Interacting Urn Processes with Multiple Drawings

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A dissertation submitted for the partial fulfilment of BS-MS dual degree in Science

Indian Institute of Science Education and Research Mohali

May 2020

Certificate of Examination

This is to certify that the dissertation titled **Interacting Urn Processes with Multiple Drawings** submitted by **Yogesh** (Reg. No. MS15127) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Neeraja Sahasrabudhe at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

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Acknowledgment

First and foremost, I would like to thank my thesis supervisor Dr. Neeraja Sahasrabudhe for her constant guidance and support. Without her persistent help and valuable feedback, this thesis would not have been possible.

Secondly, I would like to thank Nikhil Tanwar for helping me with computer simulations which are an important part of my work. I would also like to pay special regards to my friends and family. In particular, I am indebted to Mamta, Yateendra Sandeep and Ramandeep for their moral support all through the thesis year.
I am also thankful to IISER Mohali for providing me the opportunity to study in a research oriented environment all through my BS-MS years. Finally, I would like to acknowledge DST INSPIRE, Government of India for financial support.

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Introduction

Urn processes has been an active area of research in mathematics for a very long time. They are a special kind of random processes with reinforcement. In general, these are systems with either one single component which evolves according to some reinforcement rules or systems with multiple components that update randomly in such a way that the evolution of each component depends on all or some of the other components. In 2007, R. Pemantle published a survey titled 'Random Processes with Reinforcement' (also the title of his doctoral thesis in 1998) in Probability Surveys [1]. There, he notes that:

In 1988 I wrote a Ph.D. thesis entitled "Random Processes with Reinforcement". The first section was a survey of previous work: it was under ten pages. Twenty years later, the field has grown substantially. In some sense it is still a collection of disjoint techniques. The few difficult open problems that have been solved have not led to broad theoretical advances.

Since the publication of that survey more than ten years ago, the field of random processes with reinforcement has grown further into different directions, however, the tools to study these problems are more or less the same. In this thesis, we focus on one such direction, namely, the 'interacting urn models'. As noted above, such models consist of multiple components/urns that are reinforced randomly at every time step such that the reinforcement of each urn depends on some of the other (or all) urns in the system. In the last few years, a considerable attention has been given to the study of such systems. Several models of interacting urns have been studied (see [2], [3] and [4]).

A classical urn process consists of an urn with balls of two colours where a ball is drawn from the urn uniformly at random at each time-step and depending on the colour of ball drawn, more balls of the same or opposite colour are added to the urn. Such an operation is performed repeatedly. Typically, we are interested in understanding the asymptotic properties of the urn. That is, does the fraction of balls of a particular colour converge to a (possibly random) limit and is that limit same for all the urns as $t \to \infty$? What is the rate of convergence? What are the distributional properties of the limit if it is random? Finally, how do the fluctuations of fraction of balls of each colour around the limit behave? Early classical urn models have been extended to random reinforcement, infinite colour models etc in the recent past (for instance, see [5] and [6]).

In this thesis, we study *interacting urns with multiple drawings*. Urns with multiple drawings have been studied before in [7]. In these models, instead of drawing one ball at each time-step, a finite number of balls, say s, are drawn from the urn. The urn is then reinforced depending on the composition (in terms of colours) of balls drawn. The questions of interest are similar to those discussed above. Interacting urn processes consist of a set of urns (we restrict our discussion to finite number of urns) such that the reinforcement of each urn depends on the rest of the urns or a non-trivial subset of all the other urns (for instance, see [8]). In this case, we are interested in questions of synchronization. That is, do balls of each colour converge to the same limit across all the urns? One may also impose spatial structures or conditions on such models. Recently, some work has been focused on graph based interactions, where each vertex of the graph represents an urn and each urn interacts only with its neighbours in the graph (see [3] and [9]).

The thesis is organized as follows. Chapter 1 is a general discussion on urn models, its extensions and the known results in this area of research. In Chapter 2, we consider a two-colour model (consisting of white and black balls) with N interacting urns. At each time step t and for each individual urn, a biased coin is tossed with probability of head being p. Depending on the result of the toss, we draw s balls from that same urn in case of a heads and from the super urn (an imaginary urn formed by combining all the individual urns together) in case of a tails. Then, we reinforce the urns with two different types of reinforcement schemes. Due to this duality in the type of scheme, we call one of the models, the 'Pólya-type model' and the other, the 'Friedman-type model' . The difference is that in case of the former, for every ball drawn out while sampling for an urn, we put back C balls of the same colour back in that urn; while in case of the latter, we put back C balls of the opposite colour back in that urn. Note that this model consists of two qualitative aspects: it is an interacting urn model because of the possibility of balls being drawn from the super urn and it is also a multiple drawing model. This makes this model a straightforward generalization of the models studied in [7] (a single urn model with multiple drawings) and [8] (consisting of N interacting urns but no multiple drawings). We prove synchronization results and the Central Limit Theorems for both the models and calculate $\mathcal{L}^2$ convergence rates for the 'Pólya-type model' using ideas and methods used in [8], [10] and [11].

For both kinds of models discussed above, synchronization of colours is observed to occur across the urns (that is, the fraction of balls of a particular colour in each urn converges to the same limit), but the results are consistent with classical urn models: in case of the 'Pólya-type model', the common limit is random while in case of 'Friedman-type model', the fraction of balls of each colour converges to $1/2$ almost surely. This contrast motivates us to expand the two-colour model to a general d-colour model, which we study in Chapter 3. We present a synchronization result in order to prove that cross-reinforcement of colours (reinforcement of a colour by another colour which is a feature of the two-colour 'Friedman-type model') leads to fraction of balls of those colours converging to a common random variable almost surely even in a general d-colour setup while fraction of balls of a colour which reinforces itself converges to a different random variable.

Finally, we consider graph based models in Chapter 4. We place urns containing balls of two colours on the vertices of a fixed deterministic undirected graph with self loops. The reinforcement in each urn now depends only on its neighbouring urns. Depending on the type of reinforcement (Pólya or Friedman type), we again have two kinds of models. We prove synchronization results for both the models. As expected, for Pólya-type model, the synchronization of colours occurs across the connected components of the graph G ; whereas, in case of the Friedman-type reinforcement, the fraction of balls of each colour converges to a common fraction of 1/2 in all the urns, regardless of the graph structure. In Chapter 5, we present some of the simulations we performed for our models.

Chapter 1

Urn Models: An introduction

An Urn process is a Markov Random Process consisting of an imaginary exercise involving urns containing balls of different colours. The system evolves at discrete times. At each time instant, balls are added or removed from the urn depending on the kind of model and based upon the sample of balls drawn. Typically, one is interested in the convergence points of the proportion of balls of different colours in the urn. Urn models are used to model real world problems in many diverse fields like genetics, ecology, physics, and economics. In particular, they can be used to model disease spread and/or opinion dynamics. Urn processes have been extensively used to model clinical trials.

1.1 Classical Urn Models

One of the first urn models to be studied was the classical Pólya urn model introduced by George Pólya in 1923. The model consisted of an urn which initially contains a finite number of balls of different colours. At any given time instant t, a ball is drawn from the urn uniformly at random and after noting down its colour, it is replaced along with another ball of the same colour. This reinforcement is carried out at every time-step and is repeated ad-infinitum. For a two-colour Pólya-Eggenberger urn model (a slight variant of the classical Pólya urn model with the difference that s balls of the same colour are added for every ball drawn instead of one), a celebrated result states that the fraction of balls of either colour converges to a random limit as $t \to \infty$ and the distribution of the random limit is given by beta distribution, with parameters depending upon the initial state (initial number of balls of either colour) of the system. More precisely:

Theorem 1.1.1 (*Eggenberger and Pólya, 1923*)**.** *Let $\tilde{W}_n$ be the number of white ball drawings in the P'olya-Eggenberger urn after n draws. Then, as $n \to \infty$,*

$$\frac{\tilde{W}_n}{n} \xrightarrow{a.s.} W$$

such that $W \sim \beta\left(\frac{W_0}{s}, \frac{B_0}{s}\right)$, where $\beta(\cdot,\cdot)$ denotes Beta distribution.

Bernard Friedman generalized the Pólya urn model in 1949. In a two-colour Friedman's urn model, consisting of say white and black balls, at every time instant, a ball is drawn uniformly at random and is replaced with α balls of the same colour and β balls of the other colour. It is known that the fraction of balls of both the colours converges to 1/2 provided that α and β are both strictly positive. That is,

Theorem 1.1.2. *Let $\tilde{W}_n$ be the number of white balls in a Friedman's urn after n draws. Then, as $n \to \infty$,*

$$\frac{\tilde{W}_n}{n} \xrightarrow{a.s.} \frac{1}{2}.$$

In [12], D. Freedman used method of moments to obtain the fluctuating limit theorem for two-colour Friedman urns. He proved the following (W_n and B_n refer to the number of white and black balls respectively in the urn after n draws):

Theorem 1.1.3 (Theorem 3.1 in [1]). *Let $\rho := \alpha - \beta/\alpha + \beta$. Then*

(a) If $\rho > 1/2$ then $n^{-\rho}(W_n - B_n)$ converges almost surely to a nontrivial random variable;

(b) If $\rho = 1/2$ then $(nlogn)^{-1/2}(W_n - B_n)$ converges in distribution to a normal with mean zero and variance $(\alpha - \beta)^2$;

(c) If $0 \neq \rho < 1/2$ then $n^{-1/2}(W_n - B_n)$ converges in distribution to a normal with mean zero and variance $(\alpha - \beta)^2/1 - 2\rho$.

The classical urn models have been studied extensively (see [13] for other classical urn models like Ehrenfest urns, Bagchi-Pal urns and OK-Corral process). The study of urn processes falls under the umbrella of random processes with reinforcement. These also include reinforced random walks. Some of the well-known methods to study random processes with reinforcement, that are also used frequently in this thesis, are described briefly in the Appendix. In the next section we talk about two types of generalizations of classical urn models, namely, urns with multiple drawings and interacting urn models.

1.2 Urn Models with multiple drawings

Unlike the classical urn models discussed above, where only one ball is sampled at a particular time instant, we now describe models with multiple drawings (ones that involve drawing of a fixed number of finite balls). For example, authors in [7] look at a model called Generalized Friedman's urn (a single urn model with two kinds of balls: white and blue), from which samples of a given size,say s (≥ 1 balls) are taken out of the urn at each time instant, and the colours of the balls are noted down. The drawn sample is returned back to the urn, and a "Friendman-type" reinforcement occurs: if there are $0 \leq k \leq s$ white balls in the sample, the urn is reinforced with $Ck \in \mathbb{N}$ blue balls and $C(s - k)$ white balls. Here, C is a fixed constant. The sampling of balls in such models can be done in two ways:

(i) With replacement: a ball is drawn, its colour is noted and is replaced back in the urn, a second ball is drawn, its colour noted, again replaced and so on until s balls have been drawn.

(ii) Without replacement: s balls are drawn together.

Clearly, the distribution of the number of balls drawn in both the cases differ: in the former, the distribution is binomial, while in the latter, the distribution is hypergeometric. Note that no matter how many balls of any one colour are drawn in a particular sample, at any time instant a total of Cs balls are added back to the urn. Such an urn where the total number of balls at any time instant is deterministic and not random, is called a *balanced urn*. Note that all the urn models we will be considering in this thesis will necessarily consist of balanced urns and the sampling will always be one with replacement.

In [14] and [15], a similar multiple drawing model is considered with “Pólya type” reinforcement. That is, once s balls are drawn such that k of them are white, the urn is reinforced with $Ck \in \mathbb{N}$ white balls and $C(s-k)$ blue balls.

We refer the readers to [7], [8] and [10] for results on these models. We combine the multiple drawing models with interacting urns models and consequently obtain some of the results from these papers as special cases of our results.

1.3 Interacting Urn Models

As noted before, an interacting urn model is one which consists of several urns which may influence each other's reinforcement. So, the reinforcement for a particular urn at any time instant depends not only on the state of that urn at that instant, but also on that of the other urns. In [3], the authors describe a two-colour interacting urn process as follows:

We define a general two-colour interacting urn model as follows: suppose that there are N urns with configurations (W_i^t, B_i^t) for $1 \leq i \leq N$ and $t \geq 0$, where W_i^t and B_i^t denote the number of white balls and black balls respectively, at time t, in the i^{th} urn. The reinforcement scheme of each urn depends on all urns or on a non-trivial subset of the given set of N urns. Let $Z_i^t := \frac{W_i^t}{W_i^t+B_i^t}$ be the proportion of white balls in the i^{th} urn and I_i^t be the number of white balls added to the i^{th} urn at time t. We write: $W_i^{t+1} = W_i^t + I_i^{t+1}$. Suppose the evolution of the i^{th} urn depends on urns $\{i_1, \dots i_{k_i}\} \subseteq [N] := \{1, \dots, N\}$. We call this set the dependency set of the i^{th} urn. If we think of urns as nodes of a network, a natural choice for the dependency set of a vertex is the neighbourhood of that vertex. Then the random process $I^t = (I_1^t, \dots, I_N^t)$, that defines the reinforcement scheme, evolves as follows:

$$P(I_i^{t+1} = \alpha_i^{t+1} | \mathcal{F}_t) \propto f_{i,\alpha_i^{t+1}}\left(Z_{i_1}^t, \dots, Z_{i_{k_i}}^t\right), \tag{1.1}$$

where for all $t > 0$, $\alpha_i^t \in \mathbb{Z}^+$ and $f_{i,\alpha_i^{t+1}} : \mathbb{Z}^+ \times [0,1]^{k_i} \longrightarrow [0,1]$, for every $i \in [N]$.

Similarly, in [8], the authors consider a two-colour model with N urns where at every time instant, a ball is sampled for each urn depending on the outcome of the toss of a biased coin (say $P(TAILS) = \alpha$). One ball is sampled from that same urn in case of a heads, and from the *super urn* (an imaginary urn formed by combining all the urns together) in case of a tails. That is, the probability of i^{th} urn getting a white ball at time t is given by $(1-\alpha)Z_i^t + \alpha\frac{1}{N}\sum_{j=1}^{N} Z_j^t$ (Z_j^t denotes the fraction of balls of the concerned colour in the j^{th} urn at time t for $1 \leq j \leq N$). Several special cases of the above set-up of interacting urns with reinforcement of the form (1.1) have been studied recently.

In this thesis, we combine the two types of generalizations described above in sections 1.2 and 1.3 to study interacting urn models with multiple drawings. We restrict ourselves to positive reinforcement, that is, at every time-step non-negative number of balls are added to the urns (in other words, balls are never thrown out of the urns). We also only consider the case of finitely many urns and balls of finitely many colours. From the next chapter, we start with a detailed study of the models that we have looked at in this thesis. The following definitions will be useful throughout:

Definition 1.3.1 (Reinforcement of a colour by another colour)**.** *For a model with drawing of s balls, we say that a colour J is reinforced by a colour L(denoted by $L \to J$) if given that $0 \leq k \leq s$ balls of colour L are drawn, we put back Ck balls of colour J back in the concerned urn.*

Definition 1.3.2 (Self reinforcement of colours)**.** *We say that a model exhibits self reinforcement of colours, if all the colours reinforce themselves (and are not reinforced by any other colour).*

For a two colour model, we define:

Definition 1.3.3 (Mutual reinforcement of colours)**.** *We say that a two colour model exhibits mutual reinforcement of colours if the two colours reinforce each other (and none of them is reinforced by itself).*

Note that interchangeably, we call the Pólya-type model as self reinforcement model and the Friedman-type model as mutual reinforcement model.

Chapter 2

Interacting two-colour Urns with Multiple Drawings

2.1 Pólya-type model (Self reinforcement of colours)

The model consists of N urns, initially containing non-zero number of balls of two colours: white and black. The system evolution happens in discrete time. Initially (at time $t = 0$), the i^{th} urn contains a_i white and b_i black balls such that $a_i + b_i = m$ for all $1 \leq i \leq N$ (m is a fixed positive integer). An imaginary urn is formed by combining all the N urns. We call this the super urn. At each time instant t, a biased coin is tossed (with probability of heads $= p$) for each urn. Depending on the outcome of the toss , s number of balls are sampled from each urn (in case of a heads, that is, with probability p) or from the super urn (in case of a tails, that is, with probability $1 - p$). If k out of s balls sampled are white ($0 \leq k \leq s$), then Ck number of white and $C(s-k)$ number of black balls are added to the urn, where C is a fixed positive integer.

This reinforcement is done for each urn at every (discrete) time instant and is repeated over and over. Note that at every time-step a total of Cs balls are added to each urn. So, while the number of white balls added is random, the total number of balls added is deterministic (this is an example of what is known in the literature as a balanced reinforcement scheme). At time t, the total number of balls in each urn is $m + Cst$.

We fix some notation that is used throughout the thesis.

- $W_t(i)$: number of white balls in the i^{th} urn at time t.
- Z_t(i): fraction of white balls in the i^{th} urn at time t.
- $\bar{Z}_t$: fraction of white balls in the super urn at time t.
- Y_{t+1}(i): number of white balls added to the i^{th} urn at time t (note that $Y_t(i)$' s are conditionally independent).
- We define: $\alpha = \frac{\sum_{i=1}^{N} a_i}{mN}$ and $Z_t = (Z_t(1), Z_t(2) \ldots Z_t(N))$
- The σ-field $\mathcal{F}_t$ is defined as: $\mathcal{F}_t = \sigma(Z_0(1), Z_0(2) \ldots Z_0(N) \ldots Z_t(1), Z_t(2) \ldots Z_t(N))$. For $t \geq 0, \{\mathcal{F}_t\}_{t\geq 0}$ defines a filtration.

Theorem 2.1.1 below has two parts: the first part gives an explicit formula for $E[Z_t(i)]$ for $1 \le i \le N$ and the second part is a synchronisation result. It states the fact that the the fraction of white balls in each urn converges to a common fraction.

Theorem 2.1.1. *The following holds for* $1 \le i \le N$

(a) $E[Z_t(i)] = \left(\frac{a_i}{m} - \alpha\right) \frac{\binom{\frac{m}{Cs}+p+t-1}{t}}{\binom{\frac{m}{Cs}+t}{t}} + \alpha.$

(b) $\lim_{t\to\infty}(Z_t(i) - \bar{Z}_t) = 0 \ \forall\ 1 \le i \le N$ *a.s.*

Proof For $0 \le k \le s, 1 \le i \le N$

$$P(Y_{t+1}(i) = Ck|\mathcal{F}_t) = p\binom{s}{k}(Z_t(i))^k(1 - Z_t(i))^{s-k} + (1-p)\binom{s}{k}(\bar{Z}_t)^k(1-\bar{Z}_t)^{s-k}$$

Since the total number of balls is deterministic, we have :

$$\bar{Z}_t = \frac{\sum_{i=1}^N Z_t(i)}{N}$$

We now write the evolution of $Z_t(i)$ as a Stochastic Approximation Scheme. For $1 \le i \le N$:

$$\begin{aligned} Z_{t+1}(i) &= \frac{W_t(i) + Y_{t+1}(i)}{m + Cs(t+1)} \\ &= Z_t(i) - \frac{CsZ_t(i)}{m + Cs(t+1)} + \frac{Y_{t+1}(i)}{m + Cs(t+1)} \\ &= Z_t(i) + \frac{1}{m + Cs(t+1)} \left(Y_{t+1}(i) - E[Y_{t+1}(i)|\mathcal{F}_t] + h(Z_t(i))\right) \end{aligned} \tag{2.1}$$

Here, $h(Z_t(i)) = E[Y_{t+1}(i)|\mathcal{F}_t] - CsZ_t(i)$. Now,

$$\begin{aligned} E[Y_{t+1}(i)|\mathcal{F}_t] &= \sum_{k=0}^{s} Ck\left[p\binom{s}{k}(Z_t(i))^k(1 - Z_t(i))^{s-k} + (1-p)\binom{s}{k}(\bar{Z}_t)^k(1-\bar{Z}_t)^{s-k}\right] \\ &= \sum_{k=0}^{s} Cpk\binom{s}{k}(Z_t(i))^k(1 - Z_t(i))^{s-k} + \sum_{k=0}^{s} C(1-p)k\binom{s}{k}(\bar{Z}_t)^k(1-\bar{Z}_t)^{s-k} \\ &= Cps(Z_t(i))\sum_{k=1}^{s}\binom{s-1}{k-1}(Z_t(i))^{k-1}(1 - Z_t(i))^{s-1-(k-1)} \\ &+ C(1-p)s(\bar{Z}_t)\sum_{k=1}^{s}\binom{s-1}{k-1}(\bar{Z}_t)^{k-1}(1-\bar{Z}_t)^{s-1-(k-1)} \\ &= CpsZ_t(i) + C(1-p)s\bar{Z}_t \\ &= Cs\left[pZ_t(i) + (1-p)\bar{Z}_t\right] \end{aligned}$$

Hence,

$$\begin{aligned} h(Z_t(i)) &= E[Y_{t+1}(i)|\mathcal{F}_t] - CsZ_t(i) \\ &= CpsZ_t(i) + C(1-p)s\bar{Z}_t - CsZ_t(i) \\ &= Cs(1-p)[\bar{Z}_t - Z_t(i)] \end{aligned}$$

Taking conditional expectation with respect to $\mathcal{F}_t$ on both sides of (2.1), we get:

$$\begin{aligned} E[Z_{t+1}(i)|\mathcal{F}_t] &= Z_t(i) + \frac{h(Z_t(i))}{m + Cs(t+1)} \\ &= Z_t(i) + \frac{Cs(1-p)[\bar{Z}_t - Z_t(i)]}{m + Cs(t+1)} \end{aligned} \tag{2.2}$$

