Galois Cohomology for Lubin-Tate (φ_q, Γ_{LT})-Modules

Neha Kwatra

A thesis submitted for the partial fulfillment of

the degree of Doctor of Philosophy

Mathematical Sciences

Indian Institute of Science Education and Research Mohali

Knowledge city, Sector 81, SAS Nagar, Manauli PO, Mohali 140306, Punjab, India.

February 2020

Dedicated

to

My Parents

Declaration

The work presented in this thesis has been carried out by me under the guidance of Dr. Chandrakant S. Aribam at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Neha Kwatra

In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Chandrakant S. Aribam
(Supervisor)

Acknowledgements

It is my pleasure to acknowledge with gratitude all those who have helped and encouraged me at various stages.

First and foremost, I would like to express my deep and sincere gratitude to my research supervisor, *Dr. Chandrakant S. Aribam*, for his patience and invaluable guidance throughout my work. He suggested the topic and guided me to the relevant papers. Week after week, he tirelessly answered my questions as I struggled through the literature. I have matured much from his insight and ideas. The chance to witness mathematics from his vantage point has undoubtedly been the highlight of my thesis writing experience. Without his help and support, this thesis would not have taken its present shape. It was a great privilege and honor to work and study under his guidance.

Besides my supervisor, I would like to convey my gratitude to the referees of this thesis for going through it carefully and for providing valuable comments and suggestions. I would also like to acknowledge the rest of my thesis monitoring committee: *Dr. Varadharaj R. Srinivasan* and *Dr. Abhik Ganguli*, for their encouragement and insightful comments on my research work. My sincere thanks also go to *Prof. Sudesh Kaur Khanduja* and *Prof. Kapil Hari Paranjape* for their guidance and motivation and to *Dr. Amit Kulshrestha*, the Head of the Department of Mathematics, for being supportive. I am also grateful to *Dr. Suman Ahmed* for his constant advice and help.

I would like to thank the administrative system of IISER Mohali for supporting me with the excellent infrastructure and facilities. A sincere thanks to *Dr. P. Visakhi* for the library facility and for providing the subscription to various journals. I wholeheartedly acknowledge the financial support provided by IISER Mohali in the form of JRF/SRF.

I would like to thank all my teachers starting from my school days to date for guiding me in the right direction. In particular, I would take this opportunity to thank *Mrs. Saroj Jain*, who motivated me to pursue Mathematics during my college days.

I would like to thank all my school friends, batch-mates at Panjab University and IISER Mohali, with whom I shared good times and bad times as well. Many of them will recognize themselves. I thank all of them for their support and for the fun we had together. A special acknowledgement to *Manpreet Singh* for his help in formatting the entire thesis and to *Pronay Kumar Karmakar* for giving talks on "Class Field Theory".

Dear ones form the backbone of our life. Everything that I have accomplished in the years since has only been possible because of my parent's unwavering support, infinite patience and unconditional love. Thank you, *Mumma* and *Papa*: these pages are for you. I also want to thank my younger brothers, *Anil* and *Anuj*, for their love and care. Words are not enough to express my gratitude for them.

I conclude by thanking *the Divine power* for all the help that I got from expected and unexpected sources so far and for all situations and people I came across that helped me in some or the other way.

There must be few others whom I have failed to mention here, who have been there for me at some point of time during my Ph.D. period. I offer my sincere apology to them.

Neha Kwatra

Contents

Abstract

The classification of the local Galois representations using (φ, Γ)-modules by Fontaine has been generalized by Kisin and Ren over the Lubin-Tate extensions of local fields using the theory of (φ_q, Γ_{LT})-modules. In this thesis, we extend the work of (Fontaine) Herr by introducing a complex which allows us to compute cohomology over the Lubin-Tate extensions and compare it with the Galois cohomology groups. We further extend that complex to include certain non-abelian extensions. We then deduce some relations of this cohomology with those arising from (ψ_q, Γ_{LT})-modules. We also compute the Iwasawa cohomology over the Lubin-Tate extensions in terms of ψ_q-operator acting on étale (φ_q, Γ_{LT})-module attached to the local Galois representation. Moreover, we generalize the notion of (φ_q, Γ_{LT})-modules over the coefficient ring R and show that the equivalence given by Kisin and Ren extends to the Galois representations over R. This equivalence allows us to generalize our results to the case of coefficient rings.

Chapter 1

Introduction

It is well-known that the field $\mathbb{R}$ of real numbers is a completion of $\mathbb{Q}$. Besides this, associated to each prime number p, there is a field $\mathbb{Q}_p$ (the field of p-adic numbers), which is the p-adic completion of $\mathbb{Q}$. So along with the real world, there is a p-adic world for each prime number p. Ostrowski's theorem says that these are all the possible worlds. But the way these fields interact with each other is very mysterious.

The p-adic Hodge theory entirely lies in the p-adic world. It has two aspects: an arithmetic aspect to describe and classify the p-adic representations of the absolute Galois group G_K for a finite extension K of $\mathbb{Q}_p$ and a geometrical aspect to understand the p-adic representations of G_K from geometry.

In 1990, Fontaine introduced the notion of (φ, Γ)-modules in [10]. These (φ, Γ)-modules, which are utterly algebraic objects, give a remarkable and useful description of all the p-adic representations of G_K in terms of semi-linear algebra, and it is often easier to work in terms of semi-linear algebra than to work solely with Galois representations. A ring, namely $\widehat{\mathcal{E}^{ur}}$, is the base of the theory of (φ, Γ)-modules. This theory is the most powerful tool, which is currently available to study the p-adic representations of G_K. For instance, in [14], Ghate-Kumar used a similar technique to understand p-adic Galois representations.

Let K be a field complete with respect to a discrete valuation whose residue field

k is finite and of characteristic p, where p is a fixed prime number. In other words, K is a local field, and we denote by $G_K = \mathrm{Gal}(\bar{K}/K)$ the local Galois group. Recall that a $\mathbb{Z}_p$-adic representation of G_K is a $\mathbb{Z}_p$-module of finite rank with a continuous and linear action of G_K. For the Witt ring $W(k)$, let $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of $W(k)((u))$ with the field of fractions $\mathcal{E}$. Here $u = \varepsilon - 1$ and $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$ is a generator of the p-adic Tate-module of the multiplicative group $\mathbb{G}_m$, i.e., $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$ such that $\varepsilon_0 = 1, \varepsilon_1 \neq 1$, and $\varepsilon_{n+1}^p = \varepsilon_n$, (see [11, Chapter 4, Lemma 4.13]). Let K_{cyc} be the cyclotomic $\mathbb{Z}_p$-extension of K in $\bar{K}$ obtained by adjoining the p^n-th roots of unity to K, $H = \mathrm{Gal}(\bar{K}/K_{cyc})$ and $\Gamma = G_K/H = \mathrm{Gal}(K_{cyc}/K)$. Then there is a natural action of Γ and a Frobenius φ on $\mathcal{O}_{\mathcal{E}}$.

Fontaine's paper [10] mainly deals with the study of the $\mathbb{Z}_p$-adic representations of the absolute Galois group of a local field K by studying the (φ, Γ)-module attached to it. In the equal characteristic case ((p, p) case), he constructed a category of étale φ-modules over $\mathcal{O}_{\mathcal{E}}$ and proved that this category is equivalent to the category of $\mathbb{Z}_p$-adic representations of G_K. Then using the theory of the field of norms due to Fontaine and Wintenberger [36], he deduced the mixed characteristic case ($(0, p)$ case) from the equal characteristic case. In this case, the category of $\mathbb{Z}_p$-adic representations of G_K is equivalent to the category of étale (φ, Γ)-modules over $\mathcal{O}_{\mathcal{E}}$.

This equivalence is a deep result that allows the computation of Galois cohomology. In [17], Herr gave a technique to calculate the Galois cohomology by introducing a complex, namely, the *Herr complex*. The Herr complex is defined on the category of étale (φ, Γ)-modules and the cohomology groups of this complex turn out to match with the Galois cohomology groups on the category of $\mathbb{Z}_p$-adic representations of G_K. The results of Fontaine, along with this complex, play a crucial role in all the works pertaining to the computation of the Galois cohomology. In [34], Floric further extended the Herr complex to the False-Tate type curve extensions to include certain non-abelian extensions over the cyclotomic $\mathbb{Z}_p$-extension.

The well-known works of Colmez, Berger, Wach, and many others has centered on the case when K_{cyc} is the cyclotomic $\mathbb{Z}_p$-extension of K. Note that the extension K_{cyc} is obtained by adjoining the p^n-th roots of unity to K, and they are the p^n-torsion points of the multiplicative Lubin-Tate formal group $\mathbb{G}_m$ on $\mathbb{Q}_p$, with respect to the uniformizer p. Thus the cyclotomic $\mathbb{Z}_p$-extension is the same as the extension associated by Lubin-Tate theory to the multiplicative Lubin-Tate formal group. It is natural to try to carry out this theory for an arbitrary Lubin-Tate formal group $\mathcal{F}$ defined over K. In this direction, there has been a lot of activity in recent years to develop Fontaine theory for Lubin-Tate formal groups [20], [2], [12], [3], [29], [4], and [5], where the base field is a finite extension K of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and uniformizer π.

In [20], Kisin-Ren classified the local Galois representations using the extensions arising from division (torsion) points of the Lubin-Tate formal group over K. More precisely, consider a Lubin-Tate formal group $\mathcal{F}$ over a finite extension $K/\mathbb{Q}_p$, and for $n \geq 1$, let $K_n \subset K$ be the subfield generated by the π^n-torsion points of $\mathcal{F}$, where π is a uniformizer of $\mathcal{O}_K$. Define $K_\infty := \cup_{n\geq 1} K_n$ and $\Gamma_{LT} := \mathrm{Gal}(K_\infty/K)$. Then they obtained a classification of G_K-representations on finite $\mathcal{O}_K$-modules via étale (φ_q, Γ_{LT})-modules, where étale (φ_q, Γ_{LT})-modules are analogues of étale (φ, Γ)-modules ([20, Theorem 1.6]).

1.1 Main results

In this thesis, we compute the Galois cohomology of representations defined over $\mathcal{O}_K$, and the theorem of Kisin and Ren aids us in this computation. As a generalization of the Herr complex, we define a complex, namely, the *Lubin-Tate Herr complex* on the category of étale (φ_q, Γ_{LT})-modules over $\mathcal{O}_{\mathcal{E}}$ (Definition 5.4.1). Then using the Lubin-Tate Herr complex, we compute the Galois cohomology groups of representations defined over $\mathcal{O}_K$.

Our work depends heavily on the classification of G_K-representations given by

Kisin and Ren. First, we observe that Kisin-Ren theorem [20, Theorem 1.6] holds for $\mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$ the category of discrete π-primary abelian groups with a continuous and linear action of G_K. It is crucial to work with this category as this category has enough injectives, and this category is equivalent to the category of injective limits of π-power torsion objects in the category of étale (φ_q, Γ_{LT})-modules over $\mathcal{O}_{\mathcal{E}}$ (Proposition 5.1.1). Then we have the following result.

Theorem 1.1.1.(Theorem 5.5.2) *For a discrete π-primary abelian group V with a continuous and linear action of G_K, we have a natural isomorphism*

$$H^i(G_K, V) \cong \mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V))) \quad \textit{for } i \geq 0.$$

The cohomology groups on the right hand side are the cohomology groups of the Lubin-Tate Herr complex defined for étale (φ_q, Γ_{LT})-module attached to V, while the left hand side denotes the usual Galois cohomology groups of the representation V.

Then we show that both the cohomology functors commute with the inverse limits and deduce the above theorem for the case when V is a representation defined over $\mathcal{O}_K$ (Theorem 5.5.5).

We further extend the equivalence of categories of Kisin and Ren to include certain non-abelian extensions over the Lubin-Tate extension (Theorem 6.1.2) and show that the construction of the Lubin-Tate Herr complex for (φ_q, Γ_{LT})-modules can be generalized to $(\varphi_q, \Gamma_{LT,FT})$-modules over non-abelian extensions, and we call it the *False-Tate type Herr complex* (Definition 6.3.1). In this case, we establish the following theorem.

Theorem 1.1.2.(Theorem 6.4.1) *Let $V \in \mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$. Then we have*

$$H^i(G_K, V) \cong \mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT,FT}(\mathbb{D}_{LT,FT}(V))) \quad \textit{for } i \geq 0.$$

In other words, the False-Tate type Herr complex $\Phi\Gamma^{\bullet}_{LT,FT}(\mathbb{D}_{LT,FT}(V))$ *computes the Galois cohomology of* G_K *with coefficients in* V.

Next, we define an operator ψ_q acting on étale (φ_q, Γ_{LT})-modules, and then we prove the following result.

Theorem 1.1.3.(Theorem 7.2.6) *Let* $V \in \mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$. *Then we have a well-defined homomorphism*

$$\mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V)) \to \mathcal{H}^i(\Psi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V)) \quad \text{for } i \geq 0.$$

Further, the homomorphism

$$\mathcal{H}^0(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V)) \to \mathcal{H}^0(\Psi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V))$$

is injective.

In particular, if the action of $\gamma_1 - id$ is bijective on $\operatorname{Ker}\psi_M$ and M is a π-divisible module in the category $\varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}_{/\mathcal{O}_{\mathcal{E}}}$ of injective limits of π-power torsion étale (φ_q, Γ_{LT})-modules over $\mathcal{O}_{\mathcal{E}}$, then the co-chain complexes $\Phi\Gamma^{\bullet}_{LT}(M)$ and $\Psi\Gamma^{\bullet}_{LT}(M)$ are quasi-isomorphic (Remark 7.2.8). Moreover, we prove similar result in the case of False-Tate type extensions (Theorem 7.2.11). Next, we describe the Iwasawa cohomology in terms of the complex associated with ψ_q. We prove the following theorem.

Theorem 1.1.4.(Theorem 8.2.3) *For any* $V \in \mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$, *the complex*

$$\underline{\Psi}^{\bullet}(\mathbb{D}_{LT}(V(\chi_{cyc}^{-1}\chi_{LT}))) : 0 \to \mathbb{D}_{LT}(V(\chi_{cyc}^{-1}\chi_{LT})) \xrightarrow{\psi - id} \mathbb{D}_{LT}(V(\chi_{cyc}^{-1}\chi_{LT})) \to 0,$$

where $\psi = \psi_{\mathbb{D}_{LT}(V(\chi_{cyc}^{-1}\chi_{LT}))}$, *computes* $H^i_{Iw}(K_\infty/K, V)_{i\geq 1}$ *the Iwasawa cohomology groups.*

Thereafter, we use our techniques to deal with the case of coefficient rings. A coefficient ring is a complete local Noetherian ring with finite residue field.

In [9], Dee generalized Fontaine theory to the case of a general complete Noetherian local ring R, whose residue field is a finite extension of $\mathbb{F}_p$. He extended Fontaine's [10] results to the category of R-modules of finite type with a continuous R-linear action of G_K. He constructed a category of étale φ-modules (resp., étale (φ, Γ)-modules) over K parameterized by R and proved that this category is equivalent to the category of R-linear representations of G_K in the equal characteristic case (resp., mixed characteristic case) ([9, Theorem 2.1.27 and Theorem 2.2.1]). The category of étale φ-modules (resp., étale (φ, Γ)-modules) is defined to be a module of finite type over the completed tensor product $\mathcal{O}_{\mathcal{E}}\hat{\otimes}_{\mathbb{Z}_p}R$ with an action of φ (resp., φ and Γ) as in the case of Fontaine. The core point of the proof is Lemma 2.1.5. and Lemma 2.1.6. in [9]. Crucial in the proof of the equivalence of categories stated above, he used the results of Fontaine [10] for the case when the representation V has finite length. Then the general case was deduced by taking the inverse limits.

We also extend a result of Kisin and Ren ([20, Theorem 1.6]) to give a classification of the category of R-representations of G_K. We consider the category of étale (φ_q, Γ_{LT})-modules over the completed tensor product $\mathcal{O}_R := \mathcal{O}_{\mathcal{E}}\hat{\otimes}_{\mathcal{O}_K}R$, where the ring $\mathcal{O}_{\mathcal{E}}$ is constructed using the periods of Tate-module of $\mathcal{F}$. Then we prove that this category is equivalent to the category of R-representations of G_K. In the equal characteristic case, we show the following result.

Theorem 1.1.5.(Theorem 9.2.22) *The functor $V \mapsto \mathbb{D}_R(V)$ is an exact equivalence of categories between $\mathbf{Rep}_R(G_K)$ the category of R-representations of G_K and $\mathbf{Mod}^{\varphi_q,ét}_{/\mathcal{O}_R}$ the category of étale φ_q-modules over $\mathcal{O}_R$ with quasi-inverse functor $\mathbb{V}_R$.*

In the case of mixed characteristic, we have the following theorem which gives a classification of R-representations of the local Galois group in terms of étale

(φ_q, Γ_{LT})-modules over $\mathcal{O}_R$.

Theorem 1.1.6. (Theorem 9.2.25) *The functor $\mathbb{D}_R$ is an equivalence of categories between $\mathbf{Rep}_R(G_K)$ the category of R-linear representations of G_K and $\mathbf{Mod}_{/\mathcal{O}_R}^{\varphi_q,\Gamma_{LT},\acute{e}t}$ the category of étale (φ_q, Γ_{LT})-modules over $\mathcal{O}_R$. The functor $\mathbb{V}_R$ is a quasi-inverse of the functor $\mathbb{D}_R$.*

Then we generalize Theorem 1.1.1, Theorem 1.1.3 and Theorem 1.1.4 to the case of coefficient rings. The generalization of Theorem 1.1.4 to the case of coefficient rings allows us to generalize the dual exponential map

$$\mathrm{Exp}^* : H^1_{Iw}(K_\infty/K, \mathcal{O}_K(\chi_{cyc}\chi_{LT}^{-1})) \xrightarrow{\sim} \mathbb{D}_{LT}(\mathcal{O}_K)^{\psi_{\mathbb{D}_{LT}(\mathcal{O}_K)}=id}$$

defined in [29] over coefficient rings (Corollary 10.2.2). It is possible that this leads to the construction of Coates-Wiles homomorphisms for the Galois representations defined over R.

1.2 Outline of the thesis

In Chapter 2 and Chapter 3, we recall some necessary background that will be used in subsequent chapters. In Chapter 4, we give a sketch of the proof of the Kisin-Ren theorem. In Chapter 5, we define the Lubin-Tate Herr complex and compute the Galois cohomology groups of representations defined over $\mathcal{O}_K$. In the next chapter, we extend the Lubin-Tate Herr complex to include certain non-abelian extensions and show results in the computation of Galois cohomology. In Chapter 7, we define an operator ψ_q acting on the category of étale (φ_q, Γ_{LT})-modules and prove some results, which give a relationship between the cohomology groups of the Lubin-Tate Herr complex for φ_q and ψ_q. We also present our results, which give a relationship between the False-Tate type Herr complex for φ_q and ψ_q. In Chapter 8, we compute Iwasawa cohomology in terms of the complex associated with ψ_q. Then in Chapter 9, we record some significant results on coefficient rings, and we generalize a theorem of Kisin and Ren over coefficient rings, which in turn allows us to extend our results over coefficient rings, and these results appear in Chapter 10.

Chapter 2

Some Homological Algebra

In this chapter, we introduce some basic notions of homological algebra to make the thesis self-contained. In particular, we introduce the notion of double complex, total complex, and the spectral sequence associated to a double complex. These notions are very useful in studying the cohomology of the Galois groups. Most of these notions can be found in [25], [27], and [35].

2.1 Complexes and cohomology

Definition 2.1.1. A *co-chain complex* $(C^\bullet, d^\bullet)$ is a family $\{C^n\}_{n\in\mathbb{Z}}$ of abelian groups or modules together with the maps $d^n : C^n \to C^{n+1}$ such that each composite map $d^{n+1} \circ d^n : C^n \to C^{n+2}$ is zero.

The maps d^n are called the *differentials* of $C^\bullet$. The elements in the kernel of d^n are called *n-cocycles*, and the elements in the image of d^{n-1} are called *n-coboundaries*.

Definition 2.1.2. Let $(C^\bullet, d^\bullet)$ be a co-chain complex. Then the *n-th cohomology group* of $C^\bullet$ is the group of cocycles modulo coboundaries in degree n, i.e.,

$$H^n(C^\bullet) = \frac{\mathrm{Ker}(d^n)}{\mathrm{Im}(d^{n-1})}.$$

A co-chain complex $(C^\bullet, d^\bullet)$ is called *bounded* if almost all the C^n are zero. A co-chain complex $(C^\bullet, d^\bullet)$ is called *bounded above* (resp., *bounded below*) if there is a bound b (resp., a) such that $C^n = 0$ for all $n > b$ (resp., $n < a$). Clearly, a complex is bounded if and only if the complex is both bounded above and below.

Definition 2.1.3. A *co-chain complex map* between co-chain complexes $(C^\bullet, d^{C,\bullet})$ and $(D^\bullet, d^{D,\bullet})$ is a family $u^\bullet = \{u^n\}_{n\in\mathbb{Z}}$ of homomorphisms $u^n : C^n \to D^n$ such that the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C^{n-1} & \xrightarrow{d^{C,n-1}} & C^n & \xrightarrow{d^{C,n}} & C^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle u^{n-1}} & & \downarrow{\scriptstyle u^n} & & \downarrow{\scriptstyle u^{n+1}} & & \\ \cdots & \longrightarrow & D^{n-1} & \xrightarrow[d^{D,n-1}]{} & D^n & \xrightarrow[d^{D,n}]{} & D^{n+1} & \longrightarrow & \cdots \end{array}$$

commutes, i.e., $d^{D,n} \circ u^n = u^{n+1} \circ d^{C,n}$ for all n.

A co-chain complex map sends cycles to cycles and boundaries to boundaries, and thus induces a map on cohomology groups

$$H^\bullet(u^\bullet) : H^\bullet(C^\bullet, d^{C,\bullet}) \to H^\bullet(D^\bullet, d^{D,\bullet}).$$

Remark 2.1.4. The co-chain complexes form an abelian category. The objects in this category are, of course, co-chain complexes and the morphisms are given by the co-chain complex maps.

Definition 2.1.5. A morphism $u^\bullet : C^\bullet \to D^\bullet$ of co-chain complexes is called a *quasi-isomorphism* if the map $H^n(u^\bullet) : H^n(C^\bullet) \to H^n(D^\bullet)$ is an isomorphism for each $n \in \mathbb{Z}$.

Definition 2.1.6. A co-chain complex $(C^\bullet, d^\bullet)$ is called *acyclic* if $H^n(C^\bullet) = 0$ for all n.

Example 2.1.7. An exact sequence is always acyclic. □

2.2 δ-functor

Definition 2.2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories. Then a (covariant) *cohomological δ-functor* between $\mathcal{A}$ and $\mathcal{B}$ is a family $T = \{T^n\}_{n\geq 0}$ of additive functors $T^n : \mathcal{A} \to \mathcal{B}$ together with the homomorphisms

$$\delta^n : T^n(C) \to T^{n+1}(A)$$

defined for each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ with the following properties:

(i) δ is functorial, i.e., if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

is a commutative diagram of short exact sequences in $\mathcal{A}$, then

$$\begin{array}{ccc} T^n(C) & \xrightarrow{\delta^n} & T^{n+1}(A) \\ \downarrow & & \downarrow \\ T^n(C') & \xrightarrow[\delta^n]{} & T^{n+1}(A') \end{array}$$

is a commutative diagram in $\mathcal{B}$.

(ii) For each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, there is a long exact sequence

$$\cdots \to T^n(A) \to T^n(B) \to T^n(C) \xrightarrow{\delta^n} T^{n+1}(A) \to \cdots$$

in $\mathcal{B}$.

Example 2.2.2. The cohomology functor $(H^n)_{n\geq 0}$ is a cohomological δ-functor from the category of co-chain complexes to the category of abelian groups. □

Let T and S be two covariant cohomological δ-functors. Then a *morphism* $F : T \to S$ is a family $F^n : T^n \to S^n$ of natural transformations such that for every short exact sequence

$$0 \to A \to B \to C \to 0,$$

the diagram

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\delta^{n-1}} & T^n(A) & \longrightarrow & T^n(B) & \longrightarrow & T^n(C) & \xrightarrow{\delta^n} & T^{n+1}(A) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle F^n(A)} & & \downarrow{\scriptstyle F^n(B)} & & \downarrow{\scriptstyle F^n(C)} & & \downarrow{\scriptstyle F^{n+1}(A)} & & \\
\cdots & \xrightarrow[\delta^{n-1}]{} & S^n(A) & \longrightarrow & S^n(B) & \longrightarrow & S^n(C) & \xrightarrow[\delta^n]{} & S^{n+1}(A) & \longrightarrow & \cdots
\end{array}$$

is commutative.

Definition 2.2.3. A cohomological δ-functor T is *universal*, if given a cohomological δ-functor S and $F^0 : T^0 \to S^0$, there exists a morphism $F : T \to S$ of δ-functors extending F^0.

Example 2.2.4.

1. The cohomology functors $(H^n)_{n \geq 0}$ are universal δ-functors from the category of co-chain complexes to the category of abelian groups.
2. Let G be a pro-finite group. Then the cohomology functors $(H^n(G, -))_{n \geq 0}$ are universal δ-functors from the category of G-modules to the category of abelian groups. □

2.3 Dimension shifting

Definition 2.3.1. Let $\mathcal{A}$ be an abelian category. An object I in $\mathcal{A}$ is *injective* if, for every monomorphism $f : X \to Y$ and every morphism $g : X \to I$, there exists a morphism $h : Y \to I$ such that the diagram

$$\begin{array}{ccc}
X & \xhookrightarrow{f} & Y \\
{\scriptstyle g}\downarrow & \swarrow{\scriptstyle h} & \\
I & &
\end{array}$$

commutes, i.e., $g = h \circ f$.

Definition 2.3.2. An abelian category $\mathcal{A}$ is said to have *enough injectives* if, for every object X of $\mathcal{A}$, there exists an injective object I and a monomorphism $X \to I$.

Proposition 2.3.3. [27, Corollary 6.49] *Let $(T^n)_{n\geq 0}$ and $(S^n)_{n\geq 0}$ be families of additive covariant functors between $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $\mathcal{A}$ has enough injectives. If*

(i) for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, there are long exact sequences

$$\cdots \to T^n(A) \to T^n(B) \to T^n(C) \xrightarrow{\delta^n} T^{n+1}(A) \to \cdots$$

and

$$\cdots \to S^n(A) \to S^n(B) \to S^n(C) \xrightarrow{\delta^n} S^{n+1}(A) \to \cdots$$

in $\mathcal{B}$ with natural connecting homomorphisms,

(ii) T^0 is naturally isomorphic to S^0,

(iii) $T^n(I) = 0 = S^n(I)$ for all injective objects I and all $n \geq 1$,

then T^n is naturally isomorphic to S^n for all $n \geq 0$.

Remark 2.3.4. The technique of Proposition 2.3.3 is known as *dimension shifting*.

2.4 Total complex associated to double complex

In this section, we define the total complex associated to a double complex.

Definition 2.4.1. Let $\mathcal{A}$ be an abelian category. A *double complex* $(C^{\bullet\bullet}, \partial^v, \partial^h)$ (with commuting differentials) is a family of objects $\{C^{p,q} \in \mathcal{A} \mid p, q \in \mathbb{Z}\}$ together with differentials: $\partial^v : C^{p,q} \to C^{p+1,q}$ called the vertical differentials; and $\partial^h : C^{p,q} \to C^{p,q+1}$ called the horizontal differentials, satisfying $\partial^v \circ \partial^v = 0 = \partial^h \circ \partial^h$ and $\partial^h \circ \partial^v = \partial^v \circ \partial^h$.

We may picture a double complex $C^{\bullet\bullet}$ as a lattice, where each row $C^{*,q}$ and each column $C^{p,*}$ is a co-chain complex (see Figure 2.1).

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle \partial^v} & & \uparrow{\scriptstyle \partial^v} & & \uparrow{\scriptstyle \partial^v} & & \\
\cdots & \xrightarrow{\partial^h} & C^{p+1,q-1} & \xrightarrow{\partial^h} & C^{p+1,q} & \xrightarrow{\partial^h} & C^{p+1,q+1} & \xrightarrow{\partial^h} & \cdots \\
 & & \uparrow{\scriptstyle \partial^v} & & \uparrow{\scriptstyle \partial^v} & & \uparrow{\scriptstyle \partial^v} & & \\
\cdots & \xrightarrow{\partial^h} & C^{p,q-1} & \xrightarrow{\partial^h} & C^{p,q} & \xrightarrow{\partial^h} & C^{p,q+1} & \xrightarrow{\partial^h} & \cdots \\
 & & \uparrow{\scriptstyle \partial^v} & & \uparrow{\scriptstyle \partial^v} & & \uparrow{\scriptstyle \partial^v} & & \\
\cdots & \xrightarrow{\partial^h} & C^{p-1,q-1} & \xrightarrow{\partial^h} & C^{p-1,q} & \xrightarrow{\partial^h} & C^{p-1,q+1} & \xrightarrow{\partial^h} & \cdots \\
 & & \uparrow{\scriptstyle \partial^v} & & \uparrow{\scriptstyle \partial^v} & & \uparrow{\scriptstyle \partial^v} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Figure 2.1: Double complex $C^{\bullet\bullet}$

Definition 2.4.2. Let $(C^{\bullet\bullet}, \partial^v, \partial^h)$ be a double complex. Then its associated *total complex*, denoted by $\mathrm{Tot}(C^{\bullet\bullet})$, is a single co-chain complex with n-th term

$$\mathrm{Tot}(C^{\bullet\bullet})^n = \bigoplus_{p+q=n} C^{p,q}$$

and whose differential maps $d^n : \mathrm{Tot}(C^{\bullet\bullet})^n \to \mathrm{Tot}(C^{\bullet\bullet})^{n+1}$ are given by

$$d^n = \sum_{p+q=n} \partial^v + (-1)^{\text{vertical degree}} \partial^h.$$

Proposition 2.4.3. *Let* $(C^{\bullet\bullet}, \partial^v, \partial^h)$ *be a double complex. Then its associated total complex* $(\mathrm{Tot}(C^{\bullet\bullet}), d^\bullet)$ *is a complex.*

Proof. It is easy to check, see [27, Lemma 10.5]. ■

Example 2.4.4. Let $(A^\bullet, d^{A,\bullet})$ and $(B^\bullet, d^{B,\bullet})$ be two co-chain complexes of abelian groups. Then their *tensor product* is the double complex

$$C^{\bullet\bullet} = A^{\bullet} \otimes B^{\bullet}$$

with $C^{pq} = A^p \otimes B^q, p, q \in \mathbb{Z}$, and with the following differentials:

$$\partial^v = d^{A,p} \otimes id_{B^q}, \quad \partial^h = id_{A^p} \otimes d^{B,q}.$$

The associated total complex $\mathrm{Tot}(A^{\bullet} \otimes B^{\bullet})$ is usually called the tensor product of $A^{\bullet}$ and $B^{\bullet}$. □

2.5 Spectral sequence associated to double complex

Quite often, spectral sequences arise from double complexes. The spectral sequence of a double complex is a tool for computing the cohomology of a total complex, hence for computing the total cohomology of a double complex.

Definition 2.5.1. Let $C^{\bullet\bullet}$ be a double complex. Then we have the *natural filtration* $F^p \,\mathrm{Tot}(C^{\bullet\bullet})$ of the total complex defined by

$$F^p \,\mathrm{Tot}(C^{\bullet\bullet})^n = \bigoplus_{i \geq p} C^{i,n-i}.$$

For a double complex $(C^{\bullet\bullet}, \partial^v, \partial^h)$, assume that there are only finitely many non-zero $C^{p,q}$ on the line $p + q = n$ for each n. Then the above filtration on $\mathrm{Tot}(C^{\bullet\bullet})$ induces a spectral sequence

$$E_1^{pq} \Rightarrow H^{p+q}(\mathrm{Tot}(C^{\bullet\bullet}))$$

converging to the cohomology of $\mathrm{Tot}(C^{\bullet\bullet})$. The initial terms E_1^{pq} are obtained by taking the cohomology in direction q, i.e.,

$$E_1^{pq} = H^q(C^{p,\bullet}, \partial^h).$$

The E_1 terms give a complex

$$H^q(C^{\bullet\bullet}) : \cdots \xrightarrow{\partial^v} H^q(C^{p-1,\bullet}) \xrightarrow{\partial^v} H^q(C^{p,\bullet}) \xrightarrow{\partial^v} H^q(C^{p+1,\bullet}) \xrightarrow{\partial^v} \cdots,$$

whose cohomology yields the E_2 terms:

$$E_2^{pq} = H^p(H^q(C^{\bullet\bullet})).$$

One often forgets the E_1-page and calls the spectral sequence

$$E_2^{pq} = H^p(H^q(C^{\bullet\bullet})) \Rightarrow H^{p+q}(\mathrm{Tot}(C^{\bullet\bullet}))$$

the *spectral sequence associated to the double complex* $C^{\bullet\bullet}$.

Chapter 3

Lubin-Tate Theory

This chapter is an elementary and, at the same time, an essential part of this thesis. In this chapter, we provide an introduction to the theory of Lubin-Tate formal groups, and we closely follow the exposition given in [6], [16] and [23].

3.1 Formal group laws

Formal groups arise in Number Theory, Algebraic Topology and Lie Theory. In fact, their origin lies in the theory of Lie groups. In modern number theory, formal group laws play a crucial role in the study of elliptic curves and the Dirichlet series of L-functions.

Definition 3.1.1. Let $\mathcal{O}$ be a commutative ring. A one-dimensional *formal group law* $\mathcal{F}$ over $\mathcal{O}$ is a formal power series $\mathcal{F}(X,Y) \in \mathcal{O}[[X,Y]]$ in two variables with coefficients in $\mathcal{O}$ such that

(i) $\mathcal{F}(X,0) = X$ and $\mathcal{F}(0,Y) = Y$, i.e., $\mathcal{F}(X,Y) \equiv X + Y \bmod \deg 2$,

(ii) $\mathcal{F}(X,\mathcal{F}(Y,Z)) = \mathcal{F}(\mathcal{F}(X,Y),Z)$.

Moreover, if $\mathcal{F}$ satisfies

$$\mathcal{F}(X,Y) = \mathcal{F}(Y,X)$$

then $\mathcal{F}$ is said to be a *commutative formal group law*.

If $\mathcal{O}$ has no element which is both torsion and nilpotent, then every one-dimensional formal group law over $\mathcal{O}$ is commutative [16, Chapter I, §6].

In this thesis, a formal group law always means a one-dimensional commutative formal group law.

Example 3.1.2. Some natural examples of one-dimensional commutative formal group laws are:

1. $\mathbb{G}_a(X, Y) = X + Y$ (the additive formal group law).
2. $\mathbb{G}_m(X, Y) = X + Y + XY$ (the multiplicative formal group law). □

Example 3.1.3. (Formal group law of an elliptic curve) Let E be an elliptic curve given by the Weierstrass equation with coefficient in $\mathcal{O}$, i.e.,

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with $a_i \in \mathcal{O}$. Then the power series

$$F(z_1, z_2) = z_1 + z + 2 - a_1z_1z_2 - a_2(z_1^2z_2 + z_1z_2^2) + \\ (2a_3z_1^3z_2 + (a_1a_2 - 3a_3)z_1^2z_2^2 + 2a_3z_1z_2^3) + \cdots$$

defines a formal group law on E. For the construction of the power series $F(z_1, z_2)$, see [31, Chapter IV, §1]. □

Sometimes, formal group laws are also referred to as formal groups. A formal group resembles a group operation, with no actual underlying group. A group, in the usual sense, can be obtained from a formal group law by selecting a domain on which the power series converges (see [23, Chapter III, §6]).

Example 3.1.4. If $\mathcal{O}$ is a complete discrete valuation ring with maximal ideal $\mathfrak{m}$, then

$$x \underset{\mathcal{F}}{+} y := \mathcal{F}(x, y) \quad \text{for } x, y \in \mathfrak{m}$$

defines a new abelian group structure on $\mathfrak{m}$. □

3.2 The logarithm

Definition 3.2.1. [23, p.56] A *homomorphism* $h : \mathcal{F} \to \mathcal{G}$ between two formal group laws $\mathcal{F}$ and $\mathcal{G}$ is a formal power series $h(Z) \in \mathcal{O}[[Z]]$ such that

$$h(0) = 0 \quad \text{and} \quad h(\mathcal{F}(X, Y)) = \mathcal{G}(h(X), h(Y)).$$

This condition means precisely that h is a homomorphism between the groups that one can induce from $\mathcal{F}$ and $\mathcal{G}$.

Definition 3.2.2. A homomorphism $h : \mathcal{F} \to \mathcal{G}$ is said to be an *isomorphism* if there exists a homomorphism $h^{-1} : \mathcal{G} \to \mathcal{F}$ such that

$$h(h^{-1}(Z)) = Z = h^{-1}(h(Z)).$$

Example 3.2.3. Let $\mathcal{F}$ be a formal group law defined over $\mathcal{O}$. We define the homomorphisms

$$[m] : \mathcal{F} \to \mathcal{F}$$

for all $m \in \mathbb{Z}$ such that

$$\begin{aligned} [0](Z) &= 0, \\ [m+1](Z) &= \mathcal{F}([m](Z), Z), \\ [m-1](Z) &= \mathcal{F}([m](Z), i(Z)), \end{aligned}$$

where $i(Z) \in \mathcal{O}[[Z]]$ is the unique power series satisfying $\mathcal{F}(Z, i(Z)) = 0$.

