Classification of Pairs of Quaternionic Hyperbolic Isometries

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A thesis submitted for the partial fulfilment of the degree of Doctor of Philosophy

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April 2019

Dedicated

to

Bai and Namrata

Declaration

The work presented in this thesis has been carried out by me under the guidance of Dr. Krishnendu Gongopadhyay at Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Date:

Place:

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In my capacity as the supervisor of the candidate's thesis work, I certify that theabove statements by the candidate are true to the best of my knowledge.

Dr. Krishnendu Gongopadhyay

(Supervisor)

Acknowledgements

First and foremost, I would like to express my gratitude to my Ph.D. supervisor Dr. Krishnendu Gongopadhyay for accepting me as his Ph.D. student. He is an inspirational source for doing research in mathematics. During the course of my research, he was ever present to discuss doubts and ideas. Due to his friendly demeanor, I never hesitated to ask even simple questions. He always motivated me to discuss with mathematics with the subject experts I met in various conferences and workshops. Krishnendu always encouraged me to think and develop new ideas. From Krishnendu, I not only learned mathematics but also a positive attitude towards life. He is a good human being who genuinely cares about his student. I am very much indebted to him.

I would like to thank Dr. Pranab Sardar and Dr. Chanchal Kumar, who are in the research advisory committee. I have learned a lot of mathematics from Pranab Sir. He always encouraged me to give talks and coordinate seminars. Because of his initiative, we have a very active Geometric Group Theory (GGT) group, which provided me a platform to present my work and discuss new topics in and around Geometry and Topology. Dr. Chanchal Kumar built the foundation of Algebra during course work and for this, I am thankful to him. I owe my understanding of mathematics to many mathematicians at IISER Mohali, especially Prof. Kapil Hari Paranjape and Prof. I.B.S. Passi. During the initial period of Ph.D., I had many mathematical discussions with Prof. Kapil.

My life changed when I cleared the IIT-JAM exam. I'm from a rural background with very little educational opportunity, so I never thought that I would get a chance to study in an IIT. The credit of my admission to IIT Madras goes to Mr. Sidhharth Bhalerao and Dr. V.M. Sholapurkar of S.P. Collge, Pune. They guided my mathematical development in the right direction. I must express my gratitude to all the faculty members from S.P.College, Pune. I am also thankful to the faculty members of IIT Madras, especially Dr. Venkata Balaji T.E. and Prof. Ponnusamy.

I am grateful to the Students Welfare Association (SWA), Pune for providing me affordable accommodation and food during my B.Sc. SWA played an important role in my character building. It provided me the opportunity to interact with many altruistic and accomplished people and this helped me become a better human being. SWA also provided me very good friends, one of them being Sanjay who motivated me to do mathematics. I'm much indebted to Disha Pariwar Charitable Trust, Pune for providing me with economical support during B.Sc and M.Sc. Mr.Rajabhau Chavan, founder of Disha Pariwar always encouraged me to pursue higher education. I am grateful to all the donors of Disha for their faith in me.

Thanks are also due to UGC, Government of India for fully funding my Ph.D., and also to LMS-CMI for providing the grant to attend a workshop at Warwick University, England. I thank the administrative system of IISER Mohali for supporting me with the excellent infrastructure and facilities.

I must express my deepest gratitude to my parents for providing constant support and encouragement throughout my Ph.D. They gave me the freedom to take my own decisions. They stood by me during my rough patches. My mother's life gave me the strength to face difficulties in life. She works very hard to support my dreams and happiness. I feel lucky to have such parents and a Ph.D. guide in my life. I miss my grandma for her caring and loving nature. I mention special thanks to Namrata, one of my cousin for her constant support throughout my Ph.D. She taught me moral values and ethics of life. She is one of my inspirations. She always pointed me in the right direction in the journey of my life. I am thankful to Dada and Kaki for there constant support in my education. I am thankful to Kaku for her caring nature. I am thankful to my brothers and sisters who always supported me: Santosh, Vishal, Asha, Bhavana, Punam, Akshada, Nilam.

At IISER Mohali, I found a very good circle of friends. Dr. Abhay Soman, Dr. Sushil Bhunia, Dr. Soumya Dey, Kedar, Pinka, Jitendra, Swathi, Amit, and Jithin made the mathematical as well as non-mathematical part of my life in Mohali joyful. I am thankful to all of them for fruitful discussions. Especially, I am thankful to Swathi for correcting my English grammar in the thesis. Our frequent badminton matches were fun and also gave me the boost of energy to do my Ph.D. work. I always miss the company of friends from IIT Madras, especially Prabhat, Prachi, Abhyendra, Pawan, Aastha, Pal Pandi and Koushik, where we had fun and learned a lot of new things together. I am thankful to Suresh, Abhijit Ghanwat and Vishal Ghule for there abiding friendship and support. Last but not least, I would like to thank my wife, Reshma for her constant encouragement and her patience.

Kalane Sagar Balasaheb

Abstract

We consider the Lie groups $\mathrm{SU}(n,1)$ and $\mathrm{Sp}(n,1)$ that act as isometries of the complex and the quaternionic hyperbolic spaces respectively. We classify pairs of semisimple elements in $\mathrm{Sp}(n,1)$ and $\mathrm{SU}(n,1)$ up to conjugacy. This gives local parametrization of the representations ρ in $\mathrm{Hom}(\mathrm{F}_2, G)/G$ such that both $\rho(x)$ and $\rho(y)$ are semisimple elements in G, where $\mathrm{F}_2 = \langle x, y\rangle$, $G = \mathrm{Sp}(n,1)$ or $\mathrm{SU}(n,1)$. We use the $\mathrm{PSp}(n,1)$-configuration space $\mathrm{M}(n, i, m-i)$ of ordered m-tuples of distinct points in $\overline{\mathbf{H}^n_{\mathbb{H}}}$, where the first i points in an m-tuple are boundary points, to classify the semisimple pairs.

Further, we also classify points on $\mathrm{M}(n, i, m-i)$. Particularly interesting coordinates occur for lower values of n. The conjugacy classification of pairs is then applied geometrically to obtain Quaternionic hyperbolic Fenchel-Nielsen type parameters for generic representations of surface groups into $\mathrm{Sp}(2,1)$ and $\mathrm{Sp}(1,1)$.

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CHAPTER 1

Summary of the thesis

A problem of potential interest is to understand the geometry and topology of the deformation space, or the conjugation orbit space $\mathfrak{D}(\Gamma, G) = \mathrm{Hom}(\Gamma, G)/G$, where $\Gamma = \pi_1(\Sigma_g)$ and G is the isometry group of a rank one symmetric space of non-compact type. Here G acts on $\mathrm{Hom}(\Gamma, G)$ by conjugation via inner automorphisms, Σ_g denotes a closed connected orientable surface of genus $g \geq 2$ and $\pi_1(\Sigma_g)$, or simply π_1, denotes the fundamental group of Σ_g. It is well known that the rank one symmetric spaces of non-compact types are the real, complex, quaternionic hyperbolic spaces and the Cayley plane. The respective isometry groups in the first three types are given by $\mathrm{SO}(n, 1)$, $\mathrm{SU}(n, 1)$ and $\mathrm{Sp}(n, 1)$ respectively. The deformation space is geometrically interesting when G is one of these groups.

When $G = \mathrm{SL}(2, \mathbb{R})$, it acts on the two-dimensional real hyperbolic space by real Möbius transformations. The space $\mathfrak{D}(\pi_1(\Sigma_g), \mathrm{SL}(2, \mathbb{R}))$ contains the classical Teichmüller space as one of its components, and has been studied widely in the literature, though not completely understood even today, for example see the survey [**Gol06**] or the recent work [**MW16**]. When $G = \mathrm{SL}(2, \mathbb{C})$, the deformation space contains the so called quasi-Fuchsian space that is related to Thurston's program on three-manifolds and Kleinian groups, see Marden [**Mar16**]. In the case of $G = \mathrm{SL}(2, \mathbb{R})$, this leads to the Fenchel-Nielsen coordinates on the Teichmüller space, see [**Wol82**]. This was generalized by Kourounitis [**Kou94**] and Tan [**Tan94**] for $G = \mathrm{SL}(2, \mathbb{C})$.

A starting point in all these works is the problem of classifying pairs of hyperbolic elements up to conjugacy. For $G = \mathrm{SL}(2, \mathbb{R})$ or $\mathrm{SL}(2, \mathbb{C})$, it is well-known from the work of Fricke and Vogt that the group generated by a pair of elements is completely classified up to conjugacy by the traces of the generators and the trace of their product, see [**Gol09**]. It is not an easy problem to generalize the work of Fricke and Vogt to higher dimensions, or in other rank one isometry groups. There can be two approaches to handle the problem. In the first approach one uses algebraic invariant theory, and in the other geometric methods driven by underlying geometric structures.

Using invariant theory, there are attempts to classify conjugation orbits of pairs of elements in $\mathrm{SL}(n, \mathbb{C})$ using polynomials involving traces. The work of Procesi **[Pro76]** has given a set of trace coordinates for classifying conjugation orbits of free group representations into $\mathrm{GL}(n, \mathbb{C})$. Procesi's coordinate system can be restricted to $\mathrm{SL}(n, \mathbb{C})$, but a minimal family of such coordinates is known only for lower values of n, see **[Law07, Law08, Dok07]**. Since $\mathrm{SU}(n, 1)$ is a real form of $\mathrm{SL}(n+1, \mathbb{C})$, the pairs of elements in $\mathrm{SU}(n, 1)$ may be associated to certain trace parameters, though may not be minimal. Using the work of Lawton **[Law07]**, Will **[Wil09]** has obtained a set of minimal trace parameters to classify conjugation orbits of F_2 representations in $\mathrm{SU}(2, 1)$, also see Parker **[Par12]**. In an attempt to generalize this work, Gongopadhyay and Lawton **[GL17]** have classified the polystable pairs (that is, the pairs whose conjugation orbits are closed) in $\mathrm{SU}(3, 1)$ using 39 real parameters. At the same time, it has been shown that the real dimension of the smallest possible system of such real parameters to determine any polystable pair is 30. As evident from **[GL17]**, the complexity of the trace parameters increases with n. An explicit set of trace parameters for pairs in $\mathrm{SU}(n, 1)$, $n \geq 4$, is still missing in the literature.

Using geometric methods, there are attempts to classify 'geometric' pairs in rank one isometry groups, mostly $\mathrm{SU}(n, 1)$. Recall that an isometry of the complex or the quaternionic hyperbolic space is called *hyperbolic* if it fixes exactly two points on the boundary. Parker and Platis **[PP08]**, Falbel **[Fal07]** and Cunha and Gusevskii **[CG10]**, independently obtained classifications of the hyperbolic pairs in $\mathrm{SU}(2, 1)$. A common idea in these works is to associate the congruence classes of fixed points of the hyperbolic pairs to a topological space. It follows from these works that the traces of the hyperbolic elements along with a point on the respective topological spaces classify the hyperbolic pairs. Parker and Platis applied their result to construct Fenchel-Nielsen parameters on the complex hyperbolic quasi-Fuchsian space. Falbel and Platis **[FP08]** obtained geometric structures of the space constructed by Falbel. In **[GP17]**, Gongopadhyay and Parsad have generalized the work of Parker and Platis to classify generic hyperbolic pairs in $\mathrm{SU}(3, 1)$, and then to obtain Fenchel-Nielsen type coordinates on a special component of the $\mathrm{SU}(3, 1)$ deformation space of surface group representations. Recently, Gongopadhyay and Parsad have given a geometric classification of the conjugation orbits of the hyperbolic pairs in $\mathrm{SU}(n,1)$. An advantage of the approach in **[GP18a]** is that the complexity for larger n can be handled successfully to provide a classification in arbitrary dimension.

In this thesis, we have classified pairs of semisimple elements in $\mathrm{Sp}(n,1)$, up to conjugacy. In other words, we want to classify elements in the space $\mathfrak{X}_{\mathfrak{ss}}(\mathrm{F}_2, \mathrm{Sp}(n,1))$. Here $\mathfrak{X}_{\mathfrak{ss}}(\mathrm{F}_2, \mathrm{Sp}(n,1))$ is the subset of deformation space $\mathfrak{D}(\mathrm{F}_2, \mathrm{Sp}(n,1))$ consisting of representations ρ such that $\rho(x)$ and $\rho(y)$ are semisimple. Not much is known about the local structure of this space. A key obstruction in generalizing the above mentioned works to pairs in $\mathrm{Sp}(n,1)$ is the lack of conjugacy invariants due to the non-commutativity of the quaternions. Because of this, neither the classical invariant theoretic approach nor the geometric approach has a straight-forward generalization for elements in $\mathfrak{X}_{\mathfrak{ss}}(\mathrm{F}_2, \mathrm{Sp}(n,1))$. We have resolved this difficulty by associating certain spatial invariants, along with the linear algebraic invariants available in this setup. This has given us a local parametrization of the space $\mathfrak{X}_{\mathfrak{ss}}(\mathrm{F}_2, \mathrm{Sp}(n,1))$. As a byproduct, we have classified points in $\mathfrak{X}_{\mathfrak{ss}}(\mathrm{F}_2, \mathrm{SU}(n,1))$. These classifications give us a system of local parameters to points in $\mathfrak{X}_{\mathfrak{ss}}(\mathrm{F}_2, G)$, where $G = \mathrm{Sp}(n,1)$ or $\mathrm{SU}(n,1)$. Particularly interesting coordinates occur for lower values of n. The conjugacy classification of pairs is then applied geometrically to obtain Fenchel-Nielsen type parameters for generic representations of surface groups into $\mathrm{Sp}(2,1)$ and $\mathrm{Sp}(1,1)$. We briefly summarize the thesis in the following sections.

1. Moduli of ordered tuples of distinct points on $\overline{\mathbf{H}_{\mathbb{H}}}^n$

In our understanding of the conjugation orbits of the semisimple pairs, the moduli space of $\mathrm{PSp}(n,1)$-congruence classes of ordered m-tuples of distinct points on $\overline{\mathbf{H}_{\mathbb{H}}}^n$, $m \geq 4$, is used. We project a pair of isometries onto this space to associate the spatial invariants. We classified points on this space that provides some understanding of its topology. The problem to obtain configuration space of ordered tuples of points on a topological space is a problem of independent interest. The general problem may be stated as follows:

Let X be a topological space and G be a group acting diagonally on the ordered m-tuples of points on X: for $p = (p_1, \dots, p_m)$ in X^m, g in G,

$$(g, p) \mapsto (gp_1, gp_2, \dots, gp_m).$$

In general, this is a difficult problem to understand the orbit space X/G under this action. However, there are cases when this can be done using the underlying structure of X. A basic example is a case when X is the circle σ^1 and one considers ordered quadruple of points on σ^1 under the action of the group $\mathrm{SL}(2, \mathbb{R})$ that acts by the Möbius transformations on the circle. In this case, the cross ratios of four points essentially determine the orbit space. When $\mathbb{X}$ is the Riemann sphere and G is the group $\mathrm{SL}(2, \mathbb{C})$ of the Möbius transformations, a similar result also happens.

Let $\mathbf{H}_{\mathbb{C}}^n$ denote the n-dimensional complex hyperbolic space and $\partial\mathbf{H}_{\mathbb{C}}^n$ be its boundary. In this case, $X = \mathbf{H}_{\mathbb{C}}^n \cup \partial\mathbf{H}_{\mathbb{C}}^n$ and $G = \mathrm{SU}(n, 1)$. For $k = 3$, this problem is related to the classification of the congruence classes of triangles, and it was solved by Cartan, e.g. [**Gol99**]. The Cartan's angular invariants determine these classes completely. Another work along this direction was given by Brehm [**Bre90**] who associated shape invariants to such triples. The $\mathrm{SU}(n, 1)$-congruence classes of ordered tuples of points on $\mathbf{H}_{\mathbb{C}}^n \cup \partial\mathbf{H}_{\mathbb{C}}^n$ was obtained by Hakim and Sandler [**HS03**], also see, Brehm and Et-Taoui [**BE98**]. However, neither of these works gave a complete picture of the moduli space. Cunha and Gusevskii completely solved this problem for $\mathrm{SU}(n, 1)$-congruence classes of points on $\partial\mathbf{H}_{\mathbb{C}}^n$ and obtained a clear description of the moduli space in [**CG12b**]. Gusevskii et. al. also obtained the moduli space of $\mathrm{SU}(n, 1)$-congruence classes of points on $\mathbf{H}_{\mathbb{C}}^n$ and on the polar space, see [**CDG12**], [**Gus10**]. There are several works on classifications of the $\mathrm{SU}(2, 1)$-congruence classes of quadruples of distinct points on $\partial\mathbf{H}_{\mathbb{C}}^2$, see [**Fal07**], [**CG10**], [**PP08**]. All these works are independent of each other and have used different approaches.

The above works motivate the same problem when $X = \mathbf{H}_{\mathbb{H}}^n \cup \partial\mathbf{H}_{\mathbb{H}}^n$ and $G = \mathrm{Sp}(n, 1)$. Here $\mathbf{H}_{\mathbb{H}}^n$ denote the n-dimensional quaternionic hyperbolic space and $\partial\mathbf{H}_{\mathbb{H}}^n$ be its boundary. The case $k = 3$, in this case, follow from the work of Apanasov and Kim [**AK07**], who used angular invariants similarly as in the complex hyperbolic case. Recently, the $\mathrm{Sp}(2, 1)$-congruence classes of quadruples of points on $\partial\mathbf{H}_{\mathbb{H}}^n$ has been classified by Cao [**Cao16**]. This classification has been applied by us in [**GK18**]. There have been several recent works to obtain the moduli space of $\mathrm{Sp}(n, 1)$-congruence classes of ordered k-tuples on $\partial\mathbf{H}_{\mathbb{H}}^n$, see [**Cao17**], [**GJ17**].

We generalized the above works to classify points on $\mathrm{M}(n,i,m-i)$. Here $\mathrm{M}(n,i,m-i)$ is the space of $\mathrm{PSp}(n,1)$-congruence classes of ordered m-tuples of distinct points on $\overline{\mathbf{H}_{\mathbb{H}}}^n$, where the first i elements in the m-tuples belong to $\partial\mathbf{H}_{\mathbb{H}}^n$, and the remaining others are from $\mathbf{H}_{\mathbb{H}}^n$. We use the same framework following the work of Brehm and Et-Taoui in **[BET01]**, or Höfer **[Höf99]**, using Gram matrices. Let $p=(p_1,p_2,\dots,p_m)$ be an ordered m-tuple of distinct points in $\overline{\mathbf{H}_{\mathbb{H}}^n}$. The Gram matrix associated to p is the matrix $G=(g_{ij})$, denoted by $G(\mathfrak{p})$, where $g_{ij}=\langle \mathbf{p}_j,\mathbf{p}_i\rangle$ with $\mathfrak{p}=(\mathbf{p}_1,\mathbf{p}_2,\dots,\mathbf{p}_m)$ is a chosen lift of p. The Gram matrix associated to p depends on the chosen lift of p. We associate certain numerical invariants to the congruent classes of gram matrices in order to describe the points on $\mathrm{M}(n,i,m-i)$. Let $p=(p_1,\dots,p_m)$ be an ordered m-tuple of points on $\overline{\mathbf{H}_{\mathbb{H}}^n}$ such that first i elements are from $\partial\mathbf{H}_{\mathbb{H}}^n$ and the remaining ones from $\mathbf{H}_{\mathbb{H}}^n$. Let $G(p)=(g_{ij})$ be the Gram matrix associated to p. We associate the following invariants to p.

Cross-ratios: Given an ordered quadruple of pairwise distinct points (z_1,z_2,z_3,z_4) on $\mathbf{H}_{\mathbb{H}}^n \cup \partial\mathbf{H}_{\mathbb{H}}^n$, their Korányi-Reimann quaternionic cross ratio is defined by

$$\mathbb{X}(z_1,z_2,z_3,z_4)=[z_1,z_2,z_3,z_4]=\langle \mathbf{z}_3,\mathbf{z}_1\rangle\langle \mathbf{z}_3,\mathbf{z_2}\rangle^{-1}\langle \mathbf{z}_4,\mathbf{z}_2\rangle\langle \mathbf{z}_4,\mathbf{z}_1\rangle^{-1},$$

where, for $i=1,2,3,4$, $\mathbf{z}_i$ is a lift of z_i. We associate cross ratios to $p=(p_1,\dots,p_m)$ as follows:

$$\mathbb{X}_{1r}=\mathbb{X}(p_2,p_1,p_3,p_r),\ \mathbb{X}_{2s}=\mathbb{X}(p_1,p_2,p_3,p_s),$$

$$\mathbb{X}_{3s}=\mathbb{X}(p_1,p_3,p_2,p_s),\ \mathbb{X}_{ks}=\mathbb{X}(p_1,p_k,p_2,p_s),$$

for $(i+1)\le r\le m,\ 4\le s\le m,\ 4\le k\le i,\ k<s.$

A simple count shows that there are a total d number of cross ratios in the above list, where $d=\frac{i(i-3)}{2}+i(m-i)$ with i is the number of null points in p. For simplicity of notation, we shall denote them by $(\mathbb{X}_1,\dots,\mathbb{X}_d)$ unless otherwise required.

Distance Invariants: Let p_i and p_j be two distinct negative points in $\mathbf{H}_{\mathbb{H}}^n$. We define distance invariant d_{ij} by: $d_{ij}=\dfrac{\langle \mathbf{p}_j,\mathbf{p}_i\rangle\langle \mathbf{p}_i,\mathbf{p}_j\rangle}{\langle \mathbf{p}_j,\mathbf{p}_j\rangle\langle \mathbf{p}_i,\mathbf{p}_i\rangle}$. The quantity d_{ij} is $\mathrm{PSp}(n,1)$ invariant and it is independent of the chosen lifts of the points.

Angular invariants: The quaternionic Cartan's angular invariant associated to a triple (z_1, z_2, z_3) on $\mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$ is given by the following, see **[AK07]**, **[Cao16]**,**[Cao17]**,

$$\mathbb{A}(z_1,\ z_2,\ z_3) = \arccos \frac{\Re(-\langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\rangle)}{|\langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\rangle|}.$$

where $\langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\rangle = \langle \mathbf{z}_1, \mathbf{z}_2\rangle\langle \mathbf{z}_2, \mathbf{z}_3\rangle\langle \mathbf{z}_3, \mathbf{z}_1\rangle$. We associate angular invariants to p as: $\mathbb{A}_{ij} = \mathbb{A}(\mathbf{p}_1, \mathbf{p}_i, \mathbf{p}_j)$.

Rotation invariants: If $\mathbb{A}_{ij}$ is non-zero, we further associate a numerical invariant u_{ij} given by: $u_{ij} = \frac{\Im(g_{ij})}{|\Im(g_{ij})|}$. If $\mathbb{A}_{ij}$ is zero, then we shall assume $u_{ij} = 0$. The $\mathrm{Sp}(1)$-conjugacy class of u_{ij} is called a *rotation invariant* of p. Here $\mathrm{Sp}(1)$ is the unit sphere of quaternions and isomorphic to $\mathrm{SU}(2)$. For simplicity of notation, we shall denote them as $u_0, u_1, u_2, \ldots, u_t$, with the understanding that u_i denotes only non-zero rotation invariant for $1 \le i \le t$ and $u_0 = u_{23}$.

With the above notions, we have the following.

THEOREM 1.1. **[GK19**, Theorem 1.4**]** *A point $[p]$ in $\mathrm{M}(n, i, m-i)$, $p = (p_1, \ldots, p_m)$, is determined by the $\mathrm{Sp}(1)$ congruence class of the $(d+t+1)$-tuple*

$$W = (u_0, u_1, \ldots, u_t, \mathbb{X}_1, \ldots, \mathbb{X}_d),\ d = \frac{i(i-3)}{2} + i(m-i),\ m \ge 4,\ t = \frac{(m-i)^2 - (m-i)}{2} - l,$$

the angular invariants $\mathbb{A}_{23}$, $\mathbb{A}_{i_1 j_1}$ and the distance invariants $d_{i_1 j_1}$, for $i < i_1, j_1 \le m$, where l is the number of zero valued rotation invariants u_{ij}.

We have rotation invariants of the cross ratios

$$\eta_i = \frac{\Im(\mathbb{X}_i)}{|\Im(\mathbb{X}_i)|}.$$

If $\mathbb{X}_i$ is a real number for some i, we assume $\eta_i = 0$.

Let $p = (p_1, \ldots, p_m)$ be such that $\mathbb{A}(p_1, p_2, p_3) \neq 0$. Note that the $\mathrm{Sp}(1)$-congruence class of the ordered tuple $F = (u_0, u_1, \ldots, u_t, \mathbb{X}_1, \ldots, \mathbb{X}_d)$ is associated to the $\mathrm{Sp}(1)$-congruence class of the ordered tuple $\mathfrak{n} = (u_0, u_1, \ldots, u_t, \eta_1, \ldots, \eta_k)$ of points on σ^2, where k is the number of non-real cross ratios. Hence the above theorem can be restated in the following form.

