus \phi(H_1) \\ \downarrow{\scriptstyle \phi(u_1)} & & \downarrow{\scriptstyle \phi(u_2)} \\ \phi(H_2) & \xrightarrow{\quad \eta_2 \quad} & \phi(H_2) \end{array}$$

So, $(E, E, \beta') \oplus (H_1, H_1, \eta_1) \approx (H_2, H_2, \eta_2)$.
Now, we have $[E', \beta'] = [\phi(E) \oplus \phi(H_1), \beta' \oplus \eta_1] = [\phi(H_2), \phi(u_1)\phi(u_2^{-1})] = [\phi(H_2), \phi(u_1 \cdot u_2^{-1})] = j'([H, u_1 \cdot u_2^{-1}])$.

The rest is done by theorem 4.5.1.

4.7 Canonical line bundle over $\mathbb{C}P^1$

Here, we want to prove a technical result about the canonical line bundle H over $\mathbb{C}P^1 \cong S^2$, which is used throughout the text.

Theorem 4.7.1 *Let,* 1 *be the trivial line bundle. Then* $(H \otimes H) \oplus 1 \approx H \oplus H$*.*

Let's look at the clutching functions (here these are maps $S^1 \to GL_2(\mathbb{C})$) of $(H \otimes H) \oplus 1$ and $H \oplus H$. These are $z \to \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix}$ and $z \to \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$ respectively. As, $GL_2(\mathbb{C})$ is path connected, we have a path γ between Id and the matrix that interchanges the elements of the main diagonal of any matrix in $GL_2(\mathbb{C})$. Now, the path $(z \oplus 1)\gamma(1 \oplus z) : I \to GL_2(\mathbb{C})$ gives a required homotopy between the two clutching functions. So, we are done.

4.8 Module structure on $\tilde{K}^*(X, A)$

Let's consider a compact pair (X, A). We have defined and used the ring $\tilde{K}^*(X)$. Here, at first, we will give $\tilde{K}^*(X, A)$ and $\tilde{K}^*(A)$, $\tilde{K}^*(X)$ module structure.

Consider the inclusion map $i : A \hookrightarrow X$. For $\alpha \in \tilde{K}^*(A)$ and $\epsilon \in \tilde{K}^*(X)$, $\epsilon \cdot \alpha := i^*(\epsilon)\alpha$.

Let's consider the diagonal map: $X \to X \wedge X$. It induces a well defined map $X/A \to X \wedge X$. So, we get an external product $\tilde{K}^*(X) \otimes \tilde{K}^*(X, A) \to \tilde{K}^*(X, A)$.

Proposition 4.8.1 *The following exact sequence is an exact sequence of $\tilde{K}^*(X)$ modules and module homomorphisms.*

$$\begin{array}{ccc} \tilde{K}^*(X, A) & \longrightarrow & \tilde{K}^*(X) \\ & \nwarrow \quad \swarrow & \\ & \tilde{K}^*(A) & \end{array}$$

4.9 Some associated fiber bundles

Let's consider a vector bundle E over X. We can associate a fiber bundle $P(E)$ over X, whose fiber $P(E)_x$ is the projective space associated to the vector space E_x, $P(E_x)$. Here, $x \in X$, any arbitrary point.
This is the **projective bundle**, associated to E over X.
This space can be topologized as the quotient space of the sphere bundle

$S(E)$, where we factor out scalar multiplication in each fiber.
For any $x \in X$, we get a neighborhood U of x, such that $E_U \cong U \times k^n$, for some non-negative integer n. So, $P(E)_U \cong U \times kP^{n-1}$. Here, $k = \mathbb{C}$ or $\mathbb{R}$.

Now, again we consider an n-dimensional vector bundle E over X. We construct the associated **flag bundle** $F(E)$ over X.
Here, $F(E)$ is the subspace of the n-fold product $P(E) \times P(E) \times ... \times P(E)$, consisting of n-tuples of orthogonal lines from the fiber of E. The fiber of $F(E)$ consists of n-tuples of orthogonal lines through origins in k^n. Local triviality follows using the argument given for $P(E)$.

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