Then by using induction, it is easy to check that $[m]$ is a homomorphism. This homomorphism is called *the multiplication-by-m* map. Moreover, if m is a unit in $\mathcal{O}$, then

$$[m] : \mathcal{F} \to \mathcal{F}$$

is an isomorphism. This follows from [31, Chapter IV, Proposition 2.3] □

Example 3.2.4. Let K be a field of characteristic zero. Then the power series $\log(1+Z)$ and $\exp(Z) - 1$ define a mutually inverse isomorphism between $\mathbb{G}_m$ and $\mathbb{G}_a$ over K. □

Moreover, we can define $\log$ and $\exp$ for any formal group law over a field of characteristic zero. Let $\mathcal{F}$ be a formal group law over a field K of characteristic zero. Then there exists a power series

$$\log_{\mathcal{F}}(Z) \equiv Z \bmod \deg 2$$

with coefficients in K such that $\log_{\mathcal{F}} : \mathcal{F} \to \mathbb{G}_a$ is a homomorphism of formal groups, i.e.,

$$\log_{\mathcal{F}}(\mathcal{F}(X, Y)) = \log_{\mathcal{F}}(X) + \log_{\mathcal{F}}(Y).$$

Since the linear term of $\log_{\mathcal{F}}(Z)$ is Z, so the inverse of $\log_{\mathcal{F}}(Z)$ exists and it is denoted by $\exp_{\mathcal{F}}$. The existence of such a power series follows from [24, Chapter V, §4].

Proposition 3.2.5. [23, Chapter III, Proposition 6.3] *The set* $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ *of homomorphisms from* $\mathcal{F}$ *to* $\mathcal{G}$ *is an abelian group with respect to the addition*

$$(h_1 \underset{\mathcal{F}}{+} h_2)(Z) := \mathcal{F}(h_1(Z), h_2(Z))$$

with zero element 0. *The abelian group* $\mathrm{End}_{\mathcal{O}}(\mathcal{F}) := \mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{F})$ *is a ring with* *respect to the multiplication*

$$(h_1.h_2)(Z) := h_1(h_2(Z))$$

with unit element Z.

3.3 Formal modules

Definition 3.3.1. A *formal $\mathcal{O}$-module* is a formal group $\mathcal{F}$ over $\mathcal{O}$ together with a ring homomorphism

$$\begin{aligned} \mathcal{O} &\to \mathrm{End}_{\mathcal{O}}(\mathcal{F}) \\ a &\mapsto [a]_{\mathcal{F}}(Z) \end{aligned}$$

such that $[a]_{\mathcal{F}}(Z) \equiv aZ \bmod \deg 2$.

Definition 3.3.2. A *homomorphism* of formal $\mathcal{O}$-modules is a homomorphism $h : \mathcal{F} \to \mathcal{G}$ of formal groups such that

$$h([a]_{\mathcal{F}}(Z)) = [a]_{\mathcal{G}}(h(Z)) \quad \text{for } a \in \mathcal{O}.$$

Example 3.3.3. The multiplicative formal group law $\mathbb{G}_m$ is a formal $\mathbb{Z}_p$-module with respect to the map

$$\begin{aligned} \mathbb{Z}_p &\to \mathrm{End}_{\mathbb{Z}_p}(\mathbb{G}_m) \\ a &\mapsto [a]_{\mathbb{G}_m}(Z) \\ &= (1+Z)^a - 1 \\ &= \sum_{n=1}^{\infty} \binom{a}{n} Z^n. \end{aligned}$$

□

3.4 Lubin-Tate modules

In 1965, Lubin and Tate introduced a group law, namely the Lubin-Tate formal group law. A Lubin-Tate group law is a formal group over a local field of characteristic zero with an endomorphism of a specific form. This group law plays an essential role in constructing the totally ramified extensions of local fields.

Throughout this thesis, we fix a local field $(K, |.|)$ of characteristic 0 with the ring of integers $\mathcal{O}_K$, maximal ideal $\mathfrak{m}_K$, and residue field k of characteristic $p > 0$. Let π be a prime element of $\mathcal{O}_K$, $\mathrm{card}(k) = q$ and $q = p^r$ for some fixed r. Let $\bar{K}$ be a fixed algebraic closure of K with the ring of integers $\mathcal{O}_{\bar{K}}$ and maximal ideal $\mathfrak{m}_{\bar{K}}$.

Definition 3.4.1. A *Lubin-Tate module* over $\mathcal{O}_K$ for the prime element π is a formal $\mathcal{O}_K$-module $\mathcal{F}$ such that

$$[\pi]_{\mathcal{F}}(Z) \equiv Z^q \bmod \pi.$$

Example 3.4.2. The multiplicative group $\mathbb{G}_m$ is a Lubin-Tate module for the prime p, since

$$[p]_{\mathbb{G}_m}(Z) = (1 + Z)^p - 1 \equiv Z^p \bmod p. \qquad \square$$

Remark 3.4.3. Any two Lubin-Tate modules for one prime element π are isomorphic over $\mathcal{O}_K$. Also, two Lubin-Tate modules associated with two different primes π_1 and π_2 can never be isomorphic over $\mathcal{O}_K$. This follows from [23, Chapter III, Theorem 6.7].

We can obtain an $\mathcal{O}_K$-module, in the usual sense, from a formal $\mathcal{O}_K$-module by choosing a domain on which power series converges.

Proposition 3.4.4. [23, Chapter III, Proposition 7.1] *Let $\mathcal{F}$ be a formal $\mathcal{O}_K$-module. Then the set $\mathfrak{m}_{\bar{K}}$ together with the operations*

$$x \underset{\mathcal{F}}{+} y := \mathcal{F}(x, y) \quad \textit{and} \quad a.x = [a]_{\mathcal{F}}(x),$$

where $x, y \in \mathfrak{m}_{\bar{K}}, a \in \mathcal{O}_K$, is an $\mathcal{O}_K$-module in the usual sense, which we denote by $\mathcal{F}(\mathfrak{m}_{\bar{K}})$.

Remark 3.4.5. If $h : \mathcal{F} \to \mathcal{G}$ is a homomorphism (resp., isomorphism) of formal $\mathcal{O}_K$-modules, then

$$\tilde{h} : \mathcal{F}(\mathfrak{m}_{\bar{K}}) \to \mathcal{G}(\mathfrak{m}_{\bar{K}}),$$
$$x \mapsto h(x)$$

is a homomorphism (resp., isomorphism) of $\mathcal{O}_K$-modules.

3.5 Tate-module of a Lubin-Tate module

Let $\mathcal{F}$ be a Lubin-Tate module for a prime element π of $\mathcal{O}_K$.

Definition 3.5.1. The *group of π^n-division points* of $\mathcal{F}$ is defined as

$$\begin{aligned}\mathcal{F}(n) &= \{\lambda \in \mathcal{F}(\mathfrak{m}_{\bar{K}}) | \pi^n.\lambda = 0\} \\ &= \{\lambda \in \mathcal{F}(\mathfrak{m}_{\bar{K}}) | [\pi^n]_{\mathcal{F}}(\lambda) = 0\} \\ &= \mathrm{Ker}([\pi^n]_{\mathcal{F}}).\end{aligned}$$

Note that the group $\mathcal{F}(n)$ is an $\mathcal{O}_K$-submodule of $\mathcal{F}(\mathfrak{m}_{\bar{K}})$, and since it is annihilated by $\pi^n\mathcal{O}_K$, so it is an $\mathcal{O}_K/\pi^n\mathcal{O}_K$-module. Moreover, $\mathcal{F}(n)$ is a free $\mathcal{O}_K/\pi^n\mathcal{O}_K$-module of rank 1 [23, Chapter III, Proposition 7.2].

Next, we define the Tate-module of a Lubin-Tate module $\mathcal{F}$. Consider the exact sequence

$$0 \to \mathcal{F}(n) \to \mathcal{F}(\mathfrak{m}_{\bar{K}}) \xrightarrow{a \mapsto [\pi^n]_{\mathcal{F}} a} \mathcal{F}(\mathfrak{m}_{\bar{K}}) \to 0,$$

where $\mathcal{F}(n) = \mathrm{Ker}([\pi^n]_{\mathcal{F}})$. Then the Tate-module of $\mathcal{F}$ is defined as the following

$$\mathcal{TF} := \varprojlim_{n \in \mathbb{N}} \mathcal{F}(n),$$

with the transition maps

$$\mathcal{F}(n+1) \to \mathcal{F}(n)$$
$$a \mapsto [\pi]_{\mathcal{F}} a.$$

The Tate-module $\mathcal{TF}$ is a free $\mathcal{O}_K$-module of rank 1. Let $v = (v_n)_{n \in \mathbb{N}}$ be a generator of $\mathcal{TF}$. Then $\mathcal{TF} = \mathcal{O}_K.v$ is equipped with the following $\mathcal{O}_K$-action

$$\lambda.v = (\lambda_n.v_n)_{n \in \mathbb{N}}, \lambda_n \in \mathcal{O}_K, \lambda \equiv \lambda_n \text{ mod } \pi^n \mathcal{O}_K.$$

3.6 Lubin-Tate extensions

For a Lubin-Tate module $\mathcal{F}$ for a prime element π of $\mathcal{O}_K$, we have

$$\mathcal{F}(1) \subseteq \mathcal{F}(2) \subseteq \ldots \mathcal{F}(n) \subseteq \ldots.$$

Then by adjoining these subsets to K, we get a chain of algebraic extensions

$$K \subseteq K_1 := K(\mathcal{F}(1)) \subseteq \ldots \subseteq K_n := K(\mathcal{F}(n)) \subseteq \ldots \subseteq K_\infty := \bigcup_{n=1}^{\infty} K_n \subseteq \bar{K}.$$

Definition 3.6.1. A *Lubin-Tate extension* of a local field K is an abelian extension of K obtained by adjoining the group of π-division points of the Lubin-Tate module to K.

Remark 3.6.2. The Lubin-Tate extensions depend only on the choice of the prime element π, not on the Lubin-Tate module $\mathcal{F}$ for π. For more details see [23, Chapter

III, §7].

Proposition 3.6.3. [23, Chapter III, Theorem 7.4] *For all $n \geq 1$, the extension K_n/K is totally ramified abelian extension of degree $(q-1)q^{n-1}$ with the Galois group*

$$\mathrm{Gal}(K_n/K) \cong (\mathcal{O}_K/\pi^n\mathcal{O}_K)^\times.$$

Moreover, this isomorphism fits into the following commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(K_{n+1}/K) & \xrightarrow{\cong} & (\mathcal{O}_K/\pi^{n+1}\mathcal{O}_K)^\times \\ {\scriptstyle restriction}\downarrow & & \downarrow{\scriptstyle projection} \\ \mathrm{Gal}(K_n/K) & \xrightarrow[\cong]{} & (\mathcal{O}_K/\pi^n\mathcal{O}_K)^\times. \end{array}$$

Since $\mathrm{Gal}(K_\infty/K) = \varprojlim_n \mathrm{Gal}(K_n/K)$, therefore by passing to the projective limits, we obtain the isomorphism

$$\mathrm{Gal}(K_\infty/K) \cong \mathcal{O}_K^\times.$$

Chapter 4

Galois Representations

In this chapter, first, we give an overview of Fontaine's theory of (φ, Γ)-modules. Next, we prove a theorem, which is due to Kisin and Ren [20]. This theorem is a generalization of Fontaine's theory of (φ, Γ)-modules.

Recall that K is a finite extension of $\mathbb{Q}_p$ with its algebraic closure $\bar{K}$. Let $G_K := \mathrm{Gal}(\bar{K}/K)$ be the absolute Galois group of K.

4.1 Fontaine's theory: an overview

The possibility of converting the study of the p-adic representations of G_K into the investigation of the (φ, Γ)-modules is the founding point of Fontaine's theory. A *p-adic representation* of G_K is a finite dimensional $\mathbb{Q}_p$-vector space with a continuous and linear action of G_K.

In [10], Fontaine introduced a new technique to study the p-adic representations of the absolute Galois group. He proved that these representations could be investigated by the study of the étale (φ, Γ)-module, which is an algebraic object attached to the representation. To achieve this, he decomposed the Galois group along a totally ramified extension of K, by using the theory of the field of norms. This extension was obtained by using the cyclotomic tower associated with the multiplicative group

$\mathbb{G}_m$.

Then he constructed a functor $\mathbb{D}$ from the category $\mathbf{Rep}_{\mathbb{Q}_p}(G_K)$ of p-adic representations of G_K to the category $\mathbf{Mod}_{/\mathcal{E}}^{\varphi,\Gamma,ét}$ of étale (φ, Γ) modules over the discrete valued field $\mathcal{E}$. Recall that a φ-module over $\mathcal{E}$ is an $\mathcal{E}$-vector space M with a map $\varphi_M : M \to M$, which is semi-linear with respect to φ. We say that M is étale if $\dim_{\mathcal{E}} M < \infty$ and if there exists an $\mathcal{O}_{\mathcal{E}}$-lattice M' of M such that M' is an étale φ-module over $\mathcal{O}_{\mathcal{E}}$ and is stable under the action of φ. An étale (φ, Γ)-module over $\mathcal{E}$ is an étale φ-module over $\mathcal{E}$ together with a continuous and semi-linear action of Γ, which commutes with the action of φ. The functor $\mathbb{D}$ is defined as follows.

Let V be a p-adic representation of G_K. Define

$$\mathbb{D}(V) := (\widehat{\mathcal{E}^{ur}} \otimes_{\mathbb{Q}_p} V)^H,$$

where $\widehat{\mathcal{E}^{ur}}$ is the completion of the maximal unramified extension of $\mathcal{E}$. The group H is the kernel of the cyclotomic character $\chi_{cyc} : G_K \to \mathbb{Z}_p^\times$, i.e., $H = \mathrm{Gal}(\bar{K}/K_{cyc})$, where K_{cyc} is the cyclotomic $\mathbb{Z}_p$-extension of K obtained by adjoining all p^n-th roots of unity to K and roots of unity to K. The group Γ is the image of χ_{cyc}, i.e., $\Gamma = G_K/H = \mathrm{Gal}(K_{cyc}/K)$. The residue field E of $\mathcal{E}$ is the same as the field of norms of the extension K_{cyc}/K. There is an action of the group Γ and the Frobenius φ on $\mathcal{E}$, and these actions commute with each other. Note that φ is a lift of absolute Frobenius on E. Then he proved that this functor $\mathbb{D}$ gives an equivalence of categories between the category of p-adic representations of G_K and the category of étale (φ, Γ)-modules over $\mathcal{E}$.

4.2 The Kisin-Ren equivalence

For any $n \in \mathbb{N}$, the p^n-th roots of unity are the p^n-torsion points of the multiplicative Lubin-Tate formal group $\mathbb{G}_m$ on $\mathbb{Q}_p$, with respect to the uniformizer p. Therefore the cyclotomic extension is the same as the extension associated with the multiplicative

Lubin-Tate formal group $\mathbb{G}_m$. It is natural to try to carry out this theory for an arbitrary Lubin-Tate formal group defined over any finite extension K of $\mathbb{Q}_p$. In this direction, Kisin and Ren established a significant result [20, Theorem 1.6], which is a base of this thesis. For the convenience, we recall the construction of the equivalence of categories proved by Kisin and Ren.

For a local field K of characteristic 0 with residue field k, let $W = W(k)$ be the ring of Witt vectors over k and $K_0 = W[\frac{1}{p}]$ be the field of fractions of W. Then K_0 is a maximal unramified extension of $\mathbb{Q}_p$ contained in K. For an $\mathcal{O}_{K_0}$-algebra A, we write $A_K = A \otimes_{\mathcal{O}_{K_0}} \mathcal{O}_K$.

Let $\mathcal{F}$ be the Lubin-Tate group over K corresponding to the uniformizer π of $\mathcal{O}_K$. As in [20], we fix a local co-ordinate Z on $\mathcal{F}$ such that the formal Hopf algebra $\mathcal{O}_{\mathcal{F}}$ is identified with $\mathcal{O}_K[[Z]]$. For any $a \in \mathcal{O}_K$, write $[a]_{\mathcal{F}}(Z) \in \mathcal{O}_K[[Z]] = \mathcal{O}_{\mathcal{F}}$ for the power series giving the endomorphism of $\mathcal{F}$.

Let K_∞ be the Lubin-Tate extension of K. Let $H_K = \mathrm{Gal}(\bar{K}/K_\infty)$ and $\Gamma_{LT} = G_K/H_K = \mathrm{Gal}(K_\infty/K)$. Let $\mathcal{TF}$ be the p-adic Tate-module of $\mathcal{F}$. Then $\mathcal{TF}$ is a free $\mathcal{O}_K$-module of rank 1. The action of G_K on $\mathcal{TF}$ factors through Γ_{LT} and induces an isomorphism $\chi_{LT} : \Gamma_{LT} \to \mathcal{O}_K^\times$. This follows from section 3.6.

Let $\mathcal{R} = \varprojlim \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$, where the transition maps in the inverse limit are given by the Frobenius φ. The ring $\mathcal{R}$ can also be identified with $\varprojlim \mathcal{O}_{\bar{K}}/\pi\mathcal{O}_{\bar{K}}$, and the transition maps being given by the q-Frobenius $\varphi_q = \varphi^r$, where φ is the Frobenius map and $r = \log_p q$, i.e., $q = p^r$. The ring $\mathcal{R}$ is a complete valuation ring, and it is perfect of characteristic p. The fraction field $\mathrm{Fr}(\mathcal{R})$ of $\mathcal{R}$ is a complete, algebraically closed non-archimedean perfect field of characteristic p. Then we have a map $\iota : \mathcal{TF} \to \mathcal{R}$, which is induced by the evaluation of Z at π-torsion points. Let $v = (v_n)_{n\geq 0} \in \mathcal{TF}$ with $v_n \in \mathcal{F}(n)$ and $\pi.v_{n+1} = v_n$, then $\iota(v) = (v_n^*(Z) + \pi\mathcal{O}_{\bar{K}})_{n\geq 0}$. Moreover, we have the following lemma, which follows from [7, Lemma 9.3]. More details are given in [28, §2.1].

Lemma 4.2.1. [20, Lemma 1.2] *There is a unique map* $\{\} : \mathcal{R} \to W(\mathcal{R})_K$ *such* *that $\{x\}$ is a lifting of x and $\varphi_q(\{x\}) = [\pi]_{\mathcal{F}}(x)$. Moreover, $\{\}$ respects the action of G_K. In particular, if $v \in \mathcal{TF}$ is an $\mathcal{O}_K$-generator, there is an embedding $\mathcal{O}_K[[Z]] \hookrightarrow W(\mathcal{R})_K$ sending Z to $\{\iota(v)\}$, which identifies $\mathcal{O}_K[[Z]]$ with a G_K-stable, φ_q-stable subring of $W(\mathcal{R})_K$ such that $\{\iota(\mathcal{TF})\}$ lies in the image of $\mathcal{O}_K[[Z]]$.*

Note that the G_K-action on $\mathcal{O}_K[[Z]]$ factors through Γ_{LT}, and we have $\varphi_q(Z) = [\pi]_{\mathcal{F}}(Z)$ and $\sigma_a(Z) = [a]_{\mathcal{F}}(Z)$, where $\sigma_a = \chi_{LT}^{-1}(a)$ for any $a \in \mathcal{O}_K^\times$, which is well-defined as $\chi_{LT} : \Gamma_{LT} \to \mathcal{O}_K^\times$ is an isomorphism. Now, fix an $\mathcal{O}_K$-generator $v \in \mathcal{TF}$ and by using Lemma 4.2.1, identify $\mathcal{O}_K[[Z]]$ with a subring of $W(\mathcal{R})_K$ by sending Z to $\{\iota(v)\}$.

Let $\mathcal{O}_{\mathcal{E}}$ be the π-adic completion of $\mathcal{O}_K[[Z]][\frac{1}{Z}]$. Then $\mathcal{O}_{\mathcal{E}}$ is a complete discrete valuation ring with uniformizer π and the residue field $k((Z))$. Since $W(\mathcal{R})$ is π-adically complete, we may view

$$\mathcal{O}_{\mathcal{E}} \subset W(\mathcal{R})_K \subset W(\mathrm{Fr}(\mathcal{R}))_K.$$

Let $\mathcal{O}_{\mathcal{E}^{ur}} \subset W(\mathrm{Fr}(\mathcal{R}))_K$ denote the maximal integral unramified extension of $\mathcal{O}_{\mathcal{E}}$, and $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$ the π-adic completion of $\mathcal{O}_{\mathcal{E}^{ur}}$, which is again a subring of $W(\mathrm{Fr}(\mathcal{R}))_K$. Let $\mathcal{E}, \mathcal{E}^{ur}$ and $\widehat{\mathcal{E}^{ur}}$ denote the field of fractions of $\mathcal{O}_{\mathcal{E}}, \mathcal{O}_{\mathcal{E}^{ur}}$ and $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$, respectively. These rings are all stable under the action of φ_q and G_K. Also, the G_K-action factors through Γ_{LT}.

Lemma 4.2.2. [20, Lemma 1.4] *The residue field of $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$ is a separable closure of $k((Z))$, and there is a natural isomorphism*

$$\mathrm{Gal}(\mathcal{E}^{ur}/\mathcal{E}) \xrightarrow{\sim} \mathrm{Gal}(\bar{K}/K_\infty).$$

Proof. Let F be a finite extension of K. Define

$$X_K(F) := \varprojlim_n (F.K_n),$$

where $F.K_n$ is the composite of F and K_n, and the transition maps in the inverse system are given by the norm maps. Then $X_K(F)$ is a local field of characteristic p, which is a finite separable extension of $X_K(K)$ ([36, Theorem 2.1.3]). Put

$$X_K(\bar{K}) = \cup_F X_K(F),$$

where the union runs over all the finite extensions F of K contained in $\bar{K}$. Then $X_K(\bar{K})$ is a separable closure of $X_K(K)$, and the functor X_K induces an isomorphism ([36, Theorem 3.2.2])

$$\mathrm{Gal}(X_K(\bar{K})/X_K(K)) \xrightarrow{\sim} \mathrm{Gal}(\bar{K}/K_\infty).$$

Note that we have well-defined maps of rings

$$\varprojlim \mathcal{O}_{K_n} \to \varprojlim \mathcal{O}_{K_n}/(v_1) \hookrightarrow \varprojlim \mathcal{O}_{\bar{K}}/\pi\mathcal{O}_{\bar{K}} = \mathcal{R}, \tag{4.1}$$

where the transition maps in the first two inverse limits are given by the norm maps and in the final inverse limit by $x \mapsto x^q$. Thus the field $X_K(K)$ is naturally embedded in Fr($\mathcal{R}$), i.e., $X_K(K) \hookrightarrow$ Fr($\mathcal{R}$). For more details, see [36, §4]. The image of (4.1) is $k[[Z]] \subset \mathcal{R}$ thus $k((Z)) \subset$ Fr($\mathcal{R}$). Now identify $E := k((Z)) = \mathcal{O}_{\mathcal{E}}/\pi\mathcal{O}_{\mathcal{E}}$ with $X_K(K)$, then it follows that $E^{sep} := \mathcal{O}_{\mathcal{E}^{ur}}/\pi\mathcal{O}_{\mathcal{E}^{ur}} \subset$ Fr($\mathcal{R}$) is identified with $X_K(\bar{K})$. Therefore

$$\mathrm{Gal}(E^{sep}/E) \xrightarrow{\sim} \mathrm{Gal}(\bar{K}/K_\infty).$$

Since $\mathcal{E}^{ur}$ is unramified extension of $\mathcal{E}$, we have

$$\mathrm{Gal}(\mathcal{E}^{ur}/\mathcal{E}) = \mathrm{Gal}(E^{sep}/E).$$

This completes the proof of the lemma. ■

Next, the following lemma is an easy consequence of the definition of $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$.

Lemma 4.2.3. *Under the natural actions of G_E and φ_q on $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$, we have*

(i) $(\mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{G_E} = \mathcal{O}_{\mathcal{E}}$,

(ii) $(\mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{\varphi_q = id} = \mathcal{O}_K$.

Proof. (i) Clearly, we have $\mathcal{O}_{\mathcal{E}} \hookrightarrow (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{G_E}$. Moreover, this is an inclusion of π-adically complete and separated local rings, so it is sufficient to prove surjectivity modulo π^n for $n \geq 1$. We use induction on n. We have an exact sequence

$$0 \to (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}) \xrightarrow{\pi} (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}) \to E^{sep} \to 0$$

of $\mathcal{O}_{\mathcal{E}}$-modules, and by taking G_E-invariants, this induces an injection

$$(\mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{G_E}/(\pi) \hookrightarrow (E^{sep})^{G_E} = E$$

of $\mathcal{O}_{\mathcal{E}}/(\pi) = E$-modules. Since E is a field, so the inclusion map is a bijection, i.e., $\mathcal{O}_{\mathcal{E}} \hookrightarrow (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{G_E}$ is surjective modulo π. Now assume that the map $\mathcal{O}_{\mathcal{E}} \hookrightarrow (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{G_E}$ is surjective modulo π^{n-1}. Then we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathcal{E}}/\pi\mathcal{O}_{\mathcal{E}} & \longrightarrow & \mathcal{O}_{\mathcal{E}}/\pi^n\mathcal{O}_{\mathcal{E}} & \longrightarrow & \mathcal{O}_{\mathcal{E}}/\pi^{n-1}\mathcal{O}_{\mathcal{E}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{G_E}/\pi & \longrightarrow & (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{G_E}/\pi^n & \longrightarrow & (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{G_E}/\pi^{n-1} & \longrightarrow & 0 \end{array}$$

with exact rows. By using the Five lemma ([21, Chapter III]), it follows that the map $\mathcal{O}_{\mathcal{E}} \hookrightarrow (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{G_E}$ is surjective modulo π^n.

(ii) First, we show that the sequence

$$0 \to \mathcal{O}_K/\pi^n\mathcal{O}_K \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi^n\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \xrightarrow{\varphi_q - id} \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi^n\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \to 0 \tag{4.2}$$

is exact for $n \geq 1$. Since we have an exact sequence

$$0 \to k \to E^{sep} \xrightarrow[x \mapsto x^q - x]{\varphi_q - id} E^{sep} \to 0,$$

i.e., the sequence (4.2) is exact for $n = 1$. Assume that the sequence given by (4.2) is exact for $n-1$. Now consider the following diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_K/\pi\mathcal{O}_K & \longrightarrow & \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} & \xrightarrow[x\mapsto x^q - x]{\varphi_q - id} & \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_K/\pi^n\mathcal{O}_K & \longrightarrow & \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi^n\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} & \xrightarrow{\varphi_q - id} & \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi^n\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_K/\pi^{n-1}\mathcal{O}_K & \longrightarrow & \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi^{n-1}\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} & \xrightarrow{\varphi_q - id} & \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi^{n-1}\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

This is a commutative diagram of abelian groups such that each column is exact. Moreover, the top and the bottom rows are also exact by induction hypothesis. Since $\varphi_q(x) = x$ for $x \in \mathcal{O}_K$, then $\mathcal{O}_K/\pi^n\mathcal{O}_K \subseteq \mathrm{Ker}(\varphi_q - id)$, i.e., the middle row is a complex. Then by using 3×3 lemma ([35, §1.3]), we deduce that the middle row is also exact. Therefore we have an exact sequence

$$0 \to \mathcal{O}_K/\pi^n\mathcal{O}_K \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi^n\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \xrightarrow{\varphi_q - id} \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi^n\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \to 0, \text{ for } n \geq 1.$$

Since the projective system $\{\mathcal{O}_K/\pi^n\mathcal{O}_K\}_{n\geq 1}$ has surjective transition maps, passing to the projective limit is exact and gives us an exact sequence

$$0 \to \mathcal{O}_K \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \xrightarrow{\varphi_q - id} \mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \to 0.$$

Hence, $(\mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{\varphi_q = id} = \mathcal{O}_K$. ■

Remark 4.2.4. Since it follows from Lemma 4.2.2 that $\mathrm{Gal}(E^{sep}/E) \cong H_K$. Then we have

$$(\mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{H_K} = \mathcal{O}_{\mathcal{E}}.$$

Definition 4.2.5. Let κ be a field of characteristic $p > 0$. Then the *Cohen ring* $\mathcal{C}(\kappa)$ of κ is the unique (up to isomorphism) absolutely unramified discrete valuation ring of characteristic 0 with residue field κ.

The subring $\mathcal{O}_{\mathcal{E}} \subset W(\mathrm{Fr}(\mathcal{R}))$, which is constructed using the periods of $\mathcal{TF}$, is naturally a Cohen ring for $E = X_K(K)$. Moreover, by Lemma 4.2.2, we have

$$H_K \xrightarrow{\sim} G_E.$$

The G_K-action on $\mathcal{R}$ induces a G_K-action on $W(\mathrm{Fr}(\mathcal{R}))_K$ and the rings $\mathcal{O}_{\mathcal{E}}, \mathcal{O}_{\mathcal{E}^{ur}}$ and $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$ are stable under the action of G_K. On the other hand, G_E acts on $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$ by continuity and functoriality and these actions are compatible with the identification of Galois groups $H_K \xrightarrow{\sim} G_E$.

Definition 4.2.6. A *φ_q-module over* $\mathcal{O}_{\mathcal{E}}$ is an $\mathcal{O}_{\mathcal{E}}$-module M with a map $\varphi_M : M \to M$, which is semi-linear with respect to q-Frobenius φ_q, i.e.,

$$\varphi_M(x + y) = \varphi_M(x) + \varphi_M(y),$$
$$\varphi_M(\lambda x) = \varphi_q(\lambda)\varphi_M(x),$$

for all $x, y \in M$ and $\lambda \in \mathcal{O}_{\mathcal{E}}$.

For a φ_q-module M over $\mathcal{O}_{\mathcal{E}}$, let $M_{\varphi_q} := M \ {}_{\varphi_q}\!\otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}}$ denote the base change of M by $\mathcal{O}_{\mathcal{E}}$ via φ_q. Then a semi-linear map $\varphi_M : M \to M$ is equivalent to an $\mathcal{O}_{\mathcal{E}}$-linear map $\Phi_M^{lin} : M_{\varphi_q} \to M$.

Definition 4.2.7. A φ_q-module M over $\mathcal{O}_{\mathcal{E}}$ is *étale* if M is an $\mathcal{O}_{\mathcal{E}}$-module of finite type and the map $\Phi_M^{lin} : M_{\varphi_q} \to M$ is an isomorphism.

Let V be an $\mathcal{O}_K$-module of finite rank with a continuous and linear action of G_K. Then consider the φ_q-module:

$$\mathbb{D}_{LT}(V) := (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V)^{H_K} = (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V)^{G_E}.$$

The action of G_K on $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V$ induces a semi-linear action of $G_K/H_K = \Gamma_{LT} = \mathrm{Gal}(K_\infty/K)$ on $\mathbb{D}_{LT}(V)$.

Definition 4.2.8. A (φ_q, Γ_{LT})*-module* M over $\mathcal{O}_{\mathcal{E}}$ is a φ_q-module over $\mathcal{O}_{\mathcal{E}}$ together with a continuous and semi-linear action of Γ_{LT}, which commutes with the endomorphism φ_M of M. We say that M is *étale* if it is étale as a φ_q-module.

Now we consider the following categories:

- $\mathbf{Rep}_{\mathcal{O}_K}(G_K)$ = the category of finitely generated free $\mathcal{O}_K$-modules with a continuous and linear action of G_K.
- $\mathbf{Rep}_{\mathcal{O}_K-tor}(G_K)$ = the category of finitely generated torsion $\mathcal{O}_K$-modules with a continuous and linear action of G_K.
- $\mathbf{Mod}^{\varphi_q,\Gamma_{LT},ét}_{/\mathcal{O}_{\mathcal{E}}}$ = the category of finitely generated free étale (φ_q, Γ_{LT})-modules over $\mathcal{O}_{\mathcal{E}}$.
- $\mathbf{Mod}^{\varphi_q,\Gamma_{LT},ét,tor}_{/\mathcal{O}_{\mathcal{E}}}$ = the category of finitely generated torsion étale (φ_q, Γ_{LT})-modules over $\mathcal{O}_{\mathcal{E}}$.

Then $\mathbb{D}_{LT}$ is a functor from $\mathbf{Rep}_{\mathcal{O}_K}(G_K)$ (resp., $\mathbf{Rep}_{\mathcal{O}_K-tor}(G_K)$) to $\mathbf{Mod}^{\varphi_q,\Gamma_{LT},ét}_{/\mathcal{O}_{\mathcal{E}}}$ (resp., $\mathbf{Mod}^{\varphi_q,\Gamma_{LT},ét,tor}_{/\mathcal{O}_{\mathcal{E}}}$), and this follows from [20]. Moreover, $\mathbb{D}_{LT}$ is an additive functor and this follows from [10, A1, 1.2.2].

Let $M \in \mathbf{Mod}^{\varphi_q,\Gamma_{LT},ét}_{/\mathcal{O}_{\mathcal{E}}}$ (resp., $\mathbf{Mod}^{\varphi_q,\Gamma_{LT},ét,tor}_{/\mathcal{O}_{\mathcal{E}}}$). Then consider the G_K-representation:

$$\mathbb{V}_{LT}(M) := (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M)^{\varphi_q \otimes \varphi_M = id}.$$

Here G_K acts on $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$ as before and acts via Γ_{LT} on M. The diagonal action of G_K on $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M$ is $\varphi_q \otimes \varphi_M$-equivariant, which induces a G_K-action on $\mathbb{V}_{LT}(M)$.

Remark 4.2.9. Using similar proof as that in [10, A1, Proposition 1.2.4 and 1.2.6], we have that the functors $\mathbb{D}_{LT}$ and $\mathbb{V}_{LT}$ are exact functors and the natural maps

$$\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}_{LT}(V) \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V,$$

$$\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} \mathbb{V}_{LT}(M) \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M$$

are isomorphisms.

In particular, we have the following result, which is established in [20, Theorem 1.6].

Theorem 4.2.10 (Kisin-Ren)**.** *The functors*

$$V \mapsto \mathbb{D}_{LT}(V) = (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V)^{H_K} \quad \textit{and} \quad M \mapsto \mathbb{V}_{LT}(M) = (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_\mathcal{E}} M)^{\varphi_q \otimes \varphi_M = id}$$

are quasi-inverse equivalence of categories between the category $\mathbf{Rep}_{\mathcal{O}_K}(G_K)$ (*resp.,* $\mathbf{Rep}_{\mathcal{O}_{K-tor}}(G_K)$) *and* $\mathbf{Mod}^{\varphi_q,\Gamma_{LT},\acute{e}t}_{/\mathcal{O}_\mathcal{E}}$ (*resp.,* $\mathbf{Mod}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}_{/\mathcal{O}_\mathcal{E}}$).

Sketch of the Proof. Note that for any $V \in \mathbf{Rep}_{\mathcal{O}_K}(G_K)$, we have

$$V \cong \varprojlim_n V/\pi^n V.$$

Using this isomorphism, it is enough to show that the functors $\mathbb{D}_{LT}$ and $\mathbb{V}_{LT}$ are quasi-inverse equivalence of categories between the category $\mathbf{Rep}_{\mathcal{O}_{K-tor}}(G_K)$ and $\mathbf{Mod}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}_{/\mathcal{O}_\mathcal{E}}$ and then the general case follows by passing to the inverse limits.

Now let $V \in \mathbf{Rep}_{\mathcal{O}_{K-tor}}(G_K)$. Then by using Remark 4.2.9, we identify $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_\mathcal{E}} \mathbb{D}_{LT}(V)$ with $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V$. Then

$$\begin{aligned} \mathbb{V}_{LT}(\mathbb{D}_{LT}(V)) =& (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_\mathcal{E}} \mathbb{D}_{LT}(V))^{\varphi_q \otimes \varphi_{\mathbb{D}_{LT}(V)} = id} \\ \cong& (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V)^{\varphi_q \otimes id = id} \\ =& (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{\varphi_q = id} \otimes_{\mathcal{O}_K} V \\ =& \mathcal{O}_K \otimes_{\mathcal{O}_K} V \\ \cong& V. \end{aligned}$$

Here the second isomorphism follows from the above identification and the fact that φ_q acts trivially on V. The fourth equality follows from part (ii) of Lemma 4.2.3.

Similarly, for any $M \in \mathbf{Mod}_{/\mathcal{O}_{\mathcal{E}}}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}$, we have

$$\begin{aligned}
\mathbb{D}_{LT}(\mathbb{V}_{LT}(M)) =& (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} \mathbb{V}_{LT}(M))^{H_K} \\
\cong& (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M)^{H_K} \\
=& (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{H_K} \otimes_{\mathcal{O}_{\mathcal{E}}} M \\
=& \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_{\mathcal{E}}} M \\
\cong& M,
\end{aligned}$$

where the second isomorphism follows from Remark 4.2.9 and the fourth equality comes from Remark 4.2.4. This proves the theorem. ■

Remark 4.2.11. Instead of K if we choose any finite extension, say F, of K then we also have the above equivalence of categories for $\mathcal{O}_F$-modules.

Lemma 4.2.12. *The category of finitely generated G_K-modules defined over $\mathcal{O}_K$ has no non-zero injectives.*

Proof. Let I be an injective object in the category of finitely generated G_K-modules over $\mathcal{O}_K$. Then I is a finitely generated $\mathcal{O}_K$-module with a continuous and linear action of G_K. Now by structure theorem for finitely generated modules over a principal ideal domain, we have

$$I \cong \mathcal{O}_K^{\oplus r} \oplus T,$$

where T is a finite torsion module over $\mathcal{O}_K$. Note that I is injective as an $\mathcal{O}_K$-module. Since $\mathcal{O}_K$ is a principal ideal domain, so I is also π-divisible. Let π^e be an annihilator of T. Now consider the map

$$I \xrightarrow{\times \pi^e} I.$$

Then this map sends T to zero. So multiplication by π^e map can not be surjective, unless $T = 0$. Thus

$$I \cong \mathcal{O}_K^{\oplus r}.$$

Again, multiplication by π map is not surjective on I, unless $r = 0$, as we know the sequence

$$0 \to \mathcal{O}_K \xrightarrow{\pi} \mathcal{O}_K \to \mathcal{O}_K/\pi\mathcal{O}_K \to 0$$

is exact. Therefore

$$I = 0.$$

■

Chapter 5

Galois Cohomology over the Lubin-Tate Extensions

This chapter is a part of [1]. In this chapter, we define a complex, namely, *the Lubin-Tate Herr complex* (Definition 5.4.1), and then using that complex, we compute the Galois cohomology groups in terms of étale (φ_q, Γ_{LT})-modules. To achieve this, first, we decompose the Galois group G_K along the Lubin-Tate extension of K, and then we establish Proposition 5.2.5 and Proposition 5.3.7. These two results help us to deduce our main result (Theorem 5.5.2) of this chapter.

