Group actions on Dold and Milnor manifolds

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Dedicated to

My family

Declaration

The work presented in this thesis has been carried out by me under the guidance of Dr. Mahender Singh at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Pinka Dey

In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Mahender Singh
(Supervisor)

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Abstract

The Dold manifold $P(m,n)$ is the quotient of $\mathbb{S}^m \times \mathbb{C}P^n$ by the free involution that acts antipodally on the sphere $\mathbb{S}^m$ and by complex conjugation on the complex projective space $\mathbb{C}P^n$. In this thesis, we investigate free actions of finite groups on products of Dold manifolds. We show that if a finite group G acts freely and mod 2 cohomologically trivially on a finite-dimensional CW-complex homotopy equivalent to $\prod_{i=1}^{k} P(2m_i, n_i)$, then $G \cong (\mathbb{Z}_2)^l$ for some $l \leq k$. This is achieved by first proving a similar assertion for $\prod_{i=1}^{k} \mathbb{S}^{2m_i} \times \mathbb{C}P^{n_i}$. We also determine the possible mod 2 cohomology algebra of orbit spaces of arbitrary free involutions on Dold Manifolds, and give an application to $\mathbb{Z}_2$-equivariant maps from spheres to Dold manifolds.

We also study free $\mathbb{Z}_2$ and $\mathbb{S}^1$-actions on cohomology real and complex Milnor manifolds. A real Milnor manifold $\mathbb{R}H_{r,s}$ is a non-singular hypersurface of degree $(1,1)$ in the product $\mathbb{R}P^r \times \mathbb{R}P^s$. A complex Milnor manifold $\mathbb{C}H_{r,s}$ is defined analogously. We compute the mod 2 cohomology algebra of the orbit space of an arbitrary free $\mathbb{Z}_2$ and $\mathbb{S}^1$-action on a compact Hausdorff space with mod 2 cohomology algebra of a real or a complex Milnor manifold. As applications, we deduce some Borsuk-Ulam type results for equivariant maps between spheres and these spaces. For the complex case, we obtain a lower bound on the Schwarz genus, which further establishes the existence of coincidence points for maps from Milnor manifolds to the Euclidean plane.

Notation

B_G : classifying space of group G

$\mathbb{C}H_{r,s}$: complex Milnor manifold

$\mathbb{Z}_p$: cyclic group of order p

$P(m,n)$: Dold manifold

$\mathrm{rk}(V)$: dimension of the vector space V over the finite field $\mathbb{F}_2$ of two elements

$\chi(X)$: Euler characteristic of space X

$\mathbb{F}_p$: finite field of p-elements

$\mathrm{frk}_p(X)$: free p-rank of symmetry of X

$\mathrm{frk}(X)$: free rank of symmetry of X

G_x : isotropy subgroup of G at x

$X * Y$: join of X and Y

$\tau(f)$: Lefschetz number of f

$E_\infty^{*,*}$: limit term of the spectral sequence $\{E_r^{p,q}, d_r\}$

$\mathbb{S}^n$: n-dimensional sphere

$\mathbb{C}P^n$: $2n$-dimensional complex projective space

$\mathbb{R}P^n$: n-dimensional real projective space

X/G : orbit space of X by action of G

$\mathbb{R}H_{r,s}$: real Milnor manifold

$g_{\mathrm{free}}(X,G)$: Schwarz genus of the free G-space X

$\mathcal{H}^l(X)$: system of local coefficients

E_G : total space of the universal principal G-bundle

$X \simeq Y$: X and Y are homotopy equivalent

$X \simeq_2 Y$: X and Y have isomorphic mod 2 cohomology algebras

Chapter 1

Introduction

In his famous 1926 paper [36], Hopf stated the following topological spherical space form problem:

Classify all manifolds with universal cover homeomorphic to the n-dimensional sphere $\mathbb{S}^n$ for $n > 1$.

This is equivalent to determining all finite groups that can act freely on $\mathbb{S}^n$. Smith [74] showed that such a group G must have periodic cohomology, i.e., G does not contain a subgroup of the form $\mathbb{Z}_p \oplus \mathbb{Z}_p$ for any prime p.

A natural generalization of the above problem is to classify all finite groups that can act freely on products of finitely many spheres. More generally, one can ask a similar question for arbitrary topological spaces. This led to the concept of free rank of symmetry of a topological space, defined as follows:

Definition 1.0.1. *Let X be a finite-dimensional CW-complex and p be a prime number. The free p-rank of symmetry of X, denoted by $\operatorname{frk}_p(X)$, is the largest r such that $(\mathbb{Z}_p)^r$ acts freely on X. The smallest value of r over all primes p is called the free rank of symmetry of X and denoted by $\operatorname{frk}(X)$.*

By Smith's result, we get

$$\operatorname{frk}_p(\mathbb{S}^n) = \begin{cases} 1 & \text{if } n \text{ is odd and } p \text{ is arbitrary,} \\ 1 & \text{if } n \text{ is even and } p = 2, \\ 0 & \text{if } n \text{ is even and } p > 2. \end{cases}$$

For products of finitely many spheres, the following statement appears as a question [1, Question 7.2] and as a conjecture [2, p. 28].

Conjecture 1.0.2. *If $(\mathbb{Z}_p)^r$ acts freely on $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \times \cdots \times \mathbb{S}^{n_k}$, then $r \leq k$.*

Carlsson [12, 13] proved the above conjecture for homologically trivial actions on products of equidimensional spheres. Adem and Browder [1] proved that $\operatorname{frk}_p((\mathbb{S}^n)^k) = k$ with the only remaining cases as $p = 2$ and $n = 1, 3, 7$. Later, Yalçin [82] proved that $\operatorname{frk}_2((\mathbb{S}^1)^k) = k$. The most general result which settles a stable form of the above conjecture is due to Hanke [33, Theorem 1.3], which states that

$$\operatorname{frk}_p(\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k}) = k_0 \quad \text{if } p > 3(n_1 + \cdots + n_k),$$

where k_0 is the number of odd-dimensional spheres in the product $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k}$.

In a recent work [60, Theorem 1.2], Okutan and Yalçin proved the conjecture in the case when the dimensions $\{n_i\}$ of the spheres are high compared to the differences $|n_i - n_j|$ between the dimensions. The general case is still open.

For free actions of arbitrary finite groups, there is a more general conjecture due to Benson-Carlson [6]. For a finite group, define the rank of G, denoted by $\operatorname{rk}(G)$, as

$$\operatorname{rk}(G) = \max\{r \mid (\mathbb{Z}_p)^r \leqslant G \text{ for some prime } p\}$$

and define

$$h(G) = \min\{k \mid G \text{ acts freely on } \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k}\}.$$

Conjecture 1.0.3. (Benson-Carlson) *If G is a finite group, then $rk(G) = h(G)$.*

Note that, $h(G)$ is well-defined since a result of Oliver [61] states that every finite group acts freely on some products of spheres. Note that the Conjecture 1.0.3 implies Conjecture 1.0.2.

For products of even-dimensional spheres, Cusick [19,20] proved that if a finite group G acts freely on $\mathbb{S}^{2n_1} \times \cdots \times S^{2n_k}$ such that the induced action on $H_*(X; \mathbb{Z}_2)$ is trivial, then $G \cong (\mathbb{Z}_2)^r$ for some $r \leq k$.

Though an extensive amount of work has been done for spheres and their products, the free rank of many other interesting spaces is still unknown. A few results are known such as for products of lens spaces and products of projective spaces. For lens spaces, Allday [4, Conjecture 5.2] conjectured that

$$\operatorname{frk}_p \left(L_p^{2n_1-1} \times \cdots \times L_p^{2n_k-1}\right) = k,$$

and Yalçin [83] proved the equidimensional case of the above conjecture.

For products of real projective spaces, Cusick [18] conjectured that if $(\mathbb{Z}_2)^r$ acts freely and mod 2 cohomologically trivially on $\mathbb{R}P^{n_1} \times \cdots \times \mathbb{R}P^{n_k}$, then $r \leq \eta(n_1) + \cdots + \eta(n_k)$, where η is a function defined on $\mathbb{N}$ by

$$\eta(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \equiv 1 \pmod 4, \\ 2 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Cusick proved the conjecture when $n_i \not\equiv 3 \pmod 4$ for all i. Later, Yalçin [82, Theorem 8.3] proved the conjecture when n_i is odd for each $0 \leq i \leq k$. The general case is still open.

For products of complex projective spaces, Cusick [21, Theorem 4.13] proved that if a finite group G acts freely on $(\mathbb{C}P^m)^k$, then G is a 2-group of order at most 2^k and the exponent of G cannot exceed $2k$.

Viewing the product of two projective spaces as a trivial bundle, it is interesting to determine the free rank of symmetry of twisted projective space bundles over projective spaces. Milnor and Dold manifolds are fundamental examples of such spaces. It is well-known that Dold and Milnor manifolds give generators for the unoriented cobordism algebra of smooth manifolds. Therefore, determining various invariants of these manifolds is an interesting problem. The free 2-rank of symmetry of products of Milnor manifolds has been investigated in [73], wherein some bounds have been obtained for the same.

In this thesis, we investigate the free rank problem for products of Dold manifolds. A Dold manifold $P(m,n)$ is the quotient of $\mathbb{S}^m \times \mathbb{C}P^n$ by the free involution that acts antipodally on $\mathbb{S}^m$ and by complex conjugation on $\mathbb{C}P^n$. To compute the free rank of $\prod_{i=1}^{k} P(2m_i, n_i)$, we first determine the free rank of $\prod_{i=1}^{k} \mathbb{S}^{2m_i} \times \mathbb{C}P^{n_i}$. Since a Dold manifold is a twisted complex projective space bundle over a real projective space, our arguments also give the free rank of $\prod_{i=1}^{k} \mathbb{R}P^{2m_i} \times \mathbb{C}P^{n_i}$. To state our main results, we define $\mu : \mathbb{N} \to \{0,1\}$ by

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{1.1}$$

Theorem 1.0.4. (Theorem 3.3.6) *Let G be a finite group acting freely and mod 2 cohomologically trivially on a finite-dimensional CW-complex X homotopy equivalent to $\prod_{i=1}^{k} \mathbb{S}^{2m_i} \times \mathbb{C}P^{n_i}$. Then $G \cong (\mathbb{Z}_2)^l$ for some l satisfying*

$$l \leq k_0 + \mu(n_1) + \mu(n_2) + \cdots + \mu(n_k),$$

where k_0 is the number of positive-dimensional spheres in the product.

Theorem 1.0.5. (Theorem 3.3.10) *Let G be a finite group acting freely and mod 2 cohomologically trivially on a finite-dimensional CW-complex X homotopy equivalent to $\prod_{i=1}^{k} P(2m_i, n_i)$. Then $G \cong (\mathbb{Z}_2)^l$ for some l satisfying*

$$l \leq \mu(n_1) + \mu(n_2) + \cdots + \mu(n_k).$$

Since $P(0,n) = \mathbb{C}P^n$, by taking all $m_i = 0$, we retrieve the result of Cusick [21, Theorem 4.15].

For an arbitrary finite free G-CW-complex, there is a more general conjecture due to Carlsson [14, Conjecture I.3]:

Conjecture 1.0.6. *Suppose $G = (\mathbb{Z}_p)^k$ and X is a finite free G-CW-complex. Then*
$$2^k \leq \sum_{i=0}^{\dim X} \dim_{\mathbb{F}_p} H_i(X; \mathbb{Z}_p).$$

In particular, the above conjecture is applicable for CW-complexes. A direct check shows that

$$\sum_{i=0}^{\dim X} \dim_{\mathbb{F}_p} H_i(X; \mathbb{Z}_p) \geq \begin{cases} 2^{k_0+\mu(n_1)+\mu(n_2)+\cdots+\mu(n_k)} & \text{if } X = \prod_{i=1}^{k} \mathbb{S}^{2m_i} \times \mathbb{C}P^{n_i}, \\ 2^{\mu(n_1)+\mu(n_2)+\cdots+\mu(n_k)} & \text{if } X = \prod_{i=1}^{k} P(2m_i, n_i), \end{cases}$$

for mod 2 cohomologically trivial actions. Thus, Conjecture 1.0.6 is verified for our spaces under these conditions.

We also investigate group actions on Dold and Milnor manifolds from another point of view. Once we know that a group acts freely on a given space, a natural problem is to determine all actions of the group up to conjugation. Determining the homeomorphism or homotopy type of the orbit space is, in general, a difficult problem. A non-trivial result of Oliver [61] states that the orbit space of any action of a compact Lie group on an Euclidean space is contractible. For spheres, Milnor [35] proved that for any free involution on $\mathbb{S}^n$, the orbit space has the homotopy type of $\mathbb{R}P^n$. Free actions of finite groups on spheres, particularly $\mathbb{S}^3$, have been well-studied in the past, see for example [65, 66, 68]. But, not many results are known for compact manifolds other than spheres. In [56], Myers investigated orbit spaces of free involutions on three-dimensional lens spaces. In [76], Tao determined orbit spaces of free involutions on $\mathbb{S}^1 \times \mathbb{S}^2$, and Ritter [67] extended these results to free actions of cyclic groups of order 2^n. Tollefson [78] proved that there are precisely four conjugacy classes of involutions on $\mathbb{S}^1 \times \mathbb{S}^2$. Fairly recently, Jahren and Kwasik [38] classified, up to conjugation, all free involutions on $\mathbb{S}^1 \times \mathbb{S}^n$ for $n \geq 3$, by showing that there are exactly four possible homotopy types of orbit spaces.

Determining the orbit space up to homeomorphism or homotopy type is often difficult, and hence we try to determine the possible cohomology algebra of orbit spaces. Dotzel et al. [30] determined the cohomology algebra of orbit spaces of free $\mathbb{Z}_p$ (p prime) and $\mathbb{S}^1$-actions on cohomology product of two spheres. Orbit spaces of free involutions on cohomology lens spaces were investigated by Singh [72]. The cohomology algebra of orbit spaces of free involutions on product of two projective spaces was computed in [70]. Recently, Pergher et al. [63] and Mattos et al. [23] considered free $\mathbb{Z}_2$ and $\mathbb{S}^1$-actions on spaces of type (a, b), which are certain products or wedge sums of spheres and projective spaces. As applications, they also established some bundle theoretic analogues of the Borsuk-Ulam theorem for these spaces.

We determine the possible mod 2 cohomology algebra of orbit spaces of free involutions on Dold manifolds $P(m, n)$ (see Theorem 4.1.1). The case $P(1, n)$, where n is odd, has been addressed in a recent work [54, Theorem 1] of Morita et al. In this thesis, we consider the case $P(m, n)$, where n is even.

