Hilbert's 16-th Problem

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Certificate of Examination

This is to certify that the dissertation titled **"Hilbert's 16-th Problem"** submitted by **Mr.Abhay P S** (Reg. No. MS15027) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: June 15, 2020

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Shane D'Mello at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of expository work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Shane D'Mello

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Abstract

Real projective plane algebraic curves and their classification problems are introduced. Topological classification problem and its solution is discussed. Isotopy classification problem is discussed and the isotopy classification of curves upto degree 5 is looked upon. Isotopy types of curves upto degree 5 are constructed using marking method or small perturbation method.

Chapter 1

Preliminaries

1.1 Real Projective plane

Real projective plane, denoted by $\mathbb{R}P^2$ is the set of all lines passing through origin in $\mathbb{R}^3$. The topology given to $\mathbb{R}P^2$ is the quotient topology induced by the surjective map q : $\mathbb{R}^3 \to \mathbb{R}P^2$ that maps a point in $\mathbb{R}^3$ to the line passing through that point and the origin. A line in $\mathbb{R}P^2$ passing through a point (x_0,x_1,x_2) is denoted as $(x_0 : x_1 : x_2)$. Two points $(x_0 : x_1 : x_2)$ and $(y_0 : y_1 : y_2) \in \mathbb{R}P^2$ are equal iff (x_0,x_1,x_2) = c(x_0,x_1,x_2) for c $\in \mathbb{R}$. So they are called as homogeneous coordinates. Consider the unit sphere $\mathbb{S}^2$ centered in origin in $\mathbb{R}^3$. Then every line passing through the origin intersects $\mathbb{S}^2$ in two antipodal points. Hence we can also get $\mathbb{R}P^2$ with a quotient map on $\mathbb{S}^2$ that identifies antipodal points of $\mathbb{S}^2$. $\mathbb{R}P^2$ is compact and connected since it is the continuous image of a compact and connected space. $\mathbb{R}P^2$ is a smooth manifold of dimension 2. $\mathbb{R}_2$ is embedded in $\mathbb{R}P^2$ via the map i : $\mathbb{R}^2 \to \mathbb{R}P^2$ defined by i(x,y) = (x:y:1).

Chapter 2

Real Algebraic Curves

2.1 Curves

Definition 2.1.1 *A real projective algebraic plane curve of degree n is an equivalence class of homogeneous polynomials $f(x_0,x_1,x_2)$ of degree n in three variables, in $\mathbb{R}[x_0,x_1,x_2]$, with the equivalence relation $f \equiv \lambda f$, for $\lambda \in \mathbb{R} \setminus 0$.*

An important source of most of the topic on curves discussed here is [Vir]. When we say real affine plane curve, we mean the zero set of a polynomial in two variables in $\mathbb{R}[x_0,x_1]$. As mentioned in the preliminaries, $\mathbb{R}_2$ is embedded in $\mathbb{R}P^2$ via the map i : $\mathbb{R}^2 \to \mathbb{R}P^2$ defined by i(x,y) = (x:y:1). In all discussions, we use the word curve to mean a real projective algebraic plane curve, unless stated otherwise.

Example 2.1.2 *1. Consider the polynomial $a_1(x_0,x_1,x_2) = x_0+x_1+x_2$. The equivalence class of a_1 consisting of all λa_1, $\lambda \in \mathbb{R} \setminus 0$ is a degree 1 curve. Let us denote it by A_1.*
2. Consider the polynomial $a_2(x_0,x_1,x_2) = x_0^2+x_1^2-x_2^2$. The equivalence class of a_2 is a degree 2 curve, we denote it by A_2
3. Consider the polynomial $a_3(x_0,x_1,x_2) = x_2x_1^2-x_0^3-x_2x_0^2$. The class of a_3 is a degree 3 curve. We denote it by A_3.

See [Fis01] for details on projective algebraic plane curves

Definition 2.1.3 *If C is a curve with an underlying homogeneous polynomial f, then the set $\{(x_0:x_1:x_2) \in \mathbb{R}P^2 : f(x_0,x_1,x_2)=0\}$, denoted by $\mathbb{R}C$, is called the set of real points of the curve C.*

Note that f being homogeneous is necessary for the well definedness of set of real points. Let us look at the sets of real points of the examples mentioned earlier. We know that $\mathbb{R}^2$ is embedded in $\mathbb{R}P^2$ via the map $i(x_1,x_2) = (x_1 : x_2 : 1)$. It is easier to visualise the set of real points with their projections in this embedded plane.

1. $\mathbb{R}A_1$ is a line in the plane. It is called a projective line.

2. $\mathbb{R}A_2$ is a circle in the plane, since a_2 is the homogenisation into 3 variables of the circle $x_0^2 + x_1^2 = 1$.

3. The polynomial a_3 is the homogenisation of the nodal cubic $x_1^2 = x_0^3 - x_0^2$. Therefore $\mathbb{R}A_3$ is a nodal cubic in the plane.

Definition 2.1.4 (Singular Point) *A point $(x_0 : x_1 : x_2) \in \mathbb{R}P^2$ is called a singular point of the curve C with a corresponding polynomial f, if all the partial derivatives of f vanish at $(x_0,x_1,x_2) \in \mathbb{R}^3$. A curve with no singular points is called a non-singular curve.*

For a curve to have a singular point, partial derivatives of underlying polynomial has to be zero at a non-zero point of $\mathbb{R}^3$. The nodal cubic $\mathbb{R}A_3$ is not a non-singular curve.

$\frac{\partial a_3}{\partial x_0} = \frac{\partial}{\partial x_0}(x_2x_1^2 - x_0^3 - x_2x_0^2) = -3x_0^2 - 2x_2x_0$

$\frac{\partial a_3}{\partial x_1} = \frac{\partial}{\partial x_1}(x_2x_1^2 - x_0^3 - x_2x_0^2) = 2x_2x_1$

$\frac{\partial a_3}{\partial x_2} = \frac{\partial}{\partial x_2}(x_2x_1^2 - x_0^3 - x_2x_0^2) = x_1^2 - x_0^2$

The partial derivatives vanish at (0,0,1). Hence (0 : 0: 1) is a singular point of A_3. But the curves $\mathbb{R}A_1$ and $\mathbb{R}A_2$ are non-singular.

$\frac{\partial a_1}{\partial x_0} = \frac{\partial}{\partial x_0}(x_0 + x_1 + x_2) = 1$, $\frac{\partial a_1}{\partial x_1} = \frac{\partial}{\partial x_1}(x_0 + x_1 + x_2) = 1$,

$\frac{\partial a_1}{\partial x_2} = \frac{\partial}{\partial x_2}(x_0 + x_1 + x_2) = 1$

$\frac{\partial a_2}{\partial x_0} = \frac{\partial}{\partial x_2}(x_0^2 + x_1^2 - x_2^2) = 2x_0$, $\frac{\partial a_2}{\partial x_1} = \frac{\partial}{\partial x_1}(x_0^2 + x_1^2 - x_2^2) = 2x_1$

$\frac{\partial a_2}{\partial x_2} = \frac{\partial}{\partial x_2}(x_0^2 + x_1^2 - x_2^2) = 2x_2$

As we can see, partial derivatives is nowhere zero for a_1 and only vanish at (0,0,0) for a_2.

Two curves are said to be homeomorphic if their respective sets of real points are homeomorphic as subspaces of $\mathbb{R}P^2$. Two curves C_1 and C_2 are said to be isotopic or equivalently $(\mathbb{R}P^2, \mathbb{R}C_1)$ and $(\mathbb{R}P^2, \mathbb{R}C_2)$ are said to be homeomorphic as pairs if there exists a homeomorphism of $\mathbb{R}P^2$ that maps $\mathbb{R}C_1$ homeomorphically to $\mathbb{R}C_2$. When we say two curves A_1 and A_2 are situated in $\mathbb{R}P^2$ in topologically distinct ways, we mean that the pairs $(\mathbb{R}P^2, \mathbb{R}C_1)$ and $(\mathbb{R}P^2, \mathbb{R}C_2)$ are not homeomorphic or equivalently the curves are not isotopic.

2.1.1 Classification Problems

Problem 2.1.5 *Topological Classification Problem : For $n \in \mathbb{N}$, find the possible sets of real points of curves of degree n, upto homeomorphism.*

Problem 2.1.6 *Isotopy Classification Problem : For $n \in \mathbb{N}$, find possible sets of real points of curves of degree n, upto isotopy.*

There are general versions of both classification problems above, extending to singular curves also. But classification of non-singular curves is a lot more easier because non-singularity guarantees theorems like Bezout's theorem and Harnack's theorem that simplifies classification to a large extend.