COROLLARY 1.2. **[GK19**, Corollary 1.5] *A point* $[p]$ *in* $\mathrm{M}(n,i,m-i)$, $p=(p_1,\dots,p_m)$, *is determined by the angular invariants, the distance invariants, and a point on the* $\mathrm{Sp}(1)$ *configuration space of ordered* $(k+t+1)$ *tuple of points on* σ^2, *where* k *is the number of non-real (similarity classes of) cross ratios, and* $t+1$ *is the number of non-zero angular invariants.*

The description of the $\mathrm{Sp}(1)$-configuration space of ordered tuples of points can be obtained from the work **[BET01]**. Restricting to the special cases of the boundary points and the points on $\mathbf{H}_{\mathbb{H}}^n$ respectively, the above theorem gives the following.

COROLLARY 1.3. **[GK19**, Corollary 1.6] *A point* $[p]$ *in* $\mathrm{M}(n,m,0)$, $p=(p_1,\dots,p_m)$, *is determined by the* $\mathrm{Sp}(1)$ *congruence class of the* $(d+1)$*-tuple* $F=(u_0,\mathbb{X}_1,\mathbb{X}_2,\dots,\mathbb{X}_d)$, $d=\frac{m(m-3)}{2}$, $m\geq 4$, *and the angular invariant* $\mathbb{A}_{23}$.

COROLLARY 1.4. **[GK19**, Corollary 1.7] *A point* $[p]$ *in* $\mathrm{M}(n,0,m)$, $p=(p_1,\dots,p_m)$, *is determined by the angular invariants, distance invariants and* $\mathrm{Sp}(1)$ *congruence class of the unit pure quaternions associated to* p.

Let $\mathrm{M}_c(n,i,m-i)$ denote the $\mathrm{SU}(n,1)$-configuration space of ordered tuples of points on $\overline{\mathbf{H}_{\mathbb{C}}}^n$. As an application of the above theorem, we obtain a classification of points on $\mathrm{M}_c(n,i,m-i)$. Over the complex numbers, conjugacy invariants like the traces and the cross ratios are well-defined. Accordingly, it is much simpler to classify the points on the space $\mathrm{M}_c(n,i,m-i)$. The following corollary is an extension of Theorem 3.1 of Cunha and Gusevskii in **[CG12b]**.

COROLLARY 1.5. **[GK19**, Corollary 1.8] *A point* $[p]$ *in* $\mathrm{M}_c(n,i,m-i)$, $p=(p_1,\dots,p_m)$, *is determined by the complex cross ratios* $\mathbb{X}_1,\dots,\mathbb{X}_d,\ d=\frac{i(i-3)}{2}+i(m-i),\ m\geq 4$, *the angular invariants* $\mathbb{A}_{23}$, $\mathbb{A}_{i_1j_1}$, *and the distance invariants* $d_{i_1j_1}$, *for* $i<i_1,j_1\leq m$.

Using these results, it is not hard to obtain an explicit description of $\mathrm{M}(n,i,m-i)$ following similar arguments as in **[CG12b]** for the complex case and **[Cao17]** in the quaternion case, also see **[GJ17]**.

2. Conjugation orbits of semisimple pairs in rank one

The group $\mathrm{Sp}(n,1)$ acts on the n-dimensional quaternionic hyperbolic space $\mathbf{H}_{\mathbb{H}}^n$ by isometries. Elements of $\mathrm{Sp}(n,1)$ are $(n+1)\times(n+1)$ matrices over the division ring $\mathbb{H}$ of quaternions preserving a quaternionic Hermitian form of signature $(n,1)$. Given an element g in $\mathrm{Sp}(n,1)$, it has a representation $g_{\mathbb{C}}$ in $\mathrm{GL}(2n+2,\mathbb{C})$. The coefficients of the characteristic polynomials of $g_{\mathbb{C}}$ are certain conjugacy invariants of g and the collection of such coefficients is called the *real trace* of g. The real traces serve as a set of conjugacy invariants that may be associated to a pair.

Along with these real traces, another idea in our approach is to associate tuple of points on $\overline{\mathbf{H}_{\mathbb{H}}}^n = \mathbf{H}_{\mathbb{H}}^n \cup \partial\mathbf{H}_{\mathbb{H}}^n$ to a semisimple pair $(\rho(x),\rho(y))$, and then project the $\mathrm{Sp}(n,1)$-conjugation orbit of the pair to the moduli space $\mathrm{M}(n,i,m-i)$. Intuitively, the semisimple pairs are seen here as equivalence classes of 'moving frames'. To each pair, we associate points coming from the closure of the totally geodesic quaternionic lines given by these 'frames'. This association is not well-defined. However, given a semisimple pair, the orbit of such points under the group action induced by the change of eigenframes gives a well-defined association, and we denote the space of such orbits by $\mathcal{QL}_n$. A point on this space that corresponds to a given semisimple pair is called the *canonical orbit* of the pair. This space has a topological structure that comes from the topological structure of the moduli space $\mathrm{M}(n,i,m-i)$.

However, the canonical orbits along with the real traces do not give the complete set of invariants that classify the pairs. To complete the classification, we associate certain spatial invariants to the semisimple pairs and is a crucial ingredient in the classification. Let T be a semisimple element in $\mathrm{Sp}(n,1)$. Let $\lambda \in \mathbb{H}\setminus\mathbb{R}$ be a chosen eigenvalue representative in the similarity class of eigenvalues $[\lambda]$ of T with multiplicity m, $m \le n$, that is, the eigenspace of $[\lambda]$ can be identified with $\mathbb{H}^m$. The $[\lambda]$-eigenspace decomposes into a space of the complex m-dimensional subspaces of $\mathbb{H}^m$, that may be identified with the complex Grassmannian manifold $G_{m,2m}$. We call it the *eigenvalue Grassmannian* of T corresponding to the eigenvalue class $[\lambda]$. Each point on this Grassmannian corresponds to an 'eigenset' of $[\lambda]$. If $m=1$, a point on eigenvalue Grassmannian is called a *projective point* of T corresponding to the eigenvalue class $[\lambda]$.

With the above notions, one of the main results of thesis is the following.

THEOREM 1.6. **[GK19**, Theorem 1.3] *Let* (A, B) *be a semisimple pair in* $\mathrm{Sp}(n, 1)$ *such that* A *and* B *do not have a common fixed point. Then* (A, B) *is determined up to conjugation in* $\mathrm{Sp}(n, 1)$*, by the real traces* $\mathrm{tr}_{\mathbb{R}}(A)$*,* $\mathrm{tr}_{\mathbb{R}}(B)$*, the canonical orbit of* (A, B) *on* $\mathcal{QL}_n$*, and a point on each of the eigenvalue Grassmannians of* A *and* B*.*

An immediate corollary is the following.

COROLLARY 1.7. **[GK19**, Corollary 1.2] *Let* ρ *be an element in* $\mathfrak{X}_{\mathfrak{ss}}(\mathrm{F}_2, \mathrm{Sp}(n, 1))$*,* $\mathrm{F}_2 = \langle x, y\rangle$*. Then* ρ *is determined uniquely by* $\mathrm{tr}_{\mathbb{R}}(\rho(x))$*,* $\mathrm{tr}_{\mathbb{R}}(\rho(y))$*, the canonical orbit of* $(\rho(x), \rho(y))$ *on* $\mathcal{QL}_n$*, and a point on each of the eigenvalue Grassmannians of* $\rho(x)$ *and* $\rho(y)$*.*

The methods that we have followed to establish the above results, also carry over to the case of a pair of semisimple elements in $\mathrm{SU}(n, 1)$. For elements of $\mathrm{SU}(n, 1)$, the underlying hyperbolic space is defined over the complex numbers, and hence there is no ambiguity regarding the conjugacy invariants. The coefficients of the characteristic polynomials serve as well-defined conjugacy invariants for individual elements. Geometric invariants like the cross-ratios are also well-defined. Accordingly, we have the following special case of Theorem 4.11. This was proved for the hyperbolic pairs in **[GP17]**. The following is an extension of **[GP17**, Theorem 1.1] to semisimple pairs. Here $\mathrm{Tr}(A)$ denote the usual trace of a complex matrix.

COROLLARY 1.8. **[GK19**, Corollary 1.3] *Let* ρ *be an element in* $\mathfrak{X}_{\mathfrak{ss}}(\mathrm{F}_2, \mathrm{SU}(n, 1))$*,* $\mathrm{F}_2 = \langle x, y\rangle$*. Then* ρ *is determined uniquely by* $\mathrm{Tr}(\rho(x)^i)$*,* $\mathrm{Tr}(\rho(y)^i)$*,* $1 \le i \le \lfloor (n + 1)/2\rfloor$*, and the canonical orbit of* $(\rho(x), \rho(y))$*.*

For lower values of n, we can get a much better co-ordinate system that is minimal and the degrees of freedom adds up to the dimension of the group. We summarize these coordinates in the following sections.

3. Pairs of hyperbolic elements in $\mathrm{Sp}(2,1)$

A hyperbolic element A in $\mathrm{Sp}(2,1)$ will be called *loxodromic* if the similarity classes of eigenvalues of A have representatives in the set $\mathbb{H} \setminus \mathbb{R}$. If all the eigenvalues of A are real numbers, it is called *strictly hyperbolic*.

One of the main results of this thesis is a classification of pairs of hyperbolic elements in $\mathrm{Sp}(2,1)$. A pair (A, B) in $\mathrm{Sp}(2,1)$ is called *totally loxodromic* if both A and B are loxodromic. It is called *strictly hyperbolic* if both A and B are strictly hyperbolic. Recall that a pair (A, B) of $\mathrm{Sp}(2,1)$ is *irreducible* if it neither fixes a point nor preserves a proper totally geodesic subspace of $\mathbf{H}^2_{\mathbb{H}}$. The subset of $\mathfrak{D}(\mathrm{F}_2, \mathrm{Sp}(2,1))$ consisting of irreducible representations ρ such that both $\rho(x)$ and $\rho(y)$ are loxodromic, is denoted by $\mathfrak{D}_{\mathfrak{L}o}(\mathrm{F}_2, \mathrm{Sp}(2,1))$.

Let (A, B) be a pair in $\mathrm{Sp}(2,1)$ such that both A and B are hyperbolic elements. For a hyperbolic element A in $\mathrm{Sp}(2,1)$, its fixed points are denoted by a_A and r_A. There is a point $p = (a_A, r_A, a_B, r_B)$ associated to (A, B) in the space $\mathrm{M}(2,4,0)$. There are numerical invariants like the quaternionic cross ratios and the angular invariants associated to this point p. These invariants are defined as follows.

Let A and B be two loxodromic elements in $\mathrm{Sp}(2,1)$. The fixed points of A and B on $\mathbb{HP}^2 \setminus (\mathbf{H}^2_{\mathbb{H}} \cup \partial \mathbf{H}^2_{\mathbb{H}})$ are denoted by x_A and x_B respectively. Let $\mathbf{a}_A$, $\mathbf{r}_A$, $\mathbf{x}_A$ denote the lifts of a_A, r_A, x_A. For the pair (A, B) we define the following invariants.

The cross ratios of (A, B).

$$\mathbb{X}_1(A,B) = \mathbb{X}(a_A, r_A, a_B, r_B),\ \mathbb{X}_2(A,B) = \mathbb{X}(a_A, r_B, a_B, r_A),$$

$$\mathbb{X}_3(A,B) = \mathbb{X}(r_A, r_B, a_B, a_A).$$

The angular invariants of (A, B).

$$\mathbb{A}_1(A,B) = \mathbb{A}(a_A, r_A, a_B),\ \mathbb{A}_2(A,B) = \mathbb{A}(a_A, r_A, r_B),\ \mathbb{A}_3(A,B) = \mathbb{A}(r_A, a_B, r_B).$$

Recently, Cao has proved in **[Cao16]** that these invariants completely determined the $\mathrm{Sp}(2,1)$-congruence class. Cao has also obtained a topological description of the space $\mathrm{M}(2,4,0)$ in **[Cao16]**. The space $\mathrm{M}(2,4,0)$ is locally parametrized by seven real parameters consisting of a point on the four (real) dimensional 'cross ratio variety' and three real angular invariants.

However, the real traces of A and B, along with the cross ratios and angular invariants of p, are not enough to classify the pair (A, B) up to conjugacy by $\mathrm{Sp}(2,1)$. A main challenge in the quaternionic set up is to look for the 'missing' invariants to classify a pair. Here, we associate certain spatial invariants, called the *projective points* of an isometry, to determine the $\mathrm{Sp}(2,1)$ conjugation orbit of (A, B). These invariants arise naturally from an understanding of the eigenvalue classes of an isometry. It is not needed to associate projective point to a real eigenvalue of a hyperbolic element. A choice of projective points, along with 'attracting' and 'repelling' fixed points, and real trace, determine a hyperbolic element in $\mathrm{Sp}(2,1)$. We prove the following.

THEOREM 1.9. **[GK18**, Theorem 1.1] *Let (A, B) be a totally loxodromic pair in $\mathrm{Sp}(2,1)$. Then (A, B) is determined up to conjugation by the following parameters: $tr_{\mathbb{R}}(A)$, $tr_{\mathbb{R}}(B)$, the similarity classes of cross ratios of (A, B) or, a point on the four-dimensional cross ratio variety, the three angular invariants of (A, B), and the projective points $(p_1(A), p_2(A))$, $(p_1(B), p_2(B))$.*

Alternatively, associate to (A, B) an ordered quadruple of its fixed points. This gives us the following version of the above theorem.

THEOREM 1.10. **[GK18**, Theorem 1.2] *Let (A, B) be a totally loxodromic pair in $\mathrm{Sp}(2,1)$. Then (A, B) is uniquely determined up to conjugation by the following parameters: $tr_{\mathbb{R}}(A)$, $tr_{\mathbb{R}}(B)$, a point on $\mathrm{M}(2,4,0)$ corresponding to (a_A, r_A, a_B, r_B) and the projective points correspoding to A and B, i.e., $(p_1(A), p_2(A))$, $(p_1(B), p_2(B))$.*

We note that the above theorems are true for any pairs of loxodromics in $\mathrm{Sp}(2,1)$. However, when the pair (A, B) is irreducible and totally loxodromic, the degrees of freedom of the parameters add up to 21 (for each real traces 3 contributing $2 \times 3 = 6$, for the point on the cross ratio variety 4, for three angular invariants 3 and for the four projective points $8 = 2 \times (4$ projective points)), which is the same as the dimension of the group $\mathrm{Sp}(2,1)$. When the pair (A, B) is reducible, the parameter system will not be minimal and the degrees of freedom will be lesser than 21.

There are four types of hyperbolic elements in $\mathrm{Sp}(2,1)$, which depends on a number of real eigenvalues. Depending on the types of A and B, there are ten types of mixed hyperbolic pairs. Except for the totally loxodromic pairs, the degrees of freedom of the parameter systems of the other types, are always strictly lesser than 21, even if the pair is irreducible. A reason for these lesser degrees of freedom is the presence of real eigenvalues that commute with every other quaternionic number. Accordingly, we would not require projective points for the real eigenvalues present in the mixed pair. For each of these cases, the parameter system can be seen using similar arguments used in the proof of the above theorem. We only mention the following two cases.

COROLLARY 1.11. **[GK18**, Corollary 1.3] *Let (A,B) be a mixed hyperbolic pair with A is loxodromic and B is strictly hyperbolic in $\mathrm{Sp}(2,1)$. Then (A,B) is uniquely determined up to conjugation by the parameters: $tr_{\mathbb{R}}(A)$, $tr_{\mathbb{R}}(B)$, a point on the cross ratio variety, the three angular invariants and projective points $p_1(A)$, $p_2(A)$.*

Noting that for a strictly hyperbolic element A in $\mathrm{Sp}(2,1)$, $tr_{\mathbb{R}}(A)$ belongs to a one real parameter family, the degrees of freedom of the associated parameters for an irreducible pair (A,B) in the above corollary is 15, and in the following case, it is 9.

COROLLARY 1.12. **[GK18**, Corollary 1.4] *Let (A,B) be a strictly hyperbolic pair in $\mathrm{Sp}(2,1)$. Then (A,B) is uniquely determined up to conjugation by the parameters: $tr_{\mathbb{R}}(A)$, $tr_{\mathbb{R}}(B)$, a point on the cross ratio variety and three angular invariants.*

As an application of Theorem 5.6, we have a parametric description of the subset $\mathfrak{D}_{\mathfrak{L}o}(\mathrm{F}_2,\mathrm{Sp}(2,1))$ of the deformation space consisting of irreducible totally loxodromic representations.

COROLLARY 1.13. **[GK18**, Corollary 1.5] *The space $\mathfrak{D}_{\mathfrak{L}o}(\mathrm{F}_2,\mathrm{Sp}(2,1))$ is parametrized by a $(\mathbb{CP}^1)^4$ bundle over the topological space $\mathrm{D}_2\times\mathrm{D}_2\times\mathcal{M}(2)$ where, for $G=27(c-a)-9ab+2a^3$, $H=3(b-3)-a^2$,*

$$\mathrm{D}_2=\{(a,b,c)\in\mathbb{R}^3 \mid G^2+4H^3>0,\ |2a+c|\neq|2b+2|\},$$

and $\mathcal{M}(2)$ is the orbit space of the natural S_4 action on the configuration space of $\mathrm{Sp}(2,1)$-congruence classes of ordered quadruples of distinct points on $\partial\mathbf{H}^2_{\mathbb{H}}$.

3.1. Quaternionic hyperbolic Fenchel-Nielsen coordinates. Another application of Theorem 5.6 gives us local parametrization of generic elements in $\mathfrak{D}(\pi_1(\Sigma_g), \mathrm{Sp}(2,1))$. Let Σ_g be a closed, connected, orientable surface of genus $g \geq 2$. Let $\pi_1(\Sigma_g)$ be the fundamental group of Σ_g. Specify a curve system $\mathcal{C}$ of $3g-3$ closed curves γ_j on Σ_g. The complement of such curve system decomposes the surface into $2g-2$ three-holed spheres P_i. Let $\rho : \pi_1(\Sigma_g) \to \mathrm{Sp}(2,1)$ be a discrete, faithful representation such that the image of each γ_j is loxodromic. For each i, the fundamental group of P_i gives a representation φ_i in $\mathfrak{D}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{Sp}(2,1))$, induced by ρ, such that the image of φ_i is a $(0,3)$ subgroup of $\mathrm{Sp}(2,1)$. If each of these representations φ_i is irreducible, we call the representation ρ as *geometric*. We recall that a $(0,3)$ subgroup represents the fundamental group of a three-holed sphere, where the generators and their product correspond to the three boundary curves. With this terminology, we have the following.

THEOREM 1.14. [**GK18**, Theorem 1.6] *Let Σ_g be a closed surface of genus g with a curve system $\mathcal{C} = \{\gamma_j\}$, $j = 1, \dots, 3g-3$. Let $\rho : \pi_1(\Sigma_g) \to \mathrm{Sp}(2,1)$ be a geometric representation of the surface group $\pi_1(\Sigma_g)$. Then we need $42g-42$ real parameters to determine ρ in the deformation space $\mathfrak{D}(\pi_1(\Sigma_g), \mathrm{Sp}(2,1))/\mathrm{Sp}(2,1)$.*

The parameters in the above theorem may be thought of as 'Fenchel-Nielsen coordinates' in this setup. Locally this gives us the degrees of freedom that a geometric representation can move. But the complete structure of this space is still not clear to us. It would be interesting to obtain an embedding of the geometric representations into a topological space.

3.2. Reducible pairs. Now, we briefly mention the case when the pair of hyperbolic elements (A, B) is reducible. In this case, the parameters obtained in Theorem 5.6 or the subsequent corollaries, would not be minimal. As in the complex case, e.g. see [**GL17**, Section 4.1], it is possible to further reduce the degrees of freedom. If a hyperbolic pair (A, B) is reducible, it follows from [**CG74**, Proposition 4.2] that the group G generated by A, B, is a product of the form $\mathcal{A} \times \mathcal{B}$, where several possibilities of $\mathcal{A}$ and $\mathcal{B}$ are listed in [**CG74**, p.77]. When the G-invariant totally geodesic subspace is a copy of the real or complex hyperbolic space, parameters to determine them can be obtained easily from the classical Fenchel-Nielsen coordinates, and by the work of Parker and Platis **[PP08]**. Essentially, conjugacy classification of hyperbolic pairs in $\mathrm{Sp}(1,1)$ remains the only case in order to have an understanding of the reducible pairs. For this reason, instead of addressing the reducible cases in detail, we have focused only on the irreducible hyperbolic pairs in $\mathrm{Sp}(1,1)$.

4. Pairs of loxodromic elements in $\mathrm{Sp}(1,1)$

Let $\mathrm{M}(1,4,0)$ denote the configuration space of $\mathrm{Sp}(1,1)$-congruence classes of ordered quadruples of points on $\partial\mathbf{H}^1_{\mathbb{H}}$. We prove the following.

THEOREM 1.15. **[GK18**, Theorem 1.7**]** *Let (A,B) be an irreducible totally loxodromic pair in $\mathrm{Sp}(1,1)$. Then (A,B) is determined up to conjugation by the following parameters: $tr_{\mathbb{R}}(A)$, $tr_{\mathbb{R}}(B)$, a point on $\mathrm{M}(1,4,0)$ corresponding to (a_A, r_A, a_B, r_B), and two projective points $p_1(A)$, $p_1(B)$.*

The above theorem shows that the degrees of freedom of the coordinates required for a irreducible pair add up to 10 (for each real traces 2 contributing $2 \times 2 = 4$, for the point on the cross ratio variety 2 and for the two projective points $2+2$, thus totaling 4+2+4=10, which is the same as the dimension of the Lie group $\mathrm{Sp}(1,1)$.

Further note that, irreducibility is not necessary for the above theorem. But if (A,B) is reducible, the parameter system above is not minimal and the degrees of freedom can be reduced further.

The following corollaries follow immediately from the above theorem.

COROLLARY 1.16. **[GK18**, Corollary 1.8**]** *Let A be a loxodromic and B be a strictly hyperbolic element in $\mathrm{Sp}(1,1)$. Let (A,B) be irreducible. Then (A,B) is determined up to conjugation by the following parameters: $tr_{\mathbb{R}}(A)$, $tr_{\mathbb{R}}(B)$, a point on $\mathrm{M}(1,4,0)$ corresponding to (a_A, r_A, a_B, r_B), and a projective point $p_1(A)$.*

COROLLARY 1.17. **[GK18**, Corollary 1.9**]** *Let $A,\ B$ be strictly hyperbolic elements in $\mathrm{Sp}(1,1)$. Let (A,B) be irreducible. Then (A,B) is determined up to conjugation by the following parameters: $tr_{\mathbb{R}}(A)$, $tr_{\mathbb{R}}(B)$, and a point on $\mathrm{M}(1,4,0)$ corresponding to (a_A, r_A, a_B, r_B).*

The group $\mathrm{Sp}(1,1)$ may also be viewed as the isometry group of the real hyperbolic 4-space $\mathbf{H}^4_{\mathbb{R}}$. The space $\mathfrak{D}_{\mathfrak{L}o}(\mathrm{F}_2, \mathrm{Sp}(1,1))$ is of importance to understand four-dimensional real hyperbolic geometry as well. In **[TWZ12]**, Tan et. al. proved a counterpart of the classical Delambre-Gauss formula for right-angled hexagons in the real hyperbolic 4-space keeping this deformation space in mind. In this paper we show that $\mathfrak{D}_{\mathfrak{L}o}(\mathrm{F}_2, \mathrm{Sp}(1,1))$ is a $\mathbb{CP}^1 \times \mathbb{CP}^1$ bundle over a topological space that is locally embedded in $\mathbb{R}^6$. As a consequence of this description, it follows that a geometric representation in $\mathfrak{D}(\pi_1(\Sigma_g), \mathrm{Sp}(1,1))$ is determined by $20g - 20$ real parameters.

CHAPTER 2

Preliminaries

1. The quaternions

Let $\mathbb{H}$ denote the division ring of quaternions. Any $q \in \mathbb{H}$ can be uniquely written as $q = r_0 + r_1 i + r_2 j + r_3 k$, where $r_0, r_1, r_2, r_3 \in \mathbb{R}$, and i, j, k satisfy relations: $i^2 = j^2 = k^2 = ijk = -1$. The real number r_0 is called the real part of q and we denote it by $\Re(q)$. The imaginary part of q is $\Im(q) = r_1 i + r_2 j + r_3 k$. The conjugate of q is defined as $\overline{q} = r_0 - r_1 i - r_2 j - r_3 k$. The norm of q is $|q| = \sqrt{r_0^2 + r_1^2 + r_2^2 + r_3^2}$. We identify the sub-algebra $\mathbb{R} + \mathbb{R}i$ with the standard complex plane $\mathbb{C}$.

