Riemann Surfaces

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Certificate of Examination

This is to certify that the dissertation titled **"Riemann Surfaces"** submitted by **Mr. Kabeer Manali Rahul** (Reg. No. MS15152) for the partial fulfilment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: May 4, 2020

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Chetan Balwe at the Indian Institute of Science Education and Research Mohali.
This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of the study done by me and all sources listed within have been detailed in the bibliography.

Kabeer manali Rahul
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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Chetan Balwe
(Supervisor)

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Contents

Chapter 1

Introduction

1.1 Why study Riemann Surfaces ?

Riemann surface can be first thought of as generalization of the complex plane. For those who have studied manifolds, it is nothing but a one-dimensional complex manifold, which essentially means it locally "looks" like the complex plane, and that it has sufficient structure to define suitable notion of holomorphicity/meromorphicity.
Further, people who have had a course in Complex analysis might have encountered Riemann surfaces in a discussion on multi-valued functions. Riemann surfaces are the natural domains for various multi-valued functions such as the logarithm and the square root function.
A very important reason to study Riemann surfaces is in relation with Algebraic Geometry. The richer structure(i.e. the analytic structure) that is there on the Riemann surface, compared to the case on a general field (or a general algebraically closed field), makes for an interesting setting for results in Algebraic Geometry (such as Riemann-Roch Theorem), which can be proven using the help of complex analytic and the differential geometric structure of the complex numbers.

1.2 History of Riemann surface

Riemann surfaces were introduced in Bernhard Riemann's PhD thesis titled "Foundations of a general theory of functions of one complex variable" in 1851. Riemann's work was building on the work of many mathematicians, such as Augustin-Louis Cauchy's pioneering work in complex analysis (he is said to have founded the area), the works of Leonhard Euler, Adrien-Marie Legendre and Carl Friedrich Gauss (who was also Riemann's thesis advisor) on elliptic functions and integrals. Further, many deep results (such as Abel's theorem) in the theory of Riemann surfaces were proved earlier(albeit not framed in the language of Riemann surfaces) by the likes of Carl Gustav Jacob Jacobi and

Niels Henrik Abel. Works on the topology of Riemann surfaces were done later, in the 1880's, by Georg Cantor and Karl Schwarz. The work of Henri Poincaré on automorphic functions and Fuchsian groups in the last decades of the 19th century were also of importance in the development of the theory of Riemann surfaces. Hermann Weyl's book on Riemann surfaces in 1913 put the concepts regarding Riemann surfaces on stable ground, with rigorous definitions and proofs, and brought the notion to the wider mathematical community. Modern study of Riemann surfaces is divided into two different areas, Complex Manifold theory and Algebraic Geometry[1].

1.3 What we will study about Riemann Surfaces

We will begin by extending various concepts and results from Complex Analysis to Riemann surfaces. We will go on to describe the consequences of these concepts and results on a particular class of Riemann surfaces, the complex torus. We will then define various objects related to the Riemann surfaces,such as differential forms, divisors and spaces related to divisors. Finally, we shall discuss the Riemann-Roch theorem, Serre Duality and Abel's theorem.
Above all, we will look at the relation between the geometric/topological structure (genus, homology) and the analytic structure (holomorphic maps, meromorphic functions and related spaces, differential forms) of the Riemann Surface.

1.4 What we shall assume

The basic results and definitions in a first course on Complex Analysis and Topology shall be assumed. [SS10] and [Mun00] are standard references for these respectively. The knowledge of the definition of a manifold, and maps between manifolds is highly recommended, for which the reader can look at [Lee03]. The main reference for this thesis is [Mir95]. Further, we also require the classification theorem of compact, orientable surfaces. For a discussion on this, the reader can look at the relevant chapter of [Mun00].
The version of the result we require is as stated below.

Theorem 1.4.1. *Any orientable, connected, compact surface with no boundary (i.e. a compact real manifold of dimension 2) has the following properties,*
1. It is triangulable, that is, it is homeomorphic to a polyhedra with triangular faces only.
2. It is homeomorphic to either a sphere or a sphere with g-handles (i.e. the connected sum of g tori).
The genus of such a surface is defined to be the number of handles, for example the torus has genus 1.
Further, for a polyhedron R corresponding to such a surface, we define the Euler characteristic as $\chi(R) = V - E + F$, *where* V = *number of vertices of* R, E =

[1] [Rem98] is the main reference for this section.

number of edges of R and F = number of triangular faces in R.
Then, if the genus of the surfaces is g, we have that,

$$\chi(R) = V - E + F = 2 - 2g$$

Other than the above mentioned prerequisites, whenever we need some results from other areas, reference are provided when we encounter those results in the text.

Chapter 2

Definitions and the Hurwitz formula

In this chapter, we will define Riemann surfaces, maps and functions on them and some related results.

2.1 Riemann Surface and its topological structure

Definition. *A Riemann Surface is a 1-dimensional complex manifold*[1]*, that is, it is a Hausdorff*[2]*, second countable*[3]*, connected*[4] *topological space with a complex structure*[5]*.*

Riemann Surfaces are orientable[6] surfaces because the transition maps are holomorphic and injective and hence conformal[7], and therefore the local orientation we get from a chart can be consistently extended globally to the Riemann Surface. So, by the classification of compact, orientable surfaces without boundaries, every compact Riemann Surface is topologically a g-holed torus, or equivalently a sphere with g handles attached. We denote the genus of a Riemann surface R by $g(R)$. The discussion on the classification theorem is done in the Introduction.
Topologically, there is a canonical way a g-holed torus can be realised as a gluing of a $4g$ sided polygon, as shown in the figure. The polygon with this particular

[1] see for reference [Lee03]

[2] i.e., every pair of points has disjoint neighbourhoods.

[3] i.e., there exists a countable basis

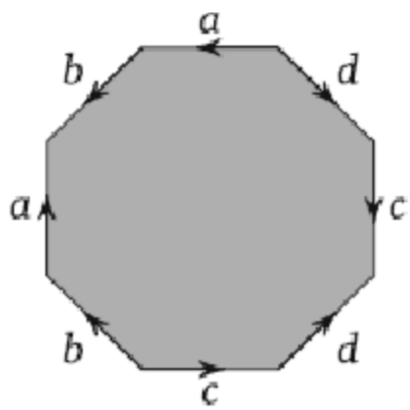
[4] i.e. $\nexists$ two disjoint open sets which cover the whole space.

[5] For a discussion on complex structure, which is nothing but the differential structure of a complex manifold, refer to Chapter 1 of [Mir95]

[6] Orientability is a property of surfaces which states that it is possible to make a consistent choice of surface normal vector at every point. Refer to [Mun00].

[7] a conformal map is a map that locally preserves angles.

gluing pattern (that is, ababcdcd..., gluing the side $4k-3$ to $4k-1$ and $4k-2$ to $4k$ for $1 \leq k \leq g$) is called the Standard Identified Polygon.



Notice that for the resulting surface R, we have the Euler characteristic,

$$\chi(R) = V - E + F = 1 - 2g + 1 = 2 - 2g$$

and hence genus g. This particular construction shall be useful later , so note that every compact Riemann surface can be topologically realised in this manner.
We now give an example of a Riemann surface.
Consider the two-dimensional sphere S^2 in $\mathbb{R}^3$. We shall give charts covering the surface, and leave it to the reader to verify that it indeed gives a Riemann surface structure on the sphere. If we denote the North pole $(0,0,1)$ and South pole $(0,0,-1)$ as N and S respectively, the charts are,

$$\phi_1 : S^2/\{N\} \to \mathbb{C},\ \phi_2 : S^2/\{S\} \to \mathbb{C}$$

defined as,

$$\phi_1(x,y,z) = \frac{x}{1-z} + \frac{iy}{1-z},\ \phi_2(x,y,z) = \frac{x}{1+z} - \frac{iy}{1+z}$$

The resulting Riemann surface is called the Riemann sphere, denoted by $\mathbb{C}_\infty$, as it can be thought of as $\mathbb{C} \cup \{\infty\}$, where $\mathbb{C}$ is represented by the domain of one of the above defined charts, and the remaining point being the point "at infinity".

2.2 Functions, maps and singularities

We now define the notion of a holomorphic function in the setting of Riemann surfaces. We do this by locally transporting the function to the complex plane using the charts. By the way complex structure is defined, this will give us a consistent definition, irrespective of the choice of the chart.

Definition. *Let R be a Riemann Surface, then a function $g : R \to \mathbb{C}$ is said to be holomorphic at a point r in R if there exist a chart $\psi : U \to \mathbb{C}$ centered at r such that $g \circ \psi^{-1}$ is holomorphic at 0.*

We can define a similar concept for a map between two Riemann surfaces, again using charts(as the definition of holomorphicity is a local one, same as in the case of the Complex plane).

Definition. *Let R,S be two Riemann Surface, then a map $G : R \to S$ is holomorphic at a point r in R if there exist charts $\psi : U \to \mathbb{C}$ and a chart $\phi : V \to \mathbb{C}$ centered at r and $G(r)$ respectively such that $\phi \circ f \circ \psi^{-1}$ is holomorphic at 0.*

We shall now define singularities, leading to the definition of a meromorphic function on a Riemann surface.

Definition. *Let $g : U/\{r\} \to \mathbb{C}$ be a holomorphic function on a punctured neighbourhood of a point r. Then, we say g has a removable singularity, a pole or an essential singularity at r if there exists a chart ψ centered at r such that $g \circ \psi^{-1}$ has a removable singularity, a pole or an essential singularity at 0 respectively.*

Definition. *A function g, holomorphic in a punctured neighbourhood of a point r, is said to be meromorphic at r if it has a removable singularity or a pole at r.*

2.3 Results inherited from Complex Analysis

Now we state a few important results for Riemann surfaces, which follow almost immediately from their counterparts in Complex analysis. The proofs are left to the reader.

1. Let f be a meromorphic function on a Riemann Surface R. If f is not identically zero, then the zeroes and poles of f form discrete subsets of R, that is, they have no limit points.

2. (Identity Therorm) Let f and g be meromorphic functions on a Riemann Surface R which agree on a set containing a limit point, then they are identically equal on R.

3. (Maximum Modulus Theorem) Let f be holomorphic on an open connected set W of a Riemann Surface R. If there exists a point r in W such that $|f(x)| \leq |f(r)|\ \forall x \in W$, then f is constant on W.

4. (Open Mapping Theorem) Any non-constant holomorphic map between Riemann Surfaces is an open map.

5. Any injective holomorphic map between Riemann Surfaces is a biholomorphism onto the image, i.e. the inverse map exists and is also holomorphic.

2.3.1 Corollaries

1. The only holomorphic functions on a compact Riemann surface are constant functions.

Proof. Let g be a holomorphic function on a compact Riemann surface R, then $|g|$ is a continuous function on R. As R is a compact space, $|g|$ attains maxima on R, and hence by the Maximum Modulus Theorem, g is constant on R. □

2. Let R be a compact Riemann Surface, and $G : R \to S$ be a non-constant holomorphic map onto another Riemann surface, then G is onto and S is compact.

Proof. G is an open map (by Open Mapping Theorem), and so G(R) is open in S. Also, as R is compact, and G continuous, $G(R)$ is compact. As S is Hausdorff, compact subsets are closed, and hence G(R) is closed. So, $G(R)$ is a non-empty set which is both open and closed in S, and hence is the full space, as S is connected. □

3. Let $G : R \to S$ be a non-constant holomorphic map between Riemann Surfaces. Then $\forall s \in S$, $G^{-1}(s)$ is a discrete subset of R.

Proof. Fix $s \in S$. Choose a local coordinate centered at s, then in the local coordinates, $G^{-1}(s)$ is just the zero set of a holomorphic function, and hence is discrete. □

2.4 Local normal form and the degree of a holomorphic map

The very existence of a holomorphic map between two compact Riemann surfaces gives a certain constraint on the topology of the two surfaces involved. To show that, we shall define an important global constant associated to a holomorphic map, the degree of the holomorphic map.
To that end, we first show that every holomorphic map locally "looks" like the power map.

Theorem 2.4.1. *Let $G : R \to S$ be a map between Riemann Surfaces holomorphic at r. Then, there exists a unique natural number m such that for any chart $\psi : V \to \mathbb{C}$ centered at G(r), there exists a chart $\phi : U \to \mathbb{C}$ centered at r so that $\psi(G(\phi^{-1}(z))) = z^m$.*
This is called the local normal form.