Now,

$$\begin{aligned} E[\bar{Z}_{t+1}|\mathcal{F}_t] &= E\left[\frac{\sum_{i=1}^N Z_{t+1}(i)}{N}|\mathcal{F}_t\right] \\ &= \frac{1}{N}\sum_{i=1}^N E[Z_{t+1}(i)|\mathcal{F}_t] \\ &= \frac{1}{N}\sum_{i=1}^N \left(Z_t(i) + \frac{Cs(1-p)[\bar{Z}_t - Z_t(i)]}{m + Cs(t+1)}\right) \\ &= \bar{Z}_t \end{aligned}$$

Hence, $\bar{Z}_t$ is a Martingale w.r.t $\mathcal{F}_t$. Taking expectation on both sides of (2.2), we get:

$$E[Z_{t+1}(i)] = E[Z_t(i)] + \frac{Cs(1-p)[E[\bar{Z}_t] - E[Z_t(i)]]}{m + Cs(t+1)}$$

Since,$\bar{Z}_t$ is a Martingale,we have :

$$E[\bar{Z}_t] = E[\bar{Z}_0] = \frac{\sum_{i=1}^N a_i}{mN} = \alpha$$

Hence,

$$\begin{aligned} E[Z_{t+1}(i)] &= E[Z_t(i)]\left(1 - \frac{Cs(1-p)}{m + Cs(t+1)}\right) + \frac{Cs(1-p)\alpha}{m + Cs(t+1)} \\ &= E[Z_t(i)]f_t + g_t \end{aligned}$$

Here, $f_t = 1 - \frac{Cs(1-p)}{m+Cs(t+1)}$ and $g_t = \frac{Cs(1-p)\alpha}{m+Cs(t+1)}$. The reader is directed to the appendix for a discussion on how to solve the above kind of recurrence relations. Now,

$$\begin{aligned} \prod_{k=0}^{m'} f_k &= \prod_{k=0}^{m'}\left(1 - \frac{Cs(1-p)}{m + Cs(k+1)}\right) = \prod_{k=0}^{m'} \frac{\frac{m}{Cs} + k + p}{\frac{m}{Cs} + k + 1} \\ &= \frac{(\frac{m}{Cs} + p + m')!(\frac{m}{Cs})!}{(\frac{m}{Cs} + m' + 1)!(\frac{m}{Cs} + p - 1)!} = \frac{\binom{\frac{m}{Cs}+p+m'}{m'+1}}{\binom{\frac{m}{Cs}+1+m'}{m'+1}} \end{aligned}$$

Hence,

$$\begin{aligned}
E[Z_t(i)] &= (\prod_{k=0}^{t-1} f_k)\left(E[Z_0(i)] + \sum_{m'=0}^{t-1} \frac{g'_m}{\prod_{k=0}^{m'} f_k}\right) \\
&= \frac{\binom{\frac{m}{Cs}+p+t-1}{t}}{\binom{\frac{m}{Cs}+t}{t}} \left[\frac{a_i}{m} + \sum_{m'=0}^{t-1} \frac{Cs(1-p)\alpha}{(\prod_{k=0}^{m'} f_k)(m+Cs(m'+1))}\right] \\
&= \frac{\binom{\frac{m}{Cs}+p+t-1}{t}}{\binom{\frac{m}{Cs}+t}{t}} \left[\frac{a_i}{m} + \sum_{m'=0}^{t-1} \frac{Cs(1-p)\alpha\binom{\frac{m}{Cs}+m'+1}{m'+1}}{(m+Cs(m'+1))\binom{\frac{m}{Cs}+p+m'}{m'+1}}\right] \\
&= \frac{\binom{\frac{m}{Cs}+p+t-1}{t}}{\binom{\frac{m}{Cs}+t}{t}} \left[\frac{a_i}{m} + (1-p)\alpha \sum_{m'=0}^{t-1} \frac{\binom{\frac{m}{Cs}+m'+1}{m'+1}}{(\frac{m}{Cs}+m'+1)\binom{\frac{m}{Cs}+p+m'}{m'+1}}\right]
\end{aligned}$$

Now,

$$\frac{\binom{\frac{m}{Cs}+m'+1}{m'+1}}{\frac{m}{Cs}+m'+1} = \frac{(\frac{m}{Cs}+m'+1)!}{(\frac{m}{Cs})!(m'+1)!(\frac{m}{Cs}+m'+1)} = \frac{(\frac{m}{Cs}+m')!}{(\frac{m}{Cs})!(m'+1)!} = \frac{Cs}{m}\binom{\frac{m}{Cs}+m'}{m'+1}$$

Hence,

$$E[Z_t(i)] = \frac{\binom{\frac{m}{Cs}+p+t-1}{t}}{\binom{\frac{m}{Cs}+t}{t}m} \left[a_i + (1-p)\alpha Cs \sum_{m'=0}^{t-1} \frac{\binom{\frac{m}{Cs}+m'}{m'+1}}{\binom{\frac{m}{Cs}+p+m'}{m'+1}}\right]$$

Using Lemma 5.2.1(see appendix), we calculate the following:

$$\sum_{m'=0}^{t-1} \frac{\binom{\frac{m}{Cs}+m'}{m'+1}}{\binom{\frac{m}{Cs}+p+m'}{m'+1}} = \sum_{k=1}^{t} \frac{\binom{\frac{m}{Cs}+k-1}{k}}{\binom{\frac{m}{Cs}+p-1+k}{k}}$$

Let $y = \frac{m}{Cs} + p - 1$, $s = t$ and $x = \frac{m}{Cs} - 1$ So,

$$\begin{aligned}
\sum_{m'=0}^{t-1} \frac{\binom{\frac{m}{Cs}+m'}{m'+1}}{\binom{\frac{m}{Cs}+p+m'}{m'+1}} &= \frac{(t+1+\frac{m}{Cs}+p-1)\binom{t+1+\frac{m}{Cs}-1}{t+1}}{(\frac{m}{Cs}-\frac{m}{Cs}-p+1)\binom{t+1+\frac{m}{Cs}+p-1}{t+1}} - \frac{m}{Cs(1-p)} \\
&= \frac{(t+\frac{m}{Cs}+p)\binom{t+\frac{m}{Cs}}{t+1}}{(1-p)\binom{t+\frac{m}{Cs}+p}{t+1}} - \frac{m}{Cs(1-p)}
\end{aligned}$$

Hence,

$$E[Z_t(i)] = \frac{\binom{\frac{m}{Cs}+p+t-1}{t}}{\binom{\frac{m}{Cs}+t}{t}m} \left[a_i + (1-p)\alpha Cs \left(\frac{(t+\frac{m}{Cs}+p)\binom{t+\frac{m}{Cs}}{t+1}}{(1-p)\binom{t+\frac{m}{Cs}+p}{t+1}} - \frac{m}{Cs(1-p)}\right)\right]$$

We can simplify this expression further to get:

$$E[Z_t(i)] = \left(\frac{a_i}{m} - \alpha\right) \frac{\binom{\frac{m}{Cs}+p+t-1}{t}}{\binom{\frac{m}{Cs}+t}{t}} + \alpha$$

The above equation holds for all $1 \leq i \leq N$. This completes the proof for the first part of the theorem.

For the second part, we write the Stochastic Approximation Scheme for Z_t in order to prove the second part. Let $Y_{t+1} = (Y_{t+1}(1), Y_{t+1}(2) \dots Y_{t+1}(N))$. We have:

$$\begin{aligned} Z_{t+1} &= Z_t - \frac{CsZ_t}{m + Cs(t+1)} + \frac{Y_{t+1}}{m + Cs(t+1)} \\ &= Z_t + \frac{1}{m + Cs(t+1)} \left(Y_{t+1} - E[Y_{t+1}|\mathcal{F}_t] + h(Z_t)\right) \end{aligned}$$

Here,

$$h(Z_t) = (h(Z_t(1), h(Z_t(2) \dots h(Z_t(N)) = Cs(1-p)(\bar{Z}_t - Z_t(1), \bar{Z}_t - Z_t(2) \dots \bar{Z}_t - Z_t(N))$$

Since the zeroes of $h(Z_t)$ give the limit points of Z_t, we are done (see appendix for details about the stochastic approximation theorem and how it has been applied to our models all through the thesis).

□

As noted before, our model is a generalization of the model studied in [8]. The next theorem has been proven using the same techniques as used there, with minor modifications. It is interesting that the asymptotic estimates calculated below are the same as those calculated there, despite the added complexity of multiple drawings which is missing in their model. Also, note that the probability of heads in our case is p while the analogous variable used for the same probability by the authors in [8] is $(1-\alpha)$ for $\alpha \in [0,1]$.

Theorem 2.1.2. *Assume that $a_i = a$ for all $1 \leq i \leq N$. In the following statement, for two positive sequences a_t and b_t we write $a_t \sim b_t$ if*

$$0 < \liminf_{t\to\infty} \frac{a_t}{b_t} \leq \limsup_{t\to\infty} \frac{a_t}{b_t} < +\infty$$

The following asymptotic equations hold:

$$Var(Z_t(i) - \bar{Z}_t) \sim \begin{cases} t^{-2(1-p)} & \frac{1}{2} < p < 1 \\ t^{-1} logt & p = \frac{1}{2} \\ t^{-1} & 0 \leq p < \frac{1}{2} \end{cases}$$

Proof Since $a_i = a$ for all $1 \leq i \leq N$, we therefore have:

$$E[Z_t(i) - \bar{Z}_t]) = 0 \ \forall \ 1 \leq i \leq N$$

Let

$$\begin{aligned} x_t &= Var(Z_t(i) - \bar{Z}_t) = E[(Z_t(i) - \bar{Z}_t)^2] - (E[Z_t(i) - \bar{Z}_t])^2 \\ &= E[(Z_t(i) - \bar{Z}_t)^2] \end{aligned}$$

Then,

$$x_{t+1} = Var(Z_{t+1}(i) - \bar{Z}_{t+1}) = E[Var(Z_{t+1}(i) - \bar{Z}_{t+1}|\mathcal{F}_t)] + Var(E[Z_{t+1}(i) - \bar{Z}_{t+1}|\mathcal{F}_t)])$$

Now,

$$Z_{t+1}(i) = Z_t(i) - \frac{CsZ_t(i)}{m + Cs(t+1)} + \frac{Y_{t+1}(i)}{m + Cs(t+1)}$$

Hence,

$$\bar{Z}_{t+1} = \bar{Z}_t - \frac{Cs\bar{Z}_t}{m + Cs(t+1)} + \sum_{i=1}^{N} \frac{Y_{t+1}(i)}{N(m + Cs(t+1))} \tag{2.3}$$

Now,

$$\begin{aligned}
Var(Z_{t+1}(i) - \bar{Z}_{t+1}|\mathcal{F}_t) &= E\left[(Z_{t+1}(i) - \bar{Z}_{t+1} - E[Z_{t+1}(i)|\mathcal{F}_t] + E[\bar{Z}_{t+1}|\mathcal{F}_t])^2|\mathcal{F}_t\right] \\
&= E\left[Z_t(i) - \frac{CsZ_t(i)}{m + Cs(t+1)} + \frac{Y_{t+1}(i)}{m + Cs(t+1)}\right. \\
&+ \frac{Cs\bar{Z}_t}{m + Cs(t+1)} - \sum_{i=1}^{N} \frac{Y_{t+1}(i)}{N(m + Cs(t+1))} - Z_t(i) \\
&+ \frac{CsZ_t(i)}{m + Cs(t+1)} - \frac{E[Y_{t+1}(i)|\mathcal{F}_t]}{m + Cs(t+1)} + \bar{Z}_t - \frac{Cs\bar{Z}_t}{m + Cs(t+1)} \\
&+ \left.\sum_{i=1}^{N} \frac{E[Y_{t+1}(i)|\mathcal{F}_t]}{N(m + Cs(t+1))})^2|\mathcal{F}_t\right] \\
&= E\left[\left(\frac{Y_{t+1}(i) - E[Y_{t+1}(i)|\mathcal{F}_t]}{m + Cs(t+1)} + \sum_{j=1}^{N} \frac{E[Y_{t+1}(j)|\mathcal{F}_t] - Y_{t+1}(j)}{N(m + Cs(t+1))}\right)^2 |\mathcal{F}_t\right] \\
&= \frac{1}{(m + Cs(t+1))^2} E\left[\left(\left(1 - \frac{1}{N}\right)(Y_{t+1}(i) - E[Y_{t+1}(i)|\mathcal{F}_t])\right.\right. \\
&+ \left.\left.\frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} (E[Y_{t+1}(j)|\mathcal{F}_t] - Y_{t+1}(j))\right)^2 |\mathcal{F}_t\right] \\
&= \frac{1}{(m + Cs(t+1))^2} \left\{E[(1 - \frac{1}{N})^2(Y_{t+1}(i) - E[Y_{t+1}(i)|\mathcal{F}_t])^2|\mathcal{F}_t]\right. \\
&+ \frac{1}{N^2} E[(\sum_{\substack{j=1 \\ j \neq i}}^{N} (E[Y_{t+1}(j)|\mathcal{F}_t] - Y_{t+1}(j))^2|\mathcal{F}_t] \\
&+ 2\left(1 - \frac{1}{N}\right)\left(\frac{1}{N}\right)(E[(Y_{t+1}(i) - E[Y_{t+1}(i)|\mathcal{F}_t]) \\
&\times \left.(\sum_{\substack{j=1 \\ j \neq i}}^{N} (E[Y_{t+1}(j)|\mathcal{F}_t] - Y_{t+1}(j)))|\mathcal{F}_t]\right\}
\end{aligned}$$

For i $\neq$ j, we have (since $Y_{t+1}(i)$ and $Y_{t+1}(j)$ are conditionally independent):

$$\begin{aligned}
E[(Y_{t+1}(i) - E[Y_{t+1}(i)|\mathcal{F}_t])(E[Y_{t+1}(j)|\mathcal{F}_t] - Y_{t+1}(j))|\mathcal{F}_t] &= E[Y_{t+1}(i)|\mathcal{F}_t]E[Y_{t+1}(j)|\mathcal{F}_t] \\
&\quad -E[Y_{t+1}(i)Y_{t+1}(j)|\mathcal{F}_t] \\
&= 0
\end{aligned}$$

Hence,

$$
\begin{aligned}
Var(Z_{t+1}(i) - \bar{Z}_{t+1}|\mathcal{F}_t) &= \frac{1}{(m+Cs(t+1))^2}\left\{\left(1-\frac{1}{N}\right)^2 Var(Y_{t+1}(i)|\mathcal{F}_t)\right. \\
&+ \left.\frac{1}{N^2}E\left[\left(\sum_{\substack{j=1 \\ j\neq i}}^{N}(E[Y_{t+1}(j)|\mathcal{F}_t] - Y_{t+1}(j)\right)^2 \middle| \mathcal{F}_t\right]\right\}
\end{aligned}
$$

Due to conditional independence, we again have that :

$$
\begin{aligned}
Var(Z_{t+1}(i) - \bar{Z}_{t+1}|\mathcal{F}_t) &= \frac{1}{(m+Cs(t+1))^2}\left\{\left(1-\frac{1}{N}\right)^2 Var(Y_{t+1}(i)|\mathcal{F}_t)\right. \\
&+ \left.\frac{1}{N^2}\sum_{\substack{j=1 \\ j\neq i}}^{N}\left(E[(E[Y_{t+1}(j)|\mathcal{F}_t] - Y_{t+1}(j))^2|\mathcal{F}_t]\right)\right\}
\end{aligned}
$$

Now,

$$
\begin{aligned}
E[(Y_{t+1}(i))^2|\mathcal{F}_t] &= \sum_{k=0}^{s} C^2k^2\left[p\binom{s}{k}(Z_t(i))^k(1-Z_t(i))^{s-k} + (1-p)\binom{s}{k}(\bar{Z}_t)^k(1-\bar{Z}_t)^{s-k}\right] \\
&= C^2p\left[s(s-1)(Z_t(i))^2 + sZ_t(i)\right] + C^2(1-p)\left[s(s-1)(\bar{Z}_t)^2 + s\bar{Z}_t\right]
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Var(Y_{t+1}(i)|\mathcal{F}_t) &= E[(Y_{t+1}(i))^2|\mathcal{F}_t] - (E[Y_{t+1}(i))|\mathcal{F}_t])^2 \\
&= C^2p[s(s-1)(Z_t(i))^2 + sZ_t(i)] + C^2(1-p)[s(s-1)(\bar{Z}_t)^2 + s\bar{Z}_t] \\
&- C^2s^2[p^2(Z_t(i))^2 + (1-p)^2(\bar{Z}_t)^2 + 2p(1-p)Z_t(i)\bar{Z}_t] \qquad (2.4)
\end{aligned}
$$

Now,

$$
E[Var(Y_t(i)] = E[Var(Y_t(j)] \ \forall \ i \neq j
$$

So,

$$
\begin{aligned}
E[Var(Z_{t+1}(i) - \bar{Z}_{t+1}|\mathcal{F}_t)] &= \frac{1}{(m+Cs(t+1))^2}\left\{\left(1-\frac{1}{N}\right)^2 + \frac{N-1}{N^2}\right\}E[Var(Y_{t+1}(i)|\mathcal{F}_t] \\
&= \frac{1}{(m+Cs(t+1))^2}\left(\frac{N-1}{N}\right)E[Var(Y_{t+1}(i)|\mathcal{F}_t]
\end{aligned}
$$

Taking expectation on both sides of (2.4), we get:

$$
E[Var(Y_{t+1}(i)|\mathcal{F}_t)] = C^2ps(s(1-p)-1)E[(Z_t(i))^2] + E[(\bar{Z}_t)^2]C^2s(1-p)(-sp-1) + C^2s\alpha
$$

Now,

$$\begin{aligned} E[Z_{t+1}(i) - \bar{Z}_{t+1}|\mathcal{F}_t] &= E\Bigg[Z_t(i) - \frac{CsZ_t(i)}{m+Cs(t+1)} + \frac{Y_{t+1}(i)}{m+Cs(t+1)} \\ &- \bar{Z}_t + \frac{Cs\bar{Z}_t}{m+Cs(t+1)} - \sum_{i=1}^{N} \frac{Y_{t+1}(i)}{N(m+Cs(t+1))}|\mathcal{F}_t\Bigg] \\ &= \left(1 - \frac{Cs(1-p)}{m+Cs(t+1)}\right)\left(Z_t(i) - \bar{Z}_t\right) \end{aligned}$$

Hence,

$$Var(E[Z_{t+1}(i) - \bar{Z}_{t+1}|\mathcal{F}_t]) = \left(1 - \frac{Cs(1-p)}{m+Cs(t+1)}\right)^2 x_t$$

Thus,

$$\begin{aligned} x_{t+1} &= \frac{1}{(m+Cs(t+1))^2}\left(\frac{N-1}{N}\right)E[Var(Y_{t+1}(i)|\mathcal{F}_t] + \left(1 - \frac{Cs(1-p)}{m+Cs(t+1)}\right)^2 x_t \\ &= \frac{N-1}{N(m+Cs(t+1))^2}\left\{C^2ps(s(1-p)-1)E[(Z_t(i))^2]\right. \\ &+ \left. E[(\bar{Z}_t)^2]C^2s(1-p)(-sp-1) + C^2s\alpha\right\} + \left(1 - \frac{Cs(1-p)}{m+Cs(t+1)}\right)^2 x_t \end{aligned} \tag{2.5}$$

Now,

$$\begin{aligned} x_t &= Var(Z_t(i) - \bar{Z}_t) = E[(Z_t(i) - \bar{Z}_t)^2] = E[(Z_t(i))^2] + E[(\bar{Z}_t)^2] - 2E[Z_t(i)\bar{Z}_t] \\ &= E[(Z_t(i))^2] + E[(\bar{Z}_t)^2] - 2\frac{\sum_{i=1}^{N} E[Z_t(i)\bar{Z}_t]}{N} = E[(Z_t(i))^2] + E[(\bar{Z}_t)^2] - 2E[(\bar{Z}_t)^2] \\ &= E[(Z_t(i))^2] - E[(\bar{Z}_t)^2] \end{aligned}$$

So, $E[(Z_t(i))^2] = E[(\bar{Z}_t)^2] + x_t$. Substituting this value of E$[(Z_t(i))^2]$ in (2.5) and simplifying, we get the following:

$$x_{t+1} = x_t\left[\left(1 - \frac{Cs(1-p)}{m+Cs(t+1)}\right)^2 + \frac{(N-1)C^2ps[s(1-p)-1]}{N(m+Cs(t+1)^2}\right] + \frac{(N-1)C^2s(\alpha - E[(\bar{Z}_t)^2])}{N(m+Cs(t+1))^2}$$

We show that the term $\alpha - E[(\bar{Z}_t)^2]$is strictly positive. Since $\bar{Z}_t$ is a Martingale, we have:

$$\begin{aligned} Var(\bar{Z}_{t+1}) = \quad & Var(E[\bar{Z}_{t+1}|\mathcal{F}_t]) + E[Var(\bar{Z}_{t+1}|\mathcal{F}_t)] \\ & = Var(\bar{Z}_t) + E[Var(\bar{Z}_{t+1}|\mathcal{F}_t)] \end{aligned} \tag{2.6}$$

Now, recall (2.3):

$$\bar{Z}_{t+1} = \bar{Z}_t - \frac{Cs\bar{Z}_t}{m+Cs(t+1)} + \sum_{i=1}^{N} \frac{Y_{t+1}(i)}{N(m+Cs(t+1))}$$