2.2 Bezouts Theorem And Corollaries

Theorem 2.2.1 *The set of real points of a curve is a smooth, one-dimensional, closed submanifold of the projective plane.*

Proof Let C be a curve with associated polynomial f. We know that $\mathbb{R}P^2$ can be covered by an atlas of three charts $\{U_i, \psi_i\}$, $\psi_i : U_i \to \mathbb{R}^2$ where,

$U_0 = \{(x_0 : x_1 : x_2) : x_0 \neq 0\}$, $\psi_0((x_0 : x_1 : x_2) = (\frac{x_1}{x_0}, \frac{x_2}{x_0})$

$U_1 = \{(x_0 : x_1 : x_2) : x_1 \neq 0\}$, $\psi_0((x_0 : x_1 : x_2) = (\frac{x_0}{x_1}, \frac{x_2}{x_1})$

$U_2 = \{(x_0 : x_1 : x_2) : x_2 \neq 0\}$, $\psi_0((x_0 : x_1 : x_2) = (\frac{x_0}{x_2}, \frac{x_1}{x_2})$

We can consider the functions $f_i : \mathbb{U}_i \to \mathbb{R}$ defined as,

$f_0(x_0 : x_1 : x_2) = f(1.\frac{x_1}{x_0}, \frac{x_2}{x_0})$

$f_1(x_0 : x_1 : x_2) = f(\frac{x_0}{x_1}, 1, \frac{x_2}{x_1})$

$f_2(x_0 : x_1 : x_2) = f(\frac{x_0}{x_2}, \frac{x_1}{x_2}, 1)$

The zero set of f_i is precisely the intersection of the zero set of f, i.e, $\mathbb{R}C$ with U_i. The union of zero sets of f_i gives $\mathbb{R}C$. Non-singularity of C guarantees that f has no critical points. By regular level set theorem, we know that $f_i^{-1}(0)$ is a submanifold of U_i of co-dimension 1 , i.e, of dimension 1. It follows that for any point $p \in \mathbb{R}C$, there is an open set U containing p and coordinate chart ψ of $\mathbb{R}P^2$ such that $\psi(U \cap \mathbb{R}C) = \psi_i(U) \cap \mathbb{R}$. This way we can show that $\mathbb{R}C$ is a one dimensional closed submanifold of $\mathbb{R}P^2$.

https://www.overleaf.com/project/5ee71c342d7ae30001ed4541

Theorem 2.2.2 *A component of the set of real points of a curve is homeomorphic to a circle.The number of components of the set of real points of a curve determines it's topology.*

Proof From the classification of 1-dimensional manifolds we know that a connected closed 1-dimensional manifold is homeomorphic to a circle. By the previous theorem set of real points of a curve is a closed 1-dimensional manifold. See [LJ03] for details on Manifolds.

Let us look at two different types of components of a curve. In the quotient map from $\mathbb{S}^2$ to $\mathbb{R}P^2$, a circle in $\mathbb{R}P^2$ is the image of either a circle, or a union of two antipodal circles on the sphere. The former is called a one-sided component and the latter is called a two-sided component. Although both are homeomorphic, the pairs $(\mathbb{R}P^2, \mathbb{R}A)$ of both are not homeomorphic. The complement of a 'two-sided component' has two components; a 'disc' which is called an *inside* component and a 'mobius strip' which is called an *outside* component. A two-sided component of a curve is also called an *oval* of the curve. An oval that is contained in the inside component of another is said to be *inside* the second.

Definition 2.2.3 *A set of ovals of a curve is called a 'nest', if in any pair of ovals in the set, one of the ovals is inside the other one.*

Theorem 2.2.4 *A curve has at most one one-sided component.*

Proof We prove that two one-sided components of a curve intersects each other. The first homology group of the projective plane with coefficients in $\mathbb{Z}_2$, $H_1(\mathbb{R}P_2, \mathbb{Z}_2)$, is isomorphic to $\mathbb{Z}_2$. A projective line realises the non-zero element of $H_1(\mathbb{R}P_2, \mathbb{Z}_2)$. If there are two one-sided components, they have to intersect because in $H_1(\mathbb{R}P_2, \mathbb{Z}_2)$ intersection product of non-zero element with itself is non-zero. But two distinct components of a curve cannot intersect. Therefore atmost one one-sided component is possible for a curve.

A curve with a one-sided component is called a one-sided curve. A curve that does not have a one-sided component is called a two-sided curve. We are using some properties of intersection product in $H_1(\mathbb{R}P_2, \mathbb{Z}_2)$ in proofs like the one above. See [Hut11] for details on intersection product. See [Bre13] for details on homology groups.

Definition 2.2.5 *Two submanifolds of a given finite-dimensional smooth manifold are said to intersect transversally if at every point of intersection, their separate tangent spaces at that point together generate the tangent space of the ambient manifold at that point.*

Theorem 2.2.6 (Bezout's Theorem) *If the intersection of sets of real points of two non-singular curves is finite, then the intersection contains atmost n_1n_2 points, where n_1 and n_2 are the degrees of the curves. For transversal curves number of points of intersection is congruent to n_1n_2 modulo 2.*

Using Bezout's theorem we can prove a lot of corollaries on the relation between degree of a curve and number of components. Harnack's Inequality is the most important corollary. A curve C can be identified with the preimage of $\mathbb{R}C$ under the embedding i : $\mathbb{R}^2 \to \mathbb{R}P^2$ defined by i(x,y) = (x:y:1). For the purpose of geometrical proofs, curve is considered as an affine plane curve in $\mathbb{R}^2$ using the above map. We pictorially depict a curve in plane as it's preimage under the above map.

Corollary 2.2.7 (Harnack's Inequality) *A degree n curve has atmost $\frac{1}{2}(n-1)(n-2)+1$ components.*

Lemma 2.2.8 *For any given set of $\frac{1}{2}t(t+3)$ points in $\mathbb{R}^3$, there exists a degree t homogeneous non-zero polynomial in $\mathbb{R}[x,y,z]$ passing through all of those points.*

Proof A degree n homogeneous polynomial in three variables is determined by $\frac{1}{2}n(n+3)+1$ coefficients of monomials. One can show this with the following combinatorial way. The monomials $x^iy^jz^k$ of a degree n homogeneous polynomial are such that i+j+k = n. So fixing i and j will automatically fix k.
i can range from 0 to n.
if i is 0, j can range from 0 to n.
if i is $0 \leq r \leq n$, j can range from 0 to (n-r).
if i is n, j can range from 0 to 0.
So there is a total of 0+1+2+....+ (n+1) possible arrangements for (i,j,k). 1+2+.... + (n+1) = $\frac{1}{2}(n+1)(n+2)$ = $\frac{1}{2}(n)(n+3)+1$.
The set A_n of all n degree homogeneous polynomials in $\mathbb{R}[x,y,z]$ forms a vector space with the usual addition and scalar multiplication in polynomials. A basis for this vector space is the set of monomials $\{x^iy^jz^k : (i+j+k) = n\}$. The dimension of the space is $\frac{1}{2}n(n+3)+1$. If we take a non-zero point $p \in \mathbb{R}^3$ and consider the subset consisting of all polynomials in A_n passing through p, we get a subspace of A_n. The co-dimension of this subspace is one. Choosing a non-zero point p gives a linear equation of our coefficients

which reduces dimension by one. If we intersect two such subspaces the dimension of the resulting subspace is reduced atmost by 1. Given $\frac{1}{2}n(n+3)+1$ points in $\mathbb{R}^3$, Consider the subspaces of polynomials passing through each point. We have $\frac{1}{2}n(n+3)$ subspaces. If we take their intersection the dimension decreases atmost by $\frac{1}{2}n(n+3)-1$. Therefore the intersection of all the subspaces has a dimension not less than 1. Any non-zero polynomial in this subspace passes through all of the given $\frac{1}{2}n(n+3)+1$ points.

Before proceeding to a general proof of Harnack's inequality, let us see why this is true for smaller degree curves. According to Harnack's inequality curves of degree 1 and 2 can have at most one component, a degree 3 curve can have atmost 2 components, and a degree 4 curve can have atmost 3 components. Suppose a degree 1 curve has more than one component. We already know the one essential component, one-sided component. The remaining components have to be ovals. Now choose a point inside one oval and a point on the line (one can easily visualize this for the affine part in the embedded plane). Consider the projective line passing through these two points. This line intersects our degree 1 curve at minimum 3 points. This is because the intersection of line and oval must be even and we have chosen the line such that it intersects te oval at least at one point. But the product of their degrees is one . Since both are non-singular we have a violation of Bezout's theorem. Let us look at a degree 2 curve. Suppose a degree 2 curve has more than one components. All components have to be ovals. Take two points inside any two ovals and consider the line passing through them. We have minimum 4 intersections of the line and the curve while product of their degrees is 2. A violation of bezout's theorem. The same proof works for degree 3. For a degree 4 curve we need a conic instead of a line. Suppose a degree 4 curve has more than 4 ovals. Take 5 points inside 5 distinct ovals. We can construct a degree 2 curve passing trough these points. This curve intersects the degree 4 curve at least at 10 points, but product of their degrees is 8.