Proof. Fix a chart $\psi : V \to \mathbb{C}$ centered at G(r). Let $\eta : U \to \mathbb{C}$ be any chart centered at r, and let $T(w) = \psi(G(\eta^{-1}(w)))$. Then, T can be written as a Taylor series around zero,

$$T(w) = \sum_{m}^{\infty} a_i w^i, \; a_m \neq 0, m \in \mathbb{Z}$$

So, we can write $T(w) = w^m Q(w)$ near 0, for a holomorphic function $Q(w)$ which is non-zero at 0. Because $Q(w) \neq 0$ near 0, locally we can define a logarithm, and hence can take the m^{th} root. That is, near zero there exists a holomorphic function $P(w)$ such that $Q(w) = P(w)^m$ and $P(0) \neq 0$. Let, $\gamma(w) = wP(w)$, and note that $\gamma'(0) \neq 0$ and hence by Inverse Function Theorem $\phi := \gamma \circ \eta$ is also a chart near p. Now, if $z = \gamma(w)$,

$$\psi(G(\phi^{-1}(z))) = \psi(G(\eta^{-1}(\gamma^{-1}(z)))) = \psi(G(\eta^{-1}(w))) = T(w) = (wP(w))^m = z^m$$

Now, for any point near G(r), there are m pre-images under G, and hence the number m is a topologically determined and hence is unique. □

This natural number m is called the multiplicity of G at r, denoted by $mult_r(G)$.
Note that, if G is given by the holomorphic function h in local coordinates, then

$$mult_r(F) = 1 + ord_z \frac{dh}{dz}$$

if r corresponds to z in local coordinates. And hence, points where multiplicity is greater than 1 form a discrete set, as they correspond to the zeroes of a holomorphic function, the derivative of h.
We now define the notion of ramification points and branch points.

Definition. *Let $G : R \to S$ be a holomorphic map between Riemann Surfaces, then $r \in R$ is called a ramification point if the multiplicity at r is greater than one. A point $s \in S$ is called a branch point if there is a ramification point r such that $G(r) = s$.*

There is a relation between order of a meromorphic function and the concept of multiplicity. Any meromorphic function can be thought of as a holomorphic map to the Riemann Sphere by sending the poles of the map to the point at infinity of the Riemann Sphere.
Let $g : R \to \mathbb{C}$ be a meromorphic function on a Riemann surface, and let $G : R \to \mathbb{C}_\infty$ be the corresponding map to the Riemann Sphere, then :
1. $mult_p(G) = ord_r(g - g(r))$, if r is not a pole of g.
2. $mult_r(G) = -ord_r(g)$, if r is a pole of g.

Using the concept of multiplicity, we can define the following important invariant of holomorphic maps between compact Riemann surface.

Theorem 2.4.2. *Let $G : R \to S$ be a non-constant holomorphic map between compact Riemann Surfaces. We define a map d from S to $\mathbb{Z}$ as*

$$d(s) = \sum_{r \in G^{-1}(s)} mult_r(G)$$

Then, this map is identically constant, and that constant integer is called the degree of the map G, denoted by deg(G).

Proof. We will show that the map d is locally constant. As $\mathbb{Z}$ has the discrete topology and S is connected, we will have that d is a constant map.

First consider the holomorphic map on the unit disk $g : \mathbb{D} \to \mathbb{D}$ given by $g(z) = z^m$. Then for this holomorphic map, 0 is the only ramification point, and the multiplicity is clearly m. For any non-zero point in the range, there are m non-zero preimages. Hence, for this map g, d is a constant map.

Coming back to the map G, fix an arbitrary point $s \in S$, and let $G^{-1}(s) = \{r_1, \cdots, r_n\}$. Note that this set is finite by the Corollary 3 from subsection 2.3.1. Then, for a local coordinate w around s, there exist local coordinates z_i around r_i such that $w = z_i^{m_i}$, by Local Normal Form. Therefore, locally these look like power maps on the disk, like the map g we studied before. So, all we need to show is that near s, there are no preimages not in a disks around one of the r_i's.

This is only possible if there exists a sequence b_n in S converging to s with preimages a_n which do not belong to the neighborhoods around r_i's. But as R is compact, there exists a subsequence of a_n which converges to a point in R, say a. But the corresponding subsequence of b_n will converge to s, whose preimages are exactly the r_i's, and hence a = r_i for some i, which is a contradiction as all a_n lie outside the neighborhoods around the r_i's. □

The above theorem gives us a result about the sum of the order of meromorphic functions on compact Riemann surfaces.

Theorem 2.4.3. *For any non-constant meromorphic function g on a compact Riemann surface R, we have that*

$$\sum_r ord_r(g) = 0$$

Proof. Consider the corresponding holomorphic map to the Riemann sphere, $G : R \to \mathbb{C}_\infty$. Then, the result follows immediately from the Theorem 2.4.2 and the result before which gave us the relation between order and multiplicity for a meromorphic function, and its corresponding holomorphic map to the Riemann sphere. All we noeed to note is that $d(0) = d(\infty)$ □

Now we are in position to prove the Hurwitz formula (also known as the Riemann-Hurwitz formula). It gives us a relation between the degree of the map, the Euler number of the Riemann surfaces involved, and the ramification points.

Theorem 2.4.4. *(Hurwitz Formula) Let $G : R \to S$ be a non-constant holomorphic map between compact Riemann Surfaces. Then, we have that*

$$deg(G)\chi(S) = \chi(R) + \sum_{r\in R}[mult_r(G) - 1]$$

(Note that this sum makes sense as $mult_r(G) - 1$ is non-zero only for ramification points, and due to the R being compact, that set is finite.)

Proof. Take a triangulation of S so that each branch point is a vertex. By a refinement of the triangulation if necessary, we can lift this triangulation to R via the map G, as using the local normal form, the map is a local homeomorphism except near the branch points. Let the triangulation of S be (V, E, F) and the corresponding one for R, (V', E', F'). Then, as the branch points are a subset of the set of vertices, we have that $F' = deg(G)F$ and $E' = deg(G)E$ For the vertices, we have that

$$\begin{aligned}
V' &= \sum_{\substack{s\ ver- \\ tex\ of\ S}} |G^{-1}(s)| \\
&= \sum_{\substack{s\ ver- \\ tex\ of\ S}} \sum_{r \in G^{-1}(s)} 1 \\
&= \sum_{\substack{s\ ver- \\ tex\ of\ S}} \sum_{r \in G^{-1}(s)} [1 - mult_r(G)] + \sum_{\substack{s\ ver- \\ tex\ of\ S}} \sum_{r \in G^{-1}(s)} mult_r(G) \\
&= \sum_{\substack{r\ ver- \\ tex\ of\ R}} [1 - mult_r(G)] + V deg(G)
\end{aligned}$$

and hence we get that,

$$\begin{aligned}
\chi(R) &= V' - E' + F' \\
&= \sum_{\substack{r\ vertex \\ of\ R}} [1 - mult_r(G)] + deg(G)(V - E + F) \\
&= \sum_{r \in R} [1 - mult_r(G)] + deg(G)(\chi(S)
\end{aligned}$$

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∎

Here we see a first example of the relation between analytic structure(holomorphic map) and the geometric structure(Euler number/genus) of a Riemann surface.

Chapter 3

The Complex Torus

In this chapter we will take a look at a particular class of Riemann surfaces, the complex torus. We begin with the definition.

Definition. *Given linearly independent (over $\mathbb{R}$) complex numbers u and v we can define the lattice generated by them as the $\mathbb{Z}$-linear combination of u and v, denoted by $L_{(u,v)}$. So, we have,*

$$L_{(u,v)} = \{c_1 u + c_2 v : c_1, c_2 \in \mathbb{R}\}$$

Let R = $\mathbb{C}/L_{(u,v)}$ be the quotient with the canonical projection map p. So, the map p sends a complex number c to its coset in R.

$$p : \mathbb{C} \to R, \quad p(c) = c + L_{(u,v)}$$

This map is clearly a covering map[1](in fact, this is the universal cover[2] as the complex plane is simply connected.), which gives us a canonical complex structure on R, giving us a Riemann surface. Such a Riemann surface is called a complex torus.

Note that the quotient surface is topologically the torus, as can be easily seen by looking at the fundamental parallelogram (i.e. the parallelogram with vertices {0, u, v, u+v }), and looking at the gluing induced on that due to the quotient map.

We now show equivalence of certain types of complex tori.

Lemma 3.0.1. *For any $w \in \mathbb{C}/\{0\}$, the complex tori $\mathbb{C}/L_{(u,v)}$ and $\mathbb{C}/L_{(wu,wv)}$ are isomorphic as Riemann surfaces.*

[1]a covering map is a continuous map such that each point has an evenly covered neighbourhood, i.e. a neighbourhood whose pullback is a collection of disjoint open sets, each homeomorphic to it.

[2]that is, a cover where the covering space is simply connected.
A simply connected space is a connected topological spaces in which every loop is continuously contractible to a point.

Proof. It is easy to see that the map $G : \mathbb{C}/L_{(u,v)} \to \mathbb{C}/L_{(wu,wv)}$ sending $z + L_{(u,v)}$ to $wz + L_{(wu,wv)}$ is an isomorphism of Riemann surfaces. □

Hence, every complex torus is isomorphic to a torus of the form $\mathbb{C}/L_{(1,t)}$, with $Im(t) > 0$. So, we only need to work with lattices generated by 1 and t for some complex number t in the upper half plane. Here on, we denote $L_{(1,t)}$ by L_t, for t in the upper half plane.

3.1 Theta function

We will now give an example of a class of meromorphic functions on the torus. For that, we start with a holomorphic function on the complex plane, and try to construct a L_t-periodic meromorphic function using ratios of the original function with certain modifications.
To this end, we define the theta function as follows,

$$\theta(z) = \sum_{-\infty}^{\infty} e^{\pi i [n^2 t + 2nz]}$$

As the series converges absolutely and uniformly in compact subsets of the complex plane, we have that $\theta(z)$ is a holomorphic function on the complex plane. Further, it is easy to see that

$$\theta(z+1) = \theta(z) \text{ and } \theta(z+t) = e^{-\pi i (t+2z)} \theta(z)$$

Using the integral representation of the theta function[3], we get that the zeroes are all simple, and exactly at points of the form $1/2 + t/2 + n + mt$ for integers n and m.
We define the translated theta function as follows

$$\theta_r(z) = \theta(z - 1/2 - t/2 - r)$$

which has zeroes exactly at $r + L_t$
Now, consider the following function,

$$S(z) = \frac{\prod_{i=1}^{m} \theta_{r_i}(z)}{\prod_{j=1}^{n} \theta_{s_j}(z)}$$

[3]the theta function can be written in terms of line integrals of some trigonometric functions. Then, using the properties of contour integration, the results about the zeroes follows.

Then, $S(z+1) = S(z)$ and further, if we consider the case m = n and $\sum r_i - \sum s_j \in \mathbb{Z}$, then

$$\begin{aligned} S(z+t) &= \frac{\prod_{i=1}^{m} \theta_{r_i}(z+t)}{\prod_{j=1}^{n} \theta_{s_i}(z+t)} \\ &= (-1)^{m-n} \frac{\prod_{i=1}^{m} e^{-2\pi i(z-r_i)} \theta_{r_i}(z)}{\prod_{j=1}^{n} e^{-2\pi i(z-s_j)} \theta_{s_i}(z)} \\ &= e^{-2\pi i \left[(m-n)z + \sum s_j - \sum r_i\right]} S(z) \\ &= S(z) \end{aligned}$$

and hence this function is L_t-periodic. In fact, this gives us a meromorphic function on the complex torus $\mathbb{C}/L_t$, with zeroes at the points $r_i + L_t$ and poles at the points $s_j + L_t$. Further, as we prove in the following result, any meromorphic function on the complex torus is of this form.

Theorem 3.1.1. *Any non-constant meromorphic function g(z) on the complex torus $\mathbb{C}/L_t$ is of the form,*

$$g(z + L_t) = c \frac{\prod_{i=1}^{n} \theta_{r_i}(z)}{\prod_{j=1}^{n} \theta_{s_j}(z)}$$

where $\sum r_i - \sum s_j \in \mathbb{Z}$ and c is a constant.