We also investigate free $\mathbb{Z}_2$ and $\mathbb{S}^1$-actions on mod 2 cohomology real and complex Milnor manifolds. More precisely, we determine the possible mod 2 cohomology algebra of orbit spaces of free $\mathbb{Z}_2$ and $\mathbb{S}^1$-actions on these spaces (see Theorems 5.3.1, 5.3.2 and 5.3.3). We also find necessary and sufficient conditions for the existence of free actions on these spaces (see corollaries 5.2.6 and 5.2.8). As applications, we obtain some Borsuk-Ulam type results for these spaces. We also determine a lower bound on the genus of these manifolds.

The problems have been investigated using equivariant cohomology theory introduced by Borel [8] and other well-known methods from equivariant topology [3, 9]. Let G be a group and X a G-space. Let

$$G \hookrightarrow E_G \longrightarrow B_G$$

be the universal principal G-bundle and

$$X \stackrel{i}{\hookrightarrow} X_G \stackrel{\pi}{\longrightarrow} B_G$$

the associated Borel fibration [8, Chapter IV]. Our main computational tool is the Leray-Serre spectral sequence associated to the Borel fibration [51, Theorem 5.2]. The E_2-term of this spectral sequence is given by

$$E_2^{k,l} = H^k(B_G; \mathcal{H}^l(X; R)),$$

the cohomology of the base B_G with locally constant coefficients $\mathcal{H}^l(X; R)$ twisted by a canonical action of $\pi_1(B_G)$, where R is a commutative ring with unity. Further, the spectral sequence converges to $H^*(X_G; R)$ as an algebra. For details, we refer the reader to [4, 9, 51].

Throughout the thesis, we denote the cyclic group of order p by $\mathbb{Z}_p$. For topological spaces X and Y, $X \simeq Y$ would mean that they are homotopy equivalent and $X \simeq_2 Y$ would mean that X and Y have isomorphic mod 2 cohomology algebras, not necessarily induced by a map between X and Y. We say that G acts mod 2 cohomologically trivially on a space X if the induced action on $H^*(X; \mathbb{Z}_2)$ is trivial. Throughout, we use singular cohomology unless otherwise specified.

The thesis is organized as follows. In Chapter 2, we recall some necessary background that will be used in subsequent chapters. In Chapter 3, we recall the definition and cohomology of Dold manifolds and prove our results on the free rank of symmetry of products of Dold manifolds. In Chapter 4, we present our results on the mod 2 cohomology algebra of orbit spaces of free involutions on Dold manifolds. In Chapter 5, we recall the definition and cohomology of Milnor manifolds and present our results on free actions of $\mathbb{Z}_2$ and $\mathbb{S}^1$ on real and complex Milnor manifolds. Finally, in Chapter 6, we give some applications of our results to equivariant maps between spheres and these manifolds.

Chapter 2

Review of transformation groups and spectral sequences

In this chapter, we recall some basic definitions and results that will be used in the subsequent chapters. Most of these results can be found, for example, in Allday-Puppe [3], Bredon [9] and McCleary [51].

2.1 Group actions and their properties

A **topological transformation group** is a triple (G, X, θ), where G is a topological group, X is a Hausdorff topological space and $\theta : G \times X \to X$ is a continuous map such that:

1. $\theta(1, x) = x$ for all $x \in X$, where 1 is the identity of G;

2. $\theta(g, \theta(h, x)) = \theta(gh, x)$ for all $g, h \in G$ and $x \in X$.

We say θ is an **action** of G on X. When the action is clear from context we write gx to denote $\theta(g, x)$ and say X is a G-space. By an **involution** we mean an action of the cyclic group $\mathbb{Z}_2$ of order 2.

For each $x \in X$, the subspace $\{gx \mid g \in G\}$ is called the orbit of x. Let X/G denote the set of all orbits and $\pi : X \to X/G$ be the canonical map taking a point to its orbit. The set X/G equipped with the quotient topology induced by π is called the **orbit space**. The next result gives some basic properties of orbit spaces.

Theorem 2.1.1. [9, Theorem 3.1] *Let G be a compact group and X a G-space. Then the following holds:*

1. *X/G is Hausdorff.*

2. *$\pi : X \to X/G$ is closed.*

3. *X is compact if and only if X/G is compact.*

4. *X is locally compact if and only if X/G is locally compact.*

For each $x \in X$, the **isotropy** subgroup of G at x is the subgroup $G_x := \{g \in G \mid gx = x\}$. These subgroups play an important role in the theory of topological transformation groups, in particular, in equivariant cohomology theory. We say an action is free if $G_x = 1$ for all $x \in X$. The subspace $X^G = \{x \in X \mid gx = x \text{ for all } g \in G\}$ of X is called the **fixed-point** set of the action. An equivariant map $f : X \to Y$ between two G-spaces is a map that commutes with the group actions, that is,

$$f(gx) = gf(x),$$

for $g \in G$ and $x \in X$. Note that, an **equivariant** map f induces a map $\bar{f} : X/G \to Y/G$ given by $\bar{f}(\bar{x}) = \overline{f(x)}$. We now state some important results regarding group actions on compact Hausdorff spaces.

Theorem 2.1.2. [9, Chapter III, Theorem 7.9] *Let $G = \mathbb{Z}_p$ be the cyclic group of prime order p acting on the compact Hausdorff space X with fixed-point set X^G. Then for each $n \geq 0$,*

$$\sum_{i \geq n} \dim_{\mathbb{F}_p} H^i(X^G; \mathbb{Z}_p) \leq \sum_{i \geq n} \dim_{\mathbb{F}_p} H^i(X; \mathbb{Z}_p).$$

As a consequence of Theorem 2.1.2 we get the following result which is originally due to Floyd.

Theorem 2.1.3. [9, Chapter III, Theorem 7.10] *Let $G = \mathbb{Z}_p$ be the cyclic group of prime order p acting on the compact Hausdorff space X with fixed-point set X^G. If $\dim_{\mathbb{F}_p} H^*(X;\mathbb{Z}_p) < \infty$, then*

$$\chi(X) + (p-1)\chi(X^G) = p\chi(X/G).$$

By Theorem 2.1.3 it follows that when n is even $\mathbb{Z}_2$ is the only non-trivial group acting freely on the sphere $\mathbb{S}^n$. When n is odd, it was shown by Madsen-Thomas-Wall [45, 77] that a finite group acts freely on $\mathbb{S}^n$ if and only if it has no subgroup isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$ and every order two element is central. We now state a well-known result which is originally due to Smith [74].

Theorem 2.1.4. [9, Chapter III, Theorem 8.1] *If p is a prime, then the group $\mathbb{Z}_p \oplus \mathbb{Z}_p$ cannot act freely on a mod p-cohomology n-sphere.*

Next, we recall some results regarding free $\mathbb{Z}_2$ and $\mathbb{S}^1$-actions on compact Hausdorff spaces. For free actions, vanishing of $H^*(X;\mathbb{Z}_2)$ implies vanishing of $H^*(X/G;\mathbb{Z}_2)$ in higher range [9, p. 374, Theorem 1.5].

Proposition 2.1.5. *Let $G = \mathbb{Z}_2$ act freely on a compact Hausdorff space X. Suppose that $H^j(X;\mathbb{Z}_2) = 0$ for all $j > n$, then $H^j(X/G;\mathbb{Z}_2) = 0$ for all $j > n$.*

For $G = \mathbb{S}^1$, one can derive an analogue of the preceding result by using the Gysin-sequence for the principle bundle $X \to X/G$.

Proposition 2.1.6. *Let $G = \mathbb{S}^1$ act freely on a compact Hausdorff space X. Suppose that $H^j(X;\mathbb{Z}_2) = 0$ for all $j > n$, then $H^j(X/G;\mathbb{Z}_2) = 0$ for all $j > n$.*

The group cohomology of a discrete group G is the singular cohomology of its classifying space B_G, which is the base space of the universal principal G-bundle $G \hookrightarrow E_G \to B_G$ [10, p. 40]. We use the well-known facts that $H^*(B_{\mathbb{Z}_2};\mathbb{Z}_2) = \mathbb{Z}_2[t]$ and $H^*(B_{\mathbb{S}^1};\mathbb{Z}_2) = \mathbb{Z}_2[u]$, where $\deg(t) = 1$ and $\deg(u) = 2$, respectively.

2.2 Borel fibration and Leary-Serre spectral sequence

Spectral sequences are one of the most important tools in algebraic topology to compute homology, cohomology and homotopy groups of topological spaces. For a fibration, the homotopy groups of the fiber, the base, and the total space are related by the associated homotopy long exact sequence, but for homology or cohomology, the relation is much more complicated and can be computed using spectral sequences. Before defining the spectral sequence, we recall the following definition.

Definition 2.2.1. *A differential bigraded module over a ring R, is a collection of R-modules $\{E^{s,t}\}$, where s and t are integers, together with R-linear mapping $d : E^{*,*} \to E^{*,*}$, the differential of bidegree $(r, 1-r)$ or $(-r, r-1)$ for some integer r which satisfies $d \circ d = 0$.*

We can take the homology of the differential bigraded module:

$$H^{s,t}(E^{*,*}; d) = \big(\ker d : E^{s,t} \to E^{s+r,t-r+1}\big)/\big(\operatorname{Im} d : E^{s-r,t+r-1} \to E^{s,t}\big).$$

Next, we give the definition of a spectral sequence.

Definition 2.2.2. *A spectral sequence is a collection of differential bigraded R-modules $\{E_r^{*,*}, d_r\}$, where $r = 1, 2, \dots$; the differentials are either all of bidegree $(-r, r-1)$ (for a homology type spectral sequence) or all of bidegree $(r, 1-r)$ (for a cohomology type spectral sequence) and $E_{r+1}^{s,t} \cong H^{s,t}(E_r^{*,*}; d_r)$ for all s, t, r.*

The r^{th}-term of the collection $\{E_r^{*,*}, d_r\}$ is called the r^{th}-page. An important remark about the differentials is that the knowledge of $E_r^{*,*}$ and d_r determines $E_{r+1}^{*,*}$ but not d_{r+1}.

Next, we discuss the convergence of a spectral sequence. For this first consider a spectral sequence of first quadrant type, that is for all r, $E_r^{s,t} = 0$ if s or t is negative. Observe that if we consider $E_r^{p,q}$ for $r > \max(p, q+1)$; the differentials d_r

become trivial. Thus, we get $E_{r+k}^{p,q} = E_r^{p,q}$ for $k \geq 0$ and we denote this common term by $E_\infty^{p,q}$. But, without the first quadrant restriction the convergence of a general spectral sequence is less obvious to define. To determine the target, we write a spectral sequence as a tower of submodules of a given module. From this, we can determine where the algebraic information is converging. For the sake of simplicity, we suppress the bigrading. We start with $E_2^{*,*}$. Let

$$Z_2 = \ker d_2 \quad \text{and} \quad B_2 = \operatorname{Im} d_2.$$

The criterion, $d_2 \circ d_2 = 0$, gives $B_2 \subset Z_2 \subset E_2$, and by definition, $E_3 \cong Z_2/B_2$. Write $\bar{Z}_3$ for $\ker d_3 : E_3 \to E_3$. Since, $\bar{Z}_3$ is a submodule of E_3, we can write it as Z_3/B_2, where Z_3 is a submodule of Z_2. Similarly, $\bar{B}_3 = \operatorname{Im} d_3$ is isomorphic to B_3/B_2, and hence

$$E_4 \cong \bar{Z}_3/\bar{B}_3 \cong (Z_3/B_2)/(B_3/B_2) \cong Z_3/B_3.$$

We can present this data as follows: $B_2 \subset B_3 \subset Z_3 \subset Z_2 \subset E_2$. Continuing this process, we can write the spectral sequence as an infinite tower of submodules of E_2:

$$B_2 \subset B_3 \subset \cdots B_n \subset \cdots \quad \cdots \subset Z_n \subset Z_3 \subset Z_2 \subset E_2 \tag{2.1}$$

such that $E_{n+1} \cong Z_n/B_n$, and the differential d_{n+1} can be considered as a mapping $Z_n/B_n \to Z_n/B_n$, which has kernel Z_{n+1}/B_n and image B_{n+1}/B_n. The short exact sequence

$$0 \longrightarrow Z_{n+1}/B_n \longrightarrow Z_n/B_n \longrightarrow B_{n+1}/B_n \longrightarrow 0$$

induced by d_{n+1} gives rise to isomorphisms $B_{n+1}/B_n \cong Z_n/Z_{n+1}$ for all n. The converse is also true: Given a tower of submodules of E_2, together with such a set of isomorphisms, determines a spectral sequence.

The submodule Z_r of E_2 consists of those elements that lie in the kernel of the previous $r-2$ differentials and the submodule B_r of E_2 is the set of elements that are boundaries by the r^{th}-page. Let $Z_\infty = \cap_n Z_n$ be the submodule of E_2 of elements that are cycles at every stage. The submodule $B_\infty = \cup_n B_n$ consists of those elements that eventually bound. From equation (2.1) we get $B_\infty \subset Z_\infty$. We define E_∞ to be the bigraded module Z_∞/B_∞ and it consists of those elements that remains after the computation of the infinite sequence of successive homologies. This E_∞-term is the main goal of a computation.

To define the notion of convergence of a spectral sequence, we need the notion of a filtration. We use only cohomology type spectral sequences.

Definition 2.2.3. *A filtration F^* of an R-module A is a collection of submodules $\{F^n A\}$ for $n \in \mathbb{Z}$ such that*

$$\cdots \subset F^{n+1}A \subset F^n A \subset F^{n-1}A \subset \cdots \subset A \qquad \text{(decreasing filtration)}$$

or

$$\cdots \subset F^{n-1}A \subset F^n A \subset F^{n+1}A \subset \cdots \subset A \qquad \text{(increasing filtration).}$$

If H^* is a filtered graded R-module, then using the degree of H^*, we can define $F^n H^p = F^n H^* \cap H^p$. Thus, the associated graded module is bigraded if we consider

$$F^n H^{p+q}/F^{n+1}H^{p+q}; \quad \text{if } F^* \text{ is decreasing}$$

or

$$F^n H^{p+q}/F^{n-1}H^{p+q}; \quad \text{if } F^* \text{ is increasing.}$$

Definition 2.2.4. *A spectral sequence $\{E_r^{*,*}, d_r\}$ is said to converge to a graded R-module H^* if there is a filtration F of H^* such that*

$$E_\infty^{p,q} \cong \begin{cases} F^n H^{p+q}/F^{n+1}H^{p+q} & \text{if } F^* \text{ is decreasing} \\ F^n H^{p+q}/F^{n-1}H^{p+q} & \text{if } F^* \text{ is increasing,} \end{cases}$$

where $E_\infty^{*,*}$ is the limit term of the spectral sequence.

