Hyperbolicity, Complexes of Groups and Cannon-Thurston Maps

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Declaration

The work presented in this thesis has been carried out by me under the guidance of Dr. Krishnendu Gongopadhyay at Indian Institute of Science Education and Research, Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Krishnendu Gongopadhyay
(Supervisor)

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Contents

Abstract

Complexes of groups describe the action of groups on simply connected polyhedral complexes. These are a natural generalisation of the concept of graphs of groups introduced by Bass and Serre. In this thesis, we address some questions associated to the complexes of groups. We first show that the palindromic width of HNN extension of a group by associated proper subgroups and the palindromic width of the amalgamated free product of two groups via a proper subgroup is infinite (except when the amalgamated subgroup has index two in each of the factors). As a corollary of these, the palindromic width of the fundamental group of a graph of groups is mostly infinite.

Next, we prove a limit set intersection theorem for a relatively hyperbolic group G that admits a decomposition into a finite graph of relatively hyperbolic groups structure with quasi-isometrically (qi) embedded condition. We prove that the set of conjugates of all the vertex groups and edge groups satisfy the limit set intersection property for conical limit points. This result is motivated by the work of Sardar for graph of hyperbolic groups [49].

Finally, we study the existence of Cannon-Thurston maps for certain subfamily of complex of hyperbolic groups. Let G be the fundamental group of a complex of hyperbolic groups $G(\mathcal{Y})$ with respect to a maximal tree $\mathcal{T}$ of $\mathcal{Y}$. Suppose $G(\mathcal{Y})$ is developable and the monomorphisms $G_e \to G_{o(e)}$ and $G_e \to G_{t(e)}$ have finite index images in the target groups. Let $\mathcal{Z}$ be a connected subcomplex of $\mathcal{Y}$ and H be its fundamental group with respect to

a maximal subtree $\mathcal{T}_1 \subset \mathcal{T}$ of $\mathcal{Z}$. If the natural homomorphism $i : H \to G$ is injective and the natural map from the development of $G(\mathcal{Z})$ to that of $G(\mathcal{Y})$ is a qi-embedding, then H is also hyperbolic and i admits a Cannon-Thurston map $\partial i : \partial H \to \partial G$.

Notations

$\mathbb{H}^n$: n-dimensional hyperbolic space

$\mathbb{E}^n$: n-dimensional Euclidean space

$\mathbb{S}^n$: n-dimensional sphere

$\mathrm{Isom}(\mathbb{H}^n)$: Isometry group of $\mathbb{H}^n$

$\mathbb{R}$: set of real numbers

$\mathbb{N}$: set of natural numbers

$\mathbb{R}^+$: set of positive real numbers

For a metric space X, the metric will be denoted by d_X.

A geodesic joining $x, y \in X$ will be denoted by $[x, y]$.

For $Y \subset X$ geodesic in Y joining $x, y \in Y$ will be denoted by $[x, y]_Y$.

For $A, B \subset X$, Hausdorff distance between A, B will denoted by $\mathrm{Hd}(A, B)$.

For $A \subset X$ and $R \geq 0$, $N_R(A)$ will denote the set $\{x \in X \mid d(x, A) \leq R\}$.

Chapter 1

Introduction

Groups can be studied by their action on geometric objects. Bass-Serre theory provides one such tool to study the structure of infinite groups from their action on simplicial trees. Associated to such an action, one has the notion of the graph of groups. A graph of groups consists of a finite graph and groups associated to every vertex and edge, along with monomorphisms from each edge group to the groups associated to its initial and terminal vertices. A natural generalisation of the graph of groups is the complex of groups. In this thesis, we aim to understand certain aspects of complexes of groups.

There are three problems addressed in this thesis. In Chapter 3, we study graph of groups. We look at the notion of the palindromic width of graph of groups and prove the infiniteness of such widths in most of the cases.

After obtaining some understanding of the palindromic widths, we prove a limit set intersection property for the vertex groups of the graph of relatively hyperbolic groups in Chapter 4.

In the final chapter of the thesis, i.e.Chapter 5, we look at certain subfamily of complex of hyperbolic groups and study the existence of Cannon-Thurston (CT) map for such subcomplexes.

We elaborate on each of these topics in the following three sections.

1.1 Palindromic width of graph of groups

Bardakov, Shpilrain and Tolstykh [12] initiated the investigation of palindromic width and proved that the palindromic width of a non-abelian free group is infinite. Bardakov and Gongopadhyay proved finiteness of palindromic width of finitely generated free nilpotent groups and certain solvable groups in [9–11]. In [8], finiteness of palindromic width of nilpotent products was proved. The palindromic width of Grigorchuk groups and wreath products was investigated by Fink [25,26]. Riley and Sale investigated palindromic width in certain wreath products and solvable groups in [47] using finitely supported functions from $\mathbb{Z}^r$ to the given group. Fink and Thom [27] studied palindromic width in non-abelian finite simple groups and yielded the first examples of groups with finite palindromic width but infinite commutator width.

This was generalised to the free product of groups by Bardakov and Tolstykh in [13]. They proved that a free product of two groups, except $\mathbb{Z}_2 * \mathbb{Z}_2$, has infinite palindromic width. We investigate the palindromic width for HNN extensions and amalgamated free products of groups. For HNN extensions we have the following.

Theorem 1.1.1. [29, Theorem 1.1] *Let G be a group. Let A and B be proper isomorphic subgroups of G and $\phi : A \to B$ be the isomorphism. The* HNN *extension*

$$G_* = \langle G, t \mid t^{-1}at = \phi(a),\ a \in A\rangle$$

of G with associated subgroups A and B has infinite palindromic width with respect to the generating set $G \cup \{t, t^{-1}\}$.

For amalgamated free product of groups, we prove the following theorem.

Theorem 1.1.2. [29, Theorem 1.2] *Let $G = A *_C B$ be the free product of two groups A and B with amalgamated proper subgroup C and $|A : C| \geq 3$, $|B : C| \geq 2$. Then $pw(G, A \cup B)$ is infinite.*

As an application of the above two theorems, we determine the palindromic width for the fundamental group of a graph of groups.

Corollary 1.1.3. [29, Corollary 1.3] *Let Y be a non-empty, connected graph. Let $(\mathcal{G}, Y)$ be a graph of groups and $\pi_1(\mathcal{G}, Y)$ be its fundamental group. Then the palindromic width of $\pi_1(\mathcal{G}, Y)$ is infinite if one of the following holds:*

1. *Y is a loop with a vertex P and edge e, and the image of G_e is a proper subgroup of G_P.*

2. *Y is a tree and has an oriented edge $e = [P_1, P_2]$ such that removing e, while retaining P_1 and P_2, gives two disjoint graphs Y_1 and Y_2 with $P_i \in V(Y_i)$ satisfying the following: extending $G_e \to G_{P_i}$ to $\phi_i : G_e \to \pi_1(\mathcal{G}, Y_i)$, $i = 1, 2$, we get $[\pi_1(\mathcal{G}, Y_1) : \phi_1(G_e)] \geq 3$ and $[\pi_1(\mathcal{G}, Y_2) : \phi_2(G_e)] \geq 2$.*

3. *Y has an oriented edge $e = [P_1, P_2]$ such that removing the edge, while retaining P_1 and P_2 does not separate Y and gives a new graph Y' satisfying the following: extending $G_e \to G_{P_i}$ to $\phi_i : G_e \to \pi_1(\mathcal{G}, Y')$, $i = 1, 2$, we have $H_i := \phi_i(G_e)$ and H_1, H_2 are proper subgroups of $\pi_1(\mathcal{G}, Y')$.*

1.2 A limit set intersection theorem for graph of relatively hyperbolic groups

Limit set intersection theorem first appeared in the work of Susskind [52], in the context of geometrically finite subgroups of Kleinian groups. Later, Susskind and Swarup [51] proved it for geometrically finite purely hyperbolic subgroups of a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$. The works of Susskind and Swarup were followed by the work of J.W. Anderson in [2–4] for some classes of subgroups of Kleinian groups. Susskind asked the following question:

Question: Let G be a non-elementary Kleinian group acting on $\mathbb{H}^n$ for some $n \geq 2$, and let H, K be non-elementary subgroups of G, then is $\Lambda_c(H) \cap \Lambda_c(K) \subset \Lambda(H \cap K)$ true?

In an attempt to answer this, in [5], Anderson showed that if G is a non-elementary purely loxodromic Kleinian group acting on $\mathbb{H}^n$ for some $n \geq 2$ and H and K are non-elementary subgroups of G, then $\Lambda_c(H) \cap \Lambda_c^u(K) \subset \Lambda_c(H \cap K)$, where $\Lambda_c^u(K)$ denotes the uniform conical limit set of K. But in [20], Das and Simmons constructed a non-elementary Fuchsian group G that admits two non-elementary subgroups $H, K \leq G$ such that $H \cap K = \{e\}$ but $\Lambda_c(H) \cap \Lambda_c(K) \neq \emptyset$, thus providing a negative answer to Susskind's question.

However, this prompts the following question in the context of hyperbolic and relatively hyperbolic groups:

Question: Suppose G is a hyperbolic (resp. relatively hyperbolic) group and H, K are subgroups of G, then is $\Lambda_c(H) \cap \Lambda_c(K) \subset \Lambda(H \cap K)$ true?

In 2012, Yang [53] proved a limit set intersection theorem for relatively quasiconvex subgroups of relatively hyperbolic groups. Limit set intersection theorem is not true for general subgroups of hyperbolic groups, and it was known to hold only for quasiconvex subgroups until the recent work of Sardar [48, 49]. In the paper, he proves that the set of conjugates of vertex and edge groups of G satisfy a limit set intersection property for conical limit points. We generalise this to graph of relatively hyperbolic groups in the following:

Theorem 1.2.1. [34, Theorem 1.1] *Let G be a group admitting a decomposition into a finite graph of relatively hyperbolic groups $(\mathcal{G}, Y)$ satisfying* *the qi-embedded condition. Further, suppose the monomorphisms from edge groups to vertex groups are strictly type-preserving, and that induced tree of coned-off spaces also satisfy the qi-embedded condition. If G is hyperbolic relative to the family of maximal parabolic subgroups $\mathcal{C}$, then the set of conjugates of vertex and edge groups of G satisfy a limit set intersection property for conical limit points.*

Our proof relies on the existence of Cannon-Thurston map for the vertex group of a graph of relatively hyperbolic groups, proved by Mj and Pal in [43].

1.3 Pullbacks of metric bundles and Cannon-Thurston maps

In a seminal work [19], Cannon and Thurston proved the following:

Theorem 1.3.1. *Let M be a closed hyperbolic 3-manifold fibering over the circle with fiber F. Let $\widetilde{F}$ and $\widetilde{M}$ denote the universal covers of F and M respectively. After identifying $\widetilde{F}$ (resp. $\widetilde{M}$) with $\mathbb{H}^2$ (resp. $\mathbb{H}^3$), we obtain the compactification $D^2 = \mathbb{H}^2 \cup S^1$ (resp. $D^3 = \mathbb{H}^3 \cup S^2$) by attaching S^1 (resp. S^2) at infinity. Let $i : F \to M$ denote the inclusion map of the fiber and $\tilde{i} : \widetilde{F} \to \widetilde{M}$ denote the lift to the universal cover. Then $\tilde{i}$ extends to a continuous map $\bar{i} : D^2 \to D^3$.*

Further, they raised the following question:

Question: Suppose that a closed surface group $\pi_1(S)$ acts freely and properly discontinuously on $\mathbb{H}^3$ by isometries such that the quotient manifold has no accidental parabolics. Then does the inclusion $i : \widetilde{S} \to \mathbb{H}^3$ extend continuously to the boundary?

If the continuous extension does exist, then it is called a **Cannon-Thurston** (CT) map. Mj [42] gave a positive answer to this question. In the endeavour to answer this, he generalised the question to the context of geometric group theory. Let X be a hyperbolic metric space and G be a hyperbolic group acting on X properly discontinuously and freely by isometries. Let Γ denote the Cayley graph of G and $i : \Gamma \to X$ be the natural orbit map. Let ∂X (resp. $\partial\Gamma$) denote the Gromov boundary and $\overline{X}$ (resp. $\overline{\Gamma}$) denote the Gromov compactification of X (resp. Γ). Then he asked the following question:

Question: Does $i : \Gamma \to X$ extend continuously to $\bar{i} : \overline{\Gamma} \to \overline{X}$?

Baker and Riley [6] provided a counterexample for the existence of Cannon-Thurston maps, thus answering Question 1.3 in negative. In fact, in [38] Matsuda and Oguni showed that for any non-elementary hyperbolic group H, there exists a hyperbolic group G with $H \subset G$ such that no Cannon-Thurston map exists for the inclusion $H \hookrightarrow G$. Over the time, existence of Cannon-Thurston map has been proved for many special cases. Motivated by the works of Mj in [40, 41], Mj and Sardar in [45], and Kapovich and Sardar in [33], we address the following question:

Question: Suppose we have an exact sequence of hyperbolic groups: $1 \to N \to G \to Q \to 1$. Let Q_1 be a subgroup of Q and G_1 denote the inverse image of Q_1 in G. This is a subgroup of G. If Q_1 is hyperbolic, then does the pair (G, G_1) admit a CT map?

We prove a much more general result.

Theorem 1.3.2. *Suppose $G(\mathcal{Y})$ is a complex of groups where $\mathcal{Y}$ is a finite complex. Let T be a maximal tree in the 1-skeleton of the first barycentric subdivision of $\mathcal{Y}$. Suppose $G = \pi_1(G(\mathcal{Y}), T)$ is hyperbolic. Furthermore, suppose the following conditions hold:*

1. *$G(\mathcal{Y})$ is developable with development B.*

2. *All groups G_σ, for $\sigma \in V(\mathcal{Y})$, and G_e, for $e \in E(\mathcal{Y})$, are hyperbolic and the injective homomorphisms $G_e \to G_{i(e)}$ and $G_e \to G_{t(e)}$ have finite*

index images in the target groups.

Let $\mathcal{Z}$ be any connected subcomplex of $\mathcal{Y}$ with maximal tree $T_1 \subset T$ in the 1-skeleton of the first barycentric subdivision of $\mathcal{Z}$, $H = \pi_1(G(\mathcal{Z}), T_1)$ and A be a development of $G(\mathcal{Z})$. If the natural homomorphism $i : H \to G$ is injective and the natural map $A \to B$ is a qi-embedding, then H is also hyperbolic and i admits a Cannon-Thurston map $\partial i : \partial H \to \partial G$.

The following is the main theorem in chapter 5:

Theorem 1.3.3. [35] *Suppose X is a metric bundle over a hyperbolic metric space B such that X is hyperbolic and all the fibers are uniformly hyperbolic and non-elementary. Suppose $i : A \to B$ is a qi embedding and (Y, A, π_Y) is the pullback of X under i. Then $i^* : Y \to X$ admits the CT map.*

We prove this using the structure of geodesics described by Mj and Sardar in [45]. As an immediate application of Theorem 1.3.3, we have the corresponding theorem for groups.

Theorem 1.3.4. *Let $1 \to N \to G \xrightarrow{\pi} Q \to 1$ be a short exact sequence of hyperbolic groups. Suppose $Q_1 < Q$ is qi-embedded and $G_1 := \pi^{-1}(Q_1)$. Then the inclusion $G_1 < G$ admits a CT map.*

The thesis is organised as follows. In Chapter 2, we recall some necessary background to be used in the subsequent chapters. In Chapter 3, we compute the palindromic width of amalgamated free products and HNN extensions and, consequently, get the palindromic width of graph of groups. In Chapter 4, we prove a limit set intersection theorem for graph of relatively hyperbolic groups. In Chapter 5, we recall the definition metric bundles and we construct a pullback for a metric bundle and finally, show the existence of Cannon-Thurston map for the pullback of a hyperbolic metric bundle.

Chapter 2

Preliminaries

2.1 Complex of groups

Complexes of groups were introduced with the intent to understand the action of a group on a polyhedral complex. They are a natural generalisation of graphs of groups. In this section, we collect some basic definitions and few results pertaining to complex of groups from [18, Chapter III.$\mathcal{C}$] and graph of groups from [50].