Proof. Let g be a non-constant meromorphic function on the torus $\mathbb{C}/L_t$, with zeroes and poles at $\{r_i + L_t\}_1^n$ and $\{s_i + L_t\}_1^n$ respectively. We already know that the number of zeroes and poles is equal (counting the order) as the complex torus is a compact Riemann surface. We shall further show that $\sum_1^n r_i = \sum_1^n s_i$ mod L_t.
Suppose not, then choose $r_0 + L_t$ and $s_0 + L_t$ such that $\sum_0^n r_i = \sum_0^n s_i$ mod L_t, and $\sum_0^n r_i - \sum_0^n s_i \in \mathbb{Z}$. Let $S(z) = \frac{\prod_{i=0}^{n} \theta_{r_i}(z)}{\prod_{j=0}^{n} \theta_{s_j}(z)}$, and consider the meromorphic function $h = S/g$, and note that it has zero and pole exactly at r_0 and s_0 respectively. Then the corresponding holomorphic map $G : \mathbb{C}/L_t \to \mathbb{C}_\infty$ has a single simple zero and pole, and hence has degree 1, which implies it is an injective holomorphic map between compact Riemann surfaces and hence an isomorphism. But this is impossible as the Riemann sphere has genus 0 and the complex torus genus 1.

Now, as $\sum_1^n r_i = \sum_1^n s_i$ mod L_t, we have that $\sum_1^n r_i - \sum_1^n s_i = a \in L_t$, then, replacing r_1 by $r_1 - a$ (note that $r_1 = r_1 - a$ mod L_t), we have that $\sum_1^n r_i = \sum_1^n s_i$ and consider the function $H(z) = \frac{\prod_{i=1}^{n} \theta_{r_i}(z)}{\prod_{j=1}^{n} \theta_{s_j}(z)}$. Then H/g has no poles (or zeroes) and hence is holomorphic function. As the complex torus is compact, H/g is a constant, and we are done. □

We shall now classify all possible complex tori up to isomorphism classes, in the next few results, and in fact we will see in the next section that they have

a nice geometric structure.
First, note that any holomorphic map between tori is a covering map , as it has no ramification points by Hurwitz formula (because they have Euler number 0). Let L_t and L_u be two lattices with corresponding tori $R = \mathbb{C}/L_t$ and $S = \mathbb{C}/L_u$. Let $H : R \to S$ be a holomorphic map, which we compose with a suitable translation so that $H(0) = 0$. Now, note that as H is a covering map, so is $H \circ p : \mathbb{C} \to S$. Since $\mathbb{C}$ is simply connected, this map is isomorphic to the universal cover of S, i.e. isomorphic to $p : \mathbb{C} \to S$, and hence we have the following commutative diagram,

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{G} & \mathbb{C} \\ \downarrow p & & \downarrow p \\ R & \xrightarrow{H} & S \end{array}$$

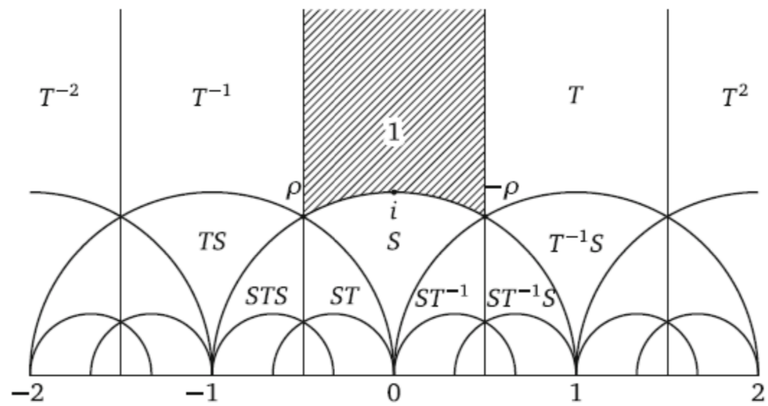
The map G is holomorphic, and without loss of generality we can assume $G(0) = 0$. Now, because of the commutative diagram, $G(z + r) - G(z) \equiv 0 \mod L_s$ $\forall r \in L_t$. Hence the function $v_r(z) = G(z + r) - G(z)$ goes into L_s. But the domain is connected and the range discrete and hence as G is continuous, it is constant. Therefore $v_r'(z) = 0 \implies G'(z + r) = G'(z)$, and hence the image of G, is the same as the image of G restricted to the fundamental parallelogram. But G is continuous (holomorphic in fact) and hence the image is bounded, but then by Liouville's theorem it is a constant, say d. And hence $dL_t \subset L_s$. And so, if it is a isomorphism, then $dL_t = L_s$. But this implies $d \cdot 1 = as + b$ and $d \cdot t = es + f$ and hence $t = (as + b)/(es + f)$. Further, as $d \cdot 1$ and $d \cdot t$ generate L_s, we have that $af - be = 1$ (not -1 because t,s belong to the upper-half plane). Hence, we have that,

Theorem 3.1.2. *Two complex tori $R = \mathbb{C}/L_t$ and $S = \mathbb{C}/L_s$ are isomorphic iff there exists a matrix $\begin{pmatrix} a & b \\ e & f \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $t = (as + b)/(es + f)$.*

3.2 Moduli space of Complex tori

Now, we will study the geometric structure of the set of isomorphism classes of complex tori (that is, the moduli space of complex tori). Note that any complex tori $R_t = \mathbb{C}/L_t$ can be represented by a complex number t in the upper half plane. Further, $R_t \cong R_s$ if they are related by the action of an element of $SL_2(\mathbb{Z})$, hence, we get that the moduli space is the upper half plane under the action of $SL_2(\mathbb{Z})$.

Theorem 3.2.1. *The fundamental domain of the action of $SL_2(\mathbb{Z})$ on the upper half plane $\mathbb{H}$ is the hyperbolic triangle with angles $(\pi/3, 0, \pi/3)$ shaded in the figure. We will denote that region by F here on.*



Proof. Consider the elements $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ giving us the transformations $S(z) = -1/z$ and $T(z) = z + 1$. We define the group H as follows, $H =< S, T >\subset SL_2(\mathbb{Z})$.

Now for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

$$Im(gz) = Im\left(\frac{az+b}{cz+d}\right) = Im\left(\frac{adz + bc\overline{z}}{|cz+d|^2}\right) = \frac{Im(z)}{|cz+d|^2}$$

and hence $Im(gz) \geq Im(z)$ iff $|cz+d| \leq 1$ which happens for only finitely many $g \in SL_2(\mathbb{Z})$.
The group $H \subset SL_2(\mathbb{Z})$ and hence for any point $z \in \mathbb{H}$, we can choose an element $h \in H$ with largest possible value of $Im(hz)$.
Now, as $T(z) = z + 1$ has the same imaginary part as z, hence we can choose an element $h_1 = T^m h \in H$ such that the imaginary part of $h_1 z$ is still maximal, with $Re(h_1 z) \leq 1/2$.
This implies, $Im(w) \geq Im(-1/w) = Im\left(\frac{Im(w)}{|w|^2}\right)$, and thus,$|w| \geq 1$, which along with $Re(h_1 z) \leq 1/2$ implies that $h_1 z \in F$.
So, for any $z \in \mathbb{H}$, there exists $h \in H$ such that $hz \in F$. Let z be a point in the interior of F, and $g \in \mathbb{Z}$ such that $gz = (az+b)/(cz+d) \in F$.
If $c = 0$, then $ad = 1$, and hence $gz = z \pm b$. If $b \neq 0$, then image of F has intersection with F only if b = 1, but in this case the intersection is only on the boundary, and is not possible for any interior point z. Hence $b = 0$, and $g = 1$.
If $c \neq 0$, then

$$\left|gz - \frac{a}{c}\right| \left|z + \frac{d}{c}\right| = \frac{1}{c^2}$$

As a/c and d/c are real, the imaginary parts of $gz - \frac{a}{c}$ and $z + \frac{d}{c}$ are the same as that of gz and z respectively. As the imaginary parts of any point in F is at least $\sqrt{3}/2$, so are the absolute values of $gz - \frac{a}{c}$ and $z + \frac{d}{c}$. Hence, $|c| \leq 2/\sqrt{3}$, and hence $c = \pm 1$ as it has to be a non-zero integer. So, we have that the above equation looks like,

$$|gz \mp a||z \pm d| = 1$$

But as gz an interior point of F, $|gz \mp a| > 1$ and as $z \in F$, we have that $|z \pm d| \geq 1$ which gives us a contradiction.

Hence, if z is a point in the interior of F, and $g \in \mathbb{Z}$ such that $gz \in F$, then $g = 1$. Therefore, if $h_1, h_2 \in H$, then $h_1 F$ and $h_2 F$ do not have any interior point in the intersection. And so, F is the fundamental domain of the action of H.

So, all we need to prove is that $H = SL_2(\mathbb{Z})$. To this end, let $g \in SL_2(\mathbb{Z})$ and z be any interior point of F. Then, gz lies in the upper half plane, and hence there exists $h \in H$ such that $(hg)z \in F$, but from above, this implies $hg = 1$, and hence $g = h^{-1} \in H$, and we are done.

$\square$

Chapter 4

Differential forms and Integration

4.1 Defining differential forms

One of the most important operations we did on the complex plane was contour integration. In this chapter we will build the required machinery to perform contour integration on Riemann surfaces. The main problem lies in getting a well-defined integrand, because there may exist multiple charts at a single point.
We define differential forms keeping this in mind, and they will serve as the integrand. We first define them for an open subset of the complex plane, and then for an arbitrary Riemann surface.

Definition. *A holomorphic(resp. meromorphic) 1-form on an open subset U of the complex plane is a formal expression of the type $w = f(z)dz$ for a holomorphic(resp. meromorphic) function f.*
Let $w_1 = f(z_1)dz_1$ and $w_2 = g(z_2)dz_2$ be two holomorphic(resp. meromorphic) 1-forms on two open sets, and let $T(z_2) = z_1$ be a holomorphic map between the two open sets. Then we say T transforms w_1 to w_2 if $g(z_2) = f(T(z_2))T'(z_2)$

Definition. *A holomorphic(resp. meromorphic) 1-form on a Riemann Surface is a collection of 1-forms $\{u_\psi\}$ for each chart ψ on R, such that the forms transform into one other under the change of coordiante maps.*
For a point p with a local chart ψ centered at p, if $u_\psi = g(z)dz$,

$$ord_p(u) := ord_0(g)$$

We can give a corresponding definition of smooth 1-forms as follows. We define them in terms of the formal symbols dz and d$\bar{z}$ using the fact that $x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i}$ and $dx = \frac{dz+d\bar{z}}{2}, dy = \frac{dz-d\bar{z}}{2i}$

Definition. *A smooth 1-form on an open subset U of the complex plane is of the form*

$$w = h(z,\overline{z})dz + g(z,\overline{z})d\overline{z}$$

Now, let $w_1 = h_1(z_1,\overline{z_1})dz_1 + g_1(z_1,\overline{z_1})d\overline{z_1}$ *and* $w_2 = h_2(z_2,\overline{z_2})dz_2 + g_2(z_2,\overline{z_2})d\overline{z_2}$ *be smooth 1-forms on two open sets. Let* $T(z_2) = z_1$ *be a holomorphic map between the two open sets, then we say T transforms* w_1 *to* w_2 *if* $h_2(z_2,\overline{z_2}) = h_1(T(z_2),\overline{T(z_2)})T'(z_2)$ *and* $g_2(z_2,\overline{z_2}) = g_1(T(z_2),\overline{T(z_2)})\overline{T'(z_2)}$.

We extend the definition to Riemann surfaces as before.

Definition. *A smooth 1-form on a Riemann Surface is a collection of smooth 1-forms* $\{w_\psi\}$ *for each chart* ψ *on R, such that the forms transform into one other under the change of coordiante maps.*
A smooth 1-form is of type (1,0) if it is locally of the form $h(z,\overline{z})dz$*, and of type (0,1) if it is locally of the form* $g(z,\overline{z})d\overline{z}$.

We now define 2-forms which will help us define surface integrals on Riemann surfaces.

Definition. *A smooth 2-form on an open subset U of the complex plane is a formal expression of the type* $w = h(z,\overline{z})dz \wedge d\overline{z}$ *for a smooth function f.*
The formal wedge products satisfy,

$$dz \wedge d\overline{z} = -d\overline{z} \wedge dz \text{ and } dz \wedge dz = d\overline{z} \wedge d\overline{z} = 0$$

Given smooth 2-forms $w_1 = f(z_1,\overline{z_1})dz_1 \wedge d\overline{z_1}$ *and* $w_2 = g(z_2,\overline{z_2})dz_2 \wedge d\overline{z_2}$ *on two open set and a holomorphic map between the two open sets* $T(z_2) = z_1$*, we say T transforms* w_1 *to* w_2 *if* $g(z_2,\overline{z_2}) = f(T(z_2),\overline{T(z_2)})||T'(z_2)||^2$

Like before, we make the definition for Riemann surfaces.