Lemma 2.2.9 *For any given set of $\frac{1}{2}t(t+3)$ points in $\mathbb{R}^3$, there exist a degree t homogeneous polynomial in three variables passing through all of those points.*

Proof (of Harnack inequality)

Let $n>2$ be an integer. Assume that a degree n curve C has more than $r=\frac{(n-1)(n-2)}{2}+1$ components. At least r components has to be ovals, as there can only be atmost one one-sided component. We use the fact that one can construct a degree t curve passing through any $\frac{t(t+3)}{2}$ points. Consider r points one inside each oval on r ovals and $(n-3)$ points on one of the remaining components. We have a total of $\frac{(n-2)(n-2+3)}{2}$ points, So there is a degree $(n-2)$ curve C_2 passing through these points. We have chosen the curve such that it intersects each oval at least at one point.But by the intersection product, we know that number of intersections of an oval and any transversal curve must be zero modulo 2. This curve intersects curve C at least at $n(n-2)+1$ points because it intersects each oval twice. This is a contradiction to Bezout's theorem.

Corollary 2.2.10 *A degree n curve, has a one-sided component if and only if n is odd.*

Proof We again make use of intersection product in $H_1(\mathbb{R}P_2, \mathbb{Z}_2)$. A projective line and oval realise the non-zero and zero elements of $H_1(\mathbb{R}P_2, \mathbb{Z}_2)$ respectively. The number of intersections of a projective line and an oval is congruent to mod 2 and the intersection is transversal. A two-sided curve only has two-sided components, i.e ovals. The intersection number of a two-sided curve with the projective line must be even. Let the degree of the curve be n. Projective line has degree 1. But since the intersection is transversal, we know by Bezout's theorem, that the intersecton number is congruent to n modulo 2, where n is the product of their degrees. Hence n has to be an even number.
Conversely if the degree n of a curve is even. Then the intersection number with projective line has to be even. But this means that the curve has no one-sided component.

Corollary 2.2.11 *For a degree n curve, union of two nests contains atmost $n/2$ ovals.*

Proof Suppose there exist curve C of degree n with two nests N_1 and N_2 such that their union has more than $n/2$ ovals. Let O_1 be the most interior oval (.i.e the one that is inside any other oval in the nest) in N_1 and O_2 that in N_2. Let P_1 and P_2 be points lying in the inside components of O_1 and O_2 respectively.Then a line passing through the two points will intersect the curve C at least at n+1 points, which contradicts Bezout's theorem.

Corollary 2.2.12 *For a degree n curve, union of five nests contains atmost n ovals, if there is no oval enveloping all the other oval.*

Proof Proof is similar to the previous one. Instead of 2 interior points take 5 interior points in each nest. Then a degree 2 curve C can be constructed that passes through these 5 points.This curve intersects each oval twice. Assume that union of nests has more than n ovals. Then C intersects the curve at least at $2(n+1)$ points. This contradicts bezout's theorem, since the product of degrees of the curves is $2n$.

From the discussions of this section, we can conclude that topological type of a curve is determined by the number of components and isotopy type is determined by number of components and the relative arrangement of ovals. The general notation used for denoting different arrangement of components is as follows. A one-sided component is denoted as $[P]$. An empty curve is denoted as [0]. A set of n ovals such that no oval lies in the inside component of any oval is depicted as $[n]$. For some arrangement of ovals X, $[1[X]]$ denotes the arrangement X enveloped inside another oval. For X and Y that are two arrangements of ovals, $[X+Y]$ denotes their disjoint union.

2.3 Isotopy Classification Of Curves Of Degree 5 and Less

To find the possible isotopy types of curves we consider all possible arrangement of ovals which is finite due to Harnack's theorem. From them the ones that contradict Bezout's theorem are ruled out. Then we construct the remaining curves to prove their existence.

2.3.1 Restrictions on possible arrangements

Degree 1

By Harnack's inequality there is at most $\frac{1}{2}(1-1)(1-2)+1=1$ component. Since the degree is odd there exists a one-sided component. Therefore a degree 1 curve has one and only one component which is a projective line.

Degree 2

The upper bound for number of components equals $\frac{1}{2}(2-1)(2-2)+1=1$. As the degree is even components are all ovals and least possible number of components is zero. Therefore zero ovals, i.e an empty curve and one oval are the two possibilities for a degree 2 curve.

These are listed in table 2.3 and are constructed in the next section.

In fact we can show this without using Harnack's inequality.

Theorem 2.3.1 *The set of real points of any degree 2 curve is either empty or an oval, upto isotopy.*

Proof let $f \in \mathbb{R}[X_0, X_1, X_2]$ be a degree 2 homogeneous polynomial of a degree 2 curve.
$f = a_{00}X_0^2 + a_{01}X_0X_1 + a_{02}X_0X_2 + a_{10}X_1X_0 + a_{11}X_1^2 + a_{02}X_1X_2 + a_{01}X_2X_0 + a_{02}X_2X_1 + a_{00}X_2^2.$

f can be described in the matrix form,

$$\begin{bmatrix} X_0, X_1, X_2 \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

The coefficient matrix can be made symmetric in the following way,

$$\begin{bmatrix} X_0, X_1, X_2 \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix}$$

where $b_{ij} = b_{ji} = \frac{1}{2}(a_{ij} + a_{ji})$.

Consider the matrix equation,

$$\begin{bmatrix} X_0, X_1, X_2 \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \end{bmatrix} = 0 \tag{2.1}$$

Let us call the coefficient matrix B. The solution set $[x_0, x_1, x_2]$ considered in $\mathbb{R}P^2$ is the curve associated with f. An important observation that we use here is that, if we linear transform the solution set with an invertible matrix, we get a curve isotopic to our original curve. This follows from the fact that $SO(n)$ is connected.

We can diagonalize a symmetric matrix with an orthogonal matrix. So $B = P^T * M * P$ where M is a diagonal matrix. We can rewrite the matrix equation as $V^T(P^T * M * P)V = 0$, where V is the column vector $[X_0, X_1, X_2]^T$. This can be further modified as $(P*V)^T*M*(P*V)$= 0 . Let us denote $P*V$ by W.

$$W^T * \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} * W$$

The corresponding polynomial is $d_1X_0^2 + d_2X_1^2 + d_3X_2^2 = 0$.

If we transform W with the matrix J given by,

$$J = \begin{bmatrix} \sqrt{\frac{1}{|d_1|}} & 0 & 0 \\ 0 & \sqrt{\frac{1}{|d_2|}} & 0 \\ 0 & 0 & \sqrt{\frac{1}{|d_1|}} \end{bmatrix}$$

Then the new transformed curve has the following posssible equations

$\pm X_0^2 \pm X_1^2 \pm X_2^2 = 0$.

We can further transform it, if needed, to make the first coefficient +1. The posibilities are

$X_0^2 + X_1^2 + X_2^2 = 0$ which gives an empty curve.

$X_0^2 + X_1^2 - X_2^2 = 0$ which gives a circle in the affine plane.

$X_0^2 - X_1^2 + X_2^2 = 0$ which gives a hyperbola in the affine plane.

$X_0^2 - X_1^2 - X_2^2 = 0$ which gives a hyperbola in the affine plane.

But all of the three non-empty among these are ovals in the projective plane. Therefore any degree 2 curve is an oval, upto isotopy.

Degree 3

By Harnack's inequality, there are at most $\frac{1}{2}(3-1)(3-2)+1 = 2$ components. One necessarily has to be a projective line. Therefore two possibilities are one projective line component, $[P]$ only or two components were one is a projective line and other is an oval, $[P+1]$. Both are constructed in the next section.

Degree 4

Maximum Possible number of components is $\frac{1}{2}(4-1)(4-2)+1 = 4$, by Harnack's inequality. All components must be ovals. Let us look at all arrangements, that are combinatorialy possible. See table 2.1.

Number of components	Possible arrangements
0	[0]
1	[1]
2	[2] , [1[1]]
3	[3], [1[2]], [1[1[1]]], $[1+1[1]]$
4	[4],[1[3]], [1[1[2]]], $[1[1+1[1]]]$,[1[1[1[1]]]] $[1+1[2]]$, $[2+[1[1]]]$, $[1+[1[1[1]]]$, $[1[1]+1[1]]$

Table 2.1: All combinatorial possibilities for degree 4

But out of these [1[2]], [1[1[1]]], $[1+1[1]]$, [1[3]], [1[1[2]]], $[1[1+1[1]]]$,[1[1[1[1]]]], $[1+1[2]]$, $[2+[1[1]]]$, $[1+[1[1[1]]]$, $[1[1]+1[1]]$ cannot be realised as sets of real points of a non-singular curve. To see this, observe that all of them have, in common, one oval that is contained inside another. Suppose one of them can be realised with a non-singular curve. Choosing a point inside the inner oval and another point inside an oval not in this pair, we can construct a line that intersects the curve at more than 5 points. This contradicts Bezout's theorem, as the product of their degrees is 4. An example is illustrated in figure2.1. The line intersects the arrangement of ovals at 6 points.