Let $M^n_{\mathcal{K}}$ be the model space of dimension n and curvature $\mathcal{K}$, i.e, $M^n_0 = \mathbb{E}^n$, $M^n_{\mathcal{K}} = \mathbb{S}^n$ for $\mathcal{K} > 0$ and $M^n_{\mathcal{K}} = \mathbb{H}^n$ for $\mathcal{K} < 0$.

Definition 2.1.1. ***Polyhedral complex:*** *A finite-dimensional CW complex K is a $M_{\mathcal{K}}$-*polyhedral complex *if*

(1) every open cell of dimension n is isometric to the interior of a compact convex polyhedron in $M^n_{\mathcal{K}}$.

(2) For a cell σ of K, restricting the attaching map to each open face of σ codimension 1 gives an isometry onto an open cell of K.

Definition 2.1.2. ***Small category without loops:*** *A* small category without loops (scwol) $\mathcal{X}$ *is a pair of disjoints sets $V(\mathcal{X}) \sqcup E(\mathcal{X})$, where $V(\mathcal{X})$ is the vertex set and $E(\mathcal{X})$ is an edge set, along with the maps:*

$o : E(\mathcal{X}) \to V(\mathcal{X})$; $t : E(\mathcal{X}) \to V(\mathcal{X})$.

Let $E^2(\mathcal{X}) = \{(a, b) \in E(\mathcal{X}) \times E(\mathcal{X}) \mid o(a) = t(b)\}$. Then we have a map $E^2(\mathcal{X}) \to E(\mathcal{X})$ given by $(a, b) \longmapsto ab$ such that

(1) for $(a, b) \in E^2(\mathcal{X})$, $o(ab) = o(b), t(ab) = t(a)$.

(2) for $a, b, c \in E(\mathcal{X})$ with $t(c) = o(b), t(b) = o(a)$, $a(bc) = (ab)c$.

(3) if $a \in E(\mathcal{X})$, then $o(a) \neq t(a)$.

For any $a \in E(\mathcal{X})$, $o(a)$ and $t(a)$ are called the initial vertex and the terminal vertex of a respectively.

Given a polyhedral complex K, one can associate a scwol $\mathcal{X}$ to it. The vertex set $V(\mathcal{X})$ is the set of barycenters of the cells of K and the edge set $E(\mathcal{X})$ is the set of 1-simplices of the barycentric subdivision of K, i.e, each $a \in E(\mathcal{X})$ corresponds to a pair of cells T, S with say, $T \subsetneq S$, where S is the initial vertex and T is the final vertex.

Conversely, the geometric realisation of a scwol $\mathcal{X}$ is a polyhedral complex. For $j > 1$, let $E^j(\mathcal{X})$ be the set of sequences $(a_1, \ldots, a_j)$ such that $(a_i, a_{i+1}) \in E^2(\mathcal{X})$ for $1 \leq i \leq j-1$. Let $E^1(\mathcal{X}) = E(\mathcal{X})$ and $E^0(\mathcal{X}) = V(\mathcal{X})$. Then, a cell of dimension k is the standard k-simplex indexed by an element of $E^k(\mathcal{X})$.

Definition 2.1.3. ***Morphism of scwols:*** *Let $\mathcal{X}, \mathcal{X}'$ be two scwols. A map $\phi : \mathcal{X} \to \mathcal{X}'$ sending $V(\mathcal{X})$ to $V(\mathcal{X}')$ and $E(\mathcal{X})$ to $E(\mathcal{X}')$ is a* morphism *if*

(1) for $a \in E(\mathcal{X}), o(\phi(a)) = \phi(o(a))$ and $t(\phi(a)) = \phi(t(a))$.

(2) For $(a, b) \in E^2(\mathcal{X})$, $\phi(ab) = \phi(a)\phi(b)$.

A morphism is **nondegenerate** if for every $\sigma \in V(\mathcal{X})$, ϕ restricted to the set of edges a with $\sigma = o(a)$ is a bijection onto the set of edges b with $o(b) = \phi(\sigma)$. An *isomorphism* is a morphism with an inverse.

Definition 2.1.4. ***Group action on scwols:*** *Let G be a group. An action of G on a scwol $\mathcal{X}$ is a homomorphism $G \to \mathrm{Aut}(\mathcal{X})$ such that for any $a \in E(\mathcal{X})$ and $g \in G$, we have $g \cdot o(a) \neq t(a)$, and if $g \cdot o(a) = o(a)$ then $g \cdot a = a$.*

Action of G on $\mathcal{X}$ induces a quotient scwol $G\backslash\mathcal{X} =: \mathcal{Y}$ with $V(\mathcal{Y}) = G\backslash V(\mathcal{X})$ and $E(\mathcal{Y}) = G\backslash E(\mathcal{X})$. For any $a \in E(\mathcal{X})$, $o(Ga) = Go(a)$, $t(Ga) = Gt(a)$ and $G(ab) = G(a)G(b)$ for any $(a,b) \in E^2(\mathcal{X})$ and the natural projection map $\pi : \mathcal{X} \to \mathcal{Y}$ is a nondegenerate morphism of scwols.

Definition 2.1.5. ***Complexes of groups:*** *A* complex of groups $G(\mathcal{Y}) = (G_\sigma, \psi_a, g_{a,b})$ *consists of a scwol* $\mathcal{Y}$ *and for each* $\sigma \in V(\mathcal{Y})$ *there is a group* G_σ*, called the* local group *at* σ*, along with monomorphisms:* $\psi_a : G_{o(a)} \to G_{t(a)}$ *and for each* $(a,b) \in E^2(\mathcal{Y})$*, there exists* $g_{a,b} \in G_{t(a)}$ *satisfying the following:*

(1) $\mathrm{Ad}(g_{a,b})\psi_{ab} = \psi_a\psi_b$*, where* $\mathrm{Ad}(g_{a,b})$ *is the conjugation by* $g_{a,b}$*, and*

(2) $\psi_a(g_{b,c})g_{a,bc} = g_{a,b}g_{ab,c}$ *for* $(a,b,c) \in E^3(\mathcal{Y})$*.*

Definition 2.1.6. ***Morphism of complexes of groups:*** *Let* $G(\mathcal{Y}), G(\mathcal{Y}')$ *be complexes of groups over* $\mathcal{Y}, \mathcal{Y}'$ *respectively. Let* $\phi : \mathcal{Y} \to \mathcal{Y}'$ *be a morphism of scwols. Then, a morphism* $\Phi = (\Phi_\sigma, \Phi(a)) : G(\mathcal{Y}) \to G(\mathcal{Y}')$ *of complexes of groups over* ϕ *consist of*

(1) a group homomorphism $\Phi_\sigma : G_\sigma \to G_{\phi(\sigma)}$ *for each* $\sigma \in V(\mathcal{Y})$*,*

(2) an element $\Phi(a) \in G'_{t(\phi(a))}$ *for each* $a \in E(\mathcal{Y})$ *such that*

$\mathrm{Ad}(\Phi(a))\psi_{\phi(a)}\Phi_{i(a)} = \Phi_{t(a)}\psi_a$ *and* $\Phi_{t(a)}(g_{a,b})\Phi(ab) = \Phi(a)\psi_{\phi(a)}(\Phi(b))g_{\phi(a),\phi(b)}$ *for every* $(a,b) \in E^2(\mathcal{Y})$*.*

The group homomorphisms Φ_σ for $\sigma \in V(\mathcal{Y})$ is called a **local map** at σ. The morphism Φ is an *isomorphism* if ϕ is an isomorphism and Φ_σ is a group isomorphism for each $\sigma \in V(\mathcal{Y})$. The morphism Φ is *injective on local groups* if each Φ_σ is injective.

Now, let G act on a scwol $\mathcal{X}$ as in Definition 2.1.4. We associate a complex of groups to this group action in the following way:

Let $\mathcal{Y} = G \setminus \mathcal{X}$ and $\pi : \mathcal{X} \to \mathcal{Y}$ be the natural projection map. For each $\sigma \in V(\mathcal{Y})$, $G_\sigma := Stab_G(\tilde{\sigma})$, where $\tilde{\sigma}$ is a lift of σ in $\mathcal{X}$ under π. Now for each $\sigma \in V(\mathcal{Y})$ and some $a \in E(\mathcal{Y})$ with $o(a) = \sigma$ and a fixed lift $\tilde{\sigma}$ of σ, there exists a unique $\tilde{a} \in E(\mathcal{X})$ such that $\pi(\tilde{a}) = a$ and $o(\tilde{a}) = \tilde{\sigma}$. Let $h_a \in G$ such that $h_a \cdot t(\tilde{a}) = \widetilde{t(a)}$. Then, for $a \in E(\mathcal{Y})$, we define $\psi_a : G_{o(a)} \to G_{t(a)}$ to be the conjugation by h_a, i.e., $\psi_a(g) = h_a g h_a^{-1}$. Finally, for any $(a, b) \in E^2(\mathcal{Y})$, $g_{a,b} := h_a h_b h_{ab}^{-1}$.

For a complex of groups $G(\mathcal{Y})$ associated to a group action of some group G, there is a natural morphism $\Phi = (\Phi_\sigma, \Phi(a))$, where Φ_σ is the inclusion map $G_\sigma \hookrightarrow G$ and $\Phi(a) = h_a$.

Definition 2.1.7. ***Developable complex of groups:*** *A complex of groups $G(\mathcal{Y})$ is* developable *if it is isomorphic to a complex of groups arising from an action of a group G on a scwol $\mathcal{X}$ such that $\mathcal{Y} = G \setminus \mathcal{X}$.*

Proposition 2.1.8. [18, Corollary 2.15] *A complex of groups $G(\mathcal{Y})$ is developable if and only if there exists a morphism Φ from $G(\mathcal{Y})$ to some group G which is injective on local groups.*

Definition 2.1.9. ***Fundamental group of complex of groups:*** *Let $G(\mathcal{Y})$ be a complex of groups over a scwol $\mathcal{Y}$. Let T be a maximal tree in $\mathcal{Y}$. Then its fundamental group $G = \pi_1(G(\mathcal{Y}), T)$ is defined in terms of generators and relators as:*

The generating set is $\left(\bigsqcup_{\sigma \in V(\mathcal{Y})} G_\sigma \right) \bigsqcup E^{\pm}(\mathcal{Y})$.

Relators are the following:

· *the relators of the groups G_σ*

· $(a^+)^{-1} = a^-, (a^-)^{-1} = a^+$

· $a^+ b^+ = g_{a,b}(ab)^+$ *for every $(a, b) \in E^2(\mathcal{Y})$*

· $\psi_a(g) = a^+ g a^-$ *for every $g \in G_{o(a)}$ and*

· $a^+ = 1$ *for every edge $a \in T$.*

Definition 2.1.10. ***Development:*** *Let $G(\mathcal{Y})$ be a complex of groups and G be a group. Let $\Phi : G(\mathcal{Y}) \to G$ be a morphism. Then the* development *of $G(\mathcal{Y})$ with respect to Φ is a scwol $D(\mathcal{Y}, \Phi)$ defined as follows:*

Vertex set: $V(D(\mathcal{Y}, \Phi)) = \{([g], \sigma) \mid \sigma \in V(\mathcal{Y}), [g] \in G/\phi_\sigma(G_\sigma)\}$, *Edge set:* $E(D(\mathcal{Y},\Phi)) = \{([g],a) \mid a \in E(\mathcal{Y}), [g] \in G/\phi_{o(a)}(G_{o(a)})\}$.

The maps $o : E(D(\mathcal{Y},\Phi)) \to V(D(\mathcal{Y},\Phi))$ *and* $t : E(D(\mathcal{Y},\Phi)) \to V(D(\mathcal{Y},\Phi))$ *are given by* $o([g],a) = ([g],o(a))$ *and* $t([g],a) = ([g],t(a))$. *Further, the composition of edges* $([g],a)([h],b) = ([h],ab)$ *is defined for* $(a,b) \in E^2(\mathcal{Y})$, $g,h \in G$ *such that* $g^{-1}h\phi(b)^{-1} \in \phi_{o(a)}(G_{o(a)})$.

There is a natural action of G on $D(\mathcal{Y},\Phi)$ given by $h \cdot ([g],\alpha) = ([hg],\alpha)$ for $g,h \in G, \alpha \in \mathcal{Y}$.

Definition 2.1.11. ***Universal cover of a developable complex of groups:*** *Let* $G(\mathcal{Y})$ *be a developable complex of groups over a connected scwol* $\mathcal{Y}$. *Let* $T \subset \mathcal{Y}$ *be a maximal subtree. Let* $i_T : G(\mathcal{Y}) \to \pi_1(G(\mathcal{Y}),T)$ *be the morphism of complex of groups mapping the local group* G_σ *to its image in* $\pi_1(G(\mathcal{Y}),T)$, *and edge* a *to the image* a^+ *in* $\pi_1(G(\mathcal{Y}),T)$. *The development* $D(\mathcal{Y},T) := D(\mathcal{Y},i_T)$ *is called a* universal cover *of* $G(\mathcal{Y})$.

Theorem 2.1.12. [18, Theorem 3.13] *The universal cover* $D(\mathcal{Y},T)$ *is connected and simply connected.*

Proposition 2.1.13. [36, Proposition 27] *Let* G *be a group acting on a simply connected scwol* $\mathcal{X}$ *and* $G(\mathcal{Y})$ *be the induced complex of groups for* $\mathcal{Y} = G\backslash\mathcal{X}$. *Choose a maximal subtree* T *in* $\mathcal{Y}$ *and* $\sigma_0 \in V(\mathcal{Y})$. *For* $e \in E^{\pm}(\mathcal{Y})$, *let* $h_e = h_a$ *for* $e = a^+$ *and* $h_e = h_a^{-1}$ *for* $e = a^-$. *For* $\sigma \in V(\mathcal{Y})$, *let* $c_\sigma = (e_1, e_2, \ldots, e_n)$ *be the unique edge path contained in* T, *without backtracking, joining* σ_0 *to* σ, *and let* $h_\sigma = h_{e_1}h_{e_2}\ldots h_{e_n}$. *Then there is a group isomorphism* $\Lambda_T : \pi_1(G(\mathcal{Y}),T) \to G$ *defined on generators by* $g \mapsto h_\sigma g h_\sigma^{-1}$ *for* $g \in G_\sigma$ *and* $a^+ \mapsto h_{t(a)}h_\sigma h_{o(a)}^{-1}$ *and a* Λ_T*-equivariant isomorphism of scwols* $\tilde{L}_T : D(\mathcal{Y},T) \to \mathcal{X}$ *mapping* $([g],\alpha)$ *to* $\Lambda_T(g)h_{o(a)} \cdot \tilde{\alpha}$.

2.1.1 Graph of groups

In this section, we follow [50].

Definition 2.1.14. ***Graph:*** *A* graph Y *is an ordered pair of sets* (V, E) *with* $V = V(Y)$*, the set of vertices of* Y *and a set* $E = E(Y)$*, the set of edges of* Y*, and a pair of maps*

$$E \to V \times V \qquad e \mapsto (o(e), t(e)); \quad \text{and} \quad E \to E \qquad e \mapsto \bar{e}$$

satisfying the following conditions: $o(\bar{e}) = t(e)$*,* $t(\bar{e}) = o(e)$ *and* $\bar{\bar{e}} = e$ *for all* $e \in E$*. Here,* $o(e)$ *is the initial vertex of the edge* e *and* $t(e)$ *is the terminal vertex;* $\bar{e}$ *is the inverse of* e*, i.e., the edge* e *with the opposite orientation.*

Definition 2.1.15. ***Graph of groups:*** *A* graph of groups $(\mathcal{G}, Y)$ *consists of a finite graph* Y *with vertex set* V *and edge set* E*, and for each vertex* $v \in V$*, there is a group* G_v *(vertex group) and for each edge* $e \in E$*, there is a group* G_e *(edge group), along with the monomorphisms:*

$$\phi_{o(e)} : G_e \to G_{o(e)}, \qquad \phi_{t(e)} : G_e \to G_{t(e)}$$

with the extra condition that $G_{\bar{e}} = G_e$*.*

Definition 2.1.16. ***Fundamental group of a graph of groups:*** *Let* $(\mathcal{G}, Y)$ *be a graph of groups. Let* T *be a maximal subtree of* Y*. Then the* fundamental group $G = \pi_1(\mathcal{G}, Y, T)$ *of* $(\mathcal{G}, Y)$ *is defined in terms of generators and relators as:*

The generating set is the disjoint union of generating sets of the vertex groups G_v *and the set* $E(Y)$ *of oriented edges of* Y*.*

Relators are the following:

· *relators from the vertex groups* G_v

· $\bar{e} = e^{-1}$

· $e\phi_{t(e)}(g)e^{-1} = \phi_{o(e)}(g)$ *for all edge* e and $g \in G_e$

· $e = 1$ *if* $e \in E(T)$.