Definition. *A smooth 2-form on a Riemann Surface is a collection of smooth 2-forms* $\{w_\psi\}$ *for each chart* ψ *on R, such that the forms transform into one other under the change of coordinate maps.*

4.2 Operations on forms

Now that we have defined 1-forms and 2-forms, we will look at a few ways of defining new forms.
We can define new differential forms using smooth functions and other smooth forms, which we will do in this section.

1. Given a smooth function f on a Riemann surface, we can define the following smooth 1-forms,

$$\partial f = \frac{\partial f}{\partial z}dz,\ \overline{\partial} f = \frac{\partial f}{\partial \overline{z}}d\overline{z},\ df = \partial f + \overline{\partial} f$$

2. Given two smooth 1-forms w and u locally given by $w = h_1(z, \overline{z})dz + g_1(z, \overline{z})d\overline{z}$ and $u = h_2(z, \overline{z})dz + g_2(z, \overline{z})d\overline{z}$, we can define a 2-form, locally given by,

$$(h_1 g_2 - h_2 g_1) dz \wedge d\overline{z}$$

which is called the wedge product of the two 1-forms, denoted by $w_1 \wedge w_2$.
3. Given a 1-form locally given by $w = h(z, \overline{z})dz + g(z, \overline{z})d\overline{z}$, we can define the following 2-forms,

$$\partial w = \frac{\partial g}{\partial z} dz \wedge d\overline{z}, \ \overline{\partial} w = -\frac{\partial h}{\partial \overline{z}} dz \wedge d\overline{z}, \ dw = \partial w + \overline{\partial} w$$

4. Given two Riemann surfaces R and S and a non-constant holomorphic function $H : R \to S$, we can use this map to pull back 1-forms from S to R. Let u be a 1-form on S. Suppose the map looks like $z = h(w)$ in local coordinates and the 1-form like $u = f(z, \overline{z})dz + g(z, \overline{z})d\overline{z}$, then the pull back is locally defined as

$$\text{H*u} = f(h(w), \overline{h(w)})h'(w)dw + g(h(w), \overline{h(w)})\overline{h'(w)}d\overline{w}$$

So given a known Riemann surfaces with differential forms (such as the Riemann sphere), and a holomorphic map from an arbitrary Riemann surface to it (in the case of the Riemann sphere, this will in fact exactly correspond to meromorphic functions), we can use the map to pull back the form to the arbitrary Riemann surface from the known one.
Further, the order of the form we pulled back can be exactly determined, which we do in the following result.

Theorem 4.2.1. *Given two Riemann surfaces R and S and a non-constant holomorphic function $H : R \to S$ and a 1-form u on S, let H^*u be the pull back. Then*

$$ord_r(H^*u) = (1 + ord_{H(r)}(u))mult_r(H) - 1$$

Proof. By local normal form, we can choose local coordinates such that locally H is of the form $z = w^n$, where $n = mult_r(H)$. In this local variable z, let u be of the form $(az^l + \text{terms of order greater than l})dz$, $l = ord_{H(r)}(u)$, then locally

$$\text{H*u} = (aw^{nl} + \text{terms of order greater than nl})(nw^{n-1})dw$$

and hence we see that the order of H^*u at r is $(l + 1)n - 1$ as we required. □

Now, we will prove some results related to integrating 1-forms and 2-forms in the setting of Riemann surfaces.
The 1-form is defined in a manner that it gives a well defined integrand and hence can be integrated along paths. Further, it can also be integrated along chains, which are finite formal sum of paths.
An important concept in Complex analysis related to contour integration is that of the residue of a meromorphic function. We can define a similar concept for a 1-form which is meromorphic at a point r, which we do now.

Let u be a 1-form, which is meromorphic at a point r, then given a local coordinate z, it is locally given by a Laurent series, $u = \sum_{-n}^{\infty} a_m z^m$, then the residue of u at r, $Res_r(u) = a_{-1}$.
Note that a priori the residue depends on the choice of local coordinate. So, we need to show it is independent of the particular local coordinate, which we do now.

Lemma 4.2.1. *Let u be a meromorphic 1-form in the neighborhood of r, and γ a small path around r, not enclosing any pole (except possible r)*[1]*, then*

$$\frac{1}{2\pi i}\int_{\gamma} u = Res_r(u)$$

Proof. Let ψ be a local chart containing the image of the path γ and let $u = f(z)dz$, where $f(z) = \sum_{-n}^{\infty} a_k z^k$, then

$$\int_{\gamma} u = \int_{\psi\circ\gamma} f(z)dz$$

and the RHS is equal to $2\pi a_{-1}$ by the usual Residue Theorem, and we are done. □

Like we integrate 1-forms on chains, we can also integrate 2-form on closed sets which are triangulable.
We can now prove the Stoke's theorem for Riemann surfaces, which we get by the Green's theorem on the plane.

Theorem 4.2.2. *Stoke's theorem : For a smooth 1-form w and a triangulable closed subset D of a Riemann surface, we have*

$$\int_{\partial D} w = \iint_D dw$$

Proof. Triangulate D in a manner so that each triangle is fully contained in the domain of a chart (this can be done by Lebesgue number lemma[2]). Then, note that the LHS is an integration over the chain of boundaries of the triangles, and the RHS is a sum over the interior of the triangles. So, if we prove the result for the case of D being a triangle in the complex plane, we are done, because we can pull each triangle back using the charts.
The proof in the plane is directly from the Greene's theorem. We prove it for the case that w is a 1-form of the type (1,0). Suppose in a local chart, $w = g(z)dz$, then we have that

$$dw = -\frac{\partial g}{\partial \bar{z}} dz \wedge d\bar{z}$$

[1] i.e. a path which entirely inside the domain of a local chart $\{V, \psi\}$, such that $\psi \circ \gamma$ has winding number 1 around $\psi(r)$, and 0 around any other pole. This can be done as the zeroes form a discrete set.

[2] It states that for any open cover of a compact set, there exists a positive number, say δ, such that any set having diameter less than δ is contained in an open set part of the open cover.

If we write everything in terms of the variable x and y, treating the function g(z) = G(x,y) in their terms, we get that

$$w = G(x,y)dx + iG(x,y)dy, dw = \frac{1}{2}\left(\frac{\partial G}{\partial x} - i\frac{\partial G}{\partial y}\right)(2idxdy)$$

Then, the required result follows immediately by Greene's theorem[3]. □

Stoke's theorem helps us prove an important result related to residues of meromorphic 1-forms, which is an analog of Cauchy's Residue theorem for compact Riemann surfaces.

Theorem 4.2.3. *Residue theorem : For any meromomrphic 1-form u on a compact Riemann surface R, we have*

$$\sum_{r\in R} Res_r(u) = 0$$

Proof. As R is compact, the poles of the meromorphic 1-form u form a finite set, say $\{r_1, \cdots, r_n\}$. For each of these points r_i, choose a small path α_i enclosing it, and none of the other poles of u. Note that we can choose this path to be such that the interior is triangulable. Now, let V_i be the interior of the path α_i. Then, by the Stoke's theorem (as we have chosen the paths in such a way that the interior, and hence the complement is triangulable), we have for the domain $D = R - \bigcup V_i$

$$\int_{\partial D} u = \iint_D du = 0$$

as u is holomorphic in an open subset containing D (note that locally if $u = h(z,\overline{z})dz + g(z,\overline{z})d\overline{z}$, then we have $du = \left(\frac{\partial g}{\partial z} - \frac{\partial h}{\partial \overline{z}}\right)dz \wedge d\overline{z}$, which is 0 if u is a holomorphic 1-form).
Now, we have,

$$\int_{\partial D} u = \int_{-\sum_i \gamma_i} u = -\sum \int_{\gamma_i} u = (2\pi i)\sum_i Res_{r_i}(u)$$

where the last equality is by the Residue Theorem in the complex plane. □

[3]Green's theorem state that for any positively oriented, piece-wise smooth simple closed curve which which forms the boundary ∂D of a region D, and G,H are functions on an open set containing D, having continuous partial derivatives on D, then

$$\int_{\partial D}(Gdx + Hdy) = \iint_D \left(\frac{\partial H}{\partial x} - \frac{\partial G}{\partial y}\right) dxdy$$

Chapter 5

Divisors

In this chapter we will study the concept of Divisors, which, as we will find out, is a very useful concept in the study of Riemann surfaces.
What we will study has a lot do with the space of meromorphic functions, so we first give a notation to it, and then define divisors.

Definition. *Let M(R) be the space of meromorphic functions on a Riemann surface R.*

Definition. *A divisor is a formal $\mathbb{Z}$- linear sum of a discrete set of points on a Riemann Surface.*

A general divisor is of the form

$$D = \sum_r D(r) \cdot r$$

It is clear that we can also think of divisors also as integer valued functions on the Riemann surface with discrete support.
Now, as the support of a divisor on a compact Riemann surface is finite, we can define the concept of the degree of a divisor on a compact Riemann Surface as $deg(D) = \sum_r D(r)$.
Note that the set of all divisors on a Riemann surface R in fact forms a group under pointwise addition. We denote this group by D(R). A further structure we can put on this set is a partial ordering[1] For that, we need to define what it means for a divisor to be less than another, which we do as follows,
1. $D \geq 0$, if $D(r) \geq 0 \ \forall r \in R$
2. $D \geq D'$ iff $D - D' \geq 0$
We leave it to the reader to verify that this indeed gives a partial order on D(R).
Further, we can define another relation, an equivalence relation on divisors, as follows.

[1] a binary relation which is reflexive, transitive and anti-symmetric (i.e. $a \leq b$ and $b \leq a$ implies $b = a$).

Definition. *We say that two divisors are linearly equivalent if their difference is a Principal divisor. If D and D' are linearly equivalent, we denote it as* $D \sim D'$.

Given a divisor D, we can associate the a set of divisors to it using the above relation, called the complete linear set of D, given by,

$$|D| = \{D' : D' \sim D \text{ and } D \geq 0\}$$

Now that we have the definition of the divisor, and some structure on the set, we give a few examples of divisors we can form using objects we have defined earlier, namely meromorphic functions and 1-forms.

Definition. *Given* $f \in M(R)$, *we define the divisor*

$$D(f) = \sum_r ord_r(f) \cdot r$$

Such divisors are called principal divisors, and their group denoted by PD(R). Similarly, given a meromorphic 1-form u, we define the divisor,

$$D(u) = \sum_r ord_r(u) \cdot r$$

Such divisors are called canonical divisors, and their collection is denoted by KD(R).

Note that the degree of any Principal divisor on any compact Riemann surface is zero as we have proved that the sum of orders of a meromorphic function on a compact Riemann surface is zero.
The relation between the two sets PD(R) and KD(R) is given in the following result.

Lemma 5.0.1. *The sets PD(R) and KD(R) are related as follows,*

$$KD(R) = PD(R) + div(u)$$

for any meromorphic 1-form u.

Proof. Suppose we are given two meromorphic 1-forms u and v. In a neighbourhood if $u = h(z)dz$ and $v = g(z)dz$, then we can locally define the meromorphic function by $f(z) = h(z)/g(z)$. This locally defined function in fact gives us a well-defined meromorphic function f on the whole of R.
Then, we have that $div(v) = div(u) + div(f)$, and hence we are done. □

Hence, the degree of a meromorphic 1-form on a compact Riemann surface is fixed, independent of the particular divisor (as the degree of any principal divisor is 0).
For example, consider the form defined locally by dz on the Riemann sphere.

Then, near infinity, it is of the form $-\frac{1}{w^2}dw$, and hence, the degree of this 1-form, and hence of any meromorphic 1-form on the Riemann sphere is -2, as it does not have any zero or pole anywhere else except ∞.
In fact we can explicitly give the degree of a meromorphic 1-form on any compact Riemann surface, and it depends only on the genus of the particular Riemann surface.