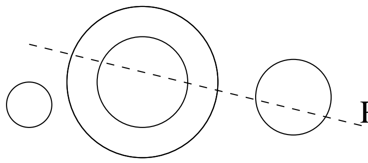

Figure 2.1: Illustration of $[2+1[1]]$

All possible arrangement of ovals except the ones ruled out so far, are listed in table2.2. All of these can be realised as sets of real points of curves of degree 4.

Degree 5

Maximum of $\frac{1}{2}(5-1)(5-2)+1=7$ components possible. One component is a projective line. The only possible arrangement that has a nest of 2 ovals,which can be realised as set of real point of a degree 5 curve is the case with 3 components. Out of the possible arrangements of 4 or more components those those arrangements having an oval contained inside another is ruled out. To show this, suppose one of them can be realised with a non-singular curve. We proceed in the same way as in the case of degree 4. Choosing a point inside the inner oval and another point inside an oval not in this pair, we can construct a line that intersects the curve at more than 5 points. This contradicts Bezout's theorem, as the product of their degrees is 5.

Number of components	Possible arrangements
0	0
1	$[P]$
2	$[P+1]$
3	$[P+2]$, $[P+1[1]]$
4	$[P+3]$, $[P+1[2]]$, $[P+1[1[1]]]$, $P+[1+1[1]]$
5	$[P+4]$,$[P+1[3]]$, $[P+1[1[2]]]$, $[P+1[1+1[1]]]$ $[P+1[1[1[1]]]]$,$[P+1+1[2]]$,$[P+2[1]]$ $[P+2+[1[1]]]$, $[P+1+[1[1[1]]]$
6	[P+5] and rest
7	[P+6] and rest

Table 2.2: All combinatorial possibilities for degree 5

All of the remaining possible arrangements are listed in table2.3. In fact all of these possible arrangements listed in table2.3 can be realised as sets of real points of the corresponding degree curves. To prove this, we have to construct curves of those degrees with the respective arrangement of components.

Degree	Isotopy types
1	$[P]$
2	[0],[1]
3	$[P]$, $[P+1]$
4	[0], [1], [2], [1[1]],[3],[4]
5	$[P]$, $[P+1]$, $[P+2]$, $[P+1[1]]$, $[P+3]$, $[P+4]$, $[P+5]$, $[P+6]$

Table 2.3: Isotopy Types of curves of degree 5 and less

2.3.2 Construction of Isotopy Types

Degree 1

Consider the polynomial $a(x_0,x_1,x_2) = x_0 + x_1 + x_2$. A curve with associated polynomial a gives the isotopy type $[P]$. The polynomial a is the homogenisation of the line $x_0 + x_1 = -1$ in $\mathbb{R}_2$.

Degree 2

Consider the polynomial $a(x_0,x_1,x_2) = {x_0}^2 + x_1^2 + x_3^2$. This gives the empty curve [0].
Consider the polynomial $a(x_0,x_1,x_2) = {x_0}^2 + x_1^2 - x_3^2$. It is a degree 2 homogenisation of the affine plane curve ${x_0}^2 + x_1^2 = 1$. It gives the isotopy type [1].
For degree 3 and above we use the marking method also known as small perturbation method to construct the curves.

The Marking Method

Theorem 2.3.2 (The Marking Method) *Let C_n and C_m be curves of degrees n and m with associated polynomials c_n and c_m, respectively. Suppose that RC_n and RC_m intersect in mn distinct points, l of which are real: $z_j, 1 < j < l$. Let B_{n+m} be a curve(not necessarily non-singular) of degree n+m, with associated polynomial b_{n+m} such that RB_{n+m} intersects $RC_n \cup RC_m$ in $(n+m)^2$ distinct points, r of which are real: A_i, $1 < i < r$. Also there is no point of intersection of all three of RC_n, RC_m and RB_{n+m}. Then for sufficiently small $t > 0$ the curve C_{n+m} given by the polynomial $c_nc_m + tb_{n+m}$ is non-singular, lies (in $\mathbb{R}P_2$) in a small neighbourhood of $RC_n \cup RC_m$ and intersects $RC_n \cup RC_m$ only at A_i, $1 < i < r$, without tangency. The components of the curve can be constructed by the following method:*
1. We construct RC_n and RC_m and , whose intersection points z_j are nodes of the curve given by c_nc_m.
2. We draw RB_{n+m} in neighbourhoods of the intersection points A_i.
3. Near each arc of $RC_n \cup RC_m$ bounded by two A_is, we place the signs of the polynomials c_nc_m and b_{n+m} on each side of the arc.
4. Take any point a of $RC_n \cup RC_m$ except A_is and z_j, and we take a point a^* in a sufficiently small regular neighbourhood of a of $RC_n \cup RC_m \cup RB_{n+m}$ in a domain where $c_nc_m(a)$ and $b_{n+m}(a)$ have different signs.
5. A component of C_{n+m} passes through a^*, is slightly shifted from the nearby arc of $RC_n \cup RC_m$, and intersects (without tangency) $RC_n \cup RC_m \cup RB_{n+m}$ only at the A_i
6. All components of C_{m+k} are constructed in this manner.

The curve B_{n+m} is also called as auxiliary curve used for the perturbation. We need an auxiliary curve of the degree same as the degree of c_nc_m for the resulting polynomial $c_nc_m + tb_{n+m}$ to be homogeneous. Marking method guarantees that the curve C_{n+m} given by

$c_nc_m + tb_{n+m}$ is non-singular. Let us see why the resulting curve passes through A_is and why it lies in regions where $c_nc_m(a)$ and $b_{m+k}(a)$ have different signs. A_is are intersecting points of $RC_n \cup RC_m$ and RB_{n+m}. Let $A_i = (a_{i_1} : a_{i_2} : a_{i_3})$. Since A_i belongs to $RC_n \cup RC_m$ and RB_{n+m} polynomials $c_nc_m(a_{i_1}, a_{i_2}, a_{i_3})$ and $b_{m+k}(a_{i_1}, a_{i_2}, a_{i_3})$ are zero. Hence $(c_nc_m + tb_{n+m})(a_{i_1} : a_{i_2} : a_{i_3})$ is zero. Therefore $A_i \in \mathrm{RC}_{n+m}$. Suppose RC_{n+m} intersects one of $RC_n \cup RC_m$ and RB_{n+m} at a point. Then c_nc_m or b_{m+k} is zero at the corresponding point in $\mathbb{R}^3$. But one of them being zero forces the other to be zero, as $c_nc_m + tb_{n+m}$ is already zero at the point. That means the point of intersection is one of the A_i.

The new curve lies in regions where $c_nc_m(a)$ and $b_{m+k}(a)$ have different signs, simply because t is greater than zero. Hence at any point where c_nc_m+ tb_{m+k} is zero, except A_i, c_nc_m and b_{n+m} have different signs. This theorem on marking method is taken from [Gud74]. See [Gud74] for more details on marking method.

Degree 3

Consider the following curves: C and L given by polynomials $c(x_0, x_1, x_2) = {x_0}^2 + x_1^2 - x_3^2$ and $l(x_0, x_1, x_2) = x_1$.

L_1 given by $l_1(x_0, x_1, x_2) = x_0 + x_3$, L_2 given by $l_2(x_0, x_1, x_2) = x_0 - 2x_3$, L_3 given by $l_3(x_0, x_1, x_2) = x_0 - 2x_3$, L_4 given by $l_4(x_0, x_1, x_2) = x_0$.

Using marking method we can prove that the curve given by $cl + tl_1l_2l_4$, for small enough t, is of isotopy type $[P]$.