Two quaternions q_1, q_2 are said to be *similar* if there exists a non-zero quaternion z such that $q_2 = z^{-1} q_1 z$ and we write it as $q_1 \backsim q_2$. It is easy to verify that $q_1 \backsim q_2$ if and only if $\Re(q_1) = \Re(q_2)$ and $|q_1| = |q_2|$. Thus the similarity class of a quaternion q contains a pair of complex conjugates with absolute-value $|q|$ and real part equal to $\Re(q)$. The multiplicative group $\mathbb{H} \setminus 0$ is denoted by $\mathbb{H}^*$.

LEMMA 2.1. *Every quaternionic element has a polar coordinate representation.*

PROOF. Let $a = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ be a quaternion as above. We want to write its polar form. We can have $a = a_0 + v$ where $v = a_1\mathrm{i} + a_2\mathrm{j} + a_3\mathrm{k}$. Now observe that $|a|^2 = {a_0}^2 + |v|^2$, where $|v| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$. So if $a \neq 0$ we get $1 = \big(\frac{a_0}{|a|}\big)^2 + \big(\frac{|v|}{|a|}\big)^2$. Now put $\frac{a_0}{|a|} = \cos(\theta)$ and $\frac{|v|}{|a|} = \sin(\theta)$ where $\theta \in [0, \pi]$. If $v \neq 0$ then we have $a = a_0 + v = |a|\big(\frac{a_0}{|a|} + \frac{|v|}{|a|}\frac{v}{|v|}\big)$. Thus we get $a = |a|(\cos(\theta) + \eta \sin(\theta))$ where $\eta = \frac{v}{|v|}$ *and* $\theta \in [0, \pi]$. If $v = 0$ then $\theta = 0$. $\square$

REMARK 2.2. In Lemma 2.1, $x = \cos(\theta) + \eta \sin(\theta)$ is a unitary quaternion number. Since $x\overline{x} = 1 = \overline{x}x$.

Commuting quaternions. Two non-real quaternions commute if and only if their imaginary parts are scaled by a real number, see **[CG74**, Lemma 1.2.2] for a proof. Let $Z(q)$

denotes the centralizer of $q \in \mathbb{H} \setminus \mathbb{R}$. Then $Z(q) = \mathbb{R} + \mathbb{R}q$. For some non-zero $\alpha \in \mathbb{H}$, $Z(q) = \alpha\mathbb{C}\alpha^{-1}$, where $\mathbb{C} = Z(re^{i\theta})$, $r = |q|$, $\Re(q) = r\cos\theta$. Given a quaternion $q \in \mathbb{H} \setminus \mathbb{R}$, we call $Z(q)$ the *complex line* passing through q. Note that if $q \in \mathbb{H} \setminus \mathbb{R}$, then $Z(\overline{q}^{-1}) = Z(q)$.

1.1. Matrices over quaternions. Let V be a right vector space over $\mathbb{H}$. Let T be a right linear map on V. After choosing a basis of V, such a linear map can be represented by an $n \times n$ matrix M_T over $\mathbb{H}$, where $n = \dim \mathrm{V}$. Thus, one may identify the right linear maps on V with $n \times n$ quaternionic matrices. Here the quaternionic matrices act on V from the left. The map T is invertible if and only if M_T is invertible. The group of all invertible (right) linear maps on V is denoted by $\mathrm{GL}(n, \mathbb{H})$.

Suppose $\lambda \in \mathbb{H}^*$ is a (right) eigenvalue of T, i.e. there exists $v \in \mathrm{V}$ such that $T(v) = v\lambda$. Observe that for $\mu \in \mathbb{H}^*$,

$$T(v\mu) = T(v)\mu = (v\lambda)\mu = (v\mu)\mu^{-1}\lambda\mu.$$

This shows that the eigenvalues of T occur in similarity classes and if v is a λ-eigenvector, then $v\mu \in v\mathbb{H}$ is a $\mu^{-1}\lambda\mu$-eigenvector. Thus the eigenvalues are no more conjugacy invariants of T, but the similarity classes of eigenvalues are conjugacy invariants. Note that each similarity class of eigenvalues contains a unique pair of complex conjugate numbers. We shall choose one of these complex eigenvalues $re^{i\theta}$, $\theta \in [0, \pi]$, to be the representative of its similarity class. Often we shall refer them as 'eigenvalues', though it should be understood that our reference is towards their similarity classes. In places, where we need to distinguish between the similarity class and a representative, we shall denote the similarity class of an eigenvalue representative λ by $[\lambda]$.

For more information on quaternionic linear algebra, we refer to the book **[Rod14]**.

2. Quaternionic hyperbolic space

Let $\mathrm{V} = \mathbb{H}^{n,1}$ be the n dimensional right vector over $\mathbb{H}$ equipped with the Hermitian form of signature $(n, 1)$ given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* H \mathbf{z} = \bar{w}_{n+1} z_1 + \bar{w}_2 z_2 + \cdots + \bar{w}_n z_n + \bar{w}_1 z_{n+1},$$

where $*$ denotes conjugate transpose. The matrix of the Hermitian form is given by

$$H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

where I_{n-1} is the identity matrix of rank $n-1$. We consider the following subspaces of $\mathbb{H}^{n,1}$:

$$\mathrm{V}_{-} = \{\mathbf{z} \in \mathbb{H}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\},\ \mathrm{V}_{+} = \{\mathbf{z} \in \mathbb{H}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle > 0\},$$

$$\mathrm{V}_0 = \{\mathbf{z} - \{\mathbf{0}\} \in \mathbb{H}^{n,1} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\}.$$

A nonzero vector $\mathbf{z}$ in $\mathbb{H}^{n,1}$ is called *positive, negative* or *null* depending on whether $\mathbf{z}$ belongs to V_+, V_- or V_0. Let $\mathbb{P} : \mathbb{H}^{n,1} - \{0\} \longrightarrow \mathbb{HP}^n$ be the right projection onto the quaternionic projective space. Image of a vector $\mathbf{z}$ will be denoted by z. The quaternionic hyperbolic space $\mathbf{H}^n_{\mathbb{H}}$ is defined to be $\mathbb{P}\mathrm{V}_-$. The ideal boundary $\partial\mathbf{H}^n_{\mathbb{H}}$ is defined to be $\mathbb{P}\mathrm{V}_0$. So we can write $\mathbf{H}^n_{\mathbb{H}} = \mathbb{P}(\mathrm{V}_-)$ as

$$\mathbf{H}^n_{\mathbb{H}} = \{(w_1, \dots, w_n) \in \mathbb{H}^n \ : \ 2\Re(w_1) + |w_2|^2 + \cdots + |w_n|^2 < 0\},$$

where for a point $\mathbf{z} = \begin{bmatrix} z_1 & z_2 & \dots & z_{n+1} \end{bmatrix}^T \in \mathrm{V}_- \cup \mathrm{V}_0$, $w_i = z_i z_{n+1}^{-1}$ for $i = 1, \dots, n$. This is the Siegel domain model of $\mathbf{H}^n_{\mathbb{H}}$. Similarly one can define the ball model by replacing H with an equivalent Hermitian form given by the diagonal matrix having one diagonal entry -1 and the other entries 1. We shall mostly use the Siegel domain model here.

There are two distinguished points in V_0 which we denote by $\mathbf{o}$ and ∞, given by

$$\mathbf{o} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},\ \infty = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then we can write $\partial\mathbf{H}^n_{\mathbb{H}} = \mathbb{P}(\mathrm{V}_0)$ as

$$\partial\mathbf{H}^n_{\mathbb{H}} - \infty = \{(z_1, \dots, z_n) \in \mathbb{H}^n : 2\Re(z_1) + |z_2|^2 + \cdots + |z_n|^2 = 0\}.$$

Note that $\overline{\mathbf{H}^n_{\mathbb{H}}} = \mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$.

Given a point z of $\overline{\mathbf{H}^n_{\mathbb{H}}} - \{\infty\} \subset \mathbb{HP}^n$ we may lift $z = (z_1, \dots, z_n)$ to a point $\mathbf{z}$ in V, called the *standard lift* of z. It is represented in projective coordinates by

$$\mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \\ 1 \end{bmatrix}.$$

The Bergman metric in $\mathbf{H}^n_{\mathbb{H}}$ is defined in terms of the Hermitian form given by:

$$ds^2 = -\frac{4}{\langle \mathbf{z}, \mathbf{z} \rangle^2} \det \begin{bmatrix} \langle \mathbf{z}, \mathbf{z} \rangle & \langle d\mathbf{z}, \mathbf{z} \rangle \\ \langle \mathbf{z}, d\mathbf{z} \rangle & \langle d\mathbf{z}, d\mathbf{z} \rangle \end{bmatrix}.$$

If z and w in $\mathbf{H}^n_{\mathbb{H}}$ correspond to vectors $\mathbf{z}$ and $\mathbf{w}$ in V_-, then the Bergman metric is also given by the distance ρ:

$$\cosh^2\big(\frac{\rho(z,w)}{2}\big) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle}.$$

More information on the basic formalism of the quaternionic hyperbolic space may be found in **[CG74]**, **[KP03]**.

3. Isometries

Let $\mathrm{Sp}(n,1)$ be the isometry group of the Hermitian form $\langle ., . \rangle$. Each matrix A in $\mathrm{Sp}(n,1)$ satisfies the relation $A^{-1} = H^{-1} A^* H$, where A^* is the conjugate transpose of A. The isometry group of $\mathbf{H}^n_{\mathbb{H}}$ is $\mathrm{PSp}(n,1) = \mathrm{Sp}(n,1)/\{\pm I\}$. However, we shall mostly deal with $\mathrm{Sp}(n,1)$.

Based on their fixed points, isometries of $\mathbf{H}^n_{\mathbb{H}}$ are classified as follows:

(1) An isometry is *elliptic* if it fixes a point on $\mathbf{H}^n_{\mathbb{H}}$.
(2) An isometry is *parabolic* if it fixes exactly one point on $\partial \mathbf{H}^n_{\mathbb{H}}$.
(3) An isometry is *hyperbolic* if it fixes exactly two points on $\partial \mathbf{H}^n_{\mathbb{H}}$.

The elliptic and hyperbolic isometries are semisimple. Now we define the following terminology for describing conjugacy classification of semisimple isometries. Let g be a semisimple element in $\mathrm{Sp}(n,1)$. Let λ be an eigenvalue of g, counted without multiplicities. Then λ is called negative, resp. null, resp. positive if the corresponding λ-eigenvector is negative, resp. null, resp. positive. Accordingly a similarity class of eigenvalues is negative, null or positive according as its representative is negative, null or positive. For details about conjugacy classification of the isometries we refer to Chen-Greenberg **[CG74]**. We note the following fact that is useful for our purposes.

LEMMA 2.3. (Chen-Greenberg)**[CG74]**

(1) *Two elliptic elements in* $\mathrm{Sp}(n,1)$ *are conjugate if and only if they have the same negative class of eigenvalues, and the same positive classes of eigenvalues.*
(2) *Two hyperbolic elements in* $\mathrm{Sp}(n,1)$ *are conjugate if and only if they have the same similarity classes of eigenvalues.*

LEMMA 2.4. *The group* $\mathrm{Sp}(n,1)$ *can be embedded in the group* $\mathrm{GL}(2n+2,\mathbb{C})$*.*

PROOF. Write $\mathbb{H}=\mathbb{C}\oplus \mathbf{j}\mathbb{C}$. For $A\in \mathrm{Sp}(n,1)$, express $A=A_1+\mathbf{j}A_2$, where $A_1, A_2\in M_{n+1}(\mathbb{C})$. This gives an embedding $A\mapsto A_{\mathbb{C}}$ of $\mathrm{Sp}(n,1)$ into $\mathrm{GL}(2n+2,\mathbb{C})$, where

$$A_{\mathbb{C}}=\begin{pmatrix} A_1 & -\overline{A_2} \\ A_2 & \overline{A_1}\end{pmatrix}. \tag{3.1}$$

□

The following lemma follows from the above.

PROPOSITION 2.5. *Let* A *be an element in* $\mathrm{Sp}(n,1)$*. Let* $A_{\mathbb{C}}$ *be the corresponding element in* $\mathrm{GL}(2n+2,\mathbb{C})$*. The characteristic polynomial of* $A_{\mathbb{C}}$ *is of the form*

$$\chi_A(x)=\sum_{j=0}^{2n+2} a_j x^{2(n+1)-j},$$

where $a_0=1=a_{2n+2}$ *and for* $1\leq j\leq n+1$, $a_j=a_{2(n+1)-j}$*. If* A *is hyperbolic, then the conjugacy class of* A *is determined by the real numbers* a_j*,* $1\leq j\leq n+1$*. If* A *is elliptic then the conjugacy class of* A *is determined by the real numbers* a_j*,* $1\leq j\leq n+1$*, along with the negative-type eigenvalue of* A*.*

In the special case of $\mathrm{Sp}(1,1)$, counterparts of the above theorem may be obtained from the works **[GP13]** or **[CPW04]**, also see **[GPP15**, Section 5.3].

DEFINITION 2.6. Let A be semisimple element in $\mathrm{Sp}(n,1)$. The real n-tuple $(a_1,\dots,a_n)$ as in Proposition 2.5 will be called the *real trace* of A and we shall denote it by $tr_{\mathbb{R}}(A)$.

4. The cross ratios

Given an ordered quadruple of pairwise distinct points (z_1, z_2, z_3, z_4) on $\overline{\mathbf{H}^n_{\mathbb{H}}}$, their Korányi-Reimann (quaternionic) cross ratio is defined by

$$\mathbb{X}(z_1, z_2, z_3, z_4) = [z_1, z_2, z_3, z_4] = \langle \mathbf{z}_3, \mathbf{z}_1 \rangle \langle \mathbf{z}_3, \mathbf{z_2} \rangle^{-1} \langle \mathbf{z}_4, \mathbf{z}_2 \rangle \langle \mathbf{z}_4, \mathbf{z}_1 \rangle^{-1},$$

where, for $i = 1, 2, 3, 4$, $\mathbf{z}_i$, are lifts of z_i. Unlike the complex case, quaternionic cross ratios are not independent of the chosen lifts. However, similarity classes of the cross ratios are independent of the chosen lifts. In other words, the conjugacy invariants obtained from the cross ratios are $\Re(\mathbb{X})$ and $|\mathbb{X}|$. Under the action of the symmetric group S_4 on a tuple, there are exactly three orbits, see [**Pla14**, Prop 3.1]. For a given quadruple of distinct points in $\partial\mathbf{H}^n_{\mathbb{H}}$, the moduli and the real parts of all 24 quaternionic cross ratios are real analytic functions of the moduli and the real parts respectively of the following three cross ratios:

$$\mathbb{X}_1(z_1, z_2, z_3, z_4) = \mathbb{X}(z_1, z_2, z_3, z_4) = [\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4],$$

$$\mathbb{X}_2(z_1, z_2, z_3, z_4) = \mathbb{X}(z_1, z_4, z_3, z_2) = [\mathbf{z}_1, \mathbf{z}_4, \mathbf{z}_3, \mathbf{z}_2],$$

$$\mathbb{X}_3(z_1, z_2, z_3, z_4) = \mathbb{X}(z_2, z_4, z_3, z_1) = [\mathbf{z}_2, \mathbf{z}_4, \mathbf{z}_3, \mathbf{z}_1].$$

Usually, cross ratios are defined for boundary points but we can generalize it for $\overline{\mathbf{H}^n_{\mathbb{H}}}$ and we will use it in chapter 3. Platis defined cross ratios for boundary points and proved that these cross ratios satisfy the following real relations:

$$|\mathbb{X}_2| = |\mathbb{X}_1||\mathbb{X}_3|, \text{ and, } 2|\mathbb{X}_1|\Re(\mathbb{X}_3) \geq |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 - 2\Re(\mathbb{X}_1) - 2\Re(\mathbb{X}_2) + 1,$$

with equality holds if and only if certain conditions hold, see [**Pla14**, Proposition 3.4].

Furthermore, in the case $n = 2$, Platis proved that these cross ratios satisfy the following real relations:

$$|\mathbb{X}_2| = |\mathbb{X}_1||\mathbb{X}_3|, \text{ and, } 2|\mathbb{X}_1|\Re(\mathbb{X}_3) = |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 - 2\Re(\mathbb{X}_1) - 2\Re(\mathbb{X}_2) + 1.$$

For a given quadruple (z_1, z_2, z_3, z_4) of $\partial\mathbf{H}^2_{\mathbb{H}}$, the triple of cross ratios $(\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3)$ corresponds to a point on the four dimensional real variety $\mathbb{R}^6$ subject to the above two real equations. This is called the *cross ratio variety*.

However, unlike the complex case, a point on the quaternionic cross ratio variety does not determine the $\mathrm{Sp}(2,1)$-congruence classes of ordered quadruples of points on $\partial\mathbf{H}^2_{\mathbb{H}}$.

This has been proven by Cao in [**Cao16**]. We shall recall Cao's result in Theorem 2.8 below.

W. Cao also defined it for $\overline{\mathbf{H}^n_{\mathbb{H}}}$, for more details, see [**Cao16**, Section 3].

5. Cartan's angular invariant

Let $z_1,\ z_2,\ z_3$ be three distinct points of $\overline{\mathbf{H}^n_{\mathbb{H}}} = \mathbf{H}^n_{\mathbb{H}} \cup \partial\mathbf{H}^n_{\mathbb{H}}$, with lifts $\mathbf{z}_1,\ \mathbf{z}_2$ and $\mathbf{z}_3$ respectively. The quaternionic Cartan's angular invariant associated to the triple (z_1, z_2, z_3) was defined by Apanasov and Kim in [**AK07**] and is given by the following:

$$\mathbb{A}(z_1,\ z_2,\ z_3) = \arccos \frac{\Re(-\langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\rangle)}{|\langle \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\rangle|}.$$

The angular invariant is an element in $[0, \frac{\pi}{2}]$. It is independent of the chosen lifts and also $\mathrm{Sp}(n,1)$-invariant. The following proposition shows that this invariant determines any triple of distinct points on $\partial\mathbf{H}^n_{\mathbb{H}}$ up to $\mathrm{Sp}(n,1)$-equivalence. For a proof see [**AK07**].

PROPOSITION 2.7. [**AK07**] *Let $z_1,\ z_2,\ z_3$ and $z'_1,\ z'_2,\ z'_3$ be triples of distinct points of $\partial\mathbf{H}^n_{\mathbb{H}}$. Then $\mathbb{A}(z_1,\ z_2,\ z_3) = \mathbb{A}(z'_1,\ z'_2,\ z'_3)$ if and only if there exist $A \in \mathrm{Sp}(n,1)$ so that $A(z_j) = z'_j$ for $j = 1, 2, 3$.*

Further, it is proved in [**AK07**] that (z_1, z_2, z_3) lies on the boundary of an $\mathbb{H}$-line, resp. a totally real subspace, if and only if $\mathbb{A} = \frac{\pi}{2}$, resp. $\mathbb{A} = 0$.

5.1. $\mathrm{Sp}(2,1)$-congruence classes of ordered quadruples of points on $\partial\mathbf{H}^2_{\mathbb{H}}$. We shall use the following result by Cao that determines ordered quadruples of points on $\partial\mathbf{H}^2_{\mathbb{H}}$, see [**Cao16**].

THEOREM 2.8. [**Cao16**] *Let $Z = (z_1, z_2, z_3, z_4)$ and $W = (w_1, w_2, w_3, w_4)$ be two ordered quadruples of pairwise distinct points in $\partial\mathbf{H}^2_{\mathbb{H}}$. Then there exists an isometry $h \in \mathrm{Sp}(2,1)$ such that $h(z_i) = w_i$, $i = 1, 2, 3, 4$, if and only if the following conditions hold:*

1. *For $j = 1, 2, 3$, $\mathbb{X}_j(z_1, z_2, z_3, z_4)$ and $\mathbb{X}_j(w_1, w_2, w_3, w_4)$ belong to the same similarity class.*
2. *$\mathbb{A}(z_1, z_2, z_3) = \mathbb{A}(w_1, w_2, w_3)$, $\mathbb{A}(z_1, z_2, z_4) = \mathbb{A}(w_1, w_2, w_4)$,*
   $\mathbb{A}(z_2, z_3, z_4) = \mathbb{A}(w_2, w_3, w_4)$.

Cao has described the moduli of $\mathrm{Sp}(2,1)$-congruence classes of ordered quadruples of distinct points on $\partial\mathbf{H}^2_{\mathbb{H}}$. We recall his result here.

THEOREM 2.9. **[Cao16]** *Let* $\mathrm{M}(2,4,0)$ *be the configuration space of ordered quadruples of distinct points on* $\partial\mathbf{H}^2_{\mathbb{H}}$. *Then* $\mathrm{M}(2,4,0)$ *is homeomorphic to the semi-analytic subspace of* $\mathbb{C}^3 \times [0,\infty) \times [0,\frac{\pi}{2}]$ *given by points* $(c_1, c_2, c_3, t, \mathbb{A})$ *subject to the relations*

$$\Re(c_1\bar{c}_2) + t\Re(c_3) \leq 0,\ \Re(c_2) \leq 0, |c_1|^2 + t^2 \neq 0,\ |c_2|^2 + |c_3|^2 \neq 0,$$

$$1 + |c_1|^2 + |c_2|^2 + |c_3|^2 + t^2 - 2\Re(c_1) + 2\Re(c_2 e^{-i\mathbb{A}}) + 2\Re((\bar{c}_1 c_2 + tc_3)e^{i\mathbb{A}}) = 0.$$

Note that there is a natural action of the symmetric group S_4 on $\mathrm{M}(2,4,0)$, coming from the S_4 action on an ordered tuple:

$$g.[(p_1, p_2, p_3, p_4)] = [(p_{g(1)}, p_{g(2)}, p_{g(3)}, p_{g(4)})].$$

The orbit space of $\mathrm{M}(2,4,0)$ under this action will be denoted by $\mathcal{M}(2)$.

CHAPTER 3

Moduli space of $\mathrm{PSp}(n, 1)$ congruence classes of points

In this chapter, we investigate about the Moduli space of $\mathrm{PSp}(n, 1)$ congruence classes of distinct points from $\overline{\mathbf{H}^n_{\mathbb{H}}}$.

1. The Gram matrix

A common feature in the works **[CG12b]**, **[Cao17]**, and the one formulated in this section is the use of the Gram matrices. This is motivated by the ideas of Brehm and Et-Taoui in **[BET01]** and Höfer in **[Höf99]**.

In this section, without loss of generality, we will assume that first i elements from $p = (p_1, p_2, \ldots, p_m)$ are null and the remaining other elements are negative, for $3 \leq i \leq m$. The following proposition is in **[Cao16**, Prop.1.1].

PROPOSITION 3.1. *If* $\mathbf{z}, \mathbf{w} \in \mathbb{H}^{n,1} - \{0\}$ *with* $\langle \mathbf{z}, \mathbf{z} \rangle \leq 0$ *and* $\langle \mathbf{w}, \mathbf{w} \rangle = 0$ *then either* $\mathbf{w} = \mathbf{z}\lambda$ *for some* $\lambda \in \mathbb{H}$ *or* $\langle \mathbf{z}, \mathbf{w} \rangle \neq 0$.

From this Proposition 3.1 we can see that $\langle \mathbf{z}, \mathbf{w} \rangle \neq 0$, for $z \in \partial\mathbf{H}^n_{\mathbb{H}}$ and $w \in \mathbf{H}^n_{\mathbb{H}}$. Also we will have $\langle \mathbf{z}, \mathbf{w} \rangle \neq 0$, for $z \neq w$ together with the condition that either $z, w \in \partial\mathbf{H}^n_{\mathbb{H}}$ or $z, w \in \mathbf{H}^n_{\mathbb{H}}$.

DEFINITION 3.2. We say that two Hermitian $m \times m$ matrices G and K are equivalent if there exist non-singular diagonal matrix D such that $G = D^*KD$.

Now by using these observations together with an appropriate chosen lift of $p = (p_1, p_2, \ldots, p_m)$, the following lemma follows using a similar argument as in **[CG12b]**.

LEMMA 3.3. *Let* $p = (p_1, p_2, \ldots, p_m)$ *be a* m*-tuple of distinct points in* $\overline{\mathbf{H}^n_{\mathbb{H}}}$. *Then the equivalence class of Gram matrices associated to* p *contains a matrix* $G = (g_{kj})$ *with* $g_{kk} = 0$ *for*

$k = 1,2,\ldots,i$, $|g_{23}| = 1$, $g_{1j} = 1$ for $j = 2,3,\ldots,i$, $g_{kk} = -1$ for $k = i+1, i+2,\ldots,m$, and $g_{1k} = r_{1k}$ for $k = i+1, i+2,\ldots,m$, where r_{1k} are real positive numbers.