Theorem 5.0.1. *If R is a compact Riemann surface which has a non-constant meromorphic function, then canonical divisors on R have degree $2g(R) - 2$.*

Proof. Let g be a non-constant meromorphic function on R, and $G : R \to \mathbb{C}_\infty$ be the corresponding holomorphic map, say of degree d. By Hurwitz formula, we have,

$$2g - 2 + deg(f) = \sum_{r \in R}(mult_r(G) - 1)$$

Let u be the 1-form on the Riemann sphere locally defined as dz, and $v = G^*(u)$ be the pullback to R. Then,

$$\begin{aligned}
\text{deg(div(v))} &= \sum_{r \in R} ord_r(v) = \sum_{r \in R} ord_r(G^*u) \\
&= \sum_{r \in R}[(1 + ord_{G(r)}u)mult_r G - 1] \qquad \text{(by Theorem 4.2.1)} \\
&= \sum_{r \in G^{-1}(s)} \sum_{s \in \mathbb{C}_\infty/\{\infty\}} [mult_r G - 1] + \sum_{r \in G^{-1}(\infty)} [-mult_r G - 1] \\
&= \sum_{r \in R}[mult_r G - 1] - \sum_{r \in G^{-1}(\infty)} 2 \cdot mult_r G \\
&= 2g - 2 + 2deg(G) - 2deg(G) \qquad \text{(by Hurwitz formula)} \\
&= 2g - 2
\end{aligned}$$

□

It is in fact true (but non-trivial) that every compact Riemann surface has a non-constant meromorphic function, and hence this result holds for all compact Riemann surfaces. We shall assume this result regarding existence of meromorphic functions on compact Riemann surfaces as a fact from now on.

5.1 Spaces related to a divisor

There is a natural subspace of the space of meromorphic functions associated to a divisor. Given a divisor D, we define the space

$$L(D) := \{f \in M(X) : div(f) + D \geq 0\}$$

The partial order relation corresponds to an inclusion relation on the level of this associated space, that is $D \leq D' \implies L(D) \subseteq L(D')$. For compact Riemann surfaces we can easily show that the space of meromorphic functions associated to negative divisors is trivial.

Lemma 5.1.1. *For any compact Riemann surface R, if any divisor D has negative degree, then* $L(D) = \{0\}$

Proof. Suppose not, then there exists $g \in L(D)/\{0\}$. Then, consider the divisor $div(g) + D =: D' \geq 0$, as $g \in L(D)$. Hence $deg(D') \geq 0$, but since $deg(div(g)) = 0$, we have that $deg(D') = deg(D) < 0$, which is a contradiction. □

We constructed the space L(D) by associating a certain space of meromorphic functions to the divisor D. Similarly, we can also associate a space of meromorphic 1-forms to the divisor D as follows.

Definition. *Given a divisor D, we define*

$$L^1(D) = \{u \text{ a meromorphic 1-form} : div(u) + D \geq 0\}$$

The equivalence relation we defined earlier on the space of divisors behaves well with respect to the spaces we have defined, which we record in the following lemma.

Lemma 5.1.2. *Given linearly equivalent divisors D and D', we have that,*

$$L(D) \cong L(D') \text{ and } L^1(D) \cong L^1(D')$$

Proof. We can give an explicit map in both the cases, namely the map corresponding to multiplication by the meromorphic function corresponding to the principal divisor $D - D'$. We leave it to the reader to fill in the details. □

We now give a relation between the two vector spaces we have associated to any divisor D.

Theorem 5.1.1. *Let D be a divisor. Fix a canonical divisor $U = div(u)$. Then the map $\mu_u : L(D + U) \to L^1(D)$ is defined by sending g to the meromorphic 1-form gu. Then, this map is an isomorphism of vector spaces.*

Proof. First of all we need to check if the map indeed lands inside $L^1(D)$. Given $g \in L(D + U)$, we have that

$$0 \leq div(g) + D + U = div(g) + div(u) + D = div(gu) + D$$

and so it indeed lands inside $L^1(D)$.
Now, injectivity and linearity is apparent by the very construction of the map. Further, given $v \in L^1(D)$, we know there exists a meromorphic fucntion g such that $v = gu$. We have that,

$$0 \leq div(v) + D = div(gu) + D = div(g) + div(u) + D = div(g) + D + U$$

and hence $g \in L(D + U)$ and $\mu_u(g) = v$. So, the map is surjective too. □

Hence, the dimensions of the two vector spaces is equal.
We now give a crude bound on the dimension of L(D), in the case of a compact Riemann surface, which will be useful later. For this, we nee the following lemma.

Lemma 5.1.3. *Let R be a Riemann surface, and D a divisor on R. Then for any point $r \in R$, the space $L(D-r) = L(D)$, or it has co-dimension 1 in L(D).*

Proof. Consider a chart centered at r, corresponding to the local coordinate z. Let D(r) = -n, then any $g \in L(D)$, is locally of the form $a_g z^n +$ higher order terms. Consider the map ϕ sending $g \in L(D)$ to a_g, the coefficient of z^n. Then this map $\phi : L(D) \to \mathbb{C}$ is a linear map. Note that the kernel of the map is exactly $L(D-r)$, and as the range is a one-dimensional space, the kernel, that is $L(D-r)$, is either the full space, in case the map goes to the subspace $\{0\}$ of $\mathbb{C}$, or it is of codimension 1 if the map is surjective. □

Using this lemma, we can get a bound on the dimension of the space L(D) for a divisor D on a compact Riemann surface, in terms of the degree of a particular divisor associated to D.

Theorem 5.1.2. *Let R be a compact Riemann surface, and D a divisor on R. Then, if we write $D = P - N$ as a difference of its positive and negative parts* [2]*, then*

$$dim(L(D)) \leq deg(P) + 1$$

In particular it is a finite-dimensional space.

Proof. We shall use induction on deg(P).
If deg(P) = 0 : It implies P = 0, which implies L(P) = {constant functions } and hence deg(L(P)) = 1. Now, as $D = P - N = 0 - N \leq P \implies L(D) \subset L(P)$ and hence

$$dim(L(D)) \leq dim(L(P)) = 1 = 1 + deg(P)$$

Now, assume true for $deg(P) \leq n-1$: Let D be a divisor such that D = P - N with deg(P) = n. Then, as $n \geq 1$, let r be a point in the support of P, i.e. $P(r) > 0$. Then, consider the divisor $D - r = (P - r) - N$, which has positive part P of degree n-1. Hence, by Induction hypothesis we have

$$dim(L(D-r)) \leq 1 + deg(P-r) = deg(P)$$

But by the previous lemma, we have that $dim(L(D-r)) \geq dim(L(D)) - 1$, and hence

$$dim(L(D)) \leq dim(L(D-r)) + 1 \leq deg(P) + 1$$

and hence the claim is true. So, the result holds by Principal of mathematical induction. □

[2] Note that

$$P := \sum_{r \in R} max\{D(r), 0\} \cdot r \text{ and } N := \sum_{r \in R} -min\{D(r), 0\} \cdot r$$

Chapter 6

Riemann-Roch theorem and Serre Duality

6.1 Introduction

In this chapter we shall prove the Riemann-Roch theorem and Serre duality. The Riemann-Roch theorem tells us precisely the dimension of the space L(D) for a divisor D on a compact Riemann surface R, in terms of the genus of R, and a new object that we shall introduce, the space $H^1(D)$. To complete our understanding of the dimension of L(D), Serre duality helps us relate the dimension of the space $H^1(D)$ to familiar quantities. These results are fundamental in Algebraic Geometry, specifically in the study of Algebraic curves. We begin this discussion by defining Algebraic curves.

Definition. *A Riemann Surface R is said to be an Algebraic Curve if the space of meromorphic functions M(X) separates points and tangents.*
The space of meromorphic functions is said to separate points of R, if for every pair of points r,s in R, there exists a meromorphic function g such that $g(r) \neq g(s)$.
The space of meromorphic functions is said to separate tangents of R, if for every point r in R there exists a meromorphic function g which has multiplicity one at r.

By our work in chapter 2, we know that any complex torus is an algebraic curve, as we know the exact form of meromorphic functions on any complex torus.
We shall use the following result without proof. It is a non-trivial result, requiring techniques from functional analysis.

Theorem 6.1.1. *Every compact Riemann surface is an algebraic curve.*

Hence we have many examples of algebraic curves.
We shall now prove a sequence of lemmas, which express the above existence

of meromorphic functions, giving us statements which will be useful in proving results later in the chapter.

Lemma 6.1.1. *Let R be an algebraic curve, and r a point in R. Then for any natural number n there exists a meromorphic function g, such that* $ord_r(g) = n$.

Proof. As R is an algebraic curve, there exists a function g such that it has multiplicity 1 at r. Hence, either $1/h$ or $h - h(r)$ has a zero of order 1 at r (depending on whether g has a pole at r or not). Hence, if we define $g = h^n$, then $ord_r(g) = n$. □

We will now define Laurent tail of a Laurent series, which will help us write the generalized version of the previous lemma.

Definition. *Given a Laurent series* $f(z) = \sum_{k=n}^{\infty} b_k z^k$, *any Laurent polynomial of the form* $\sum_{k=n}^{m} b_k z^k$ *is called a Laurent tail of* $f(z)$

We now give a generalization of the previous lemma.

Lemma 6.1.2. *Let R be an algebraic curve, r a point in R, and let z be a local coordinate centered at r. Fix a Laurent polynomial* $p(z) = \sum_{k=n}^{m} b_k z^k$, *then there exists a meromorphic function g such that the Laurent series of g at r has the Laurent tail p(z).*

Proof. Note that the Laurent polynomial has $m - n + 1$ terms. We will prove the required statement via induction on $m - n + 1$.
If $m - n + 1 = 1$, then, we are done by the previous lemma.
Now suppose $m-n+1 \geq 2$. By the previous lemma, we can find a meromorphic function h with tail $b_n z^n$ at r. Let q(z) be the Laurent tail of the Laurent series of $h - p(z)$ in the coordinate z up to the term of order m, and notice that it has at least one fewer term than p(z). Therefore, by induction hypothesis, there exists a meromorphic function f such that it's Laurent tail around r is q(z). Then, $g = h - f$ has Laurent tail p(z). □

We now extend the above results to get meromorphic functions with specific properties at several points.

Lemma 6.1.3. *Let R be an algebraic curve containing points* $r, s_1, \cdots, s_n$. *Then there exists a meromorphic function g which has a zero at r and poles at* $s_1, \cdots, s_n$.

Proof. We prove this be induction on n.
For n = 1 : Since R is an algebraic curve, there exists a meromoprhic function g such that $g(r) \neq g(s)$. If g has a pole at r, we work with $1/g$. So, WLOG, g does not have a pole at r. Then $f = g - g(p)$ has a zero at r. If it has a pole at s, we are done. Else, $h = \frac{f}{f - f(s)}$ gives us the required meromorphic function.
If the statement is true for $k < n$, then for n : By the previous case, there exists a meromorphic function f having a zero at r and a pole at s. Also, by induction hypothesis there exists a meromorphic function h having a zero at r and poles

at $s_1, \cdots, s_{n-1}$. Then, $g = h - f^t$ has a zero at r and poles at $s_1, \cdots, s_n$, for any $t > max\{|ord_{s_i} h|\}_1^n$. Clearly, it has a zero at r, as both h and f do. Further, for $1 \leq i \leq n-1$, if f does not have a pole at s_i, then g does because h does. If f has a pole at s_i, then by the choice of t, we have that the order of pole of f^t is larger than that of h, and hence g has a pole at s_i.
By a similar argument, g has a pole at s_n also, and we are done. □

The following lemma is a refinement of the above result.

Lemma 6.1.4. *Let R be an algebraic curve containing points r,$s_1, \cdots, s_n$. Let m be an integer. Then there exists a meromorphic function g such that $ord_r(g-1) \geq m$ and $ord_{s_i}(g) \geq m$ $\forall i$.*

Proof. By previous lemma, there exists a function h having a zero at r and poles at $s_1, \cdots, s_n$. Then, the meromorphic function $g = \frac{1}{1+h^m}$ satisfies the required properties. □

All of the above results combined give us the following result, known as the Laurent Series Approximation lemma.