Following the notations of Chapter 4, by Theorem 4.2.10, we know that the functor $\mathbb{D}_{LT}$ is an equivalence of categories between the category of finitely generated $\mathcal{O}_K$-modules with a continuous and linear action of G_K and the category of étale (φ_q, Γ_{LT})-modules over $\mathcal{O}_{\mathcal{E}}$. Since injective objects do not exist in the category of finitely generated $\mathcal{O}_K$-modules with a continuous and linear action of G_K (see Lemma 4.2.12). Thus the category $\mathbf{Rep}_{\mathcal{O}_K}(G_K)$ does not have enough injectives. Therefore we extend the functor $\mathbb{D}_{LT}$ to a category that has enough injectives as we are going to use injective objects to compute cohomology groups.

5.1 Equivalence of categories

Let $\mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$ be the category of discrete π-primary abelian groups with a continuous action of G_K. Then any object in this category is the filtered direct limit of π-power torsion objects in $\mathbf{Rep}_{\mathcal{O}_K-tor}(G_K)$. In fact, if $V \in \mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$, then

$$V = \bigcup_n V[\pi^n],$$

where $V[\pi^n]$ is the kernel of the multiplication by π^n map on V. Here each $V[\pi^n]$ is G_K-stable. Since G_K is a pro-finite group, the category $\mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$ has enough injectives [25, Chapter II, Lemma 2.6.5].

Now we extend the functor $\mathbb{D}_{LT}$ to the category $\mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$. Let V be a discrete π-primary abelian group on which the Galois group G_K acts continuously, then we define

$$\mathbb{D}_{LT}(V) := (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V)^{H_K}.$$

Since V is the filtered direct limit of π-power torsion objects in $\mathbf{Rep}_{\mathcal{O}_K-tor}(G_K)$ and both the tensor product and taking H_K-invariants commute with the filtered direct limits, so the functor $\mathbb{D}_{LT}$ commutes with the filtered direct limits. Therefore $\mathbb{D}_{LT}$ is an exact functor into the category $\varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}_{/\mathcal{O}_{\mathcal{E}}}$ of injective limits of π-power torsion objects in $\mathbf{Mod}^{\varphi_q,\Gamma_{LT},\acute{e}t}_{/\mathcal{O}_{\mathcal{E}}}$. Now for any object $M \in \varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}_{/\mathcal{O}_{\mathcal{E}}}$, define

$$\mathbb{V}_{LT}(M) := (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M)^{\varphi_q \otimes \varphi_M = id}.$$

The functor $\mathbb{V}_{LT}$ also commutes with the direct limits. Then we have the following proposition, which shows that the equivalence of Theorem 4.2.10 extends to the category of discrete π-primary representations of G_K, and this is an important step towards our main theorem.

Proposition 5.1.1. *The functor $\mathbb{D}_{LT}$ is a quasi-inverse equivalence of categories between $\mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$ and $\varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}_{/\mathcal{O}_{\mathcal{E}}}$ with quasi-inverse $\mathbb{V}_{LT}$.*

Proof. By Theorem 4.2.10, we know that the functors $\mathbb{D}_{LT}$ and $\mathbb{V}_{LT}$ are quasi-inverse equivalence of categories between $\mathbf{Rep}_{\mathcal{O}_K-tor}(G_K)$ and $\mathbf{Mod}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}_{/\mathcal{O}_{\mathcal{E}}}$. Moreover for any $V \in \mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$, we have

$$V = \varinjlim_n V_n$$

with $V_n \in \mathbf{Rep}_{\mathcal{O}_K-tor}(G_K)$ for $n \geq 1$. Since the functor $\mathbb{D}_{LT}$ commutes with the direct limits, i.e.,

$$\mathbb{D}_{LT}(V) = \varinjlim_n \mathbb{D}_{LT}(V_n).$$

Moreover, the functor $\mathbb{V}_{LT}$ also commutes with the direct limits. Then the proposition follows from Theorem 4.2.10 by taking direct limits. ■

5.2 The $\Phi^\bullet$ complex

Let p an odd prime number. Define $D^{sep} := \mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V$. Since $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V \cong \mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}_{LT}(V)$ (Remark 4.2.9). Thus $D^{sep} \cong \mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}_{LT}(V)$.

Definition 5.2.1. Define the *co-chain complex* $\Phi^\bullet(D^{sep})$ as follows:

$$\Phi^\bullet(D^{sep}) : 0 \to D^{sep} \xrightarrow{\varphi_q \otimes \varphi_D - id} D^{sep} \to 0,$$

where $\varphi_D = \varphi_{\mathbb{D}_{LT}(V)}$.

Remark 5.2.2. Throughout this thesis, we assume that each complex has the first term in degree -1 unless stated otherwise.

Lemma 5.2.3. *For any $V \in \mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$, let $V[0]$ be the complex with V in degree 0 and 0 everywhere else. Then the augmentation map $V[0] \to \Phi^\bullet(D^{sep})$ is a quasi-isomorphism of co-chain complexes.*

Proof. By part (ii) of Lemma 4.2.3, we know that the complex $\Phi^\bullet(E^{sep})$ is acyclic in non-zero degrees with 0-th cohomology equal to k, the augmentation map

$$k[0] \to \Phi^\bullet(E^{sep})$$

is a quasi-isomorphism. Next, observe that $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi^n$ is flat as an $\mathcal{O}_K/\pi^n$-module. This follows as we know $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$ is flat as an $\mathcal{O}_K$-module, and we also have a ring homomorphism $\mathcal{O}_K \to \mathcal{O}_K/\pi^n$. Now by using [21, Chapter XVI, Proposition 4.2], $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi^n$ is flat as an $\mathcal{O}_K/\pi^n$-module. By tensoring the short exact sequence $0 \to \mathcal{O}_K \xrightarrow{\pi^n} \mathcal{O}_K \to \mathcal{O}_K/\pi^n \to 0$ with $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$, we have $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi^n \cong \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi^n$. Thus $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi^n$ is flat as an $\mathcal{O}_K/\pi^n$-module. Then by dévissage (by using Five lemma as explained in part (i) of Lemma 4.2.3), the augmentation map

$$\mathcal{O}_K/\pi^n[0] \to \Phi^\bullet(\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi^n) \tag{5.1}$$

is also a quasi-isomorphism as each term in both complexes is a flat $\mathcal{O}_K/\pi^n$-module. If V is a finite abelian π-group then it is killed by some power of π, and we have $\Phi^\bullet(D^{sep}) = \Phi^\bullet(\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi^n) \otimes_{\mathcal{O}_K/\pi^n} V$. Since V is free $\mathcal{O}_K/\pi^n$-module, tensoring with V is an exact functor. Thus tensoring (5.1) with V, we get

$$V[0] \to \Phi^\bullet(D^{sep})$$

is a quasi-isomorphism. As the direct limit functor is exact, the general case follows by taking direct limits. ■

Lemma 5.2.4. $H^i(H_K, \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi^n) = 0$ *for all* $n \geq 1$ *and* $i \geq 1$.

Proof. Since we have an exact sequence

$$0 \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi^n\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \to \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}/\pi^{n-1}\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \to 0.$$

Then by dévissage (by the long exact cohomology sequence), we are reduced to the case $n = 1$, i.e., we only need to prove that $H^i(H_K, E^{sep}) = 0$ for all $i \geq 1$. But this is a standard fact of Galois cohomology [25, Proposition 6.1.1]. ■

Proposition 5.2.5. *For any $V \in \mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$, we have $\mathcal{H}^i(\Phi^\bullet(\mathbb{D}_{LT}(V))) \cong H^i(H_K, V)$ as Γ_{LT}-modules. In other words, the complex $\Phi^\bullet(\mathbb{D}_{LT}(V))$ computes the H_K-cohomology of V.*

Proof. Assume that V is finite. By definition,

$$\mathbb{D}_{LT}(V) = (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V)^{H_K} = (D^{sep})^{H_K}.$$

So the complex $\Phi^\bullet(\mathbb{D}_{LT}(V))$ is the H_K-invariant part of $\Phi^\bullet(D^{sep})$. Since V is finite, the terms of $\Phi^\bullet(D^{sep})$ are of the form $D^{sep} = E^{sep} \otimes_E \mathbb{D}_{LT}(V)$ and are acyclic objects for the H_K-cohomology by using Lemma 5.2.4. Then it follows from Lemma 5.2.3 that $\mathcal{H}^i(\Phi^\bullet(\mathbb{D}_{LT}(V))) \cong H^i(H_K, V)$ as Γ_{LT}-modules. As both the functors $\mathcal{H}^i(\Phi^\bullet(\mathbb{D}_{LT}(-)))$ and $H^i(H_K, -)$ commute with the filtered direct limits, the general case follows by taking the direct limits. ■

Let Δ be the torsion subgroup of Γ_{LT} and H_K^* the kernel of the quotient map $G_K \twoheadrightarrow \Gamma_{LT} \twoheadrightarrow \Gamma_{LT}^* := \Gamma_{LT}/\Delta$. Then Δ is isomorphic to μ_{q-1}.

If the order of Δ is not prime to p, then we choose a finite p-extension F of K such that torsion part of $\mathrm{Gal}(K_\infty/F)$ is prime to p. In that case, Kisin-Ren theorem allows us to compute $G_F = \mathrm{Gal}(\bar{K}/F)$-cohomology of V. Therefore, without any loss of generality, we can assume that the order of Δ is prime to p.

Proposition 5.2.6. *The complex $\Phi^\bullet(\mathbb{D}_{LT}(V)^\Delta)$ computes the H_K^*-cohomology of V.*

Proof. Since the order of Δ is prime to p, the p-cohomological dimension of Δ is zero. Moreover, the isomorphism $H_K^*/H_K \cong \Delta$ gives the following short exact sequence

$$0 \to H_K \to H_K^* \to \Delta \to 0.$$

Now the Hochschild-Serre spectral sequence gives an exact sequence

$$0 \to H^1(\Delta, V^{H_K}) \to H^1(H_K^*, V) \to H^1(H_K, V)^{\Delta} \to H^2(\Delta, V^{H_K}) \to H^2(H_K^*, V).$$

Since the p-cohomological dimension of Δ is zero, i.e., $H^i(\Delta, V^{H_K}) = 0$ for $i \geq 1$. Now the result follows from Proposition 5.2.5. ■

5.3 The $\Gamma_{LT}^{\bullet}$ complex

Note that Γ_{LT}^* is torsion-free. Assume that $\Gamma_{LT}^* \cong \bigoplus_{i=1}^{d} \mathbb{Z}_p$ as a $\mathbb{Z}_p$-module, where d is the degree of K over $\mathbb{Q}_p$. Let Γ_{LT}^* be topologically generated by the set $\mathfrak{X} := \langle \gamma_1, \gamma_2, \ldots, \gamma_d \rangle$.

Definition 5.3.1. Let A be an abelian group with a continuous and linear action of Γ_{LT}^*. Then we define $\Gamma_{LT}^{\bullet}(A)$ as follows:

$$\Gamma_{LT}^{\bullet}(A) : 0 \to A \to \bigoplus_{i_1 \in \mathfrak{X}} A \to \cdots \to \bigoplus_{\{i_1, \ldots, i_r\} \in \binom{\mathfrak{X}}{r}} A \to \cdots \to A \to 0,$$

where $\binom{\mathfrak{X}}{r}$ denotes choosing r-indices at a time from the set $\mathfrak{X}$, and for all $0 \leq r \leq |\mathfrak{X}| - 1$, the map $d_{i_1, \ldots, i_r}^{j_1, \ldots, j_{r+1}} : A \to A$ from the component in the r-th term corresponding to $\{i_1, \ldots, i_r\}$ to the component corresponding to the $(r+1)$-tuple $\{j_1, \ldots, j_{r+1}\}$ is given by

$$d_{i_1, \ldots, i_r}^{j_1, \ldots, j_{r+1}} = \begin{cases} 0 & \text{if } \{i_1, \ldots, i_r\} \nsubseteq \{j_1, \ldots, j_{r+1}\}, \\ (-1)^{s_j}(\gamma_j - id) & \text{if } \{j_1, \ldots, j_{r+1}\} = \{i_1, \ldots, i_r\} \cup \{j\}. \end{cases}$$

Here s_j is the number of elements in the set $\{i_1, \ldots, i_r\}$, which are smaller than j.

Remark 5.3.2. The above definition is motivated by the definition of Koszul complex.

Proposition 5.3.3. *Let A be an arbitrary representation of the group Γ_{LT}^{*}. Then $\Gamma_{LT}^{\bullet}(A)$ is a complex.*

Proof. Since A is a representation of Γ_{LT}^{*}, A is also an $\mathcal{O}_K[[\Gamma_{LT}^{*}]]$-module. Note that $A \otimes_{\mathcal{O}_K[[\Gamma_{LT}^{*}]]} \mathcal{O}_K[[\Gamma_{LT}^{*}]] \cong A$. First, we show that $\Gamma_{LT}^{\bullet}(\mathcal{O}_K[[\Gamma_{LT}^{*}]])$ is a complex. Let $\bigoplus_{i_1 \in \mathfrak{X}} \mathcal{O}_K[[\Gamma_{LT}^{*}]]$ be a free module with ordered basis $\{e_1, e_2, \dots, e_d\}$ and $\bigoplus_{\{i_1,\dots,i_r\} \in \binom{\mathfrak{X}}{r}} \mathcal{O}_K[[\Gamma_{LT}^{*}]]$ be a free module with basis $\{e_{i_1} \wedge \dots \wedge e_{i_r}\}$ with $i_1 < i_2 < \dots < i_r$. Then the boundary maps are given by the following

$$d : \bigoplus_{\{i_1,\dots,i_r\} \in \binom{\mathfrak{X}}{r}} \mathcal{O}_K[[\Gamma_{LT}^{*}]] \longrightarrow \bigoplus_{\{j_1,\dots,j_{r+1}\} \in \binom{\mathfrak{X}}{r+1}} \mathcal{O}_K[[\Gamma_{LT}^{*}]]$$

$$\begin{aligned} d(e_{i_1} \wedge \dots \wedge e_{i_r}) &= \sum d_{i_1,\dots,i_r}^{j_1,\dots,j_{r+1}} (e_{j_1} \wedge \dots \wedge e_{j_r} \wedge e_{j_{r+1}}) \\ &= \sum_{j=1}^{d} (-1)^{s_j} (\gamma_j - 1)(e_{i_1} \wedge \dots \wedge e_{s_j} \wedge e_j \wedge \dots \wedge e_{i_r}), \end{aligned}$$

where s_j is the number of elements in the set $\{i_1, \dots, i_r\}$ smaller than j, and the maps $d_{i_1,\dots,i_r}^{j_1,\dots,j_{r+1}}$ are as defined in Definition 5.3.1. Then

$$\begin{aligned} (d \circ d)(e_{i_1} \wedge \dots \wedge e_{i_r}) =& d\left(\sum_{j=1}^{d} (-1)^{s_j} (\gamma_j - 1)(e_{i_1} \wedge \dots \wedge e_{s_j} \wedge e_j \wedge \dots \wedge e_{i_r}) \right) \\ =& \sum_{t=1}^{d} (-1)^{s_t} (\gamma_t - 1) \sum_{j=1}^{d} (-1)^{s_j} (\gamma_j - 1)(e_{i_1} \wedge \dots \wedge e_{s_j} \wedge e_j \\ & \wedge \dots \wedge e_{s_t} \wedge e_t \wedge \dots \wedge e_{i_r}). \end{aligned}$$

Here s_t is the number of elements in the set $\{i_1, \dots, i_r\}$, which are smaller than t. Note that in the above summation, each term appears twice, so each term has two coefficients. Now for $j < t$, we compare the coefficients of the term $e_{i_1} \wedge \dots \wedge e_{s_j} \wedge e_j \wedge \dots \wedge e_{s_t} \wedge e_t \wedge \dots \wedge e_{i_r}$. The coefficient is $(-1)^{s_j+s_t+1}(\gamma_t - id)(\gamma_j - id)$ if we introduce e_j first and e_t second. Similarly, by introducing e_t first and then e_j, we get

$(-1)^{s_t+s_j}(\gamma_j - id)(\gamma_t - id)$ as the coefficient of the term $e_{i_1} \wedge \cdots \wedge e_{s_j} \wedge e_j \wedge \cdots \wedge e_{s_t} \wedge e_t \wedge \cdots \wedge e_{i_r}$. Since γ_j commutes with γ_t, and the coefficients are of opposite parity. Then it follows that $d \circ d = 0$, i.e., $\Gamma^{\bullet}_{LT}(\mathcal{O}_K[[\Gamma^*_{LT}]])$ is a complex.

Note that

$$\Gamma^{\bullet}_{LT}(A) = \Gamma^{\bullet}_{LT}(\mathcal{O}_K[[\Gamma^*_{LT}]]) \otimes_{\mathcal{O}_K[[\Gamma^*_{LT}]]} A.$$

Therefore, $\Gamma^{\bullet}_{LT}(A)$ is also a complex. ■

We write the complex $\Gamma^{\bullet}_{LT}(A)$ for the case $d = 2$ and 3.

Example 5.3.4. Let $d = 2$. Then the complex $\Gamma^{\bullet}_{LT}(A)$ is defined as follows:

$$\Gamma^{\bullet}_{LT}(A) : 0 \to A \xrightarrow{x \mapsto A_0 x} A \oplus A \xrightarrow{x \mapsto A_1 x} A \to 0,$$

where

$$A_0 = \begin{bmatrix} \gamma_1 - id \\ \gamma_2 - id \end{bmatrix}, A_1 = \begin{bmatrix} -(\gamma_2 - id) & \gamma_1 - id \end{bmatrix}.$$ □

Example 5.3.5. Let $d = 3$. Then the complex $\Gamma^{\bullet}_{LT,FT}(A)$ looks like as the following

$$\Gamma^{\bullet}_{LT}(A) : 0 \to A \xrightarrow{x \mapsto A_0 x} A^{\oplus 3} \xrightarrow{x \mapsto A_1 x} A^{\oplus 3} \xrightarrow{x \mapsto A_2 x} A \to 0,$$

where

$$A_0 = \begin{bmatrix} \gamma_1 - id \\ \gamma_2 - id \\ \gamma_3 - id \end{bmatrix}, A_1 = \begin{bmatrix} -(\gamma_2 - id) & \gamma_1 - id & 0 \\ 0 & -(\gamma_3 - id) & \gamma_2 - id \\ -(\gamma_3 - id) & 0 & \gamma_1 - id \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \gamma_3 - id & \gamma_1 - id & -(\gamma_2 - id) \end{bmatrix}.$$ □

Lemma 5.3.6. *The functor $A \mapsto \mathcal{H}^i(\Gamma^{\bullet}_{LT}(A))_{i \geq 0}$ is a cohomological δ-functor. Moreover, if A is a discrete representation of Γ^*_{LT}, then $\mathcal{H}^0(\Gamma^{\bullet}_{LT}(A)) = A^{\Gamma^*_{LT}}$.*

Proof. Let

$$0 \to A \to B \to C \to 0 \tag{5.2}$$

be a short exact sequence of representations of Γ^*_{LT}. Then we have a short exact sequence

$$0 \to \Gamma^{\bullet}_{LT}(A) \to \Gamma^{\bullet}_{LT}(B) \to \Gamma^{\bullet}_{LT}(C) \to 0 \tag{5.3}$$

of co-chain complexes. The long exact sequence of (5.3) gives maps

$$\delta^i : \mathcal{H}^i(\Gamma^{\bullet}_{LT}(C)) \to \mathcal{H}^{i+1}(\Gamma^{\bullet}_{LT}(A)),$$

which are functorial in (5.2). Therefore $A \mapsto \mathcal{H}^i(\Gamma^{\bullet}_{LT}(A))$ is a cohomological δ-functor. The second part follows from the fact that the action of Γ^*_{LT} on A factors through a finite quotient. Since the classes of the elements of γ_i ($i \in \mathfrak{X}$) generate finite quotients of Γ^*_{LT}, $A^{\Gamma^*_{LT}} = \cap_{i\in\mathfrak{X}} \operatorname{Ker}(\gamma_i - id) = \mathcal{H}^0(\Gamma^{\bullet}_{LT}(A))$. ■

Proposition 5.3.7. *Let A be a discrete π-primary representation of Γ^*_{LT}. Then $\mathcal{H}^i(\Gamma^{\bullet}_{LT}(A)) \cong H^i(\Gamma^*_{LT}, A)$ for $i \geq 0$. In other words, the complex $\Gamma^{\bullet}_{LT}(A)$ computes the Γ^*_{LT}-cohomology of A.*

Proof. We prove the proposition by using induction on the number of generators of Γ^*_{LT}. First, assume that Γ^*_{LT} is topologically generated by $\langle\gamma_1, \gamma_2\rangle$. Let $\Gamma^*_{\gamma_1}$ denote the subgroup of Γ^*_{LT} generated by γ_1 and $\Gamma^*_{\gamma_2}$ the quotient of Γ^*_{LT} by $\Gamma^*_{\gamma_1}$. We denote by $\Gamma^{\bullet}_{\gamma_i}(A)$ the co-chain complex

$$\Gamma^{\bullet}_{\gamma_i}(A) : 0 \to A \xrightarrow{\gamma_i - id} A \to 0.$$

Then the co-chain complex $\Gamma^{\bullet}_{LT}(A)$ is the total complex of the double complex $\Gamma^{\bullet}_{\gamma_2}(\Gamma^{\bullet}_{\gamma_1}(A))$, and associated to the double complex $\Gamma^{\bullet}_{\gamma_2}(\Gamma^{\bullet}_{\gamma_1}(A))$, there is a spectral sequence

$$E_2^{mn} = \mathcal{H}^m(\Gamma^{\bullet}_{\gamma_2}(\mathcal{H}^n(\Gamma^{\bullet}_{\gamma_1}(A)))) \Rightarrow \mathcal{H}^{m+n}(\Gamma^{\bullet}_{LT}(A)). \tag{5.4}$$

Moreover, associated to the group Γ^*_{LT}, we have the Hochschild-Serre spectral sequence

$$E_2^{mn} = H^m(\Gamma^*_{\gamma_2}, H^n(\Gamma^*_{\gamma_1}, A)) \Rightarrow H^{m+n}(\Gamma^*_{LT}, A). \tag{5.5}$$

Now assume that A is injective object in the category of discrete π-primary abelian groups with a continuous action of Γ^*_{LT}. Then the complex $\Gamma^\bullet_{\gamma_1}(A)$ is acyclic in non-zero degrees with 0-th cohomology isomorphic to $H^0(\Gamma^*_{\gamma_1}, A) = A^{\Gamma^*_{\gamma_1}}$ [27, Corollary 6.41], i.e, the map $A^{\Gamma^*_{\gamma_1}}[0] \to \Gamma^\bullet_{\gamma_1}(A)$ is a quasi-isomorphism. But $A^{\Gamma^*_{\gamma_1}}$ is an injective object in the category of discrete π-primary abelian groups with a continuous action of $\Gamma^*_{\gamma_2}$. Now by using step 1 and step 2 of [26, Proposition 2.1.7], the map $A^{\Gamma^*_{LT}}[0] \to \Gamma^\bullet_{\gamma_2}(A^{\Gamma^*_{\gamma_1}})$ is a quasi-isomorphism of co-chain complexes.

Note that $H^i(\Gamma^*_{LT}, -)$ is a universal δ-functor, and $\mathcal{H}^i(\Gamma^\bullet_{LT}(-))$ is a cohomological δ-functor such that $H^0(\Gamma^*_{LT}, -) \cong \mathcal{H}^0(\Gamma^\bullet_{LT}(-))$. Therefore, we have a natural transformation $H^i(\Gamma^*_{LT}, -) \to \mathcal{H}^i(\Gamma^\bullet_{LT}(-))$ of δ-functors. Then by using spectral sequences (5.4) and (5.5), we have

$$H^i(\Gamma^*_{LT}, A) \cong \mathcal{H}^i(\Gamma^\bullet_{LT}(A)) \quad \text{for } i \geq 0.$$

Now the case for general A follows from Lemma 5.3.6 by using dimension shifting (Proposition 2.3.3), i.e., the proposition holds when Γ^*_{LT} is topologically generated by γ_1 and γ_2. Then by induction assume that the result is true when Γ^*_{LT} is topologically generated by $\langle \gamma_1, \gamma_2, \ldots, \gamma_{d-1} \rangle$. Now we want to prove the proposition when Γ^*_{LT} has d generators, i.e., $\Gamma^*_{LT} = \langle \gamma_1, \gamma_2, \ldots, \gamma_d \rangle$. Consider the complexes

$$\Gamma^\bullet_{\gamma_d}(A) : 0 \to A \xrightarrow{\gamma_d - id} A \to 0,$$

and

$$\Gamma^\bullet_{LT \setminus \gamma_d}(A) : 0 \to A \to \bigoplus_{i_1 \in \mathfrak{X}'} A \to \cdots \to \bigoplus_{\{i_1, \ldots, i_r\} \in \binom{\mathfrak{X}'}{r}} A \to \cdots \to A \to 0,$$

where $\mathfrak{X}' = \{\gamma_1, \ldots, \gamma_{d-1}\}$, and for all $0 \leq r \leq |\mathfrak{X}'| - 1$, the map $d^{j_1, \ldots, j_{r+1}}_{i_1, \ldots, i_r} : A \to A$ from the component in the r-th term corresponding to $\{i_1, \ldots, i_r\}$ to the component corresponding to the $(r+1)$-tuple $\{j_1, \ldots, j_{r+1}\}$ is given by the following

$$d^{j_1,\ldots,j_{r+1}}_{i_1,\ldots,i_r} = \begin{cases} 0 & \text{if } \{i_1, \ldots, i_r\} \not\subseteq \{j_1, \ldots, j_{r+1}\}, \\ (-1)^{s_j}(\gamma_j - id) & \text{if } \{j_1, \ldots, j_{r+1}\} = \{i_1, \ldots, i_r\} \cup \{j\}, \end{cases}$$

and s_j is the number of elements in the set $\{i_1, \ldots, i_r\}$ smaller than j. Note that the complex $\Gamma^{\bullet}_{LT}(A)$ is the total complex of the double complex $\Gamma^{\bullet}_{\gamma_d}(\Gamma^{\bullet}_{LT\setminus\gamma_d}(A))$. Since the result is true for $\Gamma^{\bullet}_{LT\setminus\gamma_d}(A)$ by using induction hypothesis. The proof follows by using similar techniques as explained in the case when Γ^{*}_{LT} is generated by γ_1 and γ_2. ■

For any representation A of Γ^{*}_{LT}, clearly, the complex $\Gamma^{\bullet}_{LT}(A)$ depends on the choice of generators of Γ^{*}_{LT}.

Proposition 5.3.8. *The Galois cohomology groups computed using the complex $\Gamma^{\bullet}_{LT}(A)$ are independent of the choice of generators of Γ^{*}_{LT}.*

Proof. To prove this, we use induction on the number of generators of Γ^{*}_{LT}. Assume that Γ^{*}_{LT} has only two generators γ_1 and γ_2. Let $\Gamma^{*}_{LT} = \langle \gamma_1', \gamma_2 \rangle$ be another set of generators. Define $\frac{1}{id-\gamma_1}(a) := \lim\limits_{n\to\infty} \sum_{j=0}^{n} \gamma_1^j(a)$ for $a \in A$, where the series on the right hand side is convergent as Γ^{*}_{LT} acts continuously on A. Then $\frac{\gamma_1'-id}{\gamma_1-id}$ is unit in $\mathcal{O}_K[[\Gamma^{*}_{LT}]]$, and we have the following diagram

$$\begin{array}{ccccccccc} \Gamma^{\bullet}_{LT,\gamma_1,\gamma_2}(A) : 0 & \longrightarrow & A & \xrightarrow{x\mapsto A_0x} & A \oplus A & \xrightarrow{x\mapsto A_1x} & A & \longrightarrow & 0 \\ & & \big\downarrow id & & \big\downarrow \frac{\gamma_1'-id}{\gamma_1-id}\oplus id & & \big\downarrow \frac{\gamma_1'-id}{\gamma_1-id} & & \\ \Gamma^{\bullet}_{LT,\gamma_1',\gamma_2}(A) : 0 & \longrightarrow & A & \xrightarrow[x\mapsto A_0'x]{} & A \oplus A & \xrightarrow[x\mapsto A_1'x]{} & A & \longrightarrow & 0, \end{array}$$

where

$$A_0 = \begin{bmatrix} \gamma_1 - id \\ \gamma_2 - id \end{bmatrix}, A_1 = \begin{bmatrix} -(\gamma_2 - id) & \gamma_1 - id \end{bmatrix},$$

$$A_0' = \begin{bmatrix} \gamma_1' - id \\ \gamma_2 - id \end{bmatrix}, A_1' = \begin{bmatrix} -(\gamma_2 - id) & \gamma_1' - id \end{bmatrix}.$$

It is easy to check that the above diagram is commutative. Since $\frac{\gamma_1'-id}{\gamma_1-id}$ is unit in $\mathcal{O}_K[[\Gamma^*_{LT}]]$, then by passing to the cohomology, it induces a natural isomorphism of $\mathcal{H}^i(\Gamma^\bullet_{LT,\gamma_1,\gamma_2}(A))$ on $\mathcal{H}^i(\Gamma^\bullet_{LT,\gamma'_1,\gamma_2}(A))$. Similarly, it is easy to show that $\mathcal{H}^i(\Gamma^\bullet_{LT,\gamma'_1,\gamma_2}(A))$ is naturally isomorphic to $\mathcal{H}^i(\Gamma^\bullet_{LT,\gamma'_1,\gamma'_2}(A))$. Therefore there is a natural isomorphism between $\mathcal{H}^i(\Gamma^\bullet_{LT,\gamma_1,\gamma_2}(A))$ and $\mathcal{H}^i(\Gamma^\bullet_{LT,\gamma'_1,\gamma'_2}(A))$. Now the general case follows by induction on the number of generators of Γ^*_{LT}. ■

5.4 Lubin-Tate Herr complex

Now we define a complex, namely, the Lubin-Tate Herr complex, which is a generalization of the Herr complex [17].

Definition 5.4.1. Let $M \in \varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}_{/\mathcal{O}_\mathcal{E}}$. Define the *co-chain complex* $\Phi\Gamma^\bullet_{LT}(M)$ as the total complex of the double complex $\Gamma^\bullet_{LT}(\Phi^\bullet(M^\Delta))$, and we call it the *Lubin-Tate Herr complex* for M.

Explicitly for the cases $d = 2$ and 3, the Lubin-Tate Herr complex looks like as in the following examples. Note that in the following examples $M = M^\Delta$. We write M only for the simplicity.

Example 5.4.2. Let $d = 2$. Then the Lubin-Tate Herr complex $\Phi\Gamma^\bullet_{LT}(M)$ is defined as follows:

$$0 \to M \xrightarrow{x \mapsto A_{0,\varphi_q}x} M^{\oplus 3} \xrightarrow{x \mapsto A_{1,\varphi_q}x} M^{\oplus 3} \xrightarrow{x \mapsto A_{2,\varphi_q}x} M \to 0,$$

where

$$A_{0,\varphi_q} = \begin{bmatrix} \varphi_M - id \\ \gamma_1 - id \\ \gamma_2 - id \end{bmatrix}, A_{1,\varphi_q} = \begin{bmatrix} -(\gamma_1 - id) & \varphi_M - id & 0 \\ -(\gamma_2 - id) & 0 & \varphi_M - id \\ 0 & -(\gamma_2 - id) & \gamma_1 - id \end{bmatrix},$$

$$A_{2,\varphi_q} = \begin{bmatrix} \gamma_2 - id & -(\gamma_1 - id) & \varphi_M - id \end{bmatrix}.$$ □

Example 5.4.3. For $d = 3$, the complex $\Phi\Gamma^\bullet_{LT}(M)$ is defined as follows:

$$0 \to M \xrightarrow{x \mapsto A_{0,\varphi_q}x} M^{\oplus 4} \xrightarrow{x \mapsto A_{1,\varphi_q}x} M^{\oplus 6} \xrightarrow{x \mapsto A_{2,\varphi_q}x} M^{\oplus 4} \xrightarrow{x \mapsto A_{3,\varphi_q}x} M \to 0,$$

where

$$A_{0,\varphi_q}\begin{bmatrix}\varphi_M - id\\ \gamma_1 - id\\ \gamma_2 - id\\ \gamma_3 - id\end{bmatrix}, A_{1,\varphi_q} = \begin{bmatrix}-(\gamma_1 - id) & \varphi_M - id & 0 & 0\\ -(\gamma_2 - id) & 0 & \varphi_M - id & 0\\ -(\gamma_3 - id) & 0 & 0 & \varphi_M - id\\ 0 & -(\gamma_2 - id) & \gamma_1 - id & 0\\ 0 & 0 & -(\gamma_3 - id) & \gamma_2 - id\\ 0 & -(\gamma_3 - id) & 0 & \gamma_1 - id\end{bmatrix},$$

$$A_{2,\varphi_q} = \begin{bmatrix}\gamma_2 - id & -(\gamma_1 - id) & 0 & \varphi_M - id & 0 & 0\\ 0 & \gamma_3 - id & -(\gamma_2 - id) & 0 & \varphi_M - id & 0\\ \gamma_3 - id & 0 & -(\gamma_1 - id) & 0 & 0 & \varphi_M - id\\ 0 & 0 & 0 & \gamma_3 - id & \gamma_1 - id & -(\gamma_2 - id)\end{bmatrix},$$

$$A_{3,\varphi_q} = \begin{bmatrix}-(\gamma_3 - id) & -(\gamma_1 - id) & \gamma_2 - id & \varphi_M - id\end{bmatrix}.$$ □

Next, we compute the Galois cohomology groups using this Lubin-Tate Herr complex.

5.5 Galois cohomology via Lubin-Tate Herr complex

Lemma 5.5.1. *Let $V \in \mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$. Then $V \mapsto \mathcal{H}^i(\Phi\Gamma^\bullet_{LT}(\mathbb{D}_{LT}(V)))_{i\geq 0}$ is a cohomological δ-functor from the category of discrete π-primary representations of G_K to the category of abelian groups. Moreover, we have*

$$\mathcal{H}^0(\Phi\Gamma^\bullet_{LT}(\mathbb{D}_{LT}(V))) \cong V^{G_K}.$$

Proof. Let

$$0 \to V_1 \to V_2 \to V_3 \to 0 \tag{5.6}$$

be a short exact sequence of discrete π-primary representations of G_K. Since the functor $\mathbb{D}_{LT}$ is exact, we have a short exact sequence

$$0 \to \mathbb{D}_{LT}(V_1) \to \mathbb{D}_{LT}(V_2) \to \mathbb{D}_{LT}(V_3) \to 0$$

in $\varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT},ét,tor}_{/\mathcal{O}_\mathcal{E}}$. By using Acyclic Assembly Lemma [35, Lemma 2.7.3], we get a short exact sequence

$$0 \to \Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V_1)) \to \Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V_2)) \to \Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V_3)) \to 0 \tag{5.7}$$

of co-chain complexes. Then the long exact sequence of (5.7) gives maps

$$\delta^i : \mathcal{H}^i\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V_3)) \to \mathcal{H}^{i+1}\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V_1)),$$

which are functorial in (5.6). Therefore $V \mapsto \mathcal{H}^i\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V))_{i\geq 0}$ is a cohomological δ-functor from the category $\mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$ to the category of abelian groups.

For the second part, we know that by definition

$$\mathbb{D}_{LT}(V) = (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V)^{H_K}.$$

Since φ_q acts trivially on V and it commutes with the action of G_K, we have:

$$\begin{aligned}
\mathbb{D}_{LT}(V)^{\varphi_{\mathbb{D}_{LT}(V)}=id} &= ((\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V)^{H_K})^{\varphi_{\mathbb{D}_{LT}(V)}=id} \\
&= (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}^{\varphi_q=id} \otimes_{\mathcal{O}_K} V)^{H_K} \\
&= (\mathcal{O}_K \otimes_{\mathcal{O}_K} V)^{H_K} \\
&\cong V^{H_K},
\end{aligned}$$

where the third equality follows from Lemma 4.2.3. Therefore

$$\mathbb{D}_{LT}(V)^{\varphi_{\mathbb{D}_{LT}(V)}=id,\Gamma_{LT}=id} \cong (V^{H_K})^{\Gamma_{LT}=id} = V^{G_K}.$$

On the other hand, by definition of the Lubin-Tate Herr complex, we have

$$\begin{aligned}
\mathcal{H}^0(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V))) =& (\mathbb{D}_{LT}(V)^{\Delta})^{\varphi_{\mathbb{D}_{LT}(V)}=id,\Gamma^*_{LT}=id} \\
=& \mathbb{D}_{LT}(V)^{\varphi_{\mathbb{D}_{LT}(V)}=id,\Gamma_{LT}=id}.
\end{aligned}$$

Hence

$$\mathcal{H}^0(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V))) \cong V^{G_K}. \qquad \blacksquare$$

Theorem 5.5.2. *Let V be a discrete π-primary abelian group with a continuous action of G_K, i.e., $V \in \mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$. Then*

$$H^i(G_K, V) \cong \mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V)))$$

for $i \geq 0$. In other words, the Lubin-Tate Herr complex $\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V))$ computes the Galois cohomology of G_K with coefficients in V.