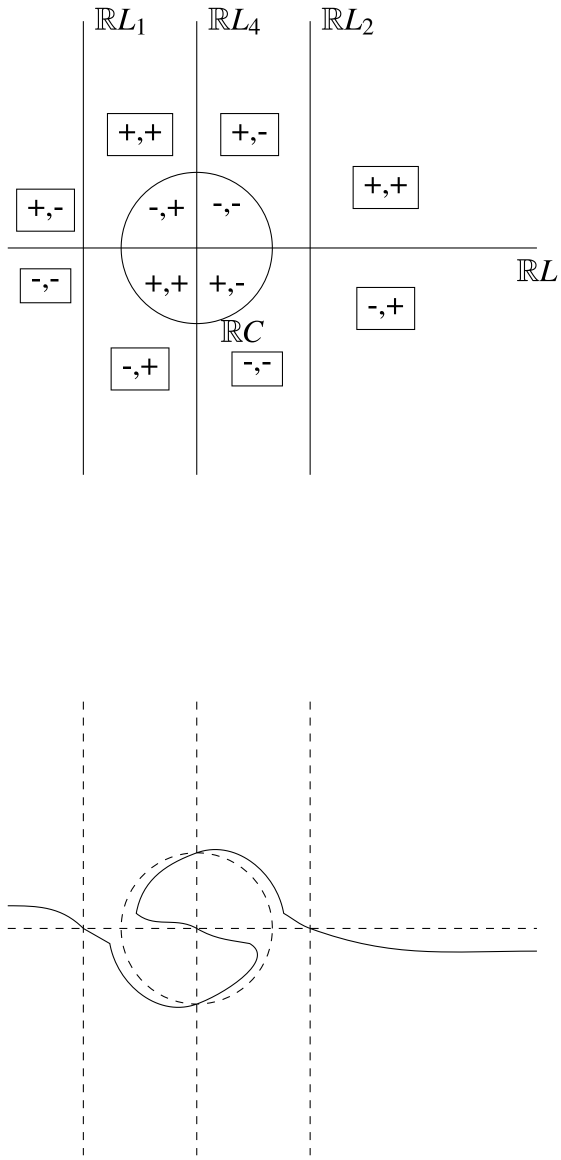

Figure 2.2: Isotopy type $[P]$ by perturbing union of conic and line.

Using marking method we can prove that the curve given by $cl + tl_1l_2l_3$, for small enough t, is of isotopy type $[P+1]$.

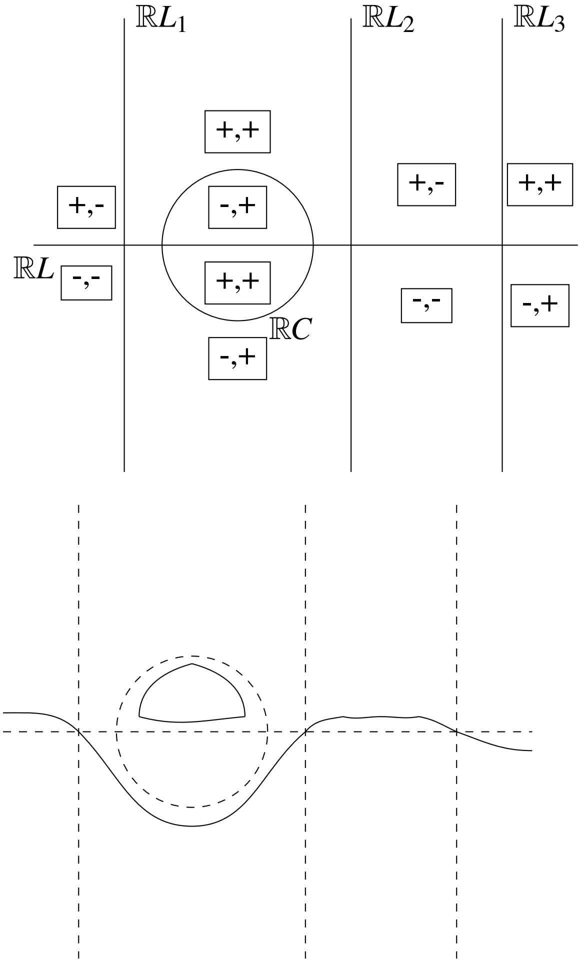

Figure 2.3: $[P+1]$ by perturbing union of conic and line

Degree 4

An empty curve [0] is given by $a = x_0^4 + x_1^4 + x_2^4$. All other isotopy types can be constructed by perturbing union of two conics that intersects in 4 points. The auxiliary curve used is union of 4 lines.

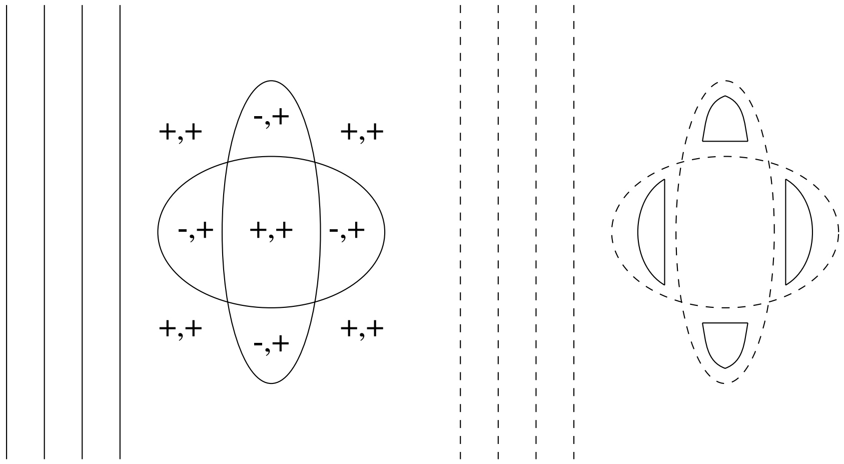

Figure 2.4: Isotopy type [4]

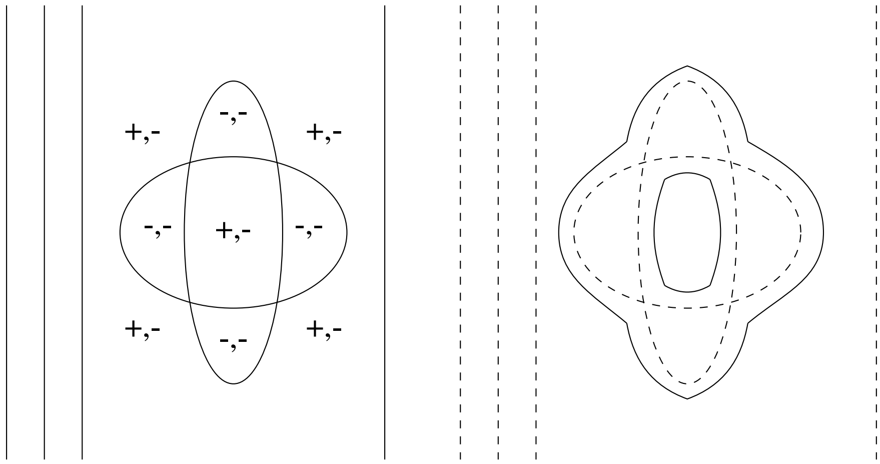

Figure 2.5: Isotopy type [1[1]]

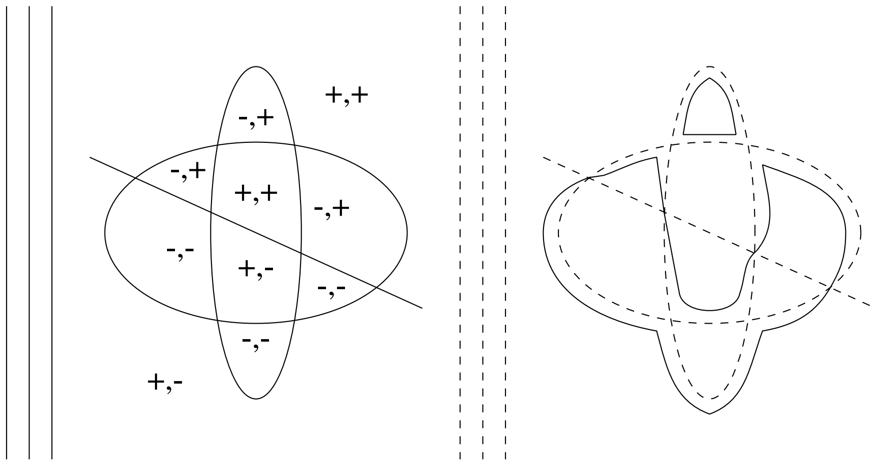

Figure 2.6: Isotopy type [2]

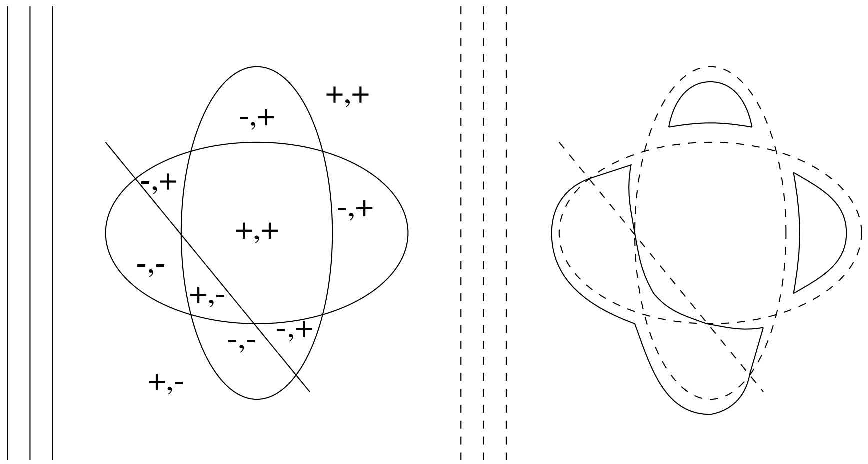

Figure 2.7: Isotopy type [3]