Lemma 6.1.5. *Let R be an algebraic curve containing points $s_1, \cdots, s_n$. Around each of these points, first choose and fix local coordinates z_i respectively. Further, choose Laurent polynomials $p_i(z_i)$ for each point. Then there exists a meromorphic function g having Laurent tail $p_i(z_i)$ at s_i for each i.*

Proof. Fix an integer M larger than every exponent in each $p_i(z_i)$. We can then think of each of these Laurent polynomials, as Laurent polynomials with terms up till $M-1$ (with certain coefficients zero, if necessary). We know there exists meromorphic functions f_i with Laurent tails p_i at s_i.
Let $A = min\{ord_{s_i} p_i\} = min\{ord_{s_i} f_i\}$. By the previous lemma, there exists meromorphic functions h_i such that,$\forall i$ $ord_{s_i}(h_i - 1) \geq M - A$ and $ord_{s_j}(h_i) \geq M - A$.
Then, $g = \sum_i f_i h_i$ is the required function, as the term $f_i h_i$ has Laurent tail p_i, and the rest of the terms have Laurent tail with lowest exponent greater than or equal to M. □

6.2 The function field M(R)

In this section we shall prove some results regarding the "size" of the field of meromorphic functions M(R).
As a reference for the requisite concepts about fields, take a look at [Lan05].
M(R) (for an algebraic curve R) is a finitely generated $\mathbb{C}$-algebra[1]. Further, it

[1] K is a finitely generated $\mathbb{C}$-algebra if there exists surjective ring homomorphism,

$$\psi : \mathbb{C}[X_1, \cdots, X_n] \to K$$

is a transcendental extension[2] of $\mathbb{C}$, of transcendence degree[3] 1. The proof for the finite generation will be skipped for now. The idea is to prove that for any non-constant meromorphic function(whose existence is guaranteed as R is an algebraic curve) g on R, M(R) is a finitely generated field extension of $\mathbb{C}(g)$. In fact, the degree of the extension is known too.

Theorem 6.2.1. *Let g be a non-constant meromorphic function, then*

$$[M(X) : \mathbb{C}(g)] = deg(D)$$

We now prove the result regarding the transcendence degree.

Theorem 6.2.2. *Let R be an algebraic curve. Then M(R) is a transcendental extension of $\mathbb{C}$, of transcendence degree 1.*

Proof. As $\mathbb{C}$ is algebraically closed, M(R) has to be a transcendental extension as R has non-constant meromorphic functions. Now suppose, contrary to what we want to prove, the transcendence degree is greater than equal to two. Then there exist algebraically independent elements $g, h \in M(R)$.
Note that we can write the corresponding principal divisors as,

$$div(g) = P_1 - N_1, div(h) = P_2 - N_2$$

Fix a point r in R, and define the divisor

$$D = (1 + max\{deg(N_1), deg(N_2)\}) \cdot r$$

Then we have that

$$g^i h^j \in L(mD), \forall i, j \geq 0, i + j \leq m$$

Each of these are linearly independent, as g and h are algebraically independent, and hence we have that,

$$dim(L(mD)) \geq (m + 1)(m + 2)/2$$

But, from the crude bound on divisors by previous chapter, using the fact that $D \geq 0$, we have,

$$dim((L(mD))) \leq 1 + deg((mD)) = 1 + deg(D) \cdot m$$

These two bounds are clearly incompatible for large values of m, and we get a contradiction. □

[2]A field extension L/K is transcendental, if there exists an element $l \in L$ which is not the root of any non-zero polynomial with coefficients in K.

[3]If L/K is a transcendental extension, then transcendence degree is the cardinality of the smallest subset S of L, such that L/K(S) is an algebraic extension.

6.3 Laurent tail divisors and the Mittag-Leffler problem

We shall now define a generalisation of divisors, from an integer linear combination of points, to that of Laurent tail divisors.

Definition. *Let R be a compact Riemann surface. For every point r in R, fix a local coordinate z_r throughout this discussion. A Laurent tail divisor is a finite formal sum of the form,*

$$\sum_r p_r(z_r) \cdot r$$

where $p_r(z_r)$ are Laurent polynomials. The set of all Laurent tail divisors on R, denoted by $T(R)$, forms a group. We can define a subgroup of this space, associated to any divisor as follows,

$$T[D](R) = \{\textstyle\sum_r p_r(z_r) \in T(R) : \ \forall r \text{ with } p_r \text{ non-zero, the leading term of } p_r \text{ has degree strictly less than } -D(r)\}$$

We now define certain natural maps involving the above defined spaces. Let D_1 and D_2 be two divisors on R such that $D_1 \leq D_2$, then we can define the truncation map

$$t = t_{D1}^{D_2} : T[D_1](X) \to T[D_2](X)$$

by truncating the Laurent tails when necessary, that is, removing terms of degree $-D_2(r)$ to $-D_1(r) - 1$. Similarly, given a divisor D, we have the map

$$\alpha_D : M(R) \longrightarrow T[D](R)$$

by removing terms of degree greater than or equal to $-D(r)$.

As a refinement of the series of lemma proved in the first section of this chapter, we will now consider the problem of constructing meromorphic functions with specified tails at a finite collection of points, and no poles elsewhere. This problem is known as the Mittag-Leffler problem. With the above defined maps, the problem boils down to the surjectivity of the map α_D.
With this in mind, we give a name to the cokernel of the map,

$$H^1(D) = coker(\alpha_D) = T[D](R)/image(\alpha_D)$$

Then there is an exact sequence associated to any divisor D on R,

$$0 \longrightarrow L(D) \longrightarrow M(R) \xrightarrow{\alpha_D} T[D](R) \longrightarrow H^1(D) \longrightarrow 0$$

Given two divisors D_1 and D_2, such that $D_1 \leq D_2$, we have that,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M(R)/L(D_1) & \xrightarrow{\alpha_{D_1}} & T[D_1](R) & \longrightarrow & H^1(D_1) & \longrightarrow & 0 \\ & & \downarrow \psi_1 & & \downarrow t_{D_1}^{D_2} = \psi_2 & & \downarrow \psi_3 & & \\ 0 & \longrightarrow & M(R)/L(D_2) & \xrightarrow[\alpha_{D_2}]{} & T[D_2](R) & \longrightarrow & H^1(D_2) & \longrightarrow & 0 \end{array}$$

For this diagram, we have used the fact that $D_1 \leq D_2$ implies $L(D_1) \subseteq L(D_2)$, that α commutes with the truncation maps along with the short exact sequence corresponding to the exact sequence we defined for any divisor D.
This commutative diagram is exactly the setting for the Snake lemma [4]. As the ψ_i's are all surjective, the corresponding cokernels are zero, and hence we obtain the following exact sequence by the Snake lemma,

$$0 \longrightarrow ker(\psi_1) \longrightarrow ker(\psi_2) \longrightarrow ker(\psi_3) \longrightarrow 0$$

and hence we can have a relation between their dimensions. We compute the dimensions of the kernels,

$$ker(\psi_1) = L(D_1)/L(D_2) \implies dim(ker(\psi_1)) = dim(L(D_1)) - dim(L(D_2))$$

For ψ_2, note that the kernel is obtained by removing terms of degree $-D_2(p)$ to $-D_1(p) - 1$. Hence, we have,

$$dim((ker(\psi_2)) = dim(ker(t_{D_1}^{D_2})) = \sum_{r \in R}(D_2(r) - D_1(r)) = deg(D_2) - deg(D_1)$$

By the finiteness of the dimension of the kernel of ψ_2, we have that the kernel of ψ_3 is also finite dimensional. We denote it by a special symbol,

$$ker(\psi_3) =: H^1(D_1/D_2)$$

By the above exact sequence, and the finite dimensionality of all the kernels, we have the following result.

Lemma 6.3.1. *For divisors $D_1 \leq D_2$ on a compact Riemann surface R,*

$$dim\left(H^1(D_1/D_2)\right) = [deg(D_2) - dim(L(D_2))] - [deg(D_1) - dim(L(D_1))]$$

So, $H^1(D_1/D_2)$ is a finite dimensional vector space for all divisors $D_1 \leq D_2$.

6.4 Dimension of $H^1(D)$ and the Riemann-Roch theorem

We will now prove that the space $H^1(D)$ is "small", that is, it is a finite dimensional space. This will directly lead to us to the Riemann-Roch theorem.

Lemma 6.4.1. *Let g be a non-constant meromorphic function on R, an algebraic curve. Then, consider the divisor of poles,*

$$D = div_p(g) = \sum_{r \in R} -min\{ord_r(f), 0\} \cdot r$$

Then $dim\left(H^1(0/nD)\right)$ is constant for large positive integer values of n.

[4] Refer to [AM94]

Proof. We shall give a sketch of this proof.
We use the following fact without proof : The number of linearly independent elements in M(R) over $\mathbb{C}(g)$ gives us a lower bound[5] on the dimension of L(nD) for large values of n. That is, there exists an number N, such that for all $n > N$

$$dim(L(nD)) \geq (n - N + 1)deg(D)$$

On the other hand, by the Lemma 3.1, we have,

$$dim\left(H^1(0/nD)\right) = [deg(nD) - dim(L(nD))] - [deg(0) - dim(L(0))]$$

Using the lower bound, we get,

$$dim\left(H^1(0/nD)\right) \leq 1 + deg(D)(N - 1)$$

Hence, as the spaces $H^1(0/nD)$ are non-decreasing in n, we get that for large n, their dimension is constant □

The above lemma is used to prove the following generalisation. We omit the proof here.

Lemma 6.4.2. *There exists a number M(R) for any algebraic curve R, such that for any divisor D on R,*

$$deg(D) - dim(L(D)) \leq M(R)$$

By the above bound, we have that there exists a divisor D_m such that $deg(D_m) - dim(L(D_m))$ is maximum possible for any divisor on R.

Lemma 6.4.3. $H^1(D_m) = 0$

Proof. Suppose not. That is, the map α_{D_m} is not surjective.
Let $A \in T[D_m](R) - \alpha_{D_m}(M(R))$. Then, there is a divior $D > D_m$ such that $t^D_{D_m}(A) = 0$. Hence, the kernel of $t^D_{D_m}$, that is, $H^1(D_m/D)$ is non-zero, and hence $dim\left(H^1(D_m/D)\right) \geq 1$
But, we know by lemma 3.1,

$$dim\left(H^1(D_m/D)\right) = [deg(D) - dim(L(D))] - [deg(D_m) - dim(L(D_m))] \leq 0$$

by the way we chose D_m. This is a contradiction, and hence we have that contrary to our initial assumption, $H^1(D_m) = 0$ □

This lemma allows us to prove the finite-dimensionality of $H^1(D)$.

Theorem 6.4.1. *$H^1(D)$ is finite dimensional as a $\mathbb{C}$-vector space for any divisor D on an algebraic curve R.*

[5] For the proof, refer to [Mir95].

Proof. Consider the divisor $D - D_m$. Recall, we can write any divisor as the difference of non-negative divisors, its positive and negative parts. Let $D - D_m = P - N$. Then, as $D_m + P \geq D_m$, we know that $t_{D_m}^{D_m+P}$ gives a surjection from $H^1(D_m)$ onto $H^1(D_m + P)$. But, $H^1(D_m) = 0$, and hence we have, $H^1(D_m + P) = 0$.
From this, we get that, $H^1(D_m + P - N) \cong H^1(D_m + P - N/D_m + P)$. But, by lemma 3.1 $H^1(D_m + P - N/D_m + P)$ is finite dimensional, and as $D = D_m + P - N$, we are done.

□

This brings us to the Riemann-Roch theorem.