Remark 2.2.5. *Two general situations in which spectral sequences arise quite naturally are when one has a filtered differential module and when one has an exact couple* [51, p. 31].

Remark 2.2.6. *We say a spectral sequence of algebras converges to H^* as an al-*

gebra if the algebra structure on $E_\infty^{*,*}$ is isomorphic to the induced algebra structure on the associated bigraded algebra.

Next, we recall an important construction due to Borel [8, Chapter IV]. For a compact Lie group G, let $G \hookrightarrow E_G \to B_G$ be the universal principal G-bundle, where B_G is the classifying space for principal G-bundles and E_G is a contractible free G-space. Let X be a G-space. Then the diagonal action of G on $X \times E_G$ is free. Let $X_G = (X \times E_G)/G$ be the orbit space of the diagonal action. Then the projection $\mathrm{pr}_2 : X \times E_G \to E_G$ is G-equivariant and gives a fibration

$$X \hookrightarrow X_G \to B_G$$

called the **Borel fibration** associated to the G-space X.

Under some mild hypotheses, namely, if the base space of a fibration is path connected and the fiber is connected, there is a spectral sequence associated to a fibration, called the **Leray-Serre spectral sequence**.

Theorem 2.2.7. [51, Theorem 5.2] *Let R be a commutative ring with unity. Suppose $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ is a fibration such that B is path connected and F is connected. Then there is a first quadrant spectral sequence of algebras, $\{E_r^{*,*}, d_r\}$, converging to $H^*(E;R)$ as an algebra, with*

$$E_2^{k,l} \cong H^k(B; \mathcal{H}^l(F;R)),$$

the cohomology of the base B with local coefficients in the cohomology of the fiber of π. This spectral sequence is natural with respect to fiber-preserving maps of fibrations.

Under some additional hypothesis, the spectral sequence takes the following simpler form.

Proposition 2.2.8. *Suppose that the system of local coefficients on B determined by the fiber is simple and $H_i(B;\mathbb{Z})$and $H_i(F;\mathbb{Z})$ are finitely generated for all i.*

Then for a field k, we have

$$E_2^{k,l} \cong H^k(B;k) \otimes H^l(F;k).$$

Remark 2.2.9. *Note that $H^*(E;R)$ is a $H^*(B;R)$-module with the multiplication given by $b \cdot c = \pi^*(b) \cup c$ for $b \in H^*(B;R)$ and $c \in H^*(E;R)$.*

The next result is useful in determining the ring structure of $H^*(E;R)$.

Theorem 2.2.10. [51, Theorem 5.9] *Consider the fibration $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ and suppose that the system of local coefficients is simple, then the edge homomorphisms*

$$H^k(B;R) = E_2^{k,0} \longrightarrow E_3^{k,0} \longrightarrow \cdots \longrightarrow E_k^{k,0} \longrightarrow E_{k+1}^{k,0} = E_\infty^{k,0} \subset H^k(E;R)$$

and

$$H^l(E;R) \longrightarrow E_\infty^{0,l} = E_{l+1}^{0,l} \subset E_l^{0,l} \subset \cdots \subset E_2^{0,l} = H^l(F;R)$$

are the homomorphisms

$$\pi^* : H^k(B;R) \to H^k(E;R) \quad \text{and} \quad i^* : H^l(E;R) \to H^l(F;R),$$

respectively.

For a fibration $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$, we say that F is **totally non-homologous to zero** in E with respect to the field k if the map $i^* : H^*(E;k) \to H^*(F;k)$ is onto. As a consequence of Theorem 2.2.10, we get the following:

Corollary 2.2.11. [51, p. 148] *Let $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ be a fibration. Then F is totally non-homologous to zero in E with respect to k if and only if the associated Leary-Serre spectral sequence degenerates at the E_2-page.*

In this direction, we have the following important result which is originally due to Serre [69, Chapter III, Proposition 9].

Theorem 2.2.12. *Let $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$ be a fibration such that F and B are path* *connected, and k be a field. Suppose that the following conditions are satisfied: (a) The fiber F is totally non-homologous to zero in E with respect to k; (b) The space $H^i(F;k)$ or $H^i(B;k)$ has finite dimension over k for all $i \geq 0$. Then an isomorphism of algebras $H^*(B;k) \otimes H^*(F;k) \to H^*(E;k)$ inducing, by passage to associated graded algebras, the isomorphism of E_2 to E_∞ exists if and only if there is a homomorphism of algebras $q^* : H^*(F;k) \to H^*(E;k)$ such that its composition with the natural homomorphism $i^* : H^*(E;k) \to H^*(F;k)$ is the identity automorphism of $H^*(F;k)$.*

Our main computational tool is the Leray-Serre spectral sequence associated to the Borel fibration $X \overset{i}{\hookrightarrow} X_G \overset{\pi}{\to} B_G$. The results mentioned in this section will be used in the subsequent chapters.

Chapter 3

Groups acting freely on products of Dold manifolds

Dold manifolds were introduced by Dold [28] in search for generators for the unoriented cobordism algebra of smooth manifolds, and he showed that the cobordism classes of Dold manifolds together with the even-dimensional real projective spaces generate the unoriented cobordism algebra. Dold manifolds have been well-studied in the past. In [31], Fujii determined the integral cohomology groups of Dold manifolds and used them to calculate K_U-groups of these manifolds. Ucci [79] characterized Dold manifolds which admit codimension-one embeddings in the Euclidean space. Korbaš [44] studied parallelizability of Dold manifolds. In a very recent work [58], Nath and Sankaran defined generalized Dold manifolds and obtained results on stable parallelizability of such manifolds. Some results were obtained in [62] concerning the possible $\mathbb{Z}_2$ cohomologies of components of the fixed-point set of a smooth involution on a Dold manifold. Orbit spaces of free involutions on Dold manifolds of type $P(1, n)$ have been investigated recently in [54]. In this chapter, we investigate the free rank of symmetry of products of Dold manifolds. The results of this chapter are proved in [26].

This chapter is organized as follows. In Section 3.1, we recall the definition and cohomology of Dold manifolds. In Section 3.2, we construct free involutions on these manifolds. In Section 3.3, we compute the free rank of symmetry of products of Dold manifolds for mod 2 cohomologically trivial actions.

3.1 Dold manifolds

Definition 3.1.1. *Let m, n be non-negative integers such that $m + n > 0$. The Dold manifold $P(m, n)$ is a closed connected smooth manifold of dimension $m+2n$, obtained as the quotient of $\mathbb{S}^m \times \mathbb{C}P^n$ under the free involution*

$$\Big((x_0, x_1, \ldots, x_m),\ [z_0, z_1, \ldots, z_n]\Big) \mapsto \Big((-x_0, -x_1, \ldots, -x_m),\ [\overline{z}_0, \overline{z}_1, \ldots, \overline{z}_n]\Big).$$

It was shown by Dold [28] that, for suitable values of m and n, the cobordism classes of $P(m, n)$ serve as generators in odd degrees for the unoriented cobordism algebra of smooth manifolds.

The projection $\mathbb{S}^m \times \mathbb{C}P^n \to \mathbb{S}^m$ induces the map

$$p : P(m, n) \to \mathbb{R}P^m,$$

which is a locally trivial fiber bundle with fiber $\mathbb{C}P^n$. It admits a cross-section $s : \mathbb{R}P^m \to P(m, n)$ defined as

$$s([x_0, x_1, \ldots, x_m]) = \big[(x_0, x_1, \ldots, x_m),\ [1, 0, \ldots, 0]\big].$$

Consider the cohomology classes $a = p^*(x) \in H^1(P(m, n); \mathbb{Z}_2)$, where x is the generator of $H^1(\mathbb{R}P^m; \mathbb{Z}_2)$, and $b \in H^2(P(m, n); \mathbb{Z}_2)$, where b is characterized by the property that its restriction to a fiber is non-trivial and $s^*(b) = 0$. The classes $a \in H^1(P(m, n); \mathbb{Z}_2)$ and $b \in H^2(P(m, n); \mathbb{Z}_2)$ generate $H^*(P(m, n); \mathbb{Z}_2)$. In particular, Dold [28] proved that

$$H^*(P(m, n); \mathbb{Z}_2) \cong \mathbb{Z}_2[a, b]/\langle a^{m+1}, b^{n+1}\rangle,$$

where $\deg(a) = 1$ and $\deg(b) = 2$.

3.2 Examples of free actions on Dold manifolds

In this section, we give some examples of free involutions on Dold manifolds. If both m and n are even, then by Euler characteristic argument we see that $\mathbb{Z}_2$ cannot act freely on $P(m,n)$.

(i) When n is odd: Define $T_1 : \mathbb{C}P^n \to \mathbb{C}P^n$ as the free involution given by

$$T_1\Big([z_0, z_1, \ldots, z_{n-1}, z_n]\Big) = [-\overline{z}_1, \overline{z}_0, \ldots, -\overline{z}_n, \overline{z}_{n-1}].$$

Then

$$id \times T_1 : S^m \times \mathbb{C}P^n \to S^m \times \mathbb{C}P^n$$

gives a free involution on $S^m \times \mathbb{C}P^n$. Since T_1 commutes with the action defining the Dold manifold, we obtain a free involution on $P(m,n)$.

(ii) When both m and n are odd: Note that the preceding free involution is defined in this case as well. Further, for $m \equiv 1 \pmod 4$, we define the action T_2 on $S^m \times \mathbb{C}P^n$ by

$$T_2\Big((x_0, x_1, \ldots, x_{m-1}, x_m),\ [z_0, z_1, \ldots, z_n]\Big) =$$

$$\Big((-x_1, x_0, \ldots, -x_m, x_{m-1}),\ [-\overline{z}_1, z_0, \ldots, -\overline{z}_n, z_{n-1}]\Big).$$

This gives a free involution on $P(m,n)$ by passing to the quotient.

For $m \equiv 3 \pmod 4$, there is another free involution on $P(m,n)$. For this, we define the action T_3 on $\mathbb{S}^m \times \mathbb{C}P^n$ by

$$T_3\Big((x_0, x_1, x_2, x_3, \ldots, x_{m-3}, x_{m-2}, x_{m-1}, x_m), [z_0, z_1, \ldots, z_{n-1}, z_n]\Big) =$$

$$\Big((-x_3, x_2, -x_1, x_0, \ldots, -x_m, x_{m-1}, -x_{m-2}, x_{m-3}), [-\overline{z}_1, z_0, \ldots, -\overline{z}_n, z_{n-1}]\Big),$$

which gives a free involution on $P(m,n)$.

(iii) When m is odd and n is even: We know that if a closed smooth manifold does not bound, then it cannot admit free involutions. If m is odd and n is even with $m < n+1$, then $P(m,n)$ does not bound (see [41, Proposition 2.4]). Hence

$P(m,n)$ does not admit any free involution in this situation.

3.3 Free rank of symmetry of products of Dold manifolds

In this section, we determine some results regarding the free rank of symmetry of $\prod_{i=1}^{k} P(2m_i, n_i)$ for cohomologically trivial actions. For this we first determine the free rank of symmetry of its universal cover $\prod_{i=1}^{k} \mathbb{S}^{2m_i} \times \mathbb{C}P^{n_i}$. Our main computational tool is the Leray-Serre spectral sequence associated to the Borel fibration.

Our first observation is an application of a well-known result of Minkowski.

Lemma 3.3.1. *Let p be an odd prime. If $\mathbb{Z}_p$ acts freely and mod 2 cohomologically trivially on a finite-dimensional CW-complex X homotopy equivalent to $\prod_{i=1}^{k} S^{2m_i} \times \mathbb{C}P^{n_i}$, then the induced action on $H^*(X;\mathbb{Z})$ is trivial.*

Proof. Let $\nu : GL(n,\mathbb{Z}) \to GL(n,\mathbb{Z}_2)$ be the reduction mod 2 homomorphism. By [53] or [59, p. 176, Theorem IX.7]), if $A \in \ker \nu$ such that $A^k = I$, then $A^2 = I$. The assertion of the lemma now follows immediately. □

Lemma 3.3.2. *If p is an odd prime, then $\mathbb{Z}_p$ cannot act freely and $\mathbb{Z}$-cohomologically trivially on a finite-dimensional CW-complex X homotopy equivalent to $\prod_{i=1}^{k} \mathbb{S}^{2m_i} \times \mathbb{C}P^{n_i}$.*

Proof. Since the Euler characteristic of $X \simeq \prod_{i=1}^{k} \mathbb{S}^{2m_i} \times \mathbb{C}P^{n_i}$ is non-zero, $\mathbb{Z}_p$ cannot act freely on X by Lefschetz fixed-point theorem [34, p. 179]. □

Lemma 3.3.1 and Lemma 3.3.2 together imply the following result.

Lemma 3.3.3. *If p is an odd prime, then $\mathbb{Z}_p$ cannot act freely and mod 2 cohomologically trivially on a finite-dimensional CW-complex X homotopy equivalent to $\prod_{i=1}^{k} \mathbb{S}^{2m_i} \times \mathbb{C}P^{n_i}$.*

The following observation is of independent interest and will be used in the proof of the main theorem.

Lemma 3.3.4. *A finite group G acts freely and cohomologically trivially on the disjoint union $\coprod_{i=1}^{k} X$ of k-copies of a connected space X if and only if it acts freely and cohomologically trivially on X.*

Proof. Suppose G acts freely and cohomologically trivially on X. Then we can define a free G-action on $\coprod_{i=1}^{k} X = \bigcup_{i=1}^{k} X \times \{i\}$ by setting

$$g(x, i) = (gx, i).$$

Since the cohomology algebra of a disjoint union is the direct product of the cohomology algebras of the components, it follows that the induced action of G on cohomology algebra of the disjoint union is also trivial. Conversely, if G acts freely and cohomologically trivially on $\coprod_{i=1}^{k} X$, then the action preserves each component and acts as desired. □

The proof of the next theorem follows from [20, Theorem A]. For the sake of completeness, we give an alternate proof generalising ideas from [19].