The fundamental group $\pi_1(\mathcal{G}, Y, T)$ is independent of the choice of the maximal subtree T, up to isomorphism. So we simply denote the fundamental group by $\pi_1(\mathcal{G}, Y)$.

Definition 2.1.17. [50, Sections 5.3, 5.4] ***Bass-Serre tree:*** *Let* $(\mathcal{G}, Y)$ *be a graph of groups defined above and* G *be its fundamental group. The* Bass-Serre tree *is the tree* $\mathcal{T}$ *with vertex set* $\bigsqcup_{v \in V(Y)} G/G_v$ *and edge set* $\bigsqcup_{e \in E(Y)} G/G_e^e$. *Here,* $G_e^e = \phi_{t(e)}(G_e) \leq G_{t(e)}$.

So, for an edge gG_e^e, $o(gG_e^e) = gG_{o(e)}$ *and* $t(gG_e^e) = geG_{t(e)}$.

The group G acts on $\mathcal{T}$ without inversion of edges such that $G \setminus \mathcal{T} \cong Y$. Conversely, if a group G acts on a tree X without inversion of edges and if $Y = G \setminus X$, then one can associate a graph of groups $(\mathcal{G}, Y)$ to Y. Fixing a lift of a maximal subtree of Y to X, for each $v \in V(Y)$ and $e \in E(Y)$, we define the vertex group G_v and the edge group G_e respectively, to be the stabiliser subgroup of the lift of v and e to X.

2.2 Hyperbolic metric spaces

We begin by recalling some basic notions from large scale geometry and hyperbolic metric spaces. Let X, Y be metric spaces and let $k \geq 1$, $\epsilon \geq 0$.

1. A map $\phi : X \to Y$ is a **proper embedding** if for all $N \geq 0$, there exists $M = M(N) \geq 0$ such that for all $x, y \in X$, $d(\phi(x), \phi(y)) \leq N$ implies $d(x, y) \leq M$.

   Suppose $\{(X_\alpha, d_{X_\alpha})\}$ and $\{(Y_\alpha, d_{Y_\alpha})\}$ are families of metric spaces. For any function $f : \mathbb{R}^+ \to \mathbb{R}^+$, a family of maps $\phi_\alpha : X_\alpha \to Y_\alpha$ is said to be **uniformly metrically proper as measured by** f if for all α and $x, y \in X_\alpha$, $d_{Y_\alpha}(\phi_\alpha(x), \phi_\alpha(y)) \leq N$ implies $d_{X_\alpha}(x, y) \leq f(N)$. If such an f exists we say that the family of maps ϕ_α is uniformly metrically proper, or simply uniformly proper.

2. Suppose A is a set. A map $\phi : A \to Y$ is said to be ϵ-**coarsely surjective** if Y is contained in the ϵ-neighborhood $\phi(A)$.

Suppose $\{A_\alpha\}$ and $\{Y_\alpha\}$ are respectively a family of sets and a family of metric spaces. A family of maps $\phi_\alpha : A_\alpha \to Y_\alpha$ is said to be **uniformly coarsely surjective** if there is a constant $D \geq 0$, such that for all α, Y_α is contained in the D-neighborhood of $\phi_\alpha(A_\alpha)$.

3. A map $\phi : X \to Y$ is said to be L-**coarsely Lipschitz** if for every $x_1, x_2 \in X$, we have $d(\phi(x_1), \phi(x_2)) \leq L \cdot d(x_1, x_2) + L$. A map ϕ is coarsely Lipschitz if it is L- coarsely Lipschitz for some $L \geq 1$.

4. (i) (cf. [28,30]) A map $\phi : X \to Y$ is said to be a (k, ϵ)-**quasiisometric embedding** (qi embedding) if for every $x_1, x_2 \in X$, we have

$$\frac{1}{k} d(x_1, x_2) - \epsilon \leq d(\phi(x_1), \phi(x_2)) \leq k.d(x_1, x_2) + \epsilon.$$

A map $\phi : X \to Y$ will simply be referred to as a quasiisometric embedding if it is a (k, ϵ)-quasiisometric embedding for some $k \geq 1$ and $\epsilon \geq 0$. A (k, k)-quasiisometric embedding will be referred to as a k-**quasiisometric embedding**.
(ii) A map $\phi : X \to Y$ is said to be a (k, ϵ)-**quasiisometry** (resp. k-**quasiisometry**) (qi) if it is a (k, ϵ)-quasiisometric embedding (resp. k-quasiisometric embedding) and if ϕ is $D-$coarsely surjective for some $D \geq 0$.
(iii) A (k, ϵ)-**quasigeodesic** (resp. a k-**quasigeodesic**) in a metric space X is a (k, ϵ)-quasiisometric embedding (resp. a k-quasiisometric embedding) $\gamma : I \to X$, where $I \subseteq \mathbb{R}$ is an interval.
(iv) Let $I \subseteq \mathbb{R}$ be a closed interval with endpoints in $\mathbb{Z} \cup \{\infty, -\infty\}$. Let $J = \mathbb{Z} \cap I$ with the restricted metric from $\mathbb{R}$. Then a (k, ϵ)-qi embedding $\alpha : J \to X$ will be called a **dotted** (k, ϵ)-**quasigeodesic**. If I is a finite interval, say $I = [m, n]$ then $\sum_{i=m}^{n-1} d_X(\alpha(i), \alpha(i+1))$ will be called the *length* of the dotted quasigeodesic α.

5. A map $\psi : Y \to X$ is said to be an ϵ-**coarse inverse** of a map $\phi : X \to Y$ if for all $x \in X$ and $y \in Y$ one has $d_X(\psi \circ \phi(x), x) \leq \epsilon$ and $d_Y(\phi \circ \psi(y), y) \leq \epsilon$.

6. Suppose $A \subset X$. Then the **nearest point projection** of X on A is a map $P_A : X \to A$ such that $d(x, P_A(x)) = inf\{d(x,y) : y \in A\}$ for all $x \in X$.

   Further, given $r \geq 0$, an r-**approximate nearest point projection** of X on A is a map $X \to A$, still denoted by P_A, such that $d(x, P_A(x) \leq r + inf\{d(x,y) : y \in A\}$ for all $x \in X \setminus A$ and $P_A(x) = x$ for all $x \in A$.

Lemma 2.2.1. [45, Lemma 1.1] *For every $K_1, K_2 \geq 1$ and $D \geq 0$ there exists $K_{2.2.1} = K_{2.2.1}(K_1, K_2, D)$ such that the following holds.*

A K_1-coarsely Lipschitz map with a K_2-coarsely Lipschitz, D-coarse inverse is a $K_{2.2.1}$-quasiisometry.

Lemma 2.2.2. *Given $K \geq 1$, $\epsilon \geq 0$ and $R \geq 0$ there are constants $C_{2.2.2} = C_{2.2.2}(K, \epsilon, R)$ and $D_{2.2.2} = D_{2.2.2}(K, \epsilon, R)$ such that the following holds:*

Suppose X, Y are any two metric spaces and $f : X \to Y$ is a (K, ϵ)-quasiisometry which is R-coarsely surjective. Then there is a $(K, C_{2.2.2})$-quasiisometric $D_{2.2.2}$-coarse inverse of f.

The following lemma follows from a simple calculation.

Lemma 2.2.3. *(1) Suppose we have a sequence of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ where f, g are coarsely L_1-Lipschitz and L_2-Lipschitz respectively. Then $g \circ f$ is coarsely $(L_1L_2, L_1L_2 + L_2)$-Lipschitz.*

(2) Suppose $f : X \to Y$ is a (K_1, ϵ_1)-qi embedding and $g : Y \to Z$ is a (K_2, ϵ_2)-qi embedding. Then $g \circ f : X \to Z$ is a $(K_1K_2, \epsilon_1 + \epsilon_2)$-qi embedding.

Moreover, if f is coarsely D_1-surjective and g is coarsely D_2-surjective then $g \circ f$ is coarsely $(K_2D_1 + \epsilon_2 + D_2)$-surjective.

In particular, composition of finitely many quasiisometries is a quasiisometry.

The following lemma appears in [33]. We include a proof for the sake completeness.

Lemma 2.2.4. *Suppose X is any metric space, $x, y \in X$, γ is a (dotted) k-quasigeodesic joining x, y and $\alpha : [0, l] \to X$ is a (dotted) coarsely L-Lipschitz path joining x, y. Suppose moreover, α is a proper embedding as measured by a function $f : [0, \infty) \to [0, \infty)$ and that $Hd(\alpha, [x, y]) \leq D$ for some $D \geq 0$. Then α is $K_{2.2.4} = K_{2.2.4}(k, f, D, L)$-quasigeodesic in X.*

Proof. Suppose γ is defined on an interval J. Let $a_1, a_2 \in I$. Since α is coarsely L-Lipschitz we have (1) $d(\alpha(a_1), \alpha(a_2)) \leq L|a_1 - a_2| + L$. Now let $b_1, b_2 \in J$ be such that for $i = 1, 2$, $d(\alpha(a_i), \gamma(b_i)) \leq D$. Let $R = d(\alpha(a_1), \alpha(a_2))$. Then by triangle inequality $d(\gamma(b_1), \gamma(b_2)) \leq 2D + R$. Since γ is a k-quasigeodesic, we have $-k + \frac{1}{k}|b_1 - b_2| \leq d(\gamma(b_1), \gamma(b_2)) \leq k|b_1 - b_2| + k$. Hence, $|b_1 - b_2| \leq k(2D + R) + k^2$. Without loss of generality, suppose $b_1 \leq b_2$. Consider the sequence of points. Consider the sequence of points $b_1 = s_0, s_1, \ldots, s_n = b_2$ in J such that for $0 \leq i \leq n - 2$, $s_{i+1} - s_i = 1$ and $s_n - s_{n-1} \leq 1$. We note that $n \leq 1 + k(2D + R) + k^2$. Let $t_i \in I$ with $t_0 = a_1, t_n = a_2$ such that $d(\gamma(s_i), \alpha(t_i)) \leq D$. Then again by triangle inequality, since γ is a k-quasigeodesic, we have, $d(\alpha(t_i), \alpha(t_{i+1}) \leq 2D + 2k$, for $0 \leq i \leq n - 1$. As α is properly embedded as measured by f, $|t_i - t_{i+1}| \leq f(2D + 2k)$. Hence,

$$|a_1 - a_2| \leq \sum_{i=0}^{n-1} |t_i - t_{i+1}| \leq nf(2D + 2k) \leq (1 + k(2D + R) + k^2)f(2D + 2k).$$

Thus,

$$(2) \quad -\frac{1 + 2kD + k^2}{k} + \frac{1}{kf(2D + 2k)}|a_1 - a_2| \leq R = d(\alpha(a_1), \alpha(a_2)).$$

Hence by (1) and (2), we take $K_{2.2.4} = 1 + 2D + k + kf(2D + 2k) + L$. $\square$

Lemma 2.2.5. *Suppose X is a length space and Y is any metric space. A map $f : X \to Y$ is coarsely Lipschitz if and only if there is a constant $C > 0$ such that for all $x_1, x_2 \in X$, $d_X(x_1, x_2) \leq 1$ implies $d_Y(f(x_1), f(x_2)) \leq C$.*

Lemma 2.2.6. *Suppose X is a length space. (1) Given $\epsilon > 0$, any pair of points of X can be joined by a $(1, \epsilon)$-quasigeodesic.*
(2) Any pair of points in X can be joined by a dotted 1-quasigeodesic in X.

Proof. (1) Let $x, y \in X$. Given $\epsilon > 0$, there is a rectifiable arc-length parametrised path $\gamma : [0, l] \to X$ such that $\gamma(0) = x, \gamma(l) = y$ and $l(\gamma) = l$ where, $l - \epsilon \leq d(x, y) \leq l$. We claim that γ is a $(1, \epsilon)$-quasigeodesic connecting x, y. Given $s \leq t \in [0, l]$ we have $d(\gamma(s), \gamma(t)) \leq l(\gamma|_{[s,t]}) = t - s = d(s, t)$. We need to show that $d(\gamma(s), \gamma(t)) \geq l(\gamma|_{[s,t]}) - \epsilon$. However, if $d(\gamma(s), \gamma(t)) < l(\gamma|_{[s,t]}) - \epsilon$, then we can replace the portion of γ from $\gamma(s)$ to $\gamma(t)$ by another path, say α, whose length will be smaller than $l(\gamma|_{[s,t]}) - \epsilon$. This implies that the length of the concatenation $\gamma|_{[0,s]} * \alpha * \gamma|_{[t,l]}$ is less than $l(\gamma) - \epsilon$. This is impossible since $d(x, y) \geq l(\gamma) - \epsilon$.

(2) Let $x, y \in X$. By (1), there exists a continuous $(1, 1)$-quasigeodesic $\gamma : [0, l] \to X$ joining x, y. If $l \in \mathbb{N}$, we restrict γ to $[0, l] \cap \mathbb{Z}$ to get the dotted 1-quasigeodesic.

Suppose l is not an integer. Let $n \in \mathbb{Z}$ such that $n < l < n + 1$. We define $\gamma' : [0, n + 1] \cap \mathbb{Z} \to X$ by setting $\gamma'(i) = \gamma(i)$ for $0 \leq i \leq n$ and $\gamma'(n+1) = \gamma(l)$. We claim that γ' is a $(1, 1)$-quasigeodesic. Let $i, j \in [0, n] \cap \mathbb{Z}$. Then $-1+|i-j| \leq d(\gamma'(i), \gamma'(j)) = d(\gamma(i), \gamma(j)) \leq |i-j|+1$. For $i \in [0, n] \cap \mathbb{Z}$, $-1+(l-i) \leq d(\gamma'(i), \gamma'(n+1)) = d(\gamma(i), \gamma(l)) \leq (l-i)+1$. Since $n < l < n+1$, $-2 + (n + 1 - i) \leq d(\gamma'(i), \gamma'(n + 1)) \leq (n + 1 - i) + 1$. The lemma follows from this. $\square$

Lemma 2.2.7. *Given a length space X there is a $(1,1)$-quasiisometry $f : X \to Y$ where Y is a metric graph.*

Proof. Let X be any length space. We define a metric graph Y as follows. We take the vertex set $V(Y) = X$. We join $x, y \in Y$ by an edge if and only if $d_X(x, y) \leq 1$. Let $f : X \to V(Y) \subset Y$ be the identity map. We claim that it is a $(1,1)$-quasiisometry. Let $x, y \in X$. By Lemma 2.2.6, there exists a $(1,\epsilon)$-quasigeodesic γ joining x and y. Without loss of generality, assume $\epsilon \leq 1$. Clearly, $d_Y(x, y) \leq d_X(x, y) + 1$. Suppose $d_Y(x, y) = n$. Let $x = x_0, x_1, \ldots, x_n = y$ be the consecutive vertices on a geodesic in Y joining x, y. Then we know that $d_X(x_i, x_{i+1}) \leq 1$. Thus, $d_X(x, y) \leq \sum_{i=1}^{n} d_X(x_{i-1}, x_i) \leq n$. Therefore, we get $d_X(x, y) \leq d_Y(f(x), f(y)) \leq d_X(x, y) + 1$. Finally, Y lies in a 1-neighbourhood of $f(X)$. Hence, f is a $(1,1)$-quasiisometry. □

Definition 2.2.8. ***Gromov inner product:*** *Let (X, d) be any metric space and $p, x, y \in X$. Then the* Gromov inner product *of x, y with respect to p is defined to be the number $\frac{1}{2}\{d(p, x) + d(p, y) - d(x, y)\}$. It is denoted by $(x.y)_p$.*

Lemma 2.2.9. *Let X be a metric space and suppose $x, y, p, x', y', p' \in X$. The following holds.*

(a) $|(x.y)_p - (x.y')_p| \leq d(y, y')$.