Theorem 6.4.2. *Riemann-Roch theorem : For any divisor D on an algebraic curve X,*

$$dim(L(D)) - dim(H^1(D)) = 1 + deg(D) - dim(H^1(0))$$

Proof. As $H^1(D)$ is finite dimensional for any divisor, we have that for divisors $D_1 \leq D_2$,

$$dim(H^1(D_1/D_2)) = dim(H^1(D_2)) - dim(H^1(D_1))$$

Hence, by lemma 3.1 we have that,

$$dim(L(D_2)) - deg(D_2) - dim(H^1(D_2)) = dim(L(D_1)) - deg(D_1) - dim(H^1(D_1))$$

Note that for any divisor D, we can write it as $P - N$, and hence we have that,

$$dim(L(D)) - deg(D) - dim(H^1(D)) = dim(L(P)) - deg(P) - dim(H^1(P))$$

as $D \leq P$. Further, as $0 \leq P$, we have,

$$dim(L(0)) - dim(H^1(0)) = dim(L(P)) - deg(P) - dim(H^1(P))$$

and by noting that $dim(L(0)) = 1$, we get that required equality. □

6.5 Serre Duality

The formula in the statement of Riemann-Roch theorem involves the dimension of $H^1(D)$, whose calculation is done using the subject matter of this section, the Serre Duality theorem.
For this, we shall begin by defining a function on T[D](R) using the Residue theorem. Fix a canonical divisor $u \in L^1(-D)$ for a divisor D on R. Then, $ord_r(u) \geq D(r)$ for any point r in R, and hence we can write u around r as,

$$u = \left(\sum_{D(r)}^{\infty} a_n z_r^n \right) dz_r$$

Now for a meromorphic function g, near a point r we can write it as the Laurent series $g = \sum b_k z^k$. Then the residue of the 1-form gu at r is exactly the coefficient of z^{-1} in the product, which is given by,

$$Res_r(gu) = \sum_{k=D(r)}^{\infty} a_n b_{-1-n}$$

So, it only depends on the information contained in $\alpha_D(g)$.
We can thus define the map, for $u \in L^1(-D)$,

$$Res_u : T[D](R) \to \mathbb{C}$$

by $Res\left(\sum p_r \cdot r\right) = \sum_r Res_r(p_r u)$. Then, in terms of this map, our earlier observation on the calculation of the residue of gu for a meromorphic function g translates to,

$$Res_u(\alpha_D(g)) = \sum_r Res_r(gu) = 0$$

by the Residue theorem, as gu is a meromorphic 1-form.
Hence, Res_u is 0 on the space $\alpha_D(M(R))$, and hence, it can be thought of as a linear functional on the quotient space $H^1(D)$ too.
Hence, we get a map,

$$Res : L^1(-D) \to H^1(D)^*$$

by sending the 1-form u to the linear functional Res_u in the dual space $H^1(D)$. This map is called the Residue map. The Serre duality theorem tells us that this map is in fact an isomoprhism of vector spaces, giving us a way to compute the dimension of $H^1(D)$. We shall prove the injectivity of the map, leaving the proof of surjectivity, which although not very difficult, is involved and based on a couple of technical lemmas.

Theorem 6.5.1. *The map,*

$$Res : L^1(-D) \to H^1(D)^*$$

is an isomorphism of $\mathbb{C}$ *vector spaces.*

Proof. Injectivity : Suppose the map is not injective, then there exists a non-zero meromorphic 1-form $u \in L^1(-D)$ such that for all $\sum p_r \cdot r \in T[D](R)$,

$$Res_w\left(\sum p_r \cdot r\right) = \sum_r Res_r(p_r u) = 0$$

As $u \neq 0$, there exists a point r such that near r, $w = \left(\sum_{k=n}^{\infty} a_k z_r^k\right) dz_r$, where $n = ord_r(u)$ and $a_n \neq 0$.
As $n \geq D(r)$, we have that $-n-1 < D(r)$ and hence, we have that $z_r^{-n-1} \cdot r \in T[D](R)$. But, this implies,

$$Res_u(z_r^{-n-1} \cdot r) = Res_r\left(z_r^{-n-1} \sum_{k=n}^{\infty} a_k z_r^k\right) = a_n$$

But $a_n \neq 0$, and we arrive at a contradiction.

□

Hence, as this is an isomorphism of vector spaces, we have that for any canonical divisor U,

$$dim(L(U-D)) = dim(L^1(-D)) = dim(H^1(D)^*) = dim(H^1(D))$$

as we had proven that $L(U+D) \cong L^1(D)$ for any divisor D and canonical divisor U. Now, we shall use the Serre Duality theorem to get the dimension of $H^1(0)$.
Recall that we had computed the degree of any canonical divisor on an compact Riemann surface of genus g, which was $2g-2$. Also, by Serre Duality, we have for any canonical divisor U,

$$dim(H^1(U)) = dim(L(U-U)) = dim(L(0)) = 1$$

Also, as $dim(H^1(0)) = dim(L(U))$ by Serre Duality, we have by Riemann-Rcoh theorem that,

$$dim(L(U)) - dim(H^1(U)) = deg(U) + 1 - dim(H^1(0))$$

Substituting in the above relations, we get,

$$dim(H^1(0)) - 1 = (2g-2) + 1 - dim(H^1(0))$$

So, we have that,

$$dim(H^1(0)) = dim(L(U)) = dim(L^1(0)) = g$$

Using this, we can write the Riemann-Roch theorem in the following form,

Theorem 6.5.2. *For any divisor D, and conical divisor U on an algebraic curve R of genus g, we have that*

$$dim(L(D)) - dim(L(U-D)) = deg(D) + 1 - g$$

In the case that $deg(D) \geq 2g-1$, then $dim(H^1(D)) = dim(L(U-D)) = 0$, and hence,

$$dim(L(D)) = deg(D) + 1 - g$$

6.6 Genus 1 curves are complex tori

As an application of the Riemann-Roch theorem, we shall give a sketch of the proof that every curve of genus 1 is isomorphic as a Riemann surface to a complex torus.
Let R be a curve of genus 1. Then topologically it is a torus, and hence we know that it has a universal cover, say S, with the projection map $p : S \to R$. Topologically the space S is isomorphic to R^2, and the fundamental group of R,

which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, acts as a pair of linearly independent translations on S. Further, S has a canonical Riemann surface structure using the map p, as it is a covering map.

If we can prove that this Riemann surface S, is isomorphic to the Riemann surface $\mathbb{C}$, then (recall this is the same method we used to define a complex torus) we will have that R is a quotient of $\mathbb{C}$ by a lattice, that is, it is a complex torus.

If $U = div(u)$ is a canonical divisor on R, then we have that $deg(U) = 2g(R) - 2 = 0$ and $dim(L(U)) = g(R) = 1$.

Let $g \in L(U)$, then $ord_r(gu) \geq 0$ for every point r in R, and hence gu is a holomorphic 1-form on R, with $deg(gu) = 0$, and hence it can have no zeroes(or poles, as it is holomorphic). Let us denote $gu = u'$.

Consider the pullback p^*u, and by the calculation we had done of the order of the pullback of a 1-form, we have that it also is holomorphic and has no zeroes on S. Fix a point s_0 on S. Then, we define the map $G : S \to \mathbb{C}$ as,

$$G(s) = \int_{s_0}^{s} p^*u$$

where the integral is over any path joining s_0 and s. This is well-defined as S is simply connected and p^*w is a holomorphic 1-form. Because S is simply connected, any two paths are homotopic. Further, to prove that the integral remains the same over the homotopy, we break the path into small regions, which we can transport to the complex plane using the charts, and there, as the 1-form is holomorphic we can apply the same methods as we use to prove the corresponding result in coomplex analysis. This map gives us the required isomorphism of Riemann surfaces.

Chapter 7

Jacobian and Abel's theorem

7.1 Introduction

Recall that for a meromorphic function g on a compact Riemann surface R, $\sum_{r\in R} ord_r(g) = 0$. Equivalently, the degree of any principal divisor is zero. But the converse is not necessarily true. In this chapter, we will look at a necessary and sufficient condition for a divisor of degree zero to be principal.

Recall the Standard Identified Polygon with $4g$ sides. It gives us a compact, orientable surface of genus g, say R. Note that every vertex of the polygon corresponds to a single point in R. Let us relabel the 4g pair of sides as $\{a_1, b_1, a'_1, b'_1, \cdots b'_g\}$. Then, the $2g$ cycles (closed loops) in R, corresponding to $\{a_i\}_1^g$ and $\{b_i\}_1^g$ generate the free group $H_1(R)$, the first homology group[1] of R (with integer coefficients).
Recall that for any holomorphic 1-form u, the 2-form $du = 0$. Therefore, for any triangulable subset D of a Riemann surface R, we have by Stoke's theorem that,

$$\int_{\partial D} u = \iint_D du = 0$$

Hence, the integral of a holomorphic 1-form is zero on any boundary. And therefore, the integral of a holomorphic 1-form is well defined for a homology class $[a] \in H_1(R)$, as any two cycles representing a homology class differ by a boundary.
Hence, if we denote the space of holomorphic 1-forms as $\Omega^1(R)$, we have a linear functional belonging to $\Omega^1(R)^*$ corresponding to every homology class $[a]$, sending holomorphic 1-form to the integral of the 1-form over the cycle a.

[1] refer to [Mun84] for a discussion on homology. Specifically, here we are working with simplicial homology.

Such linear functionals are called periods, and their group, which is a subgroup of $\Omega^1(R)^*$, is denoted by Λ.

Definition. *For a compact Riemann surface R, we define the Jacobian of R as,*

$$J(R) = \frac{\Omega^1(R)^*}{\Lambda}$$

For a complex torus, the Jacobian turns out to be isomorphic to the space as a group.

Theorem 7.1.1. *For a complex torus R, $J(R) \cong R$ as a group.*

Proof. Let $R = \mathbb{C}/L_t$ be a complex torus. We know that the space of holomorphic 1-forms $\Omega^1(R) = L^1(0)$, and hence, as we had calculated $dim(L^1(0)) = g(R)$, we get that $dim(\Omega^1(R)) = 1$.
Let $R = \mathbb{C}/L_t$ be a complex torus. We can think of it as the identified parallelogram with vertices $\{0, 1, t, 1+t\}$.
Note that the form locally defined as dz generates the space of holomorphic 1-forms, and as the space is 1-dimensional, we have that $\Omega^1(R)^* \cong \mathbb{C}$. Now, as we have discussed earlier, the space of periods is generated by the linear functionals corresponding to the sides of the identified polygon. So, we have that $\int_0^t$ and $\int_0^1$ generate the group Λ. Evaluating them on the 1-form dz, we get that, $J(R) \cong \mathbb{C}/L_t \cong R$, and we are done. □

We now define a map from the space to its Jacobian.

Definition. *Fix a base point r_0 in R. For every point r in R, choose and fix a path d_r joining r_0 and r. We define the map Abel-Jacobi map,*

$$A : R \to J(R)$$

by sending the point r, to the coset of the functional $\int_{d_r}$. This is well defined, as, if we take a different path e_r joining r_0 and r, we have that the difference of the two functionals is the functional given by integration along the cycle $d_r - e_r$, which is a period.

We have defined the Abel-Jacobi map for points as of now, but we can easily extend it to divisors linearly to get a map

$$A : D(R) \to J(R)$$

defined as follows,

$$A\left(\sum m_r r\right) = \sum m_r A(r)$$

We are interested in the restriction of the map to the divisors of degree zero, as they are the object of interest in this chapter. The following lemma shows that the map is base point independent on this space.

Lemma 7.1.1. *Let us denote the space of divisors of degree zero by $D_0(R)$. Then the Abel-Jacobi map restricted to $D_0(R)$, denoted by A_0 is base-point independent.*

Proof. Let $\{u_i\}_1^g$ be a basis of $\Omega^1(R)$, then any element of $\Omega^1(R)^*$ can be uniquely determined by their evaluation over this basis.
Now, choose a new base point r_0', and denote the corresponding map by A_0'. Let d be a path joining r_0 to r_0', then we have for any point r in R,

$$A_0'(r) - A_0(r) = \left(\int_d u_1, \cdots, \int_d u_g \right) \text{mod}\Lambda = l$$

Note that $l \in J(R)$ is independent of the point r. And hence, for $(\sum m_r r) \in D_0(R)$,

$$(A_0 - A_0') \left(\sum m_r r \right) = \sum m_r r \cdot l = \left(\sum m_r r \right) \cdot l = 0$$

□

We shall now state the main result of this chapter, the Abel's theorem.

Theorem 7.1.2. *Consider a compact Riemann surface R. Then a divisor D having degree zero is a principal divisor iff $A_0(D) = 0 \in J(R)$.*

We shall prove the sufficiency of the criterion $A_0(D) = 0$, leaving the necessity, the proof of which is based on the properties of Trace of a functions and 1-form, which is a converse operation to the pullback.

7.2 Riemann Bilinear relations and proof of sufficiency in Abel's theorem

Given a 1-form u, we define the following quantities,

$$A_i(u) = \int_{a_i} u, \; B_i(u) = \int_{b_i} u$$

for i between 1 and g, where a_i and b_i are the closed paths corresponding to the sides of the polygon. These are called the a-periods and the b-periods corresponding to the form u respectively.
Recall the Standard Identified Polygon P_R corresponding to a compact Riemann surface R of genus g. Any 1-form on R, may be considered as a form on P_R too. Consider a smooth 1-form u such that the corresponding 2-form $du = 0$. Such forms are called closed. Fix a point y in the interior of P_R. Then, for a closed form u, we define the function g_u on P_r,

$$g_u(x) = \int_{d_x} u$$

for any point $x \in P_R$ and any path d_r joining y and x in P_R. This function is well-defined (that is, independent of the choice of the path) as, given a different path d'_x, we have that,

$$\int_{d_x} u - \int_{d'_x} u = \int_{d_x - d'_x} u = \int_{int(d_x - d'_x)} du = 0$$

where we have used the Stoke's theorem, and the fact that P_R is simply connected.
The function g_u is smooth on P_R and by the Fundamental Theorem of Calculus, $dg_u = u$. The following lemma gives us a useful technical result related to this function.