Proof. Since $(H^i(G_K, -))_{i\geq 0}$ is a universal δ-functor and $(\mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(-))))_{i\geq 0}$ is a cohomological δ-functor such that $\mathcal{H}^0(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(-))) \cong H^0(G_K, -)$. Thus we have a natural transformation

$$H^i(G_K, -) \to \mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(-)))$$

of δ-functors. First, assume that V is an injective object in $\mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$. Then there is a spectral sequence

$$E_2^{mn} = \mathcal{H}^m(\Gamma^{\bullet}_{LT}(\mathcal{H}^n(\Phi^{\bullet}(\mathbb{D}_{LT}(V)^{\Delta})))) \Rightarrow \mathcal{H}^{m+n}(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V))) \qquad (5.8)$$

associated to the double complex $\Gamma^{\bullet}_{LT}(\Phi^{\bullet}(\mathbb{D}_{LT}(V)^{\Delta}))$, and associated to the group G_K, we have the Hochschild-Serre spectral sequence

$$E_2^{mn} = H^m(\Gamma^*_{LT}, H^n(H^*_K, V)) \Rightarrow H^{m+n}(G_K, V). \qquad (5.9)$$

Since V is injective, it follows from Proposition 5.2.6 that the augmentation map

$$V^{H^*_K}[0] \to \Phi^{\bullet}(\mathbb{D}_{LT}(V)^{\Delta})$$

is a quasi-isomorphism. Also, $V^{H_K^*}$ is injective as a discrete representation of Γ_{LT}^*. Then by using Proposition 5.3.7, the map

$$V^{G_K}[0] \to \Gamma_{LT}^{\bullet}(V^{H_K^*})$$

is also a quasi-isomorphism of co-chain complexes. Now the natural transformation $H^i(G_K, -) \to \mathcal{H}^i(\Phi\Gamma_{LT}^{\bullet}(\mathbb{D}_{LT}(-)))$ and the spectral sequences (5.8) and (5.9) give the following isomorphism

$$H^i(G_K, V) \cong \mathcal{H}^i(\Phi\Gamma_{LT}^{\bullet}(\mathbb{D}_{LT}(V))) \quad \text{for } i \geq 0.$$

As we know that the category $\mathbf{Rep}_{\mathcal{O}_K-tor}^{dis}(G_K)$ has enough injectives, the general case follows from Lemma 5.5.1 by using dimension shifting (Proposition 2.3.3). ■

Remark 5.5.3. The cohomology groups, computed using the Lubin-Tate Herr complex, do not depend on the generators of Γ_{LT}^*, i.e., they are independent of the choice of the generators of Γ_{LT}^*.

Next, we show that the Lubin-Tate Herr complex computes the Galois cohomology of objects in the category $\mathbf{Rep}_{\mathcal{O}_K}(G_K)$. Let $V \in \mathbf{Rep}_{\mathcal{O}_K}(G_K)$. Then V is a finitely generated free $\mathcal{O}_K$-module with a continuous and linear action of G_K and we have

$$\begin{aligned} V &= \varprojlim V \otimes_{\mathcal{O}_K} \mathcal{O}_K/\pi^n\mathcal{O}_K \\ &\cong \varprojlim V/\pi^n V, \end{aligned}$$

where each $V/\pi^n V$ is π-power torsion and it is also discrete as $V/\pi^n V$ is finite. This means that any object in $\mathbf{Rep}_{\mathcal{O}_K}(G_K)$ is the inverse limit of objects in the category $\mathbf{Rep}_{\mathcal{O}_K-tor}^{dis}(G_K)$.

Lemma 5.5.4. *Let $V \in \mathbf{Rep}_{\mathcal{O}_K}(G_K)$. Then the functor $H^i(G_K, -)$ commutes with* *the inverse limits, i.e.,* $H^i(G_K, V) \cong \varprojlim_n H^i(G_K, V/\pi^n V)$.

Proof. Since k is finite, the cohomology groups $H^i(G_K, V/\pi^n V)$ are finite for all n ([32, Theorem 2.1]). Now the result follows from [33, Corollary 2.2]. ∎

Theorem 5.5.5. *For any* $V \in \mathbf{Rep}_{\mathcal{O}_K}(G_K)$*, we have*

$$H^i(G_K, V) \cong \mathcal{H}^i(\Phi\Gamma^\bullet_{LT}(\mathbb{D}_{LT}(V))) \quad \textit{for } i \geq 0.$$

Proof. Firstly we show that the functor $\mathcal{H}^i(\Phi\Gamma^\bullet_{LT}(\mathbb{D}_{LT}(-)))$ commutes with the inverse limits. Since the transition maps are surjective in the projective system $(\Phi\Gamma^\bullet_{LT}(\mathbb{D}_{LT}(V/\pi^n V)))_n$ of co-chain complexes of abelian groups, the first hyper-cohomology spectral sequence degenerates at E_2. Moreover, it follows from Lemma 5.5.4 that $\varprojlim_n{}^1 \mathcal{H}^i(\Phi\Gamma^\bullet_{LT}(\mathbb{D}_{LT}(V/\pi^n V))) = 0$. Therefore the second hypercohomology spectral sequence

$$\varprojlim_n{}^i \mathcal{H}^j(\Phi\Gamma^\bullet_{LT}(\mathbb{D}_{LT}(V/\pi^n V))) \Rightarrow \mathcal{H}^{i+j}(\Phi\Gamma^\bullet_{LT}(\mathbb{D}_{LT}(V)))$$

also degenerates at E_2. Thus

$$\varprojlim_n \mathcal{H}^i(\Phi\Gamma^\bullet_{LT}(\mathbb{D}_{LT}(V/\pi^n V))) = \mathcal{H}^i(\Phi\Gamma^\bullet_{LT}(\mathbb{D}_{LT}(V))).$$

Now

$$\begin{aligned} H^i(G_K, V) &\cong \varprojlim_n H^i(G_K, V/\pi^n V) \\ &\cong \varprojlim_n \mathcal{H}^i(\Phi\Gamma^\bullet_{LT}(\mathbb{D}_{LT}(V/\pi^n V))) \\ &\cong \mathcal{H}^i(\Phi\Gamma^\bullet_{LT}(\mathbb{D}_{LT}(V))), \end{aligned}$$

where the first isomorphism follows from Lemma 5.5.4 and the second is induced from Theorem 5.5.2. ∎

Corollary 5.5.6. *Let V be a finite free $\mathcal{O}_K$-module with a continuous and linear action of G_K. Then*

$$\mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V))) = 0 \quad \textit{for } i \geq 3,$$

although it is not obvious from the definition of the Lubin-Tate Herr complex.

Proof. Recall the classical result that the groups $H^i(G_K, V)$ are trivial for $i \geq 3$, [30, Chapter II, Proposition 12]. Then it follows from the above theorem that

$$\mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_{LT}(V))) = 0 \quad \text{for } i \geq 3.$$

∎

Chapter 6

Galois Cohomology over the False-Tate Type Extensions

This chapter is a part of [1]. In this chapter, first, we extend a result of Kisin and Ren (Theorem 4.2.10) to certain non-abelian extensions, namely, the *False-Tate type extensions*. Then we generalize the Lubin-Tate Herr complex defined in Chapter 5 over the False-Tate type extensions, which we call the *False-Tate type Herr complex*. Further, we compute the Galois cohomology groups using the False-Tate type Herr complex.

6.1 Equivalence of categories

Recall that K is a local field of characteristic 0 with the ring of integers $\mathcal{O}_K$, maximal ideal $\mathfrak{m}_K$ and the uniformizer π. For any $x \in \mathfrak{m}_K \backslash \mathfrak{m}_K^2$, choose a system $(x_i)_{i\geq 1}$ such that $[p](x_1) = x$ and $[p](x_{i+1}) = x_i$ for all $i \geq 1$. Define $\tilde{K} := K(x_i)_{i\geq 1}$, and then the extension $\tilde{K}/K$ is not Galois. Let $L := K_\infty \tilde{K}$; then it is easy to see that the extension L/K is a Galois extension. Moreover, L/K is arithmetically pro-finite as $\mathrm{Gal}(L/K)$ is a p-adic Lie group. As in [36], we consider the field of norms for this extension. The fraction field $\mathrm{Fr}(\mathcal{R})$ contains the field of norms $E_L := X_K(L)$ of extension

L/K in a natural way, and we have $\mathrm{Gal}(\bar{K}/L) \cong \mathrm{Gal}(E^{sep}/E_L)$ ([36, Corollary 3.2.3]). Recall from Chapter 4 that the ring $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$ is a complete discrete valuation ring with residue field E^{sep} and is stable under the action of G_K and φ_q. Define $\mathcal{O}_{\mathcal{L}} := (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{\mathrm{Gal}(\bar{K}/L)}$. Since $\mathrm{Gal}(\bar{K}/L) \cong \mathrm{Gal}(E^{sep}/E_L)$, we have $E_L = (E^{sep})^{\mathrm{Gal}(\bar{K}/L)}$, and $\mathcal{O}_{\mathcal{L}}$ is a complete discrete valuation ring with residue field E_L. Moreover, the ring $\mathcal{O}_{\mathcal{L}}$ is stable under the action of G_K and φ_q. Define $\Gamma_{LT,FT} := \mathrm{Gal}(L/K)$ and $H_L = \mathrm{Gal}(\bar{K}/L)$. The following diagrams summarize the above notations.

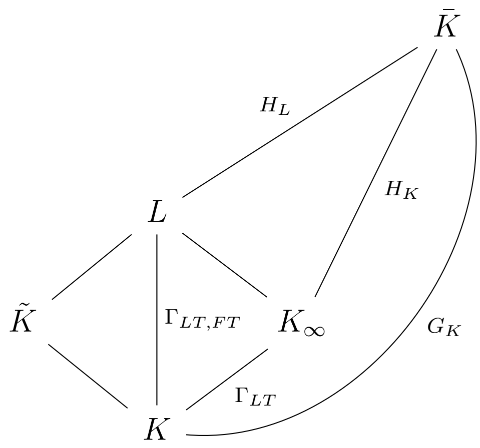

Figure 6.1: Field extensions of K

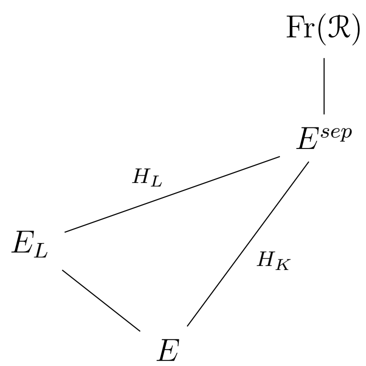

Figure 6.2: Field extensions of E

Now for any $V \in \mathbf{Rep}_{\mathcal{O}_K}(G_K)$, define

$$\mathbb{D}_{LT,FT}(V) := (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V)^{H_L}.$$

Let $\mathbf{Mod}^{\varphi_q,\Gamma_{LT,FT},\acute{e}t}_{/\mathcal{O}_{\mathcal{L}}}$ be the category of finite free étale $(\varphi_q, \Gamma_{LT,FT})$-modules over $\mathcal{O}_{\mathcal{L}}$. Then the modules $\mathbb{D}_{LT,FT}(V)$ and $\mathbb{D}_{LT}(V) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{L}}$ are in the category $\mathbf{Mod}^{\varphi_q,\Gamma_{LT,FT},\acute{e}t}_{/\mathcal{O}_{\mathcal{L}}}$, and there is a natural map $\iota : \mathbb{D}_{LT}(V) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{L}} \to \mathbb{D}_{LT,FT}(V)$.

Proposition 6.1.1. *The map ι is an isomorphism of étale $(\varphi_q, \Gamma_{LT,FT})$-modules over $\mathcal{O}_{\mathcal{L}}$.*

Proof. By Remark 4.2.9, we know that $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathbb{D}_{LT}(V) \cong \mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V$ as an étale (φ_q, Γ_{LT})-modules over $\mathcal{O}_{\mathcal{E}}$. Then

$$\begin{aligned}
\mathbb{D}_{LT}(V) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{L}} &= \mathbb{D}_{LT}(V) \otimes_{\mathcal{O}_{\mathcal{E}}} (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{\mathrm{Gal}(\bar{K}/L)} \\
&= (\mathbb{D}_{LT}(V) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\widehat{\mathcal{E}^{ur}}})^{\mathrm{Gal}(\bar{K}/L)} \\
&\cong (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V)^{\mathrm{Gal}(\bar{K}/L)} \\
&= \mathbb{D}_{LT,FT}(V),
\end{aligned}$$

where the second identity follows from the fact that $\mathrm{Gal}(\bar{K}/L) \subseteq \mathrm{Gal}(\bar{K}/K_\infty)$. Thus ι is an isomorphism of étale $(\varphi_q, \Gamma_{LT,FT})$-modules. ■

Similarly, for any $M \in \mathbf{Mod}^{\varphi_q,\Gamma_{LT,FT},\acute{e}t}_{/\mathcal{O}_{\mathcal{L}}}$, define

$$\mathbb{V}_{LT,FT}(M) := (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_{\mathcal{L}}} M)^{\varphi_q \otimes \varphi_M = id}.$$

Then we have the following theorem.

Theorem 6.1.2. *The functor $\mathbb{D}_{LT,FT}$ is an exact equivalence of categories between the category $\mathbf{Rep}_{\mathcal{O}_K}(G_K)$ (resp., $\mathbf{Rep}_{\mathcal{O}_K-tor}(G_K)$) and $\mathbf{Mod}^{\varphi_q,\Gamma_{LT,FT},\acute{e}t}_{/\mathcal{O}_{\mathcal{L}}}$ (resp., $\mathbf{Mod}^{\varphi_q,\Gamma_{LT,FT},\acute{e}t,tor}_{/\mathcal{O}_{\mathcal{L}}}$) with a quasi-inverse functor $\mathbb{V}_{LT,FT}$.*

Proof. Since the functor $\mathbb{D}_{LT,FT}$ is composite of the functor $\mathbb{D}_{LT}$ with the scalar extension $\otimes_{\mathcal{O}_{\mathcal{E}}}\mathcal{O}_{\mathcal{L}}$. Now the proof follows from Proposition 6.1.1 and Theorem 4.2.10. ■

Remark 6.1.3. The extension $\tilde{K}$ is not the canonical one. We can also define $\tilde{K}$ as follows:

1. Define $K_{cyc} := K(\mu_{p^n})_{n\geq 1}$. Let $K_{cyc} \subseteq K_\infty$ and $\tilde{K} := K(\pi^{p^{-r}}, r \geq 1)$, then $L = K_\infty\tilde{K}$ is a Galois extension of K and $\mathrm{Gal}(L/K_\infty) \cong \mathbb{Z}_p$. The case when $K_{cyc} = K_\infty$ has been considered in [34] and [18].

2. We can also define $\tilde{K} := K(y_i)_{i\geq 1}$, where $(y_i)_{i\geq 1}$ is a system satisfying $[\pi](y_1) = y$ and $[\pi](y_{i+1}) = y_i$ for all $i \geq 1$ and $y \in \mathfrak{m}_K \backslash \mathfrak{m}_K^2$. In this case, $\mathrm{Gal}(L/K_\infty)$ is isomorphic to an open subgroup of $\mathbb{Z}_p$.

Then using similar methods, as explained in Chapter 5, we extend the functor $\mathbb{D}_{LT,FT}$ to the category of discrete π-primary abelian groups with a continuous action of G_K. Then the functor $\mathbb{D}_{LT,FT}$ is an exact equivalence of categories from the category $\mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$ of discrete π-primary representations of G_K to the category $\varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT,FT},\acute{e}t,tor}_{/\mathcal{O}_{\mathcal{L}}}$ of injective limits of π-power torsion objects in $\mathbf{Mod}^{\varphi_q,\Gamma_{LT,FT},\acute{e}t}_{/\mathcal{O}_{\mathcal{L}}}$, i.e., we have the following result.

Theorem 6.1.4. *The functors $\mathbb{D}_{LT,FT}$ and $\mathbb{V}_{LT,FT}$ are quasi-inverse equivalence of categories between $\mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$ and $\varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT,FT},\acute{e}t,tor}_{/\mathcal{O}_{\mathcal{L}}}$.*

Sketch of the proof. Let $V \in \mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$. Then we have

$$V = \varinjlim_n V_n,$$

where each V_n is an object in the category $\mathbf{Rep}_{\mathcal{O}_K-tor}(G_K)$. Since the functor $\mathbb{D}_{LT}$ commutes with direct limits, then it follows from Proposition 6.1.1 that the functor $\mathbb{D}_{LT,FT}$ also commutes with direct limits. Now the result follows from Theorem 6.1.2

by taking direct limits and noting that the functor $\mathbb{V}_{LT,FT}$ also commutes with direct limits. ■

6.2 The complex $\Gamma^\bullet_{LT,FT}$

To generalize the Lubin-Tate Herr complex to the case of False-Tate type extensions; first, we extend the complex $\Gamma^\bullet_{LT}$ defined in Chapter 5 over the False-Tate type extensions. We denote this complex as $\Gamma^\bullet_{LT,FT}$ complex.

Recall that $\Gamma^*_{LT} = \langle \gamma_1, \gamma_2, \dots, \gamma_d \rangle$ as a $\mathbb{Z}_p$-module. Let $\tilde{\gamma}$ be a topological generator of $\mathrm{Gal}(L/K_\infty)$. We lift $\gamma_1, \gamma_2, \dots, \gamma_d$ to the elements of $\mathrm{Gal}(L/\tilde{K})$. Then $\Gamma^*_{LT,FT}$ is topologically generated by the set $\tilde{\mathfrak{X}} := \langle \gamma_1, \gamma_2, \dots, \gamma_d, \tilde{\gamma} \rangle$ with the relations $\gamma_i \tilde{\gamma} = \tilde{\gamma}^{a_i} \gamma_i$ such that $a_i \in \mathbb{Z}_p^\times$, where $a_i = \chi_{LT}(\gamma_i)$ for all $i = 1, \dots, d$, and χ_{LT} is the Lubin-Tate character.

Definition 6.2.1. Let A be an arbitrary representation of the group $\Gamma^*_{LT,FT}$. Then we define $\Gamma^\bullet_{LT,FT}(A)$ as follows.

$$\Gamma^\bullet_{LT,FT}(A) : 0 \to A \to \bigoplus_{i_1 \in \tilde{\mathfrak{X}}} A \to \cdots \to \bigoplus_{\{i_1,\dots,i_r\} \in \binom{\tilde{\mathfrak{X}}}{r}} A \to \cdots \to A \to 0,$$

where $\binom{\tilde{\mathfrak{X}}}{r}$ denotes choosing r-indices at a time from the set $\tilde{\mathfrak{X}}$, and for all $0 \le r \le |\tilde{\mathfrak{X}}| - 1$, the map $d^{j_1,\dots,j_{r+1}}_{i_1,\dots,i_r} : A \to A$ from the component in the r-th term corresponding to $\{i_1, \dots, i_r\}$ to the component corresponding to the $(r+1)$-tuple $\{j_1, \dots, j_{r+1}\}$ is given by

$$d^{j_1,\dots,j_{r+1}}_{i_1,\dots,i_r} = \begin{cases} 0 & \text{if } \{i_1,\dots,i_r\} \not\subseteq \{j_1,\dots,j_{r+1}\}, \\ (-1)^{s_j}(\gamma_j - id) & \text{if } \{j_1,\dots,j_{r+1}\} = \{i_1,\dots,i_r\} \cup \{j\} \\ & \text{and} \{i_1,\dots,i_r\} \text{ doesn't contain } \tilde{\gamma}, \\ (-1)^{s_j+1}\left(\gamma_j - \frac{\tilde{\gamma}^{\chi_{LT}(j)\chi_{LT}(i_1)\cdots\chi_{LT}(i_r)} - id}{\tilde{\gamma}^{\chi_{LT}(i_1)\cdots\chi_{LT}(i_r)} - id}\right) & \text{if } \{j_1,\dots,j_{r+1}\} = \{i_1,\dots,i_r\} \cup \{j\} \\ & \text{and} \{i_1,\dots,i_r\} \text{ contains } \tilde{\gamma}, \\ \tilde{\gamma}^{\chi_{LT}(i_1)\cdots\chi_{LT}(i_r)} - id & \text{if } \{j_1,\dots,j_{r+1}\} = \{i_1,\dots,i_r\} \cup \{\tilde{\gamma}\}, \end{cases}$$

where s_j is the number of elements in the set $\{i_1, \ldots, i_r\}$, which are smaller than j.

The above definition of $\Gamma^{\bullet}_{LT,FT}(A)$ is inspired by the definition of the Koszul complex.

Theorem 6.2.2. *Let A be an arbitrary representation of the group $\Gamma^{*}_{LT,FT}$. Then $\Gamma^{\bullet}_{LT,FT}(A)$ is a complex.*

Proof. Since A is a representation of $\Gamma^{*}_{LT,FT}$, A is also an $\mathcal{O}_K[[\Gamma^{*}_{LT,FT}]]$-module. Note that

$$A \otimes_{\mathcal{O}_K[[\Gamma^{*}_{LT,FT}]]} \mathcal{O}_K[[\Gamma^{*}_{LT,FT}]] \cong A.$$

First, we show that $\Gamma^{\bullet}_{LT,FT}\left(\mathcal{O}_K[[\Gamma^{*}_{LT,FT}]]\right)$ is a complex.

Let $\bigoplus_{i_1 \in \mathfrak{X}} \mathcal{O}_K[[\Gamma^{*}_{LT,FT}]]$ be a free module with ordered basis $\{e_1, e_2, \ldots, e_d, e_{d+1} = \tilde{e}\}$ and $\bigoplus_{\{i_1,\ldots,i_r\} \in \binom{\mathfrak{X}}{r}} \mathcal{O}_K[[\Gamma^{*}_{LT,FT}]]$ be a free module with basis $\{e_{i_1} \wedge \cdots \wedge e_{i_r}\}$ with $i_1 < i_2 < \cdots < i_r$. Then the boundary maps

$$d : \bigoplus_{\{i_1,\ldots,i_r\} \in \binom{\tilde{\mathfrak{X}}}{r}} \mathcal{O}_K[[\Gamma^{*}_{LT,FT}]] \to \bigoplus_{\{j_1,\ldots,j_{r+1}\} \in \binom{\tilde{\mathfrak{X}}}{r+1}} \mathcal{O}_K[[\Gamma^{*}_{LT,FT}]]$$

are given by the following

$$\begin{aligned} d(e_{i_1} \wedge \cdots \wedge e_{i_r}) &= \sum d^{j_1,\ldots,j_{r+1}}_{i_1,\ldots,i_r}(e_{j_1} \wedge \cdots \wedge e_{j_r} \wedge e_{j_{r+1}}) \\ &= \sum_{\text{Case I}} d^{j_1,\ldots,j_{r+1}}_{i_1,\ldots,i_r}(e_{j_1} \wedge \cdots \wedge e_{j_r} \wedge e_{j_{r+1}}) \\ &+ \sum_{\text{Case II}} d^{j_1,\ldots,j_{r+1}}_{i_1,\ldots,i_r}(e_{j_1} \wedge \cdots \wedge e_{j_r} \wedge e_{j_{r+1}}) \\ &+ \sum_{\text{Case III}} d^{j_1,\ldots,j_{r+1}}_{i_1,\ldots,i_r}(e_{j_1} \wedge \cdots \wedge e_{j_r} \wedge e_{j_{r+1}}), \end{aligned}$$

where the map $d^{j_1,\ldots,j_{r+1}}_{i_1,\ldots,i_r}$ is defined as in Definition 6.2.1 and the Case I, Case II, Case III are as follows.

Case I: $\{j_1, \ldots, j_{r+1}\} = \{i_1, \ldots, i_r\} \cup \{\gamma_j\}$ and $\{i_1, \ldots, i_r\}$ does not contain $\tilde{\gamma}$.

In this case,

$$\begin{aligned}d(e_{i_1}\wedge\cdots\wedge e_{i_r}) &= \sum d^{j_1,\ldots,j_{r+1}}_{i_1,\ldots,i_r}(e_{j_1}\wedge\cdots\wedge e_{j_r}\wedge e_{j_{r+1}})\\ &= \sum_{j=1}^{d}(-1)^{s_j}(\gamma_j-1)(e_{i_1}\wedge\cdots\wedge e_{s_j}\wedge e_j\wedge\cdots\wedge e_{i_r}),\end{aligned}$$

where s_j is the number of elements in the set $\{i_1,\ldots,i_r\}$ smaller than j. Then

$$\begin{aligned}(d\circ d)(e_{i_1}\wedge\cdots\wedge e_{i_r}) =& d\left(\sum_{j=1}^{d}(-1)^{s_j}(\gamma_j-1)(e_{i_1}\wedge\cdots\wedge e_{s_j}\wedge e_j\wedge\cdots\wedge e_{i_r})\right)\\ =& \sum_{t=1}^{d}(-1)^{s_t}(\gamma_t-1)\sum_{j=1}^{d}(-1)^{s_j}(\gamma_j-1)(e_{i_1}\wedge\cdots\wedge e_{s_j}\wedge e_j\\ & \wedge\cdots\wedge e_{s_t}\wedge e_t\wedge\cdots\wedge e_{i_r}).\end{aligned}$$

Here s_t is the number of elements in the set $\{i_1,\ldots,i_r\}$, which are smaller than t. Note that each term appears twice in the above summation, so each term has two coefficients. Now for $j<t$, we compare the coefficients of the term $e_{i_1}\wedge\cdots\wedge e_{s_j}\wedge e_j\wedge\cdots\wedge e_{s_t}\wedge e_t\wedge\cdots\wedge e_{i_r}$. The coefficient is $(-1)^{s_j+s_t+1}(\gamma_t-id)(\gamma_j-id)$ if we introduce e_j first and e_t second. Similarly, by introducing e_t first and then e_j, we get $(-1)^{s_t+s_j}(\gamma_j-id)(\gamma_t-id)$ as the coefficient of the term $e_{i_1}\wedge\cdots\wedge e_{s_j}\wedge e_j\wedge\cdots\wedge e_{s_t}\wedge e_t\wedge\cdots\wedge e_{i_r}$. Since γ_j commutes with γ_t. Thus, the coefficients are of opposite parity.

Case II: $\{j_1,\ldots,j_{r+1}\}=\{i_1,\ldots,i_r\}\cup\{\gamma_j\}$ and $\{i_1,\ldots,i_r\}$ contains $\tilde{\gamma}$.

Without loss of generality, we can assume that $e_{i_r}=\tilde{e}$. Then

$$\begin{aligned}d(e_{i_1}\wedge\cdots\wedge e_{i_{r-1}}\wedge e_{i_r}) =& \sum_{j=1}^{d+1}(-1)^{s_j+1}\left(\gamma_j-\frac{\tilde{\gamma}^{ha_j}-id}{\tilde{\gamma}^h-id}\right)(e_{i_1}\wedge\cdots\wedge e_{s_j}\wedge e_j\\ & \wedge\cdots\wedge e_{i_r}),\end{aligned}$$

where s_j is the number of elements in the set $\{i_1,\ldots,i_r\}$ smaller than j and $h=$

$\chi_{LT}(e_{i_1})\cdots\chi_{LT}(e_{i_{r-1}})$. Then we have

$$\begin{aligned}
&(d\circ d)(e_{i_1}\wedge\cdots\wedge e_{i_r})\\
&\quad=d\left(\sum_{j=1}^{d+1}(-1)^{s_j+1}\left(\gamma_j-\frac{\tilde{\gamma}^{ha_j}-id}{\tilde{\gamma}^h-id}\right)(e_{i_1}\wedge\cdots\wedge e_{s_j}\wedge e_j\wedge\cdots\wedge e_{i_r})\right)\\
&\quad=\sum_{t=1}^{d+1}(-1)^{s_t+1}\left(\gamma_t-\frac{\tilde{\gamma}^{ha_ja_t}-id}{\tilde{\gamma}^{ha_j}-id}\right)\sum_{j=1}^{d+1}(-1)^{s_j+1}\left(\gamma_j-\frac{\tilde{\gamma}^{ha_j}-id}{\tilde{\gamma}^h-id}\right)\\
&\quad\quad(e_{i_1}\wedge\cdots\wedge e_{s_j}\wedge e_j\wedge\cdots\wedge e_{s_t}\wedge e_t\wedge\cdots\wedge e_{i_r}).
\end{aligned}$$

Here s_t is the number of elements in the set $\{i_1,\ldots,i_r\}$, which are smaller than t. Now for $j<t$, we compare the coefficients for the term $e_{i_1}\wedge\cdots\wedge e_{s_j}\wedge e_j\wedge\cdots\wedge e_{s_t}\wedge e_t\wedge\cdots\wedge e_{i_r}$. If we introduce e_j first and e_t second then the coefficient is $(-1)^{s_j+s_t+3}\left(\gamma_t-\frac{\tilde{\gamma}^{ha_ja_t}-id}{\tilde{\gamma}^{ha_j}-id}\right)\left(\gamma_j-\frac{\tilde{\gamma}^{ha_j}-id}{\tilde{\gamma}^h-id}\right)$. Similarly, by introducing e_t first and then e_j, we get $(-1)^{s_t+s_j+2}\left(\gamma_j-\frac{\tilde{\gamma}^{ha_ta_j}-id}{\tilde{\gamma}^{ha_t}-id}\right)\left(\gamma_t-\frac{\tilde{\gamma}^{ha_t}-id}{\tilde{\gamma}^h-id}\right)$ as the coefficient of $e_{i_1}\wedge\cdots\wedge e_{s_j}\wedge e_j\wedge\cdots\wedge e_{s_t}\wedge e_t\wedge\cdots\wedge e_{i_r}$. Note that we have $\gamma_i.\tilde{\gamma}=\tilde{\gamma}^{a_i}\gamma_i$. Then

$$\begin{aligned}
&\left(\gamma_t-\frac{\tilde{\gamma}^{ha_ja_t}-id}{\tilde{\gamma}^{ha_j}-id}\right)\left(\gamma_j-\frac{\tilde{\gamma}^{ha_j}-id}{\tilde{\gamma}^h-id}\right)\\
&\quad=\left(\frac{\gamma_t(\tilde{\gamma}^{ha_j}-id)-(\tilde{\gamma}^{ha_ja_t}-id)}{\tilde{\gamma}^{ha_j}-id}\right)\left(\frac{\gamma_j(\tilde{\gamma}^h-id)-(\tilde{\gamma}^{ha_j}-id)}{\tilde{\gamma}^h-id}\right)\\
&\quad=\left(\frac{\tilde{\gamma}^{ha_ja_t}\gamma_t-\gamma_t-(\tilde{\gamma}^{ha_ja_t}-id)}{\tilde{\gamma}^{ha_j}-id}\right)\left(\frac{\tilde{\gamma}^{ha_j}\gamma_j-\gamma_j-(\tilde{\gamma}^{ha_j}-id)}{\tilde{\gamma}^h-id}\right)\\
&\quad=\left(\frac{(\tilde{\gamma}^{ha_ja_t}-id)(\gamma_t-id)}{\tilde{\gamma}^{ha_j}-id}\right)\left(\frac{(\tilde{\gamma}^{ha_j}-id)(\gamma_j-id)}{\tilde{\gamma}^h-id}\right)\\
&\quad=(\tilde{\gamma}^{ha_ja_t}-id)(\gamma_t-id)(\tilde{\gamma}^{ha_j}-id)^{-1}(\tilde{\gamma}^{ha_j}-id)(\gamma_j-id)\\
&\quad\quad(\tilde{\gamma}^h-id)^{-1}\\
&\quad=(\tilde{\gamma}^{ha_ja_t}-id)(\gamma_t-id)(\gamma_j-id)(\tilde{\gamma}^h-id)^{-1}
\end{aligned}$$

Similarly,

$$\left(\gamma_j-\frac{\tilde{\gamma}^{ha_ta_j}-id}{\tilde{\gamma}^{ha_t}-id}\right)\left(\gamma_t-\frac{\tilde{\gamma}^{ha_t}-id}{\tilde{\gamma}^h-id}\right)=(\tilde{\gamma}^{ha_ta_j}-id)(\gamma_j-id)(\gamma_t-id)(\tilde{\gamma}^h-id)^{-1}$$

Since γ_j commutes with γ_t. Thus, the coefficients of the term $e_{i_1} \wedge \cdots \wedge e_{s_j} \wedge e_j \wedge \cdots \wedge e_{s_t} \wedge e_t \wedge \cdots \wedge e_{i_r}$ are of opposite signs.

Case III: $\{j_1, \dots, j_{r+1}\} = \{i_1, \dots, i_r\} \cup \{\tilde{\gamma}\}$.

In this case, we have

$$d(e_{i_1} \wedge \cdots \wedge e_{i_r}) = (\tilde{\gamma}^h - id)(e_{i_1} \wedge \cdots \wedge \cdots \wedge e_{i_r} \wedge \tilde{e}) + \sum_{j=1}^{d} (-1)^{s_j}(\gamma_j - id)(e_{i_1} \wedge \cdots \wedge e_{s_j} \wedge e_j \wedge \cdots \wedge e_{i_r}),$$

where s_j is the number of elements in the set $\{i_1, \dots, i_r\}$ smaller than j, and $h = \chi_{LT}(e_{i_1}) \cdots \chi_{LT}(e_{i_r})$. Then

$$\begin{aligned}(d \circ d)(e_{i_1} \wedge \cdots \wedge e_{i_r}) =& d\Big((\tilde{\gamma}^h - id)(e_{i_1} \wedge \cdots \wedge \cdots \wedge e_{i_r} \wedge \tilde{e}) + \sum_{j=1}^{d} (-1)^{s_j}(\gamma_j - id) \\ & (e_{i_1} \wedge \cdots \wedge e_{s_j} \wedge e_j \wedge \cdots \wedge e_{i_r})\Big) \\ =& \sum_{j=1}^{d} (-1)^{s_j+1} \left(\gamma_j - \frac{\tilde{\gamma}^{ha_j} - id}{\tilde{\gamma}^h - id}\right)(\tilde{\gamma}^h - id) + (\tilde{\gamma}^{ha_j} - id) \\ & \sum_{j=1}^{d} (-1)^{s_j}(\gamma_j - id)(e_{i_1} \wedge \cdots \wedge e_{s_j} \wedge e_j \wedge \cdots \wedge e_{i_r} \wedge \tilde{e}),\end{aligned}$$

Then we compare the coefficients of the term $e_{i_1} \wedge \cdots \wedge e_{s_j} \wedge e_j \wedge \cdots \wedge e_{i_r} \wedge \tilde{e}$. We get $(-1)^{s_j+1}\left(\gamma_j - \frac{\tilde{\gamma}^{ha_j} - id}{\tilde{\gamma}^h - id}\right)(\tilde{\gamma}^h - id)$ as the coefficient if we introduce $\tilde{e}$ first and then e_j. Similarly, the coefficient is $(-1)^{s_j}(\tilde{\gamma}^{ha_j} - id)(\gamma_j - id)$ if we introduce e_j first and $\tilde{e}$ after. Note that

$$\begin{aligned}\left(\gamma_j - \frac{\tilde{\gamma}^{ha_j} - id}{\tilde{\gamma}^h - id}\right)(\tilde{\gamma}^h - id) =& (\tilde{\gamma}^{ha_j} - id)(\gamma_j - id)(\tilde{\gamma}^h - id)^{-1}(\tilde{\gamma}^h - id) \\ =& (\tilde{\gamma}^{ha_j} - id)(\gamma_j - id).\end{aligned}$$

This implies that the coefficients of $e_{i_1} \wedge \cdots \wedge e_{s_j} \wedge e_j \wedge \cdots \wedge e_{i_r} \wedge \tilde{e}$ are of opposite parity.

Now by combining all the three cases, notice that each term has coefficients with opposite parity. Then it follows that $d \circ d = 0$, i.e., $\Gamma^{\bullet}_{LT,FT}\left(\mathcal{O}_K[[\Gamma^*_{LT,FT}]]\right)$ is a complex. Since we have

$$\Gamma^{\bullet}_{LT,FT}(A) \cong \Gamma^{\bullet}_{LT,FT}\left(\mathcal{O}_K[[\Gamma^*_{LT,FT}]]\right) \otimes_{\mathcal{O}_K[[\Gamma^*_{LT,FT}]]} A.$$

Hence $\Gamma^{\bullet}_{LT,FT}(A)$ is a complex. ■

We explicitly write the complex $\Gamma^{\bullet}_{LT,FT}(A)$ in the case of $d = 2$ and 3.

Example 6.2.3. Let $d = 2$. Then the complex $\Gamma^{\bullet}_{LT,FT}(A)$ is defined as follows:

$$\Gamma^{\bullet}_{LT,FT}(A) : 0 \to A \xrightarrow{x \mapsto A_0 x} A^{\oplus 3} \xrightarrow{x \mapsto A_1 x} A^{\oplus 3} \xrightarrow{x \mapsto A_2 x} A \to 0,$$

where

$$A_0 = \begin{bmatrix} \gamma_1 - id \\ \gamma_2 - id \\ \tilde{\gamma} - id \end{bmatrix}, A_1 = \begin{bmatrix} -(\gamma_2 - id) & \gamma_1 - id & 0 \\ \tilde{\gamma}^{a_1} - id & 0 & -\left(\gamma_1 - \frac{\tilde{\gamma}^{a_1} - id}{\tilde{\gamma} - id}\right) \\ 0 & \tilde{\gamma}^{a_2} - id & -\left(\gamma_2 - \frac{\tilde{\gamma}^{a_2} - id}{\tilde{\gamma} - id}\right) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \tilde{\gamma}^{a_1 a_2} - id & \gamma_2 - \frac{\tilde{\gamma}^{a_1 a_2} - id}{\tilde{\gamma}^{a_1} - id} & -\left(\gamma_1 - \frac{\tilde{\gamma}^{a_1 a_2} - id}{\tilde{\gamma}^{a_2} - id}\right) \end{bmatrix}.$$ □

Example 6.2.4. Let $d = 3$. Then $\Gamma^{\bullet}_{LT,FT}(A)$ is defined as the following:

$$\Gamma^{\bullet}_{LT,FT}(A) : 0 \to A \xrightarrow{x \mapsto A_0 x} A^{\oplus 4} \xrightarrow{x \mapsto A_1 x} A^{\oplus 6} \xrightarrow{x \mapsto A_2 x} A^{\oplus 4} \xrightarrow{x \mapsto A_3 x} A \to 0$$

where

$$A_0 = \begin{bmatrix} \gamma_1 - id \\ \gamma_2 - id \\ \gamma_3 - id \\ \tilde{\gamma} - id \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -(\gamma_2 - id) & \gamma_1 - id & 0 & 0 \\ 0 & -(\gamma_3 - id) & \gamma_2 - id & 0 \\ -(\gamma_3 - id) & 0 & \gamma_1 - id & 0 \\ \tilde{\gamma}^{a_1} - id & 0 & 0 & -\left(\gamma_1 - \frac{\tilde{\gamma}^{a_1} - id}{\tilde{\gamma} - id}\right) \\ 0 & \tilde{\gamma}^{a_2} - id & 0 & -\left(\gamma_2 - \frac{\tilde{\gamma}^{a_2} - id}{\tilde{\gamma} - id}\right) \\ 0 & 0 & \tilde{\gamma}^{a_3} - id & -\left(\gamma_3 - \frac{\tilde{\gamma}^{a_3} - id}{\tilde{\gamma} - id}\right) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \gamma_3 - id & \gamma_1 - id & -(\gamma_2 - id) & 0 & 0 & 0 \\ \tilde{\gamma}^{a_1a_2} - id & 0 & 0 & \gamma_2 - \frac{\tilde{\gamma}^{a_1a_2} - id}{\tilde{\gamma}^{a_1} - id} & -\left(\gamma_1 - \frac{\tilde{\gamma}^{a_1a_2} - id}{\tilde{\gamma}^{a_2} - id}\right) & 0 \\ 0 & 0 & \tilde{\gamma}^{a_1a_3} - id & \gamma_3 - \frac{\tilde{\gamma}^{a_1a_3} - id}{\tilde{\gamma}^{a_1} - id} & 0 & -\left(\gamma_1 - \frac{\tilde{\gamma}^{a_1a_3} - id}{\tilde{\gamma}^{a_3} - id}\right) \\ 0 & \tilde{\gamma}^{a_2a_3} - id & 0 & 0 & \gamma_3 - \frac{\tilde{\gamma}^{a_2a_3} - id}{\tilde{\gamma}^{a_2} - id} & -\left(\gamma_2 - \frac{\tilde{\gamma}^{a_2a_3} - id}{\tilde{\gamma}^{a_3} - id}\right) \end{bmatrix},$$

$$A_3 = \begin{bmatrix} \tilde{\gamma}^{a_1a_2a_3} - id & -\left(\gamma_3 - \frac{\tilde{\gamma}^{a_1a_2a_3} - id}{\tilde{\gamma}^{a_1a_2} - id}\right) & \gamma_2 - \frac{\tilde{\gamma}^{a_1a_2a_3} - id}{\tilde{\gamma}^{a_1a_3} - id} & -\left(\gamma_1 - \frac{\tilde{\gamma}^{a_1a_2a_3} - id}{\tilde{\gamma}^{a_2a_3} - id}\right) \end{bmatrix}. \qquad \square$$

Then the functor $A \mapsto \mathcal{H}^i(\Gamma^\bullet_{LT,FT}(A))_{i\geq 0}$ is a cohomological δ-functor. Moreover, for a discrete π-primary abelian group A with continuous action of $\Gamma^*_{LT,FT}$, the complex $\Gamma^\bullet_{LT,FT}(A)$ computes the $\Gamma^*_{LT,FT}$-cohomology of A and $\mathcal{H}^0(\Gamma^\bullet_{LT,FT}(A)) = A^{\Gamma^*_{LT,FT}}$. The proof is similar to as that of Proposition 5.3.7.