(b) $|(x.y)_p - (x'.y')_p| \leq d(x, x') + d(y, y')$.

(c) $|(x.y)_p - (x.y)_{p'}| \leq d(p, p')$.

(d) $|(x.y)_p - (x'.y')_{p'}| \leq d(x, x') + d(y, y') + d(p, p')$.

(e) If p, x, y are points on a $(1, C)$-quasigeodesic appearing in that order then $(x.y)_p \geq d(p, x) - 5C/2$.

Proof. (a) $|(x.y)_p - (x.y')_p| = \frac{1}{2}|(d(p, y) - d(p, y')) + (d(x, y') - d(x, y))| \leq \frac{1}{2}\{|d(p, y) - d(p, y')| + |d(x, y') - d(x, y)|\} \leq d(y, y')$.

(b) $|(x.y)_p - (x'.y')_p| \leq |(x.y)_p - (x.y')_p| + |(x.y')_p - (x'.y')_p| \leq d(x, x') + d(y, y')$, by using (a).

(c) $|(x.y)_p - (x.y)_{p'}| \leq \frac{1}{2}|(d(x,p) - d(x,p')) + (d(y,p) - d(y,p'))| \leq d(p,p')$.

(d) $|(x.y)_p - (x'.y')_{p'}| \leq |(x.y)_p - (x'.y')_p| + |(x'.y')_p - (x'.y')_{p'}| \leq d(x,x') + d(y,y') + d(p,p')$, using (b) and (c).

(e) Suppose $\alpha : [0,l] \to X$ is a $(1,C)$-quasigeodesic, and $s \leq t \in [0,C]$ such that $\alpha(0) = p, \alpha(s) = x, \alpha(t) = y$. Then $2(x.y)_p = d(x,p) + d(y,p) - d(x,y) \geq s - C + t - C - (t - s + C) = 2s - 3C \geq 2d(p,x) - 5C$. Hence, $(x.y)_p \geq d(p,x) - 5C/2$. □

Lemma 2.2.10. *Suppose X is a length space and $x, y, p \in X$. Let γ be a $(1,\epsilon)$-quasigeodesic in X joining x, y. Then for any $z \in \gamma$, we have $(x.y)_p \leq d(p,z) + \frac{1}{2}\epsilon$.*

Proof. We have, $d(x,y) \geq l(\gamma) - \epsilon = l(\gamma|_{[x,z]}) + l(\gamma|_{[z,y]}) - \epsilon$

$$\geq d(x,z) + d(z,y) - \epsilon.$$

Thus, $(x.y)_p = \frac{1}{2}(d(p,x) + d(p,y) - d(x,y))$

$$\leq \frac{1}{2}(d(p,x) + d(p,y) - d(x,z) - d(z,y) + \epsilon).$$

Now, $d(p,x) - d(x,z) \leq d(p,z)$ and $d(p,y) - d(z,y) \leq d(p,z)$. Using these three inequalities we get, $(x.y)_p \leq d(p,z) + \frac{1}{2}\epsilon$. □

Definition 2.2.11. *(See also [39]) Suppose X is a length space and $Y \subset Z \subset X$. Suppose $Z = Z_1 \cup Z_2$ and $Y = Z_1 \cap Z_2$. We will say that Y* coarsely separates *Z into Z_1, Z_2 if for all $K \geq 1$ there is $R \geq 0$ such that for all points in $z_1 \in Z_1$ and $z_2 \in Z_2$ and for any K-quasigeodesic γ in X joining z_1, z_2 we have $\gamma \cap N_R(Y) \neq \emptyset$.*

The following lemma is immediate.

Lemma 2.2.12. *Suppose X is a length space, $Y \subset Z \subset X$ and Y coarsely separates Z into Z_1, Z_2. If $A \subset X$ with $Y \subset A \subset Z$ then Y coarsely separates A in to $A \cap Z_1$ and $A \cap Z_2$*

Lemma 2.2.13. *Let $\epsilon \geq 0$ and let X be a metric space and $A \subset X$. Suppose $y \in A$ is an ϵ-approximate nearest point projection of $x \in X$ on A. Let α be a $(1,1)$-quasigeodesic in X joining x, y. Then for any $x' \in \alpha$, y is an $\epsilon + 3$-approximate nearest point projection of x' on A.*

Proof. Let $z \in A$. Then $d(x,y) \leq d(x,z)+\epsilon$. Since α is a $(1,1)$-quasigeodesic, $d(x,x') + d(x',y) \leq d(x,y) + 3$. Hence, $d(x,x') + d(x',y) \leq d(x,z) + \epsilon + 3$ and so, $d(x',y) \leq d(x'z) + \epsilon + 3$. Hence, y is an $\epsilon + 3$-approximate nearest point projection of x' on A. □

2.2.1 Hyperbolic metric spaces

If X is a geodesic metric space and $x, y \in X$, then $[x,y]$ will denote a geodesic segment joining x to y. For $x, y, z \in X$, we denote a geodesic triangle with vertices x, y, z, by $\triangle xyz$. For $D \geq 0$ and $A \subset X$, $N_D(A) := \{x \in X : d(x,a) \leq D \text{ for some } a \in A\}$ will be called the D-neighborhood of A in X.

Definition 2.2.14. (1) *Suppose $\Delta x_1 x_2 x_3 \subset X$ is a geodesic triangle, and let $\delta \geq 0$, $K \geq 0$. Then, the triangle $\Delta x_1 x_2 x_3$ is δ-**slim** if any side of the triangle is contained in the δ-neighborhood of the union of the other two sides.*

(2) *Let $\delta \geq 0$ and X be a geodesic metric space. Then X is a δ-* **hyperbolic** *metric space if all geodesic triangles in X are δ-slim.*

A geodesic metric space is said to be **hyperbolic** *if it is δ-hyperbolic for some $\delta \geq 0$.*

This definition is due to E. Rips. Hence, we shall refer to the above property as **Rips hyperbolicity**. The corresponding spaces will be referred to as hyperbolic in the sense of Rips.

Definition 2.2.15. ***Gromov hyperbolicity:*** *(1) Suppose X is any metric space, not necessarily geodesic. Let $p \in X$ and $\delta \geq 0$. The Gromov inner* *product $X \times X \to \mathbb{R}$ defined by $(x, y) \mapsto (x.y)_p$ is δ-hyperbolic if*

$$(x.y)_p \geq min\{(x.z)_p, (y.z)_p\} - \delta$$

for all $x, y, z \in X$.

(2) *A metric space X, not necessarily geodesic, is called δ-hyperbolic for some $\delta \geq 0$ in the sense of Gromov if the Gromov inner product is δ-hyperbolic with respect to any point of X.*

The space is called Gromov hyperbolic if it is δ-hyperbolic for some $\delta \geq 0$.

Lemma 2.2.16. [1] *If the Gromov inner product is δ-hyperbolic with respect to a point p and then it is 2δ-hyperbolic with respect to any other point. In particular, the space is Gromov hyperbolic.*

The following is easy to verify:

Lemma 2.2.17. *Suppose X is a metric space which is δ-hyperbolic in the sense of Gromov. If $f : X \to Y$ is a coarsely R-surjective, $(1, C)$-quasiisometry then Y is $D = D(\delta, R, C)$-hyperbolic in the sense of Gromov.*

Proof. Fix an arbitrary point $p \in X$. Let $x, y \in X$. Then it is easy to verify that $|(f(x).f(y))_{f(p)} - (x.y)_p| \leq 3C/2$. Hence, for $x, y, z, p \in X$ we get

$$\begin{aligned}(f(x).f(y))_{f(p)} &\geq (x.y)_p - 3C/2 \geq min\{(x.z)_p, (y.z)_p\} - \delta - 3C/2 \\ &\geq min\{(f(x).f(z))_{f(p)}, (f(y).f(z))_{f(p)}\} - \delta - 3C.\end{aligned}$$

Let $y_1, y_2, y_3 \in Y$. Then there are $x_1, x_2, x_3 \in X$ such that $d(y_i, f(x_i)) \leq R$. It is easy to check that $|(y_i.y_j)_{f(p)} - (f(x_i).f(x_j))_{f(p)}| \leq 2R$. Thus,

$$\begin{aligned}(y_1.y_2)_{f(p)} &\geq (f(x_1).f(x_2))_{f(p)} - 2R \\ &\geq min\{(f(x_1).f(x_3))_{f(p)}, (f(x_2).f(x_3))_{f(p)}\} - \delta - 3C - 2R \\ &\geq min\{(y_1.y_3)_{f(p)}, (y_2, y_3)_{f(p)}\} - \delta - 3C - 3R.\end{aligned}$$

Now, let $y_1, y_2, y_3, y \in Y$ be arbitrary points. Let $x \in X$ be such that $d_Y(f(x), y) \leq R$. It is easy to verify that $|(y_i.y_j)_y - (y_i.y_j)_{f(x)}| \leq R$. It follows that $(y_1.y_2)_y \geq min\{(y_1.y_3)_y, (y_2.y_3)_y\} - \delta - 3C - 4R$. ☐

Lemma 2.2.18. [28] **Stability of quasigeodesics:** *For all $\delta \geq 0$ and $k \geq 1$, there is a constant $D_{2.2.18} = D_{2.2.18}(\delta, k)$ such that the following holds:*

Suppose Y is a δ-hyperbolic metric space. Then the Hausdorff distance between a geodesic and a k-quasigeodesic joining the same pair of points is less than or equal to $D_{2.2.18}$.

Lemma 2.2.19. [18, III.H, Corollary 1.8] *Suppose X is a length space. Fix $K \geq 1, \epsilon \geq 0$. Then X is δ-hyperbolic in the sense of Gromov if and only if all (K, ϵ)-quasigeodesic triangles in X are $D_{2.2.19} = D_{2.2.19}(\delta, K, \epsilon)$-slim.*

Lemma 2.2.20. *Suppose X is a δ-hyperbolic length space in the sense of Gromov. Let $x, y, p \in X$ be vertices of a triangle whose sides xy, yp, px are $(1, C)$-quasigeodesics. Then there is a constant $D = D(\delta, C)$ and a point $w \in xy$ such that $|(x.y)_p - d(p, w)| \leq D$. In particular, the difference between the distance of xy from p and $(x.y)_p$ is bounded, irrespective of x, y.*

Proof. We first join x, y, z by continuous $(1, 1)$-quasigeodesics, say α, β, γ which share common end points with xy, yp, xp respectively. Each point of α is contained in the $D_{2.2.19}(\delta, 1, 1)$-neighbourhood of $\beta \cup \gamma$. It follows by the connectedness of α that there is a point, say $z \in \alpha$ such that z is contained in the $D_{2.2.19}(\delta, 1, 1)$-neighbourhood of both β and γ. By Lemma 2.2.19, the Hausdorff distance between each pair of paths with the same end points is at most $D_{2.2.19}(\delta, 1, 1 + C)$. Thus, there is a point $w \in xy$ which is in the $(D_{2.2.19}(\delta, 1, 1) + D_{2.2.19}(\delta, 1, 1 + C))$-neighbourhood of both xp and yp. Let $R = D_{2.2.19}(\delta, 1, 1) + D_{2.2.19}(\delta, 1, 1 + C)$.

Let $x_1 \in xp, y_1 \in yp$ be such that $d(w, x_1) \leq R$ and $d(w, y_1) \leq R$. Using the δ-hyperbolicity of X we have $(x.y)_p \geq min\{(x.w)_p, (w.y)_p\} - \delta \geq min\{(x.x_1)_p, (y, y_1)_p\} - \delta - R$ by Lemma 2.2.9. Since p, x_1, x are on a $(1, C)$-quasigeodesic, again by Lemma 2.2.9 we have, $(x.x_1)_p \geq d(p, x_1) - 5C/2 \geq d(p, w) - R - 5C/2$. In the same way, $(y, y_1)_p \geq d(p, w) - R - 5C/2$. Hence,

$(x.y)_p \geq d(p, w) - \delta - 2R - 5C/2$. Finally using Lemma 2.2.10, we have $d(p, w) - \delta - 2R - 5C/2 \leq (x.y)_p \leq d(p, w) + C/2$. □

Lemma 2.2.21. [41, Lemma 3.1] *Given $\delta > 0$, there exist $D_{2.2.21}, C_{2.2.21}$ such that if a, b, c, d are vertices in a δ- hyperbolic metric space (Y, d), with $d(a, b) = d(a, [b, c])$, $d(d, c) = d(d, [b, c])$ and $d(b, c) \geq D_{2.2.21}$ then the union of geodesics $[a, b] \cup [b, c] \cup [c, d]$ is a quasigeodesic in Y and it lies in a $C_{2.2.21}$-neighbourhood of a geodesic joining a, d.*

2.2.2 Boundary of hyperbolic spaces and Cannon-Thurston maps

Given a hyperbolic metric space X, there are three ways to define a boundary-namely the geodesic boundary, the quasigeodesic boundary and the Gromov boundary or sequential boundary. We refer to [18] for these.

Definition 2.2.22. ***Geodesic boundary:*** *Suppose X is a geodesic hyperbolic metric space. Let $x_0 \in X$. Then the* geodesic boundary *∂X of X is the equivalence classes of geodesic rays α starting at x_0, where two geodesic rays α, β are said to be* equivalent *if $Hd(\alpha, \beta) < \infty$.*

The equivalence class of a geodesic ray α is denoted by $\alpha(\infty)$. The boundary ∂X does not depend on the choice of x_0.

Definition 2.2.23. ***Quasigeodesic boundary:*** *Suppose X is a hyperbolic metric space in the sense of Gromov. Then the* quasigeodesic boundary *$\partial_q X$ of X is the equivalence classes of all quasigeodesic rays α where two quasigeodesic rays α, β are said to be* equivalent *if $Hd(\alpha, \beta) < \infty$.*

For proper geodesic spaces, the above two definitions are equivalent.

Lemma 2.2.24. [18, III.H, Lemma 3.1] *If X is a δ-hyperbolic proper geodesic space, there is a natural bijection from ∂X to $\partial_q X$.*

Definition 2.2.25. ***Gromov boundary or sequential boundary:*** *Suppose X is a hyperbolic metric space in the sense of Gromov. We consider the set $\mathcal{S}$ of sequences $\{x_n\}$ in X such that* $\lim_{i,j\to\infty}(x_i.x_j)_p = \infty$*. Such a sequence is said to converge to infinity. On $\mathcal{S}$ one defines an equivalence relation where $\{x_n\} \sim \{y_n\}$ if and only if* $\lim_{i,j\to\infty}(x_i.y_j)_p = \infty$ *for some (any) base point $p \in X$. The* Gromov boundary or the sequential boundary $\partial_s X$ *of X, as a set, is defined to be $\mathcal{S}/\sim$.*

If $\xi = [\{x_n\}]$, then we write $x_n \to \xi$ and say that the sequence $\{x_n\}$ converges to ξ. The set $X \cup \partial X$ is denoted by $\overline{X}$.

There is a natural bijection between sequential boundary and the quasi-geodesic boundary for any hyperbolic geodesic metric space. See [45, Lemma 2.4]. It is easy to see that Lemma 2.2.7 implies the same for length spaces. Moreover, the geodesic and the sequential boundaries can be endowed with natural topologies. If X is a proper geodesic hyperbolic metric space then all these spaces are naturally homeomorphic (see [18, III.H, Lemma 3.13]). We recall here, the topology on the geodesic boundary and the Gromov boundary.

Topology on ∂X: Let X be a δ-hyperbolic proper geodesic space. Fix a base point $p \in X$. For any $\alpha(\infty) \in \partial X$ and $r > 0$,

$$V_r(\alpha(\infty)) := \{\beta(\infty) \in \partial X \mid \exists\, \alpha' \sim \alpha,\, \beta' \sim \beta \text{ with } \liminf_{t\to\infty}(\alpha'(t).\beta'(t))_p \geq r\}$$

Lemma 2.2.26. [18, III.H, Lemma 3.6] *Given $\alpha(\infty) \in \partial X$, the collection $\{V_n(\alpha(\infty))\}_{n\in\mathbb{N}}$ is a fundamental system of neighbourhoods of $\alpha(\infty)$ in ∂X.*

By the following lemma, the topology on ∂X does not depend on p.