Hence,

$$\begin{aligned} Var(\bar{Z}_{t+1}|\mathcal{F}_t) &= E\left[\left(\sum_{i=1}^{N}\frac{Y_{t+1}(i)}{N(m+Cs(t+1))} - \sum_{i=1}^{N}\frac{E[Y_{t+1}(i)|\mathcal{F}_t]}{N(m+Cs(t+1))}\right)^2 |\mathcal{F}_t\right] \\ &= \frac{1}{(m+Cs(t+1))^2N^2}\sum_{i=1}^{N} Var(Y_{t+1}(i)|\mathcal{F}_t) \end{aligned} \tag{2.7}$$

Taking expectation on both sides, we get:

$$E[Var(\bar{Z}_{t+1}|\mathcal{F}_t)] = \frac{1}{(m+Cs(t+1))^2N^2}\sum_{i=1}^{N} E[Var(Y_{t+1}(i)|\mathcal{F}_t)] \tag{2.8}$$

Now,

$$\begin{aligned} Var(Y_{t+1}(i)|\mathcal{F}_t) &= E[(Y_{t+1}(i))^2|\mathcal{F}_t] - (E[(Y_{t+1}(i))^2|\mathcal{F}_t])^2 \\ &= C^2ps(s-1)(Z_t(i))^2 + C^2psZ_t(i) + C^2(1-p)s(s-1)(\bar{Z}_t)^2 + C^2(1-p)s\bar{Z}_t \\ &- C^2s^2[p^2(Z_t(i))^2 + (1-p)^2(\bar{Z}_t)^2 + 2p(1-p)\bar{Z}_tZ_t(i)] \\ &= (Z_t(i))^2[C^2ps(s-1) - C^2s^2p^2] + (\bar{Z}_t)^2[C^2(1-p)s(s-1) - C^2s^2(1-p)^2] \\ &+ C^2psZ_t(i) + C^2(1-p)s\bar{Z}_t - 2C^2s^2p(1-p)\bar{Z}_tZ_t(i) \end{aligned}$$

Taking expectation on both sides,we get:

$$\begin{aligned} E[Var(Y_{t+1}(i)\mathcal{F}_t)] &= E[(Z_t(i))^2][C^2ps(s-1) - C^2s^2p^2] + E[(\bar{Z}_t)^2][C^2(1-p)s(s-1) \\ &- C^2s^2(1-p)^2] + C^2psE[Z_t(i)] + C^2(1-p)s\alpha - 2C^2s^2p(1-p)E[\bar{Z}_tZ_t(i)] \end{aligned}$$

Putting this value of $E[Var(Y_{t+1}(i)\mathcal{F}_t)]$ in (2.7), we get:

$$\begin{aligned} E[Var(\bar{Z}_{t+1}|\mathcal{F}_t)] &= \frac{1}{(m+Cs(t+1))^2N^2}\left\{E[\sum_{i=1}^{N}(Z_t(i))^2]C^2ps[s(1-p)-1] + N(Var(\bar{Z}_t)+\alpha^2)\right. \\ &\times \left. C^2s(1-p)(sp-1) + C^2sN\alpha - 2C^2s^2p(1-p)\sum_{i=1}^{N}E[\bar{Z}_tZ_t(i)]\right\} \end{aligned}$$

Since $a_i = a \; \forall \; 1 \leq i \leq N$, hence:

$$\sum_{i=1}^{N} E\left[\bar{Z}_tZ_t(i)\right] = E[\bar{Z}_t\sum_{i=1}^{N}Z_t(i)] = NE[(\bar{Z}_t)^2] = N(Var(\bar{Z}_t)+\alpha^2)$$

Recalling equation (2.6) and putting these values there, we get :

$$\begin{aligned}
Var(\bar{Z}_{t+1}) &= Var(\bar{Z}_t) + \frac{1}{(m+Cs(t+1))^2N^2}\left\{\sum_{i=1}^{N} E[(Z_t(i))^2][C^2ps(s(1-p)-1]\right. \\
&+ \left. N(Var(\bar{Z}_t)+\alpha^2)C^2s(1-p)(sp-1) + C^2sN\alpha - 2C^2s^2p(1-p)N(Var(\bar{Z}_t)+\alpha^2)\right\} \\
&= Var(\bar{Z}_t) + \frac{1}{(m+Cs(t+1))^2N^2}\left\{\sum_{i=1}^{N} E[(Z_t(i))^2][C^2ps(s(1-p)-1]\right. \\
&+ \left. N(Var(\bar{Z}_t)+\alpha^2)C^2s(1-p)(-sp-1) + C^2sN\alpha\right\} \\
&= Var(\bar{Z}_t) + \frac{C^2}{(m+Cs(t+1))^2N^2}\left\{\sum_{i=1}^{N} E[(Z_t(i))^2][ps(s(1-p)-1]\right. \\
&+ \left. N(Var(\bar{Z}_t)+\alpha^2)s(1-p)(-sp-1) + sN\alpha\right\} \\
&\leq Var(\bar{Z}_t) + \frac{C^2}{(m+Cs(t+1))^2N^2}\left\{\sum_{i=1}^{N} E[(Z_t(i))^2]ps^2(1-p) + sN\alpha\right\}
\end{aligned}$$

Note that:

$$N^2E[(\bar{Z}_t)^2] = E[(\sum_{i=1}^{N}(Z_t(i))^2] \geq \sum_{i=1}^{N} E[(Z_t(i))^2]$$

Hence,

$$Var(\bar{Z}_{t+1}) \leq Var(\bar{Z}_t) + \frac{C^2}{(m+Cs(t+1))^2N^2}\{N^2E[(\bar{Z}_t)^2]ps^2(1-p) + sN\alpha\}$$

This is same as:

$$Var(\bar{Z}_{t+1}) \leq Var(\bar{Z}_t) + \frac{C^2}{(m+Cs(t+1))^2N^2}\{N^2(Var(\bar{Z}_t)+\alpha^2)ps^2(1-p) + sN\alpha\}$$

Rearranging the terms, we get:

$$Var(\bar{Z}_{t+1}) \leq Var(\bar{Z}_t)\left(1 + \frac{p(1-p)}{(\frac{m}{Cs}+t+1)^2}\right) + \frac{\alpha^2p(1-p)}{(\frac{m}{Cs}+t+1)^2} + \frac{\alpha}{Ns(\frac{m}{Cs}+t+1)^2}$$

Since, $E[\bar{Z}_t] = \alpha$, we have:

$$E[(\bar{Z}_{t+1})^2] - \alpha^2 \leq (E[(\bar{Z}_t)^2] - \alpha^2)\left(1 + \frac{p(1-p)}{(\frac{m}{Cs}+t+1)^2}\right) + \frac{\alpha^2p(1-p)}{(\frac{m}{Cs}+t+1)^2} + \frac{\alpha}{Ns(\frac{m}{Cs}+t+1)^2}$$

Rearranging once again, we get:

$$E[(\bar{Z}_{t+1})^2] - E[(\bar{Z}_t)^2] \leq E[(\bar{Z}_t)^2]\frac{p(1-p)}{(\frac{m}{Cs}+t+1)^2} + \frac{\alpha}{Ns(\frac{m}{Cs}+t+1)^2}$$

Since $E[(\bar{Z}_t)^2]$ is bounded above by one, we have :

$$E[(\bar{Z}_{t+1})^2] - E[(\bar{Z}_t)^2] \leq \frac{p(1-p)}{(\frac{m}{Cs}+t+1)^2} + \frac{\alpha}{Ns(\frac{m}{Cs}+t+1)^2}$$

Since $\bar{Z}_0 = \alpha$, we have :

$$E[(\bar{Z}_t)^2] - \alpha^2 \le \sum_{k=1}^{t} \Big(p(1-p) + \frac{\alpha}{Ns}\Big) \frac{1}{\big(\frac{m}{Cs} + k\big)^2}$$

It follows that:

$$\alpha - E[(\bar{Z}_t)^2] \ge \alpha - \alpha^2 - \sum_{k=1}^{t} \Big(p(1-p) + \frac{\alpha}{Ns}\Big) \frac{1}{(\frac{m}{Cs} + k)^2} \ge \alpha(1-\alpha)$$

It doesn't make sense to have a model in which the balls in the urns initially are either all white or all black. So, we can assume that a $\neq 0, m$ which gives:

$$\alpha = \frac{a}{m} \neq 0, 1$$

Therefore, we have that :

$$\alpha - E[(\bar{Z}_t)^2] > 0$$

Recall:

$$x_{t+1} = f_t x_t + g_t, \tag{2.9}$$

where $f_t = \Big(1 - \frac{Cs(1-p)}{m+Cs(t+1)}\Big)^2 + \frac{(N-1)C^2ps[s(1-p)-1]}{N(m+Cs(t+1)^2} = 1 + \frac{B}{\big(\frac{m}{Cs}+t+1\big)^2} - \frac{A}{\frac{m}{Cs}+t+1}$ and $g_t = \frac{(N-1)(\alpha - E[(\bar{Z}_t)^2])}{Ns\big(\frac{m}{Cs}+t+1\big)^2}$. Here, $A = 2(1-p)$ and $B = (1-p)^2 + \frac{(N-1)p(s(1-p)-1)}{Ns}$. Now,

$$B \le (1-p)^2 + \frac{(N-1)ps(1-p)}{Ns} = (1-p)^2 + \frac{(N-1)p(1-p)}{N}$$

Since, $\frac{N-1}{N} \le 1$,we have:

$$B \le (1-p)^2 + \frac{(N-1)p(1-p)}{N} \le (1-p)^2 + p(1-p) = 1 - p \le 1$$

Now, $0 \le \frac{(N-1)p}{N} \le 1$ and $\frac{(N-1)p}{Ns} \le \frac{(N-1)p}{N} \le 1$. So,

$$\frac{-(N-1)p}{Ns} \ge -1$$

Hence, $B \ge \frac{-(N-1)p}{Ns} \ge -1$. Thus, we have shown that:

$$-1 \le B \le 1$$

Now, $(\bar{Z}_t)^2$ is a bounded submartingale since $E[(\bar{Z}_t)^2] \le \alpha$. So,

$$\lim_{t\to\infty} E[(\bar{Z}_t)^2] = E[(Z_\infty)^2] = \sup_t E[(\bar{Z}_t)^2]$$

Hence,

$$0 < \frac{(N-1)(\alpha - E[(\bar{Z}_\infty)^2])}{Ns(\frac{m}{Cs} + t + 1)^2} \le g_t \le \frac{\alpha}{(\frac{m}{Cs} + t + 1)^2} \tag{2.10}$$

We now show that $0 < f_t < 1$. Note that,

$$f(0) = 1 + \frac{(1-p)^2}{(\frac{m}{Cs}+1)^2} - \frac{2(1-p)}{\frac{m}{Cs}+1} + \frac{(N-1)p[s(1-p)-1]}{sN(\frac{m}{Cs}+1)^2}$$

Considering $f(0)$ as a function of p and differentiating we get,

$$(f(o))' = \frac{2(p-1)}{(\frac{m}{Cs}+1)^2} + \frac{2}{\frac{m}{Cs}+1} + \frac{(N-1)(s-2ps-1)}{sN(\frac{m}{Cs}+1)^2} = \frac{2Nm + (s-1)C(N-1) + 2pCs}{CsN(\frac{m}{Cs}+1)^2}$$

Since $s \geq 1$ and $N \geq 1$, we have $(s-1)C(N-1) \geq 1$. Hence, $(f(o))' \geq 0$. At $p = 0$, we have :

$$f(0) = 1 + \frac{1}{(\frac{m}{Cs}+1)^2} - \frac{2}{\frac{m}{Cs}+1} = \left(1 - \frac{1}{\frac{m}{Cs}+1}\right)^2 \geq 0$$

So, $f(0) = 0$ only if $m = 0$. But since m is a strictly positive integer, we have $f(0)(p) > 0 \; \forall \; p \in [0,1]$. Now,

$$\begin{aligned} {f_t}' &= \frac{-2B}{(\frac{m}{Cs}+t+1)^3} + \frac{A}{(\frac{m}{Cs}+t+1)^2} = \frac{1}{(\frac{m}{Cs}+t+1)^3}\left[A\left(\frac{m}{Cs}+t+1\right) - 2B\right] \\ &= \frac{2}{(\frac{m}{Cs}+t+1)^3}\left\{(1-p)\left(\frac{m}{Cs}+t+\frac{p}{N}\right) + \frac{(N-1)p}{Ns}\right\} \geq 0 \end{aligned}$$

So, f_t is increasing in t. Clearly,

$$\lim_{t\to\infty} f_t = 1$$

Hence, $0 < f_t < 1 \; \forall \; t \in N$. Now set:

$$\theta_t := \frac{x_t}{\prod_{k=0}^{t-1} f_k}$$

By (2.9), we obtain $\theta_{t+1} = \theta_t + F(t)$, where $F(t) := \frac{g(t)}{\prod_{k=0}^{t} f_k}$. Since $\theta_0 = x_0 = 0$, we get:

$$\theta_t = \sum_{i=0}^{t-1} F_i$$

The above equation is same as:

$$x_t = \left(\prod_{k=0}^{t-1} f(k)\right) \sum_{i=0}^{t-1} F_i \tag{2.11}$$

Now, as $t \to \infty$:

$$\begin{aligned} \prod_{k=0}^{t-1} f(k) &= \exp\left[\sum_{k=0}^{t-1} log\left(1 + \frac{B}{\left(\frac{m}{Cs}+k+1\right)^2} - \frac{A}{\frac{m}{Cs}+k+1}\right)\right] \\ &= \exp\left[-2(1-p)\sum_{k=0}^{t-1} \frac{1}{k+\frac{m}{Cs}+1} + O(1)\right] \\ &= \exp\left[-2(1-p) log\left(t+\frac{m}{Cs}\right) + O(1)\right] \sim t^{-2(1-p)} \end{aligned} \tag{2.12}$$

Since by (2.10), we have $g(t) \sim t^{-2}$, hence $F(t) \sim t^{2(1-p)-2}$. Therefore,

$$\theta_t = \sum_{i=0}^{t-1} F(i) \sim \begin{cases} 1 & \frac{1}{2} < p < 1 \\ logt & p = \frac{1}{2} \\ t^{2(1-p)-1} & 0 \leq p < \frac{1}{2} \end{cases} \tag{2.13}$$

By (2.11), (2.12) and (2.13) the conclusion follows. □

We now present some CLT results for our model. Such results have been proven in [10]. For the time being, we switch to the terminology that has been used in [10] in order to state our results and to perform similar calculations as those that have been performed there. This will be more convenient for the reader. Hence, we replace $\bar{Z}_t$ with Z_t and p with $1-\alpha$. Also, we assume that $Y_t(j)/Cs = I_t(j)$ and $D_t(j) = Z_t(j) - Z_t$ for all $1 \leq j \leq N, t \geq 0$. Using the synchronization result proved above, we have that :

$$\lim_{t\to\infty} Z_t(i) = \lim_{t\to\infty} Z_t = Z(\text{say}) \text{ a.s. } \forall\ 1 \leq i \leq N$$

We state the following results (note that we still retain the assumption that $a_i = a\ \forall\ 1 \leq i \leq N$):

Theorem 2.1.3. $\sqrt{t}(Z_t - Z) \xrightarrow{stably} \mathcal{N}\left(0, \frac{Z-Z^2}{Ns}\right)$
The above sequence also converges in the sense of the almost sure conditional convergence with respect to the filtration $\mathcal{F}$.

Theorem 2.1.4. *For $1/2 < \alpha \leq 1$ and $1 \leq j \leq N$, we have*

$$\sqrt{t}(Z_t(j) - Z_t) \xrightarrow{stably} \mathcal{N}\left(0, \frac{(Z - \frac{Z^2}{s})(1 - \frac{1}{N})}{s(2\alpha - 1)}\right)$$

For $\alpha = 1/2$ and $1 \leq j \leq N$, we have

$$\frac{\sqrt{t}}{\sqrt{ln(t)}}(Z_t(j) - Z_t) \xrightarrow{stably} \mathcal{N}\left(0, \frac{(Z - \frac{Z^2}{s})(1 - \frac{1}{N})}{s}\right)$$

Theorem 2.1.5. *For $1/2 < \alpha \leq 1$ and $1 \leq j \leq N$, we have*

$$\sqrt{t}(Z_t(j) - Z) \xrightarrow{stably} \mathcal{N}\left(0, \frac{1}{s}\left[\frac{Z - Z^2}{N} + \frac{(Z - \frac{Z^2}{s})(1 - \frac{1}{N})}{2\alpha - 1}\right]\right)$$

For $\alpha = 1/2$ and $1 \leq j \leq N$, we have

$$\frac{\sqrt{t}}{\sqrt{ln(t)}}(Z_t(j) - Z) \xrightarrow{stably} \mathcal{N}\left(0, \frac{(Z - \frac{Z^2}{s})(1 - \frac{1}{N})}{s}\right)$$

We refer the readers to [10] for a discussion on stable convergence. Here, we would only like to mention that the notion of stable convergence is stronger than that of convergence in distribution. Therefore, the above results imply convergence in distribution to Gaussian random variables with zero mean and variance depending on s and N as stated above. Note that putting $s = 1$ reduces the above results to Theorems $3.1, 3.3$ and 3.4 of [10].

We can use a completely similar method to prove the above results as that used in [10] since the P'olya-type urn model is a generalization of the model studied there. In fact, we will see that their proofs follow identically for our model with some minor modifications. So, instead of writing out whole proofs with all the tiny details, we sketch out some of the analogous calculations performed by the authors of [10] for our model. A complete comprehension of how to prove the results will hence only be possible if the reader is familiar with [10]. We now give a short sketch of the main ideas involved in the proofs of the theorems stated above. Note that in our model, we have:

$$\begin{aligned} Z_{t+1}(j) - Z_t(j) &= \frac{Y_{t+1}(j) - CsZ_t(j)}{m + Cs(t+1)} \\ &= \frac{\frac{Y_{t+1}(j)}{Cs} - Z_t(j)}{\frac{m}{Cs} + t + 1} \\ &= \frac{I_{t+1}(j) - Z_t(j)}{\frac{m}{Cs} + t + 1} \end{aligned} \tag{2.14}$$

Hence,

$$Z_{t+1} - Z_t = \frac{\frac{\sum_{i=1}^N I_{t+1}(i)}{N} - Z_t}{\frac{m}{Cs} + t + 1}$$

Now,

$$E[I_{t+1}(i)|\mathcal{F}_t] = \frac{E[Y_{t+1}(j)|\mathcal{F}_t]}{Cs} = \frac{Cs[pZ_t(j) + (1-p)Z_t]}{Cs} = pZ_t(j) + (1-p)Z_t$$

Hence,

$$E\left[\frac{\sum_{i=1}^N I_{t+1}(j)}{N} |\mathcal{F}_t\right] = \frac{NpZ_t + N(1-p)Z_t}{N} = Z_t$$

Also,

$$E\left[Z_{t+1}(j) - Z_t(j)|\mathcal{F}_t\right] = \frac{Cs(1-p)[Z_t - Z_t(j)]}{m + Cs(t+1)} = \frac{(1-p)[Z_t - Z_t(j)]}{\frac{m}{Cs} + t + 1} = \frac{-\alpha D_t(j)}{\frac{m}{Cs} + t + 1}$$

Comparing with [10], the reader will notice that the details of the proof for our model are almost identical to those in [10] with some minor exceptions (for example, m gets replaced with m/Cs).