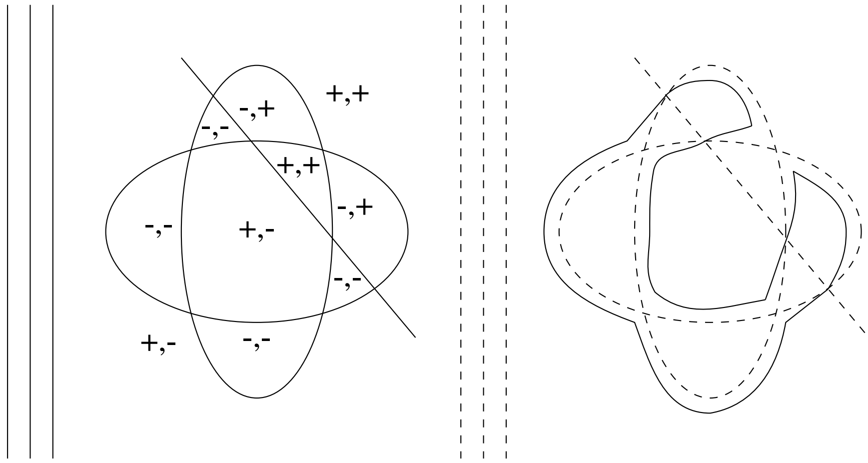

Figure 2.8: Isotopy type [1]

Degree 5

Instead of constructing all isotopic types of degree 5 curve, we can use Harnack's theorem proven in the next section, to prove their existence. Harnack's theorem guarantees the existence of a degree 5 non-singular curves with number of components ranging from 1 to 5. Since there is only one possible isotopy type for number of components 1,2,4,5,6 and 7, their existence is guaranteed by the theorem. But for a degree 5 curve with 3 components, two isotopy types are possible. We will construct both using small perturbation method on union of two curves and a line as in figure 2.9. The auxiliary curve used is a union of 5 lines.

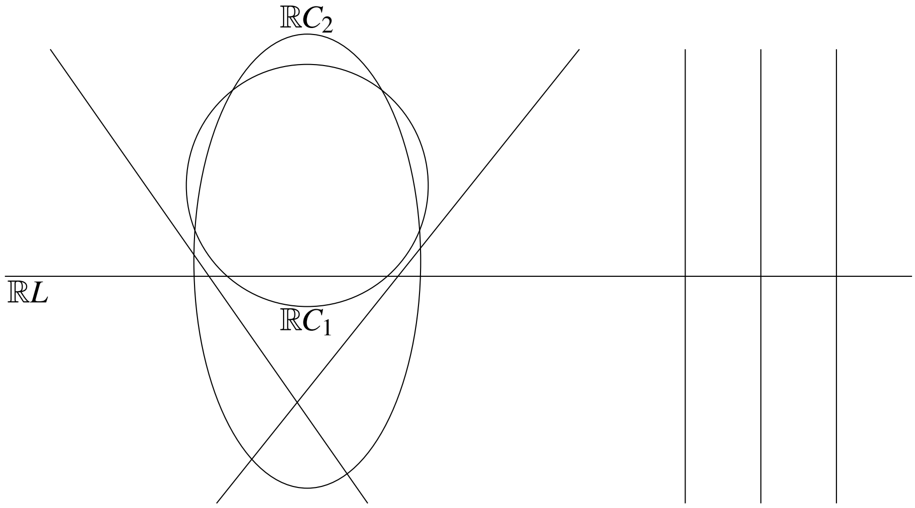

Figure 2.9: Union of two conics and a line with auxiliary curve

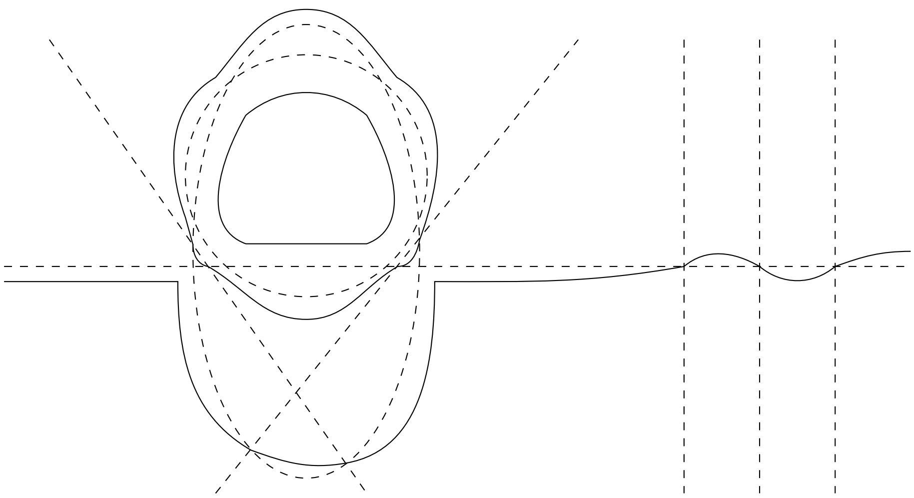

Figure 2.10: Isotopy type [P+1[1]]

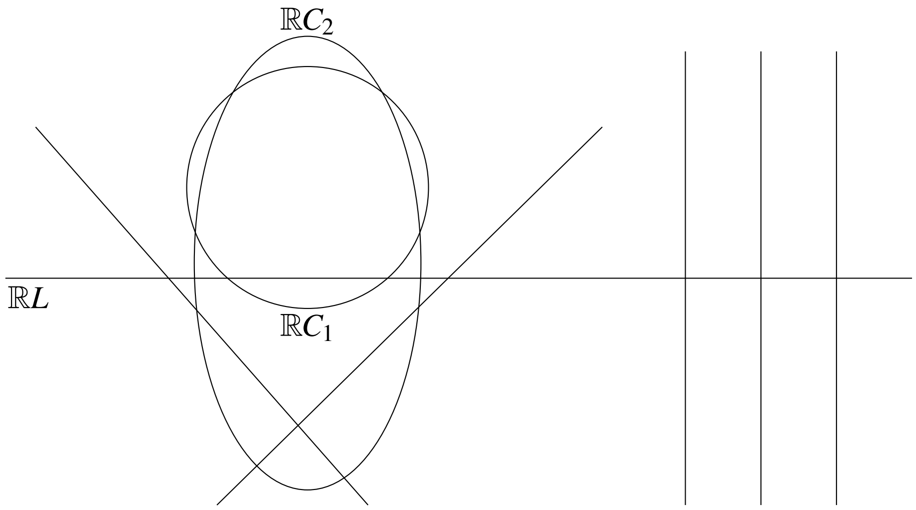

Figure 2.11: Union of two conics and a line with auxiliary curve

The other isotopy type can be constructed with a different choice of lines.

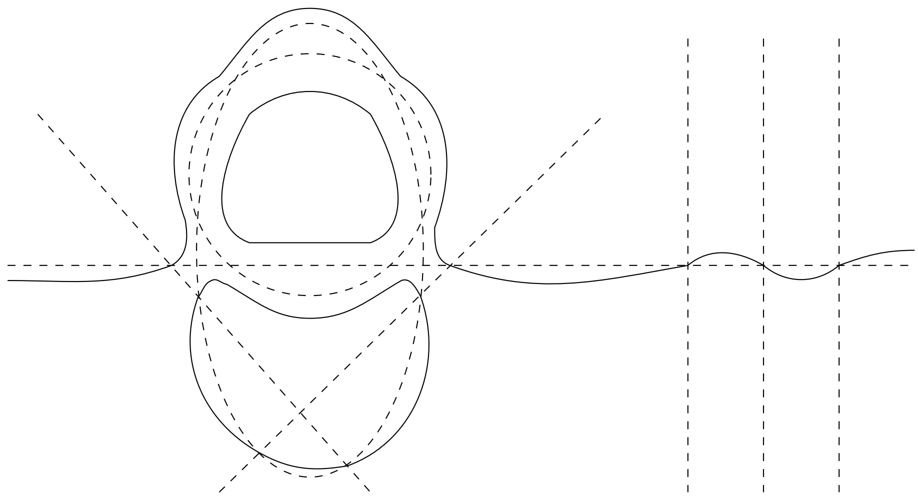

Figure 2.12: Isotopy type [P+2]

2.4 Harnack Theorem And Topological Classification Of Curves

From this section onwards, a curve is not assumed to be non-singular. If it is not explicitly stated as non-singular, a curve can be either singular or non-singular.