Theorem 3.3.5. *Let X be a finite-dimensional CW-complex homotopy equivalent to $\prod_{i=1}^{k} S^{2m_i} \times \mathbb{C}P^{n_i}$. Then $\mathbb{Z}_4$ cannot act freely and mod 2 cohomologically trivially on X.*

Proof. On the contrary, suppose $\mathbb{Z}_4$ acts freely and mod 2 cohomologically trivially on X. By Künneth formula we get

$$H^*(X; \mathbb{Z}_2) = \mathbb{Z}_2[a_1, \ldots, a_k]/\langle a_1^{n_1+1}, \ldots, a_k^{n_k+1} \rangle \otimes \bigwedge(b_1, \ldots, b_k),$$

where $\deg(a_i) = 2$ and $\deg(b_i) = 2m_i$. Consider the Leray-Serre spectral sequence $(E_r^{*,*}, d_r)$ associated to the Borel fibration

$$X \hookrightarrow X_{\mathbb{Z}_4} \xrightarrow{\pi} B_{\mathbb{Z}_4}.$$

The E_2-page of this spectral sequence looks like

$$E_2^{*,*} \cong H^*(B_{\mathbb{Z}_4}; H^*(X;\mathbb{Z}_2)).$$

Since the induced action on mod 2 cohomology is trivial, the system of local coefficients is simple. Thus, we have

$$E_2^{*,*} \cong H^*(B_{\mathbb{Z}_4};\mathbb{Z}_2) \otimes H^*(X;\mathbb{Z}_2).$$

Consider the subgroup inclusion $i : \mathbb{Z}_2 \hookrightarrow \mathbb{Z}_4$. By restricting the $\mathbb{Z}_4$ action to $\mathbb{Z}_2$, we get a free $\mathbb{Z}_2$ action on X. Then consider the Leray-Serre spectral sequence $(\widetilde{E}_r^{*,*}, \widetilde{d}_r)$ associated to the Borel fibration

$$X \hookrightarrow X_{\mathbb{Z}_2} \xrightarrow{\pi} B_{\mathbb{Z}_2}.$$

The naturality of the Leray-Serre spectral sequence gives the following commutative diagram:

$$\begin{array}{ccc} E_r^{k,l} & \xrightarrow{\quad f^* \quad} & \widetilde{E}_r^{k,l} \\ {\scriptstyle d_r}\downarrow & & \downarrow{\scriptstyle \widetilde{d}_r} \\ E_r^{k+r,l-r+1} & \xrightarrow[f^*]{} & \widetilde{E}_r^{k+r,l-r+1}, \end{array}$$

where at the E_2-page the map $f^* : (E_r^{*,*}, d_r) \to (\widetilde{E}_r^{*,*}, \widetilde{d}_r)$ is defined by

$$f^* = i^* \otimes id : H^*(B_{\mathbb{Z}_4};\mathbb{Z}_2) \otimes H^*(X;\mathbb{Z}_2) \to H^*(B_{\mathbb{Z}_2};\mathbb{Z}_2) \otimes H^*(X;\mathbb{Z}_2).$$

We know that,

$$H^*(B_{\mathbb{Z}_4};\mathbb{Z}_2) \cong \mathbb{Z}_2[x] \otimes \bigwedge(y),$$

where $\deg(x) = 2$ and $\deg(y) = 1$. We also recall that $H^*(B_{\mathbb{Z}_2};\mathbb{Z}_2) \cong \mathbb{Z}_2[t]$, where $\deg(t) = 1$. It can be seen that the map $i^* : H^*(B_{\mathbb{Z}_4};\mathbb{Z}_2) \to H^*(B_{\mathbb{Z}_2};\mathbb{Z}_2)$ is given by $i^*(x) = t^2$ and $i^*(y) = 0$. Since $i^*(y) = 0$, it follows that f^* is zero on $E_r^{odd,*}$. Since $H^*(X;\mathbb{Z}_2)$ has non-zero cohomology only in even dimension, it follows that the non-zero differentials lie in E_r-page, where r is odd. Consider an element

$z \in H^*(X; \mathbb{Z}_2)$. Then

$$\widetilde{d}_{2n+1}(1 \otimes z) = \widetilde{d}_{2n+1} f^*(1 \otimes z) = f^* d_{2n+1}(1 \otimes z).$$

As $d_{2n+1}(1 \otimes z) \in E_{2n+1}^{odd,*}$, it follows that $\widetilde{d}_{2n+1}(1 \otimes z) = 0$. Since this is true for any $z \in H^*(X; \mathbb{Z}_2)$ and for any differential $\widetilde{d}_r$, it follows that the corresponding spectral sequence collapses at the E_2-page. This contradicts the fact that $X/\mathbb{Z}_2$ is finite-dimensional. Thus, we have proved that $\mathbb{Z}_4$ cannot act freely on X.

□

Now we prove the main results of this section. For the definition of the function μ see Equation (1.1).

Theorem 3.3.6. *Let G be a finite group acting freely and mod 2 cohomologically trivially on a finite-dimensional CW-complex X homotopy equivalent to $\prod_{i=1}^{k} \mathbb{S}^{2m_i} \times \mathbb{C}P^{n_i}$. Then $G \cong (\mathbb{Z}_2)^l$ for some l satisfying*

$$l \leq k_0 + \mu(n_1) + \mu(n_2) + \cdots + \mu(n_k),$$

where k_0 is the number of positive-dimensional spheres in the product.

Proof. First, we consider the case where each $m_i > 0$. By Theorem 3.3.5 we get $\mathbb{Z}_4$ does not act freely and mod 2 cohomologically trivially on X. Next, we show that G is a finite 2-group. If G is not a finite 2-group, then there exists a prime $p > 2$ such that p divides $|G|$. Consequently, $\mathbb{Z}_p$ acts freely and mod 2 cohomologically trivially on X, which contradicts Lemma 3.3.3. Therefore, G must be a finite 2-group.

Now if $n_i \equiv 1 \pmod 4$ for all i, then the desired bound on the rank of G follows from Euler characteristic argument. But, if $n_i \equiv 3 \pmod 4$ for some i, then the Euler characteristic fails to give exact bound on the rank of G. To derive the bound in this case, we first recall some properties of Lefschetz number. For a map $f : X \to X$, the **Lefschetz number** $\tau(f)$ is defined as

$$\tau(f) = \sum_i (-1)^i \operatorname{trace}(f^* : H^n(X;\mathbb{Z}) \to H^n(X;\mathbb{Z})).$$

Suppose f^* is an automorphism of $H^*(X;\mathbb{Z})$ such that $f^*(a_i) = \xi_i a_i$ and $f^*(b_i) = \lambda_i b_i$, where a_i, b_i are the generators of $H^*(X;\mathbb{Z})$ and $\xi_i, \lambda_i \in \mathbb{Z}$. Then its Lefschetz number is

$$\tau(f) = \prod_{i=1}^{k} (1 + \xi_i + \xi_i^2 + \cdots + \xi_i^{n_i})(1 + \lambda_i). \tag{3.1}$$

Let g be an element of G and $g^*_{\mathbb{Z}}$ be the induced map on $H^*(X;\mathbb{Z})$. Since G acts freely, the Lefschetz number of $g^*_{\mathbb{Z}}$ must vanish for all $g \neq 1$. This shows that G is isomorphic to some subgroup of $\operatorname{Aut}(H^*(X;\mathbb{Z}))$. Notice that, an element of $\operatorname{Aut}(H^*(X;\mathbb{Z}))$ is completely determined by its value on the generators $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$. Since $g^*_{\mathbb{Z}}$ is an automorphism, it preserves degrees as well as cup-length, where for an element $x \in H^*(X;\mathbb{Z})$, the cup-length of x is the greatest integer n such that $x^n \neq 0$. Thus, $g^*_{\mathbb{Z}}$ can only permute those generators which have same degree and same cup-length (this follows because the cup-length of a sum of the generators is the sum of the cup-lengths of the individual generators). In other words, $g^*_{\mathbb{Z}}(a_i) = \pm a_{\sigma(i)}$ and $g^*_{\mathbb{Z}}(b_i) = \pm b_{\gamma(i)}$, where $\sigma, \gamma \in S_k$ (the symmetric group on k elements) and $a_{\sigma(i)}$ has same cup-length as a_i. Let $g^*_{\mathbb{Z}_2}$ be the induced map on $H^*(X;\mathbb{Z}_2)$. By the naturality of the cohomology functor, we get the following commutative diagram:

$$\begin{array}{ccc}
H^k(X;\mathbb{Z}) & \xrightarrow{\;g^*_{\mathbb{Z}}\;} & H^k(X;\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^k(X;\mathbb{Z}_2) & \xrightarrow[\;g^*_{\mathbb{Z}_2}\;]{} & H^k(X;\mathbb{Z}_2).
\end{array}$$

Since we assumed that G acts mod 2 cohomologically trivially, we get $g^*_{\mathbb{Z}}(a_i) = \pm a_i$ and $g^*_{\mathbb{Z}}(b_i) = \pm b_i$. This shows that G is a subgroup of $(\mathbb{Z}_2)^{2k}$. The Lefschetz number should be zero corresponding to each non-identity element of G. From Equation (3.1) we get $\tau(f) = 0$ only if either $\lambda_i = -1$ for some i or $1 + \xi_j + \xi_j^2 +$

$\cdots + \xi_j^{n_j} = 0$ for some j. The latter case is possible only if n_j is odd and in that situation we get $\xi_j = -1$. This implies that

$$G \leq (\mathbb{Z}_2)^{k+\mu(n_1)+\mu(n_2)+\cdots+\mu(n_k)},$$

where $\mu(n) = 0$ if n is even and $\mu(n) = 1$ if n is odd (note that if some n_j is even, then an automorphism g^* that maps $a_j \mapsto -a_j$ and fixes all other generators of $H^*(X;\mathbb{Z})$ is not possible). Consequently, $G \cong (\mathbb{Z}_2)^l$ for some l satisfying

$$l \leq k + \mu(n_1) + \mu(n_2) + \cdots + \mu(n_k).$$

Next, we consider the case where $m_i = 0$ for some i. Let k_0 be the number of positive-dimensional spheres in the product. Note that, $\prod_{i=1}^{k} \mathbb{S}^{2m_i} \times \mathbb{C}P^{n_i}$ can be written as disjoint union of 2^{k-k_0} copies of $\Big(\prod_{m_i>0} \mathbb{S}^{2m_i} \times \prod_{i=1}^{k} \mathbb{C}P^{n_i}\Big)$. By Lemma 3.3.4, it is enough to prove the result for $\Big(\prod_{m_i>0} \mathbb{S}^{2m_i} \times \prod_{i=1}^{k} \mathbb{C}P^{n_i}\Big)$. Applying similar argument as in the previous case, we get the desired result. □

Corollary 3.3.7. *For mod 2 cohomologically trivial actions*

$$\operatorname{frk}_p\Big(\prod_{i=1}^{k} \mathbb{S}^{2m_i} \times \mathbb{C}P^{n_i}\Big) = \begin{cases} k_0 + \mu(n_1) + \mu(n_2) + \cdots + \mu(n_k) & \text{if } p = 2, \\ 0 & \text{if } p > 2, \end{cases}$$

where k_0 is the number of positive-dimensional spheres in the product.

Proof. It is known that $\mathbb{C}P^n$ admits a free action by a finite group if and only if n is odd, and in that case the only possible group is $\mathbb{Z}_2$ [34, p. 229]. The proof follows since $\mathbb{Z}_2$ always acts freely on even-dimensional spheres. □

Remark 3.3.8. *If we take arbitrary dimensional spheres in the statement of* Theorem 3.3.6*, then our method does not work. Here we would like to emphasize that the problem of determining the free rank of symmetry of products of different dimensional spheres is a long-standing open problem.*

In view of the preceding results, we would like to propose the following conjecture.

Conjecture 3.3.9. *If $(\mathbb{Z}_2)^l$ acts freely and mod 2 cohomologically trivially on a finite-dimensional CW-complex $X \simeq \prod_{i=1}^{k} \mathbb{S}^{m_i} \times \mathbb{C}P^{n_i}$ where each $m_i > 0$, then*

$$l \leq k + \mu(n_1) + \cdots + \mu(n_k).$$

We are now in a position to obtain the free rank of symmetry of Dold manifolds.

Theorem 3.3.10. *Let G be a finite group acting freely and mod 2 cohomologically trivially on a finite-dimensional CW-complex X homotopy equivalent to $\prod_{i=1}^{k} P(2m_i, n_i)$. Then $G \cong (\mathbb{Z}_2)^l$ for some l satisfying*

$$l \leq \mu(n_1) + \mu(n_2) + \cdots + \mu(n_k).$$

Proof. First, we prove that $\mathbb{Z}_4$ cannot act freely and mod 2 cohomologically trivially on $X \simeq \prod_{i=1}^{k} P(2m_i, n_i)$. On the contrary, suppose $\mathbb{Z}_4$ acts freely and mod 2 cohomologically trivially on X, and let Y denote its orbit space. Then there is an exact sequence

$$1 \to (\mathbb{Z}_2)^k \to \pi_1(Y) \to \mathbb{Z}_4 \to 1.$$

It follows from the exactness of the sequence that $\pi_1(Y)$ is a 2-group. Also, $\pi_1(Y)$ acts freely on the universal cover $\prod_{i=1}^{k} \mathbb{S}^{2m_i} \times \mathbb{C}P^{n_i}$. By Theorem 3.3.6, $\pi_1(Y)$ must be an elementary abelian 2-group and $\pi_1(Y) \cong (\mathbb{Z}_2)^{k+2}$. But this is not possible since $(\mathbb{Z}_2)^{k+2}/(\mathbb{Z}_2)^k$ is not isomorphic to $\mathbb{Z}_4$. This implies that $\mathbb{Z}_4$ cannot act freely on $\prod_{i=1}^{k} P(2m_i, n_i)$. Similar proof also shows that there is no free $\mathbb{Z}_p$-action on $\prod_{i=1}^{k} P(2m_i, n_i)$ for odd primes p. Then using Theorem 3.3.6, one sees that $G \cong (\mathbb{Z}_2)^l$, where l is as desired. □

Examples from Section 3.2 show that the bound in preceding Theorem 3.3.10 is sharp. As a consequence, we get the following result on free rank.

Corollary 3.3.11. *For mod 2 cohomologically trivial actions*

$$\operatorname{frk}_p \big(\prod_{i=1}^{k} P(2m_i, n_i) \big) = \begin{cases} \mu(n_1) + \mu(n_2) + \cdots + \mu(n_k) & \text{if } p = 2, \\ 0 & \text{if } p > 2. \end{cases}$$

Remark 3.3.12. *Note that, the mod 2 cohomology algebras of* $\mathbb{R}P^m \times \mathbb{C}P^n$ *and that of* $P(m,n)$ *are isomorphic. Also, they have the same universal cover* $\mathbb{S}^m \times \mathbb{C}P^n$. *Hence, the proof of Theorem* 3.3.10 *also works for* $\prod_{i=1}^{k} (\mathbb{R}P^{2m_i} \times \mathbb{C}P^{n_i})$.