PROOF. Let $\mathfrak{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m)$ be a lift of p. We want to choose a lift of p such that it will satisfy required conditions of the lemma. As p_i's are distinct points in $\overline{\mathbf{H}^n_{\mathbb{H}}}$, we get $\langle \mathbf{p}_k, \mathbf{p}_k \rangle = g_{kk} = r_k$, where $r_k = 0$ for $k = 1,2,\ldots,i$ and r_k is a negative real number for $k = i+1, i+2, \ldots, m$. Now we can find out scalars λ_k such that $\langle \mathbf{p}_k\lambda_k, \mathbf{p}_k\lambda_k \rangle = 0$ for $k = 1,2,\ldots,i$ and $\langle \mathbf{p}_k\lambda_k, \mathbf{p}_k\lambda_k \rangle = -1$ for $k = i+1, i+2, \ldots, m$. In particular we can take $\lambda_k = 1$ for $k = 1,2,\ldots,i$ and $\lambda_k = \sqrt{-r_k}^{\,-1}$ for $k = i+1, i+2, \ldots, m$.

Now again rescale $\mathfrak{p}$ by $\beta_1 = 1$, $\beta_k = \overline{\langle \mathbf{p}_1\lambda_1, \mathbf{p}_k\lambda_k \rangle^{\,-1}}$ for $k = 2,3,\ldots,i$ and $\beta_k = x_{1k}$ for $k = i+1, i+2, \ldots, m$, where x_{1k} is unitary part of $\langle \mathbf{p}_1\lambda_1, \mathbf{p}_k\lambda_k \rangle$. Thus we get conditions $g_{1j} = 1$ for $j = 2,3,\ldots,i$ and $g_{1j} = r_{1k}$ for $k = i+1, i+2, \ldots, m$, where $r_{1k} = |\langle \mathbf{p}_1\lambda_1, \mathbf{p}_k\lambda_k \rangle|$ are real positive numbers. Finally for getting condition $|g_{23}| = 1$ we will further rescale $\mathfrak{p}$ by $\gamma_1 = \sqrt{r_{23}}$, $\gamma_k = \frac{1}{\sqrt{r_{23}}}$ for $k = 2,3,\ldots,i$ and $\gamma_k = 1$ for $k = i+1, i+2, \ldots, m$, where $r_{23} = |\langle \mathbf{p}_2\lambda_2\beta_2, \mathbf{p}_3\lambda_3\beta_3 \rangle|$. So appropriate lift $\mathfrak{p} = (\mathbf{p}_1\lambda_1\beta_1\gamma_1, \mathbf{p}_2\lambda_2\beta_2\gamma_2, \ldots, \mathbf{p}_m\lambda_m\beta_m\gamma_m)$ of p proves the lemma. □

REMARK 3.4. The Gram matrix $G(\mathfrak{p}) = (g_{kj})$ in the above lemma 3.3 has the form

$$G(\mathfrak{p}) = \left[\begin{array}{ccccc|c} 0 & 1 & 1 & \cdots & 1 & \\ 1 & 0 & g_{23} & \cdots & g_{2i} & \\ 1 & \overline{g_{23}} & 0 & \cdots & g_{3i} & G^* \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 1 & \overline{g_{2i}} & \overline{g_{3i}} & \cdots & 0 & \\ \hline & & & & & \\ & & \overline{G}^* & & & A \end{array}\right], \tag{1.1}$$

where $|g_{23}| = 1$, G^* is $i \times (m-i)-$ matrix with each entry nonzero together with having first row contains real positive number and A is $(m-i) \times (m-i)-$ matrix with the form,

$$A = \begin{bmatrix} -1 & g_{(i+1)(i+2)} & g_{(i+1)(i+3)} \cdots & g_{(i+1)m} \\ \overline{g_{(i+1)(i+2)}} & -1 & g_{(i+2)(i+3)} \cdots & g_{(i+2)m} \\ \vdots & \vdots & \vdots \ddots & \vdots \\ \overline{g_{(i+1)(m)}} & \overline{g_{(i+2)m}} & \overline{g_{(i+3)m}} \cdots & -1 \end{bmatrix}.$$

Now observe that the spanning set of $\mathfrak{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m)$ contains at least one negative point even if m-tuple contains all null points. So the spanning set of $\mathfrak{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m)$ is non-degenerate.

The following proposition follows using similar arguments as in the proof of [**Höf99**, Theorem 1]. It is essentially a consequence of the Witt's extension theorem.

DEFINITION 3.5. Let $p = (p_1, p_2, \dots, p_m)$ and $q = (q_1, q_2, \dots, q_m)$ be two ordered m-tuples of distinct points in $\overline{\mathbf{H}^n_{\mathbb{H}}}$. Then p and q are congruent in $\mathrm{PSp}(n,1)$ if there exist $h \in \mathrm{PSp}(n,1)$ such that $h(p_i) = q_i$ for $i = 1, 2, \dots, m$.

Here, we will recall Witt's theorem.

THEOREM 3.6 (Witt's theorem). *Any linear injective isometry $\phi : V \to W$, where V and W are linear subspaces of $\mathbb{H}^{n,1}$ can be extended to isometry of $\mathbb{H}^{n,1}$.*

PROPOSITION 3.7. *Let $p = (p_1, p_2, \dots, p_m)$ and $q = (q_1, q_2, \dots, q_m)$ be two ordered m-tuples of distinct points in $\overline{\mathbf{H}^n_{\mathbb{H}}}$. Then p and q are congruent in $\mathrm{PSp}(n,1)$ if and only if their associated Gram matrices are equivalent.*

PROOF. Suppose p and q are congruent in $\mathrm{PSp}(n,1)$. Let $G(\mathfrak{p})$ and $G(\mathfrak{q})$ are Gram matrices associated to p and q with respect to lift $\mathfrak{p}$ and $\mathfrak{q}$ respectively. Since there exist $h \in \mathrm{PSp}(n,1)$ such that $h(p_i) = q_i$ for $i = 1, 2, \dots, m$, gives that $h(\mathbf{p}_i) = \mathbf{q}_i\lambda_i$ for all $i = 1, 2, \dots, m$. So we have $G(\mathfrak{p}) = D^* G(\mathfrak{q}) D$ with $D = diag(\lambda_1, \lambda_2, \dots, \lambda_m)$, that is $G(\mathfrak{p})$ and $G(\mathfrak{q})$ are equivalent.

Conversely, suppose that $G(\mathfrak{p})$ and $G(\mathfrak{q})$ are equivalent. So there exist non-singular diagonal matrix D such that $G(\mathfrak{p}) = D^* G(\mathfrak{q}) D$. Thus we have $\langle \mathbf{p}_j, \mathbf{p}_i \rangle = \langle \mathbf{q}_j\lambda_j, \mathbf{p}_i\lambda_i \rangle$. Now suppose that V and W be subspaces of $\mathbb{H}^{n,1}$ spanned by m-tuples $(\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_m)$ and $(\mathbf{q}_1\lambda_1, \mathbf{q}_2\lambda_2, \dots, \mathbf{q}_m\lambda_m)$ respectively. Let $\phi : V \longrightarrow W$ be such that $\phi(\mathbf{p}_i) = \mathbf{q}_i\lambda_i$. Now we will prove proposition by using Witt's theorem. As V and W are non degenerate subspaces of $\mathbb{H}^{n,1}$, we can see that ϕ is linear injective isometry. Thus ϕ can be extended to linear isometry $\tilde{\phi}$ of $\mathbb{H}^{n,1}$, by Witt's theorem. So $\tilde{\phi}(p_i) = q_i$ for $i = 1, 2, \dots, m$, that is p and q are congruent in $\mathrm{PSp}(n,1)$. □

2. Semi-normalised Gram matrix

DEFINITION 3.8. We will call the matrix in lemma 3.3 as *semi-normalised Gram matrix* with respect to lift $\mathfrak{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_m)$ of p.

The following Lemma shows that semi-normalised Gram matrix is just an equivalence class.

LEMMA 3.9. *Suppose that the Gram matrix $G(\mathfrak{p})$ is semi-normalised Gram matrix for p with respect to lift $\mathfrak{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m)$. Then $G(\mathfrak{p}')$ is still a semi-normalised Gram matrix with $\mathfrak{p}' = (\mathbf{p}_1\lambda_1, \ldots, \mathbf{p}_m\lambda_m)$ if and only if $\lambda_1 = \lambda_2 = \ldots = \lambda_m$ and $\lambda_i \in \mathrm{Sp}(1)$.*

PROOF. It follows from $\langle \mathbf{p}_1\lambda_1, \mathbf{p}_k\lambda_k\rangle = 1$ that $\overline{\lambda_k}\lambda_1 = 1$, for $k = 2, 3, \ldots, i$ as $\langle \mathbf{p}_1, \mathbf{p}_k\rangle = 1$, and from $|\langle \mathbf{p}_2\lambda_2, \mathbf{p}_3\lambda_3\rangle| = 1$ that $|\overline{\lambda_3}||\lambda_2| = 1$ as $|\langle \mathbf{p}_2, \mathbf{p}_3\rangle| = 1$. Thus we have $|\lambda_1| = 1$ so $\lambda_1 \in \mathrm{Sp}(1)$. So by $\overline{\lambda_k}\lambda_1 = 1$ for $k = 2, 3, \ldots, i$ we have $\lambda_1 = \lambda_2 = \ldots = \lambda_i$ and $\lambda_i \in \mathrm{Sp}(1)$.

Also we can see that $\langle \mathbf{p}_k\lambda_k, \mathbf{p}_k\lambda_k\rangle = -1$, for $k = i+1, i+2, \ldots, m$ gives that $|\lambda_k| = 1$ for $k = i+1, i+2, \ldots, m$. Since $\langle \mathbf{p}_1\lambda_1, \mathbf{p}_j\lambda_j\rangle = r'_{1j}$, where r'_{1j} are positive real numbers for $j = i+1, i+2, \ldots, m$. Thus we have $\overline{\lambda_j}\ r_{1j}\ \lambda_1 = r'_{1j}$, where $\langle \mathbf{p}_1, \mathbf{p}_j\rangle = r_{1j}$. By using the fact $|\lambda_i| = 1$ implies $r'_{1j} = r_{1j}$ for $j = i+1, i+2, \ldots, m$. As we can commute real numbers with quaternions we get $\overline{\lambda_j}\lambda_1 = 1$ for $j = i+1, i+2, \ldots, m$. Thus we have $\lambda_j = \lambda_1$ for $j = i+1, i+2, \ldots, m$ with $|\lambda_1| = 1$ i.e., $\lambda_1 \in \mathrm{Sp}(1)$.

Conversely, we can verify that if $\mathfrak{p}' = (\mathbf{p}_1\lambda_1, \mathbf{p}_2\lambda_1, \ldots, \mathbf{p}_m\lambda_1)$ with $|\lambda_1| = 1$, then $G(\mathfrak{p}')$ is matrix of the form (1.1). □

REMARK 3.10. We can represent semi-normalised Gram matrix $G(\mathfrak{p}) = (g_{kj})$ by $V_G = (r_{1(i+1)}, r_{1(i+2)}, \ldots, r_{1m}, g_{23}, g_{24}, \ldots, g_{2m}, g_{34}, \ldots, g_{3m}, \ldots, g_{m-1m})$ in $\mathbb{H}^t$, where $|g_{23}| = 1$ and $t = \frac{(m^2-m-2i+2)}{2}$. Action of $\mathrm{Sp}(1)/\{1,-1\}$ on $\mathbb{H}^t$ by $(\mu, V_G) \to \overline{\mu}V_G\mu$ gives the orbit $O_{V_G} = \{\overline{\mu}V_G\mu : \forall\mu \in \mathrm{Sp}(1)\}$.

LEMMA 3.11. *Let G_1 and G_2 be two semi-normalised Gram matrices represented by V_{G_1} and V_{G_2} resp. Then G_1 and G_2 are equivalent if and only if $O_{V_{G_1}} = O_{V_{G_2}}$.*

PROOF. Let G_1 and G_2 be two semi-normalised Gram matrices of p and q respectively, where p and q are m-tuples of distinct points in $\overline{\mathbf{H}^n_{\mathbb{H}}}$ with lifts $\mathfrak{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m)$ and $\mathfrak{q} = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_m)$ respectively. The equivalence of G_1 and G_2 implies that there exist non-singular diagonal matrix D such that $G_1 = D^*G_2D$ where $D = diag(\lambda_1, \ldots, \lambda_m)$. So we have

(2.1) $$G_1 = (g_{ij}) = (\langle \mathbf{p}_j, \mathbf{p}_i\rangle) = (\overline{\lambda_i}\langle \mathbf{q}_j, \mathbf{q}_i\rangle\lambda_j) = (\overline{\lambda_i}g_{ij}{}'\lambda_j) = (\langle \mathbf{q}_j\lambda_j, \mathbf{q}_i\lambda_i\rangle).$$

Lemma 3.9 above gives now $\lambda_1 = \lambda_2 = \ldots = \lambda_m = \mu$ and $\mu \in \mathrm{Sp}(1)$. So from equation 2, $g_{ij} = \overline{\mu} g_{ij}' \mu$, where $\mu \in \mathrm{Sp}(1)$. So we get $O_{V_{G_1}} = O_{V_{G_2}}$.
Conversely, if $O_{V_{G_1}} = O_{V_{G_2}}$, we want to find non-singular diagonal matrix D such that $G_1 = D^* G_2 D$, where $D = diag(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m)$. As V_{G_1} lies in $O_{V_{G_1}}$, $V_{G_1} = \overline{\mu} V_{G_2} \mu$ for some $\mu \in \mathrm{Sp}(1)$ and $g_{ij} = \overline{\mu} g_{ij}' \mu$. Thus we get $D = (\mu, \mu, \ldots, \mu)$, where $\mu \in \mathrm{Sp}(1)$. $\square$

LEMMA 3.12. *Let p and q be m-tuples of distinct points in $\overline{\mathbf{H}^n_{\mathbb{H}}}$. Then p and q are congruent in $\mathrm{PSp}(n,1)$ if and only if $O_{V_{G_1}} = O_{V_{G_2}}$, where V_{G_1} and V_{G_2} are represented by semi-normalised Gram matrices associated to p and q respectively.*

PROOF. By Proposition 3.7, p and q are congruent in $\mathrm{PSp}(n,1)$ if and only if their associated Gram matrices are equivalent. Let G_1 and G_2 be the two semi-normalised Gram matrices associated to p and q respectively. We represent them by V_{G_1} and V_{G_2} respectively. Now by using 3.11, we get the result $O_{V_{G_1}} = O_{V_{G_2}}$. $\square$

3. Configuration space of ordered tuples of points

Note that $\mathbb{X}_{1j} = g_{23} r_{1j} g_{2j}^{-1}$, $\mathbb{X}_{2j} = \overline{g_{23}} g_{2j} r_{1j}^{-1}$, $\mathbb{X}_{3j} = \overline{g_{23}}^{-1} g_{3j} r_{1j}^{-1}$, $\mathbb{X}_{kj} = \overline{g_{2k}}^{-1} g_{kj} r_{1j}^{-1}$, where $r_{1j} = 1$ if $2 \leq j \leq i$. Hence the Gram matrix $G(\mathbf{p}) = (g_{ij})$ can be read off from these invariants.

Let $p_1,\ p_i,\ p_j$ be three distinct points of $\overline{\mathbf{H}^n_{\mathbb{H}}} = \mathbf{H}^n_{\mathbb{H}} \cup \partial \mathbf{H}^n_{\mathbb{H}}$, with lifts $\mathbf{p}_1,\ \mathbf{p}_i$ and $\mathbf{p}_j$, respectively. We can write $\langle \mathbf{p}_1, \mathbf{p}_j, \mathbf{p}_i \rangle = |\langle \mathbf{p}_1, \mathbf{p}_j, \mathbf{p}_i \rangle| (\cos\theta_{ij} + u_{ij} \sin\theta_{ij}) = |\langle \mathbf{p}_1, \mathbf{p}_j, \mathbf{p}_i \rangle| e^{u_{ij}\theta_{ij}}$, where $u_{ij} \in \sigma^2$ is a unit pure quaternion. If $\langle \mathbf{p}_1, \mathbf{p}_j, \mathbf{p}_i \rangle$ is a real number then u_{ij} is undefined, and we shall assume in such cases that$u_{ij} = 0$. Note that

$$\mathbb{A}(p_1,\ p_i,\ p_j) = \arg(\langle \mathbf{p}_1, \mathbf{p}_j, \mathbf{p}_i \rangle) = \arg(r_{1j} g_{ij} r_{1i}) = \arg(g_{ij}) = \theta_{ij}.$$

It follows from Remark 3.10 that the $\mathrm{Sp}(1)$-conjugacy class of u_{ij} (where it is non-zero) is an invariant of the orbit O_{V_G}.

THEOREM 3.13. **[GK19**, Theorem 1.4] *A point $[p]$ in $\mathrm{M}(n,i,m-i)$, $p = (p_1, \ldots, p_m)$, is determined by the $\mathrm{Sp}(1)$ congruence class of the $(d+t+1)$-tuple*

$$W = (u_0, u_1, \ldots, u_t, \mathbb{X}_1, \ldots, \mathbb{X}_d),\ d = \frac{i(i-3)}{2} + i(m-i),\ m \geq 4,\ t = \frac{(m-i)^2 - (m-i)}{2} - l,$$

the angular invariants $\mathbb{A}_{23}$, $\mathbb{A}_{i_1j_1}$ and the distance invariants $d_{i_1j_1}$, for $i < i_1, j_1 \leq m$, where l is the number of zero valued rotation invariants u_{ij}.

PROOF. Observe that each orbit O_{V_G} is determined up to $\mathrm{Sp}(1)$ conjugation of the semi-normalised Gram matrix represented by V_G.

Let $G = (g_{ij})$ be a semi-normalised Gram matrix represented by V_G corresponding to a chosen lift $\mathfrak{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m)$ of p. We have the following equations:

$$\mathbb{A}_{23} = \mathbb{A}(p_1, p_2, p_3) = \arccos \frac{\Re(-\langle \mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_2 \rangle)}{|\langle \mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_2 \rangle|} = \arccos \frac{\Re(-g_{23})}{|g_{23}|} = \arccos \Re(-g_{23}),$$

$$\mathbb{X}_{1j'} = g_{23} r_{1j'} {g_{2j'}}^{-1}, \mathbb{X}_{2j} = \overline{g_{23}} g_{2j} r_{1j}^{-1}, \mathbb{X}_{3j} = \overline{g_{23}}^{-1} g_{3j} r_{1j}^{-1}, \mathbb{X}_{kj} = \overline{g_{2k}}^{-1} g_{kj} r_{1j}^{-1}, u_0 = \frac{\Im(g_{23})}{|\Im(g_{23})|},$$

for $(i+1) \leq j' \leq m,\ 4 \leq j \leq m,\ 4 \leq k \leq i,\ k < j.$

Also for the negative points, we have the following equations:

$$d_{i_1j_1} = g_{i_1j_1} g_{j_1i_1} = |g_{i_1j_1}|^2, \mathbb{A}_{i_1j_1} = \arg(g_{i_1j_1}), u_{i_1j_1} = \frac{\Im(g_{i_1j_1})}{|\Im(g_{i_1j_1})|}.$$

So, given the Gram matrix G, we can determine u_0, $\mathbb{A}_{23}$, $\mathbb{A}_{i_1j_1}$, $\mathbb{X}_{ij}$, $d_{i_1j_1}$, and $u_{i_1j_1}$ by the above equations. Using Lemma 3.9 and the fact that the angular invariants $\mathbb{A}_{ij}$, distance invariants d_{ij} are independent of choices of the lifts, it determines $\mathbb{A}_{23}$, $\mathbb{A}_{i_1j_1}$, $d_{i_1j_1}$ and the $\mathrm{Sp}(1)$ congruence class of $F = (u_0, \ldots, u_t, \mathbb{X}_1, \ldots, \mathbb{X}_d)$.

It is known that each nonzero quaternion q has the unique polar form $q = |q|e^{u\theta}$.Thus, if we know $|q|$, θ and u then we will get quaternion number q uniquely. By the definition of $d_{i_1j_1}$ we have $|g_{i_1j_1}| = \sqrt{d_{i_1j_1}}$. Also by the definition of the angular invariant we get $\mathbb{A}_{i_1j_1} = \arg(g_{i_1j_1})$. Thus we can determine the matrix $A = (g_{i_1j_1})$ for negative points by $d_{i_1j_1}$, $\mathbb{A}_{i_1j_1}$ and $u_{i_1j_1}$.

Conversely, let $x = (\mu u_0 \bar{\mu}, \ldots, \mu u_t \bar{\mu}, \mu \mathbb{X}_1 \bar{\mu}, \ldots, \mu \mathbb{X}_d \bar{\mu})$ be an element from the $\mathrm{Sp}(1)$ congruence class of F for some $\mu \in \mathrm{Sp}(1)$ with respect to some lift p. By the above equations we have $\mu g_{23} \bar{\mu} = \cos \mathbb{A}_{23} + \mu u_0 \bar{\mu} \sin \mathbb{A}_{23}$ with $u_0 = \frac{\Im(g_{23})}{|\Im(g_{23})|}$. Also,

$$\mu g_{2j} \bar{\mu} = \mu g_{23} \bar{\mu}\ \mu \mathbb{X}_{2j} \bar{\mu},\ \ \mu g_{3j} \bar{\mu} = \bar{\mu}\ \mu \overline{g_{23}} \bar{\mu}\ \mu \mathbb{X}_{3j},$$

$$\mu g_{kj} \bar{\mu} = \mu \overline{g_{2k}} \bar{\mu}\ \mu \mathbb{X}_{kj} \bar{\mu}\ = \mu \overline{\mathbb{X}_{2k}} \bar{\mu}\ \mu \overline{g_{23}} \bar{\mu}\ \mu \mathbb{X}_{kj} \bar{\mu},$$

$$\mu g_{i_1j_1} \bar{\mu} = |g_{i_1j_1}| e^{\mu u_{i_1j_1} \bar{\mu} \mathbb{A}_{i_1j_1}} = \sqrt{d_{i_1j_1}} e^{\mu u_{i_1j_1} \bar{\mu} \mathbb{A}_{i_1j_1}},$$

where $g_{i_1 j_1}$ are the entries of sub-matrix A of matrix $G(\mathbf{p})$ in 1.1. So we have element $V_G = (\mu g_{23}\bar{\mu}, \mu g_{24}\bar{\mu}, \ldots, \mu g_{2m}\bar{\mu}, \mu g_{34}\bar{\mu}, \ldots, \mu g_{3m}\bar{\mu}, \ldots, \mu g_{m-1m}\bar{\mu})$ with $|\mu g_{23}\bar{\mu}| = 1$. Now using Lemma 3.12, we can determine the $\mathrm{PSp}(n, 1)$ congruence class of p. □

CHAPTER 4

Classification of pairs of semisimple isometries in $\mathrm{Sp}(n, 1)$

1. Projective points and eigenvalue Grassmannians

In this section, we introduce the notion of *projective points* and *Eigenvalue Grassmannians* of an eigenvalue. This is a crucial notion for our understanding of the pairs of semisimple elements.

1.1. Eigenvalue Grassmannians. Let T be an invertible semisimple matrix over $\mathbb{H}$ of rank n. Let $\lambda \in \mathbb{H} - \mathbb{R}$ be a chosen eigenvalue of T in the similarity class $[\lambda]$ with multiplicity m, $m \leq n$. Thus, the eigenspace of $[\lambda]$ can be identified with $\mathbb{H}^m$. Let λ be a representative of $[\lambda]$. Consider the λ-*eigenset*: $S_\lambda = \{x \in \mathbb{H}^n \mid Tx = x\lambda\}$. Note that this set can be identified with the subspace $Z(\lambda)^m$ in $\mathbb{H}^m$. Thus each eigenset of a $[\lambda]$-representative is a copy of $\mathbb{C}^m$ in $\mathbb{H}^m$. So, the set of eigensets in the $[\lambda]$-eigenspace can be identified with the set of m dimensional complex subspaces of $\mathbb{H}^m$. Identifying $\mathbb{H}^m$ with $\mathbb{C}^{2m}$, this gives us the set of $[\lambda]$-eigensets as the space complex m-dimensional subspaces of $\mathbb{C}^{2m}$, which is the complex Grassmannian manifold $G_{m,2m}$, or simply G_m. This is a compact connected smooth complex manifold of complex dimension m^2. We call it the *eigenvalue Grassmannian* of T corresponding to the eigenvalue $[\lambda]$, or simply $[\lambda]$-Grassmannian. Each point on this Grassmannian corresponds to an eigenset of $[\lambda]$. The $[\lambda]$-eigenvalue Grassmannian is a conjugacy invariant of T but individual points are not. They help us to distinguish individual isometries in the same conjugacy class.

1.2. Projective points. When $m = 1$, the eigenvalue Grassmannian is simply $\mathbb{CP}^1$, and a point on such $\mathbb{CP}^1$ has been termed as a *projective point*.