Theorem 2.4.1 *Harnack Theorem : Let n* $\in \mathbb{N}$ *. If n is odd, then for any 1<= r <=* $\frac{(n-1)(n-2)}{2} + 1$*, there exists a degree n non-singular curve with exactly r components. If n is even, then for any 0* $\leq r \leq n$*, there exists a degree n non-singular curve with exactly r components.*

The proof of Harnack theorem is done in two parts. First part is showing that for any n $\in \mathbb{N}$ there exist a non-singular curve with exactly $\frac{(n-1)(n-2)}{2} + 1$ components. This is done with the help of small perturbation method.

Lemma 2.4.2 *For any n* $\in \mathbb{N}$ *there exist a non-singular curve with exactly* $\frac{1}{2}(n-1)(n-2)+1$ *components.*

A degree n non-singular curve with $\frac{1}{2}(n-1)(n-2)+1$ components is also called an M-curve of degree n.

Proof Proof is by induction on the degree of curves. Since M-curves upto degree 5 were constructed in the previous section, we start with degree 5 M-curve. We inductively construct M-curves of higher degrees. Let us denote a degree n M-curve by M_n. M_5 is constructed using small variation method on the union of 2 conics C_1, C_2 and a line L as in figure 2.8, with union of five lines as auxiliary curve . The method of construction of M_{n+1}, for $n \geq 5$, is as follows (figures 2.9 to 2.11):

1. Take the union of M_n and L.
2. (Let us denote the auxiliary curve used for constructing M_{n+1} from M_n, as A_{n+1}). Take A_{n+1} as a union of $n+1$ lines that intersect L at $n+1$ distinct points such that,
If n is odd, $\mathbb{R}A_{n+1}$ intersects $\mathbb{R}L$ at any one component of $\mathbb{R}L \setminus \mathbb{R}A_n$.
If n is even, $\mathbb{R}A_{n+1}$ intersects $\mathbb{R}L$ only at a particular component of $\mathbb{R}L \setminus \mathbb{R}A_n$, namely, the one which intersects with $\mathbb{R}L \setminus \mathbb{R}A_{n-1}$.
3. Now construct N_{n+1} by perturbing the union of M_n and L, using the auxiliary curve A_{n+1}. The curve obtained this way has exactly $\frac{1}{2}n(n-1)+1$ components. To observe this a few

facts has to be noted. The curve M_n passes through all points of intersection of $\mathbb{R}L$ and $\mathbb{R}A_n$ and it changes direction with respect to L after passing through each of these points. Only one component of M_n passes through $\mathbb{R}L \cap \mathbb{R}A_n$. The $\frac{1}{2}(n-1)(n-2)$ components of M_n except this one, after perturbation, gives exactly the same number of components for N_{n+1}, because they undergo slight changes only. But the remaining one component after perturbation, transforms into exactly n components. This is because M_n changes direction with respect to L after passing through each of the n points in $\mathbb{R}L \cap \mathbb{R}A_n$. Therefore the total number of components of N_{n+1} equals $\frac{1}{2}(n-1)(n-2)+(n-1) = \frac{1}{2}(n-1)(n-2)+1$. Thus N_{n+1} is M_{n+1}.

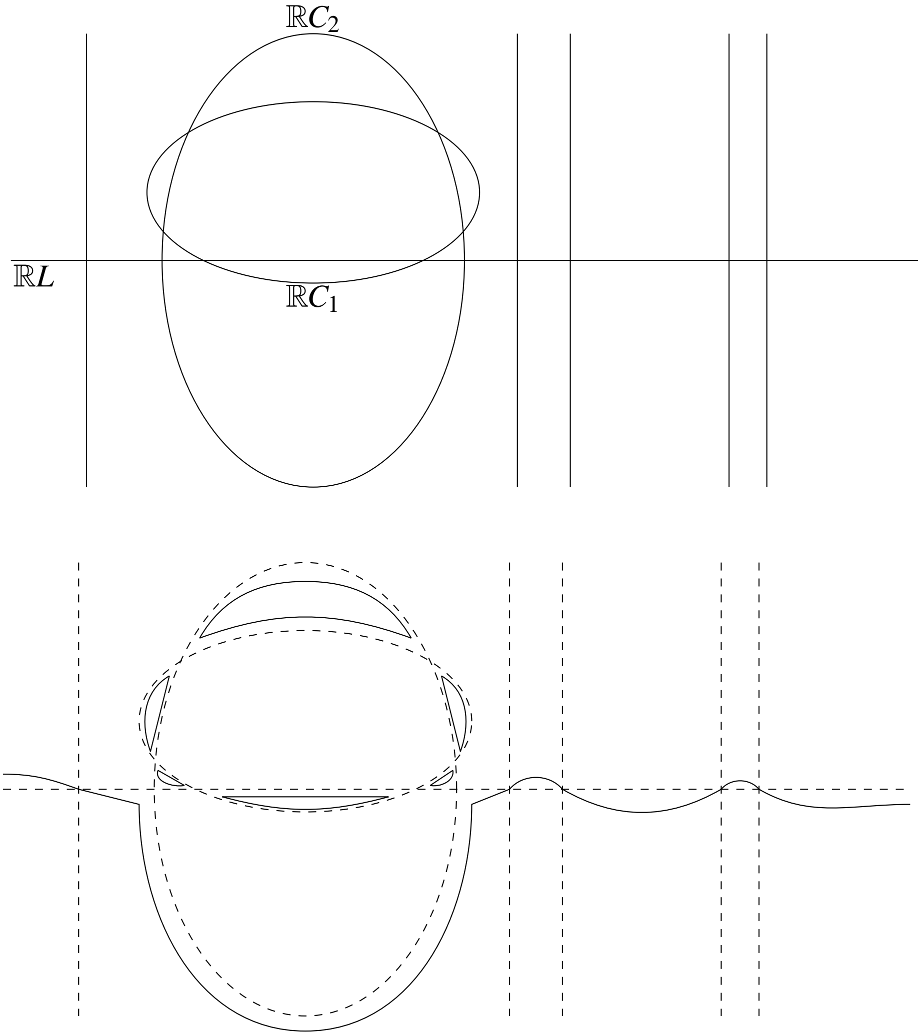

Figure 2.13: Construction of M_5

The inductive construction of M_6 is illustrated in the next page.

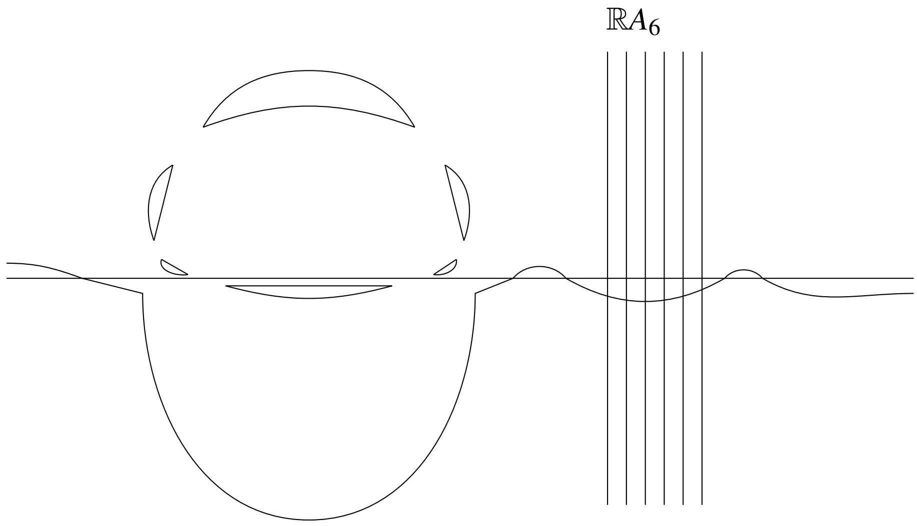

Figure 2.14: M_5 union L with $\mathbb{R}A_6$

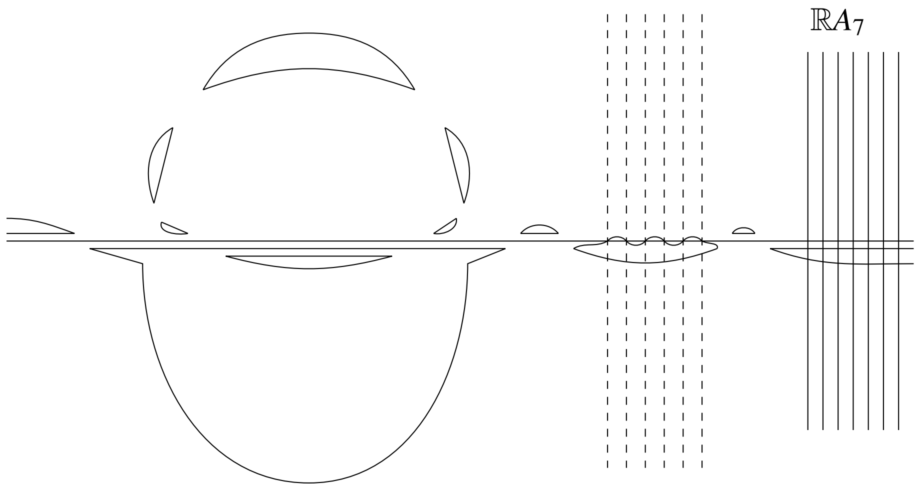

Figure 2.15: M_6 union L with $\mathbb{R}A_7$

Construction of M_7 is illustrated in the next page.