6.3 False-Tate type Herr complex

Now we define a complex, namely, the False-Tate type Herr complex on the category of $\varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT,FT},ét,tor}_{/\mathcal{O}_\mathcal{L}}$, which computes the Galois cohomology groups.

Definition 6.3.1. Let $M \in \varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT,FT},ét,tor}_{/\mathcal{O}_\mathcal{L}}$. Then we define the *co-chain complex* $\Phi\Gamma^\bullet_{LT,FT}(M)$ as the total complex of the double complex $\Gamma^\bullet_{LT,FT}(\Phi^\bullet(M^\Delta))$ and call it the *False-Tate type Herr complex* for M.

In the case of $d = 2$ and 3, the False-Tate type Herr complex looks like as in the following examples. Note that in the following examples $M = M^\Delta$. We write M only for the simplicity.

Example 6.3.2. In the case of $d = 2$, the False-Tate type Herr complex is defined as follows:

$$0 \to M \xrightarrow{x \mapsto A_{0,\varphi_q}x} M^{\oplus 4} \xrightarrow{x \mapsto A_{1,\varphi_q}x} M^{\oplus 6} \xrightarrow{x \mapsto A_{2,\varphi_q}x} M^{\oplus 4} \xrightarrow{x \mapsto A_{3,\varphi_q}x} M \to 0,$$

where

$$A_{0,\varphi_q} = \begin{bmatrix} \varphi_M - id \\ \gamma_1 - id \\ \gamma_2 - id \\ \tilde{\gamma} - id \end{bmatrix}, A_{1,\varphi_q} = \begin{bmatrix} -(\gamma_1 - id) & \varphi_M - id & 0 & 0 \\ -(\gamma_2 - id) & 0 & \varphi_M - id & 0 \\ -(\tilde{\gamma} - id) & 0 & 0 & \varphi_M - id \\ 0 & -(\gamma_2 - id) & \gamma_1 - id & 0 \\ 0 & \tilde{\gamma}^{a_1} - id & 0 & -\left(\gamma_1 - \frac{\tilde{\gamma}^{a_1} - id}{\tilde{\gamma} - id}\right) \\ 0 & 0 & \tilde{\gamma}^{a_2} - id & -\left(\gamma_2 - \frac{\tilde{\gamma}^{a_2} - id}{\tilde{\gamma} - id}\right) \end{bmatrix},$$

$$A_{2,\varphi_q} = \begin{bmatrix} \gamma_2 - id & -(\gamma_1 - id) & 0 & \varphi_M - id & 0 & 0 \\ -(\tilde{\gamma}^{a_1} - id) & 0 & \gamma_1 - \frac{\tilde{\gamma}^{a_1} - id}{\tilde{\gamma} - id} & 0 & \varphi_M - id & 0 \\ 0 & -(\tilde{\gamma}^{a_2} - id) & \gamma_2 - \frac{\tilde{\gamma}^{a_2} - id}{\tilde{\gamma} - id} & 0 & 0 & \varphi_M - id \\ 0 & 0 & 0 & \tilde{\gamma}^{a_1 a_2} - id & \gamma_2 - \frac{\tilde{\gamma}^{a_1 a_2} - id}{\tilde{\gamma}^{a_1} - id} & -\left(\gamma_1 - \frac{\tilde{\gamma}^{a_1 a_2} - id}{\tilde{\gamma}^{a_2} - id}\right) \end{bmatrix},$$

$$A_{3,\varphi_q} = \begin{bmatrix} -(\tilde{\gamma}^{a_1 a_2} - id) & -\left(\gamma_2 - \frac{\tilde{\gamma}^{a_1 a_2} - id}{\tilde{\gamma}^{a_1} - id}\right) & \gamma_1 - \frac{\tilde{\gamma}^{a_1 a_2} - id}{\tilde{\gamma}^{a_2} - id} & \varphi_M - id \end{bmatrix}.$$

□

Example 6.3.3. Let $d = 3$. Then the complex $\Phi\Gamma^{\bullet}_{LT,FT}(M)$ is defined as the following:

$$\Phi\Gamma^{\bullet}_{LT,FT}(M) : 0 \to M \xrightarrow{x \mapsto A_{0,\varphi_q} x} M^{\oplus 5} \xrightarrow{x \mapsto A_{1,\varphi_q} x} M^{\oplus 10} \xrightarrow{x \mapsto A_{2,\varphi_q} x} M^{\oplus 10} \xrightarrow{x \mapsto A_{3,\varphi_q} x} M^{\oplus 5} \xrightarrow{x \mapsto A_{4,\varphi_q} x} M \to 0,$$

where

$$A_{0,\varphi_q} = \begin{bmatrix} \varphi_M - id \\ \gamma_1 - id \\ \gamma_2 - id \\ \gamma_3 - id \\ \tilde{\gamma} - id \end{bmatrix},$$

$$A_{1,\varphi_q} = \begin{bmatrix} -(\gamma_1 - id) & \varphi_M - id & 0 & 0 & 0 \\ -(\gamma_2 - id) & 0 & \varphi_M - id & 0 & 0 \\ -(\gamma_3 - id) & 0 & 0 & \varphi_M - id & 0 \\ -(\tilde{\gamma} - id) & 0 & 0 & 0 & \varphi_M - id \\ 0 & -(\gamma_2 - id) & \gamma_1 - id & 0 & 0 \\ 0 & 0 & -(\gamma_3 - id) & \gamma_2 - id & 0 \\ 0 & -(\gamma_3 - id) & 0 & \gamma_1 - id & 0 \\ 0 & \tilde{\gamma}^{a_1} - id & 0 & 0 & -\left(\gamma_1 - \frac{\tilde{\gamma}^{a_1} - id}{\tilde{\gamma} - id}\right) \\ 0 & 0 & \tilde{\gamma}^{a_2} - id & 0 & -\left(\gamma_2 - \frac{\tilde{\gamma}^{a_2} - id}{\tilde{\gamma} - id}\right) \\ 0 & 0 & 0 & \tilde{\gamma}^{a_3} - id & -\left(\gamma_3 - \frac{\tilde{\gamma}^{a_3} - id}{\tilde{\gamma} - id}\right) \end{bmatrix},$$

$$A_{2,\varphi_q} = \left[\begin{array}{cccccccccc}
\gamma_2 - id & -(\gamma_1 - id) & 0 & 0 & \varphi_M - id & 0 & 0 & 0 & 0 & 0 \\
0 & \gamma_3 - id & -(\gamma_2 - id) & 0 & 0 & \varphi_M - id & 0 & 0 & 0 & 0 \\
\gamma_3 - id & 0 & -(\gamma_1 - id) & 0 & 0 & 0 & \varphi_M - id & 0 & 0 & 0 \\
-(\tilde{\gamma}^{a_1} - id) & 0 & 0 & \gamma_1 - \frac{\tilde{\gamma}^{a_1} - id}{\tilde{\gamma} - id} & 0 & 0 & 0 & \varphi_M - id & 0 & 0 \\
0 & -(\tilde{\gamma}^{a_2} - id) & 0 & \gamma_2 - \frac{\tilde{\gamma}^{a_2} - id}{\tilde{\gamma} - id} & 0 & 0 & 0 & 0 & \varphi_M - id & 0 \\
0 & 0 & -(\tilde{\gamma}^{a_3} - id) & \gamma_3 - \frac{\tilde{\gamma}^{a_3} - id}{\tilde{\gamma} - id} & 0 & 0 & 0 & 0 & 0 & \varphi_M - id \\
0 & 0 & 0 & 0 & \gamma_3 - id & \gamma_1 - id & -(\gamma_2 - id) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{\gamma}^{a_1 a_2} - id & 0 & 0 & \gamma_2 - \frac{\tilde{\gamma}^{a_1 a_2} - id}{\tilde{\gamma}^{a_1} - id} & -\left(\gamma_1 - \frac{\tilde{\gamma}^{a_1 a_2} - id}{\tilde{\gamma}^{a_2} - id}\right) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \tilde{\gamma}^{a_1 a_3} - id & \gamma_3 - \frac{\tilde{\gamma}^{a_1 a_3} - id}{\tilde{\gamma}^{a_1} - id} & 0 & -\left(\gamma_1 - \frac{\tilde{\gamma}^{a_1 a_3} - id}{\tilde{\gamma}^{a_3} - id}\right) \\
0 & 0 & 0 & 0 & 0 & \tilde{\gamma}^{a_2 a_3} - id & 0 & 0 & \gamma_3 - \frac{\tilde{\gamma}^{a_2 a_3} - id}{\tilde{\gamma}^{a_2} - id} & -\left(\gamma_2 - \frac{\tilde{\gamma}^{a_2 a_3} - id}{\tilde{\gamma}^{a_3} - id}\right)
\end{array}\right],$$

$$A_{3,\varphi_q} = \left[\begin{array}{cccccccccc}
-(\gamma_3 - id) & -(\gamma_1 - id) & \gamma_2 - id & 0 & 0 & 0 & \varphi_M - id & 0 & 0 & 0 \\
-(\tilde{\gamma}^{a_1 a_2} - id) & 0 & 0 & -\left(\gamma_2 - \frac{\tilde{\gamma}^{a_1 a_2} - id}{\tilde{\gamma}^{a_1} - id}\right) & \gamma_1 - \frac{\tilde{\gamma}^{a_1 a_2} - id}{\tilde{\gamma}^{a_2} - id} & 0 & 0 & \varphi_M - id & 0 & 0 \\
0 & 0 & -(\tilde{\gamma}^{a_1 a_3} - id) & -\left(\gamma_3 - \frac{\tilde{\gamma}^{a_1 a_3} - id}{\tilde{\gamma}^{a_1} - id}\right) & 0 & \gamma_1 - \frac{\tilde{\gamma}^{a_1 a_3} - id}{\tilde{\gamma}^{a_3} - id} & 0 & 0 & \varphi_M - id & 0 \\
0 & -(\tilde{\gamma}^{a_2 a_3} - id) & 0 & 0 & -\left(\gamma_3 - \frac{\tilde{\gamma}^{a_2 a_3} - id}{\tilde{\gamma}^{a_2} - id}\right) & \gamma_2 - \frac{\tilde{\gamma}^{a_2 a_3} - id}{\tilde{\gamma}^{a_3} - id} & 0 & 0 & 0 & \varphi_M - id \\
0 & 0 & 0 & 0 & 0 & 0 & \tilde{\gamma}^{a_1 a_2 a_3} - id & -\left(\gamma_3 - \frac{\tilde{\gamma}^{a_1 a_2 a_3} - id}{\tilde{\gamma}^{a_1 a_2} - id}\right) & \gamma_2 - \frac{\tilde{\gamma}^{a_1 a_2 a_3} - id}{\tilde{\gamma}^{a_1 a_3} - id} & -\left(\gamma_1 - \frac{\tilde{\gamma}^{a_1 a_2 a_3} - id}{\tilde{\gamma}^{a_2 a_3} - id}\right)
\end{array}\right],$$

$$A_{4,\varphi_q} = \left[-(\tilde{\gamma}^{a_1 a_2 a_3} - id) \quad \gamma_3 - \frac{\tilde{\gamma}^{a_1 a_2 a_3} - id}{\tilde{\gamma}^{a_1 a_2} - id} \quad -\left(\gamma_2 - \frac{\tilde{\gamma}^{a_1 a_2 a_3} - id}{\tilde{\gamma}^{a_1 a_3} - id}\right) \quad \gamma_1 - \frac{\tilde{\gamma}^{a_1 a_2 a_3} - id}{\tilde{\gamma}^{a_2 a_3} - id} \quad \varphi_M - id\right].$$

□

6.4 Galois cohomology via False-Tate type Herr complex

Now by taking the cohomology of the complex $\Phi\Gamma^{\bullet}_{LT,FT}(-)$, we have cohomological functors $(\mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT,FT}(-)))_{i\geq 0}$ from $\varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT,FT},\acute{e}t,tor}_{/\mathcal{O}_{\mathcal{L}}}$ to the category of abelian groups. Then we have the following theorem.

Theorem 6.4.1. *For any $V \in \mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$, we have a natural isomorphism*

$$H^i(G_K, V) \cong \mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT,FT}(\mathbb{D}_{LT,FT}(V))) \quad \textit{for } i \geq 0.$$

Proof. Since $\mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT,FT}(\mathbb{D}_{LT,FT}(-)))_{i\geq 0}$ is a cohomological δ-functor such that $\mathcal{H}^0(\Phi\Gamma^{\bullet}_{LT,FT}(\mathbb{D}_{LT,FT}(-))) \cong H^0(G_K, -)$. Now the proof follows as in the proof of Theorem 5.5.2. ∎

Theorem 6.4.2. *Let $V \in \mathbf{Rep}_{\mathcal{O}_K}(G_K)$. Then the False-Tate type Herr complex computes the Galois cohomology of G_K with coefficients in V.*

Proof. The proof is similar to that of Theorem 5.5.5. ∎

Chapter 7

The Operator ψ_q

In the cyclotomic case, Herr defined an operator ψ acting on the category of étale (φ, Γ)-modules, and then he proved that the φ-Herr complex and the ψ-Herr complex are quasi-isomorphic [17, Proposition 4.1]. Crucial in the proof of this fact is that "$\gamma - 1$ acts bijectively on $\operatorname{Ker}\psi$" [17, Theorem 3.8]. Then the Iwasawa cohomology groups are computed in terms of the ψ-Herr complex. Further, the isomorphism

$$\operatorname{Exp}^* : H^1_{Iw}(K_{cyc}/K, V) = \mathbb{D}(V)^{\psi=id}$$

is used to produce p-adic L-functions.

In this chapter, we define an integral operator ψ_q following [29], which acts linearly on the étale (φ_q, Γ_{LT})-module. Then we show that under some conditions on étale (φ_q, Γ_{LT})-module, the Lubin-Tate Herr complex for φ_q and ψ_q are quasi-isomorphic (see Theorem 7.2.6 and Remark 7.2.8). We also prove similar results for the False-Tate type Herr complex.

7.1 Definition of ψ_q and its properties

Recall that the residue field of $\mathcal{O}_{\mathcal{E}}$ is $E = k((Z))$, which is not perfect, so φ_q is not an automorphism but is injective. The field $\widehat{\mathcal{E}^{ur}}$, which is the fraction field of $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$, is an extension of degree q of $\varphi_q(\widehat{\mathcal{E}^{ur}})$. Put $\mathrm{tr} = \mathrm{trace}_{\widehat{\mathcal{E}^{ur}}/\varphi_q(\widehat{\mathcal{E}^{ur}})}$. Define

$$\psi_q : \widehat{\mathcal{E}^{ur}} \to \widehat{\mathcal{E}^{ur}}$$

such that

$$\varphi_q(\psi_q(x)) = \frac{1}{\pi}.\,\mathrm{tr}(x).$$

The existence of such a map ψ_q follows from [29, Remark 3.1]. Since every element $a \in E$ satisfies the purely inseparable polynomial $(x-a)^q = x^q - a^q \in \varphi_q(E)[x]$, the residue extension $E/\varphi_q(E)$ is totally inseparable. Similarly, the extension $E^{sep}/\varphi_q(E^{sep})$ is totally inseparable. Thus the map ψ_q maps $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$ to $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$ and $\mathcal{O}_{\mathcal{E}}$ to $\mathcal{O}_{\mathcal{E}}$ [29, Remark 3.2], and the trace map defined by

$$\mathrm{tr}(x) = \mathrm{trace}_{\widehat{\mathcal{E}^{ur}}/\varphi_q(\widehat{\mathcal{E}^{ur}})}(x) = \mathrm{trace}_{\varphi_q(\widehat{\mathcal{E}^{ur}})}(y \mapsto xy)$$

is trivial for these extensions. Hence if $x \in \mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$, then $\mathrm{tr}(x) \in \pi\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$. Moreover,

$$\mathrm{tr}(\varphi_q(x)) = \mathrm{trace}_{\widehat{\mathcal{E}^{ur}}/\varphi_q(\widehat{\mathcal{E}^{ur}})}(\varphi_q(x)) = q\varphi_q(x)$$

implies that

$$\psi_q(\varphi_q(x)) = \frac{q}{\pi}(x).$$

Hence

$$\psi_q \circ \varphi_q = \frac{q}{\pi} id.$$

We may extend this map ψ_q to $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V$ by trivial action on V. Since φ_q commutes with Γ_{LT}, $\varphi_q(\widehat{\mathcal{E}^{ur}})$ is stable under Γ_{LT}. Thus $\gamma \circ \mathrm{tr} \circ \gamma^{-1} = \mathrm{tr}$ for all $\gamma \in \Gamma_{LT}$. This ensures that ψ_q commutes with Γ_{LT} and it is also stable under the action of Γ_{LT}. Then it induces an operator

$$\psi_{\mathbb{D}_{LT}(V)} : \mathbb{D}_{LT}(V) \to \mathbb{D}_{LT}(V)$$

satisfying

$$\psi_{\mathbb{D}_{LT}(V)} \circ \varphi_{\mathbb{D}_{LT}(V)} = \frac{q}{\pi} id_{\mathbb{D}_{LT}(V)}.$$

Remark 7.1.1. Similarly, we also have $\psi_{\mathbb{D}_{LT,FT}(V)} : \mathbb{D}_{LT,FT}(V) \to \mathbb{D}_{LT,FT}(V)$ satisfying the above properties as ψ_q maps $\mathcal{O}_{\mathcal{L}}$ to $\mathcal{O}_{\mathcal{L}}$, i.e., we have

$$\psi_{\mathbb{D}_{LT,FT}(V)} \circ \varphi_{\mathbb{D}_{LT,FT}(V)} = \frac{q}{\pi}.id_{\mathbb{D}_{LT,FT}(V)}.$$

Next, we prove the following lemma, which we use to show the main result of this chapter.

Lemma 7.1.2. *Let A be an abelian group. Consider the following complexes $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$*

$$\mathscr{C}_i : 0 \to A \xrightarrow{d_i} A \to 0 \quad \textit{for } i = 1, 2$$

$$\mathscr{C}_3 : 0 \to A \xrightarrow{d_3} \bigoplus_{i_1 \in \mathfrak{y}} A \xrightarrow{d_3} \cdots \to \bigoplus_{\{i_1,\ldots,i_r\} \in \binom{\mathfrak{y}}{r}} A \xrightarrow{d_3} \cdots \xrightarrow{d_3} A \to 0,$$

where $\mathfrak{y}$ is a finite set and $\binom{\mathfrak{y}}{r}$ denotes choosing r-indices at a time from the set $\mathfrak{y}$. Let $\mathrm{Tot}(\mathscr{C}_i\mathscr{C}_j)$ be the total complex of the double complex $\mathscr{C}_i\mathscr{C}_j$. Then a morphism from the complex $\mathscr{C}_1$ to $\mathscr{C}_2$, which commutes with d_3, induces a natural homomorphism between the cohomology groups

$$\mathcal{H}^i(\mathrm{Tot}(\mathscr{C}_1\mathscr{C}_3)) \to \mathcal{H}^i(\mathrm{Tot}(\mathscr{C}_2\mathscr{C}_3)).$$

Proof. Given a morphism from $\mathscr{C}_1$ to $\mathscr{C}_2$, we have the following commutative diagram

$$\begin{array}{ccccccc}
\mathscr{C}_1 : 0 & \longrightarrow & A & \xrightarrow{d_1} & A & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2} & & \\
\mathscr{C}_2 : 0 & \longrightarrow & A & \xrightarrow[d_2]{} & A & \longrightarrow & 0.
\end{array}$$

This induces a morphism between the total complex $\mathrm{Tot}(\mathscr{C}_1\mathscr{C}_3)$ and $\mathrm{Tot}(\mathscr{C}_2\mathscr{C}_3)$ given by the following

$$\begin{array}{ccccccccccc}
\mathrm{Tot}(\mathscr{C}_1\mathscr{C}_3): 0 & \longrightarrow & A & \xrightarrow{(d_1,d_3)} & A\oplus\bigoplus_{i_1\in\mathfrak{y}} A & \longrightarrow \cdots \longrightarrow & \bigoplus_{\{i_1,\dots,i_{d-1}\}\in\binom{\mathfrak{y}}{d-1}} A\oplus A & \xrightarrow{d_3-d_1} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2,\bigoplus_{i_1\in\mathfrak{y}}\delta_1} & & \downarrow{\scriptstyle \bigoplus_{\{i_1,\dots,i_{d-1}\}\in\binom{\mathfrak{y}}{d-1}}\delta_2,\delta_1} & & \downarrow{\scriptstyle \delta_2} & & \\
\mathrm{Tot}(\mathscr{C}_2\mathscr{C}_3): 0 & \longrightarrow & A & \xrightarrow[(d_2,d_3)]{} & A\oplus\bigoplus_{i_1\in\mathfrak{y}} A & \longrightarrow \cdots \longrightarrow & \bigoplus_{\{i_1,\dots,i_{d-1}\}\in\binom{\mathfrak{y}}{d-1}} A\oplus A & \xrightarrow[d_3-d_2]{} & A & \longrightarrow & 0.
\end{array}$$

Since the morphism from $\mathscr{C}_1$ to $\mathscr{C}_2$ commutes with d_3, it is easy to see that each square is commutative in the above diagram, and this induces a homomorphism

$$\mathcal{H}^i(\mathrm{Tot}(\mathscr{C}_1\mathscr{C}_3)) \to \mathcal{H}^i(\mathrm{Tot}(\mathscr{C}_2\mathscr{C}_3)). \qquad \blacksquare$$

7.2 The complex $\Psi^\bullet$

Recall that $D^{sep} = \mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V \cong \mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_\mathcal{E}} \mathbb{D}_{LT}(V)$. Define a complex $\Psi^\bullet(D^{sep})$ as the following

$$\Psi^\bullet(D^{sep}) : 0 \to D^{sep} \xrightarrow{\psi_q\otimes\psi_{\mathbb{D}_{LT}(V)}-\frac{q}{\pi}id} D^{sep} \to 0.$$

7.2.1 The case of Lubin-Tate extensions

Definition 7.2.1. For any $M \in \varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}_{/\mathcal{O}_\mathcal{E}}$, the co-chain complex $\Psi\Gamma^\bullet_{LT}(M)$ is defined as the total complex of the double complex $\Gamma^\bullet_{LT}(\Psi^\bullet(M^\Delta))$. We call this complex as *the Lubin-Tate Herr complex corresponding to ψ_q*.

Note that in the following examples $M = M^\Delta$. We write M only for the simplicity.

Example 7.2.2. Let $d = 2$. Then the complex $\Psi\Gamma^\bullet_{LT}(M)$ is defined as follows

$$0 \to M \xrightarrow{x\mapsto A_{0,\psi_q}x} M^{\oplus 3} \xrightarrow{x\mapsto A_{1,\psi_q}x} M^{\oplus 3} \xrightarrow{x\mapsto A_{2,\psi_q}x} M \to 0,$$

where

$$A_{0,\psi_q} = \begin{bmatrix} \psi_M - \frac{q}{\pi}id \\ \gamma_1 - id \\ \gamma_2 - id \end{bmatrix}, A_{1,\psi_q} = \begin{bmatrix} -(\gamma_1 - id) & \psi_M - \frac{q}{\pi}id & 0 \\ -(\gamma_2 - id) & 0 & \psi_M - \frac{q}{\pi}id \\ 0 & -(\gamma_2 - id) & \gamma_1 - id \end{bmatrix},$$

$$A_{2,\psi_q} = \begin{bmatrix} \gamma_2 - id & -(\gamma_1 - id) & \psi_M - \frac{q}{\pi} id \end{bmatrix}. \qquad \square$$

Example 7.2.3. For the case of $d = 3$, the complex $\Psi\Gamma^\bullet_{LT}(M)$ looks like as the following

$$0 \to M \xrightarrow{x \mapsto A_{0,\psi_q} x} M^{\oplus 4} \xrightarrow{x \mapsto A_{1,\psi_q} x} M^{\oplus 6} \xrightarrow{x \mapsto A_{2,\psi_q} x} M^{\oplus 4} \xrightarrow{x \mapsto A_{3,\psi_q} x} M \to 0,$$

where

$$A_{0,\psi_q} \begin{bmatrix} \psi_M - \frac{q}{\pi} id \\ \gamma_1 - id \\ \gamma_2 - id \\ \gamma_3 - id \end{bmatrix}, A_{1,\psi_q} = \begin{bmatrix} -(\gamma_1 - id) & \psi_M - \frac{q}{\pi} id & 0 & 0 \\ -(\gamma_2 - id) & 0 & \psi_M - \frac{q}{\pi} id & 0 \\ -(\gamma_3 - id) & 0 & 0 & \psi_M - \frac{q}{\pi} id \\ 0 & -(\gamma_2 - id) & \gamma_1 - id & 0 \\ 0 & 0 & -(\gamma_3 - id) & \gamma_2 - id \\ 0 & -(\gamma_3 - id) & 0 & \gamma_1 - id \end{bmatrix},$$

$$A_{2,\psi_q} = \begin{bmatrix} \gamma_2 - id & -(\gamma_1 - id) & 0 & \psi_M - \frac{q}{\pi} id & 0 & 0 \\ 0 & \gamma_3 - id & -(\gamma_2 - id) & 0 & \psi_M - \frac{q}{\pi} id & 0 \\ \gamma_3 - id & 0 & -(\gamma_1 - id) & 0 & 0 & \psi_M - \frac{q}{\pi} id \\ 0 & 0 & 0 & \gamma_3 - id & \gamma_1 - id & -(\gamma_2 - id) \end{bmatrix},$$

$$A_{3,\psi_q} = \begin{bmatrix} -(\gamma_3 - id) & -(\gamma_1 - id) & \gamma_2 - id & \psi_M - \frac{q}{\pi} id \end{bmatrix}. \qquad \square$$

Next, the following proposition is an easy consequence of Lemma 7.1.2.

Proposition 7.2.4. *Let $M \in \varinjlim \mathbf{Mod}^{\varphi_q, \Gamma_{LT}, \acute{e}t, tor}_{/\mathcal{O}_\mathcal{E}}$. Then the morphism $\Phi^\bullet(M) \to \Psi^\bullet(M)$, which is given by the following*

$$\begin{array}{ccccccc} \Phi^\bullet(M) : 0 & \longrightarrow & M & \xrightarrow{\varphi_M - id} & M & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle -\psi_M} & & \\ \Psi^\bullet(M) : 0 & \longrightarrow & M & \xrightarrow[\psi_M - \frac{q}{\pi} id]{} & M & \longrightarrow & 0, \end{array}$$

induces a morphism

$$\Phi\Gamma^\bullet_{LT}(M) \to \Psi\Gamma^\bullet_{LT}(M).$$

Proof. Since ψ_M commutes with the action of Γ_{LT}. The proof follows from Lemma 7.1.2, by taking $\mathscr{C}_1 = \Phi^\bullet(M^\Delta), \mathscr{C}_2 = \Psi^\bullet(M^\Delta)$ and $\mathscr{C}_3 = \Gamma^\bullet_{LT}(M^\Delta)$. ■

Example 7.2.5. Let $d = 2$. Then the morphism between $\Phi\Gamma^{\bullet}_{LT}(M)$ and $\Psi\Gamma^{\bullet}_{LT}(M)$ is given by the following:

$$\begin{array}{ccccccccccc} \Phi\Gamma^{\bullet}_{LT}(M): 0 & \longrightarrow & M & \xrightarrow{A_{0,\varphi_q}} & M\oplus M\oplus M & \xrightarrow{A_{1,\varphi_q}} & M\oplus M\oplus M & \xrightarrow{A_{2,\varphi_q}} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \mathscr{F}} & & \downarrow{\scriptstyle \mathscr{F}'} & & \downarrow{\scriptstyle -\psi_M} & & \\ \Psi\Gamma^{\bullet}_{LT}(M): 0 & \longrightarrow & M & \xrightarrow[A_{0,\psi_q}]{} & M\oplus M\oplus M & \xrightarrow[A_{1,\psi_q}]{} & M\oplus M\oplus M & \xrightarrow[A_{2,\psi_q}]{} & M & \longrightarrow & 0, \end{array}$$

where

$$\begin{aligned} \mathscr{F}(x_1, x_2, x_3) =&(-\psi_M(x_1), x_2, x_3), \\ \mathscr{F}'(x_1, x_2, x_3) =&(-\psi_M(x_1), -\psi_M(x_2), x_3), \end{aligned}$$

and the maps A_{i,φ_q} and A_{i,ψ_q} are the same as defined in Example 5.4.2 and Example 7.2.2. Note that in this example, we write M only for simplicity. Indeed, $M = M^{\Delta}$. $\square$

Theorem 7.2.6. *Let $M \in \varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}_{/\mathcal{O}_{\mathcal{E}}}$. Then we have a well-defined homomorphism*

$$\mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT}(M)) \to \mathcal{H}^i(\Psi\Gamma^{\bullet}_{LT}(M)) \quad \textit{for } i \geq 0.$$

Further, the homomorphism $\mathcal{H}^0(\Phi\Gamma^{\bullet}_{LT}(M)) \to \mathcal{H}^0(\Psi\Gamma^{\bullet}_{LT}(M))$ is injective.

Proof. Since $(-\psi_M)(\varphi_M - id) = (\psi_M - \frac{q}{\pi}id)$, and ψ_M commutes with Γ_{LT}. Then we have a morphism $\Phi\Gamma^{\bullet}_{LT}(M) \to \Psi\Gamma^{\bullet}_{LT}(M)$ of co-chain complexes, which induces a well-defined homomorphism

$$\mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT}(M)) \to \mathcal{H}^i(\Psi\Gamma^{\bullet}_{LT}(M)) \quad \text{for } i \geq 0.$$

For the second part, let $\mathcal{K}$ be the kernel and $\mathcal{C}$ be the co-kernel of the morphism $\Phi\Gamma^{\bullet}_{LT}(M) \to \Psi\Gamma^{\bullet}_{LT}(M)$. Then the complex $\mathcal{K}$ and the complex $\mathcal{C}$ are given as the following:

$$\mathcal{K} : 0 \to 0 \to \operatorname{Ker} \psi_M \oplus \bigoplus_{i_1 \in \mathfrak{X}} 0 \to \cdots \to \bigoplus_{\{i_1,\ldots,i_{d-1}\} \in \binom{\mathfrak{X}}{d-1}} \operatorname{Ker} \psi_M \oplus 0 \to \operatorname{Ker} \psi_M \to 0,$$

$$\mathcal{C} : 0 \to 0 \to \operatorname{coker} \psi_M \oplus \bigoplus_{i_1 \in \mathfrak{X}} 0 \to \cdots \to \bigoplus_{\{i_1,\ldots,i_{d-1}\} \in \binom{\mathfrak{X}}{d-1}} \operatorname{coker} \psi_M \oplus 0 \to \operatorname{coker} \psi_M \to 0.$$

The morphisms of the complex $\mathcal{K}$ are the restrictions of that of the complex $\Phi\Gamma^\bullet_{LT}(M)$, and the morphisms of the complex $\mathcal{C}$ are induced from the complex $\Psi\Gamma^\bullet_{LT}(M)$. Then we have the exact sequence

$$0 \to \mathcal{K} \to \Phi\Gamma^\bullet_{LT}(M) \to \Psi\Gamma^\bullet_{LT}(M) \to \mathcal{C} \to 0,$$

which gives us the following short exact sequences

$$0 \to \mathcal{K} \to \Phi\Gamma^\bullet_{LT}(M) \to \mathbb{I} \to 0, \tag{7.1}$$

$$0 \to \mathbb{I} \to \Psi\Gamma^\bullet_{LT}(M) \to \mathcal{C} \to 0, \tag{7.2}$$

where $\mathbb{I}$ is the image of $\Phi\Gamma^\bullet_{LT}(M) \to \Psi\Gamma^\bullet_{LT}(M)$. Since $\mathcal{H}^0(\mathcal{C}) = 0$, by taking the long exact cohomology sequence of (7.2), we have $\mathcal{H}^0(\mathbb{I}) \cong \mathcal{H}^0(\Psi\Gamma^\bullet_{LT}(M))$. Also, we have a long exact sequence

$$\begin{aligned} 0 &\to \mathcal{H}^0(\mathcal{K}) \to \mathcal{H}^0(\Phi\Gamma^\bullet_{LT}(M)) \to \mathcal{H}^0(\mathbb{I}) \\ &\to \mathcal{H}^1(\mathcal{K}) \to \mathcal{H}^1(\Phi\Gamma^\bullet_{LT}(M)) \to \mathcal{H}^1(\mathbb{I}) \\ &\to \mathcal{H}^2(\mathcal{K}) \to \mathcal{H}^2(\Phi\Gamma^\bullet_{LT}(M)) \to \mathcal{H}^2(\mathbb{I}) \to \mathcal{H}^3(\mathcal{K}) \to 0 \to \cdots \end{aligned}$$

Since $\mathcal{H}^0(\mathbb{I}) \cong \mathcal{H}^0(\Psi\Gamma^\bullet_{LT}(M))$ and $\mathcal{H}^0(\mathcal{K}) = 0$, the homomorphism

$$\mathcal{H}^0(\Phi\Gamma^\bullet_{LT}(M)) \to \mathcal{H}^0(\Psi\Gamma^\bullet_{LT}(M))$$

is injective. ■

Remark 7.2.7. Let the action of $\tau_1 := \gamma_1 - id$ be bijective on $\operatorname{Ker}\psi_M$, then the homomorphism $\mathcal{H}^0(\Phi\Gamma^\bullet_{LT}(M)) \to \mathcal{H}^0(\Psi\Gamma^\bullet_{LT}(M))$ is an isomorphism. Moreover, the homomorphism $\mathcal{H}^1(\Phi\Gamma^\bullet_{LT}(M)) \to \mathcal{H}^1(\Psi\Gamma^\bullet_{LT}(M))$ is an injection.

Proof. Consider the co-chain complexes

$$\mathscr{C}_{\gamma_1}(M) : 0 \to M \xrightarrow{-(\gamma_1 - id)} M \to 0,$$

$$\Gamma^\bullet_{LT\setminus\gamma_1}(M) : 0 \to M \to \bigoplus_{i_1 \in \mathfrak{X}'} M \to \cdots \to \bigoplus_{\{i_1,\dots,i_r\}\in\binom{\mathfrak{X}'}{r}} M \to \cdots \to M \to 0,$$

where $\mathfrak{X}' = \{\gamma_2, \dots, \gamma_d\}$, and $\binom{\mathfrak{X}'}{r}$ denotes choosing r-indices at a time from the set $\mathfrak{X}'$. For all $0 \le r \le |\mathfrak{X}'| - 1$, the map $d^{j_1,\dots,j_{r+1}}_{i_1,\dots,i_r} : M \to M$ from the component in the r-th term corresponding to $\{i_1, \dots, i_r\}$ to the component corresponding to the $(r+1)$-tuple $\{j_1, \dots, j_{r+1}\}$ is given by the following

$$d^{j_1,\dots,j_{r+1}}_{i_1,\dots,i_r} = \begin{cases} 0 & \text{if } \{i_1,\dots,i_r\} \not\subseteq \{j_1,\dots,j_{r+1}\}, \\ (-1)^{s_j+1}(\gamma_j - id) & \text{if } \{j_1,\dots,j_{r+1}\} = \{i_1,\dots,i_r\} \cup \{j\}, \end{cases}$$

and s_j is the number of elements in the set $\{i_1, \dots, i_r\}$ smaller than j.