Now,

$$
\begin{aligned}
\sum_{i=1}^{N} Var[I_k(i)|\mathcal{F}_{k-1}] &= \sum_{i=1}^{N} \frac{1}{C^2 s^2}[E[(Y_k(i))^2|\mathcal{F}_{k-1}] - (E[Y_k(i)|\mathcal{F}_{k-1}])^2] \\
&= \frac{1}{C^2 s^2}\sum_{i=1}^{N}[C^2 ps(s-1)(Z_k(i))^2 + C^2 psZ_k(i) + C^2(1-p)s(s-1)(Z_k)^2 \\
&+ C^2(1-p)sZ_k - C^2 s^2(pZ_k(j) + (1-p)Z_k)^2] \\
&\xrightarrow[a.s.]{k\to\infty} \frac{1}{s^2}\sum_{i=1}^{N}[s(s-1)Z^2 + sZ - s^2 Z^2] \\
&= \frac{N(sZ - s^2 Z^2)}{s^2} \\
&= \frac{N(Z - Z^2)}{s}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{k^2}{(\frac{m}{Cs}+k)^2} E\left[\left(\frac{\sum_{i=1}^{N} I_k(i)}{N} - Z_{k-1}\right)^2 |\mathcal{F}_{k-1}\right] &= \frac{k^2}{(\frac{m}{Cs}+k)^2 N^2}\sum_{i=1}^{N} Var[I_k(i)|\mathcal{F}_{k-1}] \\
&\xrightarrow[a.s.]{k\to\infty} \frac{(Z-Z^2)N}{sN^2} = \frac{Z-Z^2}{sN}
\end{aligned}
$$

Therefore (for Theorem 2.1.3),

$$V = \frac{Z - Z^2}{sN}$$

For simplicity, let $D_t = D_t(j)$. Then,

$$
\begin{aligned}
E[D_{k+1}|\mathcal{F}_k] &= E[Z_{k+1}(j) - Z_{k+1}|\mathcal{F}_k] \\
&= Z_k(j) + \frac{Cs(1-p)[Z_k - Z_k(j)]}{m + Cs(k+1)} - Z_k \\
&= (Z_k(j) - Z_k)\left(1 - \frac{Cs(1-p)}{m + Cs(k+1)}\right) \\
&= D_k\left(1 - \frac{\alpha}{\frac{m}{Cs} + k + 1}\right)
\end{aligned}
$$

Using (2.14), we have:

$$
E[k^2(Z_{k+1}(j) - Z_k(j))^2|\mathcal{F}_k] = \frac{k^2}{(\frac{m}{Cs}+k+1)^2} E[(I_{k+1}(j) - Z_k(j))^2 \mathcal{F}_k]
$$

Now,

$$
\begin{aligned}
& E[(I_{k+1}(j) - Z_k(j))^2 \mathcal{F}_k] \\
= \quad & E[(I_{k+1}(j))^2 \mathcal{F}_k] + (Z_k(j))^2 - 2Z_k(j)E[I_{k+1}(j)|\mathcal{F}_k] \\
= \quad & \frac{C^2(1-p)sZ_k + C^2ps(s-1)(Z_k(j))^2 + C^2psZ_k(j) + C^2(1-p)s(s-1)(Z_k)^2}{C^2s^2} \\
+ \quad & (Z_k(j))^2 - \frac{2Z_k(j)Cs[pZ_k(j) + (1-P)Z_k]}{Cs} \\
\xrightarrow[a.s.]{k\to\infty} \quad & \frac{(1-p)sZ + ps(s-1)Z^2 + psZ + (1-p)s(s-1)Z^2}{s^2} + Z^2 - 2Z[pZ + (1-p)Z] \\
= \quad & \frac{(1-p)sZ + s(s-1)Z^2 + psZ + s^2Z^2 - 2s^2Z[pZ + (1-p)Z]}{s^2} \\
= \quad & \frac{Z - Z^2}{s}
\end{aligned}
$$

Hence,

$$
\lim_{k\to\infty} E[k^2(Z_{k+1}(j) - Z_k(j))^2|\mathcal{F}_k] = \frac{Z - Z^2}{s}
$$

Note that:

$$
\begin{aligned}
& E\left[(I_{k+1}(j) - Z_k(j))\left(\frac{\sum_{i=1}^N I_{k+1}(i)}{N} - Z_k\right)|\mathcal{F}_k\right] \\
= \quad & E[\frac{(I_{k+1}(j))^2}{N} + \sum_{\substack{i=1 \\ i\neq j}}^{N} \frac{I_{k+1}(i)I_{k+1}(j)}{N} - Z_kI_{k+1}(j) \\
- \quad & Z_k(j)\frac{\sum_{i=1}^N I_k(i)}{N} + Z_kZ_k(j)|\mathcal{F}_k]
\end{aligned} \tag{2.15}
$$

Now,

$$
E[I_{k+1}(j)|\mathcal{F}_k] = pZ_k(j) + (1-p)Z_k \xrightarrow[a.s.]{k\to\infty} Z
$$

Also,

$$
\begin{aligned}
& E[(I_{k+1}(j))^2|\mathcal{F}_k] \\
= \quad & \frac{E[(Y_{k+1}(j))^2|\mathcal{F}_k]}{C^2s^2} \\
= \quad & \frac{C^2ps(s-1)(Z_k(j)^2 + C^2psZ_k(j) + C^2(1-p)s(s-1)(Z_k)^2 + C^2(1-p)sZ_k}{C^2s^2} \\
\xrightarrow[a.s.]{k\to\infty} \quad & \frac{ps(s-1)Z^2 + psZ + (1-p)s(s-1)Z^2 + (1-p)sZ}{s^2} \\
= \quad & \frac{(s-1)Z^2 + Z}{s}
\end{aligned}
$$

Putting the convergent values calculated above in (2.15) after taking limit on both sides, we get:

$$
\lim_{k\to\infty} E\left[(I_{k+1}(j) - Z_k(j))\left(\frac{\sum_{i=1}^N I_{k+1}(i)}{N} - Z_k\right)|\mathcal{F}_k\right] = \frac{(s-1)Z^2 + Z}{Ns} + \frac{(N-1)Z^2}{N} - Z^2
$$

Now, we calculate (for Theorem 2.1.4):

$$V = \lim_{k\to\infty} E\left[Y_k|\mathcal{F}_k\right]$$

Here,

$$\begin{aligned} Y_k &= k^2[(Z_{k+1}(j)-Z_k(j))^2+(Z_{k+1}-Z_k)^2-2(Z_{k+1}(j)-Z_k(j))(Z_{k+1}-Z_k)] \\ &= k^2\left[\left(\frac{I_{k+1}(j)-Z_k(j)}{\frac{m}{Cs}+k+1}\right)^2+\left(\frac{\frac{\sum_{i=1}^{N} I_{k+1}(i)}{N}-Z_k}{\frac{m}{Cs}+k+1}\right)^2-2\frac{I_{k+1}(j)-Z_k(j)}{\frac{m}{Cs}+k+1}\frac{\frac{\sum_{i=1}^{N} I_{k+1}(i)}{N}-Z_k}{\frac{m}{Cs}+k+1}\right] \end{aligned}$$

From above calculations, we get that:

$$\begin{aligned} V &= \frac{Z-Z^2}{s}+\frac{Z-Z^2}{sN}+-2\left[\frac{(s-1)Z^2+Z}{sN}+\frac{(N-1)Z^2}{N}-Z^2\right] \\ &= \frac{(Z-\frac{Z^2}{s})(1-\frac{1}{N})}{s} \end{aligned}$$

2.2 Friedman-type model (Mutual reinforcement of colours)

All notations are the same as that of the previous model. The urn process is also the same except for one modification: for every k white balls drawn for a particular urn, we put back $C(s-k)$ white balls back in that same urn (instead of Ck), that is, instead of self reinforcement, mutual reinforcement of colours takes place. We now prove limit results for this model. The first part of the theorem below states a synchronization result, while the second part is the fluctuation theorem.

Theorem 2.2.1. *Let $Z_t = (Z_t(1), Z_t(2)\dots Z_t(N))$. The following holds:*

(a) For all $1 \le i \le N$,

$$\lim_{t\to\infty} Z_t(i) = \frac{1}{2} \text{ a.s.}$$

(b) $\sqrt{t}\,(Z_t-\theta)$ converges in distribution to $\mathcal{N}(0,\Sigma)$.

Here, $\theta = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2})$ and Σ is the covariance matrix and is given by the following integral:

$$\Sigma = \frac{1}{4s}\int_0^\infty \left(e^{(\nabla f(\theta)+\frac{Id}{2})u}\right)^T e^{(\nabla f(\theta)+\frac{Id}{2})u}du$$

and $\nabla f(\theta)$is the Jacobian of f evaluated at θ, where:

$$f(Z_t) = \frac{E[Z_{t+1}|F_t]}{Cs} - Z_t = \begin{bmatrix} 1-(1+p)Z_t(1)-(1-p)\bar{Z}(t) \\ 1-(1+p)Z_t(2)-(1-p)\bar{Z}(t) \\ . \\ \\ . \\ . \\ 1-(1+p)Z_t(N)-(1-p)\bar{Z}(t) \end{bmatrix}$$

We use the theory of stochastic approximation to prove the above theorem. A similar approach to get such a result has been used in [11] where the authors use a weakened version of a result by Zhang (Theorem 5 in [11], reproduced in the appendix).

Proof The Stochastic Approximation Scheme remains similar to that of the previous model, except that now we have that for $0 \le k \le s, 1 \le i \le N$:

$$P(Y_{t+1}(i) = Ck|\mathcal{F}_t) = p\binom{s}{k}(Z_t(i))^{s-k}(1 - Z_t(i))^k + (1-p)\binom{s}{k}(\bar{Z}_t)^{s-k}(1-\bar{Z}_t)^k$$

Hence,

$$\begin{aligned} E[Y_{t+1}(i)|\mathcal{F}_t] &= \sum_{k=0}^{s} C(s-k)p(Z_t(i))^k\binom{s}{k}(1-Z_t(i))^{s-k} \\ &+ \sum_{k=0}^{s} C(s-k)(1-p)(\bar{Z}_t)^k\binom{s}{k}(1-\bar{Z}_t)^{s-k} \\ &= Csp - CpsZ_t(i) + Cs(1-p) - C(1-p)s\bar{Z}_t \\ &= Cs[1 - pZ_t(i) - (1-p)\bar{Z}_t] \end{aligned}$$

Hence,

$$\begin{aligned} h(Z_t(i)) &= E[Y_{t+1}(i)|\mathcal{F}_t] - CsZ_t(i) \\ &= Cs[1 - pZ_t(i) - (1-p)\bar{Z}_t] - CsZ_t(i) \\ &= Cs[1 - (1+p)Z_t(i) - (1-p)\bar{Z}_t] \end{aligned}$$

This gives:

$$h(Z_t) = Cs\begin{pmatrix} 1-(1+p)Z_t(1)-(1-p)\bar{Z}_t \\ 1-(1+p)Z_t(2)-(1-p)\bar{Z}_t \\ \vdots \\ 1-(1+p)Z_t(N)-(1-p)\bar{Z}_t \end{pmatrix}$$

We calculate the zeroes of $h(Z_t)$:

$$1 - (1+p)Z_t(i) - (1-p)\bar{Z}_t = 0 \;\forall\; 1 \le i \le N$$

Rearranging the terms,

$$(1+p)Z_t(i) + (1-p)\bar{Z}_t = 1 \;\forall\; 1 \le i \le N$$

Adding all the N equations, we get :

$$(1+p)N\bar{Z}_t + (1-p)N\bar{Z}_t = N \;\forall\; 1 \le i \le N$$

Hence,

$$\bar{Z}_t = \frac{1}{2}$$

Putting the value of $\bar{Z}_t$ back in the N equations, we get:

$$Z_t(i) = \frac{1}{2} \; \forall \; 1 \leq i \leq N$$

Since the zeroes of $h(Z_t)$ give the limit points of Z_t, we are done proving the first part of the theorem. Now,

$$\begin{aligned}
h(Z_t(i)) &= Cs[1 - (1+p)Z_t(i) - (1-p)\bar{Z}_t] \\
&= Cs\left[1 - (1+p)Z_t(i) - \frac{(1-p)\sum_{i=1}^{N} Z_t(i)}{N}\right] \\
&= Cs\left[1 - (1 + p + \frac{1-p}{N})Z_t(i) - \frac{(1-p)\sum_{\substack{j=1 \\ j\neq i}}^{N} Z_t(j)}{N}\right]
\end{aligned}$$

Hence,

$$h(Z_t) = Cs\begin{pmatrix} 1 - (1+p+\frac{1-p}{N})Z_t(1) - \frac{1-p}{N}Z_t(2) - \frac{1-p}{N}Z_t(3)\ldots - \frac{1-p}{N}Z_t(N) \\ 1 - \frac{1-p}{N}Z_t(1) - (1+p+\frac{1-p}{N})Z_t(2) - \frac{1-p}{N}Z_t(3)\ldots - \frac{1-p}{N}Z_t(N) \\ \vdots \\ 1 - \frac{1-p}{N}Z_t(1) - \frac{1-p}{N}Z_t(2) - \frac{1-p}{N}Z_t(3)\ldots - (1+p+\frac{1-p}{N})Z_t(N) \end{pmatrix}$$

We calculate the Jacobian to get:

$$\begin{aligned}
-\nabla h =& Cs\begin{pmatrix} 1+p+\frac{1-p}{N} & \frac{1-p}{N} & \cdots & \frac{1-p}{N} \\ \frac{1-p}{N} & 1+p+\frac{1-p}{N} & \cdots & \frac{1-p}{N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1-p}{N} & \frac{1-p}{N} & \cdots & 1+p+\frac{1-p}{N} \end{pmatrix} \\
=& CsA
\end{aligned}$$

Here, A denotes the matrix above. Let $\frac{1-p}{N} = \alpha$. Then,$p = 1 - N\alpha$ and

$$1 + p + \frac{1-p}{N} = 1 + 1 - N\alpha + \alpha = 2 - \alpha(N-1)$$

So,

$$A = \begin{pmatrix} 2-\alpha(N-1) & \alpha & \cdots & \alpha \\ \alpha & 2-\alpha(N-1) & \cdots & \alpha \\ \vdots & \vdots & \vdots & \vdots \\ \alpha & \alpha & \cdots & 2-\alpha(N-1) \end{pmatrix}$$

Let $x = det(A - \lambda I)$. Then,

$$x = det\begin{pmatrix} 2-\alpha(N-1)-\lambda & \alpha & \cdots & \alpha \\ \alpha & 2-\alpha(N-1)-\lambda & \cdots & \alpha \\ \vdots & \vdots & \vdots & \vdots \\ \alpha & \alpha & \cdots & 2-\alpha(N-1)-\lambda \end{pmatrix}$$

$$= det \begin{pmatrix} 2-\alpha(N-1)-\lambda & \alpha & \cdots & \alpha \\ -(2-\alpha N-\lambda) & 2-\alpha(N-1)-\lambda & \cdots & \alpha \\ 0 & \alpha & \cdots & \vdots \\ 0 & \alpha & \cdots & \vdots \\ 0 & \alpha & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \alpha & \cdots & 2-\alpha(N-1)-\lambda \end{pmatrix}$$

Simplifying further, we get:

$$x = (2-\alpha N-\lambda)^{N-1} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & (N-1)\alpha \\ 0 & 0 & 0 & \cdots & -1 & 2-\alpha(N-1)-\lambda \end{pmatrix}$$

Hence,

$$\begin{aligned} x &= (2-\alpha N-\lambda)^{N-1}[2-\alpha(N-1)-\lambda+(N-1)\alpha] \\ &= (2-\lambda)(2-\alpha N-\lambda)^{N-1} \end{aligned}$$

Putting $x = 0$, we get $\lambda = 2, 2-\alpha N$. Now,

$$2-\alpha N = 2 - \frac{(1-p)N}{N} = 1+p$$

Hence, the eigenvalues of A are 2 and $1+p$. We use Theorem 5.1.3 (see appendix) with $f(Z_t) = h(Z_t)/Cs$ to conclude the proof. The eigen value of $-\nabla f$ with the largest real part is 2 which is greater than 1/2. For $t \geq 0$, $\mathbb{E}(\Delta\hat{M}_{t+1}\Delta\hat{M}_{t+1}{}^T|\mathcal{F}_t)$ is an $N \times N$ matrix whose non-diagonal entries are all zero because the random variables $\{Y_{t+1}(i)\}_{1\leq i\leq N}$ are pairwise independent $\forall\ t \geq 0$.
We calculate the diagonal entries of $\mathbb{E}(\Delta\hat{M}_{t+1}\Delta\hat{M}_{t+1}{}^T|\mathcal{F}_t)$ (label them $\mathbb{E}(\Delta\hat{M}_{t+1}\Delta\hat{M}_{t+1}{}^T|\mathcal{F}_t)(1), \ldots$ $\ldots, \mathbb{E}(\Delta\hat{M}_{t+1}\Delta\hat{M}_{t+1}{}^T|\mathcal{F}_t)(N)$ in the usual order). Then, for $1 \leq i \leq N$:

$$\begin{aligned} \mathbb{E}(\Delta\hat{M}_{t+1}\Delta\hat{M}_{t+1}{}^T|\mathcal{F}_t)(i) &= \frac{E[Y_{t+1}(i)^2|\mathcal{F}_t] - (E[Y_{t+1}(i)|\mathcal{F}_t])^2}{C^2s^2} \\ &= \frac{C^2s^2 + C^2s[(s-1)Z_t(i)^2 + Z_t(i)] - 2C^2s^2Z_t(i) - C^2s^2(1-Z_t(i))^2}{C^2s^2} \end{aligned}$$

Using the first part of the theorem proven above, we have (for $1 \leq i \leq N$):

$$\lim_{t\to\infty} \mathbb{E}(\Delta\hat{M}_{t+1}\Delta\hat{M}_{t+1}{}^T|\mathcal{F}_t)(i) = \frac{C^2s^2 + C^2s\left[(s-1)\left(\frac{1}{2}\right)^2 + \frac{1}{2}\right] - 2C^2s^2\left(\frac{1}{2}\right) - C^2s^2\left(1-\frac{1}{2}\right)^2}{C^2s^2} = \frac{1}{4s} \text{ a.s}$$

Hence,

$$\hat{\Gamma} = \frac{1}{4s}\mathbb{I}_{N\times N}$$

The second part of the theorem hence immediately follows.

□

As noted before, the above model is an extension of the Generalized Friedman's urn model studied in [7] which doesn't have the interaction aspect and has only a single urn. Note that theorem 2 of [7] follows directly from the result above by putting $N = 1$ and $p = 1$. Indeed, with these substitutions, we have:

$$-\nabla f = [2]$$

Using the above theorem, we calculate the covariance matrix (which is not a matrix anymore):

$$\sum = \frac{1}{4s}\int_0^\infty e^{\left(-2+\frac{1}{2}\right)u}e^{\left(-2+\frac{1}{2}\right)u}du = \frac{1}{12s}$$

Hence, the central limit theorem reduces to the following:

$$\sqrt{n}\left(\frac{W_n}{m+Csn} - \frac{1}{2}\right) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, \frac{1}{12s}\right)$$

Simplifying, we get :

$$\sqrt{n}\left(\frac{W_n - \frac{1}{2}(m+Csn)}{m+Csn}\right) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, \frac{1}{12s}\right)$$

Separating the terms leads to:

$$\frac{W_n - \frac{1}{2}Csn}{\frac{m}{\sqrt{n}} + Cs\sqrt{n}} - \frac{\frac{1}{2}m}{\frac{m}{\sqrt{n}} + Cs\sqrt{n}} \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, \frac{1}{12s}\right)$$

Taking limit and scaling by Cs gives:

$$\frac{W_n - \frac{1}{2}Csn}{\sqrt{n}} \xrightarrow[n\to\infty]{d} \mathcal{N}\left(0, \frac{C^2s}{12}\right)$$

The above equation is same as that in the statement of Theorem 2 in [7].

We now obtain expressions for expectation of $\bar{Z}_t$ and $Z_t(i)$ for $i \in \{1, 2, \ldots, N\}$ respectively.

Theorem 2.2.2. *(a)*

$$E[\bar{Z}_t] = \frac{\alpha(\frac{m}{Cs})(\frac{m}{Cs}-1) + \frac{mt}{Cs} + \frac{t(t-1)}{2}}{(\frac{m}{Cs}+t-1)(\frac{m}{Cs}+t)}$$

(b)

$$E[Z_t(i)] = \left(\frac{a_i}{m} - \alpha\right)\frac{(\frac{m}{Cs}-1)(\frac{m}{Cs})}{(t+\frac{m}{Cs}-1)(t+\frac{m}{Cs})} + \frac{t^2 + t(\frac{2m}{Cs}-1) + \frac{2m\alpha}{Cs}(\frac{m}{Cs}-1)}{2(t+\frac{m}{Cs})(t+\frac{m}{Cs}-1)} \quad \textit{for } 1 \le i \le N.$$

Proof Recall that:

$$Z_{t+1}(i) = Z_t(i) + \frac{Y_{t+1}(i) - CsZ_t(i)}{m + Cs(t+1)}$$

Taking conditional expectation with respect to $\mathcal{F}_t$ on both sides, we get:

$$\begin{aligned} E[Z_{t+1}(i)|\mathcal{F}_t] &= Z_t(i) + \frac{E[Y_{t+1}(i)|\mathcal{F}_t] - CsZ_t(i)}{m + Cs(t+1)} \\ &= Z_t(i) + \frac{Cs[1 - pZ_t(i) - (1-p)\bar{Z}_t] - CsZ_t(i)}{m + Cs(t+1)} \end{aligned} \tag{2.16}$$

This gives:

$$\begin{aligned} E[\bar{Z}_{t+1}|\mathcal{F}_t] &= \bar{Z}_t + \frac{Cs[1 - p\bar{Z}_t - (1-p)\bar{Z}_t] - Cs\bar{Z}_t}{m + Cs(t+1)} \\ &= \bar{Z}_t + \frac{Cs(1 - 2\bar{Z}_t)}{m + Cs(t+1)} \end{aligned}$$

Taking expectation on both sides, we get:

$$\begin{aligned} E[\bar{Z}_{t+1}] &= E[\bar{Z}_t] + \frac{Cs(1 - 2E[\bar{Z}_t])}{m + Cs(t+1)} \\ &= E[\bar{Z}_t]\left(1 - \frac{2Cs}{m + Cs(t+1)}\right) + \frac{Cs}{m + Cs(t+1)} \end{aligned}$$

Hence,

$$E[\bar{Z}_{t+1}] = f_t E[\bar{Z}_t] + g_t$$

Here, $f_t = 1 - \frac{2Cs}{m+Cs(t+1)}$and $g_t = \frac{Cs}{m+Cs(t+1)}$
Now,

$$\begin{aligned} \prod_{k=0}^{m'} f_k &= \prod_{k=0}^{m'} \left(1 - \frac{2Cs}{m + Cs(k+1)}\right) = \prod_{k=0}^{m'} \left(\frac{m + Cs(k-1)}{m + Cs(k+1)}\right) \\ &= \prod_{k=0}^{m'} \left(\frac{\frac{m}{Cs} + k - 1}{\frac{m}{Cs} + k + 1}\right) = \frac{(\frac{m}{Cs} - 1)(\frac{m}{Cs})}{(\frac{m}{Cs} + m')(\frac{m}{Cs} + m' + 1)} \end{aligned}$$