Identify $\mathbb{H}$ with $\mathbb{C}^2$. Two non-zero quaternions q_1 and q_2 are equivalent if $q_2 = q_1 c$, $c \in \mathbb{C} \setminus 0$. This equivalence relation projects $\mathbb{H}$ to the one dimensional complex projective space $\mathbb{CP}^1$.

Let v be a λ-eigenvector of T. Then $\mathbf{v}\mathbb{H}$ is the quaternionic line spanned by $\mathbf{v}$. Now, we identify $\mathbf{v}\mathbb{H}$ with $\mathbb{H}$. Then for each point on $\mathbb{CP}^1$, there is a choice of the lift $\mathbf{v}$ of v that spans a complex line in $\mathbf{v}\mathbb{H}$. The point on $\mathbb{CP}^1$ that corresponds to a specified choice of $\mathbf{v}$ is called a *projective point* of T corresponding to the eigenvalue $\lambda \in \mathbb{H} \setminus \mathbb{R}$. A $[\lambda]$-projective point of T corresponds to an eigenset of an eigenvalue representative λ, equivalently, to the centralizer $Z(\lambda)$. The $\mathbb{CP}^1$ obtained as above from the eigenspace $\mathbf{v}\mathbb{H}$ is called a $[\lambda]$-*eigensphere* of T. Since $[\lambda]$ is a conjugacy invariant of T, so is the $[\lambda]$-eigensphere $\mathbb{CP}^1$.

To see the projective points from another viewpoint, we note the following. Consider the action of $\mathrm{Sp}(1)$ on $\mathbb{H}^*$ by conjugation. The similarity class of an eigenvalue $[\lambda]$ represents an orbit under this action. The stabilizer of a point under this action is $\mathrm{SU}(1)$. Hence $[\lambda]$ is identified with $\mathbb{CP}^1$ that is the orbit space $\mathrm{Sp}(1)/\mathrm{SU}(1)$. The choice of λ on this $\mathbb{CP}^1$ is a $[\lambda]$-*projective point* of T.

If $\lambda \in \mathbb{R} \setminus \{0\}$, then it commutes with every quaternion, and hence $Z(\lambda) = \mathbb{H}$. Consequently, there is only one eigenset of λ that equals the eigenspace. We may assume the λ-eigensphere to be a single point in this case.

2. Semisimple isometries in $\mathrm{Sp}(n, 1)$

In $\mathrm{Sp}(n, 1)$, the semisimple isometries are classified as hyperbolic and elliptic.

2.1. Elliptic isometries. Let A be an elliptic element in $\mathrm{Sp}(n, 1)$. Recall that an eigenvalue of an elliptic element A always has norm 1. Let λ be an eigenvalue from the similarity class of eigenvalues $[\lambda]$ of A. Let $\mathbf{x}$ be a λ-eigenvector. Then $\mathbf{x}$ defines a point x on $\mathbb{HP}^n$, that is either a point on $\mathbf{H}^n_{\mathbb{H}}$ or a point in $\mathbb{P}(\mathrm{V}_+)$. The lift of x in $\mathbb{H}^{n,1}$ is the quaternionic line $\mathbf{x}\mathbb{H}$. We call x as *projective fixed point* of A corresponding to $[\lambda]$.

Let the eigenvalues of A be the $n+1$ unit complex numbers $e^{i\theta_1}, \dots, e^{i\theta_{n+1}}$, where $e^{i\theta_1}$ is negative and $e^{i\theta_k}$, $k = 2, \dots, n+1$, positive. Up to conjugacy, A is of the form:

$$E_A(\theta_1, \theta_2, \dots, \theta_{n+1}) = \begin{bmatrix} e^{\mathbf{i}\theta_1} & 0 & 0 & \dots & 0 & 0 \\ 0 & e^{\mathbf{i}\theta_2} & 0 & \dots & 0 & 0 \\ & & \ddots & & & \\ 0 & 0 & 0 & \dots & e^{\mathbf{i}\theta_n} & 0 \\ 0 & 0 & 0 & \dots & 0 & e^{\mathbf{i}\theta_{n+1}} \end{bmatrix}. \tag{2.1}$$

Let $C_A = \begin{bmatrix} \mathbf{x}_{1,A} & \mathbf{x}_{2,A} & \dots & \mathbf{x}_{n+1,A} \end{bmatrix}$ be the matrix corresponding to the eigenvectors of the above eigenvalue representatives. We can choose C_A to be an element of $\mathrm{Sp}(n, 1)$ by normalizing the eigenvectors:

$$\langle \mathbf{x}_{1,A}, \mathbf{x}_{1,A} \rangle = -1, \ \langle \mathbf{x}_{j,A}, \mathbf{x}_{j,A} \rangle = 1, \ j \neq 1.$$

Then $A = C_A E_A C_A^{-1}$.

2.2. Eigenspace decomposition of an elliptic element. Suppose A is an elliptic element in $\mathrm{Sp}(n, 1)$. Let the eigenvalue classes of A be represented by $e^{\mathbf{i}\theta_1}$, $e^{\mathbf{i}\theta_2}, \dots, e^{\mathbf{i}\theta_k}$, ordered so that $e^{\mathbf{i}\theta_1}$ is the negative eigenvalue. Let V_{θ_i} be the eigenspace to the eigenvalue class of $e^{\mathbf{i}\theta_i}$. Let $m_i = \dim \mathrm{V}_{\theta_i}$. We call $(m_1, \dots, m_k)$ the *multiplicity* of A. The space $\mathbb{H}^{n,1}$ has the following orthogonal decomposition into eigenspaces (here $\oplus$ denotes orthogonal sum):

$$\mathbb{H}^{n,1} = \mathrm{V}_{\theta_1} \oplus \dots \oplus \mathrm{V}_{\theta_k}, \tag{2.2}$$

Change of eigenbasis amounts to conjugation by an element C, and $CAC^{-1} = A$ if and only if $C \in Z(A)$. So, a normalised eigenbasis of A is determined up to conjugation action of $Z(A) = \prod_{i=1}^{k} Z(A|_{\mathrm{V}_{\theta_i}})$ on each of the summands.

For description of the centralizers, see **[G13]**. Let $e^{\mathbf{i}\theta_j}$ represent an eigenvalue of A with multiplicity m_j. It follows from **[G13]** that $Z(A|_{\mathrm{V}_{\theta_j}})$ can be identified with $\mathrm{U}(m_j - 1, 1)$ if the eigenvalue is negative, and with $\mathrm{U}(m_j)$ otherwise. So, if A does not have eigenvalues 1 or -1, given the multiplicity $(m_1, \dots, m_k)$, $Z(A)$ may be identified to the group

$$Z(A) = \mathrm{U}(m_1 - 1, 1) \times \mathrm{U}(m_2) \times \dots \mathrm{U}(m_k).$$

When A has an eigenvalue 1 or -1, one of the factors in the above product is replaced by $\mathrm{Sp}(m_1 - 1, 1)$ or $\mathrm{Sp}(m_i)$ depending upon the eigenvalue is negative or positive.

2.3. Hyperbolic isometries. Let A be a hyperbolic element in $\mathrm{Sp}(n,1)$. Let λ be an eigenvalue from the similarity class of eigenvalues $[\lambda]$ of A. Let $\mathbf{x}$ be a λ-eigenvector. Then $\mathbf{x}$ defines a point x on $\mathbb{HP}^n$, that is either a point on $\partial\mathbf{H}^n_{\mathbb{H}}$ or a point in $\mathbb{P}(\mathrm{V}_+)$. The lift of x in $\mathbb{H}^{n,1}$ is the quaternionic line $\mathbf{x}\mathbb{H}$. Then x is a *projective fixed point* of A. The quaternionic line $\mathbf{x}\mathbb{H}$ is the *eigenline* spanned by the eigenvector $\mathbf{x}$.

There are two eigenvalue classes of null-type and the respective eigenlines correspond to attracting and repelling fixed points. Let $r_A \in \partial\mathbf{H}^n_{\mathbb{H}}$ be the *repelling fixed point* of A that corresponds to the eigenvalue $re^{\mathbf{i}\theta}$ and let a_A be the *attracting fixed point* corresponding to the eigenvalue $r^{-1}e^{\mathbf{i}\theta}$. Let r_A and a_A lift to eigenvectors $\mathbf{r}_A$ and $\mathbf{a}_A$ respectively. Let $\mathbf{x}_{j,A}$ be an eigenvector corresponding to $e^{\mathbf{i}\phi_j}$. We may further assume that θ, ϕ_j are in $[0,\pi]$. The point $x_{j,A}$ on $\mathbb{P}(\mathrm{V}_+)$ is the *polar-point* of A. For $(r,\theta,\phi_1,\dots,\phi_{n-1})$ as above, let $E_A(r,\theta,\phi_1,\dots,\phi_{n-1})$, or simply E_A be the matrix:

$$E_A(r,\theta,\phi_1,\dots,\phi_{n-1}) = \begin{bmatrix} re^{\mathbf{i}\theta} & 0 & 0 & \dots & 0 & 0 \\ 0 & e^{\mathbf{i}\phi_1} & 0 & \dots & 0 & 0 \\ & & \ddots & & & \\ 0 & 0 & 0 & \dots & e^{\mathbf{i}\phi_{n-1}} & 0 \\ 0 & 0 & 0 & \dots & 0 & r^{-1}e^{\mathbf{i}\theta} \end{bmatrix}. \tag{2.3}$$

Let $C_A = \begin{bmatrix} \mathbf{a}_A & \mathbf{x}_{1,A} & \dots & \mathbf{x}_{n-1,A} & \mathbf{r}_A \end{bmatrix}$ be the matrix corresponding to the eigenvectors of the above eigenvalue representatives. We can choose C_A to be an element of $\mathrm{Sp}(n,1)$ by normalizing the eigenvectors:

$$\langle \mathbf{a}_A, \mathbf{r}_A\rangle = 1,\ \langle \mathbf{x}_{j,A}, \mathbf{x}_{j,A}\rangle = 1.$$

Then $A = C_A E_A C_A^{-1}$.

2.4. Eigenspace decomposition of a hyperbolic element. Suppose A is a hyperbolic element in $\mathrm{Sp}(n,1)$. Suppose also that the eigenvalue classes are represented by $re^{\mathbf{i}\theta}$, $r^{-1}e^{\mathbf{i}\theta}$, $r>1$, and $e^{\mathbf{i}\phi_1},\dots,e^{\mathbf{i}\phi_k}$. Let $m_i = \dim \mathrm{V}_{\phi_i}$. We call $m=(m_1,\dots,m_k)$ the *multiplicity* of A. Then $\mathbb{H}^{n,1}$ has the following orthogonal decomposition into eigenspaces (here $\oplus$ denotes orthogonal sum):

$$\mathbb{H}^{n,1} = \text{Ł}_r \oplus \mathrm{V}_{\phi_1} \oplus \dots \oplus \mathrm{V}_{\phi_k}, \tag{2.4}$$

where $Ł_r$ is the $(1,1)$ (right) subspace of $\mathbb{H}^{n,1}$ spanned by $\mathbf{a}_A$, $\mathbf{r}_A$. As in the elliptic case, change of normalised eigenbasis of A is determined up to the conjugation action of $Z(A)$.

2.5. Determination of the semisimple elements. In the following, we determine the semisimple isometries. We shall associate certain spatial parameters to an isometry that would determine it completely. Lemma 4.3 below will be crucial for classification of the conjugation orbits of semisimple pairs.

DEFINITION 4.1. Let A and A' be two semisimple elements with the same set of eigenvalue classes. We shall say that A and A' have the same projective fixed points if for each non-real class $[\lambda]$ they have the same projective fixed point x_λ.

DEFINITION 4.2. Let A and A' be two semisimple elements having a common non-real eigenvalue class $[\lambda]$. We shall say that A and A' have the same point on the $[\lambda]$-eigenvalue Grassmannian if they have the same point on the eigenvalue Grassmannian with respect to the representative λ of $[\lambda]$.

LEMMA 4.3. *Let A and A' be two semisimple elements in* $\mathrm{Sp}(n,1)$. *Then $A = A'$ if and only if they have the same real trace, the same projective fixed points and the same point on each of the eigenvalue Grassmannians.*

PROOF. If $A = A'$, then the statement is clear. For the converse, let A and A' be two semisimple elements of $\mathrm{Sp}(n,1)$. Since they have the same real traces, they have the same eigenvalue classes. Further, A and A' have the same projective fixed points, hence $\mathbf{x}_{j,A'} = \mathbf{x}_{j,A} q_j$ for $1 \leq j \leq n+1$, and $q_j \in \mathbb{H}^*$. Hence,

$$A = C_A E_A C_A^{-1}, \text{ and } A' = C_{A'} E_A C_{A'}^{-1}.$$

Let

$$D = \begin{bmatrix} q_1 & 0 & 0 & \dots & 0 \\ 0 & q_3 & 0 & \dots & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & q_{n-1} & 0 \\ 0 & 0 & \dots & 0 & q_2 \end{bmatrix}.$$

Then $A = C_A E_A C_A^{-1}$ and, $A' = C_A D E_A D^{-1} C_A^{-1}$. Thus $A = A'$ if and only if D commutes with E_A, which is equivalent to the condition of having the same point on each of the eigenvalue Grassmannians. □

In linear algebraic terms, the above lemma may be re-stated as follows.

COROLLARY 4.4. *Let A and A' be two semisimple elements in* $\mathrm{Sp}(n,1)$. *Then $A = A'$ if and only if the following holds.*

1. *A and A' have the same similarity classes of eigenvalues.*
2. *To each eigenvalue class $[\lambda]$, A and A' have the same eigenspace, and*
3. *to each representative λ' of $[\lambda]$, A and A' have the same eigensets.*

3. Associated points of an isometry

3.1. Associated points of a hyperbolic element.

DEFINITION 4.5. Suppose A is a hyperbolic element in $\mathrm{Sp}(n,1)$. Consider an ordered set of eigenbasis $\mathcal{B}_A = \{\mathbf{a}_A, \mathbf{x}_{1,A}, \dots, \mathbf{x}_{n-1,A}, \mathbf{r}_A\}$ corresponding to A, that is normalised so that for $1 \le l \le n-1$,

$$\langle \mathbf{a}_A, \mathbf{r}_A \rangle = \langle \mathbf{x}_{l,A}, \mathbf{x}_{l,A} \rangle = 1,\ \langle \mathbf{x}_{l,A}, \mathbf{x}_{m,A} \rangle = 0,\ l \neq m. \tag{3.1}$$

Define a set of $n+1$ boundary points associated to A as follows:

$$\mathbf{p}_{1,A} = \mathbf{a}_A,\ \mathbf{p}_{2,A} = \mathbf{r}_A,\ \mathbf{p}_{l,A} = (\mathbf{a}_A - \mathbf{r}_A)/\sqrt{2} + \mathbf{x}_{l-2,A},\ 3 \le l \le n+1. \tag{3.2}$$

The set $p_A = \{p_{1,A}, \dots, p_{n+1,A}\}$ is called a set of *associated points* of A.

LEMMA 4.6. *Let A be a hyperbolic element of* $\mathrm{Sp}(n,1)$. *Let $Z(A)$ denote the centralizer of A in* $\mathrm{Sp}(n,1)$. *Then the associated points of A is well-defined up to an orbit of the subgroup $Z(A)$.*

PROOF. Let A be a hyperbolic element in $\mathrm{Sp}(n,1)$. Let $\mathfrak{p} = (p_{1.A}, \dots, p_{n+1,A})$ be a tuple of associated points of A given by an eigenbasis $\mathcal{B}_A$ as above. If we choose another normalised eigenbasis $\mathcal{B}'_A$ of A, we get another set of associated points $\mathfrak{p}'$. The map M changing $\mathcal{B}_A$ to $\mathcal{B}'_A$ satisfies $MAM^{-1} = A$. Thus, M belongs to $Z(A)$, and $M(\mathfrak{p}) = \mathfrak{p}'$. □

3.2. Associated points of an elliptic element.

DEFINITION 4.7. Let A be an elliptic element in $\mathrm{Sp}(n,1)$. Let $\mathcal{B}_A = \{\mathbf{x}_{1,A}, \dots, \mathbf{x}_{n+1,A}\}$ be a set of eigenvectors of A chosen so that for $1 \le l \le n+1$,

$$\langle \mathbf{x}_{1,A}, \mathbf{x}_{1,_A} \rangle = -1,\ \langle \mathbf{x}_{j,A}, \mathbf{x}_{j,A} \rangle = 1,\ j \neq 1,\ \langle \mathbf{x}_{l,A}, \mathbf{x}_{m,A} \rangle = 0,\ l \neq m. \tag{3.3}$$

Define a set of $n+1$ points on $\mathbf{H}_{\mathbb{H}}^n$ as follows:

$$\mathbf{p}_{1,A} = \mathbf{x}_{1,A}, \ \mathbf{p}_{l,A} = \mathbf{x}_{1,A}\sqrt{2} + \mathbf{x}_{j,A}, \ 2 \leq j \leq n+1. \tag{3.4}$$

The set $p_A = \{p_{1,A}, \dots, p_{n+1,A}\}$ is called a set of *associated points* to A.

LEMMA 4.8. *Let A be an elliptic element of* $\mathrm{Sp}(n,1)$. *Let $Z(A)$ denote the centralizer of A in* $\mathrm{Sp}(n,1)$. *Then the associated points of A is well-defined up to an orbit of the subgroup $Z(A)$.*

The proof of the above lemma is similar to the proof of Lemma 4.6.

DEFINITION 4.9. Given a semisimple element A in $\mathrm{Sp}(n,1)$, a set of eigenbasis of the type $\mathcal{B}_A$ as given above, will be called an *eigenframe* of A.

A set of $n+1$ vectors of $\mathbb{H}^{n,1}$ alike an eigenframe will be called an *orthonormal frame*.

3.2.1. *Change of associated points amounts to a change of eigenbases.* The following lemma is easy to prove and it shows that the change of associated points amount change of eigenbases.

LEMMA 4.10. *Let A, A' be semisimple elements in* $\mathrm{Sp}(n,1)$ *with chosen eigenframes. Let $p_A = \{p_{1,A}, \dots, p_{n+1,A}\}$ and $p_{A'} = \{p_{1,A'}, \dots, p_{n+1,A'}\}$ be sets of associated points to A and A', respectively. Suppose that there exists $C \in$* $\mathrm{Sp}(n,1)$ *such that $C(p_{l,A}) = p_{l,A'}$, $1 \leq l \leq n$. Then $C(x_{j,A}) = x_{j,A'}$ for all j.*

PROOF. We prove our claim for hyperbolic isometries. The elliptic case is similar.

Let $C(\mathbf{p}_{l,A}) = \mathbf{p}_{l,A'}\alpha_l$ for $1 \leq l \leq n$. Observe that $\langle \mathbf{p}_{1,A}, \mathbf{p}_{l,A} \rangle = -1/\sqrt{2} = \langle \mathbf{p}_{1,A'}, \mathbf{p}_{l,A'} \rangle$ and $\langle \mathbf{p}_{2,A}, \mathbf{p}_{l,A} \rangle = 1/\sqrt{2} = \langle \mathbf{p}_{2,A'}, \mathbf{p}_{l,A'} \rangle$ for $3 \leq l \leq n$. Since $C \in \mathrm{Sp}(n,1)$ preserve the form $\langle .,. \rangle$, from these relations we have $\alpha_i = \bar{\alpha}_1^{-1} = \bar{\alpha}_2^{-1}$ for $3 \leq i \leq n$. Now, $\langle \mathbf{p}_{1,A}, \mathbf{p}_{2,A} \rangle = 1$ gives $|\alpha_1| = 1$. Hence, $C(x_{i-2,A}) = x_{i-2,A'}$ for $3 \leq i \leq n$. This implies $C(x_{n-1,A}) = x_{n-1,A'}$. □

4. Canonical orbit of a pair

4.1. Canonical orbit of a pair of hyperbolics.

4.1.1. *Moduli of normalised boundary points.* Consider the set $\mathcal{E}$ of ordered tuples of boundary and polar points on $(\partial\mathbf{H}^n_{\mathbb{H}})^4 \times \mathbb{P}(\mathrm{V}_+)^{2(n-1)}$ given by a pair of orthonormal frames (F_1, F_2):

$$\mathfrak{p} = (q_1, q_2, r_1, r_2, \dots, r_{n-1}, q_{n+1}, q_{n+2}, r_{n+1}, \dots, r_{2n-1}).$$

This corresponds to pair of orthonormal frames of $\mathbb{H}^{n,1}$:

$$\hat{\mathfrak{p}} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{n-1}, \mathbf{q}_{n+1}, \mathbf{q}_{n+2}, \mathbf{r}_{n+1}, \dots, \mathbf{r}_{2n-1}),$$

where $\{\mathbf{q}_1, \mathbf{q}_2\} \cap \{\mathbf{q}_{n+1}, \mathbf{q}_{n+2}\} = \emptyset$, $\langle \mathbf{q}_i, \mathbf{q}_i \rangle = 0 = \langle \mathbf{q}_{n+i}, \mathbf{q}_{n+i} \rangle$ for $i = 1, 2$, $\langle \mathbf{r}_j, \mathbf{r}_j \rangle = \langle \mathbf{r}_{n+j}, \mathbf{r}_{n+j} \rangle = 1$ for all $j = 1, \dots, n-1$, $\langle \mathbf{q}_1, \mathbf{q}_2 \rangle = \langle \mathbf{q}_{n+1}, \mathbf{q}_{n+2} \rangle = \langle \mathbf{q}_1, \mathbf{q}_{n+2} \rangle = 1$, $\langle \mathbf{q}_i, \mathbf{r}_j \rangle = 0 = \langle \mathbf{q}_{n+i}, \mathbf{r}_{n+j} \rangle$, for $i = 1, 2$, $j = 1, \dots, n-1$.

To each such point, we have an ordered tuple of boundary points, not necessarily distinct, $(p_1, \dots, p_{2n+2})$ satisfying the conditions:

$$\langle \mathbf{p}_1, \mathbf{p}_2 \rangle = \langle \mathbf{p}_{n+2}, \mathbf{p}_{n+3} \rangle = \langle \mathbf{p}_1, \mathbf{p}_{n+2} \rangle = 1, \tag{4.1}$$

$$\langle \mathbf{p}_i, \mathbf{p}_j \rangle = -1 = \langle \mathbf{p}_{n+i}, \mathbf{p}_{n+j} \rangle, i \neq j,\ i, j = 3, \dots, n-1; \tag{4.2}$$

$$\langle \mathbf{p}_1, \mathbf{p}_i \rangle = -\frac{1}{\sqrt{2}} = \langle \mathbf{p}_{n+2}, \mathbf{p}_{n+i} \rangle,\ i = 3, \dots, n-1; \tag{4.3}$$

$$\langle \mathbf{p}_2, \mathbf{p}_k \rangle = \frac{1}{\sqrt{2}} = \langle \mathbf{p}_{n+2}, \mathbf{p}_{n+k} \rangle,\ k = 3, \dots, n-1, \tag{4.4}$$

where $\mathbf{p}_i$ denotes the standard lift of p_i for each i. Note that $\mathbf{p}_i, \mathbf{p}_{n+i}$, $i = 3, \dots, n$, may not be distinct. In this case, we relabel them and write them as a ordered tuple of distinct boundary points $\hat{\mathfrak{p}} = (p_1, p_2, \dots, p_t)$, $n+3 \leq t \leq 2n+2$, so that they correspond to the original ordering of $\mathfrak{p}$.

Let $\mathcal{L}_t$ be the section of $\mathrm{M}(n, t, 0)$ defined by the equations (4.1)– (4.4), and the ordering as described above. Let $\mathcal{L}$ be the disjoint union $\mathcal{L} = \bigcup_{t=n+3}^{2n+2} \mathcal{L}_t$.