Lemma 2.2.27. [32, Proposition 2.14] *For a δ-hyperbolic proper geodesic metric space, the topology on the boundary is independent of the choice of basepoint.*

Topology on $\partial_s X$: Let X be a δ-hyperbolic metric space. We no longer assume that X is proper or a geodesic space.

Definition 2.2.28. *(1) If $\{\xi_n\}$ is a sequence of points in $\partial_s X$, we say that $\{\xi_n\}$ converges to $\xi \in \partial_s X$ if the following holds: Suppose $\xi_n = [\{x_k^n\}_k]$ and $\xi = [\{x_k\}]$. Then $\lim_{n\to\infty}(\liminf_{i,j\to\infty}(x_i.x_j^n)_p) = \infty$.*

(2) The limit set *of a subset Y of X is the set $\{\xi \in \partial_s X \mid \exists\{y_n\} \subset Y \, with \, y_n \to \xi\}$. We denote this set by $\Lambda(Y)$.*

(3) A subset $A \subset \partial_s X$ is said to be closed *if for any sequence $\{\xi_n\}$ in A, $\xi_n \to \xi$ implies $\xi \in A$.*

Lemma 2.2.29. [49, Lemma 2.3] *Suppose $\{x_n\}, \{y_n\}$ are two sequences in a hyperbolic metric space X, both converging to some points in ∂X. If $\{d(x_n, y_n)\}$ is bounded then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.*

Lemma 2.2.30. [49, Lemma 2.4]

(1) There is a natural topology on the boundary ∂X of a proper hyperbolic metric space X with respect to which, ∂X is compact.

(2) If $f : X \to Y$ is a quasiisometric embedding of proper hyperbolic metric spaces, then f induces a topological embedding $\partial f : \partial X \to \partial Y$. If f is a quasiisometry, then ∂f is a homeomorphism.

Definition 2.2.31. [41] ***Cannon-Thurston map:*** *If $f : Y \to X$ is a proper embedding of hyperbolic metric spaces, then* Cannon-Thurston (**CT**) map exists for f *if f gives rise to a continuous map $\partial f : \partial Y \to \partial X$.*

This means that given any $\xi \in \partial Y$ and any sequence of points $\{y_n\}$ in Y converging to ξ, the sequence $\{f(y_n)\}$ converges to a definite point of ∂X independent of the $\{y_n\}$ and the resulting map $\partial f : \partial Y \to \partial X$ is continuous.

Existence of Cannon-Thurston maps:

Lemma 2.2.32 (Mitra's criterion). [41, Lemma 2.1] *Suppose X, Y are geodesic hyperbolic metric spaces and $f : Y \to X$ is a proper embedding.*

Then f admits CT if the following holds:

Given $y_0 \in Y$, there exists a non-negative function $M(N)$, such that $M(N) \to \infty$ as $N \to \infty$ and for all geodesic segments λ in Y lying outside an N-ball around $y_0 \in Y$, any geodesic segment in X joining the end-points of $i(\lambda)$ lies outside the $M(N)$-ball around $i(y_0) \in X$.

Mitra's criterion (Lemma 2.2.32) holds for hyperbolic geodesic metric spaces. We give a modification for it which gives a criterion of existence of Cannon-Thurston maps in the case of length spaces.

Lemma 2.2.33. *Suppose X, Y are length spaces hyperbolic in the sense of Gromov and $f : Y \to X$ is any map. Let $p \in Y$ and $\epsilon > 0$.*

(*) Suppose for all $N > 0$, there is $M > 0$ such that $N \to \infty$ implies $M \to \infty$ with the following property: For $y_1, y_2 \in Y$ and a $(1, \epsilon)$-quasigeodesic α in Y joining y_1, y_2 and a $(1, \epsilon)$-quasigeodesic β in X joining $f(y_1), f(y_2)$, $B(p, N) \cap \alpha = \emptyset$ implies $B(f(p), M) \cap \beta = \emptyset$. Then CT map exists for $f : Y \to X$.

Proof. Suppose $\{y_n\}$ is a sequence in Y converging to infinity. Then $\lim_{i,j\to\infty}(y_i.y_j)_p = \infty$. Suppose $\alpha_{i,j}$ is a $(1, \epsilon)$-quasigeodesic in Y joining y_i, y_j. Then by Lemma 2.2.20, we have $\lim_{i,j\to\infty} d(p, \alpha_{i,j}) = \infty$. Hence by (*), if $\gamma_{i,j}$ is a $(1, \epsilon)$-quasigeodesic in X joining $f(y_i), f(y_j)$, then $\lim_{i,j\to\infty} d(f(p), \gamma_{i,j}) = \infty$. Again by Lemma 2.2.20, this implies that $\lim_{i,j\to\infty}(f(y_i).f(y_j))_{f(p)} = \infty$. Thus $\{(f(y_n)\}$ is converging to infinity in X. The same argument shows that if $\{y_n\}$ and $\{z_n\}$ are two sequences in Y representing the same point of $\partial_s Y$ then $\{f(y_n)\}$ and $\{f(z_n)\}$ also represent the same point of $\partial_s X$. Thus, we have a well-defined map $\partial f : \partial_s Y \to \partial_s X$.

The continuity of this map also follows by similar arguments. We need to show that if $\xi_n \to \xi$ in $\partial_s Y$ then $\partial f(\xi_n) \to \partial f(\xi)$. Suppose ξ_n is represented by the class of $\{y^n_k\}_k$ and ξ is the equivalence class of $\{y_k\}$. Then

$$\lim_{n\to\infty}(\liminf_{i,j\to\infty}(y^n_i.y_j)_p) = \infty.$$

By Lemma 2.2.20, we have $\lim_{n\to\infty}(\liminf_{i,j\to\infty} d(p,\alpha^n_{i,j})) = \infty$ for any $(1,\epsilon)$-quasigeodesic in Y joining y^n_i and y_j. By (*), we have

$$\lim_{n\to\infty}(\liminf_{i,j\to\infty} d(f(p),\gamma^n_{i,j})) = \infty,$$

where $\gamma^n_{i,j}$ is any $(1,\epsilon)$-quasigeodesic in X joining $f(y^n_i), f(y_j)$. This in turn implies that $\lim_{n\to\infty}(\liminf_{i,j\to\infty}(f(y^n_i).f(y_j))_{f(p)}) = \infty$.

Therefore we have, $\partial f(\xi_n) \to \partial f(\xi)$. □

Examples and remarks:

1. Suppose $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is an exponential function. The f is not coarsely Lipschitz but f admits CT.

2. The condition (*) in the above Lemma 2.2.33 is not necessary in general for the existence of CT map. Here is an example: Suppose X is a tree built in two phases. First, we have a star, i.e., a tree with one central vertex x_0 on which end points of finite intervals σ_n, for $n \in \mathbb{N}$, are glued. Let each σ_n be isometric to the interval $[0,n]$ in $\mathbb{R}$. Clearly, the lengths of the intervals are unbounded. For each $n \in \mathbb{N}, i \in [0,n] \cap \mathbb{Z}$, two distinct rays $\alpha_{n,i}, \beta_{n,i}$ are glued to the ith integer point of σ_n. Suppose Y is obtained by collapsing the central star in X and f is the quotient map. Then Y consists of $\alpha_{n,i}, \beta_{n,i}$, $n \in \mathbb{N}, i \in [0,n] \cap \mathbb{Z}$, all glued at $\alpha_{n,i}(0), \beta_{n,i}(0)$. Then clearly CT exists but (*) is violated. Let $N > 0$. Consider σ_{N+1}. Let $\alpha_{N+1,N+1}, \beta_{N+1,N+1} : [0,\infty) \to X$ be the two quasigeodesic rays glued to the $N+1$th integer point of σ_{N+1}. Then for any $s,t \in [0,\infty)$, a geodesic in X

joining $\alpha_{N+1,N+1}(s), \beta_{N+1,N+1}(t)$ is the concatenation of $\alpha_{N+1,N+1}|[0,s]$ and $\beta_{N+1,N+1}|_{[0,t]}$. Clearly, $d_X(x_0, \alpha_{N+1,N+1}|[0,s] * \beta_{N+1,N+1}|_{[0,t]}) = n+1 > n$. But in Y, $f(x_0) = \alpha_{N+1,N+1}(0) = \beta_{N+1,N+1}(0)$. Thus, $d_Y(f(x_0), \alpha_{N+1,N+1}|[0,s] * \beta_{N+1,N+1}|_{[0,t]}) = 0$.

Lemma 2.2.34. *Let X, Y be hyperbolic metric spaces and $f : Y \to X$ be a proper embedding. If the CT map exists for f then, $\Lambda(f(Y)) = \partial f(\partial Y)$.*

We mention the following lemma with brief remarks about the proof since it states some standard facts from hyperbolic geometry.

Lemma 2.2.35. *Suppose G is a hyperbolic group which acts on a hyperbolic metric space X by isometries properly. Suppose that an orbit map $f : G \to X$, $g \mapsto gx$ admits CT. If the CT map $\partial f : \partial G \to \partial X$ is injective then f is a qi embedding.*

Proof. Suppose f is not a qi embedding. Then given $n \in \mathbb{N}$, there is a geodesic $[g_n, h_n] \subset G$ such that no geodesic in X joining $g_n x, h_n x$ is contained in the n-neighborhood of $f([g_n, h_n])$. Now applying a suitable element of G, we may assume that the midpoint of $[g_n, h_n]$ is $1 \in G$. Now, passing to a subsequence we may assume without loss of generality assume that $\{g_n\}$ and $\{h_n\}$ are converging to two different points of ∂G. Clearly, these points have the same image under the CT map. □

Lemma 2.2.36. ***Functoriality of CT maps:*** *(1) Suppose X, Y, Z are hyperbolic metric spaces and $f : X \to Y$ and $g : Y \to Z$ admit CT maps. Then so does $g \circ f$ and $\partial(g \circ f) = \partial g \circ \partial f$.*

(2) If $i : X \to X$ is the identity map then it admits a CT map ∂i which is the identity map on ∂X.

(3) If two maps $f, h : X \to Y$ are at a finite distance admitting CT maps then they induce the same CT map.

(4) Suppose $f : X \to Y$ is a qi embedding of hyperbolic length spaces. There is a continuous injective CT map $\partial f : \partial_s X \to \partial_s Y$ which is a homeomorphism onto its image. Moreover, if f is a quasiisometry then ∂f is a homeomorphism.

2.2.3 Quasiconvex subspaces of hyperbolic spaces

Definition 2.2.37. *Let X be a geodesic metric space and let $A \subseteq X$. For $K \geq 0$, we say that A is K-**quasiconvex** in X if any geodesic with end points in A is contained in the K-neighborhood of A. A subset $A \subset X$ is said to be **quasiconvex** if it is K-quasiconvex for some $K \geq 0$.*

Lemma 2.2.38. *Let X be a δ-hyperbolic metric space and $\epsilon \geq 0$.*

(1) Let $p, q, r \in X$. Let α and β be $(1,1)$-quasigeodesics in X joining p, q and q, r respectively. Suppose q is an ϵ-approximate nearest point projection of $p \in X$ on β. Then $\alpha * \beta$ is a $(3, 2+\epsilon)$-quasigeodesic.

(2) Suppose X is δ-hyperbolic for $\delta \geq 0$ and $U \subset X$ is a K-quasiconvex set and $\epsilon \geq 0$. Let $p \in X$ and $q \in U$ be an ϵ-approximate nearest point projection of p on U and $r \in U$. Let α, β be k-quasigeodesics in X joining p, q and q, r respectively. Then $\alpha * \beta$ is $K_{2.2.38} = K_{2.2.38}(\delta, K, k, \epsilon)$-quasigeodesic in X.

Proof. (1) Let $p' \in \alpha, q' \in \beta$. Then $d(p', q) \leq d(p', q') + \epsilon + 3$ and $d(q', q) \leq 2d(p', q) + \epsilon + 3$. Without loss of generality, assume $\alpha(s) = p', \alpha(s+m) = q, \beta(0) = q$ and $\beta(t) = q'$. Since α, β are $(1,1)$-quasigeodesics, $m - 1 \leq d(p', q) \leq d(p', q') + \epsilon + 3$ and $t - 1 \leq d(q, q') \leq 2d(p', q) + \epsilon + 3$. Then, $m + t - 2 \leq 3d(p', q) + 2\epsilon + 6$. Also, $d(p', q') \leq d(p', q) + d(q, q') \leq m + t + 2$. Thus,

$$-\frac{8}{3} - \frac{2\epsilon}{3} + \frac{1}{3}(m+t) \leq \frac{1}{3}d(p', q') \leq m + t + 2$$

and the lemma follows.

(2) Let α, β be $(1,1)$-quasigeodesics in X joining p, q and q, r respectively.

Then q is an $\epsilon+K$-approximate nearest point projection of p on β_1. Then by (1), $\alpha_1 * \beta_1$ is a $(3, 2+\epsilon+K)$-quasigeodesic. By Lemma 2.2.18, $\mathrm{Hd}(\alpha, \alpha_1) \leq D_{2.2.18}(\delta, k), \mathrm{Hd}(\beta, \beta_1) \leq D_{2.2.18}(\delta, k)$. Hence, $\mathrm{Hd}(\alpha*\beta, \alpha_1*\beta_1) \leq D_{2.2.18}(\delta, k)$. So by Lemma 2.2.4, it is enough to show that $\alpha * \beta$ is uniformly properly embedded in X.

Let $\gamma = \alpha * \beta$ and $\gamma_1 = \alpha_1 * \beta_1$. Let $R = D_{2.2.18}(\delta, k)$. Suppose $\alpha : [0, l] \to X, \beta : [0, m] \to X$ with $\alpha(0) = p, \alpha(l) = \beta(0) = q$ and $\beta(m) = r$. Let $s \leq t \in [0, l+m]$ such that $d(\gamma(s), \gamma(t)) \leq D$. It is enough to check the case where $s \in [0, l]$ and $t \in [0, m]$ since α, β are k-quasigeodesics. In this case, $\gamma(s) = \alpha(s)$ and $\gamma(t-l) = \beta(t)$. Let $\gamma_1(s'), \gamma_1(t')$ such that $d(\gamma(s), \gamma_1(s')) \leq R$ and $d(\gamma(l-t), \gamma_1(t')) \leq R$. Also, let $\gamma_1(u) = q$. Since γ_1 is a $(3, 2+\epsilon+K)$-quasigeodesic, $|s'-t'| \leq 3d(\gamma_1(s'), \gamma_1(t')) + 3(2+\epsilon+K) \leq 3(2R+D) + 3(2+\epsilon+K) = 6R+3D+3\epsilon+3K+6$. Since $s' \leq u \leq t'$, $|s'-u| \leq 6R+3D+3\epsilon+3K+6$ and $|t'-u| \leq 6R+3D+3\epsilon+3K+6$. Then $d(\gamma_1(s'), q), d(\gamma_1(t'), q)$ are at most $3(6R+3D+3\epsilon+3K+6)+2+\epsilon+K =: D'$, say. Then, $d(\gamma(s), y), d(\gamma(l-t), y)$ are at most $D'+R$. Then $l-s \leq k(R+D')+k^2$ and $t-l \leq k(R+D')+k^2$. Hence, $|t-s| \leq 2(k(R+D')+k^2)$. Thus, γ is uniformly properly embedded in X. □

The following corollary easily follows.

Corollary 2.2.39. *Let X be a δ-hyperbolic metric space and let α be a k-quasigeodesic. Let $x \in X$ and $y \in \alpha$ be an ϵ-approximate nearest point projection of x on α. Let β be a k-quasigeodesic joinin x, y. Then $\beta * \alpha$ is a $K_{2.2.39} = K_{2.2.39}(\delta, k, \epsilon)$-quasigeodesic in X.*

The following corollary easily follows from Lemma 2.2.38 and Lemma 2.2.13.

Corollary 2.2.40. *Let X be a δ-hyperbolic metric space and $V \subset U$ are K-quasiconvex subsets of X. Let $x \in X$, $x_1 \in U, x_2 \in V$ be ϵ-approximate nearest point projections of x on U, V respectively. Also, let $x_3 \in V$ be an ϵ-*

approximate nearest point projection of x_1 *on* V*. Then* $d(x_2, x_3) \leq D_{2.2.40} = D_{2.2.40}(\delta, K, \epsilon)$.