So,

$$\begin{aligned} \frac{g'_m}{\prod_{k=0}^{m'} f_k} &= \frac{Cs}{m + Cs(m'+1)} \frac{(\frac{m}{Cs} + m')(\frac{m}{Cs} + m' + 1)}{(\frac{m}{Cs} - 1)(\frac{m}{Cs})} \\ &= \frac{\frac{m}{Cs} + m'}{(\frac{m}{Cs})(\frac{m}{Cs} - 1)} \end{aligned}$$

Now,

$$\sum_{m'=0}^{t-1} \frac{g'_m}{\prod_{k=0}^{m'} f_k} = \frac{1}{(\frac{m}{Cs})(\frac{m}{Cs}-1)} \sum_{m'=0}^{t-1} \left(\frac{m}{Cs} + m'\right)$$
$$= \frac{1}{(\frac{m}{Cs})(\frac{m}{Cs}-1)} \left[\frac{t(t-1)}{2} + \frac{mt}{Cs}\right]$$

Hence,

$$E[\bar{Z}_t] = \left(\prod_{k=0}^{t-1} f_k\right)\left(E[\bar{Z}_0] + \sum_{m'=0}^{t-1} \frac{g'_m}{\prod_{k=0}^{m'} f_k}\right)$$
$$= \frac{(\frac{m}{Cs}-1)(\frac{m}{Cs})}{(\frac{m}{Cs}+t+1)(\frac{m}{Cs}+t)} \left\{\alpha + \frac{1}{(\frac{m}{Cs}-1)(\frac{m}{Cs})}\left(\frac{t(t-1)}{2} + \frac{mt}{Cs}\right)\right\}$$
$$= \frac{\alpha(\frac{m}{Cs})(\frac{m}{Cs}-1) + \frac{mt}{Cs} + \frac{t(t-1)}{2}}{(\frac{m}{Cs}+t-1)(\frac{m}{Cs}+t)}$$

Taking expectation on both sides of (2.16), we get:

$$E[Z_{t+1}(i)] = E[Z_t(i)]\left(1 - \frac{Cs(1+p)}{m+Cs(t+1)}\right) + \frac{Cs}{m+Cs(t+1)} - \frac{Cs(1-p)E[\bar{Z}_t]}{m+Cs(t+1)}$$
$$= f_t E[Z_t(i)] + g_t$$

Here, $f_t = 1 - \frac{Cs(1+p)}{m+Cs(t+1)}$ and $g_t = \frac{Cs}{m+Cs(t+1)} - \frac{Cs(1-p)E[\bar{Z}_t]}{m+Cs(t+1)}$

Now,

$$\prod_{k=0}^{m'} f_k = \prod_{k=0}^{m'}\left(1 - \frac{Cs(1+p)}{m+Cs(k+1)}\right) = \prod_{k=0}^{m'} \frac{\frac{m}{Cs}+k-p}{\frac{m}{Cs}+k+1}$$
$$= \frac{(\frac{m}{Cs}-p+m')!(\frac{m}{Cs})!}{(\frac{m}{Cs}-p-1)!(\frac{m}{Cs}+m'+1)!} = \frac{\binom{\frac{m}{Cs}-p+m'}{m'+1}}{\binom{\frac{m}{Cs}+1+m'}{m'+1}}$$

So,

$$\frac{g'_m}{\prod_{k=0}^{m'} f_k} = \frac{\binom{\frac{m}{Cs}+1+m'}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}} \left\{\frac{Cs}{m+Cs(m'+1)} - \frac{Cs(1-p)E[\bar{Z}'_m]}{m+Cs(m'+1)}\right\}$$
$$= \frac{Cs}{m} \frac{\binom{\frac{m}{Cs}+m'}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}} [1 - (1-p)E[\bar{Z}_{m'}]]$$

Now,

$$E[\bar{Z}_{m'}] \frac{\binom{\frac{m}{Cs}+m'}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}} = \frac{\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}(\frac{m}{Cs}-1)(\frac{m}{Cs}-2)} \frac{\alpha(\frac{m}{Cs})(\frac{m}{Cs}-1) + \frac{mm'}{Cs} + \frac{m'(m'-1)}{2}}{(\frac{m}{Cs}+m'-1)(\frac{m}{Cs}+m')}$$

Hence,

$$
\begin{aligned}
\frac{g'_m}{\prod_{k=0}^{m'} f_k} &= \frac{Cs}{m}\frac{\binom{\frac{m}{Cs}+m'}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}} - \frac{(1-p)\alpha}{(\frac{m}{Cs}-2)}\frac{\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}} \\
&- \frac{(1-p)m'\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}(\frac{m}{Cs}-1)(\frac{m}{Cs}-2)} - \frac{(1-p)Csm'(m'-1)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{2m\binom{\frac{m}{Cs}-p+m'}{m'+1}(\frac{m}{Cs}-1)(\frac{m}{Cs}-2)} \\
&= \frac{Cs}{m}\frac{\binom{\frac{m}{Cs}+m'}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}} - \frac{(1-p)\alpha}{(\frac{m}{Cs}-2)}\frac{\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}} \\
&- \frac{(1-p)(\frac{m}{Cs}-p+m'-\frac{m}{Cs}+p)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}(\frac{m}{Cs}-1)(\frac{m}{Cs}-2)} \\
&- \frac{(1-p)Cs(\frac{m}{Cs}-p+m'-\frac{m}{Cs}+p)(m'-1)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{2m\binom{\frac{m}{Cs}-p+m'}{m'+1}(\frac{m}{Cs}-1)(\frac{m}{Cs}-2)} \\
&= \frac{Cs}{m}\frac{\binom{\frac{m}{Cs}+m'}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}} - \frac{(1-p)\alpha}{(\frac{m}{Cs}-2)}\frac{\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}} \\
&- \frac{(1-p)(\frac{m}{Cs}-p+m')\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}(\frac{m}{Cs}-1)(\frac{m}{Cs}-2)} + \frac{(1-p)(\frac{m}{Cs}-p)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}(\frac{m}{Cs}-1)(\frac{m}{Cs}-2)} \\
&- \frac{(1-p)Cs(\frac{m}{Cs}-p+m')(m'-1)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{2m\binom{\frac{m}{Cs}-p+m'}{m'+1}(\frac{m}{Cs}-1)(\frac{m}{Cs}-2)} + \frac{(1-p)Cs(\frac{m}{Cs}-p)(m'-1)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{2m\binom{\frac{m}{Cs}-p+m'}{m'+1}(\frac{m}{Cs}-1)(\frac{m}{Cs}-2)}
\end{aligned}
\tag{2.17}
$$

Now,

$$
\begin{aligned}
\frac{(m'-1)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}} &= \frac{(\frac{m}{Cs}-p+m')\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}} - \frac{(\frac{m}{Cs}-p+1)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}} \\
&= \frac{(\frac{m}{Cs}-p-1)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p-1+m'}{m'+1}} - \frac{(\frac{m}{Cs}-p+1)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}}
\end{aligned}
$$

Similarly,

$$
\frac{(m'-1)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p-1+m'}{m'+1}} = \frac{(\frac{m}{Cs}-p-2)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p-2+m'}{m'+1}} - \frac{(\frac{m}{Cs}-p)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p-1+m'}{m'+1}} \tag{2.18}
$$

Now,

$$
\begin{aligned}
\frac{(\frac{m}{Cs}-p+m')(m'-1)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}} &= \frac{(\frac{m}{Cs}-p-1)(m'-1)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'-1}{m'+1}} \\
&= (\frac{m}{Cs}-p-1)\left(\frac{(\frac{m}{Cs}-p-2)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p-2+m'}{m'+1}} - \frac{(\frac{m}{Cs}-p)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p-1+m'}{m'+1}}\right)
\end{aligned}
$$

(using (2.18))

Putting the values of $\frac{(\frac{m}{Cs}-p+m')(m'-1)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}}$ and $\frac{(m'-1)\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}}$ calculated above in the last two terms of (2.17) and clubbing similar terms together, we get :

$$\begin{aligned}
\frac{g'_m}{\prod_{k=0}^{m'} f_k} &= \frac{Cs}{m}\frac{\binom{\frac{m}{Cs}+m'}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}} + \frac{\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}}\left\{-\frac{(1-p)\alpha}{\frac{m}{Cs}-2} + \frac{(1-p)(\frac{m}{Cs}-p)}{(\frac{m}{Cs}-2)(\frac{m}{Cs}-1)}\right. \\
&- \left.\frac{(1-p)(\frac{m}{Cs}-p)Cs(1+\frac{m}{Cs}-p)}{2m(\frac{m}{Cs}-2)(\frac{m}{Cs}-1)}\right\} + \frac{\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p-1+m'}{m'+1}}\left\{-\frac{(1-p)(\frac{m}{Cs}-p-1)}{(\frac{m}{Cs}-2)(\frac{m}{Cs}-1)}\right. \\
&+ \left.\frac{(1-p)(\frac{m}{Cs}-p)Cs(\frac{m}{Cs}-p-1)}{2m(\frac{m}{Cs}-2)(\frac{m}{Cs}-1)} + \frac{(1-p)Cs(\frac{m}{Cs}-p-1)(\frac{m}{Cs}-p)}{2m(\frac{m}{Cs}-2)(\frac{m}{Cs}-1)}\right\}
\end{aligned}$$

Now,

$$\begin{aligned}
E[Z_t(i)] &= \left(\prod_{k=0}^{t-1} f_k\right)\left(E[Z_0(i)] + \sum_{m'=0}^{t-1}\frac{g'_m}{\prod_{k=0}^{m'} f_k}\right) \\
&= \frac{\binom{\frac{m}{Cs}-p+t-1}{t}}{\binom{\frac{m}{Cs}+t}{t}}\left(\frac{a_i}{m} + \sum_{m'=0}^{t-1}\frac{g'_m}{\prod_{k=0}^{m'} f_k}\right) \qquad (2.19)
\end{aligned}$$

Using Lemma 5.2.1, we calculate the following:

$$\sum_{m'=0}^{t-1}\frac{\binom{\frac{m}{Cs}+m'}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}} = \frac{(t+\frac{m}{Cs}-p)\binom{t+\frac{m}{Cs}}{t+1}}{(1+p)\binom{t+\frac{m}{Cs}-p}{t+1}} - \frac{m}{Cs(1+p)}$$

$$\sum_{m'=0}^{t-1}\frac{\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p+m'}{m'+1}} = \frac{(t+\frac{m}{Cs}-p)\binom{t+\frac{m}{Cs}-2}{t+1}}{(p-1)\binom{t+\frac{m}{Cs}-p}{t+1}} - \frac{\frac{m}{Cs}-2}{p-1}$$

$$\sum_{m'=0}^{t-1}\frac{\binom{\frac{m}{Cs}+m'-2}{m'+1}}{\binom{\frac{m}{Cs}-p-1+m'}{m'+1}} = \frac{(t+\frac{m}{Cs}-p-1)\binom{t+\frac{m}{Cs}-2}{t+1}}{p\binom{t+\frac{m}{Cs}-p-1}{t+1}} - \frac{\frac{m}{Cs}-2}{p}$$

Using the values of these sums, we calculate $\sum_{m'=0}^{t-1}\frac{g'_m}{\prod_{k=0}^{m'} f_k}$.
After putting the value of $\sum_{m'=0}^{t-1}\frac{g'_m}{\prod_{k=0}^{m'} f_k}$ in (2.19) and simplifying, we finally get :

$$E[Z_t(i)] = \left(\frac{a_i}{m}-\alpha\right)\frac{\binom{t+\frac{m}{Cs}-p-1}{t}}{\binom{t+\frac{m}{Cs}}{t}} + \frac{t^2+t(\frac{2m}{Cs}-1)+\frac{2m\alpha}{Cs}(\frac{m}{Cs}-1)}{2(t+\frac{m}{Cs})(t+\frac{m}{Cs}-1)}$$

□

To obtain the result in [7], we substitute $p = 1$ in the equation above.

$$\begin{aligned}
E[Z_t(i)] &= \left(\frac{a_i}{m} - \alpha\right) \frac{\binom{t+\frac{m}{Cs}-2}{t}}{\binom{t+\frac{m}{Cs}}{t}} + \frac{t^2 + t(\frac{2m}{Cs} - 1) + \frac{2m\alpha}{Cs}(\frac{m}{Cs} - 1)}{2(t + \frac{m}{Cs})(t + \frac{m}{Cs} - 1)} \\
&= \left(\frac{a_i}{m} - \alpha\right) \frac{(\frac{m}{Cs} - 1)(\frac{m}{Cs})}{(t + \frac{m}{Cs} - 1)(t + \frac{m}{Cs})} + \frac{t^2 + t(\frac{2m}{Cs} - 1) + \frac{2m\alpha}{Cs}(\frac{m}{Cs} - 1)}{2(t + \frac{m}{Cs})(t + \frac{m}{Cs} - 1)} \\
&= \frac{2ma_i - 2a_iCs + t^2C^2s^2 + 2mtCs - tC^2s^2}{2(m + Cs(t-1))(m + Cst)}
\end{aligned}$$

Switching to the notation used in [7], we make the following substitutions: $a_i = W_0$, $m = T_0$, $t = n$ and $Z_t(i) = W_n$, to obtain:

$$\frac{E[W_n]}{T_0 + Csn} = \frac{2W_0(T_0 - Cs) + C^2s^2n(n-1) + 2T_0Csn}{2(T_0 + Cs(n-1))(T_0 + Csn)}$$

That is,

$$E[W_n] = \frac{2W_0(T_0 - Cs) + C^2s^2n(n-1) + 2T_0Csn}{2(T_0 + Cs(n-1))}$$

The above expression is the same as the first moment calculated by the authors in [7].

Chapter 3

Interacting d-colour Urns with Multiple Drawings

3.1 A general d- colour urn model

3.1.1 Model description

Consider N urns each with balls of d colours, where the colours are labelled as $C_1, C_2, \ldots, C_d$. We still retain the assumption that each urn has a fixed number of total balls in the beginning of the process. We sample s balls after a coin toss like before, that is, for a particular urn, we draw from that same urn if a heads shows up and draw from the super urn in case of a tails. However, since there are more than two colours, there are multiple ways in which reinforcement can occur (unlike the two colour case where the reinforcement could take place only in two ways). We assume that each colour is reinforced by one and only one other colour and that the reinforcement scheme is fixed for each urn in terms of which colour reinforces which. The reinforcement scheme is defined below. We first introduce some definitions.

Definition 3.1.1 (Full mixing)**.** *A set of $m \geq 2$ colours, denoted by $\{C_1, C_2, \ldots, C_m\}$, is said to undergo full mixing when for each $i \in [m]$, C_i is reinforced by C_j for some $j \in [m], j \neq i$.*

Here, $[m]$ denotes the set $\{1, 2, \ldots, m\}$. Note that the assumption that each colour is reinforced by one and only one other colour forces an order of the form $C_{i_1} \to C_{i_2} \to \ldots \to C_{i_m}$, where $\{i_1, i_2, \ldots, i_m\} = \{1, 2, \ldots, m\}$ and the notation $C_{i_k} \to C_{i_j}$ means that the colour C_{i_k} reinforces the colour C_{i_j}.

Definition 3.1.2 (Non-trivial mixing)**.** *For $r, k_1, k_2, \ldots, k_r, l \in \mathbb{N} \cup \{0\}$ such that $2 \leq k_i \leq d \;\forall\; 1 \leq i \leq r$ and $\sum_{i=1}^{r} k_i + l = d$, we call the reinforcement scheme a $\{k_1, k_2, \ldots, k_r, l\}$-mixing if for every k_i $(1 \leq i \leq r)$, the set of colours $\{C_{\sum_{j=1}^{i} k_{j-1}+1}, \ldots, C_{\sum_{j=1}^{i} k_{j-1}+k_i}\}$ (assume $k_0 = 0$) are mixed fully in the sense of definition 3.1.1 and the rest of the l colours reinforce themselves, that is, if $j \in \{\sum_{i=1}^{r} k_i + 1, \ldots, d\}$, then C_j reinforces itself.*

Note that if $l = d$, we have a Pólya type reinforcement and each colour reinforces itself. For each $1 \leq i \leq r$, the set $\{C_{\sum_{j=1}^{i} k_{j-1}+1}, \ldots, C_{\sum_{j=1}^{i} k_{j-1}+k_i}\}$(where we assume that the set of colours $\{C_{\sum_{j=1}^{i} k_{j-1}+1}, \ldots, C_{\sum_{j=1}^{i} k_{j-1}+k_i}\}$ undergoes full mixing) is called a *mixing set*.

Thus, every partition of $[d]$ defines a unique number of mixing sets and therefore a different partition leads to a different reinforcement scheme for the urn system. Renumbering or renaming the colours, if necessary, we can assume that the colours reinforce each other according to the scheme given below (the following holds for all $1 \leq i \leq r$ while assuming that $k_0 = 0$):

$$C_{\sum_{j=1}^{i} k_{j-1}+1} \to C_{\sum_{j=1}^{i} k_{j-1}+2} \ldots, C_{\sum_{j=1}^{i} k_{j-1}+k_i-1} \to C_{\sum_{j=1}^{i} k_{j-1}+k_i} \to C_{\sum_{j=1}^{i} k_{j-1}+1}$$

We use the following notations: for $1 \leq i \leq N, 1 \leq j \leq d$, let $Z_{ij}(t)$ denote the fraction of balls of the colour C_j in the i^{th} urn at time t. For $1 \leq j \leq d$, let $Z_j(t)$ denote the fraction of balls of the j^{th} colour in the super urn at time t. Then, $Z_j(t) = \sum_{i=1}^{N} Z_{ij}(t)/N$ for $1 \leq j \leq d$. In order to denote a vector A with Nd entries, we divide the entries of that vector into N blocks, so that the vector can be thought of as being composed of the entries of N d-vectors placed sequentially next to each other. More precisely, $A = (A_1, A_2, A_3 \ldots A_N)$ is an Nd-dimensional vector, obtained by sequentially placing the entries of the d-vectors, where each A_i is a d-dimensional vector.

3.1.2 Main Results and Proofs

We show that if the urn process is given an infinite amount of time, not only do the colours synchronise across urns (that is, the fraction of balls of a particular colour becomes the same across all the urns), but also that the fraction of balls of colours belonging to the same mixing set become equal almost surely.

Theorem 3.1.3. *For $1 \leq i, j \leq N, i \neq j$,*

$$\lim_{t \to \infty} (Z_{il}(t) - Z_{jl}(t)) = 0 \quad a.s. \; \forall \;\; 1 \leq l \leq d.$$

Theorem 3.1.4. *Suppose $1 \leq i, j \leq d, i \neq j$ are such that C_i and C_j belong to the same mixing set. Then,*

$$\lim_{t \to \infty} (Z_{li}(t) - Z_{lj}(t)) = 0 \quad a.s. \; \forall \;\; 1 \leq l \leq N.$$

We prove both the theorems simultaneously using the method of Stochastic Approximation.

Proof Let $Z_t = (Z_{11}(t), Z_{12}(t) \ldots Z_{1d}(t), Z_{21}(t) \ldots Z_{2d}(t) \ldots Z_{N1}(t), Z_{N2}(t) \ldots Z_{Nd}(t))$.
Note the sequence in which we have numbered the elements of Z_t: we first write the fraction of balls of the d colours of the first urn, followed by the fraction of balls for the d colours of the second urn and so on. This will be important while writing the Stochastic Approximation scheme for Z_t.
For $1 \leq j \leq d$, $1 \leq i \leq N$, we define:
r_{ij}:= number of balls drawn for the i^{th} urn (j^{th} colour type) at time t.
Let $\sum_{i=1}^{r} k_i = a$.
Let Y_{t+1} denote the vector whose entries consist of the number of balls of different colours added to each urn at time t (sequenced in a manner identical to Z_t).