Chapter 4

Free involutions on Dold manifolds

In this chapter, we compute the mod 2 cohomology algebra of orbit spaces of arbitrary free involutions on Dold manifolds $P(m,n)$, where n is even (Theorem 4.1.1). For Dold manifolds $P(1,n)$, where n is odd, the problem has been considered in a recent work [54, Theorem 1] of Morita et al. We also give some examples that realize the cohomology algebras of Theorem 4.1.1. The results of this chapter are proved in [25].

4.1 Orbit spaces of free involutions on Dold manifolds

The following is the main result of this section.

Theorem 4.1.1. *Let $G = \mathbb{Z}_2$ act freely on a finite-dimensional CW-complex homotopy equivalent to the Dold manifold $P(m,n)$, where n is even. Then m is odd and $H^*(P(m,n)/G;\mathbb{Z}_2)$ is isomorphic to the graded commutative algebra*

$$\mathbb{Z}_2[x,y,z]/\langle x^2, y^{\frac{m+1}{2}}, z^{n+1}\rangle,$$

where $\deg(x) = 1$, $\deg(y) = 2$ *and* $\deg(z) = 2$.

Proof. If both m and n are even, then Euler characteristic of $P(m, n)$ is odd, and $\mathbb{Z}_2$ cannot act freely on $P(m, n)$. Thus, m must be odd. First, assume that $m > 1$. Recall that, the cohomology algebra of $P(m, n)$ is

$$H^*(P(m, n); \mathbb{Z}_2) \cong \mathbb{Z}_2[a, b]/\langle a^{m+1}, b^{n+1}\rangle, \tag{4.1}$$

where $\deg(a) = 1$ and $\deg(b) = 2$. Consider an arbitrary free involution on $P(m, n)$, and let Y denote its orbit space. Note that, $\pi_1(Y)$ can either be $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and it acts on the universal cover $\mathbb{S}^m \times \mathbb{C}P^n$ of $P(m, n)$.

Case (1): Suppose that $\pi_1(Y) = \mathbb{Z}_4$. It follows that $H^1(Y; \mathbb{Z}_2) = \mathbb{Z}_2$. Now consider the Leray-Serre spectral sequence associated to the Borel fibration

$$P(m, n) \overset{i}{\hookrightarrow} Y \overset{\pi}{\longrightarrow} B_{\mathbb{Z}_2}.$$

Since n is even, the induced action of $\pi_1(B_{\mathbb{Z}_2}) = \mathbb{Z}_2$ on $H^*(P(m, n); \mathbb{Z}_2)$ is trivial. Thus the above fibration has a simple system of local coefficients, and the spectral sequence takes the form

$$E_2^{p,q} \cong H^p(B_{\mathbb{Z}_2}; \mathbb{Z}_2) \otimes H^q(P(m, n); \mathbb{Z}_2).$$

Since Y is finite-dimensional, the spectral sequence does not degenerate at the E_2-page. As $H^1(Y; \mathbb{Z}_2) = \mathbb{Z}_2$, the only possibility for the differential is $d_2(1\otimes a) = t^2\otimes 1$ and $d_2(1 \otimes b) = 0$. By computing the ranks of $\operatorname{Ker} d_2$ and $\operatorname{Im} d_2$, we get $\operatorname{rk}(E_3^{k,l}) = 0$ for all $k \geq 2$. Thus, the spectral sequence collapses at the E_3-page and we get $E_\infty^{*,*} \cong E_3^{*,*}$. Consequently,

$$H^k(Y; \mathbb{Z}_2) \cong \bigoplus_{i+j=k} E_\infty^{i,j} = E_\infty^{0,k} \oplus E_\infty^{1,k-1}$$

for $0 \leq k \leq m + 2n$.

We see that $1 \otimes a^2 \in E_2^{0,2}$ and $1 \otimes b \in E_2^{0,2}$ are permanent cocycles. Let i^*

and π^* be the homomorphisms induced on cohomology. It is known that they are precisely the edge maps (Theorem 2.2.10). Choose $y \in H^2(Y;\mathbb{Z}_2)$ such that $i^*(y) = a^2$. Then we have $y^{\frac{m+1}{2}} = 0$. Similarly, if we choose $z \in H^2(Y;\mathbb{Z}_2)$ such that $i^*(z) = b$, then we have $z^{n+1} = 0$. Let $x = \pi^*(t) \in E_\infty^{1,0} \subseteq H^1(Y;\mathbb{Z}_2)$ be determined by $t \otimes 1 \in E_2^{1,0}$. As $E_\infty^{2,0} = 0$, we have $x^2 = 0$. In this case, we obtain

$$H^*(Y;\mathbb{Z}_2) \cong \mathbb{Z}_2[x, y, z]/\langle x^2, y^{\frac{m+1}{2}}, z^{n+1}\rangle, \tag{4.2}$$

where $\deg(x) = 1$, $\deg(y) = 2$ and $\deg(z) = 2$.

Case (2): Suppose that $\pi_1(Y) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. It follows that $H^1(Y;\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since n is even, the differentials $d_2(1 \otimes b)$ and $d_3(1 \otimes b)$ should be trivial. This implies that the differential $d_2(1 \otimes a)$ is non-zero. Hence the case $\pi_1(Y) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ cannot arise.

Finally, consider the case $m = 1$. Since n is even, we get $d_2(1 \otimes b) = 0$ and $d_3(1 \otimes b) = 0$. Proceeding as in case (1), we get the following:

$$H^*(P(1, n)/\mathbb{Z}_2;\mathbb{Z}_2) \cong \mathbb{Z}_2[x, z]/\langle x^2, z^{n+1}\rangle, \tag{4.3}$$

where $\deg(x) = 1$ and $\deg(z) = 2$. This completes the proof.

□

Since the mod 2 cohomology algebras of $\mathbb{R}P^m \times \mathbb{C}P^n$ and $P(m, n)$ are isomorphic, and both have the same universal cover $\mathbb{S}^m \times \mathbb{C}P^n$, the proof of Theorem 3.3.10 also works for $\mathbb{R}P^m \times \mathbb{C}P^n$. Hence, we obtain the following result.

Corollary 4.1.2. *Let $n > 0$ be an even integer and $G = \mathbb{Z}_2$ act freely on $\mathbb{R}P^m \times \mathbb{C}P^n$. Then m is odd and $H^*\big((\mathbb{R}P^m \times \mathbb{C}P^n)/G;\mathbb{Z}_2\big)$ is isomorphic to the graded commutative algebra*

$$\mathbb{Z}_2[x, y, z]/\langle x^2, y^{\frac{m+1}{2}}, z^{n+1}\rangle,$$

where $\deg(x) = 1$, $\deg(y) = 2$ *and* $\deg(z) = 2$.

Remark 4.1.3. *In view of Theorem* 4.1.1 *and the recent work* [54]*, the problem* *of computing the mod 2 cohomology algebra of orbit spaces of free involutions on Dold manifolds $P(m,n)$ remains open in the case when $m > 1$ and n is odd. This is an ongoing work of the author.*

4.2 Examples realizing the cohomology algebras

We conclude this chapter by giving some examples realizing the cohomology algebra of Theorem 4.1.1.

Example 4.2.1. *Recall that, $P(m,0) = \mathbb{R}P^m$. The orbit space of a free involution on $\mathbb{R}P^m$ is the lens space since any free involution on $\mathbb{R}P^m$ lifts to a free $\mathbb{Z}_4$-action on $\mathbb{S}^m$ and the cohomology algebra of a lens space is consistent with Theorem 4.1.1.*

Example 4.2.2. *We do not have examples realizing the cohomology algebra of Theorem 4.1.1 when $n > 0$ is even. However, we have an example realizing the cohomology algebra when n is odd. Let m and n be odd integers, and let g be a generator of $\mathbb{Z}_4$. Define the action of $\mathbb{Z}_4$ on $\mathbb{S}^m \times \mathbb{C}P^n$ by*

$$g\big((x_0, x_1, \ldots, x_{m-1}, x_m), [z_0, z_1, \ldots, z_{n-1}, z_n]\big) =$$

$$\big((-x_1, x_0, \ldots, -x_m, x_{m-1}), [\overline{z}_1, z_0, \ldots, \overline{z}_n, z_{n-1}]\big).$$

Let Y denote the orbit space of this action. Note that the action induces a free involution on the Dold manifold $P(m,n)$ with the same orbit space Y. The projection $\mathbb{S}^m \times \mathbb{C}P^n \to \mathbb{S}^m$ induces the map $p : Y \to L^m(4,1)$, which is a locally trivial fiber bundle with fiber $\mathbb{C}P^n$. Notice that, the system of local coefficients is trivial. Since the projection π has a section, the Leray-Serre cohomology spectral sequence associated to the fiber bundle

$$\mathbb{C}P^n \overset{i}{\hookrightarrow} Y \overset{\pi}{\longrightarrow} L^m(4,1)$$

degenerates at the E_2*-page. Consequently, the map* $i^* : H^*(Y;\mathbb{Z}_2) \to H^*(\mathbb{C}P^n;\mathbb{Z}_2)$ *is onto. By Corollary* 2.2.11 *and Theorem* 2.2.12*, we get*

$$H^*(Y;\mathbb{Z}_2) \cong \mathbb{Z}_2[x,y,z]/\langle x^2, y^{\frac{m+1}{2}}, z^{n+1}\rangle,$$

where $\deg(x) = 1$, $\deg(y) = 2$ *and* $\deg(z) = 2$.

Chapter 5

Free compact group actions on Milnor manifolds

Let r and s be integers such that $0 \leq s \leq r$. A real Milnor manifold, denoted by $\mathbb{R}H_{r,s}$, is the non-singular hypersurface of degree $(1,1)$ in the product $\mathbb{R}P^r \times \mathbb{R}P^s$ of real projective spaces. Milnor [52] introduced these manifolds in search of generators for the unoriented cobordism algebra. Clearly, $\mathbb{R}H_{r,s}$ is a $(s+r-1)$-dimensional closed smooth manifold, and can be described in terms of homogeneous coordinates of real projective spaces as

$$\mathbb{R}H_{r,s} = \Big\{([x_0, \ldots, x_r], [y_0, \ldots, y_s]) \in \mathbb{R}P^r \times \mathbb{R}P^s \mid x_0y_0 + \ \cdots \ + x_sy_s = 0\Big\}.$$

Alternatively, $\mathbb{R}H_{r,s}$ is given as the total space of the following fiber bundle

$$\mathbb{R}P^{r-1} \overset{i}{\hookrightarrow} \mathbb{R}H_{r,s} \xrightarrow{p} \mathbb{R}P^s.$$

This is the projectivization of the vector bundle

$$\mathbb{R}^r \hookrightarrow E^{\perp} \longrightarrow \mathbb{R}P^s,$$

where $E^{\perp}$ is the orthogonal complement in $\mathbb{R}P^s \times \mathbb{R}^{r+1}$ of the canonical line bundle

$$\mathbb{R} \hookrightarrow E \longrightarrow \mathbb{R}P^s.$$

Similarly, a complex Milnor manifold, denoted by $\mathbb{C}H_{r,s}$, is a $2(s+r-1)$-dimensional closed smooth manifold, given in terms of homogeneous coordinates as

$$\mathbb{C}H_{r,s} = \Big\{([z_0, \ldots, z_r], [w_0, \ldots, w_s]) \in \mathbb{C}P^r \times \mathbb{C}P^s \mid z_0\overline{w}_0 + \ \cdots \ + z_s\overline{w}_s = 0\Big\}.$$

As in the real case, $\mathbb{C}H_{r,s}$ is the total space of the fiber bundle

$$\mathbb{C}P^{r-1} \overset{i}{\hookrightarrow} \mathbb{C}H_{r,s} \overset{p}{\longrightarrow} \mathbb{C}P^s.$$

It is known due to Conner and Floyd [17, p.63] that $\mathbb{C}H_{r,s}$ is unoriented cobordant to $\mathbb{R}H_{r,s} \times \mathbb{R}H_{r,s}$. These manifolds have been well-studied in the past. See, [32, 39, 73] for some recent results. Their cohomology algebra is also well-known [11, 55], and we need it for the proof of our main results.

Theorem 5.0.1. *Let $0 \leq s \leq r$. Then the following holds:*

1. $H^*(\mathbb{R}H_{r,s}; \mathbb{Z}_2) \cong \mathbb{Z}_2[a,b]/\langle a^{s+1}, b^r + ab^{r-1} + \cdots + a^s b^{r-s}\rangle$, *where a and b are both homogeneous elements of degree one.*

2. $H^*(\mathbb{C}H_{r,s}; \mathbb{Z}_2) \cong \mathbb{Z}_2[g,h]/\langle g^{s+1}, h^r + gh^{r-1} + \cdots + g^s h^{r-s}\rangle$, *where g and h are both homogeneous elements of degree two.*

Note that, $\mathbb{R}H_{r,0} = \mathbb{R}P^{r-1}$ and $\mathbb{C}H_{r,0} = \mathbb{C}P^{r-1}$. Since orbit spaces of free involutions on real and complex projective spaces are well-known [70], henceforth, we assume that $1 \leq s \leq r$.

In this chapter, we investigate free $\mathbb{Z}_2$ and $\mathbb{S}^1$-actions on mod 2 cohomology real and complex Milnor manifolds. More precisely, we determine the possible mod 2 cohomology algebra of orbit spaces of free $\mathbb{Z}_2$ and $\mathbb{S}^1$-actions on these spaces. We also find necessary and sufficient conditions for the existence of free actions on these spaces. In Section 5.1, we construct free $\mathbb{Z}_n$ and $\mathbb{S}^1$-actions on these manifolds. Induced action on cohomology is investigated in Section 5.2. In Section 5.3, we prove our main results. The results of this chapter are proved in [27].

5.1 Examples of free actions on Milnor manifolds

5.1.1 Circle actions

We give examples of free $\mathbb{S}^1$-actions on $\mathbb{R}H_{r,s}$ in the case when both r and s are odd, and later prove that this is indeed a necessary condition for the existence of a free $\mathbb{S}^1$-action on $\mathbb{R}H_{r,s}$. We first give a free $\mathbb{S}^1$-action on $\mathbb{R}P^s$. Note that, only odd-dimensional real projective spaces admit free $\mathbb{S}^1$-actions. Let $s = 2m + 1$ and $[w_0, \ldots, w_m]$ denote an element of $\mathbb{R}P^{2m+1}$, where w_i are complex numbers.