In particular, for any two ϵ*-approximate nearest point projections* x_1, x_2 *of* x *on* U*,* $d(x_1, x_2) \leq D_{2.2.40}(\delta, K, \epsilon)$.

Corollary 2.2.41. *Given* $\delta, K, \epsilon \geq 0$*, there exists* $L_{2.2.41} = L_{2.2.41}(\delta, K, \epsilon), D_{2.2.41} = D_{2.2.41}(\delta, K, \epsilon)$ *and* $R_{2.2.41} = R_{2.2.41}(\delta, K, \epsilon)$ *such that the following hold:*
(1) Suppose X *is a* δ*-hyperbolic metric space and* U *is a* K*-quasiconvex subset of* X*. Then an* ϵ*-approximate nearest point projection map* $P : X \to U$ *is coarsely* $L_{2.2.41}$*-Lipschitz.*
(2) Let $V \subset X$ *also be a* K*-quasiconvex subset and* $v_1, v_2 \in V$*. Let* $u_i = P(v_i)$*, for* $i = 1, 2$*. If* $d(u_1, u_2) \geq D$*, then* $u_1, u_2 \in N_R(V)$*.*

In particular, if diameter of $P(V)$ *is at least* D *then* $d(U, V) \leq R$*.*

Proof. (1) Let $x, y \in X$ with $d(x, y) \leq 1$. Then $d(y, P(y)) \leq d(y, U) + \epsilon$ and $d(y, P(x)) \leq d(y, x) + d(x, P(x)) \leq d(x, U) + \epsilon + 1 \leq d(y, U) + \epsilon + 2$. Thus, $P(x)$ is an $\epsilon + 2$-approximate nearest point projection of x on U. Then by Corollary 2.2.40, $d(P(x), P(y)) \leq D_{2.2.40}(\delta, K, \epsilon + 2)$. This proves the result.

(2) Proof is similar to that of [41, Lemma 3.1].

Lemma 2.2.42. *Let* (X, d) *be a* δ*-hyperbolic geodesic metric space,* $\epsilon \geq 1$ *and* A *be a* K*-quasiconvex subset for* $K \geq 0$*. Let* $x, y \in X$ *and let* α *be a geodesic in* X *joining* x *to* y*. Let* x_1 *and* y_1 *denote the* ϵ*-approximate nearest point projections of* x *and* y *respectively in* A*. Let* β *be a geodesic in* X *joining* x_1 *and* y_1*. Then the* ϵ*-approximate nearest point projections of a point of* α *on* β *and* A *are uniformly close.*

Proof. For any $w \in \alpha$, let w_1 and w_2 denote the ϵ-approximate nearest point projections of w in β and A respectively. Since A is K-quasiconvex, there exists $w_3 \in A$ such that $d(w_1, w_3) \leq K$. Since $[x, x_1] \cup \alpha \cup [y, y_1] \cup \beta$ is a hyperbolic geodesic quadrilateral, it is 2δ-slim. So, there exists $w' \in [y, y_1] \cup$

$\beta \cup [x, x_1]$ such that $d(w, w') \leq 2\delta$. If $w' \in \beta$, then clearly $d(w, w_1) \leq 2\delta$ and moreover, $d(w, w_2) \leq d(w, w_3) + \epsilon \leq 2\delta + K + \epsilon$. Then we are done.

So let $w' \in [x, x_1] \cup [y, y_1]$. Say, $w' \in [y, y_1]$. By Lemma 2.2.13, y_1 is an $\epsilon + 3$-approximate nearest point projection of w' on A. Then by Corollary 2.2.41, $d(w_2, y_1) \leq L_{2.2.41} d(w, w') + L_{2.2.41} \leq L_{2.2.41}(2\delta + 1)$.

Now, $d(w, w_1) \leq d(w, \beta) + \epsilon$. Then, $d(w', w_1) \leq d(w', \beta) + \epsilon + 2\delta$. So, w_1 is an $\epsilon + 2\delta$-approximate nearest point projection of w' on β. By Corollary 2.2.39, $[w', w_1] * \beta|_{[w_1, y_1]}$ is a $K_{2.2.39}(\delta, \epsilon)$-quasigeodesic and by Lemma 2.2.18, there exists $z \in [w', y_1]$ such that $d(w_1, z) \leq D_{2.2.18}(\delta, K_{2.2.39}(\delta, \epsilon)) =: D_1$. So, $d(z, w_3) \leq D_1 + K$. Again by Lemma 2.2.13, y_1 is an $\epsilon + 3$-approximate nearest point projection of z on A. Thus, $d(z, y_1) \leq d(z, w_3) + \epsilon + 3 \leq D_1 + K + \epsilon + 3$. Therefore, $d(w_1, y_1) \leq 2D_1 + K + \epsilon + 3$. Finally, this gives us, $d(w_1, w_2) \leq L_{2.2.41}(2\delta + 1) + 2D_1 + K + \epsilon + 3$. Then for $D_{2.2.42} = \max\{2\delta + K + \epsilon, L_{2.2.41}(2\delta + 1) + 2D_1 + K + \epsilon + 3\}$, we have, $d(w_1, w_2) \leq D_{2.2.42}$. □

Definition 2.2.43. *Suppose Y is a metric space and $U, V \subset Y$. We say that U, V are ϵ-***separated** *if* $\inf\{d(y_1, y_2) \mid y_1 \in U, y_2 \in V\} \geq \epsilon$. *A collection of subsets $\{U_\alpha\}$ of Y is said to be* uniformly separated *if there exists an $\epsilon > 0$ such that any pair of distinct elements of the collection $\{U_\alpha\}$ is ϵ-separated.*

Definition 2.2.44. *Suppose Y is a δ-hyperbolic metric space and U_1, U_2 are two quasiconvex subsets. Let $D > 0$. We say that U_1, U_2 are* **mutually** *D-***cobounded***, or simply D-cobounded, if any nearest point projection of U_1 to U_2 has diameter at most D and vice versa.*

Lemma 2.2.45. *Suppose X is a hyperbolic metric space and A, B are K-quasiconvex, D-cobounded sets. Then $N_R(A), N_R(B)$ are also uniformly quasiconvex and uniformly cobounded.*

Lemma 2.2.46. *Given $\delta \geq 0$ and $K \geq 0$ there are constants $R = R_{2.2.46}(\delta, K)$ and $D = D_{2.2.46}(\delta, K)$ such that the following holds:*

Suppose X is a δ-hyperbolic metric space and $U, V \subset X$ are two K-quasiconvex and R-separated subsets. Then U, V are D-cobounded.

Lemma 2.2.47. *Given $\delta \geq 0$ and $K \geq 0$ there are constants $R = R_{2.2.47}(\delta, K)$ and $D = D_{2.2.47}(\delta, K)$ such that the following holds:*

Suppose X is a δ-hyperbolic metric space and $U, V \subset X$ are two K-quasiconvex and R-separated subsets. Then there are points $x_0 \in U$, $y_0 \in V$ such that $[x_0, y_0] \subset N_D([x, y])$, for all $x \in U$ and $y \in V$.

Corollary 2.2.48. *Given $\delta \geq 0$ and $D, K \geq 0$ there exists $C = C_{2.2.48}(\delta, D, K)$ such that the following holds:*

Suppose X is a δ-hyperbolic metric space and $U, V \subset X$ are two K-quasiconvex and D-cobounded subsets. Choose $a \in U, b \in V$ such that $d(a, b) = d(U, V)$, and $[c, a] \subset U$, $[b, d] \subset V$ are K-quasigeodesics, then $[c, a] \cup [a, b] \cup [b, d]$ is a C-quasigeodesic.

Lemma 2.2.49. *Given $\delta \geq 0, k > 0, D_0 > 0$, there exists $D_{2.2.49} = D_{2.2.49}(\delta, k, D_0)$ such that following holds:*

Let X be a δ-hyperbolic metric space and Y, Z be quasiconvex subsets of X. Suppose there exists $z_0 \in Z$ such that for any pair of points $y \in Y, z \in Z$, there exists a uniform k-quasigeodesic $\alpha(y, z)$ joining y, z, satisfying $z_0 \in N_{D_0}(\alpha(y, z))$. Then, the nearest point projection of Y on Z is uniformly bounded. Moreover, if there exists $y_0 \in Y$ such that $y_0 \in N_{D_0}(\alpha(y, z))$, then Y, Z are mutually cobounded.

Proof. Let $y_1, y_2 \in Y$ and let $z_1, z_2 \in Z$ be their nearest point projections in Z respectively. Then, by stability of quasigeodesics, there exists $x_i \in [y_i, z_i]$, for $i = 1, 2$, such that $d(z_0, x_i) \leq D_0 + D_{2.2.18}$. Clearly, z_i is a nearest point projection of x_i in Z. Then, $d(x_i, z_i) \leq d(x_i, z_0) \leq D_0 + D_{2.2.18}$. Thus, $d(z_1, z_2) \leq d(z_1, x_1) + d(x_1, z_0) + d(z_0, x_2) + d(x_2, z_2) \leq 4(D_0 + D_{2.2.18})$. So, for $D_{2.2.49} = 4(D_0 + D_{2.2.18})$, the nearest point projection of Y on Z is uniformly

$D_{2.2.49}$-bounded. Similarly, if there exists $y_0 \in Y$ such that $y_0 \in N_{D_0}(\alpha(y,z))$, then, Y, Z are mutually $D_{2.2.49}$-cobounded. □

2.3 Relatively hyperbolic metric spaces

Let M be a finite volume hyperbolic manifold. Such a manifold has finitely many cusps, say $E_1, ..., E_k$. Inclusion maps $E_i \hookrightarrow M$ induce injections $\pi_1(E_i) \to \pi_1(M)$. Then M is quasiisometric to a finite wedge of rays connected at their respective endpoints. This motivates Gromov's definition of relative hyperbolicity.

By Milnor-Svarc Lemma, if a group G acts on a δ-hyperbolic metric space X properly discontinuously and cocompactly by isometries, then X is quasiisometric to the Cayley graph of G and hence, G is a hyperbolic group. This can be extended to relatively hyperbolic groups in the following way:

Let G be a group acting on a proper hyperbolic geodesic space X properly discontinuously by isometries such that $V = X/G$ is quasiisometric to a union of k copies of $(-\infty, 0]$ joined at 0. For $1 \leq i \leq k$, let $\gamma_i : [0,\infty) \to X$ denote the lift of the i-th copy of $(-\infty, 0]$ to X with $\gamma_i(0) = p$ for $p \in X$. Let $H_i = Stab_G(\gamma_i(\infty))$. Then G is *hyperbolic relative to* $\{H_1, ..., H_k\}$.

Here, H_i are called *peripheral subgroups*. There are various characterisations of relative hyperbolicity and many of these are equivalent. In this section, we refer to the definitions by Farb (cf. [24]) and Gromov (cf. [30]).

Farb's definition

Definition 2.3.1. ***Coned-off space:*** *Let (X,d) be a path metric space and $\mathcal{A} = \{A_\alpha\}_{\alpha\in\Lambda}$ be a collection of uniformly separated subsets of X, i.e., there exists $\epsilon > 0$ such that $d(A_\alpha, A_\beta) > \epsilon$ for all distinct A_α, A_β in $\mathcal{A}$. For each $A_\alpha \in \mathcal{A}$, introduce a vertex $\nu(A_\alpha)$ and join every element of A_α to the vertex* *by an edge of length* $\frac{1}{2}$*. This new space is denoted by* $\widehat{X} = \mathcal{E}(X, \mathcal{A})$*. The new vertices are called cone points and* $H_\alpha \in \mathcal{H}$ *are called horosphere-like sets. The new space is called a* coned-off space of X with respect to $\mathcal{A}$.

Terminology:

1. Let X be a geodesic space. For $x, y \in X$, $d(x, y)$ or $d_X(x, y)$ denotes the distance in the original metric on X. For any two subsets $A, B \subset X$, we denote the Hausdorff distance between them by $\mathrm{Hd}(A, B)$. For $C \geq 0$, $N_C(A)$ will denote the C-neighbourhood of A in X.

2. The induced length metric on $\widehat{X}$ is called the **electric metric**.

3. For a geodesic space (X, d), let $\widehat{X}$ denote the coned-off metric space relative to a collection of horosphere-like sets $\{A_\alpha\}_{\alpha \in \Lambda}$. Then for $x, y \in \widehat{X}$, $d_{\widehat{X}}(x, y)$ denotes the distance in the electric metric.

4. Geodesics and quasigeodesics in $\widehat{X}$ are called **electric geodesics** and **electric quasigeodesics** respectively.

5. Let γ be a path in X. If γ penetrates a horosphere-like set A_α, we replace portions of γ inside A_α by edges joining the entry and exit points of γ in A_α to $\nu(A_\alpha)$. We denote the new path by $\hat{\gamma}$. If $\hat{\gamma}$ is an electric geodesic (resp. electric quasigeodesic), we call γ a **relative geodesic** (resp. **relative quasigeodesic**) in X.

6. For any electric geodesic $\hat{\alpha}$, we denote the union of subsegments of $\hat{\alpha}$ lying outside the horosphere-like sets by α^b.

7. A path γ in X is a path **without backtracking** if it does not return to any coset A_α after leaving it.

Definition 2.3.2. ***Bounded region penetration property:*** *Let* $(X, \mathcal{A})$ *be as in Definition 2.3.1. The pair* $(X, \mathcal{A})$ *satisfies* bounded region penetration property *if, for every $K \geq 1$, there exists $B = B(K)$ such that if β and γ are two relative K-quasigeodesics without backtracking and joining the same pair of points, then*

1. *if β penetrates a horosphere-like set A_α and γ does not, then the length of the portion of β lying inside A_α is at most B, with respect to the metric on X;*

2. *if both β and γ penetrate a horosphere-like set A_α, then the distance between the entry points of β and γ into A_α and the distance between the exit points of β and γ from A_α is at most B, with respect to the metric on X.*

Definition 2.3.3. ***Strongly relative hyperbolic space:*** *A metric space X is* strongly hyperbolic relative to a collection of subsets $\mathcal{A}$ *if the coned-off space $\mathcal{E}(X, \mathcal{A})$ is a hyperbolic metric space and $(X, \mathcal{A})$ satisfies the bounded region penetration property.*

Definition 2.3.4. ***Strongly relative hyperbolic group*** *A group G is* strongly hyperbolic relative to a collection of subgroups $\mathcal{H} = \{H_\alpha\}_{\alpha\in\Lambda}$ *if the Cayley graph X, of G, is strongly hyperbolic relative to the collection of subgraphs corresponding to the left cosets of H_α in G for every $\alpha \in \Lambda$.*

Examples and non-examples:

1. Let M be a torus with a cusp and let H denote the cusp subgroup. Then $(\pi_1(M), H)$ is a strongly relatively hyperbolic group.

2. Any hyperbolic group is hyperbolic relative to the identity subgroup.

3. A free product of groups $G * H$ is hyperbolic relative to $\{G, H\}$.

4. $\mathbb{Z} \oplus \mathbb{Z} =< a, b \,|\, ab = ba >$ is not strongly hyperbolic relative to $< a >$. But it is weakly hyperbolic relative to $< a >$.

Another definition of relatively hyperbolic groups is due to Gromov.