Then the complex $\mathcal{K}$ can be written as the total complex of the following double complex

$$\begin{array}{ccccccccccc}
& & 0 & & 0 & & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \operatorname{Ker}\psi_M & \longrightarrow & \bigoplus_{i_1\in\mathfrak{X}'} \operatorname{Ker}\psi_M & \longrightarrow \cdots \longrightarrow & \bigoplus_{\{i_1,\dots,i_r\}\in\binom{\mathfrak{X}'}{r}} \operatorname{Ker}\psi_M & \longrightarrow \cdots \longrightarrow & \operatorname{Ker}\psi_M & \longrightarrow & 0 \\
& & \uparrow{\scriptstyle -(\gamma_1-id)} & & \uparrow{\scriptstyle -(\gamma_1-id)} & & \uparrow{\scriptstyle -(\gamma_1-id)} & & \uparrow{\scriptstyle -(\gamma_1-id)} & & \\
0 & \longrightarrow & \operatorname{Ker}\psi_M & \longrightarrow & \bigoplus_{i_1\in\mathfrak{X}'} \operatorname{Ker}\psi_M & \longrightarrow \cdots \longrightarrow & \bigoplus_{\{i_1,\dots,i_r\}\in\binom{\mathfrak{X}'}{r}} \operatorname{Ker}\psi_M & \longrightarrow \cdots \longrightarrow & \operatorname{Ker}\psi_M & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
& & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

In other words, $\mathcal{K}$ is the total complex of $\Gamma^\bullet_{LT\setminus\gamma_1}(\mathscr{C}_{\gamma_1}(\operatorname{Ker}\psi_M))$, which is bounded double complex with exact columns as $0 \to \operatorname{Ker}\psi_M \xrightarrow{-(\gamma_1 - id)} \operatorname{Ker}\psi_M \to 0$ is exact. Therefore $\mathcal{K}$ is acyclic [35, Ex. 1.2.5]. Then by taking the long exact cohomology sequence of (7.1), we have $\mathcal{H}^i(\Phi\Gamma^\bullet_{LT}(M)) \cong \mathcal{H}^i(\mathbb{I})$. Moreover, $\mathcal{H}^0(\mathcal{C}) = 0$. Now the result follows from the long exact cohomology sequence for the short exact sequence (7.2). ■

Remark 7.2.8. Let M be a π-divisible module in $\varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}_{/\mathcal{O}_\mathcal{E}}$ such that the action of $\tau_1 := \gamma_1 - id$ is bijective on $\operatorname{Ker}\psi_M$. Then we have an isomorphism

$$\mathcal{H}^i(\Phi\Gamma^\bullet_{LT}(M)) \xrightarrow{\sim} \mathcal{H}^i(\Psi\Gamma^\bullet_{LT}(M)) \quad \text{for } i \geq 0.$$

Proof. Since M is π-divisible and $\frac{q}{\pi} = \pi^{r-1} \mod \mathcal{O}_K^\times$, the map $\frac{q}{\pi} : M \to M$ is surjective. Also, $\psi_M \circ \varphi_M = \frac{q}{\pi} id_M$. Then $\psi_M : M \to M$ is surjective, and the co-kernel complex $\mathcal{C}$ consists of zeros, i.e., $\mathcal{C}$ is a zero complex. Since the action of $\tau_1 := \gamma_1 - id$ is bijective on $\operatorname{Ker}\psi_M$, it follows from Remark 7.2.7 that the complex $\mathcal{K}$ is acyclic. Now by taking the cohomology of the following short exact sequence

$$0 \to \mathcal{K} \to \Phi\Gamma^\bullet_{LT}(M) \to \Psi\Gamma^\bullet_{LT}(M) \to 0,$$

we get the desired result. ■

7.2.2 The case of False-Tate type extensions

Definition 7.2.9. Let $M \in \varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT,FT},\acute{e}t,tor}_{/\mathcal{O}_\mathcal{L}}$. Then *the False-Tate type Herr complex* $\Psi\Gamma^\bullet_{LT,FT}(M)$ *corresponding to* ψ_q is defined as the total complex of the double complex $\Gamma^\bullet_{LT,FT}(\Psi^\bullet(M^\Delta))$.

Example 7.2.10. Let $d = 2$. Then the complex $\Psi\Gamma^\bullet_{LT,FT}(M)$ is defined as follows:

$$0 \to M \xrightarrow{x \mapsto A_{0,\psi_q}x} M^{\oplus 4} \xrightarrow{x \mapsto A_{1,\psi_q}x} M^{\oplus 6} \xrightarrow{x \mapsto A_{2,\psi_q}x} M^{\oplus 4} \xrightarrow{x \mapsto A_{3,\psi_q}x} M \to 0,$$

where $M = M^{\Delta}$, and

$$A_{0,\psi_q} = \begin{bmatrix} \psi_M - \frac{q}{\pi}id \\ \gamma_1 - id \\ \gamma_2 - id \\ \tilde{\gamma} - id \end{bmatrix}, A_{1,\psi_q} = \begin{bmatrix} -(\gamma_1 - id) & \psi_M - \frac{q}{\pi}id & 0 & 0 \\ -(\gamma_2 - id) & 0 & \psi_M - \frac{q}{\pi}id & 0 \\ -(\tilde{\gamma} - id) & 0 & 0 & \psi_M - \frac{q}{\pi}id \\ 0 & -(\gamma_2 - id) & \gamma_1 - id & 0 \\ 0 & \tilde{\gamma}^{a_1} - id & 0 & -\left(\gamma_1 - \frac{\tilde{\gamma}^{a_1} - id}{\tilde{\gamma} - id}\right) \\ 0 & 0 & \tilde{\gamma}^{a_2} - id & -\left(\gamma_2 - \frac{\tilde{\gamma}^{a_2} - id}{\tilde{\gamma} - id}\right) \end{bmatrix},$$

$$A_{2,\psi_q} = \begin{bmatrix} \gamma_2 - id & -(\gamma_1 - id) & 0 & \psi_M - \frac{q}{\pi}id & 0 & 0 \\ -(\tilde{\gamma}^{a_1} - id) & 0 & \gamma_1 - \frac{\tilde{\gamma}^{a_1} - id}{\tilde{\gamma} - id} & 0 & \psi_M - \frac{q}{\pi}id & 0 \\ 0 & -(\tilde{\gamma}^{a_2} - id) & \gamma_2 - \frac{\tilde{\gamma}^{a_2} - id}{\tilde{\gamma} - id} & 0 & 0 & \psi_M - \frac{q}{\pi}id \\ 0 & 0 & 0 & \tilde{\gamma}^{a_1 a_2} - id & \gamma_2 - \frac{\tilde{\gamma}^{a_1 a_2} - id}{\tilde{\gamma}^{a_1} - id} & -\left(\gamma_1 - \frac{\tilde{\gamma}^{a_1 a_2} - id}{\tilde{\gamma}^{a_2} - id}\right) \end{bmatrix},$$

$$A_{3,\psi_q} = \begin{bmatrix} -(\tilde{\gamma}^{a_1 a_2} - id) & -\left(\gamma_2 - \frac{\tilde{\gamma}^{a_1 a_2} - id}{\tilde{\gamma}^{a_1} - id}\right) & \gamma_1 - \frac{\tilde{\gamma}^{a_1 a_2} - id}{\tilde{\gamma}^{a_2} - id} & \psi_M - \frac{q}{\pi}id \end{bmatrix}. \qquad \square$$

Let $\mathscr{C}_1 = \Phi^{\bullet}(M^{\Delta}), \mathscr{C}_2 = \Psi^{\bullet}(M^{\Delta})$ and $\mathscr{C}_3 = \Gamma^{\bullet}_{LT,FT}(M^{\Delta})$. Then by using Lemma 7.1.2, we have a morphism

$$\Phi\Gamma^{\bullet}_{LT,FT}(M) \to \Psi\Gamma^{\bullet}_{LT,FT}(M).$$

Next, we prove a result in the case of False-Tate type extensions, which is analogous to Theorem 7.2.6. Recall that $\Gamma^{*}_{LT,FT}$ is topologically generated by $\tilde{\mathfrak{X}} = \{\gamma_1, \ldots, \gamma_d, \tilde{\gamma}\}$, and $a_i = \chi_{LT}(\gamma_i)$.

Theorem 7.2.11. *Let $M \in \varinjlim \mathbf{Mod}^{\varphi_q, \Gamma_{LT,FT}, \acute{e}t, tor}_{/\mathcal{O}_{\mathcal{L}}}$. Then the morphism*

$$\Phi\Gamma^{\bullet}_{LT,FT}(M) \to \Psi\Gamma^{\bullet}_{LT,FT}(M)$$

induces a well-defined homomorphism $\mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT,FT}(M)) \to \mathcal{H}^i(\Psi\Gamma^{\bullet}_{LT,FT}(M))$ for $i \geq 0$. Moreover, we have $\mathcal{H}^0(\Phi\Gamma^{\bullet}_{LT,FT}(M)) \hookrightarrow \mathcal{H}^0(\Psi\Gamma^{\bullet}_{LT,FT}(M))$.

Proof. Since $(-\psi_M)(\varphi_M - id) = (\psi_M - \frac{q}{\pi}id)$, and ψ_M commutes with the action of $\Gamma_{LT,FT}$. Then we have a morphism $\Phi\Gamma^{\bullet}_{LT,FT}(M) \to \Psi\Gamma^{\bullet}_{LT,FT}(M)$ of co-chain complexes, which induces a well-defined homomorphism

$$\mathcal{H}^i(\Phi\Gamma^\bullet_{LT,FT}(M)) \to \mathcal{H}^i(\Psi\Gamma^\bullet_{LT,FT}(M)) \quad \text{for } i \geq 0.$$

Let $\mathcal{K}$ be the kernel and $\mathcal{C}$ the co-kernel of the morphism $\Phi\Gamma^\bullet_{LT,FT}(M) \to \Psi\Gamma^\bullet_{LT,FT}(M)$. Then we have an exact sequence

$$0 \to \mathcal{K} \to \Phi\Gamma^\bullet_{LT,FT}(M) \to \Psi\Gamma^\bullet_{LT,FT}(M) \to \mathcal{C} \to 0.$$

Now the result follows by using the same method as in Theorem 7.2.6. ■

Remark 7.2.12. Let $\tau_1 := \gamma_1 - id$ act bijectively on $\operatorname{Ker}\psi_M$, i.e., the complex $0 \to \operatorname{Ker}\psi_M \xrightarrow{\gamma_1 - id} \operatorname{Ker}\psi_M \to 0$ is exact. Then the homomorphism

$$\mathcal{H}^0(\Phi\Gamma^\bullet_{LT,FT}(M)) \to \mathcal{H}^0(\Psi\Gamma^\bullet_{LT,FT}(M))$$

is an isomorphism. Moreover, the map $\mathcal{H}^1(\Phi\Gamma^\bullet_{LT,FT}(M)) \to \mathcal{H}^1(\Psi\Gamma^\bullet_{LT,FT}(M))$ is injective.

Proof. Let $\tilde{\mathfrak{X}}' = \{\gamma_2, \ldots, \gamma_d, \tilde{\gamma}\}$. Then consider the complex

$$\mathscr{C}(M) : 0 \to M \to \bigoplus_{i_1 \in \tilde{\mathfrak{X}}'} M \to \cdots \to \bigoplus_{\{i_1, \ldots, i_r\} \in \binom{\tilde{\mathfrak{X}}'}{r}} M \to \cdots \to M \to 0,$$

where $\binom{\tilde{\mathfrak{X}}'}{r}$ denotes choosing r-indices at a time from the set $\tilde{\mathfrak{X}}'$, and for all $0 \leq r \leq |\tilde{\mathfrak{X}}'| - 1$, the map $d^{j_1, \ldots, j_{r+1}}_{i_1, \ldots, i_r} : M \to M$ from the component in the r-th term corresponding to $\{i_1, \ldots, i_r\}$ to the component corresponding to the $(r+1)$-tuple

$\{j_1, \ldots, j_{r+1}\}$ is given by

$$d_{i_1,\ldots,i_r}^{j_1,\ldots,j_{r+1}} = \begin{cases} 0 & \text{if } \{i_1,\ldots,i_r\} \not\subseteq \{j_1,\ldots,j_{r+1}\}, \\ (-1)^{s_j+1}(\gamma_j - id) & \text{if } \{j_1,\ldots,j_{r+1}\} = \{i_1,\ldots,i_r\} \cup \{j\} \\ & \text{and} \{i_1,\ldots,i_r\} \text{ doesn't contain } \tilde{\gamma}, \\ (-1)^{s_j}\left(\gamma_j - \frac{\tilde{\gamma}^{\chi_{LT}(j)\chi_{LT}(i_1)\cdots\chi_{LT}(i_r)} - id}{\tilde{\gamma}^{\chi_{LT}(i_1)\cdots\chi_{LT}(i_r)} - id}\right) & \text{if } \{j_1,\ldots,j_{r+1}\} = \{i_1,\ldots,i_r\} \cup \{j\} \\ & \text{and} \{i_1,\ldots,i_r\} \text{ contains } \tilde{\gamma}, \\ -\left(\tilde{\gamma}^{\chi_{LT}(i_1)\cdots\chi_{LT}(i_r)} - id\right) & \text{if } \{j_1,\ldots,j_{r+1}\} = \{i_1,\ldots,i_r\} \cup \{\tilde{\gamma}\}, \end{cases}$$

and s_j is the number of elements in the set $\{i_1, \ldots, i_r\}$ smaller than j. Let $\mathscr{C}_{a_1}(M)$ denote the complex $\mathscr{C}(M)$ with $\tilde{\gamma}$ replaced by $\tilde{\gamma}^{a_1}$. Then the kernel complex $\mathcal{K}$ can be written as the total complex of the following bounded double complex

$$\begin{array}{ccccccccccccc} & & 0 & & 0 & & & & 0 & & & & 0 \\ & & \uparrow & & \uparrow & & & & \uparrow & & & & \uparrow \\ \mathscr{C}_{a_1}(\operatorname{Ker}\psi_M) : 0 & \longrightarrow & \operatorname{Ker}\psi_M & \longrightarrow & \bigoplus_{i_1 \in \tilde{\mathfrak{X}}'} \operatorname{Ker}\psi_M & \longrightarrow & \cdots & \longrightarrow & \bigoplus_{\{i_1,\ldots,i_r\} \in \binom{\tilde{\mathfrak{X}}'}{r}} \operatorname{Ker}\psi_M & \longrightarrow & \cdots & \longrightarrow & \operatorname{Ker}\psi_M \longrightarrow 0 \\ & & \uparrow{\scriptstyle -(\gamma_1 - id)} & & \uparrow & & & & \uparrow & & & & \uparrow \\ \mathscr{C}(\operatorname{Ker}\psi_M) : 0 & \longrightarrow & \operatorname{Ker}\psi_M & \longrightarrow & \bigoplus_{i_1 \in \tilde{\mathfrak{X}}'} \operatorname{Ker}\psi_M & \longrightarrow & \cdots & \longrightarrow & \bigoplus_{\{i_1,\ldots,i_r\} \in \binom{\tilde{\mathfrak{X}}'}{r}} \operatorname{Ker}\psi_M & \longrightarrow & \cdots & \longrightarrow & \operatorname{Ker}\psi_M \longrightarrow 0 \\ & & \uparrow & & \uparrow & & & & \uparrow & & & & \uparrow \\ & & 0 & & 0 & & & & 0 & & & & 0 \end{array}$$

where the vertical maps $d_{b_1,\ldots,b_r}^{c_1,\ldots,c_r} : \operatorname{Ker}\psi_M \to \operatorname{Ker}\psi_M$ from the component in the r-th term corresponding to $\{b_1, \ldots, b_r\}$ to the component corresponding to r-th component $\{c_1, \ldots, c_r\}$ is given by the following

$$d_{b_1,\ldots,b_r}^{c_1,\ldots,c_r} = \begin{cases} -(\gamma_1 - id) & \text{if } \{b_1,\ldots,b_r\} \text{ doesn't contain any term} \\ & \text{ of the form } (\tilde{\gamma} - id), \\ -\left(\dfrac{(\tilde{\gamma}^{a_1\chi_{LT}(b_1)\cdots\chi_{LT}(b_r)} - id)(\gamma_1 - id)}{\tilde{\gamma}^{\chi_{LT}(b_1)\cdots\chi_{LT}(b_r)} - id}\right) & \text{if } \{b_1,\ldots,b_r\} \text{ contains a term of the form} \\ & (\tilde{\gamma}^{\chi_{LT}(b_1)\cdots\chi_{LT}(b_r)} - id). \end{cases}$$

It is easy to see that each square is commutative in the above double complex. Since $\dfrac{\tilde{\gamma}^{a_1\chi_{LT}(b_1)\cdots\chi_{LT}(b_r)} - id}{\tilde{\gamma}^{\chi_{LT}(b_1)\cdots\chi_{LT}(b_r)} - id}$ is a unit in $\mathcal{O}_K[[\Gamma^*_{LT,FT}]]$, thus $\mathcal{K}$ is exact at every point. Now the result follows by using the same technique as in Remark 7.2.7. ■

Next, we give an illustration of the above remark by the following example.

Example 7.2.13. Let $d = 2$. Then $\Gamma^*_{LT,FT} = \langle \gamma_1, \gamma_2, \tilde{\gamma} \rangle$, and the morphism $\Phi\Gamma^\bullet_{LT,FT}(M) \to \Psi\Gamma^\bullet_{LT,FT}(M)$ is given by the following

$$\begin{array}{ccccccccccccc}
\Phi\Gamma^\bullet_{LT,FT}(M) : 0 & \longrightarrow & M & \xrightarrow{A_{0,\varphi_q}} & M^{\oplus 4} & \xrightarrow{A_{1,\varphi_q}} & M^{\oplus 6} & \xrightarrow{A_{2,\varphi_q}} & M^{\oplus 4} & \xrightarrow{A_{3,\varphi_q}} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \mathscr{F}} & & \downarrow{\scriptstyle \mathscr{F}'} & & \downarrow{\scriptstyle \mathscr{F}''} & & \downarrow{\scriptstyle -\psi_M} & & \\
\Psi\Gamma^\bullet_{LT,FT}(M) : 0 & \longrightarrow & M & \xrightarrow[A_{0,\psi_q}]{} & M^{\oplus 4} & \xrightarrow[A_{1,\psi_q}]{} & M^{\oplus 6} & \xrightarrow[A_{2,\psi_q}]{} & M^{\oplus 4} & \xrightarrow[A_{3,\psi_q}]{} & M & \longrightarrow & 0,
\end{array}$$

where

$$\begin{aligned}
\mathscr{F}(x_1, x_2, x_3, x_4) &= (-\psi_M(x_1), x_2, x_3, x_4), \\
\mathscr{F}'(x_1, x_2, x_3, x_4, x_5, x_6) &= (-\psi_M(x_1), -\psi_M(x_2), -\psi_M(x_3), x_4, x_5, x_6), \\
\mathscr{F}''(x_1, x_2, x_3, x_4) &= (-\psi_M(x_1), -\psi_M(x_2), -\psi_M(x_3), x_4),
\end{aligned}$$

and the maps A_{i,φ_q} and A_{i,ψ_q} are the same as defined in Example 6.3.2 and Example 7.2.10, respectively. Since ψ_q commutes with the action of $\Gamma_{LT,FT}$, it is easy to see that each square diagram is commutative. Thus we have a morphism of co-chain complexes, which induces a well-defined homomorphism

$$\mathcal{H}^i(\Phi\Gamma^\bullet_{LT,FT}(M)) \to \mathcal{H}^i(\Psi\Gamma^\bullet_{LT,FT}(M)) \quad \text{for } i \geq 0.$$

The kernel $\mathcal{K}$ and the co-kernel $\mathcal{C}$ of the morphism $\Phi\Gamma^\bullet_{LT,FT}(M) \to \Psi\Gamma^\bullet_{LT,FT}(M)$ are given by the following complexes:

$$\begin{aligned}
&\mathcal{K} : 0 \to 0 \to \operatorname{Ker}\psi_M \to \oplus^3 \operatorname{Ker}\psi_M \to \oplus^3 \operatorname{Ker}\psi_M \to \operatorname{Ker}\psi_M \to 0, \\
&\mathcal{C} : 0 \to 0 \to \operatorname{coker}\psi_M \to \oplus^3 \operatorname{coker}\psi_M \to \oplus^3 \operatorname{coker}\psi_M \to \operatorname{coker}\psi_M \to 0.
\end{aligned}$$

The complex $\mathcal{K}$ is a sub-complex of $\Phi\Gamma^\bullet_{LT,FT}(M)$ and the morphisms are induced from $\Phi\Gamma^\bullet_{LT,FT}(M)$ by restriction, and $\mathcal{C}$ is a quotient of $\Psi\Gamma^\bullet_{LT,FT}(M)$ and the morphisms are induced from $\Psi\Gamma^\bullet_{LT,FT}(M)$. Note that $\mathcal{K}$ can be written as the total complex of the following double complex:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \operatorname{Ker}\psi_M & \xrightarrow{x\mapsto\partial_0' x} & \operatorname{Ker}\psi_M \oplus \operatorname{Ker}\psi_M & \xrightarrow{x\mapsto\partial_1' x} & \operatorname{Ker}\psi_M & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle -(\gamma_1 - id)} & & \uparrow{\scriptstyle \mathscr{H}} & & \uparrow{\scriptstyle \mathscr{H}'} & & \\
0 & \longrightarrow & \operatorname{Ker}\psi_M & \xrightarrow[x\mapsto\partial_0 x]{} & \operatorname{Ker}\psi_M \oplus \operatorname{Ker}\psi_M & \xrightarrow[x\mapsto\partial_1 x]{} & \operatorname{Ker}\psi_M & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where

$$\partial_0' = \begin{bmatrix} -(\gamma_2 - id) \\ -(\tilde{\gamma}^{a_1} - id) \end{bmatrix}, \partial_1' = \begin{bmatrix} -(\tilde{\gamma}^{a_1 a_2} - id) & \gamma_2 - \dfrac{\tilde{\gamma}^{a_1 a_2} - id}{\tilde{\gamma}^{a_1} - id} \end{bmatrix},$$

$$\partial_0 = \begin{bmatrix} -(\gamma_2 - id) \\ -(\tilde{\gamma} - id) \end{bmatrix}, \partial_1 = \begin{bmatrix} -(\tilde{\gamma}^{a_2} - id) & \gamma_2 - \dfrac{\tilde{\gamma}^{a_2} - id}{\tilde{\gamma} - id} \end{bmatrix},$$

$$\mathscr{H}(x_1, x_2) = \left(-(\gamma_1 - id)x_1, -\left(\frac{(\tilde{\gamma}^{a_1} - id)(\gamma_1 - id)}{\tilde{\gamma} - id} \right) x_2 \right),$$
$$\mathscr{H}'(x_1) = -\left(\frac{(\tilde{\gamma}^{a_1 a_2} - id)(\gamma_1 - id)}{\tilde{\gamma}^{a_2} - id} \right) x_1.$$

Since $\dfrac{\tilde{\gamma}^{a_1} - id}{\tilde{\gamma} - id}$ and $\dfrac{\tilde{\gamma}^{a_1 a_2} - id}{\tilde{\gamma}^{a_2} - id}$ are units in $\mathcal{O}_K[[\Gamma^*_{LT,FT}]]$, the columns of the above double complex are exact. Therefore $\mathcal{K}$ is acyclic. Since we have the exact sequence

$$0 \to \mathcal{K} \to \Phi\Gamma^\bullet_{LT,FT}(M) \to \Psi\Gamma^\bullet_{LT,FT}(M) \to \mathcal{C} \to 0,$$

and $\mathcal{H}^i(\mathcal{K}) = 0$ for all $i \geq 0$. Also, $\mathcal{H}^0(\mathcal{C}) = 0$. Now it is easy to see that

(i) $\mathcal{H}^0(\Phi\Gamma^\bullet_{LT,FT}(M)) \cong \mathcal{H}^0(\Psi\Gamma^\bullet_{LT,FT}(M))$,

(ii) $\mathcal{H}^1(\Phi\Gamma^\bullet_{LT,FT}(M)) \to \mathcal{H}^1(\Psi\Gamma^\bullet_{LT,FT}(M))$ is injective. □

Remark 7.2.14. Let M be a π-divisible module in $\varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT,FT},\acute{e}t,tor}_{/\mathcal{O}_{\mathcal{L}}}$ such that $\tau_1 = \gamma_1 - id$ acts bijectively on $\operatorname{Ker}\psi_M$. Then we have

$$\mathcal{H}^i(\Phi\Gamma^\bullet_{LT,FT}(M)) \xrightarrow{\sim} \mathcal{H}^i(\Psi\Gamma^\bullet_{LT,FT}(M)) \quad \text{for } i \geq 0.$$

Proof. The proof is similar to Remark 7.2.8. ■

Chapter 8

Iwasawa Cohomology over the Lubin-Tate Extensions

In the previous chapter, we have defined an operator ψ_q acting on the étale (φ_q, Γ_{LT})-module over $\mathcal{O}_{\mathcal{E}}$. In this chapter, we define a complex, namely, $\underline{\Psi}^\bullet$ complex by using the operator ψ_q and compute the Iwasawa cohomology for the Lubin-Tate extensions in terms of this complex (Theorem 8.2.3).

8.1 The complex $\underline{\Psi}^\bullet$

For any $M \in \varinjlim \mathbf{Mod}_{/\mathcal{O}_{\mathcal{E}}}^{\varphi_q, \Gamma_{LT}, \acute{e}t, tor}$, define the complex $\underline{\Psi}^\bullet(M)$ as follows:

$$\underline{\Psi}^\bullet(M) : 0 \to M \xrightarrow{\psi_M - id} M \to 0.$$

Let $M \in \mathbf{Mod}_{/\mathcal{O}_{\mathcal{E}}}^{\varphi_q, \Gamma_{LT}, \acute{e}t, tor}$, then $\pi^n M = 0$ for some $n \geq 1$. Define

$$M^\vee := \mathrm{Hom}_{\mathcal{O}_K}(M, K/\mathcal{O}_K),$$

which can be identified with $\mathrm{Hom}_{\mathcal{O}_{\mathcal{E}}}(M, \mathcal{O}_{\mathcal{E}}/\pi^n \mathcal{O}_{\mathcal{E}}(\chi_{LT}))$. For more details see (24) in [29].

Proposition 8.1.1. *Let* $M \in \mathbf{Mod}_{/\mathcal{O}_{\mathcal{E}}}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}$. *Then the pairing*

$$\mathcal{H}^i(\Phi^\bullet(M^\vee)) \times \mathcal{H}^{1-i}(\underline{\Psi}^\bullet(M)) \to K/\mathcal{O}_K$$

is perfect.

Proof. For an étale (φ_q, Γ_{LT})-module M such that $\pi^n M = 0$ for some $n \geq 1$, we have a Γ_{LT}-invariant continuous pairing defined by the following ([29, Remark 4.7])

$$\begin{aligned} \langle \cdot, \cdot \rangle : & M \times M^\vee \to K/\mathcal{O}_K \\ & (m, F) \mapsto \pi^{-n} \operatorname{Res}(F(m) d \log_{LT}(\omega_{LT})) \bmod \mathcal{O}_K, \end{aligned}$$

where Res is the residue map and $\omega_{LT} = \{\iota(v)\}$. For the definition of $\{\iota(v)\}$, see section 4.2). Moreover, this pairing satisfy the following properties:

(i) The operator ψ_M is left adjoint to $\varphi_{M^\vee}$ under the pairing $\langle \cdot, \cdot \rangle$, i.e.,

$$\langle \psi_M(m), F \rangle = \langle m, \varphi_{M^\vee}(F) \rangle$$

for all $m \in M$ and all $F \in M^\vee$;

(ii) The operator φ_M is left adjoint to $\psi_{M^\vee}$ under the pairing $\langle \cdot, \cdot \rangle$, i.e.,

$$\langle \varphi_M(m), F \rangle = \langle m, \psi_{M^\vee}(F) \rangle$$

for all $m \in M$ and all $F \in M^\vee$.

This induces a pairing

$$\mathcal{H}^i(\Phi^\bullet(M^\vee)) \times \mathcal{H}^{1-i}(\underline{\Psi}^\bullet(M)) \to K/\mathcal{O}_K \tag{8.1}$$

of $\mathcal{O}_K$-modules. Note that the cohomology groups $\mathcal{H}^i(\Phi^\bullet(M^\vee))$ and $\mathcal{H}^i(\underline{\Psi}^\bullet(M))$

are trivial for $i \geq 2$ and for $i < 0$, it is sufficient to check only for $i = 0$ and 1. Now,

$$\begin{aligned}\mathcal{H}^0(\Phi^\bullet(M^\vee))^\vee &= ((M^\vee)^{\varphi_{M^\vee}=id})^\vee \\ &= (M^\vee)^\vee/(\varphi_{M^\vee} - id)^\vee(M^\vee)^\vee \\ &= M/(\psi_M - id)M \\ &= \mathcal{H}^1(\underline{\Psi}^\bullet(M)),\end{aligned}$$

where the first and the last equality follows from the definition of $\Phi^\bullet(M^\vee)$ and $\underline{\Psi}^\bullet(M)$, respectively. The third equality uses the property that ψ_M is left adjoint to $\varphi_{M^\vee}$ and φ_M is left adjoint to $\psi_{M^\vee}$. Similarly,

$$\begin{aligned}\mathcal{H}^1(\Phi^\bullet(M^\vee))^\vee &= ((M^\vee)/(\varphi_{M^\vee} - id)M^\vee)^\vee \\ &= ((M^\vee)^\vee)^{(\varphi_{M^\vee}=id)^\vee} \\ &= M^{\psi_M=id} \\ &= \mathcal{H}^0(\underline{\Psi}^\bullet(M)).\end{aligned}$$

Hence

$$\mathcal{H}^i(\Phi^\bullet(M^\vee))^\vee \cong \mathcal{H}^{1-i}(\underline{\Psi}^\bullet(M))$$

for all i. In other words, the pairing given by (8.1) is perfect. ■

Remark 8.1.2. Since for any $M \in \varinjlim \mathbf{Mod}_{/\mathcal{O}_\mathcal{E}}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}$, we have

$$M = \varinjlim M_n,$$

where $M_n \in \mathbf{Mod}_{/\mathcal{O}_\mathcal{E}}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}$. Also, the functors $\mathcal{H}^i(\Phi^\bullet(-))$ and $\mathcal{H}^i(\underline{\Psi}^\bullet(-))$ commute with direct limits. Then by taking direct limits in the above proposition, we have

$$\mathcal{H}^i(\Phi^\bullet(M^\vee))^\vee \cong \mathcal{H}^{1-i}(\underline{\Psi}^\bullet(M))$$

for any $M \in \varinjlim \mathbf{Mod}_{/\mathcal{O}_{\mathcal{E}}}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}$, i.e., Proposition 8.1.1 holds for any $M \in \varinjlim \mathbf{Mod}_{/\mathcal{O}_{\mathcal{E}}}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}$.

8.2 Iwasawa cohomology

Next, we describe the Iwasawa cohomology in terms of $\underline{\Psi}^{\bullet}$ complex.

Definition 8.2.1. Let $V \in \mathbf{Rep}_{\mathcal{O}_K}(G_K)$, we define

$$H^i_{Iw}(K_\infty/K, V) := \varprojlim_{L} H^i(L, V),$$

where L varies over the finite Galois extensions of K contained in K_∞, and the projective limit is taken with respect to the cohomological corestriction maps.

Note that the functor $(H^i_{Iw}(K_\infty/K, V))_{i\geq 0}$ is a cohomological δ-functor on the category of finite free $\mathcal{O}_K$-modules with a continuous and linear action of G_K. This follows from [29, Lemma 5.9].

Proposition 8.2.2. *Let $V \in \mathbf{Rep}_{\mathcal{O}_K}(G_K)$. Then $H^0_{Iw}(K_\infty/K, V) = 0$.*

Proof. Since $H^0_{Iw}(K_\infty/K, V) := \varprojlim_L H^0(L, V) = \varprojlim_L V^{G_L}$, where L varies over the finite Galois extensions of K contained in K_∞, and the transition maps are given by the norm maps. Now if V is finite, then the vanishing of $H^0_{Iw}(K_\infty/K, V)$ is obvious. Therefore, assume that V is finitely generated free $\mathcal{O}_K$-module. Then consider the exact sequence

$$0 \to V \xrightarrow{\pi} V \to V/\pi V \to 0.$$

Since $(H^i_{Iw}(K_\infty/K, V))_{i\geq 0}$ is a cohomological δ-functor, we have an exact sequence

$$0 \to H^0_{Iw}(K_\infty/K, V) \xrightarrow{\pi} H^0_{Iw}(K_\infty/K, V) \to H^0_{Iw}(K_\infty/K, V/\pi V).$$

Note that $H^0_{Iw}(K_\infty/K, V/\pi V) = 0$ as $V/\pi V$ is finite. Then

$$0 \to H^0_{Iw}(K_\infty/K, V) \xrightarrow{\pi} H^0_{Iw}(K_\infty/K, V) \to 0$$

is exact. This implies that $\pi H^0_{Iw}(K_\infty/K, V) = H^0_{Iw}(K_\infty/K, V)$. Moreover, the identity $H^0_{Iw}(K_\infty/K, V) = \varprojlim_L V^{G_L}$ implies that $H^0_{Iw}(K_\infty/K, V)$ is a pro-finite $\mathcal{O}_K$-module. Now it follows from topological Nakayama's Lemma [19, Lemma 3.2.6] that $H^0_{Iw}(K_\infty/K, V) = 0$. ■

Next, we prove the following theorem, which gives a description of Iwasawa cohomology groups in terms of $\underline{\Psi}^\bullet$ complex. Note that this theorem is already proved in [29] as Theorem 5.13 by using Local Tate duality. We express the proof of this theorem in terms of complexes.

Theorem 8.2.3. *Let $V \in \mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$. Then the complex*

$$\underline{\Psi}^\bullet(\mathbb{D}_{LT}(V(\chi_{cyc}^{-1}\chi_{LT}))) : 0 \to \mathbb{D}_{LT}(V(\chi_{cyc}^{-1}\chi_{LT})) \xrightarrow{\psi - id} \mathbb{D}_{LT}(V(\chi_{cyc}^{-1}\chi_{LT})) \to 0,$$

where $\psi = \psi_{\mathbb{D}_{LT}(V(\chi_{cyc}^{-1}\chi_{LT}))}$ and χ_{cyc} is the cyclotomic character, computes the Iwasawa cohomology groups $H^i_{Iw}(K_\infty/K, V)$ for $i \geq 1$, i.e.,

$$H^i_{Iw}(K_\infty/K, V) \cong \mathcal{H}^{i-1}(\underline{\Psi}^\bullet(\mathbb{D}_{LT}(V(\chi_{cyc}^{-1}\chi_{LT})))).$$

Proof. Since $V \in \mathbf{Rep}^{dis}_{\mathcal{O}_K-tor}(G_K)$, i.e., V is a discrete π-primary representation of G_K, we have an isomorphism

$$H^i_{Iw}(K_\infty/K, V) \cong H^{2-i}(\mathrm{Gal}(\bar{K}/K_\infty), V^\vee(\chi_{cyc}))^\vee,$$

which is induced from the Local Tate duality. For more details see [29, Remark 5.11].

Moreover,

$$\begin{aligned} H^{2-i}(\mathrm{Gal}(\bar{K}/K_\infty), V^\vee(\chi_{cyc}))^\vee &= \mathcal{H}^{2-i}(\Phi^\bullet(\mathbb{D}_{LT}(V^\vee(\chi_{cyc}))))^\vee \\ &= \mathcal{H}^{2-i}(\Phi^\bullet(\mathbb{D}_{LT}(V^\vee)(\chi_{cyc})))^\vee \\ &= \mathcal{H}^{2-i}(\Phi^\bullet(\mathbb{D}_{LT}(V)^\vee(\chi_{LT}^{-1}\chi_{cyc})))^\vee \\ &= \mathcal{H}^{2-i}(\Phi^\bullet(\mathbb{D}_{LT}(V(\chi_{cyc}^{-1}\chi_{LT}))^\vee))^\vee \\ &\cong \mathcal{H}^{i-1}(\underline{\Psi}^\bullet(\mathbb{D}_{LT}(V(\chi_{cyc}^{-1}\chi_{LT})))). \end{aligned}$$

Here the first equality follows from Proposition 5.2.5. The second and third equality uses Remark 4.6 and Remark 5.6 of [29], respectively, while the last isomorphism comes from Remark 8.1.2 . Hence

$$H^i_{Iw}(K_\infty/K, V) \cong \mathcal{H}^{i-1}(\underline{\Psi}^\bullet(\mathbb{D}_{LT}(V(\chi_{cyc}^{-1}\chi_{LT})))).$$

This proves the theorem. ∎

Corollary 8.2.4. *For any* $V \in \mathbf{Rep}_{\mathcal{O}_K}(G_K)$*, we have*

$$H^i_{Iw}(K_\infty/K, V) \cong \mathcal{H}^{i-1}(\underline{\Psi}^\bullet(\mathbb{D}_{LT}(V(\chi_{cyc}^{-1}\chi_{LT}))))$$

for all $i \geq 1$.

Proof. Since the transition maps are surjective in $(\underline{\Psi}^\bullet(\mathbb{D}_{LT}(V/\pi^n V(\chi_{cyc}^{-1}\chi_{LT}))))_{n\geq 1}$ the projective system of co-chain complexes of abelian groups, thus the first hyper-cohomology spectral sequence degenerates at E_2. Moreover,

$$\varprojlim_n{}^1 \mathcal{H}^i(\underline{\Psi}^\bullet(\mathbb{D}_{LT}(V/\pi^n V(\chi_{cyc}^{-1}\chi_{LT})))) = 0.$$

Then the second hyper-cohomology spectral sequence

$$\varprojlim_n{}^i \mathcal{H}^j(\underline{\Psi}^\bullet(\mathbb{D}_{LT}(V/\pi^n V(\chi_{cyc}^{-1}\chi_{LT})))) \Rightarrow \mathcal{H}^{i+j}(\underline{\Psi}^\bullet(\mathbb{D}_{LT}(V(\chi_{cyc}^{-1}\chi_{LT}))))$$

also degenerates at E_2. Therefore

$$\varprojlim_n \mathcal{H}^i(\underline{\Psi}^\bullet(\mathbb{D}_{LT}(V/\pi^n V(\chi_{cyc}^{-1}\chi_{LT})))) = \mathcal{H}^i(\underline{\Psi}^\bullet(\mathbb{D}_{LT}(V(\chi_{cyc}^{-1}\chi_{LT})))).$$

Moreover,

$$\begin{aligned} H^i_{Iw}(K_\infty/K, V) &= \varprojlim_L H^i(L, V) \\ &\cong \varprojlim_L \varprojlim_n H^i(L, V/\pi^n V) \\ &= \varprojlim_n \varprojlim_L H^i(L, V/\pi^n V) \\ &= \varprojlim_n H^i_{Iw}(K_\infty/K, V/\pi^n V). \end{aligned}$$

Here the first and the last equality uses the definition of the Iwasawa cohomology. The second isomorphism follows from Lemma 5.5.4.