Define a map $\mathbb{S}^1 \times \mathbb{R}P^{2m+1} \to \mathbb{R}P^{2m+1}$ by

$$(\xi, [w_0, \ldots, w_m]) = [\sqrt{\xi} w_0, \ldots, \sqrt{\xi} w_m].$$

It can be checked that the preceding map gives a free $\mathbb{S}^1$-action on $\mathbb{R}P^{2m+1}$.

Let $r = 2n+1$ and write an element of $\mathbb{R}P^r$ as $[z_0, \ldots, z_n]$, where z_i are complex numbers. Define an action of $\mathbb{S}^1$ on $\mathbb{R}P^r$ by

$$(\xi, [z_0, \ldots, z_n]) = [\sqrt{\xi} z_0, \ldots, \sqrt{\xi} z_n].$$

The diagonal action on $\mathbb{R}P^r \times \mathbb{R}P^s$ is free. Also, $\mathbb{R}H_{r,s}$ is invariant under this action and gives rise to a free $\mathbb{S}^1$-action on $\mathbb{R}H_{r,s}$. Restricting the above $\mathbb{S}^1$-action gives free $\mathbb{Z}_n$ (in particular $\mathbb{Z}_2$) action on $\mathbb{R}H_{r,s}$.

For any two spaces X and Y, we denote by $X \simeq_2 Y$, if they have isomorphic mod 2 cohomology algebras, not necessarily induced by a map between X and Y. For the complex case, we have the following result.

Proposition 5.1.1. *There is no free $\mathbb{S}^1$-action on a compact Hausdorff space $X \simeq_2 \mathbb{C}H_{r,s}$.*

Proof. Recall that, we have a fiber bundle

$$\mathbb{C}P^{r-1} \hookrightarrow \mathbb{C}H_{r,s} \longrightarrow \mathbb{C}P^s$$

with $\chi(\mathbb{C}H_{r,s}) = r(s+1)$. Suppose there is a free $\mathbb{S}^1$-action on X. Restriction of this action gives free $\mathbb{Z}_p$-actions for each prime p. By Floyd's Euler characteristic formula (see Theorem 2.1.3), we have

$$r(s+1) = \chi(X) = p\,\chi(X/\mathbb{Z}_p)$$

for each prime p, which is a contradiction. Hence, there is no free $\mathbb{S}^1$-action on a space $X \simeq_2 \mathbb{C}H_{r,s}$. $\square$

5.1.2 Involutions

If $s = r$, then interchanging the coordinates i.e.,

$$([z_0, \ldots, z_s], [w_0, \ldots, w_s]) \longmapsto ([w_0, \ldots, w_s], [z_0, \ldots, z_s])$$

gives a free involution on a Milnor manifold. But, if $1 < s < r$ and $r \not\equiv 2 \pmod 4$, then we show that $\mathbb{R}H_{r,s}$ (respectively $\mathbb{C}H_{r,s}$) admits a free involution if and only if both r and s are odd. We have seen examples of free involutions on real Milnor manifolds in the preceding subsection.

To define a free involution on complex Milnor manifold, we recall that $\mathbb{C}P^n$ admits a free action by a finite group if and only if n is odd, and in that case the only possible group is $\mathbb{Z}_2$ [34, p. 229]. If s is odd, then the map

$$[z_0, z_1, \ldots, z_{s-1}, z_s] \longmapsto [-\overline{z}_1, \overline{z}_0, \ldots, -\overline{z}_s, \overline{z}_{s-1}],$$

defines a free involution on $\mathbb{C}P^s$. Similarly, when r odd, the map

$$[z_0, z_1, \dots, z_{r-1}, z_r] \longmapsto [-\overline{z}_1, \overline{z}_0, \dots, -\overline{z}_r, \overline{z}_{r-1}],$$

is a free involution on $\mathbb{C}P^r$. Hence, the diagonal action on $\mathbb{C}P^r \times \mathbb{C}P^s$ is free and its restriction gives a free involution on $\mathbb{C}H_{r,s}$.

5.2 Induced action on cohomology

When a group acts on a topological space, in general, it is difficult to determine the induced action on cohomology. In our context, we have the following:

Proposition 5.2.1. *Let $G = \mathbb{Z}_2$ act freely on a compact Hausdorff space $X \simeq_2 \mathbb{R}H_{r,s}$, where $1 < s < r$ and $r \not\equiv 2 \pmod 4$. Then the induced action on $H^*(X; \mathbb{Z}_2)$ is trivial.*

Proof. Let $G = \langle g \rangle$ and $a, b \in H^1(X; \mathbb{Z}_2)$ be generators of the cohomology algebra $H^*(X; \mathbb{Z}_2)$. By the naturality of cup product, for all $i,\ j \geq 0$, we have

$$g^*(a^i b^j) = (g^*(a))^i (g^*(b))^j.$$

Therefore, it is enough to consider

$$g^* : H^1(X; \mathbb{Z}_2) \to H^1(X; \mathbb{Z}_2).$$

Suppose g^* is non-trivial. Then it cannot preserve both a and b. Assuming $g^*(b) \neq b$, we have $g^*(b) = a$ or $a + b$. If $g^*(b) = a$, then $g^*(b^{s+1}) = a^{s+1} = 0$, which implies $b^{s+1} = 0$. Hence, $b^r = 0$, contradicting the fact that top dimensional cohomology must be non-zero with $\mathbb{Z}_2$ coefficients. So, we must have $g^*(b) = a + b$ and $g^*(a) = a$. Suppose r is odd. Then

$$g^*(a^{s-1} b^r) = a^{s-1}(a + b)^r = r a^s b^{r-1} + a^{s-1} b^r = a^s b^{r-1} + a^{s-1} b^r = 0.$$

This gives $a^{s-1}b^r = 0$, which is a contradiction. Hence, when r odd, the induced action on $H^*(X;\mathbb{Z}_2)$ must be trivial.

Suppose $r \equiv 0 \pmod 4$. Notice that, $b^{r+1} = 0$ implies $g^*(b^{r+1}) = (a+b)^{r+1} = 0$. But, from the binomial expansion

$$(a+b)^{r+1} = a^{r+1} + \cdots + \binom{r+1}{2} a^2 b^{r-1} + (r+1)ab^r,$$

we see that the last term is non-zero and the second last term is zero modulo 2. This gives $(a+b)^{r+1} \neq 0$, a contradiction. Hence, the induced action must be trivial in this case as well. □

Remark 5.2.2. *If $s = 1$ and $r > 1$ is an odd integer, then orders of b and $a + b$ are $r + 1$ and r, respectively. Hence, in this case g^* is identity. For $s = 1$ or $r \equiv 2 \pmod 4$, the induced action on $H^*(X;\mathbb{Z}_2)$ might be non-trivial.*

Similarly, for the complex case, we have the following:

Proposition 5.2.3. *Let $G = \mathbb{Z}_2$ act freely on a compact Hausdorff space $X \simeq_2 \mathbb{C}H_{r,s}$, where $1 < s < r$ and $r \not\equiv 2 \pmod 4$. Then the induced action on $H^*(X;\mathbb{Z}_2)$ is trivial.*

Remark 5.2.4. *If $r = s$, then the free involution on $\mathbb{R}H_{s,s}$ given by*

$$([x_0, \ldots, x_s], [y_0, \ldots, y_s]) \longmapsto ([y_0, \ldots, y_s], [x_0, \ldots, x_s]),$$

and the similar free involution on $\mathbb{C}H_{s,s}$ given by

$$([z_0, \ldots, z_s], [w_0, \ldots, w_s]) \longmapsto ([w_0, \ldots, w_s], [z_0, \ldots, z_s])$$

are cohomologically non-trivial [73, Propositions 5.1 and 5.2].

Next, we determine conditions on r and s for which a compact Hausdorff space $X \simeq_2 \mathbb{R}H_{r,s}$ or $\mathbb{C}H_{r,s}$ admits a free involution.

Proposition 5.2.5. *Let $G = \mathbb{Z}_2$ act freely on $X \simeq_2 \mathbb{R}H_{r,s}$, where $1 < s < r$ and $r \not\equiv 2 \pmod 4$. Then both r and s are odd.*

Proof. Suppose $\mathbb{Z}_2$ acts freely on $X \simeq_2 \mathbb{R}H_{r,s}$. Let $a, b \in H^1(X;\mathbb{Z}_2)$ be generators of the cohomology algebra $H^*(X;\mathbb{Z}_2)$. By Proposition 5.2.1, $\pi_1(B_G) = \mathbb{Z}_2$ acts trivially on $H^*(X;\mathbb{Z}_2)$, so that the fibration $X \hookrightarrow X_G \longrightarrow B_G$ has a simple system of local coefficients. Hence, the spectral sequence has the form

$$E_2^{p,q} \cong H^p(B_G;\mathbb{Z}_2) \otimes H^q(X;\mathbb{Z}_2).$$

If $d_2 : E_2^{0,1} \to E_2^{2,0}$ is trivial, then the spectral sequence degenerates at E_2-term, and we get $H^i(X/G;\mathbb{Z}_2) \neq 0$ for infinitely many values of i. This contradicts Proposition 2.1.5. Thus, d_2 must be non-trivial. Hence, we have the following three possibilities:

(i) $d_2(1 \otimes a) = t^2 \otimes 1$ and $d_2(1 \otimes b) = 0$.

(ii) $d_2(1 \otimes a) = 0$ and $d_2(1 \otimes b) = t^2 \otimes 1$.

(iii) $d_2(1 \otimes a) = t^2 \otimes 1$ and $d_2(1 \otimes b) = t^2 \otimes 1$.

We first prove that cases (i) and (ii) are not possible.

Assuming that s is odd, first we show that case (i) is not possible. Suppose $d_2(1 \otimes a) = t^2 \otimes 1$ and $d_2(1 \otimes b) = 0$. By the derivation property of the differential, we have

$$d_2(t^k \otimes a^m b^n) = \begin{cases} t^{k+2} \otimes a^{m-1}b^n & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

Then the relation $b^r + ab^{r-1} + \cdots + a^s b^{r-s} = 0$ gives

$$\begin{aligned} 0 &= d_2(1 \otimes (b^r + ab^{r-1} + \cdots + a^s b^{r-s})) \\ &= d_2(1 \otimes b^r) + d_2(1 \otimes ab^{r-1}) + \cdots + d_2(1 \otimes a^s b^{r-s}) \\ &= 0 + t^2 \otimes b^{r-1} + \cdots + t^2 \otimes a^{s-1} b^{r-s} \\ &= t^2 \otimes (b^{r-1} + \cdots + a^{s-1} b^{r-s}), \end{aligned}$$

a contradiction. Hence, case (i) is not possible. The even case follows similarly. The same argument works for case (ii) as well.

Hence $d_2(1 \otimes a) = t^2 \otimes 1$ and $d_2(1 \otimes b) = t^2 \otimes 1$. If s is even, then $a^{s+1} = 0$ gives

$$0 = d_2(1 \otimes a^{s+1}) = t^2 \otimes a^s,$$

a contradiction. Therefore, s must be odd, and similar arguments show that r is odd as well. □

As a consequence of Proposition 5.2.5 and the previously defined $\mathbb{S}^1$-action on $\mathbb{R}H_{r,s}$, we obtain the following:

Corollary 5.2.6. *Let $1 < s < r$ and $r \not\equiv 2 \pmod 4$. Then $\mathbb{R}H_{r,s}$ admits a free involution if and only if both r and s are odd.*

We have similar observations for the complex case.

Proposition 5.2.7. *Let $\mathbb{Z}_2$ act freely on $X \simeq_2 \mathbb{C}H_{r,s}$ with $1 < s < r$ and $r \not\equiv 2 \pmod 4$. Then both r and s are odd.*

Corollary 5.2.8. *Let $1 < s < r$ and $r \not\equiv 2 \pmod 4$. Then $\mathbb{C}H_{r,s}$ admits a free involution if and only if both r and s are odd.*

For $\mathbb{S}^1$-actions, we have

Proposition 5.2.9. *Let $1 \leq s \leq r$. Then $\mathbb{S}^1$ acts freely on $\mathbb{R}H_{r,s}$ if and only if both r and s are odd.*

Proof. For $G = \mathbb{S}^1$, since $\pi_1(B_G) = 1$, the system of local coefficients is simple. Recall that, $H^*(B_{\mathbb{S}^1}; \mathbb{Z}_2) = \mathbb{Z}_2[u]$, where $\deg(u) = 2$. Hence, the spectral sequence has the form

$$E_2^{p,q} \cong H^p(B_G; \mathbb{Z}_2) \otimes H^q(X; \mathbb{Z}_2).$$

Clearly, for p odd, $E_2^{p,q} = 0$. As in the case of $\mathbb{Z}_2$-action, it can be seen that the differential d_2 must be non-zero and the only possibility for d_2 is $d_2(1 \otimes a) = u \otimes 1$ and $d_2(1 \otimes b) = u \otimes 1$. Consequently, both r and s must be odd. □

5.3 Orbit spaces of free $\mathbb{Z}_2$ and $\mathbb{S}^1$-actions on Milnor manifolds

In this section, we determine the possible mod 2 cohomology algebra of orbit spaces of arbitrary free $\mathbb{Z}_2$ and $\mathbb{S}^1$-actions on Milnor manifolds.

Theorem 5.3.1. *Suppose $G = \mathbb{Z}_2$ acts freely on a compact Hausdorff space $X \simeq_2 \mathbb{R}H_{r,s}$ such that induced action on mod 2 cohomology is trivial. Then*

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y, z, w]/I,$$

where

$$I = \Big\langle z^2,\ w^2 - \gamma_1 zw - \gamma_2 x - \gamma_3 y, x^{\frac{s+1}{2}} + \alpha_0 zwx^{\frac{s-1}{2}} + \alpha_1\, zwx^{\frac{s-3}{2}} y + \cdots + \alpha_{\frac{s-1}{2}}\, zwy^{\frac{s-1}{2}},$$

$$(w + \beta_0 z)y^{\frac{r-1}{2}} + (w + \beta_1 z)xy^{\frac{r-3}{2}} + \cdots + (w + \beta_{\frac{s-1}{2}} z)\, x^{\frac{s-1}{2}} y^{\frac{r-s}{2}} \Big\rangle,$$

with $\deg(x) = 2$, $\deg(y) = 2$, $\deg(z) = 1$, $\deg(w) = 1$ *and* $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}_2$.