So, $Y_{t+1} = C(B_1, B_2 \dots B_N)$, where for $1 \le i \le d$, each B_i is a d-vector defined as follows:

$$B_i = \big(r_{ik_1}, r_{i1}, r_{i2}, \dots r_{i(k_1-1)}, r_{i(k_1+k_2)}, r_{i(k_1+1)} \dots, r_{i(k_1+k_2-1)}, r_{i(k_1+k_2+k_3)}, r_{i(k_1+k_2+1)} \dots r_{i(k_1+k_2+k_3-1)} \dots r_{i(k_1+k_2+k_3\dots+k_r-1)}, r_{i(a+1)}, r_{i(a+2)} \dots r_{id}\big)$$

Now,

$$E[Y_{t+1}|\mathcal{F}_t] = \sum_{\substack{\sum_{j=1}^d r_{ij}=s \ \forall \ 1\le i\le N \\ 0\le r_{ij}\le s \ \forall \ 1\le i\le N, 1\le j\le d}} C(B_1, B_2 \dots B_N) \prod_{m'=1}^{N} \binom{s}{r_{m'1}, r_{m'2} \dots r_{m'd}} \times \left\{ p \prod_{q=1}^{d} (Z_{m'q}(t))^{r_{m'q}} + (1-p) \prod_{q=1}^{d} (Z_q(t))^{r_{m'q}} \right\}$$

$E[Y_{t+1}|\mathcal{F}_t]$ is itself an Nd-vector (represented as a column matrix).
Let

$$E[Y_{t+1}|\mathcal{F}_t] = (E[Y_{t+1}(1)|\mathcal{F}_t], E[Y_{t+1}(2)|\mathcal{F}_t] \dots E[Y_{t+1}(Nd)|\mathcal{F}_t])^T$$

Let us calculate $E[Y_{t+1}(1)|\mathcal{F}_t]$:

$$\begin{aligned}
E[Y_{t+1}(1)|\mathcal{F}_t] &= \sum_{\substack{\sum_{j=1}^d r_{ij}=s \ \forall \ 1\le i\le N \\ 0\le r_{ij}\le s \ \forall \ 1\le i\le N, 1\le j\le d}} C r_{1k_1} \prod_{m'=1}^{N} \binom{s}{r_{m'1}, r_{m'2} \dots r_{m'd}} \\
&\times \left\{ p \prod_{q=1}^{d} (Z_{m'q}(t))^{r_{m'q}} + (1-p) \prod_{q=1}^{d} (Z_q(t))^{r_{m'q}} \right\} \\
&= \sum_{\substack{\sum_{j=1}^d r_{1j}=s \\ 0\le r_{1j}\le s \ \forall \ 1\le j\le d}} C r_{1k_1} \binom{s}{r_{11}, r_{12} \dots r_{1d}} \left\{ p \prod_{q=1}^{d} (Z_{1q}(t))^{r_{1q}} + (1-p) \prod_{q=1}^{d} (Z_q(t))^{r_{1q}} \right\} \\
&\times \left(\sum_{\substack{\sum_{j=1}^d r_{ij}=s \ \forall \ 2\le i\le N \\ 0\le r_{ij}\le s \ \forall \ 2\le i\le N, 1\le j\le d}} \prod_{m'=2}^{N} \binom{s}{r_{m'1}, r_{m'2} \dots r_{m'd}} \left\{ p \prod_{q=1}^{d} (Z_{m'q}(t))^{r_{m'q}} \right. \right. \\
&+ \left. \left. (1-p) \prod_{q=1}^{d} (Z_q(t))^{r_{m'q}} \right\} \right)
\end{aligned}$$

Now,

$$\begin{aligned}
&\sum_{\substack{\sum_{j=1}^d r_{ij}=s \ \forall \ 2\le i\le N \\ 0\le r_{ij}\le s \ \forall \ 2\le i\le N, 1\le j\le d}} \prod_{m'=2}^{N} \binom{s}{r_{m'1}, r_{m'2} \dots r_{m'd}} \left\{ p \prod_{q=1}^{d} (Z_{m'q}(t))^{r_{m'q}} + (1-p) \prod_{q=1}^{d} (Z_q(t))^{r_{m'q}} \right\} \\
&= \prod_{m'=2}^{N} \sum_{\substack{\sum_{j=1}^d r_{ij}=s \ \forall \ 2\le i\le N \\ 0\le r_{ij}\le s \ \forall \ 2\le i\le N, 1\le j\le d}} \binom{s}{r_{m'1}, r_{m'2} \dots r_{m'd}} \left\{ p \prod_{q=1}^{d} (Z_{m'q}(t))^{r_{m'q}} + (1-p) \prod_{q=1}^{d} (Z_q(t))^{r_{m'q}} \right\} = 1
\end{aligned}$$

Hence,

$$\begin{aligned}
& E[Y_{t+1}(1)|\mathcal{F}_t] \\
= & \sum_{\substack{\sum_{j=1}^{d} r_{1j}=s \\ 0\le r_{1j}\le s\ \forall\ 1\le j\le d}} C r_{1k_1} \binom{s}{r_{11}, r_{12}\dots r_{1d}} \left\{ p\prod_{q=1}^{d} (Z_{1q}(t))^{r_{1q}} + (1-p)\prod_{q=1}^{d} (Z_q(t))^{r_{1q}} \right\} \\
= & \; C \sum_{\substack{\sum_{j=1}^{d} r_{1j}=s \\ 0\le r_{1j}\le s\ \forall\ 1\le j\le d, j\neq k_1 \\ 1\le r_{1k_1}\le s}} r_{1k_1} \binom{s}{r_{11}, r_{12}\dots r_{1d}} \left\{ p\prod_{q=1}^{d} (Z_{1q}(t))^{r_{1q}} + (1-p)\prod_{q=1}^{d} (Z_q(t))^{r_{1q}} \right\} \\
= & \; CspZ_{1k_1}(t) \sum_{\substack{\sum_{j=1}^{d} r_{1j}=s \\ 0\le r_{1j}\le s\ \forall\ 1\le j\le d, j\neq k_1 \\ 1\le r_{1k_1}\le s}} r_{1k_1} \binom{s-1}{r_{11}, r_{12}\dots r_{1k_1}-1\dots r_{1d}} \prod_{\substack{q=1 \\ q\neq k_1}}^{d} (Z_{1q}(t))^{r_{1q}}((Z_{1k_1}(t))^{r_{1k_1}-1}) \\
+ & \; Cs(1-p)Z_{k_1}(t) \sum_{\substack{\sum_{j=1}^{d} r_{1j}=s \\ 0\le r_{1j}\le s\ \forall\ 1\le j\le d, j\neq k_1 \\ 1\le r_{1k_1}\le s}} r_{1k_1} \binom{s-1}{r_{11}, r_{12}\dots r_{1k_1}-1\dots r_{1d}} \prod_{\substack{q=1 \\ q\neq k_1}}^{d} (Z_q(t))^{r_{1q}}((Z_{k_1}(t))^{r_{1k_1}-1})
\end{aligned}$$

Since $r_{1k_1} \geq 1$ and $\sum_{j=1}^{d} r_{1j} = s$, we have that : $r_{1j} \neq s$ for $1 \leq j \leq d, j \neq k_1$.
Let $r_{1k_1} - 1 = f$
Hence,

$$\begin{aligned}
& \sum_{\substack{\sum_{j=1}^{d} r_{1j} \\ 0\le r_{1j}\le s\ \forall\ 1\le j\le d, j\neq k_1 \\ 1\le r_{1k_1}\le s}} \binom{s-1}{r_{11}, r_{12}\dots r_{1k_1}-1\dots r_{1d}} \prod_{\substack{q=1 \\ q\neq k_1}}^{d} (Z_{1q}(t))^{r_{1q}}((Z_{1k_1}(t))^{r_{1k_1}-1}) \\
& = \sum_{\substack{\sum_{\substack{j=1 \\ j\neq k_1}}^{d} r_{1j}+f=s-1 \\ 0\le r_{1j}\le s-1\ \forall\ 1\le j\le d, j\neq k_1 \\ 0\le f\le s-1}} \binom{s-1}{r_{11}, r_{12}\dots f\dots r_{1d}} (\prod_{\substack{q=1 \\ q\neq k_1}}^{d} (Z_{1q}(t))^{r_{1q}})((Z_{1k_1}(t))^{f}) \\
& = 1
\end{aligned}$$

Similarly,

$$\sum_{\substack{\sum_{j=1}^{d} r_{1j}=s \\ 0\le r_{1j}\le s\ \forall\ 1\le j\le d, j\neq k_1 \\ 1\le r_{1k_1}\le s}} \binom{s-1}{r_{11}, r_{12}\dots r_{1k_1}-1\dots r_{1d}} \prod_{\substack{q=1 \\ q\neq k_1}}^{d} (Z_q(t))^{r_{1q}}((Z_{k_1}(t))^{r_{1k_1}-1}) = 1$$

Therefore, we have :

$$E[Y_{t+1}(1)|\mathcal{F}_t] = Cs(pZ_{1k_1}(t) + (1-p)Z_{k_1}(t))$$

We can calculate the other terms of $E[Y_{t+1}|\mathcal{F}_t]$ in a similar manner.

Now,

$$Z_{t+1} = \frac{Z_t(m+Cst)+Y_{t+1}}{m+Cs(t+1)} = Z_t + \frac{Y_{t+1}-CsZ_t}{m+Cs(t+1)}$$

Hence,

$$h(Z_t) = E[Y_{t+1}|\mathcal{F}_t] - CsZ_t$$

Let $h(Z_t) = Cs(E_1, E_2, E_3.....E_d)^T$
For $1 \leq i \leq d$,we have that:

$$E_i = \begin{pmatrix} pZ_{ik_1}(t) + (1-p)Z_{k_1}(t) - Z_{i1}(t) \\ pZ_{i1}(t) + (1-p)Z_1(t) - Z_{i2}(t) \\ \vdots \\ pZ_{i(k_1-1)}(t) + (1-p)Z_{(k_1-1)}(t) - Z_{ik_1}(t) \\ pZ_{i(k_1+k_2)}(t) + (1-p)Z_{(k_1+k_2)}(t) - Z_{i(k_1+1)}(t) \\ pZ_{i(k_1+1)}(t) + (1-p)Z_{(k_1+1)}(t) - Z_{i(k_1+2)}(t) \\ \vdots \\ pZ_{i(k_1+k_2-1)}(t) + (1-p)Z_{(k_1+k_2-1)}(t) - -Z_{i(k_1+k_2)}(t) \\ \vdots \\ pZ_{i(k_1+k_2+\ldots k_r-1)}(t) + (1-p)Z_{(k_1+k_2+\ldots k_r-1)}(t) - Z_{i(k_1+k_2+\ldots k_r)}(t) \\ pZ_{i(a+1)}(t) + (1-p)Z_{(a+1)}(t) - Z_{i(a+1)}(t) \\ \vdots \\ pZ_{id}(t) + (1-p)Z_d(t) - Z_{id}(t) \end{pmatrix}$$

We solve the set of linear equations obtained by setting $h(Z_t) = 0$ in order to study the limit points of Z_t.
For 1$\leq i \leq$ d, we label the d terms of E_i as follows:
Let $E_i = (E_{i1}, E_{i2}......E_{id})$.
We put the Nd linear terms equal to zero in an appropriate sequence:
Putting E_{ij} equal to zero for a+1 $\leq j \leq$ d , 1 $\leq i \leq$ N, we get:

$$Z_{ij}(t) = Z_j(t) \; \forall \; a+1 \leq j \leq d, 1 \leq i \leq N$$

Hence, we have:

$$\begin{aligned} &Z_{1(a+1)}(t) = Z_{2(a+1)}(t) = Z_{3(a+1)}(t) = \ldots Z_{N(a+1)}(t) = Z_{(a+1)}(t) \\ &Z_{1(a+2)}(t) = Z_{2(a+2)}(t) = Z_{3(a+2)}(t) = \ldots Z_{N(a+2)}(t) = Z_{(a+2)}(t) \\ &\vdots \\ &Z_{1d}(t) = Z_{2d}(t) = Z_{3d}(t) = \ldots Z_{Nd}(t) = Z_d(t) \end{aligned}$$

Hence, we have shown the synchronisation (of the colours which reinforce themselves) across the urns. Putting E_{i1} equal to 0 for $1 \leq i \leq$ N and adding up all the N equations together, we get:

$$p\sum_{i=1}^{N} Z_{ik_1}(t) + (1-p)NZ_{k_1}(t) - \sum_{i=1}^{N} Z_{i1}(t) = 0$$

That is,

$$pNZ_{k_1}(t) + (1-p)NZ_{k_1}(t) - NZ_1(t) = 0$$

This gives:

$$Z_{k_1}(t) = Z_1(t)$$

Putting E_{i2} equal to 0 for $1 \leq i \leq$ N and adding up all the N equations together, we similarly get:

$$Z_1(t) = Z_2(t)$$

Proceeding in the same way, we finally get:

$$\begin{aligned}
&Z_1(t) = Z_2(t) = Z_3(t) = \ldots = Z_{k_1}(t)\\
&Z_{(k_1+1)}(t) = Z_{(k_1+2)}(t) = Z_{(k_1+3)}(t) = \ldots = Z_{(k_1+k_2)}(t)\\
&\vdots\\
&Z_{(k_1+k_2+k_3+.....k_{r-1}+1)}(t) = Z_{(k_1+k_2+k_3+.....k_{r-1}+2)}(t) = \ldots = Z_{(k_1+k_2+k_3+.....k_{r-1}+k_r)}(t)
\end{aligned}$$

We put $E_{11} - E_{12},\ E_{12} - E_{13},\ E_{13} - E_{14} \ldots E_{1(k_1-1)} - E_{1k_1}$ equal to zero (and use the equalities above) to get the following set of equations:

$$(p+1)Z_{11}(t) = pZ_{1k_1}(t) + Z_{12}(t) \tag{A.1}$$

$$(p+1)Z_{12}(t) = pZ_{11}(t) + Z_{13}(t) \tag{A.2}$$

$$\vdots$$

$$(p+1)Z_{1(k_1-1)}(t) = pZ_{1k_1-2}(t) + Z_{1k_1}(t) \tag{A.(k_1-1)}$$

$$(p+1)Z_{1k_1}(t) = pZ_{1k_1-1}(t) + Z_{11}(t) \tag{A.k_1}$$

We perform the following operations:
Subtract $(A.1)$ from $(A.2)$, $(A.2)$ from $(A.3)\ldots(A.(k_1-1))$ from $(A.1)$ to get the following:

$$Z_{11}(t) - Z_{13}(t) = p(Z_{1k_1}(t) - Z_{12}(t)) \tag{B.1}$$

$$Z_{12}(t) - Z_{14}(t) = p(Z_{11}(t) - Z_{13}(t)) \tag{B.2}$$

$$\vdots$$

$$Z_{1(k_1-1)}(t) - Z_{11}(t) = p(Z_{1(k_1-2)}(t) - Z_{1k_1}(t)) \tag{B.(k_1-1)}$$

$$Z_{1k_1}(t) - Z_{12}(t) = p(Z_{1(k_1-1)}(t) - Z_{11})(t)) \tag{B.k_1}$$

Now, put the value of $Z_{11}(t) - Z_{13}(t)$ from $(B.1)$ in $(B.2)$ to get:

$$Z_{12}(t) - Z_{14}(t) = p^2(Z_{1k_1}(t) - Z_{12}(t)) \tag{C.1}$$

Put this value of $Z_{12}(t) - Z_{14}(t)$ in $(B.3)$ to get:

$$Z_{13}(t) - Z_{15}(t) = p^3(Z_{1k_1}(t) - Z_{12}(t)) \tag{C.2}$$

Keep proceeding in a similar manner to get:

$$Z_{14}(t) - Z_{16}(t) = p^4(Z_{1k_1}(t) - Z_{12}(t)) \tag{C.3}$$

$$\vdots$$

$$Z_{1(k_1-1)}(t) - Z_{11}(t) = p^{k_1-1}(Z_{1k_1}(t) - Z_{12}(t)) \tag{C.(k_1-2)}$$

$$Z_{1k_1}(t) - Z_{12}(t) = p^{k_1}(Z_{1k_1}(t) - Z_{12}(t)) \tag{C.(k_1-1)}$$

Assuming p $\neq$ 1, we have (using $(C.(k_1 - 1))$) that :

$$Z_{1k_1}(t) = Z_{12}(t)$$

Using equations $(C.1)$, $(C.2)$ $\dots (C.(k_1 - 1))$ and the fact that $Z_{1k_1}(t) = Z_{12}(t)$, we get that:

$$Z_{11}(t) = Z_{13}(t) = Z_{15}(t) = \dots$$
$$Z_{12}(t) = Z_{14}(t) = Z_{16}(t) = \dots$$

Using $(A.2)$ and the fact that $Z_{11}(t) = Z_{13}(t)$, we get:

$$Z_{11}(t) = Z_{12}(t)$$

Therefore, we have shown that :

$$Z_{11}(t) = Z_{12}(t) = Z_{13}(t) = \dots = Z_{1k_1}(t)$$

Hence, we have shown the synchronisation of colours of the first mixing set of the first urn.
We can use an identical procedure to show the synchronisation for other urns and for other mixing sets, so that we finally have:

$$Z_{i1}(t) = Z_{i2}(t) = Z_{i3}(t) = \dots Z_{ik_1}(t)$$
$$Z_{i(k_1+1)}(t) = Z_{i(k_1+2)}(t) = Z_{i(k_1+3)}(t) = ... = Z_{i(k_1+k_2)}(t)$$
$$\vdots$$
$$Z_{i(k_1+k_2+\dots k_{r-1}+1)}(t) = Z_{i(k_1+k_2+k_{r-1}+2)}(t) = Z_{i(k_1+k_2+\dots k_{r-1}+3)}(t) = \dots = Z_{i(k_1+k_2+\dots k_{r-1}+k_r)}(t)$$

The above equations hold for all $1 \leq i \leq N$.
Hence, the proof for synchronisation across the mixing sets is complete.
Now, we show that the first colour synchronises across all the urns:
We put E_{i1} equal to 0 (and use the fact that $Z_{ik_1}(t) = Z_{i1}(t)\ \forall\ 1 \leq i \leq N$ to get:

$$Z_{i1}(t) = Z_{k_1}(t)\ \forall\ 1 \leq i \leq N$$

We can similarly prove synchronisation across urns for all the colours which belong to the mixing sets. We had already shown synchronisation for the colours not belonging to the mixing sets.
Since the zeroes of $h(Z_t)$ give us the limit points of Z_t, we are done.

□

3.2 A model with random permutations

3.2.1 Model description

We use the same terminology as used in the above model. The entire process is the same except that the reinforcement scheme is different.

At each time instance, we randomly choose a permutation of colours for the purpose of reinforcement for all the urns. Then for each urn, we toss a coin. In case of a heads, we draw s balls from that same urn and reinforce according to that chosen permutation. In case of a tails, we do the same except that this time we draw s balls from the super urn.

For example, suppose there are three colours (say red, blue and black labelled as 1, 2 and 3 respectively). Let $s = 4$ and $C = 2$. Suppose the chosen permutation of colours is $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. At some time instance, let us say we drew 4 balls and they came out to be: red, black, blue and red (in that particular order).

According to the permutation, the red colour reinforces the blue colour, the blue colour reinforces the black colour and the black colour reinforces the red colour. Since there are 2 red balls drawn, we put back 4 blue balls back in the urn ($C = 2$). Similarly, we put 2 red and 2 black balls back in that urn.

3.2.2 Main result

We show that for this model, the fraction of balls of each colour converges to a common fraction across every urn.

Theorem 3.2.1. *For $1 \leq i \leq N$, $1 \leq j \leq d$,*

$$\lim_{t\to\infty} Z_{ij}(t) = \frac{1}{d} \quad a.s.$$

Proof The Stochastic Approximation Scheme remains similar to that of the above model. Recall that:

$$h(Z_t) = E[Y_{t+1}|\mathcal{F}_t] - CsZ_t$$

Also, recall that Z_t denotes the vector with entries consisting of fraction of balls of all the colours in different urns at time t in the sequence defined above and Y_{t+1} denotes the number of balls of different colours added to each urn at time t for different urns (in a sequence identical to that of Z_t). We will calculate $E[Y_{t+1}|\mathcal{F}_t]$ for this model, and study the zeroes of h(Z_t).
For $1 \leq i \leq d$, let $F_i = (r_{i1}, r_{i2}, \ldots r_{id})$.
Like before,we use the following notation :
r_{ij}:= number of balls drawn for the i^{th} urn (j^{th} colour type) at time t ($1 \leq i \leq N$, $1 \leq j \leq d$).
So,

$$\begin{aligned} E[Y_{t+1}|\mathcal{F}_t] &= \sum_{\substack{\sum_{j=1}^d r_{ij}=s\ \forall\ 1\leq i\leq N \\ 0\leq r_{ij}\leq s\ \forall\ 1\leq i\leq N, 1\leq j\leq d}} \frac{C}{d!}(F_1, F_2, \ldots, F_N) \sum_{\sigma\in S_d} \prod_{m'=1}^{N} \binom{s}{r_{m'1}, r_{m'2} \ldots .. r_{m'd}} \left\{ p \prod_{q=1}^{d} (Z_{m'q}(t))^{r_{m'\sigma(q)}} \right. \\ &+ \left. (1-p) \prod_{q=1}^{d} (Z_q(t))^{r_{m'\sigma(q)}} \right\} \end{aligned}$$

We first calculate $E[Y_{t+1}(1)|\mathcal{F}_t]$. By $\sum\limits_{r_{ij}}$ we denote the sum over $\{\sum_{j=1}^d r_{ij} = s\ \forall\ 1 \leq i \leq N\}$ such that $0 \leq r_{ij} \leq s\ \forall\ 1 \leq i \leq N$, $1 \leq j \leq d$.

$$\begin{aligned} E[Y_{t+1}(1)|\mathcal{F}_t] &= \sum_{r_{ij}} \frac{C}{d!} r_{11} \sum_{\sigma\in S_d} \prod_{m'=1}^{N} \binom{s}{r_{m'1}, r_{m'2}, \ldots, r_{m'd}} \left\{ p \prod_{q=1}^{d} (Z_{m'q}(t))^{r_{m'\sigma(q)}} \right. \\ &+ \left. (1-p) \prod_{q=1}^{d} (Z_q(t))^{r_{m'\sigma(q)}} \right\} \\ &= \sum_{\sigma\in S_d} \sum_{r_{ij}} \frac{C}{d!} r_{11} \binom{s}{r_{11}, r_{12}, \ldots, r_{1d}} \left[p \prod_{q=1}^{d} (Z_{1q}(t))^{r_{1\sigma(q)}} + (1-p) \prod_{q=1}^{d} (Z_q(t))^{r_{1\sigma(q)}} \right] \\ &\times \left(\sum_{r_{ij}} \prod_{m'=2}^{N} \binom{s}{r_{m'1}, r_{m'2}, \ldots, r_{m'd}} \left[p \prod_{q=1}^{d} (Z_{m'q}(t))^{r_{m'\sigma(q)}} + (1-p) \prod_{q=1}^{d} (Z_q(t))^{r_{m'\sigma(q)}} \right] \right) \\ &= \sum_{\sigma\in S_d} \sum_{\substack{\sum_{j=1}^d r_{1j}=s \\ 1\leq r_{11}\leq s, 0\leq r_{1j}\leq s\ \forall\ 2\leq j\leq d}} \frac{C}{d!} r_{11} \left[p \binom{s}{r_{11}, r_{12}, \ldots, r_{1d}} \prod_{q=1}^{d} (Z_{1q}(t))^{r_{1\sigma(q)}} \right. \\ &+ \left. (1-p) \prod_{q=1}^{d} (Z_q(t))^{r_{1\sigma(q)}} \right] \end{aligned}$$