4.1.2. *Canonical orbit of a hyperbolic pair.* Let (A, B) be a hyperbolic pair in $\mathrm{Sp}(n, 1)$. Given the pair (A, B), consider the ordered set of eigenvectors of A, B given by the tuple

$$\mathfrak{e} = (\mathbf{a}_A, \mathbf{r}_A, \mathbf{x}_{1,A}, \dots, \mathbf{x}_{n-1,A}, \mathbf{a}_B, \mathbf{r}_B, \mathbf{x}_{1,B}, \dots, \mathbf{x}_{n-1,B}),$$

with normalization as follows:

$$\langle \mathbf{a}_A, \mathbf{r}_A \rangle = \langle \mathbf{x}_{i,A}, \mathbf{x}_{i,A} \rangle = 1, \ \langle \mathbf{x}_{i,A}, \mathbf{x}_{j,A} \rangle = 0, \ i \neq j. \tag{4.5}$$

$$\langle \mathbf{a}_B, \mathbf{r}_B \rangle = \langle \mathbf{x}_{i,B}, \mathbf{x}_{i,B} \rangle = 1, \ \langle \mathbf{x}_{i,B}, \mathbf{x}_{j,B} \rangle = 0, \ i \neq j. \tag{4.6}$$

$$\langle \mathbf{r}_A, \mathbf{a}_B \rangle = 1. \tag{4.7}$$

Assign the associated boundary points to $\mathfrak{e}$ defined by (3.2):

$$\mathfrak{p} = (a_A, r_A, q_{1,A}, \dots, q_{n-1,A}, a_B, r_B, q_{1,B}, \dots, q_{n-1,B}). \tag{4.8}$$

If we change $\mathfrak{e}$ to another pair of eigenframes $\mathfrak{e}'$ of (A, B), say

$$\mathfrak{e}' = (C(\mathbf{a}_A), C(\mathbf{r}_A), C(\mathbf{x}_{1,A}), \dots, C(\mathbf{x}_{n-1,A}), D(\mathbf{a}_B), D(\mathbf{r}_B), D(\mathbf{x}_{1,B}), \dots, D(\mathbf{x}_{n-1,B})),$$

then since we are not changing (A, B), must have $C \in Z(A)$ and $D \in Z(B)$. Accordingly, there is an action of $Z(A) \times Z(B)$ by the change of eigenframes, and the point $\mathfrak{p}$ changes to a point $\mathfrak{p}'$ on some $\mathrm{M}(n, t, 0)$, where $n+3 \leq t \leq 2n+2$. Thus, $\mathfrak{e}$, and hence $\mathfrak{p}$ is determined by (A, B) up to the above action of the group $Z(A) \times Z(B)$ on $\mathfrak{p}$.

The $Z(A) \times Z(B)$ action on $\mathfrak{p}$ above defines a set of points on $\mathcal{L}$. We shall call this set as a $Z(A) \times Z(B)$ orbit on $\mathcal{L}$ and denote it by $[\mathfrak{p}]$. We call $[\mathfrak{p}]$ the *canonical orbit* of (A, B). The association of the canonical orbit $[\mathfrak{p}]$ to the conjugacy class of (A, B) is well-defined.

It follows from the description of centralizers in **[G13]** that for all pairs of hyperbolic elements (A, B) with multiplicities $(a_1, \dots, a_k; b_1, \dots, b_l)$, and without an eigenvalue 1 or -1, we can identify their centralizers. This induces an action of $Z(A) \times Z(B)$ on $\mathcal{L}$ by the above construction. The orbit space on $\mathcal{L}$ under this $Z(A) \times Z(B)$ action is denoted by $\mathcal{QL}_n(a_1, \dots, a_k; b_1, \dots, b_l)$. If either of the hyperbolic elements in the pair has an eigenvalue 1 or -1, then the group $Z(A) \times Z(B)$ changes, but the same construction goes through. Taking disjoint union of all such orbit spaces, we get a space $\mathcal{QL}_n$. Each point on $\mathcal{QL}_n$ corresponds to a conjugacy class of a hyperbolic pair (A, B).

4.2. Canonical orbit of a pair of elliptics.

4.2.1. *Moduli of normalised points.* Consider the set $\mathcal{E}$ of ordered tuples of points on $(\mathbf{H}^n_{\mathbb{H}})^2 \cup \mathbb{P}(\mathrm{V}_+)^{2n}$ given by a pair of orthonormal frames (F_1, F_2):

$$\mathbf{p} = (x_1, \dots, x_{n+1}, x_{n+2}, \dots, x_{2n+2}).$$

To each such point, we have an ordered tuple of negative points, not necessarily distinct, $(p_1, \dots, p_{2n+2})$ satisfying the conditions:

$$\langle \mathbf{p}_{1,A}, \mathbf{p}_{1,A} \rangle = -1 = \langle \mathbf{p}_{n+2}, \mathbf{p}_{n+2} \rangle \tag{4.9}$$

$$\langle \mathbf{p}_{j,A}, \mathbf{p}_{j,A} \rangle = -1 = \langle \mathbf{p}_{j,A'}, \mathbf{p}_{j,A'} \rangle \tag{4.10}$$

$$\langle \mathbf{p}_{s,A}, \mathbf{p}_{t,A} \rangle = 0 = \langle \mathbf{p}_{s,A'}, \mathbf{p}_{t,A'} \rangle, \ s \neq t, 1 \leq s, t \leq n+1, \tag{4.11}$$

$$\langle \mathbf{p}_{1,A}, \mathbf{p}_{j,A} \rangle = -\sqrt{2} = \langle \mathbf{p}_{1,A'}, \mathbf{p}_{j,A'} \rangle, \ j \neq 1, \tag{4.12}$$

$$\langle \mathbf{p}_{l,A}, \mathbf{p}_{m,A} \rangle = -2 = \langle \mathbf{p}_{l,A'}, \mathbf{p}_{m,A'} \rangle, \ l, m \neq 1, \tag{4.13}$$

where $\mathbf{p}_i$ denotes the standard lift of p_i for each i. Note that $\mathbf{p}_i, \mathbf{p}_{n+i}$, $i = 3, \dots, n$, may not be distinct. If they are not distinct, we relabel them and write them as an ordered tuple of distinct negative points $\hat{\mathbf{p}} = (p_1, p_2, \dots, p_t)$, $n+3 \leq t \leq 2n+2$, so that they correspond to the original ordering of $\mathbf{p}$.

Let $\mathcal{L}_t$ be the section of $\mathrm{M}(n, 0, t)$ defined by the equations (4.9)– (4.13), and let also the ordering as described above. Let $\mathcal{L}$ be the disjoint union $\mathcal{L} = \bigcup_{t=n+3}^{2n+2} \mathcal{L}_t$.

4.2.2. *Canonical orbit of an elliptic pair.* Let A and B are elliptic elements in $\mathrm{Sp}(n, 1)$ without a common fixed point. Given the pair (A, B), fix eigenframes of A and B so that

$$\langle \mathbf{x}_{1,A}, \mathbf{x}_{1,A} \rangle = -1 = \langle \mathbf{x}_{1,B}, \mathbf{x}_{1,B} \rangle, \ \langle \mathbf{x}_{i,A}, \mathbf{x}_{i,A} \rangle = \langle \mathbf{x}_{i,B}, \mathbf{x}_{i,B} \rangle = 1 = \langle \mathbf{x}_{1,A}, \mathbf{x}_{1,B} \rangle, 1 \leq i \leq n+1.$$

Consider the ordered tuple of eigenvectors $\mathcal{B} = (\mathbf{x}_{1,A}, \dots, \mathbf{x}_{n+1,A}, \mathbf{x}_{1,B}, \dots, \mathbf{x}_{n+1,B})$. This gives an ordered tuple of points in $\mathbf{H}^n_{\mathbb{H}}$ given by

$$\mathfrak{p} = (p_{1,A}, \dots, p_{n+1,A}, p_{1,B}, \dots, p_{n+1,B}),$$

where $\mathbf{p}_{i,A}$, $\mathbf{p}_{i,B}$ are defined by (3.4).

The tuple $\mathfrak{p}$ is determined by (A, B) up to the above action of the group $G = Z(A) \times Z(B)$ on $\mathfrak{p}$. So, to each pair (A, B), we have a $Z(A) \times Z(B)$ orbit of associated points. The $Z(A) \times Z(B)$ action on $\mathfrak{p}$ defines a set of points on $\mathcal{L}$. We shall call this set as a $Z(A) \times Z(B)$ orbit on $\mathcal{L}$ and denote it by $[\mathfrak{p}]$. We call $[\mathfrak{p}]$ the *canonical orbit* of (A, B). The canonical orbit $[\mathfrak{p}]$ corresponds uniquely to the conjugacy class of (A, B).

The above action of $Z(A) \times Z(B)$ on $\mathfrak{p}$ induces an action of $Z(A) \times Z(B)$ on $\mathcal{L}$ similarly as described in the previous section. This gives a $Z(A) \times Z(B)$ orbit $[\mathfrak{p}]$ in $\mathcal{L}$ and we call it the *canonical orbit* of (A, B). The orbit space on $\mathcal{L}$ under the above G-action will be denoted by the same symbol as in the previous section, $\mathcal{QL}_n(a_1, \dots, a_k; b_1, \dots, b_l)$. Taking disjoint union of all such orbit spaces, we get a space $\mathcal{QL}_n$. Each point on $\mathcal{QL}_n$ corresponds to a conjugacy class of an elliptic pair (A, B).

4.3. Canonical orbit of a mixed pair. In this case, the construction is very similar to the elliptic and hyperbolic case. Let A be hyperbolic and B be elliptic in $\mathrm{Sp}(n, 1)$. Given the pair (A, B), fix a pair of associated orthonormal frames $\mathcal{B} = (\mathcal{B}_A, \mathcal{B}_B)$ so that the eigenvectors are normalised as in Section 3.1 and Section 3.2. Next we choose an ordering as in the previous section and associate an ordered tuple of points $\mathfrak{p}$ on $\mathrm{M}(n, n+1, n+1)$. The $Z(A) \times Z(B)$ action gives a orbit $[\mathfrak{p}]$ on $\mathrm{M}(n, n+1, n+1)$ as earlier. Taking disjoint union of all such orbits, we get a space, still denoted by $\mathcal{QL}_n$ as in the previous section. Each point on $\mathcal{QL}_n$ corresponds to a conjugacy class of a mixed pair (A, B).

5. Classification of semisimple pairs in $\mathrm{Sp}(n, 1)$

THEOREM 4.11. **[GK19,** Theorem 1.3**]** *Let (A, B) be a semisimple pair in $\mathrm{Sp}(n, 1)$ such that A and B do not have a common fixed point. Then (A, B) is determined up to conjugation in $\mathrm{Sp}(n, 1)$, by the real traces $\mathrm{tr}_{\mathbb{R}}(A)$, $\mathrm{tr}_{\mathbb{R}}(B)$, the canonical orbit of (A, B) on $\mathcal{QL}_n$, and a point on each of the eigenvalue Grassmannians of A and B.*

PROOF. For simplicity, we shall assume that neither A nor B has an eigenvalue 1 or -1. The proof is just similar in these omitted cases.

Suppose that (A, B) and (A', B') are hyperbolic pairs having equal real trace and the same canonical orbit. Equality of real traces implies that they have the same multiplicities, say $(a_1, \ldots, a_k, b_1, \ldots, b_l)$. Following the notation in Section 2, we may assume $A = C_A E_A C_A^{-1},\ B = C_B E_B C_B^{-1}$ and similarly for A' and B'. In this case, C_A is an element in the subgroup $\mathrm{Sp}(1,1) \times \mathrm{Sp}(a_1) \times \ldots \times \mathrm{Sp}(a_k)$:

$$C_A = \begin{bmatrix} \mathbf{a}_A & L_1 & \ldots & L_k & \mathbf{r}_A \end{bmatrix},$$

where $L_i = \begin{bmatrix} \mathbf{x}_{t_i,A} & \ldots & \mathbf{x}_{t_i+a_i-1,A} \end{bmatrix}$, E_A is the diagonal matrix

$$E_A = \begin{bmatrix} re^{\mathbf{i}\theta} & 0 & & \ldots & \\ 0 & I_{a_1}\lambda_1 & 0 & 0\ldots & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & I_{a_k}\lambda_k & 0 \\ 0 & 0 & 0 & 0 & r^{-1}e^{\mathbf{i}\theta} \end{bmatrix},$$

where I_s denotes the identity matrix of rank s. Similarly E_B is a diagonal matrix

$$E_B = \begin{bmatrix} se^{\mathbf{i}\alpha} & 0 & & \ldots & \\ 0 & I_{b_1}\mu_1 & 0 & 0\ldots & 0 \\ & & \ddots & & \\ 0 & 0 & 0 & I_{b_l}\mu_l & 0 \\ 0 & 0 & 0 & 0 & s^{-1}e^{\mathbf{i}\alpha} \end{bmatrix},$$

and,

$$C_B = \begin{bmatrix} \mathbf{a}_B & K_1 & \ldots & K_l & \mathbf{r}_B \end{bmatrix},$$

where $K_i = \begin{bmatrix} \mathbf{x}_{t_j,B} & \ldots & \mathbf{x}_{t_j+b_j-1,B} \end{bmatrix}$. In the above notation, $t_i = \sum_{p=1}^{i} a_{p-1}$, $s_j = \sum_{p=1}^{j} b_{p-1}$, $a_0 = b_0 = 1$.

Since the canonical orbits are equal, by Lemma 4.10 it follows that there exists a $C \in \mathrm{Sp}(n,1)$ such that $C(a_A) = a_{A'},\ C(r_A) = r_{A'},\ C(a_B) = a_{B'},\ C(r_B) = r_B$, and for $1 \le i \le k$, $1 \le j \le l$,

$$C(\mathbf{x}_{t_i,A}, \ldots, \mathbf{x}_{t_i+a_i-1,A}) = M(\mathbf{x}_{t_i,A'}, \ldots, \mathbf{x}_{t_i+a_i-1,A'}),$$

$$C(\mathbf{x}_{t_j,B}, \ldots, \mathbf{x}_{t_j+b_j-1,B}) = N(\mathbf{x}_{t_j,B'}, \ldots, \mathbf{x}_{t_j+b_j-1,B'}),$$

where $M \in Z(A')$, $N \in Z(B')$. Since A' commutes with M, $A'M(\mathbf{x}_{t_i,A'}) = MA'(\mathbf{x}_{t_i,A'}) = M(\mathbf{x}_{t_i,A'})\lambda_i$. From the above, we also have $M(\mathbf{x}_{t_i,A'}) = C(\mathbf{x}_{t_i,A})$, which implies that CAC^{-1} and A' have the same projective fixed point given by $M(x_{t_i,A'}) = C(x_{t_i,A})$.

Thus CAC^{-1} and A' have the same projective fixed points. Since A and A' define the same point on each of the eigenvalue Grassmannians and have the same real traces, by Lemma 4.3, $CAC^{-1} = A'$. Similarly, $B' = CBC^{-1}$.

Suppose (A, B) and (A', B') are elliptic pairs with the same real traces and the same canonical orbit. Since they have the same traces, their multiplicities are also the same, say $(a_1, \ldots, a_k, b_1, \ldots, b_l)$. In this case, C_A is an element in the subgroup $\mathrm{Sp}(a_1 - 1, 1) \times \mathrm{Sp}(a_2) \times \ldots \times \mathrm{Sp}(a_k)$:

$$C_A = \begin{bmatrix} E_1 & E_2 & \ldots & E_k \end{bmatrix},$$

where $E_i = \begin{bmatrix} \mathbf{x}_{t_i,A} & \ldots & \mathbf{x}_{t_i+a_i-1,A} \end{bmatrix}$, and E_A is the diagonal matrix

$$E_A = \begin{bmatrix} \lambda_1 I_{a_1} & & \\ & \ddots & \\ & & \lambda_k I_{a_k} \end{bmatrix},$$

where I_s denotes the identity matrix of rank s. Similarly for C_B, let

$$E_B = \begin{bmatrix} \mu_1 I_{a_1} & & \\ & \ddots & \\ & & \mu_k I_{a_k} \end{bmatrix},$$

and

$$C_B = \begin{bmatrix} E'_1 & E'_2 & \ldots & E'_k \end{bmatrix},$$

where $E'_i = \begin{bmatrix} \mathbf{x}_{t_i,B'} & \ldots & \mathbf{x}_{t_i+a_i-1,B'} \end{bmatrix}$.

Since the canonical orbits are equal, by Lemma 4.10 it follows that there exists a $C \in \mathrm{Sp}(n, 1)$ such that for $1 \le i \le k$, $1 \le j \le l$,

$$C(\mathbf{x}_{t_i,A}, \ldots, \mathbf{x}_{t_i+a_i-1,A}) = M(\mathbf{x}_{t_i,A'}, \ldots, \mathbf{x}_{t_i+a_i-1,A'}),$$

$$C(\mathbf{x}_{t_j,B}, \ldots, \mathbf{x}_{t_j+b_j-1,B}) = N(\mathbf{x}_{t_j,B'}, \ldots, \mathbf{x}_{t_j+a_i-1,B'}),$$

where $M \in Z(A')$, $N \in Z(B')$. Now, using arguments as in the hyperbolic case, it follows that CAC^{-1} and A' have the same projective fixed points. Hence by Lemma 5.2, $CAC^{-1} = A'$. Similarly, $CBC^{-1} = B'$.

Suppose (A, B) and (A', B') are mixed pairs such that A, A' are hyperbolic and B, B' are elliptic. For the mixed pairs case the argument is similar. This completes the proof. $\square$

CHAPTER 5

Quaternionic hyperbolic Fenchel-Neilsen coordinates in $\mathrm{Sp}(2,1)$

In this chapter, Section 1 and Section 2 are really special cases of the work in the preceding chapter.

1. Hyperbolic isometries

Let A be a hyperbolic element in $\mathrm{Sp}(2,1)$. Let λ represents an eigenvalue from the similarity class of eigenvalues $[\lambda]$ of A. Let $\mathbf{x}$ be a λ-eigenvector. Then $\mathbf{x}$ defines a point x on $\mathbb{HP}^2$ that is either a point on $\partial\mathbf{H}^2_{\mathbb{H}}$ or, a point in $\mathbb{P}(\mathrm{V}_+)$. The lift of x in $\mathbb{H}^{2,1}$ is the quaternionic line $\mathbf{x}\mathbb{H}$. We call x as *eigen-point* of A corresponding to $[\lambda]$. Note that x is a fixed point of A in $\mathbb{HP}^2 \setminus (\mathbf{H}^2_{\mathbb{H}} \cup \partial\mathbf{H}^2_{\mathbb{H}})$.

Let A be a hyperbolic element then it has similarity classes of eigenvalues $[\lambda]$, $[\overline{\lambda}]^{-1}$ and $[\mu]$, where $|\lambda| < 1$, $|\mu| = 1$. Thus we can choose eigenvalue representatives to be the complex numbers $re^{i\theta},\ r^{-1}e^{i\theta}$, $e^{i\phi}$, $\theta, \phi \in [0,\pi]$, $0 < r < 1$.

Let $a_A \in \partial\mathbf{H}^2_{\mathbb{H}}$ be the *attracting fixed point* of A that corresponds to the eigenvalue $re^{i\theta}$ and let $r_A \in \partial\mathbf{H}^2_{\mathbb{H}}$ be the *repelling fixed point* corresponding to the eigenvalue $r^{-1}e^{i\theta}$. Let a_A and r_A lift to eigenvectors $\mathbf{a}_A$ and $\mathbf{r}_A$ respectively. Let $\mathbf{x}_A$ be an eigenvector corresponding to $e^{i\phi}$. The point x_A on $\mathbb{P}(\mathrm{V}_+)$ is the *polar-point* of A. For (r,θ,ϕ) as above, define $E_A(r,\theta,\phi)$ as

$$E_A(r,\theta,\phi) = \begin{bmatrix} re^{i\theta} & 0 & 0 \\ 0 & e^{i\phi} & 0 \\ 0 & 0 & r^{-1}e^{i\theta} \end{bmatrix}. \tag{1.1}$$

Let $C_A = \begin{bmatrix} \mathbf{a}_A & \mathbf{x}_A & \mathbf{r}_A \end{bmatrix}$ be the 3×3 matrix corresponding to the eigenvectors. We can choose C_A to be an element of $\mathrm{Sp}(2,1)$ by normalizing the eigenvectors:

$$\langle \mathbf{a}_A, \mathbf{r}_A\rangle = 1,\ \langle \mathbf{x}_A, \mathbf{x}_A\rangle = 1.$$

Then $A = C_A E_A(r,\theta,\phi) C_A^{-1}$. So, every hyperbolic element A in $\mathrm{Sp}(2,1)$ is conjugate to a matrix of the form $E_A(r,\theta,\phi)$.

2. Classification of hyperbolic elements in $\mathrm{Sp}(2,1)$.

For clarity, first we define the following notion.

DEFINITION 5.1. Let A and A' be two hyperbolic elements having a common eigenvalue $[\alpha]$, $\alpha \in \mathbb{H} \setminus \mathbb{R}$. Then A and A' are said to have the same $[\alpha]$-projective point if they have the same projective point with respect to a representative α of $[\alpha]$.

Suppose A and A' are two hyperbolic elements having the same $[\alpha]$-projective point. If A and A' have the same eigenset with respect to a chosen representative α, then they have the same eigensets with respect to all other representatives. In this case A and A' define the same projective point for all representatives of $[\alpha]$.

Now, we have the following lemma that determines hyperbolic elements in $\mathrm{Sp}(2,1)$.

LEMMA 5.2. *Let A and A' be hyperbolic elements in $\mathrm{Sp}(2,1)$. Then $A = A'$ if and only if they have the same attracting fixed point, the same repelling fixed point, the same real trace, and the same projective points.*

PROOF. If $A = A'$, then the statement is clear. We prove the converse. Without loss of generality, we can assume A and A' to be loxodromic elements. The other cases are similar.

Let A and A' be two loxodromic elements of $\mathrm{Sp}(2,1)$. Since they have the same real traces, let the eigenvalue representatives of A and A' be $re^{i\theta}$, $r^{-1}e^{i\theta}$ and $e^{i\phi}$, where $0 < r < 1$, $\theta, \phi \in (0,\pi)$. Then

$$A = C_A E_A(r,\theta,\phi) C_A^{-1}, \text{ and } A' = C_{A'} E_A(r,\theta,\phi) C_{A'}^{-1}.$$

Since A and A' have the same fixed-points, hence $\mathbf{a}_{A'} = \mathbf{a}_A q_1$, $\mathbf{r}_{A'} = \mathbf{r}_A q_2$, and $\mathbf{x}_{A'} = \mathbf{x}_A q_3$, where $q_j \in \mathbb{H} \setminus 0$. Let

$$D = \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_3 & 0 \\ 0 & 0 & q_2 \end{pmatrix}.$$

Then $A = C_A E_A(r,\theta,\phi)C_A^{-1}$ and, $A' = C_A D E_A(r,\theta,\phi)D^{-1}C_A^{-1}$. Now D commutes with $E_A(r,\theta,\phi)$ if and only if q_1, q_2, q_3 are complex numbers. This is equivalent to the condition of having the same projective points. Thus, it follows that $A = A'$. □

2.1. Projective parameters of a loxodromic. Suppose A is a loxodromic element in $\mathrm{Sp}(2,1)$. If a_A and r_A are the fixed-points of A, then they are joined by a complex line

$$\mathbf{a}_A Z(\lambda) + \mathbf{r}_A Z(\lambda),$$

and hence $(\mathbf{a}_A, \mathbf{r}_A)$ is determined by a single projective point on $\mathbb{CP}^1$ that corresponds to $Z(\lambda)$. Here we have used the fact that $Z(\lambda) = Z(\bar{\lambda}^{-1})$. Similarly, the projective point of $\mathbf{x}_A$ corresponds to the centralizer of $Z(\mu)$.

Thus given a triple $(a_A, r_A, x_A) \in \partial\mathbf{H}^2_{\mathbb{H}} \times \partial\mathbf{H}^2_{\mathbb{H}} \times \mathbb{P}(\mathrm{V}_+)$, and real number r, there is a two complex parameter family $\mathcal{H}_r$ of loxodromic elements having the same real trace r and fixing points (a_A, r_A, x_A). This two complex family of parameters correspond to the projective points of A on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Thus $\mathcal{H}_r$ is parametrized by $\mathbb{CP}^1 \times \mathbb{CP}^1$. Given $A \in \mathcal{H}_r$, the space $\mathbb{CP}^1 \times \mathbb{CP}^1$ corresponds to the conjugacy class of A in the stabilizer subgroup $\mathrm{Sp}(2,1)_{(a_A, r_A)}$.

2.2. Parametrization of conjugacy classes of loxodromics in $\mathrm{Sp}(2,1)$. When A is a loxodromic element in $\mathrm{Sp}(2,1)$, it follows that $tr_{\mathbb{R}}(A)$ belongs to an open subspace D_2 of $\mathbb{R}^3$. Using Proposition 2.5 we deduce the following lemma that provides a description of D_2. We recall that A is loxodromic if none of the eigenvalues are real.

LEMMA 5.3. *Let A be hyperbolic in $\mathrm{Sp}(2,1)$ and let $A_{\mathbb{C}}$ be its complex representative. The characteristic polynomial of $A_{\mathbb{C}}$ is of the form*

$$\chi_A(x) = x^6 - ax^5 + bx^4 - cx^3 + bx^2 - ax + 1,$$

where a, b, c are real numbers. Define

$$G = 27(a-c) + 9ab - 2a^3,$$

$$H = 3(b-3) - a^2,$$

$$\Delta = G^2 + 4H^3.$$

Then A is loxodromic if and only if $\Delta > 0$, $|2a + c| \neq |2b + 2|$.