Proof. Let $a, b \in H^1(X; \mathbb{Z}_2)$ be generators of the cohomology algebra $H^*(X; \mathbb{Z}_2)$. By similar arguments as in Proposition 5.2.5, we see that both r and s must be odd and

$$d_2(1 \otimes a) = t^2 \otimes 1 \text{ and } d_2(1 \otimes b) = t^2 \otimes 1.$$

By the derivation property of the differential, we have

$$d_2(1 \otimes a^m b^n) = \begin{cases} t^2 \otimes a^{m-1}b^n + t^2 \otimes a^m b^{n-1} & \text{if } m \text{ and } n \text{ are odd} \\ t^2 \otimes a^{m-1}b^n & \text{if } m \text{ is odd and } n \text{ is even} \\ t^2 \otimes a^m b^{n-1} & \text{if } m \text{ is even and } n \text{ is odd} \\ 0 & \text{if } m \text{ and } n \text{ are even.} \end{cases}$$

It suffices to look at

$$d_2 : E_2^{0,q} \to E_2^{2,q-1}.$$

• For $q \leq s$, a basis of $E_2^{0,q} \cong H^q(X;\mathbb{Z}_2)$ consists of

$$\{a^q, a^{q-1}b, \ldots, \ ab^{q-1}, b^q\}.$$

If q is even, then $\mathrm{rk}(\mathrm{Ker}\, d_2) = \frac{q}{2} + 1$ and $\mathrm{rk}(\mathrm{Im}\, d_2) = \frac{q}{2}$. If q is odd, then $\mathrm{rk}(\mathrm{Ker}\, d_2) = \frac{q+1}{2} = \mathrm{rk}(\mathrm{Im}\, d_2)$.

• For $s < q \leq r-1$, a basis consists of

$$\{a^s b^{q-s}, a^{s-1}b^{q-s+1}, \ldots, \ ab^{q-1}, b^q\}.$$

In this case, $\mathrm{rk}(\mathrm{Ker}\, d_2) = \frac{s+1}{2} = \mathrm{rk}(\mathrm{Im}\, d_2)$.

• For $r \leq q \leq s+r-1$, a basis consists of

$$\{a^s b^{q-s}, a^{s-1}b^{q-s+1}, \ldots, \ a^{q-r+1}b^{r-1}\}.$$

If q is odd, then $\mathrm{rk}(\mathrm{Ker}\, d_2) = \frac{s+r-1-q}{2}$ and $\mathrm{rk}(\mathrm{Im}\, d_2) = \frac{s+r+1-q}{2}$. And, if q is even, then $\mathrm{rk}(\mathrm{Ker}\, d_2) = \frac{r+s-q}{2} = \mathrm{rk}(\mathrm{Im}\, d_2)$.

From the above observation, we get for all $k \geq 2$ and for all l, $E_3^{k,l} = 0$ as $\mathrm{rk}(E_3^{k,l}) = 0$. This gives

$$E_3^{k,l} = \begin{cases} \mathrm{Ker}\{d_2 : E_2^{k,l} \longrightarrow E_2^{k+2,l-1}\} & k = 0,\, 1 \text{ and for all } l, \\ 0 & k \geq 2 \text{ and for all } l. \end{cases}$$

Note that $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is trivial for all $r \geq 3$ and for all k, l, and hence $E_\infty^{*,*} \cong E_3^{*,*}$. Since $H^*(X_G;\mathbb{Z}_2) \cong \mathrm{Tot} E_\infty^{*,*}$, the total complex of $E_\infty^{*,*}$, we have for all $0 \leq n \leq r+s-1$

$$H^n(X_G;\mathbb{Z}_2) \cong \bigoplus_{i+j=n} E_\infty^{i,j} = E_\infty^{0,n} \oplus E_\infty^{1,n-1}$$

.

Note that $t \otimes 1$ is a permanent cocycle and let $z = \pi^*(t) \in E_\infty^{1,0} \subseteq H^1(X_G;\mathbb{Z}_2)$ be determined by $t \otimes 1 \in E_2^{1,0}$. As $E_\infty^{2,0} = 0$, we have $z^2 = 0$. Also, $1 \otimes (a+b) \in E_2^{0,1}$

is a permanent cocycle. Let $w \in H^1(X_G;\mathbb{Z}_2)$ such that $i^*(w) = a+b$. Notice that, $1\otimes a^2 \in E_2^{0,2}$ and $1\otimes b^2 \in E_2^{0,2}$ are permanent cocycles, and hence they determine elements in $E_\infty^{0,2}$. Let $x, y \in H^2(X_G;\mathbb{Z}_2)$ such that $i^*(x) = a^2$ and $i^*(y) = b^2$. As $a^{s+1} = 0$, we get the following

$$x^{\frac{s+1}{2}} + \alpha_0 zwx^{\frac{s-1}{2}} + \alpha_1\, zwx^{\frac{s-3}{2}}y + \cdots + \alpha_{\frac{s-1}{2}}\, zwy^{\frac{s-1}{2}} = 0,$$

where $\alpha_i \in \mathbb{Z}_2$. Notice that,

$$i^*(wy^{\frac{r-1}{2}} + wxy^{\frac{r-3}{2}} + \cdots + wx^{\frac{s-1}{2}}y^{\frac{r-s}{2}}) = 0.$$

Hence, it satisfies

$$wy^{\frac{r-1}{2}} + wxy^{\frac{r-3}{2}} + \cdots + wx^{\frac{s-1}{2}}y^{\frac{r-s}{2}} = \beta_0 zy^{\frac{r-1}{2}} + \beta_1 zxy^{\frac{r-3}{2}} + \beta_{\frac{s-1}{2}} zx^{\frac{s-1}{2}}y^{\frac{r-s}{2}},$$

where $\beta_i \in \mathbb{Z}_2$. Note that, we can write w^2 as the following:

$$w^2 = \gamma_1 zw + \gamma_2 x + \gamma_3 y,$$

where $\gamma_i \in \mathbb{Z}_2$. Therefore,

$$H^*(X/G;\mathbb{Z}_2) \cong H^*(X_G;\mathbb{Z}_2) \cong \mathbb{Z}_2[x,y,z,w]/I,$$

where

$$I = \Big\langle z^2, w^2 - \gamma_1 zw - \gamma_2 x - \gamma_3 y, x^{\frac{s+1}{2}} + \alpha_0 zwx^{\frac{s-1}{2}} + \alpha_1\, zwx^{\frac{s-3}{2}}y + \cdots + \alpha_{\frac{s-1}{2}}\, zwy^{\frac{s-1}{2}},$$

$$(w+\beta_0 z)y^{\frac{r-1}{2}} + (w+\beta_1 z)xy^{\frac{r-3}{2}} + \cdots + (w+\beta_{\frac{s-1}{2}} z)\, x^{\frac{s-1}{2}}y^{\frac{r-s}{2}}\Big\rangle,$$

with $\deg(x) = 2$, $\deg(y) = 2$, $\deg(z) = 1$, $\deg(w) = 1$ and $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}_2$. □

For the complex case, we prove the following:

Theorem 5.3.2. *Suppose $G = \mathbb{Z}_2$ acts freely on a compact Hausdorff space*

$X \simeq_2 \mathbb{C}H_{r,s}$, such that induced action on mod 2 cohomology is trivial. Then

$$H^*(X/G;\mathbb{Z}_2) \cong \mathbb{Z}_2[x,y,z,w]/J,$$

where

$$J = \Big\langle z^3,\ w^2 - \gamma_1 z^2 w - \gamma_2 x - \gamma_3 y,\ x^{\frac{s+1}{2}} + \alpha_0 z^2 w x^{\frac{s-1}{2}} + \alpha_1 z^2 w x^{\frac{s-3}{2}} y + \cdots + \alpha_{\frac{s-1}{2}} z^2 w y^{\frac{s-1}{2}},$$

$$(w+\beta_0 z^2) y^{\frac{r-1}{2}} + (w+\beta_1 z^2) x y^{\frac{r-3}{2}} + \cdots + (w+\beta_{\frac{s-1}{2}} z^2) x^{\frac{s-1}{2}} y^{\frac{r-s}{2}} \Big\rangle,$$

with $\deg(x)=4$, $\deg(y)=4$, $\deg(z)=1$, $\deg(w)=2$ *and* $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}_2$.

Proof. Let $G=\mathbb{Z}_2$ act freely on $X \simeq_2 \mathbb{C}H_{r,s}$. Note that $E_2^{k,l}=0$ for l odd. This gives

$$d_2 : E_2^{k,l} \to E_2^{k+2,l-1}$$

is zero, and hence $E_3^{k,l} = E_2^{k,l}$ for all k, l. Let a, $b \in H^2(X;\mathbb{Z}_2)$ be generators of the cohomology algebra $H^*(X;\mathbb{Z}_2)$. As in the proof of Theorem 5.3.1, the only possibility for d_3 is

$$d_3(1\otimes a) = t^3 \otimes 1 \ \text{ and } \ d_3(1\otimes b) = t^3\otimes 1.$$

Note that r and s must be odd. For various values of l, we consider the differentials

$$d_3 : E_3^{0,2l} \to E_3^{3,2l-2}.$$

If we compute the ranks of $\operatorname{Ker} d_3$ and $\operatorname{Im} d_3$, we get $\operatorname{rk}(E_4^{k,2l}) = 0$ for all $k \geq 3$. This implies that $E_4^{k,2l}=0$ for all $k\geq 3$ and $E_4^{k,2l} = \operatorname{Ker}\{d_3 : E_3^{k,2l} \to E_3^{k+3,2l-2}\}$ for $k = 0$, 1, 2. Also,

$$d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$$

is zero for all $r \geq 4$. Hence, $E_\infty^{*,*} \cong E_4^{*,*}$. Since $H^*(X_G;\mathbb{Z}_2) \cong \operatorname{Tot}E_\infty^{*,*}$, we get

$$H^n(X_G;\mathbb{Z}_2) \cong \bigoplus_{i+j=n} E_\infty^{i,j} = E_\infty^{0,n} \oplus E_\infty^{1,n-1} \oplus E_\infty^{2,n-2}$$

for all $0 \leq p \leq 2(s+r-1)$.

Note that $t \otimes 1$ is a permanent cocycle and let $z = \pi^*(t) \in E_\infty^{1,0} \subseteq H^1(X_G; \mathbb{Z}_2)$ be determined by $t \otimes 1 \in E_2^{1,0}$. As $E_\infty^{3,0} = 0$, we have $z^3 = 0$. Also, $1 \otimes (a+b) \in E_2^{0,2}$ is a permanent cocycle. Let $w \in H^2(X_G; \mathbb{Z}_2)$ such that $i^*(w) = a + b$. Also, $1 \otimes a^2 \in E_2^{0,4}$ and $1 \otimes b^2 \in E_2^{0,4}$ are permanent cocycles, and hence they determine elements in $E_\infty^{0,4}$. Let $x, y \in H^4(X_G; \mathbb{Z}_2)$ such that $i^*(x) = a^2$ and $i^*(y) = b^2$. As $a^{s+1} = 0$, we get the following relation

$$x^{\frac{s+1}{2}} + \alpha_0 z^2 w x^{\frac{s-1}{2}} + \alpha_1 z^2 w x^{\frac{s-3}{2}} y + \cdots + \alpha_{\frac{s-1}{2}} z^2 w y^{\frac{s-1}{2}} = 0,$$

where $\alpha_i \in \mathbb{Z}_2$. Notice that,

$$i^*(wy^{\frac{r-1}{2}} + wxy^{\frac{r-3}{2}} + \cdots + wx^{\frac{s-1}{2}} y^{\frac{r-s}{2}}) = 0.$$

Hence, it satisfies

$$wy^{\frac{r-1}{2}} + wxy^{\frac{r-3}{2}} + \cdots + wx^{\frac{s-1}{2}} y^{\frac{r-s}{2}} = \beta_0 z^2 y^{\frac{r-1}{2}} + \beta_1 z^2 x y^{\frac{r-3}{2}} + \beta_{\frac{s-1}{2}} z^2 x^{\frac{s-1}{2}} y^{\frac{r-s}{2}},$$

where $\beta_i \in \mathbb{Z}_2$. Note that, w^2 satisfies the following relation

$$w^2 = \gamma_1 z^2 w + \gamma_2 x + \gamma_3 y,$$

where $\gamma_i \in \mathbb{Z}_2$. Therefore,

$$H^*(X/G; \mathbb{Z}_2) \cong H^*(X_G; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y, z, w]/J,$$

where

$$J = \Big\langle z^3,\ w^2 - \gamma_1 z^2 w - \gamma_2 x - \gamma_3 y, x^{\frac{s+1}{2}} + \alpha_0 z^2 w x^{\frac{s-1}{2}} + \cdots + \alpha_{\frac{s-1}{2}} z^2 w y^{\frac{s-1}{2}},$$

$$(w + \beta_0 z^2) y^{\frac{r-1}{2}} + (w + \beta_1 z^2) x y^{\frac{r-3}{2}} + \cdots + (w + \beta_{\frac{s-1}{2}} z^2)\, x^{\frac{s-1}{2}} y^{\frac{r-s}{2}} \Big\rangle,$$

with $\deg(x) = 4$, $\deg(y) = 4$, $\deg(z) = 1$, $\deg(w) = 2$ and $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}_2$. This completes the proof. □

For $\mathbb{S}^1$ actions, we have

Theorem 5.3.3. *Let $G = \mathbb{S}^1$ act freely on a compact Hausdorff space $X \simeq_2 \mathbb{R}H_{r,s}$. Then*

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y, w]/\langle x^{\frac{s+1}{2}},\ wy^{\frac{r-1}{2}} + xwy^{\frac{r-3}{2}} + \cdots + wx^{\frac{s-1}{2}} y^{\frac{r-s}{2}},\ w^2 - \alpha x - \beta y\rangle,$$

where $\deg(x) = 2$, $\deg(y) = 2$, $\deg(w) = 1$ *and* $\alpha, \beta \in \mathbb{Z}_2$.

Proof. By Proposition 5.2.9, the only possibility for the differential d_2 is that $d_2(1 \otimes a) = u \otimes 1$, $d_2(1 \otimes b) = u \otimes 1$ and both r and s are odd. As in the proof of Theorem 5.3.1, if we compute the ranks of $\mathrm{Ker}\, d_2$ and $\mathrm{Im}\, d_2$, we get $\mathrm{rk}(E_3^{k,l}) = 0$ and hence $E_3^{k,l} = 0$ for all $k \geq 1$ and for all l. Also, $E_3^{0,l} = \mathrm{Ker}\{d_2 : E_2^{0,l} \to E_2^{2,l-1}\}$ for all l.