Thus,

$$E[Y_{t+1}(1)|\mathcal{F}_t] = \frac{Cs}{d!}\sum_{\sigma\in S_d}\sum_{\substack{\sum_{j=1}^d r_{1j}=s \\ 1\le r_{11}\le s, 0\le r_{1j}\le s \ \forall\ 2\le j\le d}}\Bigg[p\binom{s-1}{r_{11}-1, r_{12},\ldots,r_{1d}}\prod_{q=1}^{d}(Z_{1q}(t))^{r_{1\sigma(q)}} + (1-p)\prod_{q=1}^{d}(Z_q(t))^{r_{1\sigma(q)}}\Bigg]$$

Now,

$$\sum_{\substack{\sum_{j=1}^d r_{1j}=s \\ 1\le r_{11}\le s, 0\le r_{1j}\le s \ \forall\ 2\le j\le d}}\binom{s-1}{r_{11}-1, r_{12},\ldots,r_{1d}}\prod_{q=1}^{d}(Z_{1q}(t))^{r_{1\sigma(q)}}$$

$$= Z_{1(\sigma^{-1}(1))}(t)\sum_{\substack{\sum_{j=1}^d r_{1j}=s \\ 1\le r_{11}\le s, 0\le r_{1j}\le s \ \forall\ 2\le j\le d}}\binom{s-1}{r_{11}-1, r_{12},\ldots,r_{1d}}\left(\prod_{\substack{q=1 \\ q\neq\sigma^{-1}(1)}}^{d}(Z_{1q}(t))^{r_{1\sigma(q)}}\right)(Z_{1(\sigma^{-1}(1))}(t))^{(r_{11}-1)}$$

$$= Z_{1(\sigma^{-1}(1))}(t)$$

Similarly, for the super urn:

$$\sum_{\substack{\sum_{j=1}^d r_{1j}=s \\ 1\le r_{11}\le s, 0\le r_{1j}\le s \ \forall\ 2\le j\le d}}\binom{s-1}{r_{11}-1, r_{12},\ldots,r_{1d}}\prod_{q=1}^{d}(Z_q(t))^{r_{1\sigma(q)}} = Z_{\sigma^{-1}(1)}(t)$$

Hence,

$$E[Y_{t+1}(1)|\mathcal{F}_t] = \frac{Cs}{d!}\sum_{\sigma\in S_d}\Big(pZ_{1(\sigma^{-1}(1))}(t) + (1-p)Z_{\sigma^{-1}(1)}(t)\Big)$$

Other terms of $E[Y_{t+1}|\mathcal{F}_t]$ can similarly be computed to get:

$$E[Y_{t+1}|\mathcal{F}_t] = \frac{Cs}{d!}\sum_{\sigma\in S_d}(L_1, L_2, \ldots, L_N)$$

where, for $1\le i\le N$:

$$L_i = (pZ_{i(\sigma^{-1}(1))}(t)+(1-p)Z_{\sigma^{-1}(1)}(t), pZ_{i(\sigma^{-1}(2))}(t)+(1-p)Z_{\sigma^{-1}(2)}(t)\ldots pZ_{i(\sigma^{-1}(d))}(t)+(1-p)Z_{\sigma^{-1}(d)})(t))$$

Since $\sum_{k=1}^d Z_{ik}(t) = 1\ \forall\ 1\le i\le N$, we have that:
For $1\le i\le N, 1\le j\le d$:

$$\sum_{\sigma\in S_d} Z_{i(\sigma^{-1}(j))}(t) = (d-1)!$$

Also, for $1 \leq j \leq d$:

$$\sum_{\sigma \in S_d} Z_{\sigma^{-1}(j)}(t) = \sum_{i=1}^{N} \frac{Z_{i(\sigma^{-1}(j))}(t)}{N} = (d-1)!$$

Hence, $E[Y_{t+1}|\mathcal{F}_t] = \frac{Cs}{d}(1, 1, \ldots, 1)$. This gives:

$$h(Z_t) = Cs\left(\frac{1}{d} - Z_{11}(t), \frac{1}{d} - Z_{12}(t) \ldots \frac{1}{d} - Z_{1d}(t), \frac{1}{d} - Z_{21}(t) \ldots, \frac{1}{d} - Z_{2d}(t), \ldots, \frac{1}{d} - Z_{Nd}(t)\right)$$

Since the zeroes of $h(Z_t)$ give the limit points of Z_t, we are done. $\square$

Chapter 4

Graph based models

In each of the models we have studied until now, there has been one common feature: while sampling the balls, we always drew them from the super urn in case of a tails, where super urn was formed by merging all the urns together. Now, we study a more general model where reinforcement probability of each urn depends only on a subset of the set of all urns.

Consider an undirected graph $G = (V, E)$ such that $|V| = N$. We allow the graph to have self-loops (in fact, in the model we study, we have a self loop at every vertex) but no multi-edges between any two vertices. We place an urn with non-zero number of balls of either colour at each vertex (we restrict our discussion to urns containing balls of only two colours). Let A denote the adjacency matrix of the graph. In this model, the sampling scheme and hence the reinforcement depends on the adjacency matrix: while sampling for an urn, we draw balls from all its immediate neighbours (including from the urn itself since there is always a self loop for each urn) in case of a tails, while we sample balls from only that urn in case of a heads. So, we can think that there is a different super urn corresponding to each urn in this model. More precisely, each urn has an associated super urn, formed by merging each of its neighbouring urns.

Given such a sampling scheme, we can again have two types of two-colour models depending on the type of reinforcement: we call one the graph based Pólya model (self reinforcement of colours) and the other as graph based Friedman model (mutual reinforcement of colours). For urns i and j $(1 \leq i, j \leq N)$, let $i \sim j$ represent the fact that i and j are neighbours, that is, $(i, j) \in E$. Note that:

$$A_{ij} = \begin{cases} 1 & i \sim j \\ 0 & i \nsim j \end{cases}$$

We use the same notations as before with the addition that for $1 \leq i \leq N$, we denote the fraction of white balls in the super urn corresponding to the i^{th} urn by $\widetilde{Z}_t(i)$ and denote the degree of the i^{th} urn by d_i. That is,

$$\widetilde{Z}_t(i) = \frac{\sum\limits_{j \sim i} Z_t(j)}{d_i}$$

4.1 Graph based Pólya model

Theorem 4.1.1. *Let the graph G have k connected components labelled as $C_1, C_2 \dots C_k$. For $1 \leq i \leq k$ and for $j, l \in C_i (j \neq l)$:*

$$\lim_{t \to \infty} (Z_t(j) - Z_t(l)) = 0 \quad a.s.$$

Proof We prove the theorem by writing the Stochastic Approximation Scheme which is similar to that of the ordinary Pólya urn. Recall that $Y_{t+1}(i)$ denotes the number of white balls added to the i^{th} urn at time t for $1 \leq i \leq N$. For $1 \leq i \leq N$, $0 \leq k \leq s$:

$$P(Y_{t+1}(i) = Ck|\mathcal{F}_t) = p\binom{s}{k}(Z_t(i))^k(1 - Z_t(i))^{s-k} + (1-p)\binom{s}{k}(\widetilde{Z}_t(i))^k(1 - \widetilde{Z}_t(i))^{s-k}$$

Just like the Pólya model, we therefore have:

$$h(Z_t) = Cs(1-p)\begin{bmatrix} \widetilde{Z}_t(1) - Z_t(1) \\ \widetilde{Z}_t(2) - Z_t(2) \\ \vdots \\ \widetilde{Z}_t(N) - Z_t(N) \end{bmatrix}$$

Setting $h(Z_t)$ equal to 0, we get the following matrix equation:

$$(D^{-1}A - I){Z_t}^T = 0 \tag{4.1}$$

Here,

$$D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_N \end{bmatrix}$$

Note that, since $d_i \geq 1 \; \forall \; 1 \leq i \leq N$, D^{-1} exists, is well defined and not a zero matrix. Hence, (4.1) is equivalent to:

$$(D - A){Z_t}^T = 0 \tag{4.2}$$

We first assume that $d_i \geq 2 \; \forall \; 1 \leq i \leq N$ (that is each urn has at least one more immediate neighbour except for itself).

Observe that $D - A$ is the Laplacian matrix. Since the graph is undirected, the Laplacian is symmetric. The rank of a symmetric matrix is equal to the number of its non zero eigenvalues. Also, we know that, a simple graph G has k connected components iff the algebraic multiplicity of 0 in the Laplacian is k (Theorem 3.10 in [16]). Since the graph G is not simple (it has self loops), in order to apply the above result, we need to make some adjustments:

Corresponding to A, we define another matrix A' where :

$$A'_{ij} = \begin{cases} A_{ij} & i \neq j \\ 0 & i = j \end{cases}$$

The degrees of all the vertices in the graph corresponding to A' have been reduced by one when compared to the graph G. But the number of connected components is still the same (because of the assumption above). Corresponding to A', we define:

$$D' = \begin{bmatrix} d_1 - 1 & & \\ & \ddots & \\ & & d_N - 1 \end{bmatrix}$$

Hence, we have that:

$$D' - A' = D - A$$

So, (4.1) is equivalent to :

$$(D' - A'){Z_t}^T = 0$$

The graph corresponding to A' is simple. We therefore have that the number of free variables in the solution to (4.2) is equal to the number of connected components in the graph G. Let those free variables be labelled as $r_1, r_2 \ldots r_k$. For $1 \leq j \leq k$, let us set $Z_t(l) = r_j \; \forall \; l \in C_j$. Hence, we have that:

$$\widetilde{Z}_t(i) = Z_t(i) \; \forall \; 1 \leq i \leq N$$

Therefore, setting the fraction of white balls for urns in a common connected component equal to one common free variable gives a solution of the equation $h(Z_t) = 0$. Since, we cannot have any other kind of solution, and since the zeroes of $h(Z_t)$ give the limit points of Z_t, we have proven the theorem subject to the above assumption.

Even if we relax the assumption above, it is easy to see why the theorem still holds. Assume that r out of N urns are isolated (they have only themselves as their immediate neighbour). We can leave these r urns out. Now, we have $N-r$ urns and $k-r$ connected components left. The above theorem still holds for these $k-r$ connected components because for them the assumption above still holds. As for the remaining r runs, each of these forms a connected component, and the theorem holds trivially for these connected components.

□

4.2 Graph based Friedman model

We use the same notations and assumptions as that in the Pólya graph based model except that we change the reinforcement scheme to a mutual reinforcement one.

Theorem 4.2.1. *For $1 \leq j \leq N$:*

$$\lim_{t \to \infty} Z_t(j) = \frac{1}{2} \quad a.s.$$

Proof For $1 \leq i \leq N$, $0 \leq k \leq s$:

$$P(Y_{t+1}(i) = Ck|\mathcal{F}_t) = p\binom{s}{k}(Z_t(i))^{s-k}(1 - Z_t(i))^k + (1-p)\binom{s}{k}(\widetilde{Z}_t(i))^{s-k}(1 - \widetilde{Z}_t(i))^k$$

Writing the Stochastic Approximation Scheme like we did for the Friedman model, we get:

$$h(Z_t) = Cs \begin{bmatrix} 1-(1+p)Z_t(1)-(1-p)\widetilde{Z}_t(1) \\ 1-(1+p)Z_t(2)-(1-p)\widetilde{Z}_t(2) \\ \vdots \\ 1-(1+p)Z_t(N)-(1-p)\widetilde{Z}_t(N) \end{bmatrix}$$

Setting $h(Z_t)$ equal to zero, we get:

$$\begin{bmatrix} 1-(1+p)Z_t(1) \\ 1-(1+p)Z_t(2) \\ \vdots \\ 1-(1+p)Z_t(N) \end{bmatrix} = (1-p)D^{-1}A{Z_t}^T$$

This is same as:

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = [(1+p)I + (1-p)D^{-1}A]{Z_t}^T \tag{4.3}$$

Now, assume that the matrix $(1+p)I + (1-p)D^{-1}A$ has zero as an eigenvalue. Then, there exists a non zero vector v such that:

$$[1+p)I + (1-p)D^{-1}A]v = 0$$

Rearranging,we get:

$$D^{-1}Av = \frac{1+p}{p-1}v$$

Hence, $\frac{1+p}{p-1}$ is an eigenvalue of the matrix $D^{-1}A$. Assuming that p $\neq 0$, we have :

$$\left|\frac{1+p}{p-1}\right| > 1$$

But $D^{-1}A$ is a stochastic matrix. Hence, the absolute value of its eigenvalues is less than or equal to 1. This leads to a contradiction. Therefore, the concerned matrix cannot have zero as an eigenvalue. Hence, it is invertible. Therefore, (4.3) has a unique solution. It is easily verified that $Z_t = (\frac{1}{2}, \frac{1}{2} \dots \frac{1}{2})$ satisfies it. □

We have only obtained asymptotic limit results for the graph based models. We believe that by using techniques from [7], we can also obtain fluctuation results as well as $\mathcal{L}^2$ rates of convergence for this model.

Chapter 5

Simulations

Computer simulations are a remarkable tool to verify and visualise results obtained using theoretical tools and in order to predict and dismiss some possible results that can be potentially obtained. We present some of the simulation results that we obtained for our models.

First, we verify whether the $\mathcal{L}^2$ rates obtained in Theorem 2.1.2 are correct. So, we simulate a two-colour urn model with two urns such that $Z_0(1) = Z_0(2) = 1/4$, $m = 4$, $C = 2$ and $s = 3$. We plot $\text{Var}(Z_t(1) - \bar{Z}_t)$ against time in Figure 5.1 for large values of t for three different values of $p(1/3, 1/2 \text{ and } 2/3)$. In order to better compare the slopes of the graphs of $\text{Var}(Z_t(1) - \bar{Z}_t)$ against time when $p = 1/3$ and when $p = 1/2$, we plot these two graphs separately in Figure 5.2. The results obtained in the two figures are compatible with the predictions of Theorem 2.1.2.

It has been long known that the fraction of white balls in a classical Pólya- Eggenberger converges to a random variable with distribution that of a Beta distributed random variable (Theorem 1.1.1). It is a difficult and yet unsolved problem to find out what will be the distribution of the convergent random variable in case of a model with multiple drawings because such models lack a property called exchangeability (see [13]). In absence of theoretical tools necessary to know the distribution of the convergent random variable in case of the 'Pólya-type model' we studied above, we take help of simulations. We simulate a two-colour urn model with two urns such that $Z_0(1) = 1/4, Z_0(2) = 1/2$, $m = 4$, $C = 2$ and $s = 3$. Since the colours synchronize across urns, we have that :

$$\lim_{t\to\infty} Z_t(1) = \lim_{t\to\infty} Z_t(2) = Z \quad \text{(say)} \quad \text{a.s.}$$

Since we are interested in knowing the distribution of the random variable Z, we run several simulations of the urn process and draw a histogram (plotting the values of Z on x-axis and frequency on y-axis). Then we fit the histogram with a beta distribution (Figure 5.3) and with a normal distribution (Figure 5.4). We find that beta distribution seems to give an almost perfect fit for our model compared to the normal distribution which gives a much poorer fit. However, the authors in [14] claim that the distribution for such models is not a beta distribution.

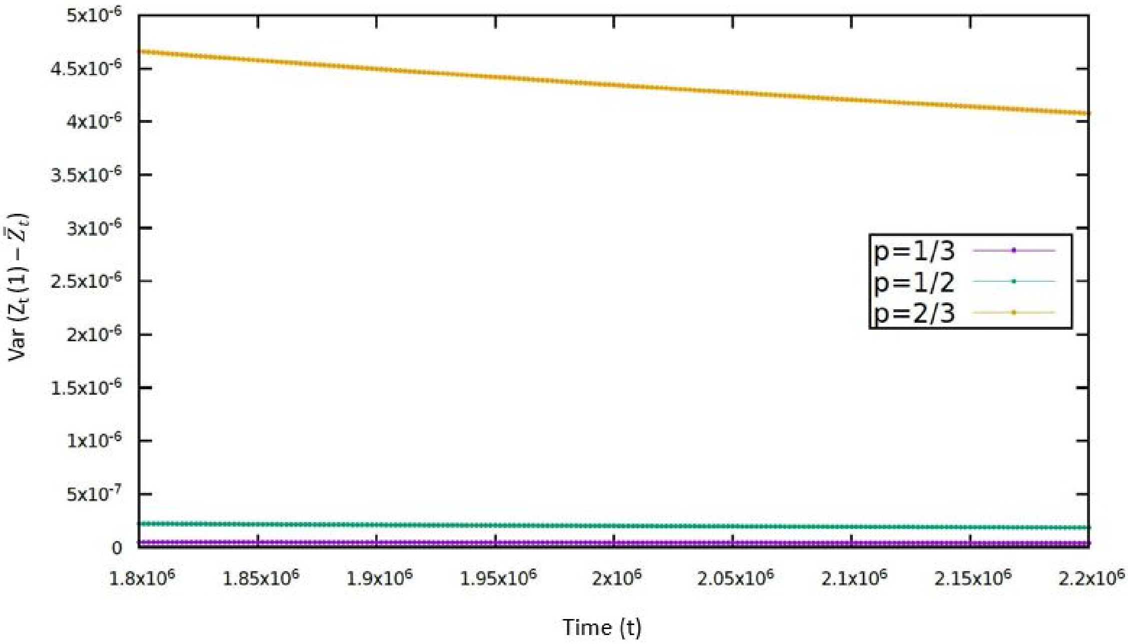

Figure 5.1: A plot of $\mathrm{Var}(Z_t(1) - \bar{Z}_t)$ against time for values of p in all the three regimes.

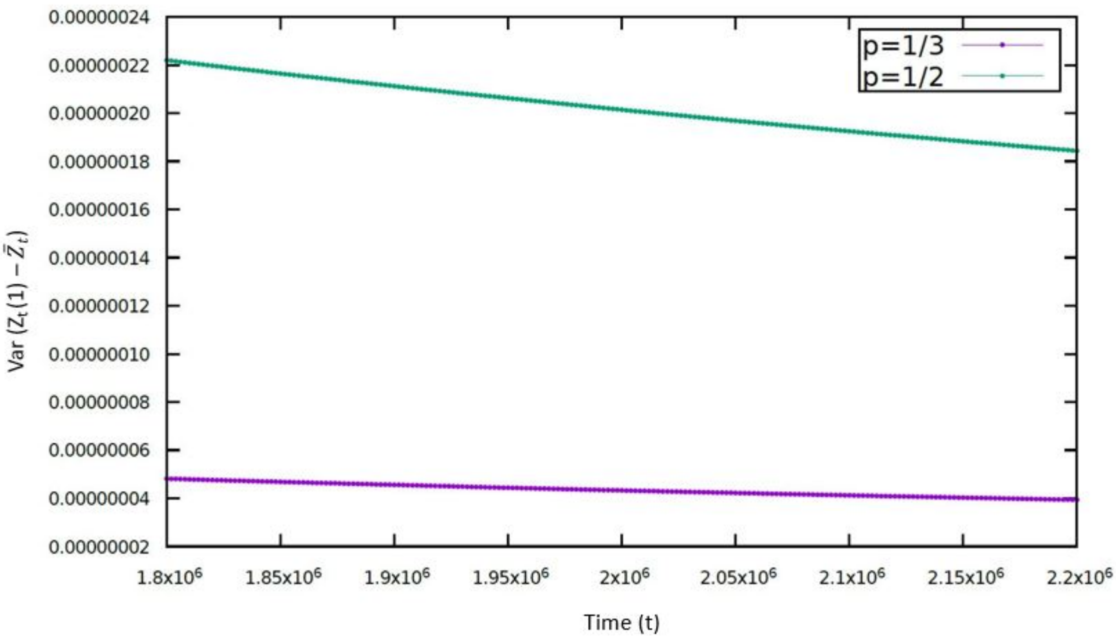

Figure 5.2: A plot of $\mathrm{Var}(Z_t(1) - \bar{Z}_t)$ against time for values of p in two of the three regimes.