Gromov's definition

Definition 2.3.5. [30] ***Hyperbolic cone:*** *Let* (Y, d) *be a geodesic space. Then the* hyperbolic cone *of* Y*,* $Y^h = Y \times [0, \infty)$ *with the path metric* d_h *is defined as follows:*

Let $\alpha = (\alpha_1, \alpha_2) : [0, 1] \to Y \times [0, \infty)$ *be a path in* Y^h*. Let* $0 = t_0 < t_1 < \ldots < t_n = 1$ *be a partition of* $[0, 1]$*. Then,*

$$l_{Y^h}(\alpha) = \lim \textstyle\sum_{1 \leq i \leq n-1} \sqrt{e^{-2\alpha_2(t_i)} d_Y(\alpha_1(t_i), \alpha_1(t_{i+1}))^2 + |\alpha_2(t_i) - \alpha_2(t_{i+1})|^2},$$

where the limit is taken over all the partitions of $[0, 1]$*. So for any* $x, y \in Y^h$*,*

$d_{Y^h}(x, y) = \inf\{l_{Y^h}(\alpha) \,|\, \alpha : [0, 1] \to Y^h$ *with* $\alpha(0) = x, \alpha(1) = y\}$*. Then we have,*

1. For $(x, t), (y, t) \in Y \times \{t\}$*,* $d_{Y,t}((x, t), (y, t)) = e^{-t} d(x, y)$*, where* $d_{Y,t}$ *is the induced path metric on* $Y \times \{t\}$*. Paths joining* (x, t) *and* (y, t) *that lie in* $Y \times [0, \infty)$ *are called horizontal paths.*

2. For $t, s \in [0, \infty)$ *and any* $x \in Y$*,* $d_h((x, t), (x, s)) = |t - s|$*. Paths joining such elements are called vertical paths.*

In general, for $x, y \in Y^h$*,* $d_h(x, y)$ *is the path metric induced by these vertical and horizontal paths.*

Definition 2.3.6. [30] ***Relatively hyperbolic space:*** *Let* X *be a geodesic metric space and* $\mathcal{A}$ *be a set of mutually disjoint subsets. For each* $A \in \mathcal{A}$*, we attach a hyperbolic cone* A^h *to* A *by identifying* $(x, 0)$ *with* x *for all* $x \in A$*. This space is denoted by* $X^h = \mathcal{G}(X, \mathcal{A})$*.* X *is said to be* hyperbolic relative to $\mathcal{A}$ *in the sense of Gromov if* $\mathcal{G}(X, \mathcal{A})$ *is a complete hyperbolic space.*

Definition 2.3.7. [30] ***Relatively hyperbolic group****: Let* G *be a finitely generated group and* $\mathcal{H} = \{H_\alpha\}_{\alpha \in \Lambda}$ *be a collection of finitely generated subgroups. Let* Γ *be the Cayley graph of* G *and let* $H_{(g,\alpha)}$ *be the sub-*

graph corresponding to the left coset gH_α *in* Γ. *We denote it by* $\Gamma^h = \mathcal{G}(\Gamma, \{H_{(g,\alpha)}\}_{\alpha\in\Lambda, g\in G})$. *G is said to be* hyperbolic relative to $\mathcal{H}$ *in the sense of Gromov, if* Γ^h *is a complete hyperbolic metric space.*

Terminology:

1. For a geodesic metric space (X, d), let X^h denote the metric space with hyperbolic cones attached to the collection of horosphere-like sets. Then for $x, y \in X^h$, $d_{X^h}(x, y)$ denotes the distance in the path metric of X^h. For any two subsets $A, B \subset X^h$, we denote the Hausdorff distance between them by $\mathrm{Hd}_{X^h}(A, B)$.

2. For $C \geq 0$, $N^h_C(Z)$ will denote a C-neighbourhood of a subset Z of (X^h, d_{X^h}).

3. A geodesic (resp. quasigeodesic) in X^h is called a **hyperbolic geodesic** (resp. **hyperbolic quasigeodesic**).

4. Let $\hat{\alpha}$ be an electric quasigeodesic without backtracking in $\widehat{X}$. For each A_α penetrated by $\hat{\alpha}$, let x, y be the entry and exit points of $\hat{\alpha}$, respectively. We join x and y by a geodesic in A^h_α. This gives a path in X^h and we call it an **electro-ambient quasigeodesic**. This path is, in fact, a quasigeodesic in X^h.

5. The electro-ambient quasigeodesic corresponding to an electric geodesic $\hat{\alpha}$ is always denoted by α.

6. Let G be hyperbolic relative to a collection of subgroups $\{H_\alpha\}$. Let Γ denote a Cayley graph of G. Then H_α and their conjugates are called **parabolic subgroups**. In Γ^h, each hyperbolic cone has a single limit point in $\partial\Gamma^h$ and it is called a **parabolic limit point**.

Remark 1. *Suppose a metric space X is strongly hyperbolic relative to a collection of subsets* $\mathcal{A}$, *then the space obtained by coning off the hyperbolic* *cones,* $\mathcal{E}(\mathcal{G}(X,\mathcal{A}),\mathcal{A}^h)$*, is quasiisometric to* $\mathcal{E}(X,\mathcal{A})$*.* $\mathcal{E}(X,\mathcal{A})$ *is isometrically embedded in* $\mathcal{E}(\mathcal{G}(X,\mathcal{A}),\mathcal{A}^h)$ *and* $\mathcal{E}(\mathcal{G}(X,\mathcal{A}),\mathcal{A}^h)$ *lies in a 1-neighbourhood of the image of* $\mathcal{E}(X,\mathcal{A})$*.*

Next result says that a relatively hyperbolic geodesic metric space X is properly embedded in X^h.

Lemma 2.3.8. [46, Lemma 1.2.19] *Let* X *be a geodesic metric space hyperbolic relative to a collection of uniformly* ϵ*-separated, uniformly properly embedded closed subsets, in the sense of Gromov. Then* X *is properly embedded in* X^h *i.e., for all* $M > 0$*, there exists* $N = N(M)$ *such that* $d_{X^h}(i(x), i(y)) \leq M$ *implies* $d(x,y) \leq N$*, for every* $x, y \in X$*. Here* $i : X \to X^h$ *is the inclusion map.*

Using Lemma 2.3.8, we prove the following result.

Lemma 2.3.9. *Let* X *be a geodesic metric space hyperbolic relative to a collection of uniformly* ϵ*-separated, uniformly properly embedded closed subsets* $\mathcal{A} = \{A_\alpha\}_{\alpha\in\Lambda}$*, in the sense of Gromov. Let* γ *be a geodesic ray in* X^h *such that* $\gamma(\infty)$ *is not a parabolic limit point. Then for any* $R > 0$*, if* $x \in X$ *such that* $x \in N^h_R(\gamma)$*, then there exists* $R_1 = R_1(R)$ *such that* $x \in N_{R_1}(\gamma \cap X)$*.*

Proof. Let $y \in \gamma$ such that $d_{X^h}(x, y) \leq R$. If $y \in \gamma \cap X$, by Lemma 2.3.8, there exists $N_1 = N(R)$ such that $d_X(x, y) \leq N_1$. Now, suppose $y \in \gamma \cap A^h_\alpha$, for some $\alpha \in \Lambda$. Let γ_1 denote the geodesic segment $\gamma|_{[a,b]}$, where a denotes the entry point of γ into A_α and b denotes the exit point of γ from A_α. Let $t \in [0,\infty)$ such that for $(a,t),(b,t) \in A_\alpha \times \{t\}$, $d_{h,t}((a,t),(b,t)) = e^{-t}d_{A_\alpha}(a,b) = 1$, where $d_{h,t}$ is the induced path metric on $A_\alpha \times \{t\}$.

Then, $d_{A_\alpha}(a,b) = e^t$ and $t = \ln d_{A_\alpha}(a,b)$. Let λ_1 and λ_2 denote the vertical paths in A^h_α joining $(a,0)$ to (a,t) and $(b,0)$ to (b,t) respectively. Let λ_0 denote the horizontal path in A^h_α joining (a,t) to (b,t). The path $\lambda = \lambda_1 * \lambda_0 * \lambda_2$ is a quasigeodesic in A^h_α and by stability of quasigeodesics, there exists $K_1 > 0$ such that $\mathrm{Hd}_{X^h}(\gamma_1, \lambda) \leq K_1$. Since $y \in \gamma_1$, there exists $z \in \lambda$ such that $d_{X^h}(y, z) \leq K_1$ and we have, $d_{X^h}(x, z) \leq R + K_1$. But length of the quasigeodesic λ is $2t + 1$ and clearly, $t \leq R + K_1$ and $d_{X^h}((a, 0), z) \leq t + 1 \leq R + K_1 + 1$. Thus, $d_{X^h}((a, 0), x) \leq 2(R + K_1) + 1$. By Lemma 2.3.8, there exists $N_2 = N(2(R + K_1) + 1)$ such that $d_X(x, y) \leq N_2$.

For $R_1 = \max\{N_1, N_2\}$, we have $x \in N_{R_1}(\gamma)$. □

Lemma 2.3.10. [46, Lemma 1.2.31] *Let $K \geq 1$, $\lambda \geq 0$, $\epsilon > 0$, $r \geq 0$. Suppose X_1, X_2 are geodesic spaces and $\mathcal{H}_{X_1}, \mathcal{H}_{X_2}$ are collections of ϵ-separated and intrinsically geodesic closed subspaces of X_1, X_2 respectively. Let $\phi : X_1 \to X_2$ be a (K, λ)-quasiisometry such that for each $H_1 \in \mathcal{H}_{X_1}$, there exists $H_2 \in \mathcal{H}_{X_2}$ such that* $\mathrm{Hd}(\phi(H_1), H_2) \leq r$ *in X_2 and* $\mathrm{Hd}(\phi^{-1}(H_2), H_1) \leq r$ *in X_1. Then $\phi : X_1 \to X_2$ induces a (K^h, λ^h)-quasiisometry $\phi^h : X_1^h \to X_2^h$, for some $K^h \geq 1$, $\lambda^h \geq 0$.*

Definition 2.3.11. [43] ***Electric projection:*** *Let Y be a space hyperbolic relative to the collection $\{A_\alpha\}_{\alpha\in\Lambda}$. Let $i : Y^h \to \mathcal{E}(\mathcal{G}(Y, \mathcal{A}), \mathcal{A}^h)$ be the inclusion map. we identify $\mathcal{E}(\mathcal{G}(Y, \mathcal{A}), \mathcal{A}^h)$ with $\widehat{Y}$. Let $\hat{\alpha}$ be an electric geodesic in $\widehat{Y}$ and α be the corresponding electro-ambient quasigeodesic. Let π_α be a nearest point projection from Y^h onto α.* Electric projection *is the map $\hat{\pi}_{\hat{\alpha}} : \widehat{Y} \to \hat{\alpha}$ given by the following: For $x \in Y$, $\hat{\pi}_{\hat{\alpha}}(x) = i(\pi_\alpha(x))$. If x is a cone point of a horosphere like set $A_\beta \in \mathcal{A}$, choose some $z \in A_\beta$ and define $\hat{\pi}_{\hat{\alpha}}(x) = i(\pi_\alpha(z))$.*

Lemma 2.3.12. [43, Lemma 1.16] *Let Y be hyperbolic relative to $\mathcal{A}$. There exists a constant $D_{2.3.12} > 0$, $C_{2.2.21}$ such that for any $A \in \mathcal{A}$ and $x, y \in A$ and a geodesic $\hat{\alpha}$ in $\widehat{Y}$, we have $d_{\widehat{Y}}(i(\pi_\alpha(x)), i(\pi_\alpha(y)) \leq D_{2.3.12}$.*

This implies that the electric projection is coarsely well-defined.

Lemma 2.3.13. [43, Lemma 1.17] *Let Y be hyperbolic relative to the collection $\mathcal{A}_Y$. For $\widehat{Y}$, there exists $D_{2.3.13} > 0$ (depending on the hyperbolic-*

ity constant of $\widehat{Y}$*) such that for all* $x, y \in \widehat{Y}$ *and an electric geodesic* $\hat{\alpha}$*,* $d_{\widehat{Y}}(\hat{\pi}_{\hat{\alpha}}(x), \hat{\pi}_{\hat{\alpha}}(y)) \leq D_{2.3.13} d_{\widehat{Y}}(x, y) + D_{2.3.13}$.

So, the electric projection is distance decreasing. The following lemma says that electric projections and quasiisometries 'almost commute'.

Lemma 2.3.14. [43, Lemma 1.18] *Let* Y_1 *and* Y_2 *be metric spaces hyperbolic relative to the collections* $\mathcal{A}_{Y_1}$ *and* $\mathcal{A}_{Y_2}$ *respectively and let* $\phi : Y_1 \to Y_2$ *be a strictly type-preserving quasiisometry. Let* $\hat{\mu}_1$ *be a quasigeodesic in* $\widehat{Y}_1$ *joining* a, b *and let* $\hat{\phi} : \widehat{Y}_1 \to \widehat{Y}_2$ *be the induced quasiisometry. Let* $\hat{\mu}_2$ *be a quasigeodesic in* $\widehat{Y}_2$ *joining* $\hat{\phi}(a)$ *and* $\hat{\phi}(b)$*. If* $p \in \widehat{Y}_1$*, then* $d_{\widehat{Y}_2}(\hat{\pi}_{\hat{\mu}_2}(\hat{\phi}(p)), \hat{\phi}(\hat{\pi}_{\hat{\mu}_1}(y)) \leq D_{2.3.14}$*, for some constant* $D_{2.3.14} > 0$.

The above two lemmas also hold for (quasi) geodesic rays. Now, we have the following theorem due to Bowditch, giving the equivalence between the two definitions of relative hyperbolicity:

Theorem 2.3.15. [16] *The following are equivalent:*

1. X is hyperbolic relative to the collection of uniformly separated subsets $\mathcal{A}$ *in* X*.*

2. X is hyperbolic relative to the collection of uniformly separated subsets $\mathcal{A}$ *in* X *in the sense of Gromov.*

3. X^h *is hyperbolic relative to the collection* $\mathcal{A}^h$*.*

Boundary of relatively hyperbolic groups:

For a proper hyperbolic metric space (Y, d), we can associate a topological space to it, i.e., its Gromov boundary ∂Y. Bowditch generalised the Gromov boundary to the context of relatively hyperbolic groups [16].

Definition 2.3.16. ***Bowditch boundary:*** *Suppose* X *is a metric space hyperbolic relative to a collection of subsets* $\{A_\alpha\}_{\alpha \in \Lambda}$*. Then the* Bowditch boundary *(or relative hyperbolic boundary) of* X *with respect to* $\{A_\alpha\}_{\alpha\in\Lambda}$ *is the boundary of* X^h*, and it is denoted by* ∂X^h*.*

So, for a group G hyperbolic relative to a collection of subgroups $\mathcal{H}$, its boundary is the boundary of Γ^h, where Γ is a Cayley graph of G.

Chapter 3

Palindromic width of graph of groups

In this chapter we prove the following two theorems:

Theorem 3.0.1. *Let G be a group and let A and B be proper isomorphic subgroups of G and $\phi : A \to B$ be an isomorphism. The HNN extension*

$$G_* = \langle G, t \mid t^{-1}at = \phi(a),\ a \in A\rangle$$

of G with associated subgroups A and B has infinite palindromic width with respect to the generating set $G \cup \{t, t^{-1}\}$.

Theorem 3.0.2. *Let $G = A *_C B$ be the free product of two groups A and B with amalgamated proper subgroup C and $|A : C| \geq 3$, $|B : C| \geq 2$. Then $pw(G, A \cup B)$ is infinite.*

The following is the layout of this chapter: In the first section, we recall the required definitions and results. In the second section, we calculate the palindromic width of an HNN Extension and in Section 3.4, we calculate the palindromic width of amalgamated free products. In the final section, using results of Section 3.3 and Section 3.4, we show that the palindromic width of a graph of groups is infinite.

3.1 Preliminaries

Let G be a group with a set of generators S. A reduced word in the alphabet $S \cup S^{-1}$ is a **palindrome** if it reads the same forwards and backwards. The **palindromic length**, $l_{\mathcal{P}}(g)$, of an element g in G is the minimum number k such that g can be expressed as a product of k palindromes. The **palindromic width** of G with respect to S is denoted by $pw(G, S)$. When there is no confusion about the underlying generating set S, we simply denote the palindromic width with respect to S by $pw(G)$.