PROOF. Since A is hyperbolic, it has eigenvalue representatives of the form $re^{i\theta}$, $r^{-1}e^{i\theta}$ and $e^{i\phi}$, where $r > 0, r \neq 1$ and $\theta, \phi \in [0, \pi]$. Further note that if α is a root of $\chi_A(x)$, then $\alpha + \alpha^{-1}$ is a root of $g_A(t)$. Thus we obtain

$$g_A(t) = t^3 - at^2 + (b-3)t - (c-2a), \tag{2.1}$$

where

$$\begin{aligned} a &= 2\big(r + \frac{1}{r}\big)\cos\theta + 2\cos\phi, \\ b &= 4\big(r + \frac{1}{r}\big)\cos\theta\cos\phi + 4\cos^2\theta + r^2 + \frac{1}{r^2} + 1, \\ c &= 4\big(r + \frac{1}{r}\big)\cos\theta + 2\big(r^2 + \frac{1}{r^2} + 4\cos^2\theta\big)\cos\phi. \end{aligned}$$

Now to detect the nature of roots of the cubic equation, we look at the discriminant sequence (G, H, Δ) of $g_A(t)$. The multiplicity of a root of $g_A(t)$ is determined by the resultant $R(g, g'')$ of $g_A(t)$ and its second derivative $g_A''(t) = 6t - 2a$. Note that

$$R(g, g'') = -8[27(a-c) + 9ab - 2a^3] = -8G.$$

When A is loxodromic, $g_A(t)$ has the following roots:

$$t_1 = re^{i\theta} + r^{-1}e^{-i\theta}, t_2 = r^{-1}e^{i\theta} + re^{-i\theta}, t_3 = 2\cos\phi.$$

Thus in this case $\Delta > 0$ and $g_A(\pm 2) \neq 0$. In all other cases, either, $\Delta > 0$ and $g_A(\pm 2) = 0$, or, $\Delta = 0$. This proves the lemma. □

For $G = 27(c-a) - 9ab + 2a^3$, $H = 3(b-3) - a^2$,

$$\mathrm{D}_2 = \{(a,b,c) \in \mathbb{R}^3 \mid G^2 + 4H^3 > 0,\ |2a + c| \neq |2b + 2|\}.$$

For each loxodromic A, $tr_{\mathbb{R}}(A)$ is an element of D_2. Conversely, given an element (a, b, c) from the set D_2, we have a conjugacy class of loxodromics A with $tr_{\mathbb{R}}(A) = (a, b, c)$. Thus we have the following consequence of the above lemma.

COROLLARY 5.4. *The set of conjugacy classes of loxodromic elements in* $\mathrm{Sp}(2,1)$ *can be identified with* D_2.

REMARK 5.5. Other than loxodromics, there are three more types of hyperbolic elements in $\mathrm{Sp}(2,1)$. These types correspond to the cases:

(i) $\Delta > 0$, $|2a + c| = |2b + 2|$, and

(ii) $\Delta = 0$, $|2a + c| \neq |2b + 2|$.

(iii) $\Delta = 0$, $|2a + c| = |2b + 2|$.

In the case (i), the hyperbolic element has only one real eigenvalue, in the case (ii), there are only two real eigenvalues, and in case (iii), all the eigenvalues are real numbers, i.e. the element is strictly hyperbolic. For hyperbolic elements of types (i) and (ii), the real traces are parametrized by a two real parameter family.

3. Classification of pair of loxodromics in $\mathrm{Sp}(2,1)$

THEOREM 5.6. **[GK18**, Theorem 1.1] *Let (A, B) be a totally loxodromic pair in $\mathrm{Sp}(2,1)$. Then (A, B) is determined up to conjugation by the following parameters: $tr_{\mathbb{R}}(A)$, $tr_{\mathbb{R}}(B)$, the similarity classes of cross ratios of (A, B) or, a point on the four-dimensional cross ratio variety, the three angular invariants of (A, B), and the projective points $(p_1(A), p_2(A))$, $(p_1(B), p_2(B))$.*

PROOF. Suppose (A, B) and (A', B') be pairs of loxodromics such that $tr(A) = tr(A')$, $tr(B) = tr(B')$, for $i = 1, 2, 3$, $[\mathbb{X}_i(A, B)] = [\mathbb{X}_i(A', B')]$, $\mathbb{A}_i(A, B) = \mathbb{A}_i(A', B')$. Since the cross ratios and angular invariants are equal, by Theorem 2.8, there is an element C in $\mathrm{Sp}(2,1)$ such that $C(a_A) = a_{A'}$, $C(r_A) = r_{A'}$, $C(a_B) = a_{B'}$ and $C(r_B) = r_{B'}$. Therefore A' and CAC^{-1} have the same fixed points. Since they also have the same real traces, and the same projective points, we have by Lemma 5.2 that $CAC^{-1} = A'$. Similarly, $CBC^{-1} = B'$. This completes the proof. □

COROLLARY 5.7. **[GK18**, Corollary 1.3] *Let (A, B) be a mixed hyperbolic pair with A is loxodromic and B is strictly hyperbolic in $\mathrm{Sp}(2,1)$. Then (A, B) is uniquely determined up to conjugation by the parameters: $tr_{\mathbb{R}}(A)$, $tr_{\mathbb{R}}(B)$, a point on the cross ratio variety, the three angular invariants and projective points $p_1(A)$, $p_2(A)$.*

PROOF. In this case A is loxodromic and B is strictly hyperbolic. The same proof as above works noting that for B we do not require any projective points, as the real eigenvalues are elements of centralizers in $\mathbb{H}$. □

Similarly, Corollary 1.12 follows, and also similar results can be deduced for other types of mixed hyperbolic pairs.

COROLLARY 5.8. **[GK18**, Corollary 1.5] *The space* $\mathfrak{D}_{\mathfrak{L}o}(\mathrm{F}_2, \mathrm{Sp}(2,1))$ *is parametrized by a* $(\mathbb{CP}^1)^4$ *bundle over the topological space* $\mathrm{D}_2 \times \mathrm{D}_2 \times \mathcal{M}(2)$ *where, for* $G = 27(c-a) - 9ab + 2a^3$, $H = 3(b-3) - a^2$,

$$\mathrm{D}_2 = \{(a,b,c) \in \mathbb{R}^3 \mid G^2 + 4H^3 > 0,\ |2a+c| \neq |2b+2|\},$$

and $\mathcal{M}(2)$ *is the orbit space of the natural* S_4 *action on the configuration space of* $\mathrm{Sp}(2,1)$*-congruence classes of ordered quadruples of distinct points on* $\partial\mathbf{H}^2_{\mathbb{H}}$.

PROOF. Let $\mathrm{F}_2 = \langle x, y\rangle$. Given a representation ρ, we have the following correspondence from $\mathfrak{D}_{\mathfrak{L}o}(\mathrm{F}_2, \mathrm{Sp}(2,1))$ onto $\mathrm{D}_2 \times \mathrm{D}_2 \times \mathcal{M}(2)$:

$$\mathfrak{f} : [\rho] \mapsto (tr_{\mathbb{R}}(\rho(x)), tr_{\mathbb{R}}(\rho(y)), [(a_{\rho(x)}, r_{\rho(x)}, a_{\rho(y)}, r_{\rho(y)})]).$$

Given a point on $\mathrm{D}_2 \times \mathrm{D}_2 \times \mathcal{M}(2)$, it follows from Lemma 5.2 that the inverse-image of the point under $\mathfrak{f}$ is $(\mathbb{CP}^1 \times \mathbb{CP}^1) \times (\mathbb{CP}^1 \times \mathbb{CP}^1)$ corresponding to the projective points of $(\rho(x), \rho(y))$. This completes the proof. $\square$

4. Quaternionic hyperbolic Fenchel-Nielsen coordinates

To prove the theorem, we need to determine the parameters that are needed while attaching two $(0,3)$ groups, or 'closing a handle'. To obtain Fenchel-Nielsen coordinates on the representation variety, we may need to attach two $(0,3)$ groups to yield an $(1,1)$ group. Here an $(1,1)$ group is a subgroup of $\mathrm{Sp}(2,1)$ generated by two loxodromic elements and their commutator. For this reason, we need to define the *twist-bend parameters* while attaching $(0,3)$ groups. We follow similar ideas as noted in the paper of Parker and Platis **[PP08]**, and will also use the same terminologies given there. For detailed information about the terminologies and ideas inbuilt in the process, we refer to **[PP08]** and **[Wol82]**. We shall only sketch those parts from the scheme of attaching two $(0,3)$ groups that are not apparent in the $\mathrm{Sp}(2,1)$ setting, and deserves mention for clarity of the exposition. In the following all the $(0,3)$ groups will be assumed to be irreducible.

4.1. Twist-bend parameters. Let $\langle A, B\rangle$ and $\langle C, D\rangle$ be two $(0,3)$ groups such that their *boundaries are compatible*, that is, $A = D^{-1}$. A *quaternionic hyperbolic twist-bend* corresponds to an element K in $\mathrm{Sp}(2,1)$ that commutes with A and conjugate $\langle C, D\rangle$. Up to conjugacy assume A fixes $0, \infty$ and it is of the form $E_A(r,\theta,\phi)$. Since K commutes with A, it is also of the form $E_K(s,\psi,\xi)$, $s \geq 1$, $\psi,\xi \in [0,\pi]$. If $s \neq 1$, from Lemma 5.2, it follows that there is a total of seven real parameters associated to K, the real trace (s,ψ,ξ), along with four real parameters associated to the projective points. If $s = 1$, then K is a boundary elliptic and the eigenvalue $[e^{i\psi}]$ has multiplicity 2. But, we can still define the projective points for these eigenvalues. There are exactly one negative-type and one positive-type eigenvalues of K in this case. Since K commutes with A, the projective points of K is determined by the projective points of A. Hence, there are two projective points of K to determine it. Consequently, we shall have seven parameters associated to a twist-bend K. We denote these parameters by $\kappa = (s,\psi,\xi,k_1,k_2)$, where $k_1 = p_1(K)$, $k_2 = p_2(K)$ are the projective points of the similarity classes of eigenvalues of K.

We further fix the convention of choosing the twist-bend parameters such that it is *oriented consistently with* A, i.e. if $A = QE_A(r,\theta,\phi)Q^{-1}$, then $K = QE_K(s,\psi,\xi)Q^{-1}$. Since $\langle A, B\rangle$ and $\langle C, A^{-1}\rangle$ are considered irreducible, without loss of generality we assume that a_B, r_C do not lie on a boundary of totally geodesic subspace joining a_A and r_A. To obtain conjugacy invariants to measure the twist-bend parameter we define the following quantities.

$$\tilde{\mathbb{X}}_1(\kappa) = \mathbb{X}_1(a_A, r_A, K(r_C), a_B), \tilde{\mathbb{X}}_2(\kappa) = \mathbb{X}_2(a_A, r_A, K(r_C), a_B),$$
$$\tilde{\mathbb{A}}_1(\kappa) = \mathbb{A}(a_A, r_A, K(r_C)),\ \tilde{\mathbb{A}}_3(\kappa) = \mathbb{A}(r_A, K(r_C), a_B).$$

LEMMA 5.9. *Let A, B, C be loxodromic transformations of $\mathbf{H}^2_{\mathbb{H}}$ such that $\langle A, B\rangle$ and $\langle A^{-1}, C\rangle$ are irreducible $(0,3)$ subgroups of $\mathrm{Sp}(2,1)$. Suppose that $K = E_K(s,\psi,\xi,k_1,k_2)$ and $K' = E_{K'}(s',\psi',\xi',k_1',k_2')$ represents twist-bend parameters that are oriented consistently with A. If*

$[\tilde{\mathbb{X}}_1(\kappa)] = [\tilde{\mathbb{X}}_1(\kappa')]$, $[\tilde{\mathbb{X}}_2(\kappa)] = [\tilde{\mathbb{X}}_2(\kappa')]$, and $\tilde{\mathbb{A}}_1(\kappa) = \tilde{\mathbb{A}}_1(\kappa')$, $\tilde{\mathbb{A}}_3(\kappa) = \tilde{\mathbb{A}}_3(\kappa')$, and $k_1 = k_1'$, $k_2 = k_2'$, then $K = K'$.

PROOF. Without loss of generality we assume $a_A = o$, $r_A = \infty$. In view of the conditions

$$[\tilde{\mathbb{X}}_1(\kappa)] = [\tilde{\mathbb{X}}_1(\kappa')],\ [\tilde{\mathbb{X}}_2(\kappa)] = [\tilde{\mathbb{X}}_2(\kappa')], \text{ and}$$
$$\tilde{\mathbb{A}}_1(\kappa) = \tilde{\mathbb{A}}_1(\kappa'),\ \tilde{\mathbb{A}}_3(\kappa) = \tilde{\mathbb{A}}_3(\kappa').$$

and by using real relations of cross ratio, we get $[\tilde{\mathbb{X}}_3(\kappa)] = [\tilde{\mathbb{X}}_3(\kappa')]$. The quantities $\tilde{\mathbb{A}}_2(\kappa)$ and $\tilde{\mathbb{A}}_2(\kappa')$ are trivially equal. Now, following similar arguments as in the proof of [**Cao16**, Theorem 5.2], we have f in $\mathrm{Sp}(2,1)$ such that $f(a_A) = a_A$, $f(r_A) = r_A$, $f(a_B) = a_B$ and $f(E_K(r_C)) = E_{K'}(r_C)$. Further, since it fixes three points on the boundary, it must be of the form

$$f = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \psi \end{bmatrix}.$$

Since a_B does not lie on a totally geodesic subspace joining a_A and r_A, we must have $\mu = \pm 1 = \psi$. Thus, it follows that $E_K(r_C) = E_{K'}(r_C)$. Now by using the fact that $E_K {E_{K'}}^{-1}$ has the three fixed points $a_A = o, r_A = \infty$ and r_C together with the condition that r_C does not lie on a totally geodesic subspace joining a_A and r_A, we have $E_K(s, \psi, \xi) = E_{K'}(s', \psi', \xi')$ i.e, $s = s', \psi = \psi', \xi = \xi'$.

Hence, K and K' are conjugate with the same attracting and the same repelling points. So, by Lemma 5.2, $K = K'$ if and only if they have the same projective points and the same fixed points. This completes the proof. □

In view of Theorem 5.6 and Lemma 5.9, we shall now proceed to prove Theorem 5.14. The strategy of the proof is similar to the proofs given by Parker and Platis in [**PP08**, Section 8], or Gongopadhyay and Parsad in [**GP17**, Section 6]. The main challenge in the quaternionic set up was the derivations of Theorem 5.6 and Lemma 5.9. After those are in place, the rest follows mimicking arguments of Parker and Platis with appropriate modifications in the quaternionic set up. These arguments are mostly group theoretic and does not involve arithmetic of the underlying quaternionic algebra. We will only mention the key points of the attaching process. For detailed ideas behind them, we refer to the original article of Parker and Platis [**PP08**].

4.1.1. *Attaching two (0,3) subgroups.* A $(0,4)$ subgroup of $\mathrm{Sp}(2,1)$ is a group with four loxodromic generators such that their product is identity. These four loxodromic maps correspond to the boundary curves of the four-holed spheres and are called *peripheral*. Thus a $(0,4)$ group is freely generated by any of these three loxodromic elements. Let $\langle A, B\rangle$ and $\langle C, D\rangle$ be two $(0,3)$ groups with $A^{-1} = D$. Algebraically, a $(0,4)$ group is constructed by the amalgamated free product of these groups with amalgamation along the common cyclic subgroup $\langle A\rangle$. Conjugating $\langle C, D\rangle$ by the twist-bend K yields a new $(0,4)$ subgroup that is depended on K. Varying this K gives the twist-bend deformation.

The following lemma goes the same way as Lemma 8.3 in [**PP08**, p.131].

LEMMA 5.10. *Suppose* $\Gamma_1 = \langle A, B\rangle$ *and* $\Gamma_2 = \langle C, D\rangle$ *are two irreducible* $(0,3)$ *groups with peripheral elements* A, B, $B^{-1}A^{-1}$ *and* C, D, $D^{-1}C^{-1}$ *respectively. Moreover suppose that* $A = D^{-1}$. *Let* K *be any element of* $\mathrm{Sp}(2,1)$ *that commutes with* $A = D^{-1}$. *The the group* $\langle A, B, KCK^{-1}\rangle$ *is a* $(0,4)$ *group with peripheral elements* B, $B^{-1}A^{-1}$, KCK^{-1} *and* $KD^{-1}C^{-1}K^{-1}$.

PROPOSITION 5.11. *Suppose that* $\langle A, B\rangle$ *and* $\langle C, A^{-1}\rangle$ *are two irreducible* $(0,3)$ *subgroups of* $\mathrm{Sp}(2,1)$. *Let* $\kappa = (s, \psi, \xi, k_1, k_2)$ *be a twist-bend parameter oriented consistently with* A *and let* $\langle A, B, KCK^{-1}\rangle$ *be the corresponding* $(0,4)$ *group. Then* $\langle A, B, KCK^{-1}\rangle$ *is uniquely determined up to conjugation in* $\mathrm{Sp}(2,1)$ *by the* Fenchel-Nielsen coordinates*:*

the real traces: $tr_{\mathbb{R}}(A),\ tr_{\mathbb{R}}(B),\ tr_{\mathbb{R}}(C)$;

the cross ratios: $[\mathbb{X}_k(A,\ B)],\ [\mathbb{X}_k(A, C)], k = 1,2$;

the angular invariants: $\mathbb{A}_j(A, B),\ \mathbb{A}_j(A, C),\ j = 1,2,3$;

six projective points: $p_i(A),\ p_i(B),\ p_i(C),\ i = 1,2$;

and the twist bend $\kappa = (s, \psi, \xi, k_1, k_2)$.

Thus we need a total of 42 real parameters to specify $\langle A, B, KCK^{-1}\rangle$ *uniquely up to conjugacy.*

In the parameter space associated to $\langle A, B, KCK^{-1}\rangle$, the parameters corresponding to the traces have real dimension 6, the cross ratios and angular invariants add up to 16 degrees of freedom, six projective points add up to 12 degrees of freedom and the twist parameter has 7 degrees of freedom adding to a total of: $(3\times3)+(2\times4)+(1\times6)+(2\times6)+7 = 42$ degrees of freedom.

PROOF. Suppose, $\langle A, B, KCK^{-1}\rangle$ and $\langle A', B', K'C'K'^{-1}\rangle$ are two $(0,4)$ subgroups having the same Fenchel-Nielsen coordinates. Let the given relations hold. Using these relations, it follows from Theorem 5.6 that there exist C_1 and C_2 in $\mathrm{Sp}(2,1)$ that conjugate $\langle A, B\rangle$ and $\langle C, A^{-1}\rangle$ respectively to $\langle A', B'\rangle$ and $\langle C', A'^{-1}\rangle$.

Now the twist-bends are defined with respect to the same initial group $\langle A, B, C\rangle$ that we fix at the beginning before attaching the two $(0,3)$ groups. So, without loss of generality, we may assume that $A = A',\ B = B',\ C = C'$, and thus, $C_1 = C_2$. Now with respect to the same initial group $\langle A, B, C\rangle$, by Lemma 5.9 it follows that $K = K'$. This implies that $\langle A, B, KCK^{-1}\rangle$ is determined uniquely up to conjugacy.

Conversely, suppose that $\langle A, B, KCK^{-1}\rangle$ and $\langle A', B', K'C'K'^{-1}\rangle$ are conjugate, and having the same projective points. Then clearly, the traces and angular invariants are equal and, the cross ratios are similar. The only thing remains to show is the equality of the twist-bends. Now by the invariance of the cross ratios it follows that

$$[\tilde{\mathbb{X}}_1(\kappa)] = [\tilde{\mathbb{X}}_1(\kappa')],\ [\tilde{\mathbb{X}}_2(\kappa)] = [\tilde{\mathbb{X}}_2(\kappa')],\ \tilde{\mathbb{A}}_1(\kappa) = \tilde{\mathbb{A}}_1(\kappa'),\ \tilde{\mathbb{A}}_3(\kappa) = \tilde{\mathbb{A}}_3(\kappa').$$

and hence by Lemma 5.9, $\kappa = \kappa'$. □

4.2. Closing a handle. The next step is to obtain a one-holed torus by attaching two holes of the same three-holed sphere in the quaternionic hyperbolic plane. The process of attaching these two holes is called *closing a handle*. Geometrically, it corresponds to attach two boundary components of the same three-holed sphere. From a group theoretic viewpoint, closing a handle is the same as taking the HNN-extension of the $(0,3)$ group $\langle A, BA^{-1}B^{-1}\rangle$ by adjoining the element B to form a $(1,1)$ group. When we take the HNN-extension, the map B is not unique. If K is any element in $\mathrm{Sp}(2,1)$ that commutes with A, then $\langle A, BK\rangle$ gives another $(1,1)$ group. Varying K corresponds to a twist-bend coordinate as above.

If $A = QE(s,\psi,\xi)Q^{-1}$ for (s,ψ,ξ), just as before, we define the twist-bend parameter κ by $K = QE(s,\psi,\xi,k_1,k_2)Q^{-1}$, and we say, $\kappa = (s,\psi,\xi,k_1,k_2)$ is oriented consistently with A. In this case also κ is defined relative to a reference group that we fix at the beginning of the attaching procedure.

LEMMA 5.12. *Let $\langle A, BA^{-1}B^{-1}\rangle$ be a irreducible $(0,3)$ group. Let B be a fixed choice of an element in $\mathrm{Sp}(2,1)$ conjugating A^{-1} to $BA^{-1}B^{-1}$. Let $\kappa = (s,\psi,\xi,k_1,k_2)$ and $\kappa' = (s',\psi',\xi',k_1,k_2')$ be twist-bend parameters oriented consistently with A. Then $\langle A, BK\rangle$ is conjugate to $\langle A, BK'\rangle$ if and only if $\kappa = \kappa'$.*

PROOF. If $\kappa = \kappa'$, then clearly $K = K'$ and hence the groups are equal.

Conversely, suppose $\langle A, BK\rangle$ is conjugate to $\langle A, BK'\rangle$. The element D that conjugates these groups, must commutes with A. Hence,

$$D(a_A) = a_A,\ D(r_A) = r_A.$$

Since $BA^{-1}B^{-1}$ has been fixed at the beginning, we have

$$BA^{-1}B^{-1} = (BK')A^{-1}(BK')^{-1} = D(BA^{-1}B^{-1})D^{-1}.$$

Thus D commutes with $BA^{-1}B^{-1}$ and fixes $a_{BA^{-1}B^{-1}} = B(r_A),\ r_{BA^{-1}B^{-1}} = B(a_A)$. Since the fixed points are distinct, D is either the identity or, the fixed points $a_A,\ r_A,\ B(a_A),\ B(r_A)$ belong to the boundary of the same totally geodesic subspace fixed by D. But the later is not possible by the irreducibility of the $(0,3)$ group. So, D must be the identity, $BK' = BK$ and hence, $K = K'$, i.e., $\kappa = \kappa'$. □

PROPOSITION 5.13. *Let $\langle A, BK\rangle$ be a $(1,1)$ group obtained from the irreducible $(0,3)$ group $\langle A, BA^{-1}B^{-1}\rangle$ by closing a handle with associated twist-bend parameter κ. Then $\langle A, BK\rangle$ is determined up to conjugation by its* Fenchel-Nielsen coordinates

$$tr_{\mathbb{R}}(A),\ [\mathbb{X}_j(A, BA^{-1}B^{-1})],\ \mathbb{A}_j(A, BA^{-1}B^{-1}),\ j = 1,2,3,\ p_1(A),\ p_2(A),$$

and the twist-bend parameter $\kappa = (s, \psi, \xi, k_1, k_2)$.

Thus, we need 21 real parameters to specify $\langle A, BK\rangle$ up to conjugacy.

PROOF. Let $\langle A, BK\rangle$ and $\langle A, B'K'\rangle$ be two $(1,1)$ groups with the same Fenchel-Nielsen coordinates. In particular $tr(A) = tr(A')$ and hence

$$tr(BA^{-1}B^{-1}) = \overline{tr(A)} = \overline{tr(A')} = tr(B'A'^{-1}B^{-1}).$$

Further using the following relations

$$[\mathbb{X}_k(A, BA^{-1}B^{-1})] = [\mathbb{X}_k(A', B'A'^{-1}B'^{-1})],$$

$$[\mathbb{X}_k(A, BA^{-1}B^{-1})] = [\mathbb{X}_k(A', B'A'^{-1}B'^{-1})],\ k = 1,2,3,$$

we see by Theorem 5.6 that the $(0,3)$ groups $\langle A, BA^{-1}B^{-1}\rangle$ and $\langle A', B'A'^{-1}B'^{-1}\rangle$ are determined by the projective points of A. Thus we can assume $A = A',\ BA^{-1}B^{-1} = B'A'^{-1}B'^{-1}$. Now using the above lemma, we see that $\kappa = \kappa'$. Hence $K = K'$. Thus, the group $\langle A, BK\rangle$ is determined uniquely up to conjugation.