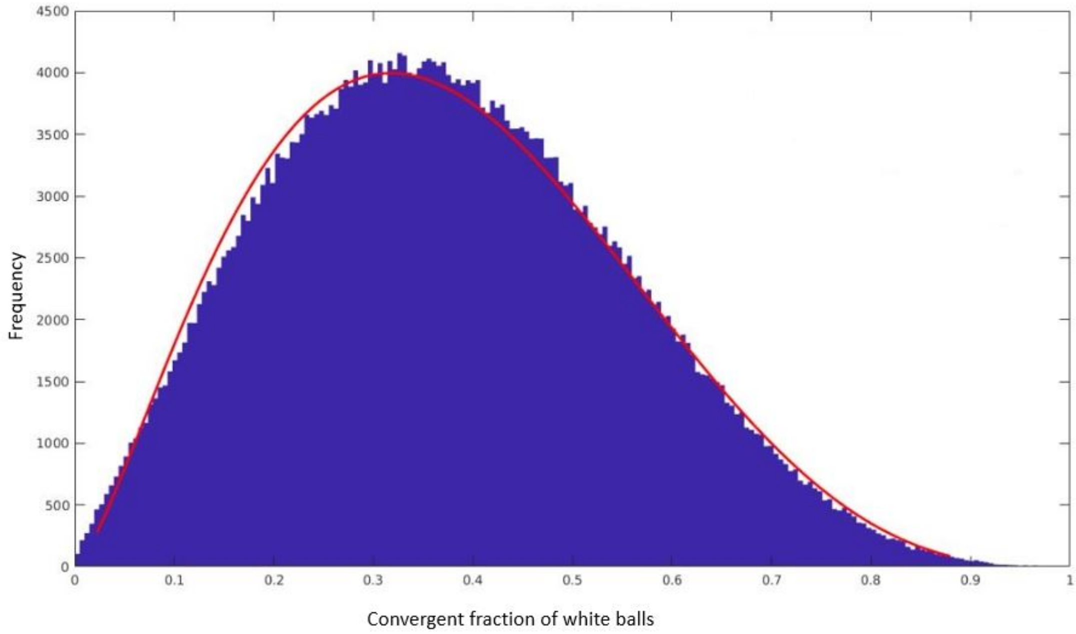

Figure 5.3: A plot of frequency against Z fitted with a beta distribution.

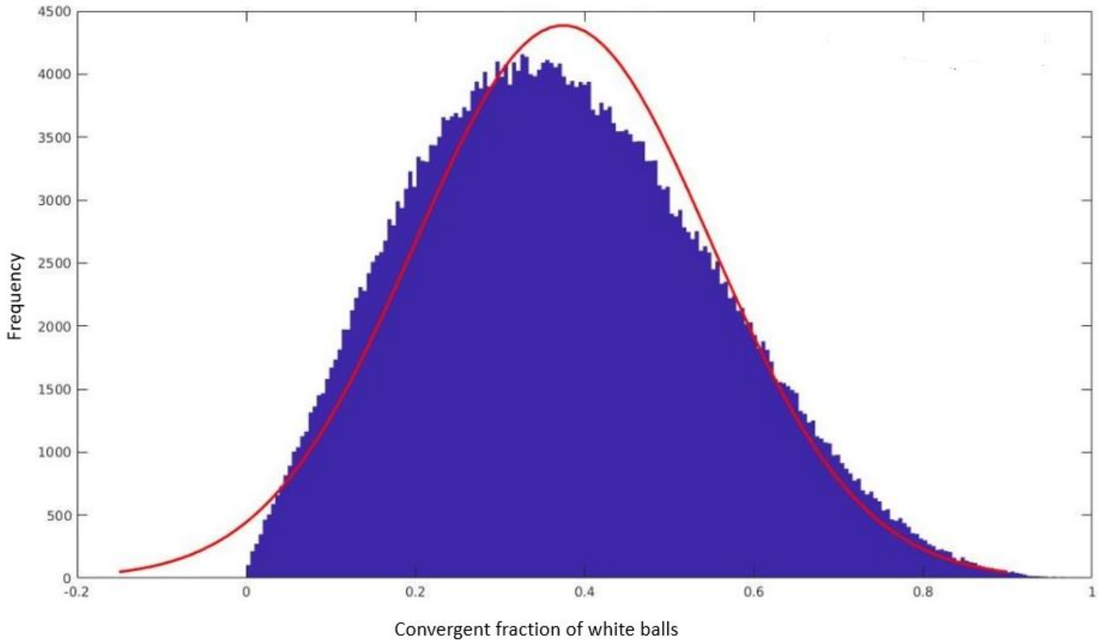

Figure 5.4: A plot of frequency against Z fitted with a normal distribution.

We also simulated a four-colour model with two urns such that $Z_{11}(0) = Z_{14}(0) = 1/8, Z_{12}(0) = 1/4$, $Z_{13}(0) = 1/2$,$Z_{21}(0) = Z_{22}(0) = Z_{23}(0) = Z_{24}(0) = 1/4$, $m = 8$, $C = 2$ and $s = 3$. In this model, we allow the first two colours to reinforce each other while the rest of the two colours reinforce each other. Since the colours synchronize across urns and across mixing sets, we have that :

$$\lim_{t\to\infty} Z_{11}(t) = \lim_{t\to\infty} Z_{12}(t) = \lim_{t\to\infty} Z_{21}(t) = \lim_{t\to\infty} Z_{22}(t) = Z_1 \quad \text{(say)} \quad \text{a.s.}$$

Also,

$$\lim_{t\to\infty} Z_{13}(t) = \lim_{t\to\infty} Z_{23}(t) = Z_2 \quad \text{(say)} \quad \text{a.s.}$$

We run several simulations of the urn process and again draw a histogram (plotting the values of Z_1 on x-axis, values of Z_2 on y-axis and the frequency on z-axis). The resulting histogram is shown in Figure 5.5 and a top view of it is shown in Figure 5.6.

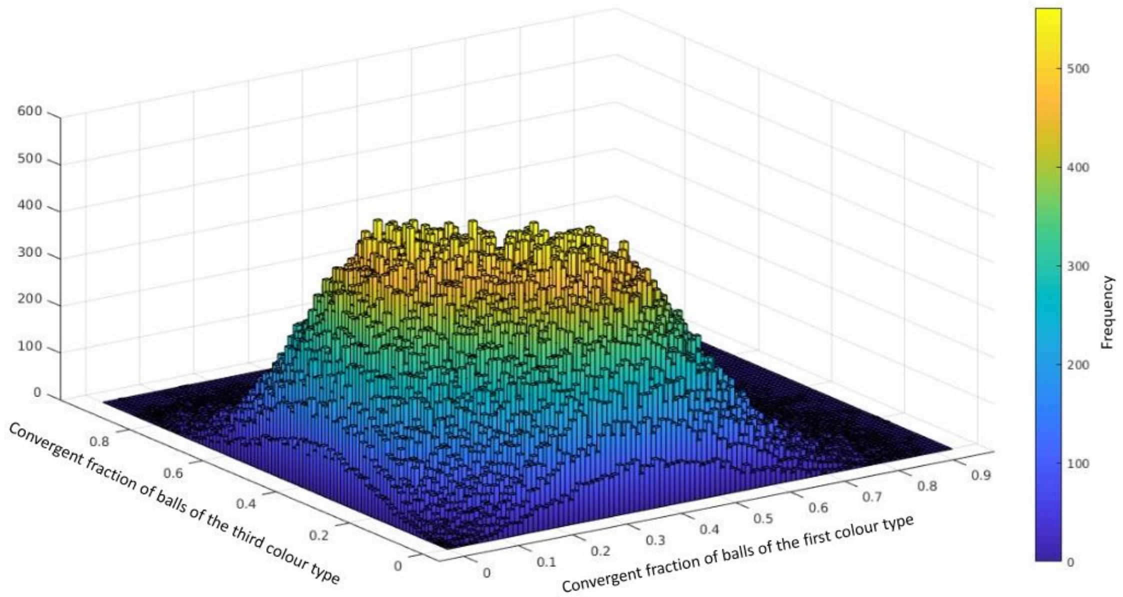

Figure 5.5: A plot of frequency on z-axis against Z_1 on x-axis and Z_2 on y-axis.

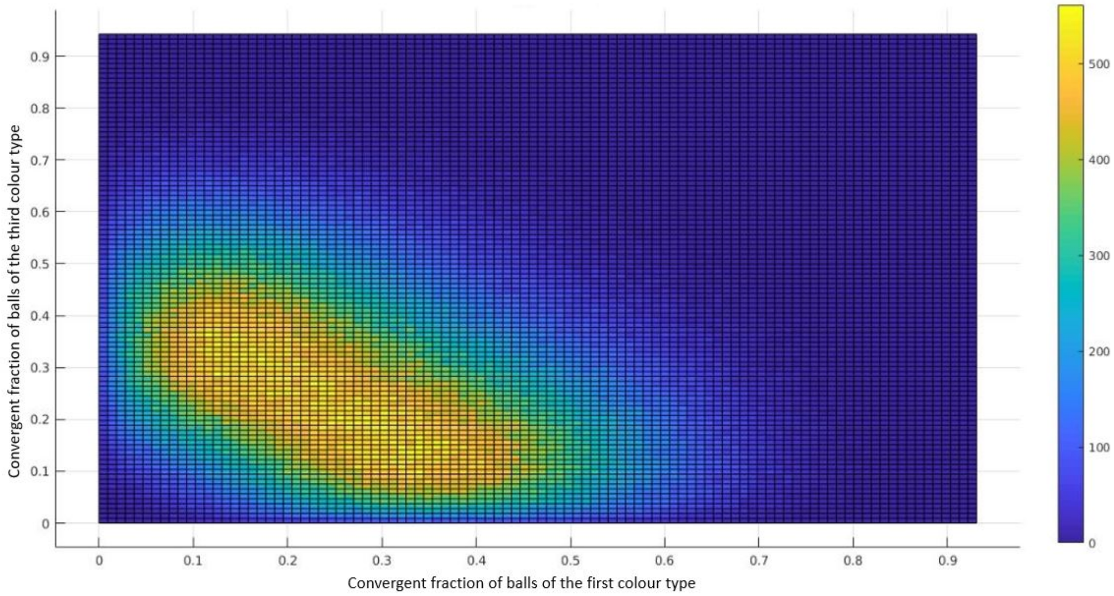

Figure 5.6: A descriptive colour diagram corresponding to the histogram in Figure 5.5.

Appendix

In [1], the author discusses various methods used to study asymptotic properties of random processes with reinforcement. We studied and used some of these methods in our analysis. In particular, we studied [7] where method of moments is used, [10] where Martingale theory is used to obtain almost sure convergence and stable convergence to a Gaussian limit in some cases and finally and [8] where stochastic approximation is used to study urn processes. We briefly explain the methods and present relevant results below.

5.1 Theory of Stochastic Approximation

A stochastic approximation scheme in $\mathbb{R}^d$ is given by :

$$x_{t+1} = x_t + a(t+1)[h(x_t) + M_{t+1}], t \geq 0$$

with the following assumptions:
A1. The map $h : R^d \to R^d$ is Lipschitz: $\|h(x) - h(y)\| \leq L\|x - y\|$ for some $0 < L < \infty$
A2. $\{a(t)\}$ are called step sizes satisfying $\sum_t a(t) = \infty$ and $\sum_t (a(t))^2 < \infty$
A3. $\{M_t\}$ is a Martingale difference sequence with respect to the increasing family of σ-fields:

$$\mathcal{F}_t := \sigma(x_m, M_m, m \leq t) = \sigma(x_0, M_1 \ldots M_t), n \geq 0$$

That is,

$$E[M_{t+1}|\mathcal{F}_t] = 0 \text{ a.s.}, t \geq 0$$

Furthermore, $\{M_t\}$ are square-integrable with $E[\|M_{t+1}\|^2|\mathcal{F}_t] \leq K(1 + \|x_t\|^2)$ a.s $t \geq 0$ for some constant $K > 0$.
A4. $sup_t\|x_t\| < \infty$ a.s.

Theorem 5.1.1. *(Theorem A.1 in [17]) For a general stochastic Approximation scheme given by*

$$x_{t+1} = x_t + a(t+1)[h(x_t) + M_{t+1}], t \geq 0$$

with the above set of assumptions:
The set Θ^∞ of limiting values of h as $t \to \infty$ is a.s. a compact connected set, stable by the flow of

$$ODE_h \equiv \dot{x} = h(x)$$

Furthermore if $x^* \in \Theta^\infty$ is a uniformly stable equilibrium on Θ^∞ of ODE_h, then

$$x_t \to x \quad a.s. \quad as \quad t \to \infty$$

Note that whenever we use the theory of stochastic approximation in the thesis, we implicitly make use of the following lemma (which we prove below):

Lemma 5.1.2. *A linear function $h : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz with respect to the Euclidean norm.*

Proof Since h is linear, there exists a $d \times d$ matrix A such that

$$h(x) = Ax \;\forall\; x \in \mathbb{R}^d$$

For $x, y \in \mathbb{R}^d$ $(x \neq y)$:

$$h(x) - h(y) = A(x - y)$$

Let $x = (x_1, x_2 \ldots x_d)$ and $y = (y_1, y_2 \ldots y_d)$ where $x_i, y_i \in \mathbb{R}^d$ for all $1 \leq i \leq d$. Also, let $A_{ij} = a_{ij}$ for $1 \leq i, j \leq d$.
Hence,

$$h(x) - h(y) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ \vdots & \vdots & \vdots & \vdots \\ a_{d1} & a_{d2} & \cdots & a_{dd} \end{pmatrix} \begin{pmatrix} x_1 - y_1 \\ \vdots \\ x_d - y_d \end{pmatrix}$$

Using Cauchy Schwartz Inequality, we have that:
For $1 \leq i \leq d$,

$$\left(\sum_{j=1}^{d} a_{ij}(x_j - y_j) \right)^2 \leq \left(\sum_{j=1}^{d} a_{ij}{}^2 \right) \left(\sum_{j=1}^{d} (x_j - y_j)^2 \right)$$

Adding up all of these d inequalities, we get that:

$$\sum_{i=1}^{d} \left(\sum_{j=1}^{d} a_{ij}(x_j - y_j) \right)^2 \leq \sum_{i=1}^{d} \left(\sum_{j=1}^{d} a_{ij}{}^2 \right) \left(\sum_{j=1}^{d} (x_j - y_j)^2 \right)$$

Hence,

$$\sum_{i=1}^{d} \left(\sum_{j=1}^{d} a_{ij}(x_j - y_j) \right)^2 \leq K^2 \left(\sum_{j=1}^{d} (x_j - y_j)^2 \right)$$

where $K = \sqrt{\sum_{i,j=1}^{d} a_{ij}{}^2}$.
Taking square root on both sides of the above equation, we get that:

$$\|h(x) - h(y)\| \leq K\|x - y\|$$

Since the above inequality holds for arbitrary $x, y \in \mathbb{R}^d (x \neq y)$, the linear function h is Lipschitz.

□

Whenever we apply the theory of stochastic approximation to our models, $h(x_t)$ is always linear. Hence, the condition A1. is always satisfied, given the above lemma. Condition A.2 is satisfied because we always have $a(t) \sim A/t$ for some $A > 0$. Also, in all our cases, we have $M_{t+1} = x_{t+1} - E[x_{t+1}|\mathcal{F}_t]$. Hence,

$$E[M_{t+1}|\mathcal{F}_t] = 0 \ \text{ a.s.}, t \geq 0$$

The rest of the conditions follow trivially since we always apply the theory to vectors which consist of fraction of balls of different colours (each element of the vector is hence bounded by one).

5.1.1 The Central Limit Theorem

The following result has been taken from [11](Theorem 5) and has been used while proving Theorem 2.2.1 above.

Theorem 5.1.3. *Consider the sequence of random vectors $(\theta_n)_{n\geq 0}$ defined by the following recursion:*

$$\forall\ n \geq n_0;\ \ \theta_{n+1} = \theta_n + \gamma_{n+1} f(\theta_n) + \gamma_{n+1}(\Delta\hat{M}_{n+1} + \hat{\epsilon}_{n+1})$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is a differentiable non-null function, θ_0 is a deterministic vector, for all $n \geq n_0$, $\Delta\hat{M}_n$ is an $\mathcal{F}_n$-increment martingale and $\hat{\epsilon}_n$ is an $\mathcal{F}_n$-adapted remainder term.
Assume in addition that

$$\hat{\epsilon}_n \to 0,\ sup_{n\geq 0}\mathbb{E}[\|\Delta\hat{M}_{n+1}\|^2|\mathcal{F}_n] < \infty$$

Assume that $\theta_n$satisfies the above recursion with $\gamma_n = \frac{1}{n}$ and that there exists $\theta \in \mathbb{R}^d$ a stable zero of f such that θ_n converges to θ with positive probability. Also assume that, for some $\delta > 0$,

$$sup_{n\geq 0}\mathbb{E}(\|\Delta\hat{M}_{n+1}\|^{2+\delta}|\mathcal{F}_n) < \infty,\ and\, \mathbb{E}(\Delta\hat{M}_{n+1}\Delta\hat{M}_{n+1}{}^t|\mathcal{F}_n) \xrightarrow{n\to\infty} \hat{\Gamma}\ \ almost\ surely,$$

where $\hat{\Gamma}$ is a deterministic symmetric positive semi-definite matrix and for some $\eta > 0$

$$n^{\frac{3}{2}}\mathbb{E}[\|\hat{\epsilon}_{n+1}\|^2 \mathbb{1}_{\|\theta_n-\theta\|\leq\eta}|\mathcal{F}_n] \xrightarrow{n\to\infty} 0.$$

Let $\hat{\Lambda}$ be the eigenvalue of $-\nabla f(\theta)$ with the largest real part. If we assume that θ_n converges almost surely to some deterministic limit θ, then:

- *If $Re(\hat{\Lambda}) > \frac{1}{2}$, then $\sqrt{n}(\theta_n - \theta) \xrightarrow{n\to\infty} \mathcal{N}\left(0, \hat{\sum}\right)$ in distribution, where*

$$\hat{\sum} = \int_0^\infty (e^{(\nabla f(\theta)+\frac{Id}{2})u})^t\hat{\Gamma}e^{(\nabla f(\theta)+\frac{Id}{2})u}du$$

  Assume additionally that f is twice differentiable, and that all Jordan blocks of $\nabla f(\theta)$ associated to $\hat{\Lambda}$ have size 1.

  Then:

- *If $Re(\hat{\Lambda}) = \frac{1}{2}$ then $\sqrt{\frac{n}{logn}}(\theta_n - \theta) \xrightarrow{n\to\infty} \mathcal{N}\left(0, \hat{\sum}\right)$, in distribution, where*

$$\hat{\sum} = \lim_{n\to\infty} \int_0^{logn} (e^{(\nabla f(\theta)+\frac{Id}{2})u})^t \hat{\Gamma} e^{(\nabla f(\theta)+\frac{Id}{2})u} du$$

- *If* $Re(\hat{\Lambda}) < \frac{1}{2}$, *then* $n^{Re(\hat{\Lambda})}(\theta_n - \theta)$ *converges almost surely to a finite random variable.*

5.2 Auxiliary Results

The following lemma is a result directly taken from [18] and is used multiple times while calculating first moments. We present it here without a proof although it follows trivially using the telescopic technique of summation.

Lemma 5.2.1.

For $x, y \in \mathbb{R}$ and $s \in \mathbb{N}$:

$$\sum_{k=1}^{s} \frac{\binom{k+x}{k}}{\binom{k+y}{k}} = \frac{(s+1+y)\binom{s+1+x}{s+1}}{(x+1-y)\binom{s+1+y}{s}+1} - \frac{x+1}{x+1-y}$$

Martingales

Definition 5.2.2 (Martingales)**.** *A stochastic process* $\{Z_n, n \geq 1\}$ *is a martingale if*

$$\mathbb{E}[Z_n] < \infty \quad and \quad \mathbb{E}[Z_n | Z_1, \ldots, Z_{n-1}] = Z_n \ \forall \ n \geq 2$$

Theorem 5.2.3 (Martingale Convergence Theorem)**.** *Let* $\mathbb{F} = (\mathcal{F}_n)_{n\geq 0}$ *be a filtration with* $\mathcal{F}_\infty = \sigma(\cup_{n\geq 0}\mathcal{F}_n)$ *Let* $\{X_n\}_{n\geq 0}$ *be a submartingale adapted to* $\mathbb{F}$ *such that*

$$sup\{\mathbb{E}[X_n{}^+] : n \geq 0\} < \infty.$$

Then there exists an $\mathcal{F}_\infty$*- measurable random variable* X_∞ *with* $\mathbb{E}[X_\infty] < \infty$ *and* $X_n \xrightarrow{n\to\infty} X_\infty$ *almost surely.*

Note that while the martingale convergence theorem helps us to show that a sequence admits an almost sure limit, it does not say anything about the distribution of the limit. In urn models, it is often useful (but also cumbersome) to compute moments of the converging sequence to determine the distribution of the limiting variable X when we know that the sequence $\{X_n\}_{n\geq 0}$ converges (almost surely) to X. This method was used by D. Freedman in [12] to prove Theorem 1.1.3.

5.3 Solving first order non homogeneous recurrence relations with variable coefficients

The concerned recurrence relation is of the form:

$$a_{n+1} = f_n a_n + g_n$$

The above equation is same as:

$$\frac{a_{n+1}}{\prod_{k=0}^{n} f_k} - \frac{f_n a_n}{\prod_{k=0}^{n} f_k} = \frac{g_n}{\prod_{k=0}^{n} f_k}$$

That is,

$$\frac{a_{n+1}}{\prod_{k=0}^{n} f_k} - \frac{a_n}{\prod_{k=0}^{n-1} f_k} = \frac{g_n}{\prod_{k=0}^{n} f_k}$$

Let

$$A_n = \frac{a_n}{\prod_{k=0}^{n-1} f_k}$$

Then, we have:

$$A_{n+1} - A_n = \frac{g_n}{\prod_{k=0}^{n} f_k}$$

Now,

$$\sum_{m=0}^{n-1}(A_{m+1} - A_m) = A_n - A_0 = \sum_{m=0}^{n-1} \frac{g_m}{\prod_{k=0}^{m} f_k}$$

Therefore:

$$a_n = (\prod_{k=0}^{n-1} f_k)\left(A_0 + \sum_{m=0}^{n-1} \frac{g_m}{\prod_{k=0}^{m} f_k}\right)$$

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