Now the result follows from Theorem 8.2.3 by taking the inverse limits. ■

Chapter 9

An Equivalence of Categories over the Coefficient Ring

Recall that the Theorem 4.2.10 gives a classification of the category of finitely generated $\mathcal{O}_K$-modules with a continuous and linear action of G_K in terms of étale (φ_q, Γ_{LT})-modules over $\mathcal{O}_{\mathcal{E}}$. In this chapter, we extend Theorem 4.2.10 to give an understanding of the category of R-modules of finite type with a continuous and R-linear action of G_K, where R is any complete Noetherian local ring whose residue field is a finite extension of $\mathbb{F}_p$. We consider a category of étale (φ_q, Γ_{LT})-modules over the completed tensor product $\mathcal{O}_{\mathcal{E}} \hat{\otimes}_{\mathcal{O}_K} R$. Then we show that this category is equivalent to the category of R-linear representations of G_K. Our method for proving this equivalence of categories is similar to as in [9]. The core point, which we use in the main result, is Lemma 9.2.6.

We divide this chapter into two sections. In section 9.1, we collect some preliminary results on coefficient rings. In section 9.2, we extend Theorem 4.2.10 to the case of coefficient rings.

9.1 Background on Coefficient Rings

In this section, we recall some significant results on coefficient rings, which we are going to use in this chapter. Most of these results are given in [9], [11], and [22].

9.1.1 Basic definitions

Definition 9.1.1. [8, Chapter 1, Definition 2.1] A *coefficient ring* R is a complete Noetherian local ring with finite residue field k_R of characteristic p, i.e., k_R is a finite extension of $\mathbb{F}_p$.

Example 9.1.2. Let $\mathbb{Q}_p$ be the field of p-adic numbers with ring of integers $\mathbb{Z}_p$. Then the ring $\mathbb{Z}_p$ and the power series ring $\mathbb{Z}_p[[X]]$ are coefficient rings. Moreover, for any finite extension K of $\mathbb{Q}_p$, the ring $\mathcal{O}_K$ and $\mathcal{O}_K[[X_1, X_2, \dots, X_n]]$ are also coefficient rings. Here $\mathcal{O}_K$ denotes the ring of integers of K. □

The ring R has a natural pro-finite topology with a base of open ideals given by the powers of its maximal ideal $\mathfrak{m}_R$. In other words, $R = \varprojlim_n R/\mathfrak{m}_R^n R$.

A *coefficient ring homomorphism* is a continuous homomorphism of coefficient rings $R' \to R$ such that the inverse image of the maximal ideal $\mathfrak{m}_R$ is the maximal ideal $\mathfrak{m}_{R'} \subset R'$ and the induced homomorphism on residue fields is an isomorphism.

Definition 9.1.3. For a fixed prime number p, a *p-ring* is a complete discrete valuation ring whose valuation ideal is generated by a prime element.

Example 9.1.4. For a given field k of characteristic p, the ring $W(k)$ is a p-ring. Here $W(k)$ denotes the ring of Witt vectors of k. In particular, the ring $\mathbb{Z}_p$ of p-adic integers is a p-ring. Moreover, for a finite extension K of $\mathbb{Q}_p$, the ring $\mathcal{O}_K$ is also a p-ring. The valuation ideal of $\mathcal{O}_K$ is generated by a prime element π of $\mathcal{O}_K$. □

Let R and S be two rings with ideals $I \subset R$ and $J \subset S$. Assume that R and S are both T-algebras for some other ring T.

Definition 9.1.5. The *completed tensor product* $R\hat{\otimes}_T S$ is defined as the completion of $R \otimes_T S$ with respect to the $(I \otimes S + R \otimes J)$-adic topology.

9.1.2 Some preliminary results

Now we define the completed tensor product of a p-ring and a coefficient ring.

Let $\mathcal{O}$ be a p-ring and R be a coefficient ring. Let $\mathcal{O}_K$ be a finite extension of $\mathbb{Z}_p$. Assume that both $\mathcal{O}$ and R are $\mathcal{O}_K$-algebras, where the maps $\mathcal{O}_K \to \mathcal{O}$ and $\mathcal{O}_K \to R$ are local homomorphisms. Define

$$\mathcal{O}_R := \mathcal{O}\hat{\otimes}_{\mathcal{O}_K} R.$$

Then by [9, Proposition 1.2.3], $\mathcal{O}_R$ is a complete Noetherian semi-local ring. Note that the residue field of $\mathcal{O}_R$ need not be finite as there is no restriction on the residue field of $\mathcal{O}$.

Proposition 9.1.6. *Let A be a Noetherian semi-local commutative ring with unity and $\mathfrak{m}_A$ be the radical (intersection of all maximal ideals) of A. Then $A/\mathfrak{m}_A^n$ is Artinian for all $n \geq 1$.*

Proof. We prove this by induction on n. Let $n = 1$. Then by using Chinese Remainder theorem, we have

$$A/\mathfrak{m}_A \cong \bigoplus_{i=1}^{n} A/\mathfrak{m}_i, \tag{9.1}$$

where $\mathfrak{m}_i$ is a maximal ideal of A for $1 \leq i \leq n$, and the map is a natural projection map. Note that each $A/\mathfrak{m}_i$ is Artinian being a field. Consequently, the right hand side of (9.1) is Artinian as it is a finite direct sum of Artinian rings. Therefore $A/\mathfrak{m}_A$ is Artinian, and the result is true for $n = 1$. Assume that the result is true for $n - 1$. For general n, the result follows from the following short exact sequence

$$0 \to \mathfrak{m}_A^{n-1}/\mathfrak{m}_A^n \to A/\mathfrak{m}_A^n \to A/\mathfrak{m}_A^{n-1} \to 0.$$

Now $A/\mathfrak{m}_A^{n-1}$ is Artinian by induction hypothesis. Note that A is Noetherian. This implies that $\mathfrak{m}_A^{n-1}/\mathfrak{m}_A^n$ is a finitely generated module over $A/\mathfrak{m}_A$. Since every finitely generated module over an Artinian ring is Artinian, $\mathfrak{m}_A^{n-1}/\mathfrak{m}_A^n$ is Artinian. Hence $A/\mathfrak{m}_A^n$ is Artinian. ■

Since $\mathcal{O}_R$ is a complete semi-local Noetherian ring with unity. Then we deduce the following result from Proposition 9.1.6.

Corollary 9.1.7. *$\mathcal{O}_R/\mathfrak{m}_R^n\mathcal{O}_R$ is Artinian for all $n \geq 1$.*

Let R and S be two coefficient rings and $\mathcal{O}$ be a p-ring (or any local ring with residue field of characteristic p). Let $\theta : R \to S$ be a coefficient ring homomorphism. Then it induces a homomorphism $\theta : \mathcal{O} \otimes_{\mathcal{O}_K} R \to \mathcal{O} \otimes_{\mathcal{O}_K} S$.

Assume that θ is local. Then we have $(\mathcal{O} \otimes \mathfrak{m}_R + \mathfrak{m}_{\mathcal{O}} \otimes R) \subset \mathcal{O} \otimes \mathfrak{m}_S + \mathfrak{m}_{\mathcal{O}} \otimes S$, and θ is continuous with respect to the obvious topologies. Therefore it induces a semi-local homomorphism

$$\theta : \mathcal{O}_R \to \mathcal{O}_S.$$

Proposition 9.1.8. [9, Proposition 1.2.6] *Let $\theta : \mathcal{O}_1 \to \mathcal{O}_2$ be a local homomorphism of p-rings and let R be a coefficient ring. If θ is flat then it induces a faithfully flat homomorphism*

$$\theta_R : \mathcal{O}_{1,R} \to \mathcal{O}_{2,R}.$$

Proof. Note that

$$(\mathfrak{m}_R\mathcal{O}_{i,R})^n = \mathfrak{m}_R^n\mathcal{O}_{i,R}, \quad \text{for } i = 1, 2.$$

Now by using [22, Theorem 22.3], it is enough to check that $\mathcal{O}_{2,R}/\mathfrak{m}_R^n\mathcal{O}_{2,R}$ is flat over $\mathcal{O}_{1,R}/\mathfrak{m}_R^n\mathcal{O}_{1,R}$. Moreover, it follows from [15, 0.19.7.1.2] that

$$\mathcal{O}_{i,R}/\mathfrak{m}_R^n\mathcal{O}_{i,R} = \mathcal{O}_i \otimes_{\mathcal{O}_K} R/\mathfrak{m}_R^n.$$

Since θ is flat and flatness is preserved under base extension ([21, Chapter XVI, Proposition 4.2]). Thus $\mathcal{O}_{2,R}/\mathfrak{m}_R^n\mathcal{O}_{2,R}$ is flat over $\mathcal{O}_{1,R}/\mathfrak{m}_R^n\mathcal{O}_{1,R}$.

Next, in order to show faithful flatness we need to prove that every maximal ideal of $\mathcal{O}_{1,R}$ is in the image of the induced map

$$\mathrm{Spec}(\mathcal{O}_{2,R}) \to \mathrm{Spec}(\mathcal{O}_{1,R}).$$

For this, consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{1,R} & \longrightarrow & \mathcal{O}_{2,R} \\ \downarrow & & \downarrow \\ k_{\mathcal{O}_1} \otimes_k k_R & \longrightarrow & k_{\mathcal{O}_2} \otimes_k k_R, \end{array}$$

which we get by dividing the radicals of $\mathcal{O}_{1,R}$ and $\mathcal{O}_{2,R}$. Recall that the base change preserves the faithful flatness. Since the lower horizontal map is the base change of a field extension, so it is faithfully flat. Now it follows that θ_R is faithfully flat. ■

9.2 An equivalence over coefficient rings

In this section, we consider a category of étale φ_q-modules (resp., étale (φ_q, Γ_{LT})-modules) over $\mathcal{O}_{\mathcal{E}}\hat{\otimes}_{\mathcal{O}_K}R$ and prove that this category is equivalent to the category of R-linear representations of G_K in the equal characteristic case (resp., mixed characteristic case).

9.2.1 The characteristic p case

Let E be a local field of characteristic $p > 0$. Then E is a finite extension of $\mathbb{F}_p((t))$. Assume that $E \cong k((t))$ such that $\mathrm{card}(k) = q$, where $q = p^r$ for some fixed r. Recall that the Cohen ring of E is the unique (up to isomorphism) absolutely unramified discrete valuation ring of characteristic 0 with residue field E.

Let $\mathcal{O}_{\mathcal{E}}$ be the Cohen ring of E with uniformizer π. Let $\mathcal{E}$ be the field of fractions of $\mathcal{O}_{\mathcal{E}}$. Then

$$\mathcal{O}_{\mathcal{E}} = \varprojlim_{n\in\mathbb{N}} \mathcal{O}_{\mathcal{E}}/\pi^n\mathcal{O}_{\mathcal{E}},\ \mathcal{O}_{\mathcal{E}}/\pi\mathcal{O}_{\mathcal{E}} = E \text{ and } \mathcal{E} = \mathcal{O}_{\mathcal{E}}\left[\frac{1}{\pi}\right].$$

The field $\mathcal{E}$ is a complete discrete valued field of characteristic 0, whose residue field is E. Moreover, if $\mathcal{E}'$ is another field with the same property, then there is a continuous homomorphism $\iota : \mathcal{E} \to \mathcal{E}'$ of valued fields inducing identity on E, and ι is always an isomorphism. If E is perfect, then we may identify $\mathcal{O}_{\mathcal{E}}$ with the ring $W(E)$ of Witt vectors with coefficients in E, and ι is unique. So, we have a p-ring $\mathcal{O}_{\mathcal{E}}$ of characteristic zero with fraction field $\mathcal{E}$ and residue field E. We fix a choice of $\mathcal{E}$.

Let $f : E \to F$ be a homomorphism of local fields of characteristic p. Then it follows from [11, Theorem A.45] that there is a unique local homomorphism $\mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{F}}$, which induces f on the residue fields. Also, for any finite separable extension F of E, there is a unique unramified extension $\mathcal{E}_F = \mathrm{Fr}(\mathcal{O}_{\mathcal{F}})$ of $\mathcal{E}$ whose residue field is F. Moreover, if F/E is Galois, then $\mathcal{E}_F/\mathcal{E}$ is also Galois with the Galois group

$$\mathrm{Gal}(\mathcal{E}_F/\mathcal{E}) = \mathrm{Gal}(F/E).$$

Let E^{sep} be the separable closure of E. Then

$$E^{sep} = \bigcup_{F\in S} F,$$

where S runs over the finite extensions of E contained in E^{sep}. If $F, F' \in S$ and $F \subset F'$, then $\mathcal{E}_F \subset \mathcal{E}'_F$. Define

$$\mathcal{E}^{ur} := \bigcup_{F\in S} \mathcal{E}_F.$$

Clearly, $\mathcal{E}^{ur}$ is a Galois extension of $\mathcal{E}$, and there is an identification of Galois groups

$$G_E = \mathrm{Gal}(E^{sep}/E) \xrightarrow{\sim} \mathrm{Gal}(\mathcal{E}^{ur}/\mathcal{E}).$$

Let $\mathcal{O}_{\mathcal{E}^{ur}}$ be the ring of integers of $\mathcal{E}^{ur}$. Then $\mathcal{O}_{\mathcal{E}^{ur}}$ is maximal unramified integral extension of $\mathcal{O}_{\mathcal{E}}$ with field of fractions $\mathcal{E}^{ur}$, and $\mathcal{O}_{\mathcal{E}^{ur}}$ has a valuation induced from $\mathcal{O}_{\mathcal{E}}$. Moreover, the valuation ring $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$ in the completion $\widehat{\mathcal{E}^{ur}}$ of $\mathcal{E}^{ur}$ is a p-ring with residue field E^{sep}. The Galois group G_E acts continuously on $\widehat{\mathcal{E}^{ur}}$.

For the rest of the thesis, R will always denote a coefficient ring, unless stated otherwise. Also, we assume that R is always an $\mathcal{O}_K$-algebra such that the map $\mathcal{O}_K \to R$ is a local ring homomorphism. Here $\mathcal{O}_K$ is the ring of integers of a p-adic field K with residue field k such that $\mathrm{card}(k) = q$.

Now define the rings

$$\mathcal{O}_R := \mathcal{O}_{\mathcal{E}} \hat{\otimes}_{\mathcal{O}_K} R,$$
$$\widehat{\mathcal{O}_R^{ur}} := \mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \hat{\otimes}_{\mathcal{O}_K} R.$$

Then it follows from Proposition 9.1.8 that $\widehat{\mathcal{O}_R^{ur}}$ is an $\mathcal{O}_R$-algebra and is faithfully flat over $\mathcal{O}_R$. Since the Galois group G_E acts continuously on $\mathcal{O}_{\mathcal{E}^{ur}}$, it induces an action of G_E on $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$. Now by taking the trivial action of G_E on R, it induces a Galois action on $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} R$. Moreover, this action is continuous as G_E acts continuously on $\mathcal{E}^{ur}$. Thus the action of G_E on $\widehat{\mathcal{O}_R^{ur}}$ is continuous with respect to the $\mathfrak{m}_R\widehat{\mathcal{O}_R^{ur}}$-adic topology.

Remark 9.2.1. It follows from [9, Proposition 1.2.3] that $\mathcal{O}_R$ and $\widehat{\mathcal{O}_R^{ur}}$ are Noetherian semi-local rings, complete with respect to the $\mathfrak{m}_R$-adic topology, and that $\mathfrak{m}_R$ generates the radical of these rings.

Let $\varphi_q := (x \mapsto x^q)$ be the q-Frobenius on E. Choose a lift of φ_q on $\mathcal{E}$ such that it maps $\mathcal{O}_{\mathcal{E}}$ to $\mathcal{O}_{\mathcal{E}}$. Then we have a ring homomorphism $\varphi_q : \mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E}}$ such that

$$\varphi_q(x) \equiv x^q \mod \pi.$$

Now assume that φ_q is flat. Then we have a R-linear homomorphism

$$\varphi_q := \varphi_q \otimes id_R : \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_K} R \to \mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_K} R.$$

Since the ideal $\mathfrak{m}_{\mathcal{O}_{\mathcal{E}}} \otimes R + \mathcal{O}_{\mathcal{E}} \otimes \mathfrak{m}_R$ in $\mathcal{O}_{\mathcal{E}} \otimes_{\mathcal{O}_K} R$ is generated by $\mathfrak{m}_R$, it is clear that φ_q maps $\mathfrak{m}_{\mathcal{O}_{\mathcal{E}}} \otimes R + \mathcal{O}_{\mathcal{E}} \otimes \mathfrak{m}_R$ to itself. Then we have the following lemma.

Lemma 9.2.2. *The homomorphism*

$$\varphi_q : \mathcal{O}_R \to \mathcal{O}_R$$

is faithfully flat.

Proof. Since φ_q is flat, the proof follows from Proposition 9.1.8. ■

Since the q-Frobenius φ_q on $\mathcal{O}_{\mathcal{E}}$ extends uniquely by functoriality and continuity to a q-Frobenius on $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$, we also have a faithfully flat homomorphism from $\widehat{\mathcal{O}^{ur}_R}$ to $\widehat{\mathcal{O}^{ur}_R}$.

Next, we define the category of R-representations of the Galois group G_E and the category of φ_q-modules over $\mathcal{O}_R$. Then we construct a functor from the category of R-representations of G_E to the category of φ_q-modules over $\mathcal{O}_R$.

Definition 9.2.3. An *R-representation* of the Galois group G_E is a finitely generated R-module with a continuous and R-linear action of G_E.

Definition 9.2.4. A *φ_q-module over $\mathcal{O}_R$* is an $\mathcal{O}_R$-module M together with a map

$$\varphi_M : M \to M,$$

which is semi-linear with respect to the q-Frobenius φ_q, i.e.,

$$\varphi_M(x+y) = \varphi_M(x) + \varphi_M(y),$$
$$\varphi_M(\lambda x) = \varphi_q(\lambda)\varphi_M(x),$$

for all $x, y \in M$ and $\lambda \in \mathcal{O}_R$.

Remark 9.2.5. Let M be a φ_q-module over $\mathcal{O}_R$. Then a semi-linear map $\varphi_M : M \to M$ is equivalent to an $\mathcal{O}_R$-linear map

$$\Phi_M^{lin} : M_{\varphi_q} \to M,$$

where $M_{\varphi_q} = M \ {}_{\varphi_q}\!\otimes_{\mathcal{O}_R} \mathcal{O}_R$ is the base change of M by $\mathcal{O}_R$ via φ_q.

Let $\mathbf{Rep}_R(G_E)$ denote the category of R-linear representations of G_E and $\mathbf{Mod}^{\varphi_q}_{/\mathcal{O}_R}$ the category of φ_q-modules over $\mathcal{O}_R$. The morphisms in $\mathbf{Mod}^{\varphi_q}_{/\mathcal{O}_R}$ are $\mathcal{O}_R$-linear homomorphisms commuting with φ.

Let V be an R-representation of G_E. Define

$$\mathbb{D}_R(V) := (\widehat{\mathcal{O}^{ur}_R} \otimes_R V)^{G_E}.$$

Here G_E acts diagonally. Moreover, the multiplication by $\mathcal{O}_R$ on $\hat{\mathcal{O}}^{ur}_R \otimes_R V$ is G_E-equivariant, thus $\mathbb{D}_R(V)$ is an $\mathcal{O}_R$-module. We extend the definition of the q-Frobenius to $\widehat{\mathcal{O}^{ur}_R} \otimes_R V$ as follows:

$$\varphi_q(\lambda \otimes v) = \varphi_q(\lambda) \otimes v \quad \text{for } \lambda \in \widehat{\mathcal{O}^{ur}_R} \text{ and } v \in V,$$

and then φ_q commutes with the action of G_E. It induces a $\mathcal{O}_R$-module homomorphism

$$\varphi_{\mathbb{D}_R(V)} : \mathbb{D}_R(V) \to \mathbb{D}_R(V),$$

which is semi-linear with respect to the q-Frobenius φ_q.

Therefore $V \mapsto \mathbb{D}_R(V)$ is a functor from $\mathbf{Rep}_R(G_E)$ to $\mathbf{Mod}^{\varphi_q}_{/\mathcal{O}_R}$. The following lemma shows that the functor $\mathbb{D}_R$ commutes with restriction of scalars.

Lemma 9.2.6. *Let V be an R-representation of G_E such that $\mathfrak{m}_R^n V = 0$ for some n. Then we have*

$$\mathbb{D}_{LT}(V) \cong \mathbb{D}_R(V)$$

as an $\mathcal{O}_K$-module.

Proof. We use induction on n. First assume that $\mathfrak{m}_R V = 0$. Since V is an R-representation of G_E, it is finitely generated as an R-module. Then $\mathfrak{m}_R V = 0$ implies that V is also finitely generated as an $R/\mathfrak{m}_R$-module. But we know that $R/\mathfrak{m}_R = k_R$ is the residue field of R and it is finite. Since R is an $\mathcal{O}_K$-algebra, then it follows that k_R is a finite extension of k, where k is the residue field of $\mathcal{O}_K$. Now by using Nakayama's lemma (for local rings), V is finitely generated as an $\mathcal{O}_K$-module. Next, suppose that the statement is true for $n-1$, i.e., if $\mathfrak{m}_R^{n-1} W = 0$ for any R-module W, then W is finitely generated as an $\mathcal{O}_K$-module. Now let $\mathfrak{m}_R^n V = 0$. Consider the exact sequence

$$0 \to \mathfrak{m}_R^{n-1} V \to V \to V/\mathfrak{m}_R^{n-1} V \to 0.$$

Then by using induction hypothesis, $\mathfrak{m}_R^{n-1} V$ and $V/\mathfrak{m}_R^{n-1} V$ are finitely generated as $\mathcal{O}_K$-modules. Thus, V is finitely generated as an $\mathcal{O}_K$-module. Hence

$$\widehat{\mathcal{O}_R^{ur}} \otimes_R V = (\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \hat{\otimes}_{\mathcal{O}_K} R) \hat{\otimes}_R V \cong \mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \hat{\otimes}_{\mathcal{O}_K} V = \mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} V.$$

Here the first equality follows from the fact that $\widehat{\mathcal{O}_R^{ur}}$ is complete and V is finitely generated as an R-module. The last one uses that $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$ is complete, and V is finitely generated as an $\mathcal{O}_K$-module. Then taking G_E-invariants, we get the desired result. ∎

Next, we show that the functor $\mathbb{D}_R$ is an exact faithful functor, and it commutes with the inverse limits. Let $V \in \mathbf{Rep}_R(G_E)$ and $\mathfrak{m}_R^n V$ be a submodule of V generated by elements of the form mv for $m \in \mathfrak{m}_R^n$ and $v \in V$. Define $V_n = V/\mathfrak{m}_R^n V \cong V \otimes_R R/\mathfrak{m}_R^n$.

Proposition 9.2.7. *For any $V \in \mathbf{Rep}_R(G_E)$, we have*

$$\mathbb{D}_R(V) \xrightarrow{\sim} \varprojlim_n \mathbb{D}_R(V_n).$$

Proof. Since $\widehat{\mathcal{O}_R^{ur}}$ is complete with respect to $\mathfrak{m}_R\widehat{\mathcal{O}_R^{ur}}$-adic topology and V is finitely generated as an R-module. Therefore $\widehat{\mathcal{O}_R^{ur}} \otimes_R V$ is complete with respect to $\mathfrak{m}_R$-adic topology, and we have

$$\begin{aligned}
\widehat{\mathcal{O}_R^{ur}} \otimes_R V &= \varprojlim_n (\widehat{\mathcal{O}_R^{ur}} \otimes_R V)/\mathfrak{m}_R^n \\
&= \varprojlim_n (\widehat{\mathcal{O}_R^{ur}} \otimes_R (V/\mathfrak{m}_R^n V)) \\
&= \varprojlim_n (\widehat{\mathcal{O}_R^{ur}} \otimes_R V_n),
\end{aligned}$$

where the second equality follows from the fact that the radical $\mathfrak{m}_R\widehat{\mathcal{O}_R^{ur}} \otimes V + \widehat{\mathcal{O}_R^{ur}} \otimes \mathfrak{m}_R V$ is generated by $\mathfrak{m}_R$. Since taking G_E-invariants commutes with the inverse limits, the proposition follows by taking G_E-invariants. ■

Lemma 9.2.8. *The functor $\mathbb{D}_R$ is an additive functor.*

Proof. Let V and W be two R-representations of G_E. Then by Proposition 9.2.7, we have

$$\begin{aligned}
\mathbb{D}_R(V \oplus W) &\cong \varprojlim \mathbb{D}_R((V \oplus W) \otimes_R R/\mathfrak{m}_R^n) \\
&\cong \varprojlim \mathbb{D}_R(V \otimes_R R/\mathfrak{m}_R^n \oplus W \otimes_R R/\mathfrak{m}_R^n) \\
&\cong \varprojlim \mathbb{D}_R(V_n \oplus W_n).
\end{aligned}$$

Since V_n and W_n are of finite length, then by Lemma 9.2.6, we have $\mathbb{D}_R(V_n \oplus W_n) =$

$\mathbb{D}_{LT}(V_n \oplus W_n)$. Moreover, it follows from [20] that the functor $\mathbb{D}_{LT}$ is an additive functor, i.e., $\mathbb{D}_{LT}(V_n \oplus W_n) = \mathbb{D}_{LT}(V_n) \oplus \mathbb{D}_{LT}(W_n)$. Then

$$\begin{aligned}\varprojlim \mathbb{D}_{LT}(V_n \oplus W_n) &\cong \varprojlim(\mathbb{D}_{LT}(V_n) \oplus \mathbb{D}_{LT}(W_n)) \\ &\cong \varprojlim \mathbb{D}_{LT}(V_n) \oplus \varprojlim \mathbb{D}_{LT}(W_n) \\ &\cong \varprojlim \mathbb{D}_{R}(V_n) \oplus \varprojlim \mathbb{D}_{R}(W_n) \\ &\cong \mathbb{D}_{R}(V) \oplus \mathbb{D}_{R}(W),\end{aligned}$$

where the third isomorphism uses Lemma 9.2.6 and the fourth one follows from Proposition 9.2.7. ■

Proposition 9.2.9. *The functor $\mathbb{D}_R$ is exact and faithful.*

Proof. Let

$$0 \to A \to B \to C \to 0$$

be an exact sequence of R-representations of G_E. Since $\widehat{\mathcal{O}_R^{ur}}$ is flat as an R-module, we have a short exact sequence

$$0 \to \widehat{\mathcal{O}_R^{ur}} \otimes_R A \to \widehat{\mathcal{O}_R^{ur}} \otimes_R B \to \widehat{\mathcal{O}_R^{ur}} \otimes_R C \to 0.$$

Taking G_E-invariants, we have a long exact sequence

$$0 \to \mathbb{D}_R(A) \to \mathbb{D}_R(B) \to \mathbb{D}_R(C) \to \cdots .$$

Thus the functor $\mathbb{D}_R$ is left exact. Now if V has finite length, then by Lemma 9.2.6, we have

$$\mathbb{D}_{LT}(V) = \mathbb{D}_R(V)$$

as an $\mathcal{O}_K$-module. Moreover, it follows from [20] that the functor $\mathbb{D}_{LT}$ is an exact functor. Hence for a short exact sequence $0 \to A \to B \to C \to 0$ of finite length representations, the sequence

$$0 \to \mathbb{D}_R(A) \to \mathbb{D}_R(B) \to \mathbb{D}_R(C) \to 0$$

is exact. Now let

$$0 \to A \to B \to C \to 0$$

be a sequence of arbitrary R-representations of G_E. On tensoring it with $R/\mathfrak{m}_R^n$, we have an exact sequence

$$A/\mathfrak{m}_R^n \to B/\mathfrak{m}_R^n \to C/\mathfrak{m}_R^n \to 0. \tag{9.2}$$

Note that the sequence (9.2) is an exact sequence of finite length representations and using the exactness of the functor $\mathbb{D}_R$ for finite length representations, we get an exact sequence

$$\mathbb{D}_R(A/\mathfrak{m}_R^n) \to \mathbb{D}_R(B/\mathfrak{m}_R^n) \to \mathbb{D}_R(C/\mathfrak{m}_R^n) \to 0 \quad \forall \quad n \geq 1. \tag{9.3}$$

Let K_n be the kernel of the map

$$\mathbb{D}_R(B/\mathfrak{m}_R^n) \to \mathbb{D}_R(C/\mathfrak{m}_R^n).$$

Since the sequence (9.3) is exact, we have a surjective homomorphism

$$\mathbb{D}_R(A/\mathfrak{m}_R^n) \to K_n.$$

Then it follows from Lemma 9.2.6 that $\mathbb{D}_R(A/\mathfrak{m}_R^n)$ is finitely generated over the Artinian ring $\mathcal{O}_R/\mathfrak{m}_R^n$. Thus the inverse system $(\mathbb{D}_R(A/\mathfrak{m}_R^n))_{n\geq 1}$ satisfies the Mittag-Leffler condition. Since the map

$$\mathbb{D}_R(A/\mathfrak{m}_R^n) \to K_n$$

is surjective, $(K_n)_{n\geq 1}$ also satisfies Mittag-Leffler condition. Also, the functor $\mathbb{D}_{LT}$ commutes with the inverse limits, thus by taking the inverse limits in (9.3), it follows that the map

$$\mathbb{D}_R(B) \to \mathbb{D}_R(C)$$

is surjective. Hence the functor $\mathbb{D}_R$ is an exact functor. Moreover, the functor $\mathbb{D}_R$ is an additive functor (Lemma 9.2.8). Then it follows from [13, Proposition 1.2.1] that the functor $\mathbb{D}_R$ is also faithful as $\mathbb{D}_R(V) \neq 0$ if $V \neq 0$. ∎

As a consequence of the above lemma, we have the following proposition.

Proposition 9.2.10. *Let V be an R-representation of G_E. Then for an ideal I of R, we have*

(i) $I.\mathbb{D}_R(V) = \mathbb{D}_R(I.V)$,

(ii) $\mathbb{D}_R(V)/I.\mathbb{D}_R(V) \xrightarrow{\sim} \mathbb{D}_R(V/I.V)$.

Proof. Since R is Noetherian, the ideal I is finitely generated. Assume that the ideal I is generated by the set $\{x_1, x_2, \dots, x_n\}$. Define the map

$$\rho : V^n \to V$$

$$(v_1, \dots, v_n) \mapsto \sum_{i=1}^{n} x_i v_i.$$

Clearly, $I.V$ is the image of ρ. Since the functor $\mathbb{D}_R$ is exact, it follows that $\mathbb{D}_R(I.V)$ is the image of $\mathbb{D}_R(\rho)$. Moreover, the Galois group G_E acts R-equivariantly. Then by identifying $\mathbb{D}_R(V^n)$ with $\mathbb{D}_R(V)^n$, the map

$$\mathbb{D}_R(\rho) : \mathbb{D}_R(V)^n \to \mathbb{D}_R(V)$$

is given by

$$(d_1, \dots, d_n) \mapsto \sum_{i=1}^{n} x_i d_i.$$

Now, its image is $I.\mathbb{D}_R(V)$. Hence

$$I.\mathbb{D}_R(V) = \mathbb{D}_R(I.V).$$

The second part follows by applying the functor $\mathbb{D}_R$ to the short exact sequence

$$0 \to I.V \to V \to V/I.V \to 0.$$ ■

Proposition 9.2.11. *Let $V \in \mathbf{Rep}_R(G_E)$. Then*

$$\mathbb{D}_R(V) \xrightarrow{\sim} \varprojlim_n (\mathbb{D}_R(V)/\mathfrak{m}_R^n \mathbb{D}_R(V)).$$

Proof. Since the functor $\mathbb{D}_R$ commutes with the inverse limits (Proposition 9.2.7). Thus

$$\mathbb{D}_R(V) \xrightarrow{\sim} \varprojlim_n \mathbb{D}_R(V_n) = \varprojlim_n \mathbb{D}_R(V/\mathfrak{m}_R^n V). \tag{9.4}$$

Moreover, $\mathfrak{m}_R^n$ is an ideal of R. Then by Proposition 9.2.10, we have

$$\mathbb{D}_R(V)/\mathfrak{m}_R^n \mathbb{D}_R(V) \xrightarrow{\sim} \mathbb{D}_R(V/\mathfrak{m}_R^n V). \tag{9.5}$$

Now by combining (9.4) and (9.5), we get the desired result. ■

Proposition 9.2.12. *Let V be an R-representation of G_E. Then $\mathbb{D}_R(V)$ is a finitely generated $\mathcal{O}_R$-module.*

Proof. By Proposition 9.2.11, we have

$$\mathbb{D}_R(V) \xrightarrow{\sim} \varprojlim_n \mathbb{D}_R(V)/\mathfrak{m}_R^n \mathbb{D}_R(V)).$$

Therefore the module $\mathbb{D}_R(V)$ is separated with respect to the $\mathfrak{m}_R$-adic topology. Recall that $V_n = V/\mathfrak{m}_R^n V$ and $\mathfrak{m}_R$ is an ideal of R, it follows that

$$\mathbb{D}_R(V)/\mathfrak{m}_R.\mathbb{D}_R(V) \xrightarrow{\sim} \mathbb{D}_R(V/\mathfrak{m}_R.V) = \mathbb{D}_R(V_1).$$

Here the first equality comes from the Proposition 9.2.10. Now V_1 is killed by $\mathfrak{m}_R$, then by [9, Lemma 2.1.5], we have

$$\mathbb{D}_R(V_1) = \mathbb{D}_{k_R}(V_1) \tag{9.6}$$

as an $\mathcal{O}_R$-module. Note that $\mathbb{D}_{k_R}(V_1)$ is finitely generated as an $\mathcal{O}_{\mathcal{E}}$-module. Then it follows from (9.6) that $\mathbb{D}_R(V)/\mathfrak{m}_R.\mathbb{D}_R(V)$ is also finitely generated as an $\mathcal{O}_R$-module. Thus by using [22, Theorem 8.4], we deduce that $\mathbb{D}_R(V)$ is finitely generated as an $\mathcal{O}_R$-module. ■

Proposition 9.2.13. *Let V be an R-representation of G_E. Then we have*

$$\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} \mathbb{D}_R(V) \xrightarrow{\sim} \widehat{\mathcal{O}_R^{ur}} \otimes_R V.$$

Proof. First assume that V is killed by $\mathfrak{m}_R^n$ for some n. Then by Lemma 9.2.6, we have $\mathbb{D}_{LT}(V) = \mathbb{D}_R(V)$ as an $\mathcal{O}_K$-module. Now by using Remark 4.2.9, the map

$$\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} \mathbb{D}_R(V_n) \to \widehat{\mathcal{O}_R^{ur}} \otimes_R V_n$$

is an isomorphism. Moreover,

$$\begin{aligned}
\varprojlim \widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} \mathbb{D}_R(V_n) &= \varprojlim \widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} \mathbb{D}_R(V/\mathfrak{m}_R^n V) \\
&\cong \varprojlim \widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} \mathbb{D}_R(V)/\mathfrak{m}_R^n \mathbb{D}_R(V) \\
&\cong \varprojlim (\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} \mathbb{D}_R(V))/\mathfrak{m}_R^n \\
&\cong \widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} \mathbb{D}_R(V),
\end{aligned}$$

where the second isomorphism uses Proposition 9.2.10, the third isomorphism comes from the fact that $\mathfrak{m}_R$ generates the radical of $\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} \mathbb{D}_R(V)$. The last isomorphism follows from Proposition 9.2.12 and the fact that $\widehat{\mathcal{O}_R^{ur}}$ is complete with respect to $\mathfrak{m}_R\widehat{\mathcal{O}_R^{ur}}$-adic topology. On the other hand,

$$\begin{aligned}\varprojlim \widehat{\mathcal{O}_R^{ur}} \otimes_R V_n =& \varprojlim \widehat{\mathcal{O}_R^{ur}} \otimes_R V/\mathfrak{m}_R^n V \\ =& \varprojlim (\widehat{\mathcal{O}_R^{ur}} \otimes_R V)/\mathfrak{m}_R^n \\ \cong& \widehat{\mathcal{O}_R^{ur}} \otimes_R V.\end{aligned}$$

Hence for general V, the result follows by taking the inverse limits. ■

Next, we define a full subcategory of $\mathbf{Mod}_{/\mathcal{O}_R}^{\varphi_q}$, which is an essential image of the functor $\mathbb{D}_R$.

Definition 9.2.14. A φ_q-module M over $\mathcal{O}_R$ is said to be *étale* if Φ_M^{lin} is an isomorphism, and M is finitely generated as an $\mathcal{O}_R$-module.

Let $\mathbf{Mod}_{/\mathcal{O}_R}^{\varphi_q,\acute{e}t}$ denote the category of étale φ_q-modules over $\mathcal{O}_R$. The morphisms of étale φ_q-modules are the morphisms of underlying φ_q-modules. Since $\mathbf{Mod}_{/\mathcal{O}_R}^{\varphi_q}$ is an abelian category. It follows from [10] that the category $\mathbf{Mod}_{/\mathcal{O}_R}^{\varphi_q,\acute{e}t}$ is also an abelian category. Moreover, it is also stable under sub-objects, quotients and tensor product.

Proposition 9.2.15. *Let $V \in \mathbf{Rep}_R(G_E)$. Then $\mathbb{D}_R(V)$ is an étale φ_q-module over $\mathcal{O}_R$.*

Proof. By Proposition 9.2.12, we know that $\mathbb{D}_R(V)$ is finitely generated as an $\mathcal{O}_R$-module. Now we only need to show that the map $\Phi_{\mathbb{D}_R(V)}^{lin}$ is an isomorphism.