Theorem 7.2.1. *Given closed smooth 1-forms u and v on a compact Riemann surface R, we have,*

$$\int_{\partial P_R} g_u v = \sum_{i=1}^{g} [A_i(u)B_i(v) - A_i(v)B_i(u)]$$

where $\partial P_r = \sum_1^g (a_j + b_j - a'_j - b'_j)$ *is the chain corresponding to the boundary of the polygon. This holds even if v is a meromorphic 1-form with no poles on the boundary of* P_R.

Proof. Given a point x on a_i, let x' be the point it is identified to on the side a'_i. Let γ be a path joining these two points. Then, the closed path γ in R is homotopic to the closed path b_i in R. Hence,

$$g_u(x') - g_u(x) = \int_\gamma g_u = \int_{b_i} g_u = B_i(u)$$

Similarly, for a point x on b_i, corresponding point x' on b'_i, and a path γ joining them, the closed path γ in R is homotopic to the closed path $a'_i = -a_i$ in R. Hence,

$$g_u(x') - g_u(x) = \int_\gamma g_u = \int_{-a_i} g_u = -A_i(u)$$

Note that since v is a 1-form on R, its values on corresponding points of a_i and a'_i and b_i and b'_i is equal.

From the above discussion, we have that,

$$\begin{aligned}
\int_{\partial P_R} g_u v &= \sum_1^g \left[\int_{a_i} g_u v + \int_{-a_i'} g_u v + \int_{b_i} g_u v + \int_{b_i'} g_u v \right] \\
&= \sum_1^g \left[\int_{a_i} g_u v - \int_{a_i'} g_u v + \int_{b_i} g_u v - \int_{b_i'} g_u v \right] \\
&= \sum_1^g \left[\int_{x \in a_i} (g_u(x) - g_u(x')v(x) \right] + \sum_1^g \left[\int_{x \in b_i} (g_u(x) - g_u(x'))v \right] \\
&= \sum_1^g -B_i(u) \int_{a_i} v + \sum_1^g A_i(u) \int_{b_i} v \\
&= \sum_{i=1}^g [A_i(u)B_i(v) - A_i(v)B_i(u)]
\end{aligned}$$

□

The above lemma gives us the following result.

Lemma 7.2.1. *Let u be a non-zero holomorphic 1-form on R. Then, we have the following inequality,*

$$Im \left(\sum_1^g A_i(u) \overline{B_i(u)} \right) < 0$$

Proof. Given a local coordinate z, if $u = h(z)dz$, then we have that $u \wedge \overline{u} = |h(z)|^2 du \wedge d\overline{u} = -2ih(x,y)dx \wedge dy$, and therefore, using the previous lemma we have,

$$\begin{aligned}
0 &> Im \left(\iint_{P_R} u \wedge \overline{u} \right) \\
&= Im \left(\iint_{P_R} (dg_u \wedge \overline{u} + g_u \wedge d\overline{u}) \right) \text{ as } dg_u = u,\, d\overline{u} = 0 \\
&= Im \left(\iint_{P_R} d(g_u \overline{u}) \right) = Im \left(\int_{\partial P_R} g_u \overline{u} \right) \\
&= Im \left(\sum_{i=1}^g [A_i(u)B_i(\overline{u}) - A_i(\overline{u})B_i(u)] \right) \\
&= Im \left[2i \cdot Im \left(\sum_{i=1}^g A_i(u)B_i(\overline{u}) \right) \right] \\
&= 2Im \left(\sum_{i=1}^g A_i(u)B_i(\overline{u}) \right)
\end{aligned}$$

and we are done. □

The above lemma immediately gives us that if for a holomorphic 1-form $A_i(u) = 0 \ \forall i$ or $B_i(u) = 0 \ \forall i$, then $u = 0$.
If we fix a basis $\{u_i\}_1^g$ of $\Omega^1(R)$, then we can define the following matrices

Definition. *Define the period matrices $\mathcal{A}$ and $\mathcal{B}$ as,*

$$(\mathcal{A})_{ij} = A_i(u_j), (\mathcal{B})_{ij} = B_i(u_j)$$

We shall now show that these matrices are invertible.

Lemma 7.2.2. *The matrices $\mathcal{A}$ and $\mathcal{B}$ are invertible.*

Proof. Suppose that $\mathcal{A}$ is not invertible, then there exists a non-zero vector $a = (a_i)_{g\times 1}$ such that $\mathcal{A}a = 0$. Then, as $a \neq 0$, the form $u = \sum_1^g a_i u_i \neq 0$. But,

$$0 = (\mathcal{A}a)_i = \sum_{j=1}^{g} A_i(u_j)c_j = A_i(u)$$

and hence $u = 0$, a contradiction. The proof for invertibility of B follows the same way. □

The period matrices satisfy the following property.

Lemma 7.2.3. *The matrices $\mathcal{A}$ and $\mathcal{B}$ satisfy,*

$$\mathcal{A}^{Tr}\mathcal{B} = \mathcal{A}\mathcal{B}^{Tr}$$

Proof. Let $1 \leq j, l \leq g$. Then by Thereom 2.1, we have that

$$\int_{\partial P_R} g_{u_j} u_l = \sum_{i=1}^{g} [A_i(u_j)B_i(u_l) - A_i(u_l)B_i(u_j)]$$

Then, note that as u_j and u_l are holomorphic 1-forms, we have that $u_j \wedge u_l = 0$ and $dw_l = 0$,

$$\int_{\partial P_R} g_{u_j} u_l = \iint_{P_R} d(g_{u_j} u_l) = \iint_{P_R} (u_j \wedge u_l + g_{u_j} du_l) = 0$$

Which implies,

$$\left(\mathcal{A}^{Tr}\mathcal{B}\right)_{jl} = \sum_{i=1}^{g} A_i(u_j)B_i(u_l) = \sum_{i=1}^{g} A_i(u_l)B_i(u_j) = \left(\mathcal{A}\mathcal{B}^{Tr}\right)_{jl}$$

□

The following definition will help prove the Riemann bilinear relations.

Definition. *As the period matrices are invertible, there exists a basis, say $\{u_i'\}_1^g$ of $\Omega^1(R)$ such that $\mathcal{A} = \mathcal{I}_{g\times g}$ in that basis.*
The matrix $\mathcal{B}$ in this basis is said to be normalized.

We will briefly show how to get such a basis. As $\mathcal{A}$ is invertible, there exist a series of elementary column transformations which when applied on $\mathcal{A}$, give us the identity matrix. Let the column vectors of $\mathcal{A}$ be $\{C_i\}_1^g$. Then corresponding to the three elementary column transformations,
1. interchange the i^{th} and the j^{th} column,
2. replace C_i with lC_i for some non-zero constant l, and
3. replace C_i with $lC_j + C_i$ for $j \neq i$, we perform a corresponding operation on the basis,
1. interchange u_i with u_j,
2. replace u_i with lu_i for the same non-zero constant l, and
3. replace u_i with $lu_j + u_i$.
The basis we get at the end of the process will be the required one, called the normalized basis.

Lemma 7.2.4. *A normalized b-period $\mathcal{B}$ matrix satisfies the following properties:*
1. $\mathcal{B}$ is symmetric.
2. The imaginary part of $\mathcal{B}$ is positive definite.
These are known as the Riemann Bilinear relations.

Proof. The first relation follows immediately from Lemma 2.3.
For the second relation, choose any non-zero vector $a = (a_i)_{g\times 1}$. Let $u = \sum_1^g a_i u_i'$, then by lemma 2.1, we have that,

$$0 > Im\left(\sum_1^g A_i(u)\overline{B_i(u)}\right) = Im\left(\sum_{i=1}^g \sum_{j=1}^g a_i a_j \overline{B_i(u)}\right)$$

Hence, we have that $Im(a^{Tr}\overline{\mathcal{B}}a < 0)$ for any vector a, and hence,

$$a^{Tr} \cdot Im(\mathcal{B}) \cdot a > 0$$

for any vector a, and hence $\mathcal{B}$ is a positive . □

These relations give us the structure of $J(R)$.

Definition. *A g-dimensional complex torus is a group of the type C^g/L, where L is a subgroup generated by $2g$ vectors independent over $\mathbb{R}$.*

Theorem 7.2.2. *J(R) is a g-dimensional complex torus.*

Proof. All we need to show is that Λ is a subgroup generated by $2g$ vectors independent over $\mathbb{R}$. But the a-periods and the b-periods generate Λ. Hence, all we need to show is that the columns of the period matrices are independent vectors.
We consider the basis in which $\mathcal{B}$ is normalized. Suppose the columns are not linearly independent over $\mathbb{R}$, then there exist vectors $a = (a_i)_{g\times 1}$ and $b = (b_i)_{g\times 1}$, not both 0, such that $a \cdot \mathcal{I} + b \cdot \mathcal{B} = 0$. But this implies, $Im(b\mathcal{B}) = b \cdot Im(\mathcal{B})0$, which by previous lemma implies $b = 0$. This further implies that $a = 0$ too, which is a contradiction. □

The following technical lemma is required to prove the sufficiency part in Abel's theorem. We leave the proof, which uses the above lemmas, and an application of Riemann-Roch theorem.

Lemma 7.2.5. *Let $D \in D_0(R)$ such that $A_0(D) = 0 \in J(R)$. Then there exists a meromorphic 1-form u satisfying the following conditions:*
1. u has a pole at r if and only if $D(r) \neq 0$
2. All poles of u are simple poles.
3. $Res_r(u) = D(r)$ for all points r.
4. The a-periods and b-periods of u are integer multiples of $2\pi i$.

We prove the sufficiency now.

Proof. Consider a divisor D of degree 0, with $A_0(D) = 0$. Then, we have a 1-form u satisfying the properties of the previous lemma. Let us choose and fix a point r_0 in R.
Using the fact that the residue at any pole of u is an integer, and that the a-periods and b-periods of u are integer multiples of $2\pi i$, we define the function g as,

$$g(r) = e^{\int_{d_r} u}$$

where d_p is any path joining r_0 and r.
Clearly, g is holomorphic away from the support of D. Now, support $D(r) = m_r \neq 0$. Then, in a local coordinate z, we have that, $u = m_r z^{-1} + h(z)$ where h is holomorphic. Hence, near r, we have that $\int_{d_z} u = m_r ln(z) + f(z)$, for a holomorphic function f. So, locally, $g(z) = z^{m_r} e^{f(z)}$, which is a meromorphic function, with $ord_r(g) = m_r = D(r)$. Hence, $D = div(g)$, where g is a meromorphic function, and hence it is a principal divisor, which was to be shown. □

We conclude this chapter by giving a proof of the fact that every curve of genus 1 is a complex torus.
We first prove that Abel-Jacobi map is injective for a large class of Riemann surfaces

Lemma 7.2.6. *The Abel-Jacobi map is injective for all Riemann surfaces R with genus greater than or equal to 1.*

Proof. Suppose not. Then there exist points s, r in R such that $A(r) = A(s)$, that is, for the divisor $s - r$, $A_0(s - r) = 0$. Then, by Abel's theorem, $s - r$ is a principal divisor, and hence there exists a meromorphic function on R having a simple zero at s, a simple pole at r, and no other zeroes or poles. Then, the associated holomorphic map to the Riemann sphere is non-constant and of degree 1, and therefore is an isomorphism. This is a contradiction, as R has genus greater than or equal to 1 while the Riemann sphere has genus zero.
Hence, the map is injective. □

Theorem 7.2.3. *Every compact Riemann surface with genus 1 is isomorphic to a complex torus.*

Proof. Let R be a compact Riemann surface with genus 1. Then, by Lemma 6.2.2, $J(R)$ is a complex torus. Further, as the Abel-Jacobi map is defined locally via an integral, it is a holomorphic map. By Lemma 6.2.6, it is injective. But we know that injective holomorphic maps between compact Riemann surfaces are isomorphisms (that is, biholomorphisms), and hence, we are done. □

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