Note that, $d_r : E_r^{k,l} \to E_r^{k+r,l-r+1}$ is trivial for all $r \geq 3$ and for all k, l. Hence $E_\infty^{*,*} = E_3^{*,*}$. Since $H^*(X_G; \mathbb{Z}_2) \cong \mathrm{Tot} E_\infty^{*,*}$, we have

$$H^n(X_G; \mathbb{Z}_2) \cong \bigoplus_{i+j=n} E_\infty^{i,j} = E_\infty^{0,n}$$

for all $0 \leq n \leq r + s - 1$.

We see that $1 \otimes (a + b) \in E_2^{0,1}$ is a permanent cocycle. Let $w \in H^1(X_G; \mathbb{Z}_2)$ such that $i^*(w) = a + b$. Also, $1 \otimes a^2 \in E_2^{0,2}$ and $1 \otimes b^2 \in E_2^{0,2}$ are permanent cocycles. Hence they determine elements in $E_\infty^{0,2}$. Let $x, y \in H^2(X_G; \mathbb{Z}_2)$ such that $i^*(x) = a^2$ and $i^*(y) = b^2$. As $a^{s+1} = 0$, we get $x^{\frac{s+1}{2}} = 0$. Note that

$$i^*(wy^{\frac{r-1}{2}} + xwy^{\frac{r-3}{2}} + \cdots + wx^{\frac{s-1}{2}} y^{\frac{r-s}{2}}) = 0.$$

Hence, we get the following relation

$$wy^{\frac{r-1}{2}} + xwy^{\frac{r-3}{2}} + \cdots + wx^{\frac{s-1}{2}} y^{\frac{r-s}{2}} = 0.$$

Note that, we can write w^2 as the following

$$w^2 = \alpha x + \beta y,$$

for some $\alpha, \beta \in \mathbb{Z}_2$. Therefore,

$$H^*(X/G; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y, w]/\langle x^{\frac{s+1}{2}}, wy^{\frac{r-1}{2}} + xwy^{\frac{r-3}{2}} + \cdots + wx^{\frac{s-1}{2}} y^{\frac{r-s}{2}}, w^2 - \alpha x - \beta y\rangle,$$

where $\deg(x) = 2$, $\deg(y) = 2$, $\deg(w) = 1$ and $\alpha, \beta \in \mathbb{Z}_2$. □

Example 5.3.4. *Take $r = 3$ and $s = 1$. Recall that, $\mathbb{R}H_{3,1}$ is a 3-dimensional closed smooth manifold. A free $\mathbb{S}^1$-action on $\mathbb{R}H_{3,1}$ gives a principal $\mathbb{S}^1$-bundle $\mathbb{R}H_{3,1} \to \mathbb{R}H_{3,1}/\mathbb{S}^1$ with compact 2-dimensional base. Now, using the Leray-Serre spectral sequence associated to the Borel fibration, one can see that $H^1(\mathbb{R}H_{3,1}/\mathbb{S}^1; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Hence, the orbit space must be $\mathbb{R}P^2$ and its cohomology algebra is given by Theorem* 5.3.3 *by taking $\beta = 1$.*

Chapter 6

Applications to $\mathbb{Z}_2$-equivariant maps

A natural problem in the theory of transformation groups is to determine the necessary and sufficient conditions for the existence of a G-equivariant map between two G-spaces. One of the most well-known result in this direction is the classical Borsuk-Ulam theorem [40], which states that there does not exist a $\mathbb{Z}_2$-equivariant map $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$, where $\mathbb{Z}_2$ acts antipodally on the spheres. This formulation of the Borsuk-Ulam theorem has a lot of equivariant generalizations. We refer the reader to [50, 75] for extensive literature on this theme. Dold [29] extended this theorem to the setting of fiber bundles. Using the techniques introduced by Dold; Mattos and Santos [22] obtained parametrized Borsuk-Ulam theorems for bundles whose fiber has the mod p cohomology algebra (resp. rational cohomology algebra) of a product of spheres with any free $\mathbb{Z}_p$ (resp. $\mathbb{S}^1$-action). In this direction, Singh [71] proved parametrized Borsuk-Ulam theorems for bundles whose fiber has the mod 2 cohomology algebra of a real or a complex projective space with any free involution. Also there are linear versions of this problem; for example, in [46–48], Marzantowicz studied Borsuk-Ulam theorem for G-equivariant maps $f : S(V) \to S(W)$, where V and W are orthogonal representations of a compact Lie group G. See [24, 37, 42, 43, 49, 57] for similar results by other authors. On the other hand, sufficient conditions for the existence of G-equivariant maps between representation spheres were investigated recently in [7]. We deduce some Borsuk-Ulam type results for equivariant maps from spheres to Dold and Milnor manifolds, thus providing some applications to our results.

6.1 Index and co-index of involutions

Let X be a compact Hausdorff space with a free involution and $\mathbb{S}^n$ the unit n-sphere equipped with the antipodal involution. In an attempt to generalize the Borsuk-Ulam theorem, Conner and Floyd [16] suggested to look for the largest integer n so that there exists a $\mathbb{Z}_2$-equivariant map from $\mathbb{S}^n$ to X. They defined the index of the involution on X as

$$\text{ind}(X) = \max\ \{\ n \mid \text{there exists a } \mathbb{Z}_2\text{-equivariant map } \mathbb{S}^n \to X\}.$$

In view of the classical Borsuk-Ulam theorem, $\text{ind}(\mathbb{S}^n) = n$. The characteristic classes with $\mathbb{Z}_2$-coefficients can be used to derive a cohomological criteria to compute the index. Let $w \in H^1(X/G; \mathbb{Z}_2)$ be the Stiefel-Whitney class of the principal G-bundle $X \to X/G$. Conner and Floyd defined

$$\text{co-ind}_{\mathbb{Z}_2}(X) = \max\ \{\ n \mid w^n \neq 0\}.$$

Notice that $\text{co-ind}_{\mathbb{Z}_2}(\mathbb{S}^n) = n$. By [16, Equation (4.5)], we obtain

$$\text{ind}(X) \leq \text{co-ind}_{\mathbb{Z}_2}(X).$$

6.2 Equivariant maps from spheres to Dold and Milnor manifolds

The indices defined in the preceding section can be used to obtain the following results.

Proposition 6.2.1. *If n is even, then there is no $\mathbb{Z}_2$-equivariant map $S^k \to P(m,n)$ for $k \geq 2$.*

Proof. Let $X = P(m,n)$ and take a classifying map

$$f : X/\mathbb{Z}_2 \to B_{\mathbb{Z}_2}$$

for the principal $\mathbb{Z}_2$-bundle $X \to X/\mathbb{Z}_2$. Consider the Borel fibration $X \hookrightarrow X_{\mathbb{Z}_2} \xrightarrow{\pi} B_{\mathbb{Z}_2}$. Let $\eta : X/\mathbb{Z}_2 \to X_{\mathbb{Z}_2}$ be a homotopy equivalence. Then $\pi \circ \eta : X/\mathbb{Z}_2 \to B_{\mathbb{Z}_2}$ also classifies the principal $\mathbb{Z}_2$-bundle $X \to X/\mathbb{Z}_2$, and hence it is homotopic to f. Therefore, it suffices to consider the map

$$\pi^* : H^1(B_{\mathbb{Z}_2}; \mathbb{Z}_2) \to H^1(X_{\mathbb{Z}_2}; \mathbb{Z}_2).$$

The image of the Stiefel-Whitney class of the universal principal $\mathbb{Z}_2$-bundle $\mathbb{Z}_2 \hookrightarrow E_{\mathbb{Z}_2} \longrightarrow B_{\mathbb{Z}_2}$ is the Stiefel-Whitney class of $X \to X/\mathbb{Z}_2$. It follows from the proof of Theorem 4.1.1 that $x \in H^1(X/\mathbb{Z}_2; \mathbb{Z}_2)$ is the Stiefel-Whitney class with $x \neq 0$ and $x^2 = 0$. This gives co-ind$_{\mathbb{Z}_2}(X) = 1$ and ind$(X) \leq 1$. Hence, there is no $\mathbb{Z}_2$-equivariant map $S^k \to P(m,n)$ for $k \geq 2$. □

For Milnor manifolds, we have the following results.

Proposition 6.2.2. *Let $X \simeq_2 \mathbb{R}H_{r,s}$ be a compact Hausdorff space, where $1 \leq s < r$. Then there is no $\mathbb{Z}_2$-equivariant map $\mathbb{S}^k \to X$ for $k \geq 2$.*

Proof. For $X \simeq_2 \mathbb{R}H_{r,s}$, using the proof of Theorem 5.3.1, we see that $x \in H^1(X/G; \mathbb{Z}_2)$ is the Stiefel-Whitney class with $x \neq 0$ and $x^2 = 0$. This gives co-ind$_{\mathbb{Z}_2}(X) = 1$ and ind$(X) \leq 1$. Hence, there is no $\mathbb{Z}_2$-equivariant map $\mathbb{S}^k \to X$ for $k \geq 2$. □

Proposition 6.2.3. *Let $X \simeq_2 \mathbb{C}H_{r,s}$ be a compact Hausdorff space, where $1 \leq s < r$. Then there is no $\mathbb{Z}_2$-equivariant map $\mathbb{S}^k \to X$ for $k \geq 3$.*

Proof. From the proof of Theorem 5.3.2, we get $x \in H^1(X/G; \mathbb{Z}_2)$ is the Stiefel-

Whitney class with $x^2 \neq 0$ and $x^3 = 0$. This gives $\text{co-ind}_{\mathbb{Z}_2}(X) = 2$ and $\text{ind}(X) \leq 2$. Hence, there is no $\mathbb{Z}_2$-equivariant map $\mathbb{S}^k \to X$ for $k \geq 3$. □

Given a G-space X, Volovikov [80] defined a numerical index $i(X)$ as the smallest r such that for some k, the differential

$$d_r : E_r^{k-r,r-1} \to E_r^{k,0}$$

in the Leray-Serre spectral sequence of the fibration $X \overset{i}{\hookrightarrow} X_G \overset{\pi}{\longrightarrow} B_G$ is non-trivial. It is clear that $i(X) = r$ if $E_2^{k,0} = E_3^{k,0} = \cdots = E_r^{k,0}$ for all k and $E_r^{k,0} \neq E_{r+1}^{k,0}$ for some k. If $E_2^{*,0} = E_\infty^{*,0}$, then $i(X) = \infty$. Thus, $i(X)$ is either an integer greater than 1 or ∞. Using this index, Coelho, Mattos and Santos proved the following [15, Theorem 1.1] result.

Proposition 6.2.4. *Let G be a compact Lie group and X, Y be path-connected compact Hausdorff spaces with free G-actions. Suppose that $i(X) \geq m + 1$ for some natural $m \geq 1$. If $H^{k+1}(Y/G; \mathbb{Z}_2) = 0$ for some $1 \leq k < m$ and* $\text{rk}\left(H^{k+1}(B_G); \mathbb{Z}_2\right) > 0$*, then there is no G-equivariant map $f : X \to Y$.*

The preceding result yields the following

Proposition 6.2.5. *Suppose $\mathbb{Z}_2$ acts freely on $X \simeq_2 \mathbb{C}H_{r,s}$ and a path-connected compact Hausdorff space Y such that $H^2(Y/G; \mathbb{Z}_2) = 0$. Then there is no $\mathbb{Z}_2$-equivariant map $X \to Y$.*

Proof. We obtained $i(X) = 3$ in the proof of Theorem 5.3.2. The result is a consequence of Proposition 6.2.4. □

6.3 Schwarz genus and coincidence-point set

Let G be a finite group considered as a 0-dimensional simplicial complex and X a paracompact space with a free G-action. The **Schwarz genus** $\mathrm{g}_{\text{free}}(X, G)$ of the free G-space X is the smallest number n such that there exists a G-equivariant map

$$X \to \underbrace{G * \cdots * G}_{n\text{-fold join of } G},$$

where the right-hand side is equipped with the diagonal G-action. Note that for $G = \mathbb{Z}_2$, the Schwarz genus is the least integer n for which there exists a $\mathbb{Z}_2$-equivariant map $f : X \to S^{n-1}$. See [81, Chapter V] for the original source and [5,80] for more details and applications of Schwarz genus. In the literature, the free genus for $G = \mathbb{Z}_2$ is known under different names, for example, co-index [16], level [64] and B-index [84].

Proposition 6.3.1. *Let $X \simeq_2 \mathbb{C}H_{r,s}$ be a compact Hausdorff space with a free $\mathbb{Z}_2$-action. Then $g_{\text{free}}(X, \mathbb{Z}_2) \geq 3$. In particular, there does not exist any $\mathbb{Z}_2$-equivariant map $X \to \mathbb{S}^1$.*

Proof. It follows from [16] that

$$g_{\text{free}}(X, \mathbb{Z}_2) \geq \text{co-ind}_{\mathbb{Z}_2}(X) + 1,$$

and hence, $g_{\text{free}}(X, \mathbb{Z}_2) \geq 3$. □

Let G be a finite group and X a G-space. Given a continuous map $f : X \to Y$, the **coincidence-point set** $A(f,\, k)$ is defined as

$$A(f, k) = \{x \in X \mid \exists \text{ distinct } g_1, \ldots, g_k \in G \text{ such that } f(g_1 x) = \cdots = f(g_k x)\}.$$

The following result of Schwarz [81, pp. 122-123] relates the free genus and the coincidence-point set.

Theorem 6.3.2. *Let X be a paracompact connected space with a free $\mathbb{Z}_p$-action. Suppose $g_{\text{free}}(X, \mathbb{Z}_p) > m(p-1)$. Then for any continuous map $f : X \to \mathbb{R}^m$*

$$g_{\text{free}}(A(f,p), \mathbb{Z}_p) \geq g_{\text{free}}(X, \mathbb{Z}_p) - m(p-1).$$

In particular, the set $A(f,p)$ is non-empty.

We conclude with a consequence of Proposition 6.3.1 and Theorem 6.3.2.

Proposition 6.3.3. *Let* $X \simeq_2 \mathbb{C}H_{r,s}$ *be a compact Hausdorff space with a free* $\mathbb{Z}_2$*-action. Then any continuous map* $X \to \mathbb{R}^2$ *has a non-empty coincidence set.*

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