If $\mathfrak{m}_R^n V = 0$ for some n, then by Lemma 9.2.6, we have $\mathbb{D}_{LT}(V) = \mathbb{D}_R(V)$ as an $\mathcal{O}_K$-module, and it follows from [20] that the map

$$\Phi_{\mathbb{D}_R(V_n)}^{lin} : \mathbb{D}_R(V_n)_{\varphi_q} \to \mathbb{D}_R(V_n)$$

is an isomorphism. Since the functor $\mathbb{D}_R$ commutes with the inverse limits, the result follows for general V by taking the inverse limits. ■

Next, we construct a functor $\mathbb{V}_R$ from the category of étale φ_q-modules over $\mathcal{O}_R$ to the category of R-representations of G_E and show that the functor $\mathbb{V}_R$ is a quasi-inverse functor to $\mathbb{D}_R$. The functor $\mathbb{V}_R$ is defined in the following way.

Let M be an étale φ_q-module over $\mathcal{O}_R$. Then view $\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M$ as a φ_q-module via

$$\varphi_{\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M}(\lambda \otimes m) = \varphi_q(\lambda) \otimes \varphi_M(m) \quad \text{for } \lambda \in \widehat{\mathcal{O}_R^{ur}}, m \in M.$$

For simplicity, we write $\varphi_q \otimes \varphi_M$ rather than $\varphi_{\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M}$.

The Galois group G_E acts on $\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M$ via its action on $\widehat{\mathcal{O}_R^{ur}}$ and the group action commutes with the action of $\varphi_q \otimes \varphi_M$. Define

$$\mathbb{V}_R(M) := (\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M)^{\varphi_q \otimes \varphi_M = id},$$

which is a sub R-module stable under the action of G_E. Therefore $M \mapsto \mathbb{V}_R(M)$ is a functor from $\mathbf{Mod}_{/\mathcal{O}_R}^{\varphi_q, ét}$ to $\mathbf{Rep}_R(G_E)$.

Next, we show that the functor $\mathbb{V}_R$ commutes with the inverse limits and it is also an exact functor. Let $M_n = M/\mathfrak{m}_R^n M$, where $\mathfrak{m}_R^n M$ is sub-module of M.

Proposition 9.2.16. *Let M be an étale φ_q-module over $\mathcal{O}_R$. Then*

$$\mathbb{V}_R(M) \xrightarrow{\sim} \varprojlim_n \mathbb{V}_R(M_n).$$

Proof. Since taking $\varphi_q \otimes \varphi_M$-invariant commutes with the inverse limits, it is sufficient to prove that the tensor product with $\widehat{\mathcal{O}_R^{ur}}$ commutes with the inverse limits.

But

$$\varprojlim_n (\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M_n) = \varprojlim_n (\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} (M/\mathfrak{m}_R^n M))$$
$$= \varprojlim_n (\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M)/\mathfrak{m}_R^n.$$

Note that $\widehat{\mathcal{O}_R^{ur}}$ is complete with respect to $\mathfrak{m}_R\widehat{\mathcal{O}_R^{ur}}$-adic topology. Also, M is finitely generated as an $\mathcal{O}_R$-module since M is étale. So $\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M$ is complete with respect to $\mathfrak{m}_R$-adic topology. Hence

$$\varprojlim_n (\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M)/\mathfrak{m}_R^n = \widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M$$

Now the proposition follows by taking $\varphi_q \otimes \varphi_M$-invariants. ■

Lemma 9.2.17. *Let V be an R-representation of G_E. Then $\varphi_q \otimes id_V - id$ is a surjective homomorphism of abelian groups acting on $\widehat{\mathcal{O}_R^{ur}} \otimes_R V$.*

Proof. Let $\mathfrak{m}_R V = 0$. Then the map

$$\varphi_q - id : E^{sep} \to E^{sep}$$

is surjective, since for all $\lambda \in E^{sep}$, the polynomial $x^q - x - \lambda$ is separable. As k_R is a finite extension of $\mathbb{F}_p$, and φ_q acts trivially on k_R, the map

$$\varphi_q - id : E^{sep} \otimes_{\mathbb{F}_p} k_R \to E^{sep} \otimes_{\mathbb{F}_p} k_R$$

is also surjective. Moreover, $\varphi_q - id$ is continuous, thus the map

$$\varphi_q - id : k_{\widehat{\mathcal{O}_R^{ur}}} = E^{sep} \hat{\otimes}_{\mathbb{F}_p} k_R \to E^{sep} \hat{\otimes}_{\mathbb{F}_p} k_R$$

is surjective. Note that $V_1 = V/\mathfrak{m}_R V$ is free over k_R, and φ_q acts on $k_{\widehat{\mathcal{O}_R^{ur}}} \otimes_{k_R} V_1$ via its action on $k_{\widehat{\mathcal{O}_R^{ur}}}$, it follows that $\varphi_q \otimes id_{V_1} - id$ is surjective on $k_{\widehat{\mathcal{O}_R^{ur}}} \otimes_{k_R} V_1$. Then by dévissage (by using Five lemma as explained in part (i) of Lemma 4.2.3),

$$\varphi_q \otimes id_{V_n} - id : \widehat{\mathcal{O}_R^{ur}}/\mathfrak{m}_R^n \otimes_R V_n \to \widehat{\mathcal{O}_R^{ur}}/\mathfrak{m}_R^n \otimes_R V_n$$

is surjective. Since $\widehat{\mathcal{O}_R^{ur}}/\mathfrak{m}_R^n$ is Artinian, and V_n has a finite length. Consequently, the Mittag-Leffler condition holds for $\widehat{\mathcal{O}_R^{ur}}/\mathfrak{m}_R^n \otimes_R V_n$. Then by passage to the inverse limits, the result holds for general V. ■

Lemma 9.2.18. *Let M be an étale φ_q-module over $\mathcal{O}_R$ such that $\mathfrak{m}_R^n M = 0$ for some $n \geq 1$. Then*

$$\mathbb{V}_{LT}(M) = \mathbb{V}_R(M),$$

as an $\mathcal{O}_K$-module.

Proof. The proof is similar to Lemma 9.2.6. ■

Proposition 9.2.19. *Let M be an étale φ_q-module over $\mathcal{O}_R$. Then the homomorphism $\varphi_q \otimes \varphi_M - id$ is surjective on $\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M$.*

Proof. If $\mathfrak{m}_R^n M = 0$ for some n, then by Lemma 9.2.18, we have $\mathbb{V}_{LT}(M) = \mathbb{V}_R(M)$ as an $\mathcal{O}_K$-module. Now by using Remark 4.2.9, it follows that the natural map

$$\widehat{\mathcal{O}_R^{ur}} \otimes_R \mathbb{V}_R(M_n) \to \widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M_n$$

is an isomorphism. Moreover, this isomorphism respects the action of $\varphi_q \otimes \varphi_M$. Then by using Lemma 9.2.17, the map $\varphi_q \otimes \varphi_M - id$ is surjective on $\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M_n$, and the general case follows by passing to the inverse limits. ■

Proposition 9.2.20. *The functor $\mathbb{V}_R$ is an exact functor.*

Proof. Let

$$0 \to M \to M' \to M'' \to 0$$

be a short exact sequence of étale φ_q-modules over $\mathcal{O}_R$. Then we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M & \longrightarrow & \widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M' & \longrightarrow & \widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_q\otimes\varphi_M - id} & & \downarrow{\scriptstyle \varphi_q\otimes\varphi_{M'} - id} & & \downarrow{\scriptstyle \varphi_q\otimes\varphi_{M''} - id} & & \\ 0 & \longrightarrow & \widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M & \longrightarrow & \widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M' & \longrightarrow & \widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M'' & \longrightarrow & 0. \end{array}$$

Now by applying the Snake lemma, we get an exact sequence

$$0 \to \mathbb{V}_R(M) \to \mathbb{V}_R(M') \to \mathbb{V}_R(M'') \to \widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M/(\varphi_q \otimes \varphi_M - id) \to \cdots .$$

By Lemma 9.2.19, we know that the map $\varphi_q \otimes \varphi_M - id$ is a surjective homomorphism acting on $\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M$, so the last term is zero, and the sequence

$$0 \to \mathbb{V}_R(M) \to \mathbb{V}_R(M') \to \mathbb{V}_R(M'') \to 0$$

is an exact sequence. Hence the functor $\mathbb{V}_R$ is exact. ■

Proposition 9.2.21. *Let M be an étale φ_q-module over $\mathcal{O}_R$. Then $\mathbb{V}_R(M)$ is finitely generated as an R-module, and the homomorphism of $\widehat{\mathcal{O}_R^{ur}}$-modules*

$$\widehat{\mathcal{O}_R^{ur}} \otimes_R \mathbb{V}_R(M) \to \widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M$$

is an isomorphism.

Proof. Since by analogue of Proposition 9.2.11 for $\mathbb{V}_R$, we have

$$\mathbb{V}_R(M) \xrightarrow{\sim} \varprojlim(\mathbb{V}_R(M)/\mathfrak{m}_R^n \mathbb{V}_R(M)).$$

Then $\mathbb{V}_R(M)$ is separated with respect to the $\mathfrak{m}_R$-adic topology. Moreover, by analogue of Proposition 9.2.10 for $\mathbb{V}_R$, we have

$$\mathbb{V}_R(M)/\mathfrak{m}_R.\mathbb{V}_R(M) \xrightarrow{\sim} \mathbb{V}_R(M/\mathfrak{m}_R.M) = \mathbb{V}_R(M_1).$$

As M_1 is killed by $\mathfrak{m}_R$, then by [9, Lemma 2.1.5], we know $\mathbb{V}_R(M_1) = \mathbb{V}_{k_R}(M_1)$ as an $\mathcal{O}_R$-modules. Now note that $\mathbb{V}_{k_R}(M_1)$ is finitely generated as an $\mathcal{O}_{\mathcal{E}}$-module, thus $\mathbb{V}_R(M)/\mathfrak{m}_R.\mathbb{V}_R(M)$ is also finitely generated as an $\mathcal{O}_R$-module. Then by theorem [22, Theorem 8.4], it follows that $\mathbb{V}_R(M)$ is finitely generated as an $\mathcal{O}_R$-module.

For the second part, if M is killed by $\mathfrak{m}_R^n$, then by Lemma 9.2.18, $\mathbb{V}_{LT}(M) = \mathbb{V}_R(M)$ as an $\mathcal{O}_K$-module. Now by using Remark 4.2.9, the map

$$\widehat{\mathcal{O}_R^{ur}} \otimes_R \mathbb{V}_R(M_n) \to \widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M_n$$

is an isomorphism, and the general case follows by taking the inverse limits as in Proposition 9.2.13. ∎

Next, the following theorem gives an equivalence of categories between the category $\mathbf{Rep}_R(G_E)$ and $\mathbf{Mod}_{/\mathcal{O}_R}^{\varphi_q,ét}$.

Theorem 9.2.22. *The functor*

$$\mathbb{D}_R : \mathbf{Rep}_R(G_E) \to \mathbf{Mod}_{/\mathcal{O}_R}^{\varphi_q,ét}$$

is an equivalence of categories with quasi-inverse functor

$$\mathbb{V}_R : \mathbf{Mod}_{/\mathcal{O}_R}^{\varphi_q,ét} \to \mathbf{Rep}_R(G_E).$$

Proof. Let V be an R-representation of G_E and M be an étale φ_q-module over $\mathcal{O}_R$. Then to prove the theorem, it is enough to show

$$\mathbb{V}_R(\mathbb{D}_R(V)) \xrightarrow{\cong} V \quad \text{and} \quad \mathbb{D}_R(\mathbb{V}_R(M)) \xrightarrow{\cong} M.$$

Since by Proposition 9.2.13, we have an isomorphism

$$\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} \mathbb{D}_R(V) \to \widehat{\mathcal{O}_R^{ur}} \otimes_R V$$

of G_E-modules. Then by taking $\varphi_q \otimes \varphi_M$-invariant, we get an isomorphism

$$\mathbb{V}_R(\mathbb{D}_R(V)) \to (\widehat{\mathcal{O}_R^{ur}} \otimes_R V)^{\varphi_q \otimes \varphi_M = id}.$$

Note that V has trivial action of $\varphi_q \otimes \varphi_M$, so there is map

$$V \to (\widehat{\mathcal{O}_R^{ur}} \otimes_R V)^{\varphi_q \otimes \varphi_M = id}.$$

Now for finite length modules, the above map is an isomorphism by using Theorem 4.2.10. By taking the inverse limits, the map will be an isomorphism for the general V. Hence

$$\mathbb{V}_R(\mathbb{D}_R(V)) \xrightarrow{\cong} V.$$

Similarly, the map

$$M \to (\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M)^{G_E}$$

is an isomorphism. Moreover, by Proposition 9.2.21, we have an isomorphism

$$\widehat{\mathcal{O}_R^{ur}} \otimes_R \mathbb{V}_R(M) \to \widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M.$$

Then taking G_E-invariants, we have

$$\mathbb{D}_R(\mathbb{V}_R(M)) = (\widehat{\mathcal{O}_R^{ur}} \otimes_R \mathbb{V}_R(M))^{G_E} \xrightarrow{\sim} (\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M)^{G_E}.$$

Therefore

$$\mathbb{D}_R(\mathbb{V}_R(M)) \to M$$

is an isomorphism, and this proves the theorem. ∎

Remark 9.2.23. The functors $\mathbb{D}_R$ and $\mathbb{V}_R$ are compatible with the tensor product, i.e., if V_1 and V_2 are R-representations of G_E, and M_1 and M_2 are étale φ_q-modules over $\mathcal{O}_R$, then

(i) The homomorphism $\mathbb{D}_R(V_1)\otimes_{\mathcal{O}_R}\mathbb{D}_R(V_2) \to \mathbb{D}_R(V_1\otimes_R V_2)$ of étale φ_q-modules is an isomorphism.

(ii) The natural homomorphism $\mathbb{V}_R(M_1) \otimes_R \mathbb{V}_R(M_2) \to \mathbb{V}_R(M_1 \otimes_{\mathcal{O}_R} M_2)$ of R-representations of G_E is an isomorphism.

9.2.2 The characteristic zero case

Let K be a local field of characteristic 0. From chapter 4, recall that the ring $\mathcal{O}_{\mathcal{E}}$ is the π-adic completion of $\mathcal{O}_K[[Z]]\left[\frac{1}{Z}\right]$, and $\mathcal{O}_{\mathcal{E}^{ur}}$ is the maximal integral unramified extension of $\mathcal{O}_{\mathcal{E}}$. The ring $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$ is the π-adic completion of $\mathcal{O}_{\mathcal{E}^{ur}}$. Also, the Galois group $H_K = \mathrm{Gal}(\bar{K}/K_\infty)$ is identified with G_E, where E is the field of norms of the extension K_∞/K and it is a field of characteristic p.

Let V be an R-representation of G_K. Then

$$\mathbb{D}_R(V) := (\widehat{\mathcal{O}_R^{ur}} \otimes_R V)^{H_K} = (\widehat{\mathcal{O}_R^{ur}} \otimes_R V)^{G_E}$$

is a φ_q-module over $\mathcal{O}_R$. The G_K-action on $\widehat{\mathcal{O}_R^{ur}} \otimes_R V$ induces a semi-linear action of $G_K/H_K = \Gamma_{LT} = \mathrm{Gal}(K_\infty/K)$ on $\mathbb{D}_R(V)$.

Definition 9.2.24. A (φ_q, Γ_{LT})*-module* M over $\mathcal{O}_R$ is a φ_q-module over $\mathcal{O}_R$ equipped with a continuous semi-linear action of Γ_{LT}, which commutes with the endomorphism φ_M of M, and a (φ_q, Γ_{LT})-module is *étale* if its underlying φ_q-module is étale.

Let $\mathbf{Mod}^{\varphi_q,\Gamma_{LT},ét}_{/\mathcal{O}_R}$ be the category of étale (φ_q, Γ_{LT})-modules over $\mathcal{O}_R$. Then $\mathbb{D}_R$ is a functor from the category $\mathbf{Rep}_R(G_K)$ of R-linear representations of G_K to the category $\mathbf{Mod}^{\varphi_q,\Gamma_{LT},ét}_{/\mathcal{O}_R}$ of étale (φ_q, Γ_{LT})-modules over $\mathcal{O}_R$.

If M is an étale (φ_q, Γ_{LT})-module over $\mathcal{O}_R$, then

$$\mathbb{V}_R(M) = (\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M)^{\varphi_q\otimes\varphi_M=id}$$

is an R-representation of G_K. The group G_K acts on $\widehat{\mathcal{O}_R^{ur}}$ as before and acts via Γ_{LT} on M. The G_K action on $\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M$ is $\varphi_q \otimes \varphi_M$-equivariant, and this induces a G_K action on $\mathbb{V}_R(M)$.

For any $V \in \mathbf{Rep}_R(G_K)$, there is a canonical R-linear homomorphism

$$V \to \mathbb{V}_R(\mathbb{D}_R(V))$$

of representations of G_K. By Theorem 9.2.22, this is an isomorphism when restricted to H_K, so it must be an isomorphism of G_K-representations. Similarly, for an étale (φ_q, Γ_{LT})-module M, the canonical homomorphism of étale (φ_q, Γ_{LT})-modules

$$M \to \mathbb{D}_R(\mathbb{V}_R(M))$$

is an isomorphism. Moreover, by using Theorem 9.2.22, the underlying map of φ_q-modules is an isomorphism, and this proves the following theorem.

Theorem 9.2.25. *The functor $\mathbb{D}_R$ is an exact equivalence of categories between $\mathbf{Rep}_R(G_K)$ the category of R-linear representations of G_K and $\mathbf{Mod}_{/\mathcal{O}_R}^{\varphi_q,\Gamma_{LT},ét}$ the category of étale (φ_q, Γ_{LT})-modules over $\mathcal{O}_R$ with quasi-inverse functor $\mathbb{V}_R$.*

Next, we extend the functor $\mathbb{D}_R$ to the category $\mathbf{Rep}_{\mathfrak{m}_R-tor}^{dis}(G_K)$ of discrete $\mathfrak{m}_R$-primary abelian groups with a continuous and linear action of G_K. Any object in $\mathbf{Rep}_{\mathfrak{m}_R-tor}^{dis}(G_K)$ is the filtered direct limit of $\mathfrak{m}_R$-power torsion objects in $\mathbf{Rep}_R(G_K)$. For any $V \in \mathbf{Rep}_{\mathfrak{m}_R-tor}^{dis}(G_K)$, define

$$\mathbb{D}_R(V) = (\widehat{\mathcal{O}_R^{ur}} \otimes_R V)^{H_K}.$$

Note that the functor $\mathbb{D}_R$ commutes with the direct limits as the tensor product and taking H_K-invariants commute with the direct limits. Then $\mathbb{D}_R(V)$ is an object into the category $\varinjlim \mathbf{Mod}_{/\mathcal{O}_R}^{\varphi_q,\Gamma_{LT},ét,tor}$ of injective limits of $\mathfrak{m}_R$-power torsion objects in

$\mathbf{Mod}_{/\mathcal{O}_R}^{\varphi_q,\Gamma_{LT},\acute{e}t}$. For any $M \in \varinjlim \mathbf{Mod}_{/\mathcal{O}_R}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}$, put

$$\mathbb{V}_R(M) = (\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_R} M)^{\varphi_q \otimes \varphi_M = id}.$$

Then the functor $\mathbb{V}_R$ also commutes with the direct limits, and we have the following result.

Proposition 9.2.26. *The functor $\mathbb{D}_R$ and $\mathbb{V}_R$ are quasi-inverse equivalences of categories between $\mathbf{Rep}_{\mathfrak{m}_R-tor}^{dis}(G_K)$ and $\varinjlim \mathbf{Mod}_{/\mathcal{O}_R}^{\varphi_q,\Gamma_{LT},\acute{e}t,tor}$.*

Proof. Since the functors $\mathbb{D}_R$ and $\mathbb{V}_R$ commute with the direct limits, the proposition follows from Theorem 9.2.25 by taking the direct limits. ■

Chapter 10

Galois Cohomology over the Coefficient Ring

This chapter is a part of [1]. By Theorem 9.2.25, it is evident that given an object in the category of R-representation of G_K, one can give its description in terms of the étale (φ_q, Γ_{LT})-module attached to it. In this chapter, we show that this is the case for the continuous cohomology groups of an R-representation V of G_K, i.e., we describe the continuous cohomology groups of V in terms of the corresponding étale (φ_q, Γ_{LT})-module. We give a generalization of most of our results to the case of the coefficient ring.

10.1 Galois cohomology

In the previous chapter, we have shown that the functor $\mathbb{D}_R$ is a quasi-inverse equivalence of categories between $\mathbf{Rep}_R(G_K)$ (resp., $\mathbf{Rep}^{dis}_{\mathfrak{m}_R-tor}(G_K)$) and $\mathbf{Mod}^{\varphi_q,\Gamma_{LT},ét}_{/\mathcal{O}_R}$ (resp., $\varinjlim \mathbf{Mod}^{\varphi_q,\Gamma_{LT},ét,tor}_{/\mathcal{O}_R}$). The following theorem is a generalization of Theorem 5.5.5 over the coefficient rings.

Theorem 10.1.1. *Let V be an R-representation of G_K. Then there is a natural* *isomorphism*

$$H^i(G_K, V) \cong \mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_R(V))) \quad \textit{for } i \geq 0.$$

Proof. First assume that the representation V has finite length, i.e., $\mathfrak{m}_R^n V = 0$ for some $n \in \mathbb{N}$. Then by Lemma 9.2.6 and 5.5.5, we have

$$H^i(G_K, V) \cong \mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_R(V))).$$

Next, it follows from [32, Theorem 2.1] and [33, Corollary 2.2] that the functor $H^i(G_K, -)$ commutes with the inverse limits. Moreover, by Proposition 9.2.7, we have

$$\mathbb{D}_R(V) \xrightarrow{\sim} \varprojlim \mathbb{D}_R(V_n).$$

Observe that the modules $\mathbb{D}_R(V_n)$ are finitely generated over the Artinian ring $\mathcal{O}_R/\mathfrak{m}_R^n\mathcal{O}_R$, so the inverse limit functor is an exact functor on the category of $\mathfrak{m}_R$-power torsion étale (φ_q, Γ_{LT})-modules over $\mathcal{O}_R$. Then we have

$$\mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_R(V))) = \varprojlim \mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_R(V_n))).$$

Hence the general case follows by passing to the inverse limits. ■

Next, in order to generalize the Theorem 7.2.6 over coefficient rings; first, we extend the operator $\psi := \psi_{\mathbb{D}_{LT}(V)}$ to $\mathbb{D}_R(V)$.

As ψ_q maps $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$ to $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}$, we extend ψ_q to $\mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes_{\mathcal{O}_K} R$ by making it trivially act on R. Moreover, it maps $\mathfrak{m}_{\mathcal{O}_{\widehat{\mathcal{E}^{ur}}}} \otimes R + \mathcal{O}_{\widehat{\mathcal{E}^{ur}}} \otimes \mathfrak{m}_R$ to itself, thus induces an R-linear map

$$\psi_q : \widehat{\mathcal{O}_R^{ur}} \to \widehat{\mathcal{O}_R^{ur}}.$$

Since ψ_q acts Galois equivariantly, so making it act on $\widehat{\mathcal{O}_R^{ur}} \otimes_{\mathcal{O}_K} V$ by its action on $\widehat{\mathcal{O}_R^{ur}}$, we have an operator $\psi_{\mathbb{D}_R(V)}$ on $\mathbb{D}_R(V)$.

Recall that $\mathbf{Rep}^{dis}_{\mathfrak{m}_R - tor}(G_K)$ is the category of discrete $\mathfrak{m}_R$-primary abelian groups with a continuous and linear action of G_K. Any object in $\mathbf{Rep}^{dis}_{\mathfrak{m}_R-tor}(G_K)$ is the filtered direct limit of $\mathfrak{m}_R$-power torsion objects in $\mathbf{Rep}_R(G_K)$. Now we have the following theorem.

Theorem 10.1.2. *Let $V \in \mathbf{Rep}^{dis}_{\mathfrak{m}_R-tor}(G_K)$. Then we have a well-defined homomorphism*

$$\mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_R(V))) \to \mathcal{H}^i(\Psi\Gamma^{\bullet}_{LT}(\mathbb{D}_R(V))) \quad \textit{for } i \geq 0.$$

Moreover, the homomorphism $\mathcal{H}^0(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_R(V))) \to \mathcal{H}^0(\Psi\Gamma^{\bullet}_{LT}(\mathbb{D}_R(V))$ is injective.

Proof. If V is a finite abelian $\mathfrak{m}_R$ group, then the theorem follows from Lemma 9.2.6 and Theorem 7.2.6. Also, the functors $\mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT}(\mathbb{D}_R(-)))$ and $\mathcal{H}^i(\Psi\Gamma^{\bullet}_{LT}(\mathbb{D}_R(-)))$ commute with the inverse limits. Hence the result follows for general V by passing to the inverse limits. ∎

Next, we compute the Iwasawa cohomology groups of V in terms of cohomology groups of $\underline{\Psi}^{\bullet}$-complex. The following theorem is a generalization of [29, Theorem 5.13] to the case of coefficient rings. It is possible that this approach leads to the construction of a Perrin-Riou homomorphism for Galois representation defined over the coefficient ring R.

Theorem 10.1.3. *Let $V \in \mathbf{Rep}_R(G_K)$. Then we have*

$$H^i_{Iw}(K_\infty/K, V) \cong \mathcal{H}^{i-1}(\underline{\Psi}^{\bullet}(\mathbb{D}_R(V(\chi_{cyc}^{-1}\chi_{LT})))) \quad \textit{for } i \geq 1.$$

Proof. Suppose that V has a finite length. Then $\mathbb{D}_R(V) = \mathbb{D}_{LT}(V)$ as an $\mathcal{O}_K$-module. Thus $\psi_{\mathbb{D}_R(V)}$ agrees with the $\psi_{\mathbb{D}_{LT}(V)}$. Now by Corollary 8.2.4, we have

$$H^i_{Iw}(K_\infty/K, V) \cong \mathcal{H}^{i-1}(\underline{\Psi}^{\bullet}(\mathbb{D}_R(V(\chi_{cyc}^{-1}\chi_{LT})))) \quad \text{for } i \geq 1.$$

Moreover, it follows from Lemma 5.5.4 that the functor $H^i_{Iw}(K_\infty/K, -)$ also commutes with the inverse limits. Then by passing to the inverse limits, we deduce the theorem for general V. ■

Remark 10.1.4. It is possible to extend Theorem 6.1.2 to the case of the coefficient ring, and using that we can prove that for any $V \in \mathbf{Rep}_R(G_K)$,

$$H^i(G_K, V) \cong \mathcal{H}^i(\Phi\Gamma^{\bullet}_{LT,FT}(\mathbb{D}_R(V))) \quad \text{for } i \geq 0.$$

This gives a generalization of Theorem 6.4.1 over the coefficient rings. We can also generalize Theorem 7.2.11 to the case of coefficient ring.

10.2 The dual exponential map

By Theorem 8.2.3, we have

$$H^1_{Iw}(K_\infty/K, \mathcal{O}_K(\chi_{cyc}\chi_{LT}^{-1})) \cong \mathbb{D}_{LT}(\mathcal{O}_K)^{\psi_{\mathbb{D}_{LT}(\mathcal{O}_K)}=id}.$$

The map

$$\text{Exp}^* : H^1_{Iw}(K_\infty/K, \mathcal{O}_K(\chi_{cyc}\chi_{LT}^{-1})) \to \mathbb{D}_{LT}(\mathcal{O}_K)^{\psi_{\mathbb{D}_{LT}(\mathcal{O}_K)}=id}$$

is called the *dual exponential map*. These dual exponential maps occur in the construction of the Coates-Wiles homomorphisms. For more details about this dual exponential map, see [29]. We generalize the dual exponential map over the coefficient ring to check if one can extend the Coates-Wiles homomorphisms to Galois representations defined over R.

Theorem 10.2.1. *Let $V \in \mathbf{Rep}_R(G_K)$. Then we have the following commutative diagram*

$$\begin{array}{ccc} H^i_{Iw}(K_\infty/K, V) & \xrightarrow{\cong} & \mathcal{H}^{i-1}(\underline{\Psi}^\bullet(\mathbb{D}_R(V(\chi_{cyc}^{-1}\chi_{LT})))) \\ \downarrow & & \downarrow \\ H^i_{Iw}(K_\infty/K, V\otimes_R \mathcal{O}_K) & \xrightarrow[\cong]{} & \mathcal{H}^{i-1}(\underline{\Psi}^\bullet(\mathbb{D}_{LT}(V\otimes_R \mathcal{O}_K(\chi_{cyc}^{-1}\chi_{LT})))). \end{array}$$

Proof. Note that the ring homomorphism $R \to \mathcal{O}_K$ induces a map from $V \to V \otimes_R \mathcal{O}_K$. By [29, Lemma 5.8], we have

$$H^i_{Iw}(K_\infty/K, V) = H^i(G_K, R[[\Gamma_{LT}]] \otimes_R V),$$

$$H^i_{Iw}(K_\infty/K, V\otimes_R \mathcal{O}_K) = H^i(G_K, \mathcal{O}_K[[\Gamma_{LT}]] \otimes_{\mathcal{O}_K} (V\otimes_R \mathcal{O}_K)).$$

Here the right hand side refers to cohomology with continuous co-chains. Therefore, we have a well-defined map

$$H^i_{Iw}(K_\infty/K, V) \to H^i_{Iw}(K_\infty/K, V\otimes_R \mathcal{O}_K).$$

Similarly, the map $V \to V\otimes_R \mathcal{O}_K$ defines a map from $\mathbb{D}_R(V) \to \mathbb{D}_{LT}(V\otimes_R \mathcal{O}_K)$, and this induces a well-defined map

$$\mathcal{H}^{i-1}(\underline{\Psi}^\bullet(\mathbb{D}_R(V(\chi_{cyc}^{-1}\chi_{LT})))) \to \mathcal{H}^{i-1}(\underline{\Psi}^\bullet(\mathbb{D}_{LT}(V\otimes_R \mathcal{O}_K(\chi_{cyc}^{-1}\chi_{LT})))).$$

Now the result follows from Theorem 8.2.3 and Theorem 10.1.3. ■

Next, we generalize the dual exponential map over coefficient rings.

Corollary 10.2.2. *There is a dual exponential map*

$$\mathrm{Exp}^*_R : H^1_{Iw}(K_\infty/K, R(\chi_{cyc}\chi_{LT}^{-1})) \xrightarrow{\sim} \mathcal{O}_R^{\psi_R = id}$$

over R, and the diagram

$$\begin{array}{ccc} H^1_{Iw}(K_\infty/K, R(\chi_{cyc}\chi_{LT}^{-1})) & \xrightarrow{\mathrm{Exp}^*_R} & \mathcal{O}_R^{\psi_R=id} \\ \downarrow & & \downarrow \\ H^1_{Iw}(K_\infty/K, \mathcal{O}_K(\chi_{cyc}\chi_{LT}^{-1})) & \xrightarrow[\mathrm{Exp}^*]{} & \mathcal{O}_{\mathcal{E}}^{\psi=id} \end{array}$$

where $\psi_R = \psi_{\mathbb{D}_R(R)}$ and $\psi = \psi_{\mathbb{D}_{LT}(\mathcal{O}_K)}$, is commutative.

Proof. Since $R(\chi_{cyc}\chi_{LT}^{-1})$ is an R-representation of G_K, by Theorem 10.1.3, we have

$$\begin{aligned} H^1_{Iw}(K_\infty/K, R(\chi_{cyc}\chi_{LT}^{-1})) &\cong \mathcal{H}^0(\underline{\Psi}^\bullet(\mathbb{D}_R(R(\chi_{cyc}\chi_{LT}^{-1})(\chi_{cyc}^{-1}\chi_{LT}))) \\ &\cong \mathcal{O}_R^{\psi_R=id}. \end{aligned}$$

Also, by Theorem 8.2.3, we have

$$\begin{aligned} H^1_{Iw}(K_\infty/K, R(\chi_{cyc}\chi_{LT}^{-1}) \otimes_R \mathcal{O}_K) &\cong H^1_{Iw}(K_\infty/K, \mathcal{O}_K(\chi_{cyc}\chi_{LT}^{-1})) \\ &\cong \mathcal{H}^0(\underline{\Psi}^\bullet(\mathbb{D}_{LT}(\mathcal{O}_K(\chi_{cyc}\chi_{LT}^{-1})(\chi_{cyc}^{-1}\chi_{LT}))) \\ &\cong \mathcal{O}_{\mathcal{E}}^{\psi=id}. \end{aligned}$$

Now, the result follows from Theorem 10.2.1 by putting $i = 1$ and $V = R(\chi_{cyc}\chi_{LT}^{-1})$.

■

Bibliography

[1] ARIBAM, C., AND KWATRA, N. Galois cohomology for Lubin-Tate (φ_q, Γ_{LT})-modules over coefficient rings. arxiv:1908.03941 (2019).

[2] BERGER, L. Multivariable Lubin-Tate (φ, Γ)-modules and filtered φ-modules. *Math. Res. Lett. 20*, 3 (2013), 409–428.

[3] BERGER, L. Multivariable (φ, Γ)-modules and locally analytic vectors. *Duke Math. J. 165*, 18 (2016), 3567–3595.

[4] BERGER, L., AND FOURQUAUX, L. Iwasawa theory and F-analytic Lubin-Tate (φ, Γ)-modules. *Doc. Math. 22* (2017), 999–1030.

[5] BERGER, L., SCHNEIDER, P., AND XIE, B. Rigid Character Groups, Lubin-Tate Theory, and (φ,Γ)-Modules. *Mem. Amer. Math. Soc. 263*, 1275 (2020), v + 79.

[6] CASSELS, J., FRÖLICH, A., SOCIETY, L. M., AND UNION, I. M. *Algebraic Number Theory: Proceedings of an Instructional Conference. Edited by J. W. S. Cassels and A. Fröhlich.* 1976.

[7] COLMEZ, P. Espaces de Banach de dimension finie. *J. Inst. Math. Jussieu 1*, 3 (2002), 331–439.

[8] CORNELL, G. A., SILVERMAN, J. H., AND STEVENS, G. Modular forms and fermat's last theorem.

[9] DEE, J. Φ-Γ modules for families of Galois representations. *J. Algebra 235*, 2 (2001), 636–664.

[10] FONTAINE, J.-M. Représentations p-adiques des corps locaux. I. In *The Grothendieck Festschrift, Vol. II*, vol. 87 of *Progr. Math.* Birkhäuser Boston, Boston, MA, 1990, pp. 249–309.

[11] FONTAINE, J.-M., AND OUYANG, Y. *Theory of p-adic Galois Representations*. Springer.

[12] FOURQUAUX, L., AND XIE, B. Triangulable $\mathcal{O}_F$-analytic (φ_q, Γ)-modules of rank 2. *Algebra Number Theory 7*, 10 (2013), 2545–2592.

[13] FU, L. *Etale cohomology theory*, revised ed., vol. 14 of *Nankai Tracts in Mathematics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.

[14] GHATE, E., AND KUMAR, N. (p, p)-Galois representations attached to automorphic forms on GL_n. *Pacific J. Math. 252*, 2 (2011), 379–406.

[15] GROTHENDIECK, A. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I. *Inst. Hautes Études Sci. Publ. Math.*, 20 (1964), 259.

[16] HAZEWINKEL, M. *Formal groups and applications*, vol. 78 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[17] HERR, L. Sur la cohomologie galoisienne des corps p-adiques. *Bull. Soc. Math. France 126*, 4 (1998), 563–600.

[18] HERR, L. Φ-Γ-modules and Galois cohomology. In *Invitation to higher local fields (Münster, 1999)*, vol. 3 of *Geom. Topol. Monogr.* Geom. Topol. Publ., Coventry, 2000, pp. 263–272.

[19] HIDA, H. *Geometric modular forms and elliptic curves*, second ed. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[20] KISIN, M., AND REN, W. Galois representations and Lubin-Tate groups. *Doc. Math. 14* (2009), 441–461.

[21] LANG, S. *Algebra*, third ed., vol. 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

[22] MATSUMURA, H. *Commutative ring theory*, second ed., vol. 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1989. Translated from the Japanese by M. Reid.

[23] NEUKIRCH, J. *Class field theory*, vol. 280 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1986.

[24] NEUKIRCH, J. *Algebraic number theory*, vol. 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[25] NEUKIRCH, J., SCHMIDT, A., AND WINGBERG, K. *Cohomology of number fields*, second ed., vol. 323 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2008.

[26] PAL, A., AND ZÁBRÁDI, G. Cohomology and overconvergence for representations of powers of galois groups. *Journal of the Institute of Mathematics of Jussieu*, 1–61.

[27] ROTMAN, J. J. *An introduction to homological algebra*, second ed. Universitext. Springer, New York, 2009.

[28] SCHNEIDER, P. *Galois Representations and* (φ, Γ)*-Modules*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2017.

[29] SCHNEIDER, P., AND VENJAKOB, O. Coates-Wiles homomorphisms and Iwasawa cohomology for Lubin-Tate extensions. In *Elliptic curves, modular forms and Iwasawa theory*, vol. 188 of *Springer Proc. Math. Stat.* Springer, Cham, 2016, pp. 401–468.

[30] SERRE, J.-P. *Cohomologie galoisienne*, fifth ed., vol. 5 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.

[31] SILVERMAN, J. H. *The arithmetic of elliptic curves*, second ed., vol. 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2009.

[32] TATE, J. Duality theorems in Galois cohomology over number fields. In *Proc. Internat. Congr. Mathematicians (Stockholm, 1962)* (1963), Inst. Mittag-Leffler, Djursholm, pp. 288–295.

[33] TATE, J. Relations between K_2 and Galois cohomology. *Invent. Math. 36* (1976), 257–274.

[34] TAVARES RIBEIRO, F. An explicit formula for the Hilbert symbol of a formal group. *Ann. Inst. Fourier (Grenoble) 61*, 1 (2011), 261–318.

[35] WEIBEL, C. A. *An introduction to homological algebra*, vol. 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[36] WINTENBERGER, J.-P. Le corps des normes de certaines extensions infinies de corps locaux; applications. *Ann. Sci. École Norm. Sup. (4) 16*, 1 (1983), 59–89.