Conversely, suppose $\langle A, BK\rangle$ and $\langle A', B'K'\rangle$ are conjugate, then it is clear that the unique conjugacy class is determined by the given invariants. $\square$

Here, we will recall definition of geometric representation. Let Σ_g be a closed, connected, orientable surface of genus $g \geq 2$. Let $\pi_1(\Sigma_g)$ be the fundamental group of Σ_g. Specify a curve system $\mathcal{C}$ of $3g-3$ closed curves γ_j on Σ_g. The complement of such curve system decomposes the surface into $2g-2$ three-holed spheres P_i. Let $\rho : \pi_1(\Sigma_g) \to \mathrm{Sp}(2,1)$ be a discrete, faithful representation such that the image of each γ_j is loxodromic. For each i, the fundamental group of P_i gives a representation φ_i in $\mathfrak{D}_{\mathfrak{L}}(\mathrm{F}_2, \mathrm{Sp}(2,1))$, induced by ρ, such that the image of φ_i is a $(0,3)$ subgroup of $\mathrm{Sp}(2,1)$. If each of these representations φ_i is irreducible, we call the representation ρ as *geometric*.

THEOREM 5.14. [**GK18**, Theorem 1.6] *Let Σ_g be a closed surface of genus g with a curve system $\mathcal{C} = \{\gamma_j\}$, $j = 1, \ldots, 3g-3$. Let $\rho : \pi_1(\Sigma_g) \to \mathrm{Sp}(2,1)$ be a geometric representation of the surface group $\pi_1(\Sigma_g)$. Then we need $42g-42$ real parameters to determine ρ in the deformation space $\mathfrak{D}(\pi_1(\Sigma_g), \mathrm{Sp}(2,1))/\mathrm{Sp}(2,1)$.*

PROOF. Let $\Sigma_g \setminus \mathcal{C}$ be the complement of the curve system $\mathcal{C}$ in Σ_g. This is a disjoint union of $2g-2$ three holed spheres. Each such three-holed sphere in $\Sigma_g \setminus \mathcal{C}$ corresponds to an irreducible $(0,3)$ subgroup of $\mathrm{Sp}(2,1)$. By Theorem 5.6, a $(0,3)$ subgroup $\langle A, B\rangle$ is determined up to conjugacy by the 21 real parameters given there. While attaching two three-holed spheres, we attach two $(0,3)$ groups subject to the compatibility condition that a peripheral element in one group is conjugate to the inverse of a peripheral element in the other group. This gives a $(0,4)$ group that is specified up to conjugacy by the 42 real parameters described in Proposition 5.11. Proceeding this way, attaching $2g-2$ of the above $(0,3)$ groups, we get a surface with $2g$ handles, and it is determined by $21(2g-2) = 42g-42$ real parameters obtained from the attaching process. The handles correspond to the g curves that in turn correspond to the two boundary components of the three-holed spheres.

Now, there are g quaternionic constraints that are imposed to close these handles: one of the peripheral elements of each of these $(0,3)$ groups must be conjugate to the inverse of the other peripheral element. Note that, corresponding to each peripheral element there are 7 natural real parameters: the real trace and two projective points. So, the number of real parameters reduces to $42g - 42 - 7g = 35g - 42$. But there are g twist-bend parameters $\kappa_i = (s_i, \psi_i, \xi_i, k_{1i}, k_{2i})$, one for each handle, and each contributes 7 real parameters. Thus, we need $35g - 30 + 7g = 42g - 42$ real parameters to specify ρ up to conjugacy.

If two representations have the same coordinates, then the coordinates of the $(0, 3)$ groups are the same and so they are conjugate. Further, it follows from Proposition 5.11 and Proposition 5.13 that the $(0, 4)$ groups and the $(1, 1)$ groups are also determined uniquely up to conjugacy while attaching the $(0, 3)$ groups. Hence, representations with the same parameters are conjugate. Conversely, if two representations are conjugate, then clearly they have the same coordinates.

This proves the theorem. □

CHAPTER 6

Quaternionic hyperbolic Fenchel-Neilsen coordinates in $\mathrm{Sp}(1,1)$ and $\mathrm{GL}(2,\mathbb{H})$

1. Hyperbolic pairs in $\mathrm{Sp}(1,1)$

The group $\mathrm{Sp}(1,1)$ acts by isometries of the quaternionic hyperbolic line $\mathbf{H}_{\mathbb{H}}^1$. Note that $\mathrm{PSp}(1,1)$ is isomorphic to the isometry group $\mathrm{PO}(4,1)$ of the real hyperbolic 4 space. In this isomorphism, the group $\mathrm{Sp}(1,1)$ acts on the quaternionic model of $\mathbf{H}_{\mathbb{R}}^4$ by linear fractional transformations. This provides a quaternionic analogue of the classical Möbius transformations of the Riemann sphere. We refer to the book **[Par08]** for more details on this action.

Here, we view $\mathrm{Sp}(1,1)$ as a subgroup of $\mathrm{Sp}(2,1)$ that preserves a one-dimensional totally geodesic quaternionic subspace, a copy of $\mathbf{H}_{\mathbb{H}}^1$, in $\mathbf{H}_{\mathbb{H}}^2$. From this viewpoint, we shall follow the framework of the previous sections and will look at the linear action of $\mathrm{Sp}(1,1)$ on $\mathbf{H}_{\mathbb{H}}^1$.

1.1. Real trace. The following is a special case of [**GP13**, Theorem 3.1].

PROPOSITION 6.1. *Let A be an element in $\mathrm{Sp}(1,1)$. Let $A_{\mathbb{C}}$ be the corresponding element in $\mathrm{GL}(4,\mathbb{C})$. The characteristic polynomials of $A_{\mathbb{C}}$ is of the form*

$$\chi_A(x) = x^4 - ax^3 + bx^2 - ax + 1 = x^2 g(x + x^{-1}),$$

where $a, b \in \mathbb{R}$. Let Δ be the negative of the discriminant of the polynomial $g_A(t) = g(x + x^{-1})$. Then the conjugacy class of A is determined by the real numbers a and b. Further, A is loxodromic if and only if $\Delta > 0$.

This gives us the following definition.

DEFINITION 6.2. Let g be a hyperbolic isometry of $\mathbf{H}_{\mathbb{H}}^1$. The real tuple (a, b) in Proposition 6.1 is called the *real trace* of g and shall be denoted by $tr_{\mathbb{R}}(g)$.

Thus the real trace of a hyperbolic element g of $\mathrm{Sp}(1,1)$ corresponds to a point on $\mathbb{R}^2$. When g is strictly hyperbolic, then $4b = a^2 + 8$. In this case, the real trace is determined by a parameter on $\mathbb{R}$.

The real traces of loxodromic elements in $\mathrm{Sp}(1,1)$ are given by the following subset of $\mathbb{R}^2$:

$$\mathrm{D}_1 = \{(a,b) \in \mathbb{R}^2 \mid b^2 > 4a\}.$$

The following result follow similarly as Lemma 5.2.

LEMMA 6.3. *Let A, A' be hyperbolic elements in* $\mathrm{Sp}(1,1)$. *Then $A = A'$ if and only if they have the same attracting (or repelling) fixed point, the same real trace and the same projective point.*

1.2. Quadruple of boundary points. Next we observe the following analogue of Cao's theorem. Cao proved Theorem 2.8 assuming $n \geq 2$. However, the same proof boils down to a much simpler form when $n = 1$. A version of this theorem follows from the work in **[GL12]**, however there the authors have defined cross ratios using quaternionic arithmetic by the identification of $\mathrm{Sp}(1,1)$ with the quaternionic linear fractional transformations. We can prove the following lemma by using similar ideas in proof of Theorem 2.8 of Cao.

Given an ordered quadruple of pairwise distinct points (z_1, z_2, z_3, z_4) on $\partial\mathbf{H}^2_{\mathbb{H}}$, their Korányi-Reimann quaternionic cross ratio is defined by

$$\mathbb{X}(z_1, z_2, z_3, z_4) = [z_1, z_2, z_3, z_4] = \langle \mathbf{z}_3, \mathbf{z}_1 \rangle \langle \mathbf{z}_3, \mathbf{z_2} \rangle^{-1} \langle \mathbf{z}_4, \mathbf{z}_2 \rangle \langle \mathbf{z}_4, \mathbf{z}_1 \rangle^{-1}.$$

For a pair of hyperbolic elements (A, B) of $\mathrm{Sp}(1,1)$, define

$$\mathbb{X}(A, B) = \mathbb{X}(a_A, r_A, a_B, r_B).$$

LEMMA 6.4. *Let $Z = (z_1, z_2, z_3, z_4)$ and $W = (w_1, w_2, w_3, w_4)$ be two quadruple of pairwise distinct points in $\partial\mathbf{H}^1_{\mathbb{H}}$. Then there exists an isometry $h \in \mathrm{Sp}(1,1)$ such that $h(z_i) = w_i$, $i = 1, 2, 3, 4$, if and only if*

$$\Re(\mathbb{X}(z_1, z_2, z_3, z_4)) = \Re(\mathbb{X}(w_1, w_2, w_3, w_4)) \text{ and}$$

$$|\mathbb{X}(z_1, z_2, z_3, z_4)| = |\mathbb{X}(w_1, w_2, w_3, w_4)|.$$

PROOF. Suppose that $\Re(\mathbb{X}(z_1, z_2, z_3, z_4)) = \Re(\mathbb{X}(w_1, w_2, w_3, w_4))$, $|\mathbb{X}(z_1, z_2, z_3, z_4)| = |\mathbb{X}(w_1, w_2, w_3, w_4)|$. We want to find $h \in \mathrm{Sp}(1,1)$ such that $h(z_i) = w_i$, $i = 1, 2, 3, 4$. Without loss of generality assume that $z_1 = w_1 = o$ and $z_2 = w_2 = \infty$. By the given hypothesis, $|\mathbb{X}(o, \infty, z_3, z_4)| = |\mathbb{X}(o, \infty, w_3, w_4)|$ implies that $\frac{|z_3|}{|w_3|} = \frac{|z_4|}{|w_4|}$. Now, by using the fact that all z_i, w_i has zero real part together with condition $\Re(\mathbb{X}(o, \infty, z_3, z_4)) = \Re(\mathbb{X}(o, \infty, w_3, w_4))$ gives equality of the pairs of angles between the vectors $\Im(\frac{z_3}{|z_3|})$, $\Im(\frac{z_4}{|z_4|})$, and $\Im(\frac{w_3}{|w_3|})$, $\Im(\frac{w_4}{|w_4|})$. So there exist $\psi \in \mathrm{Sp}(1)$ such that $z_3 = t\bar{\psi} w_3 \psi$ and $z_4 = t\bar{\psi} w_4 \psi$, where $t = \frac{|z_3|}{|w_3|} = \frac{|z_4|}{|w_4|}$. Thus, we get the required isometry

$$h = \begin{bmatrix} \frac{\psi}{\sqrt{t}} & 0 \\ 0 & \sqrt{t}\psi \end{bmatrix},$$

in $\mathrm{Sp}(1,1)$ such that $h(o) = o$, $h(\infty) = \infty$, $h(z_3) = w_3$ and $h(z_4) = w_4$. □

Thus in this case the moduli of ordered quadruple of points up to $\mathrm{Sp}(1,1)$ congruence is determined by the tuple $(\Re(\mathbb{X}(p)), |\mathbb{X}(p)|)$. We denote it by $\mathrm{M}(1,4,0)$. An explicit description of this space may be obtained from the work of **[GL12]**. From **[GL12**, Proposition 10], it follows that $\mathrm{M}(1,4,0)$ is embedded in $\mathbb{R}^4$.

THEOREM 6.5. **[GK18**, Theorem 1.7] *Let (A, B) be an irreducible totally loxodromic pair in $\mathrm{Sp}(1,1)$. Then (A, B) is determined up to conjugation by the following parameters: $tr_{\mathbb{R}}(A)$, $tr_{\mathbb{R}}(B)$, a point on $\mathrm{M}(1,4,0)$ corresponding to (a_A, r_A, a_B, r_B), and two projective points $p_1(A)$, $p_1(B)$.*

PROOF. Suppose (A, B) and (A', B') be pairs of loxodromics such that $tr_{\mathbb{R}}(A) = tr_{\mathbb{R}}(A')$, $tr_{\mathbb{R}}(B) = tr_{\mathbb{R}}(B')$, $\Re(\mathbb{X}(A, B)) = \Re(\mathbb{X}(A', B'))$ and $|\mathbb{X}(A, B)| = |\mathbb{X}(A', B')|$. By Lemma 6.4, there is an element C in $\mathrm{Sp}(1,1)$ such that $C(a_A) = a_{A'}$, $C(r_A) = r_{A'}$, $C(a_B) = a_{B'}$ and $C(r_B) = r_{B'}$. Therefore A' and CAC^{-1} have the same attracting and the same repelling fixed points. Since they also have the same real trace and the same projective point, by Lemma 6.3, $CAC^{-1} = A'$. Similarly, $CBC^{-1} = B'$. This completes the proof. □

Given a representation ρ in $\mathfrak{D}_{\mathfrak{L}o}(\mathrm{F}_2, \mathrm{Sp}(1,1))$, we associate the following tuple to it:

$$(tr_{\mathbb{R}}(\rho(x)), tr_{\mathbb{R}}(\rho(y)), \Re(\mathbb{X}(A, B)), |\mathbb{X}(A, B)|).$$

Let $\mathcal{M}(1)$ denote the orbit space of $\mathrm{M}(1,4,0)$ under the natural S_4 action on the quadruples. This gives the following.

COROLLARY 6.6. *$\mathfrak{D}_{\mathfrak{L}o}(\mathrm{F}_2, \mathrm{Sp}(1,1))$ is parametrized by a $\mathbb{CP}^1 \times \mathbb{CP}^1$ bundle over the topological space $\mathrm{D}_1 \times \mathrm{D}_1 \times \mathcal{M}(1)$.*

This implies the following theorem that follows similarly as mentioned in the previous section. A twist-bend $E_K(s, \psi, k)$ in this case would corresponds to four real degrees of freedom given by the real trace (s, ψ) and a projective point k. The rest follows similarly as in the previous chapter. Since the arguments are very similar, we omit the details.

THEOREM 6.7. *Let Σ_g be a closed surface of genus g with a curve system $\mathcal{C} = \{\gamma_j\}$, $j = 1, \ldots, 3g-3$. Let $\rho : \pi_1(\Sigma_g) \to \mathrm{Sp}(1,1)$ be a geometric representation of the surface group $\pi_1(\Sigma_g)$ into $\mathrm{Sp}(1,1)$. Then we need $20g - 20$ real parameters to determine ρ in the deformation space $\mathfrak{D}(\pi_1(\Sigma_g), \mathrm{Sp}(1,1))/\mathrm{Sp}(1,1)$.*

2. Hyperbolic pairs in $\mathrm{GL}(2, \mathbb{H})$

The group $\mathrm{GL}(2, \mathbb{H})$ acts on $\partial \mathbf{H}^5_{\mathbb{R}} = \widehat{\mathbb{H}} = \mathbb{H} \cup \{\infty\}$ by linear fractional transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto (az+b)(cz+d)^{-1},$$

and this action is extended over the hyperbolic space by Poincaré extensions. This action identifies the projective general linear group $\mathrm{PGL}(2, \mathbb{H}) = \mathrm{GL}(2, \mathbb{H})/\mathrm{Z}(\mathrm{GL}(2, \mathbb{H}))$ to the group $\mathrm{PO}(5,1)$, see **[Par08]** or **[Gon10]** for more details. Note that $Z(\mathrm{GL}(2, \mathbb{H}))$ is isomorphic to the multiplicative group $\mathbb{R}^* = \mathbb{R} \setminus 0$.

2.0.1. 3-*Simple loxodromics.* Let g be a 3-simple loxodromic element in $\mathrm{GL}(2, \mathbb{H})$. Up to conjugacy, it is of the form

$$E_{r,s,\theta,\phi} = \begin{bmatrix} re^{i\theta} & 0 \\ 0 & se^{i\phi} \end{bmatrix}, \ r > 0, \ s > 0, \ \theta, \phi \in (0, \pi).$$

As above, using the embedding of $\mathrm{GL}(2, \mathbb{H})$ into $\mathrm{GL}(4, \mathbb{C})$ one can define *real trace* of a 3-simple loxodromic in $\mathrm{GL}(2, \mathbb{H})$. It follows from the work in **[Gon10]** or **[PS09]** that the *real traces* of 3-simple loxodromic elements of $\mathrm{GL}(2, \mathbb{H})$ correspond to a three real parameter family (a, b, c), $a \neq c$, that forms an open subset of $\mathbb{R}^3$. We denote this open subset by D_3. This can be seen using ideas similar to the proof of Lemma 5.3.

2.0.2. *Cross ratios.* As mentioned in the previous case, in this case Gwynne and Libine, **[GL12]** has defined cross ratio of four boundary points analogous to the cross ratio of four points on the Riemann sphere. Gwynne and Libine have used them to obtain configuration of $\mathrm{GL}(2, \mathbb{H})$-congruence classes of quadruples of pairwise distinct points on $\widehat{\mathbb{H}}$. Let $\mathcal{C}_4$ denote the space of quadruples of points on $\partial \mathbf{H}_{\mathbb{R}}^5$ up to $\mathrm{GL}(2, \mathbb{H})$ congruence. It follows from the work **[GL12]** that the subspace of $\mathcal{C}_4$ consisting of quadruples of pairwise distinct points on $\widehat{\mathbb{H}}$ in general position (i.e. the four points do not belong to a circle), is a real two-dimensional subspace of $\mathbb{R}^4$. Let $\mathfrak{C}_4$ denote the orbit space of $\mathcal{C}_4$ under the natural S_4 action on the tuples.

2.0.3. *The deformation space.* Let $\mathfrak{D}_{\mathfrak{L}*}(\mathrm{F}_2, \mathrm{GL}(2, \mathbb{H}))$ denote the subset of the deformation space $\mathfrak{D}_{\mathfrak{L}o}(\mathrm{F}_2, \mathrm{GL}(2, \mathbb{H}))$ consisting of elements ρ such that both $\rho(x)$ and $\rho(y)$ are 3-simple loxodromics. The following is evident using similar arguments as in the previous sections.

THEOREM 6.8. *The set* $\mathfrak{D}_{\mathfrak{L}*}(\mathrm{F}_2, \mathrm{GL}(2, \mathbb{H}))$ *is a* $(\mathbb{CP}^1)^4$ *bundle over the topological space* $\mathrm{D}_3 \times \mathrm{D}_3 \times \mathfrak{C}_4$.

Thus we need a total of 16 degrees of freedom to specify an element uniquely in $\mathfrak{D}_{\mathfrak{L}*}(\mathrm{F}_2, \mathrm{GL}(2, \mathbb{H}))$. Note that the real dimension of the group $\mathrm{GL}(2, \mathbb{H})$ is also 16. However the group $\mathrm{PGL}(2, \mathbb{H})$ has 15 real dimension. The group $\mathrm{GL}(2, \mathbb{H})$ is fibered over $\mathrm{PGL}(2, \mathbb{H})$ by the punctured real line. This fibration induces a fibration of $\mathfrak{D}_{\mathfrak{L}*}(\mathrm{F}_2, \mathrm{GL}(2, \mathbb{H})$ over $\mathfrak{D}_{\mathfrak{L}*}(\mathrm{F}_2, \mathrm{PGL}(2, \mathbb{H})$ by the punctured real line, and thus we need 15 real parameters to determine a irreducible point on $\mathfrak{D}_{\mathfrak{L}*}(\mathrm{F}_2, \mathrm{PGL}(2, \mathbb{H}))$. The following theorem can be proved following similar arguments as earlier.

THEOREM 6.9. *Let* Σ_g *be a closed surface of genus* g *with a curve system* $\mathcal{C} = \{\gamma_j\}$, $j = 1, \dots, 3g-3$. *Let* $\rho : \pi_1(\Sigma_g) \to \mathrm{PGL}(2, \mathbb{H})$ *be a geometric representation of the surface group* $\pi_1(\Sigma_g)$ *into* $\mathrm{PGL}(2, \mathbb{H})$. *Then we need* $30g - 30$ *real parameters to determine* ρ *in the deformation space* $\mathfrak{D}(\pi_1(\Sigma_g), \mathrm{PGL}(2, \mathbb{H}))/\mathrm{PGL}(2, \mathbb{H})$.

CHAPTER 7

Future research directions

We recall that the group $\mathrm{PGL}(2,\mathbb{H})$ is isomorphic to $\mathrm{PO}(5,1)$ and acts on the real hyperbolic 5-space $\mathbf{H}^5_{\mathbb{R}}$ by isometries. The group $\mathrm{PSp}(1,1)$ is isomorphic to a subgroup of $\mathrm{PGL}(2,\mathbb{H})$ that preserves a copy of $\mathbf{H}^4_{\mathbb{R}}$ inside $\mathbf{H}^5_{\mathbb{R}}$. Using the approaches taken in this paper, it is also possible to state similar results for pairs of hyperbolic elements in $\mathrm{GL}(2,\mathbb{H})$. Algebraic and dynamical classification of elements in $\mathrm{GL}(2,\mathbb{H})$ are available in **[Gon10]**, **[PS09]**. There are three types of hyperbolic elements in $\mathrm{GL}(2,\mathbb{H})$ depending upon the number of 'rotation angles'. Two of these types come from $\mathrm{Sp}(1,1)$, and the third type called *2-rotatory hyperbolic* in **[Gon10]** or *3-simple loxodromic* in **[Par08]**, does not have representatives in $\mathrm{Sp}(1,1)$. Though we shall address pairs of this type briefly, and will determine them up to conjugacy in $\mathrm{GL}(2,\mathbb{H})$, they deserve separate attention. This space may be thought of as the quaternionic analogue of the complex Teichmüller space and it certainly deserves a more thorough treatment.

As it is clearly visible in all the statements above, the notion of projective points is crucial for the development in this thesis. The lack of numerical invariants over the quaternions has been supplemented by the use of these spatial invariants. We believe that there should be a classification of hyperbolic pairs in $\mathrm{Sp}(2,1)$ using purely numerical invariants. We hope that a counterpart of [**Par12**, Theorem 4.8] will be available for the group generated by pairs in $\mathrm{Sp}(2,1)$. We expect that the real traces of A, B, and several of their compositions like Procesi **[Pro76]**, should be sufficient to classify the pair itself. A starting point in this direction could be the work of Đoković and Smith in **[ĐoS]**, where a set of minimal real trace coordinates is available for $\mathrm{Sp}(2)$ conjugation orbits in $\mathrm{M}_2(\mathbb{H})$. It might be possible to extend the invariant theoretic methods in **[ĐoS]** to classify conjugacy classes in $\mathrm{Sp}(2,1)$, and also to classify the conjugation orbits of pairs of elements in $\mathrm{Sp}(2,1)$.

Also, it is interesting problem to study change of coordinates on the same three-holed sphere and 'trace' coordinates. From a group theoretic viewpoint, a three-holed sphere corresponds to a subgroup generated by two hyperbolic elements A and B whose product AB is also loxodromic. The three boundary curves corresponds to the loxodromic elements A, B and $B^{-1}A^{-1}$ respectively. A group generated by such hyperbolic elements is called a $(0, 3)$ subgroup. The classical Fenchel-Nielsen coordinates are obtained by 'gluing' such $(0, 3)$ subgroups, and the coordinates are given by several parameters associated to these subgroups and the gluing process.

There is a natural three-fold symmetry associated to a $(0, 3)$ subgroup. This is respected in the classical Fenchel-Nielsen coordinates of the Teichmüller or quasi-Fuchsian space. For example, if we change the coordinates associated to $\langle A, B\rangle$ to $\langle A, B^{-1}A^{-1}\rangle$, then the Fenchel-Nielsen coordinates remain unchanged in the classical set up. This is clearly not the case in our set up, neither it was in the set up of Parker and Platis in the complex hyperbolic set up, see [**PP08**, Section 7.2]. However, Parker and Platis rectified this problem partially by relating the traces of A, B and several of their compositions with the cross ratios, and thus by giving a real analytic change of coordinates between two $(0, 3)$ groups coming from the same three-holed sphere. Following the work of Will [**Wil09**], Parker resolved this problem by using trace parameters to determine an irreducible $(0, 3)$ subgroup of $\mathrm{SU}(2, 1)$ in [**Par12**].

We do not know how to resolve this issue in the $\mathrm{Sp}(2, 1)$ set up. We expect that by computations it should be possible to relate the real traces and the points on $\mathrm{M}(2)$ of two $(0, 3)$ subgroups coming from the same three-holed sphere. The computations to do that are too involved, and we are unable to resolve the difficulty here. It is also not clear to us that how the projective points change when we have a change of the $(0, 3)$ subgroups.

As mentioned in the Introduction, innovation of a set of 'real trace coordinates' using classical invariant theory might also be helpful to resolve the above problem concerning the three-fold symmetry of a three-holed sphere.

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