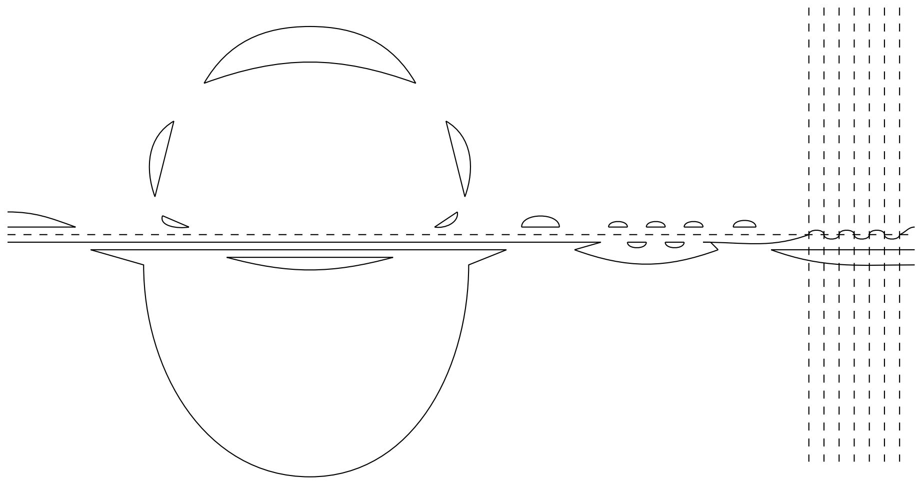

Figure 2.16: M_7

The first step of proving Harnack's theorem is done. Before proceeding to the second part let us look at some definitions.Let us denote the matrix of second partial derivatives of the associated polynomial c of a curve C at a point $(a_0 : a_1 : a_2)$ as $M[c](a_0 : a_1 : a_2)$.

Definition 2.4.3 *A singular point $(a_0 : a_1 : a_2)$ of a curve C with associated polynomial c is called a nondegenerate double point if $M[c](a_0 : a_1 : a_2)$ is of rank 2.*
A non degenerate double point $(a_0 : a_1 : a_2)$ is called a crossing, if $M[c](a_0 : a_1 : a_2)$ has a positive and a negative eigen value.
A non degenerate double point $(a_0 : a_1 : a_2)$ is called 'solitary', if $M[c](a_0 : a_1 : a_2)$ has both positive or both negative eigen values. A solitary point of a curve C is an isolated point in the set of real points of the curve.

Now let us look at the second part of the proof. This proof is taken from [Vir]. Let C_m be the set of all m-degree curves (not necessarily non-singular). C_m can be identified as a real projective space of dimension $\frac{1}{2}m(m+3)$. For a curve C with degree m, an associated polynomial c has $\frac{1}{2}m(m+3)+1$ coefficients. Once we fix a general indexing for the coefficients depending on the monomials, we can consider the mapping of a curve to the point of $\mathbb{R}P_{\frac{1}{2}m(m+3)}$ that has the coefficients,in the same order, as homogeneous coordinates. Since two associated polynomials of a curve are constant multiples of each other, the map is well-defined and a bijection. We take coefficients of $c(x_0,x_1,x_2)$ with the indexing that gives c_{ij} as the coefficient of $x_0^{m-i-j}x_1^i x_2^j$.
Consider two subsets of the space C_m,
$NC_m = \{C \in C_m :$ C is non-singular $\}$.
$SC_m = \{C \in C_m :$ C is singular $\}$.

NC_m has some important properties. Firstly NC_m is an open subset of C_m. Also every curve in NC_m has a neighbourhood of isotopic curves in NC_m, because slight modifications of coefficients of associated polynomial results in associated polynomial of smooth sections of a tubular fibration of original curve. The set of curves in SC_m having singularity at a fixed point $x \in \mathbb{R}P_2$ is a subspace of C_m of dimension $\frac{1}{2}m(m+3)-3$. Look at the subset I = $\{(x,C) \in \mathbb{R}P_2 \times C_m :$ C is a singular curve with singularity at x$\}$ of $\mathbb{R}P_2 \times C_m$. I is an algebraic subvariety of $\mathbb{R}P_2 \times C_m$. I is, in fact , a $\frac{1}{2}m(m+3)-1$ dimensional manifold.
SC_m is the image of I under a smooth map, namely, restriction of the projection $\mathbb{R}P_2 \times C_m \to \mathbb{R}P_2$ to I. Therefore SC_m is a manifold whose dimension does not exceed $\frac{1}{2}m(m+3)-1$. Consider the subset of SC_m, S_1C_m consisting of curves with only one singular point. $SC_m \setminus S_1C_m$ is a manifold whose dimension does not exceed $m(m+3-2)$. The subset of SC_m consisting of curves C such having singularity such that rank of matrix of second partial derivatives at that point is at most 1 is also a manifold whose dimension does not exceed $m(m+3-2)$. We can show this using similar arguments used above for SC_m. Let us assume that dimension of SC_m is less than or equal to $\frac{1}{2}m(m+3)-2$. A submanifold of codimension less than or equal to two, will not seperate the manifold C_m. If SC_m does not separate C_m, then there would exist only one isotopy class of degree m non-singular curves. Therefore dimension of SC_m is at least $\frac{1}{2}m(m+3)-1$ and therefore equal to $\frac{1}{2}m(m+3)-1$. From these results, the existence of an everywhere dense open subset of SC_m contained inside S_1C_m, follows. We call this subset *principal part* of SC_m, denoted by PSC_m. Principal part of SC_m can be divided into two open sets; One consisting of curves with crossing and the other consisting of curves with solitary point. We denote them as P_1SCm and P_2SC_m respectively.
A curve C, considered as a point of C_m, moving along an arc that intersects P_2SC_m transversaly has two possible modifications. It will contract one oval to a solitary point that vanishes or develops a solitary point in $\mathbb{R}C$, that gives an oval. A curve C, considered as a point of C_m, moving along an arc that intersects P_1SC_m transversaly will develop a crossing by merging arcs of the curve and then diverges. We call a line in C_m, a pencil of degree m. Consider two curves in a pencil given by polynomials c_1 and c_2. Then the associated polynomial of any curve in the pencil is a linear combination of c_1 and c_2 with non-zero real coefficients. The pencils that intersect SC_m at points of PSC_m and only transversally form an open everywhere dense subset of set of all real pencils of curves of degree m. We have already

shown that for $m \in \mathbb{N}$ there exist degree m non singular curves with maximum number of components allowed by Harnack's theorem. Also there exist degree m non singular curve with least number of components allowed by Harnack's theorem which is 0 and 1 for even and odd m respectively. The degree m non-singular curve $x_0^m + x_1^m + x_2^m = 0$ is an empty curve for even m and has only one component when m is even.
What remains to show is the existence of non-singular curves of degee m with the intermediate values of number of components. A set of isotopic non-singular curves is open in C_m. So, there exist a pencil of degree m curves that connects curves of the maximum and least number of components and intersects SC_m only transversally in only PSC_m. This pencil contains non-singular curves with all intermediate number of components. To see this, recall that moving from the curve with least number of components to that with most number of components, modifications that curves undergo change the number of components by 1.

2.4.1 Topological classification problem

Harnack's theorem, in fact, is the solution for the topological classification problem. For any n, we know the different types of non-singular curves upto homeomorphism. If n is odd, then for any 1$\leq$ r $\leq \frac{(n-1)(n-2)}{2} + 1$, there exists a degree n non-singular curve with exactly r components. If n is even, then for any $0 \leq$ r $\leq \frac{(n-1)(n-2)}{2} + 1$, there exists a degree n non-singular curve with exactly r components. Harnack's inequality shows that $\frac{(n-1)(n-2)}{2} + 1$ is the maximum number of components possible for a degree n curve. We know that the number of components determines the topological type of a curve.

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