Definition 3.1.1. ***Quasi-homomorphism:*** *Let G be a group. A map $\Delta : G \to \mathbb{R}$ is a* quasi-homomorphism *if there exists a constant $c \geq 0$ such that for any $g, h \in G$,*

$$\Delta(gh) \leq \Delta(g) + \Delta(h) + c.$$

3.1.1 HNN Extensions

Let G be a group and A and B be proper isomorphic subgroups of G with the isomorphism $\phi : A \to B$. Then the HNN extension of G is

$$G_* = \langle G, t \mid t^{-1}at = \phi(a), a \in A \rangle.$$

Definition 3.1.2. ***Reduced sequence:*** *A sequence $g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, \ldots, t^{\epsilon_n}, g_n$, $n \geq 0$, is said to be* reduced *if it does not contain subsequences of the form t^{-1}, g_i, t with $g_i \in A$ or t, g_i, t^{-1} with $g_i \in B$.*

Lemma 3.1.3 (**Britton's Lemma**). *If the sequence $g_0, t^{\epsilon_1}, \ldots, t^{\epsilon_n}, g_n$ and $n \geq 1$, then $g_0 t^{\epsilon_1} \ldots t^{\epsilon_n} g_n \neq 1$ in G_*.*

If a sequence $g_0, t^{\epsilon_1}, g_1, t^{\epsilon_2}, \ldots, g_{n-1}, t^{\epsilon_n}, g_n$ is reduced and $n \geq 1$, then $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \ldots g_{n-1} t^{\epsilon_n} g_n$ is called a **reduced word**.

Such a representation of a group element of an HNN extension is not unique but the following lemma holds:

Lemma 3.1.4. *Let $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \ldots g_{n-1} t^{\epsilon_n} g_n$, $h = h_0 t^{\theta_1} h_1 t^{\theta_2} \ldots h_{m-1} t^{\theta_m} h_m$ be reduced words, and suppose $g = h$ in G_*. Then $m = n$ and $\epsilon_i = \theta_i$ for $i = 1, \ldots, n$.*

Proof. Proof follows from [7, Lemma 3]. □

Definition 3.1.5. ***Signature:*** *The* signature *of $g \in G_*$ is the sequence $sqn(g) = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$, $\epsilon_i \in \{1, -1\}$ for $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \ldots g_{n-1} t^{\epsilon_n} g_n$.*

By Lemma 3.1.4, the signature of any $g \in G_*$ is unique, irrespective of the choice of the reduced word.

Let σ=$(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ be a signature. Then the length of the signature, $|\sigma| = n$. And the inverse signature, σ^{-1}=$(-\epsilon_n, -\epsilon_{n-1}, \ldots, -\epsilon_1)$. So, $sqn(g^{-1}) = (sqn(g))^{-1}$. Product of two signatures σ and τ, $\sigma\tau$, is obtained by writing τ after σ.

Suppose $\sigma = \sigma_1 \rho$ and $\tau = \rho^{-1} \tau_1$ with $|\rho| = r$, then we can define an r-product,

$$\sigma[r]\tau = \sigma_1 \tau_1.$$

The following lemma is immediate from the above notions.

Lemma 3.1.6. [7, Lemma 4] *For any $g, h \in G_*$, there exists an integer $r \geq 0$ such that $sqn(gh) = sqn(g)[r]sqn(h)$, with $sqn(g) = \sigma_1 \rho$ and $sqn(h) = \rho^{-1} \tau_1$ and $|\rho| = r$.*

A reduced expression is called *positive (resp. negative)* if all exponents ϵ_i are positive (resp. negative). Further, if it is either positive or negative then the reduced expression is called *homogeneous*.

3.1.2 Amalgamated free products

Let $A = \langle a_1, \ldots | R_1, \ldots \rangle$ and $B = \langle b_1, \ldots | S_1, \ldots \rangle$ be groups. Let $C_1 \subset A$ and $C_2 \subset B$ be subgroups such that there exists an isomorphism $\phi : C_1 \to C_2$. Then the *free product of A and B*, amalgamating the subgroups C_1 and C_2 by the isomorphism ϕ is the group

$$G = \langle A, B \,|\, c = \phi(c), c \in C_1 \rangle.$$

We can view G as the quotient of the free product $A * B$ by the normal subgroup generated by $\{c\phi(c)^{-1} | c \in C_1\}$. The subgroups A and B are called factors of G, and since C_1 and C_2 are identified in G, we will denote them both by C.

Definition 3.1.7. ***Reduced sequence:*** *A sequence $x_1, \ldots, x_n$, $n \geq 0$, is said to be* reduced *if*

(i) Each x_i is in one of the factors.

(ii) Successive x_i, x_{i+1} come from different factors.

(iii) If $n > 1$, no x_i is in C.

(iv) If $n = 1$, $x_1 \neq 1$.

For the normal form of elements in free products with amalgamation, see for eg. [37], if $x_1, \ldots, x_n$ is a reduced sequence, $n \geq 1$, then the product $x_1 \ldots x_n \neq 1$ is in G and it is called a *reduced word*. Such a representation of a group element is not unique but the following proposition holds:

Proposition 3.1.8. *Let $g = x_1 \ldots x_n$ and $h = y_1 \ldots y_m$ be reduced words such that $g = h$ in G. Then $m = n$.*

Proof. Since $g = h$, we have, $1 = x_1 \ldots x_n y_m^{-1} \ldots y_2^{-1} y_1^{-1}$. Since g and h are reduced, we require $x_n y_m^{-1}$ to belong to C. To reduce it further, we need $x_{n-1} x_n y_m^{-1} y_{m-1}^{-1}$ to be in C and so on. Hence, $m = n$. □

Definition 3.1.9. *Let $g = x_1 \dots x_n$ be a reduced word of $g \in G$. The elements x_k are said to be **syllables** of g. Then the **length** of g is the number of syllables of g and it is denoted by $l(g)$. Here, for $g = x_1 \dots x_n$, $l(g) = n$.*

3.2 Palindromic width for HNN extensions of groups

Let σ=$(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ be the signature of an element $g \in G_*$. We define,

$p_k(g)$ = number of $+1, +1, \dots, +1$ sections of length k,

$m_k(g)$ = number of $-1, -1, \dots, -1$ sections of length k,

$d_k(g)$= $p_k(g) - m_k(g)$,

$r_k(g)$= remainder of $d_k(g)$ divided by 2, and,

$$\Delta(g) = \sum_{k=1}^{\infty} r_k(g).$$

Clearly, $p_k(g^{-1}) = m_k(g)$ and so, $d_k(g^{-1}) + d_k(g) = 0$ for all $g \in G_*$.

We now prove that Δ is a quasi-homomorphism. We borrow the following notation from [7]: if f_k, g_k, f'_k and g'_k are functions indexed by a parameter k, then for any $n \in \mathbb{N}$,

$$f_k =_n g_k$$

means that $f_k = g_k$ for all but at most n many k's.

If $f_k =_n g_k$ and $g_k =_m f'_k$, for some $m \in \mathbb{N}$, then $f_k =_{n+m} f'_k$. Also, if $f_k =_n g_k$ and $f'_k =_m g'_k$ then, $f_k + f'_k =_{n+m} g_k + g'_k$.

The following lemma and proof are identical to [7, Lemma 9].

Lemma 3.2.1. *For any elements $g, h \in G_*$, $\Delta(gh) \leq \Delta(g) + \Delta(h) + 6$, i.e. Δ is a quasi-homomorphism.*

Proof. By Lemma 3.1.6, there exists an integer $r \geq 0$ such that $sqn(gh) =$

$sqn(g)[r]sqn(h)$, with $sqn(g) = \sigma_1\rho$ and $sqn(h) = \rho^{-1}\tau_1$ and $|\rho| = r$.

Clearly, $p_k(g) =_1 p_k(\sigma_1) + p_k(\rho)$ and $p_k(h) =_1 p_k(\rho^{-1}) + p_k(\tau_1)$.

We also have $sqn(gh) = \sigma_1\tau_1$. So, $p_k(gh) =_1 p_k(\sigma_1) + p_k(\tau_1)$. Then,

$$\begin{aligned} p_k(gh) &=_1 p_k(\sigma_1) + p_k(\tau_1) \\ &=_3 p_k(g) - p_k(\rho) + p_k(h) + p_k(\rho) \\ &=_3 p_k(g) + p_k(h). \end{aligned}$$

Similarly, $m_k(gh) =_3 m_k(g) + m_k(h)$ and so, $d_k(gh) =_6 d_k(g) + d_k(h)$.

Thus, $\Delta(gh) \leq \Delta(g) + \Delta(h) + 6$. □

Definition 3.2.2. ***Group-palindrome:*** *Let* $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \ldots g_{n-1} t^{\epsilon_{n-1}} g_n$ *be a reduced element in* G_*. *Put*

$$\bar{g} = g_n t^{\epsilon_{n-1}} g_{n-1} t^{\epsilon_{n-2}} \ldots g_1 t^{\epsilon_1} g_0.$$

We say g *is a* group-palindrome *if* $\bar{g} = g$ *and* $\bar{g}$ *depends on the reduced form.*

Lemma 3.2.3. *A group-palindrome* $g \in G_*$ *has the form*

$$g = \begin{cases} g_0 t^{\epsilon_1} g_1 \ldots g_{k-1} t^{\epsilon_k} g'_k t^{\epsilon_k} g_{k-1} \ldots g_1 t^{\epsilon_1} g_0, & \text{if } |sqn(g)| = 2k, \\ g_0 t^{\epsilon_1} g_1 \ldots t^{\epsilon_k} g_k t^{\epsilon_{k+1}} g'_k t^{\epsilon_k} \ldots g_1 t^{\epsilon_1} g_0, & \text{if } |sqn(g)| = 2k+1, \end{cases}$$

where $g'_k = x g_k$ *where* $x \in A \cup B$.

Proof. Let $g \in G_*$ is a group-palindrome.

CASE 1: $|sqn(g)| = 2k + 1$.

Let $g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \ldots g_{2k} t^{\epsilon_{2k+1}} g_{2k+1}$. We know, $g = \bar{g}$. Then, $g\bar{g}^{-1} = 1$,

$$\text{i.e., } g_0 t^{\epsilon_1} g_1 \ldots t^{\epsilon_{2k+1}} g_{2k+1} g_0^{-1} t^{-\epsilon_1} g_1^{-1} \ldots t^{-\epsilon_{2k+1}} g_{2k+1}^{-1} = 1. \quad (3.1)$$

Left side is reducible. So we have, $g_{2k+1} g_0^{-1} = x_0$, where $x_0 \in A$ (or $x_0 \in$

B) such that $t^{\epsilon_{2k+1}}x_0t^{-\epsilon_1} = y_0$, with $y_0 \in B$ (or $y_0 \in A$) and $\epsilon_{2k+1} = \epsilon_1 = -1$ (or 1). Substituting this in (3.1),

$$g_0t^{\epsilon_1}g_1 \ldots t^{\epsilon_{2k}}g_{2k}y_0g_1^{-1}t^{-\epsilon_2} \ldots g_{2k+1}^{-1} = 1. \tag{3.2}$$

Since $y_0 \in B$ (or $y_0 \in A$), $g_{2k}y_0{g_1}^{-1} = y_1$, $y_1 \in B$ (or $y_1 \in A$) such that $t^{\epsilon_{2k}}y_1t^{-\epsilon_2} = x_1$, where $x_1 \in A$ (or $x_1 \in B$) and $\epsilon_{2k} = \epsilon_2 = 1$ (or -1). Substituting this in (3.2),

$$g_0t^{\epsilon_1}g_1 \ldots t^{\epsilon_{2k-1}}g_{2k-1}x_1g_2^{-1}t^{-\epsilon_2} \ldots g_{2k+1}^{-1} = 1. \tag{3.3}$$

Since $x_1 \in A$ (or $x_1 \in B$), $g_{2k-1}x_1{g_2}^{-1} = x_2$, $x_2 \in A$ (or $x_2 \in B$) such that $t^{\epsilon_{2k-1}}x_2t^{-\epsilon_3} = y_2$, where $y_2 \in B$ (or $y_2 \in A$) and $\epsilon_{2k-1} = \epsilon_2 = -1$ (or 1). In general, $g_{2k-i}x_ig_{i+1}^{-1} = x_{i+1}$, $x_i, x_{i+1} \in A$ (or B) such that $t^{\epsilon_{2k-i}}x_{i+1}t^{-\epsilon_{i+2}} = y_{i+1}$, where $y_{i+1} \in B$ (or A) and $\epsilon_{2k-i} = \epsilon_{i+2}$, for $0 \leq i \leq k-1$.

In the expression $g = g_0t^{\epsilon_1}g_1t^{\epsilon_2} \ldots g_kt^{\epsilon_{k+1}}g_{k+1}t^{\epsilon_{k+2}}g_{k+2} \ldots t^{\epsilon_{2k+1}}g_{2k+1}$,

we put $g_{2k+1} = x_0g_0$, and for $0 \leq i \leq k-1$, $g_{2k-i} = x_{i+1}g_{i+1}x_i^{-1}$ and $\epsilon_{2k-i} = \epsilon_{i+2}$. Then,

$g = g_0 \ldots g_kt^{\epsilon_{k+1}}x_kg_kx_{k-1}^{-1}t^{\epsilon_k}y_{k-1}g_{k-1}y_{k-2}^{-1}t^{\epsilon_{k-1}} \ldots y_1g_1{y_0}^{-1}t^{\epsilon_1}x_0g_0$.

We know $t^{\epsilon_{i+2}}x_{i+1} = y_{i+1}t^{\epsilon_{i+2}}$ (or $t^{\epsilon_{i+2}}y_{i+1} = x_{i+1}t^{\epsilon_{i+2}}$) for $-1 \leq i \leq k-2$; this implies

$g = g_0 \ldots g_kt^{\epsilon_{k+1}}x_kg_kx_{k-1}^{-1}x_{k-1}t^{\epsilon_k}g_{k-1}y_{k-2}^{-1}y_{k-2}t^{\epsilon_{k-1}} \ldots x_1t^{\epsilon_2}g_1{y_0}^{-1}y_0t^{\epsilon_1}g_0$,

and so, $g = g_0 \ldots g_kt^{\epsilon_{k+1}}x_kg_kt^{\epsilon_k}g_{k-1}t^{\epsilon_{k-1}} \ldots t^{\epsilon_2}g_1t^{\epsilon_1}g_0$.

Therefore, $g = g_0 \ldots g_kt^{\epsilon_{k+1}}g_k't^{\epsilon_k}g_{k-1}t^{\epsilon_{k-1}}g_{k-2} \ldots g_1t^{\epsilon_1}g_0$; where $g_k' = x_kg_k$.

CASE 2: $|sqn(g)| = 2k$.

Let $g = g_0t^{\epsilon_1}g_1t^{\epsilon_2} \ldots g_{2k-1}t^{\epsilon_{2k}}g_{2k}$. We know, $g = \bar{g}$. This implies,

$$g_0t^{\epsilon_1}g_1 \ldots t^{\epsilon_{2k}}g_{2k}g_0^{-1}t^{-\epsilon_1}g_1^{-1} \ldots t^{-\epsilon_{2k}}g_{2k}^{-1} = 1. \tag{3.4}$$

Left side is reducible. So we have, $g_{2k}g_0^{-1} = x_0$, where $x_0 \in A$ (or $x_0 \in B$) such that $t^{\epsilon_{2k}}x_0t^{-\epsilon_1} = y_0$, with $y_0 \in B$ (or $y_0 \in A$) and $\epsilon_{2k} = \epsilon_1 = -1$ (or 1). Substituting this in (3.4),

$$g_0t^{\epsilon_1}g_1 \dots t^{\epsilon_{2k-1}}g_{2k-1}y_0g_1^{-1}t^{-\epsilon_2} \dots t^{-\epsilon_{2k}}g_{2k}^{-1} = 1. \tag{3.5}$$

Since $y_0 \in B$ (or $y_0 \in A$), $g_{2k-1}y_0g_1{}^{-1} = y_1$, $y_1 \in B$ (or $y_1 \in A$) such that $t^{\epsilon_{2k-1}}y_1t^{-\epsilon_2} = x_1$, where $x_1 \in A$ (or $x_1 \in B$) and $\epsilon_{2k-1} = \epsilon_2$. Substituting this in (3.5),

$$g_0t^{\epsilon_1}g_1 \dots t^{\epsilon_{2k-2}}g_{2k-2}x_1g_2^{-1}t^{-\epsilon_3} \dots t^{-\epsilon_{2k}}g_{2k}^{-1} = 1. \tag{3.6}$$

Similarly, since $x_1 \in A$ (or $x_1 \in B$), $g_{2k-2}x_1g_2{}^{-1} = x_2$, $x_2 \in A$ (or $x_2 \in B$) such that $t^{\epsilon_{2k-2}}x_2t^{-\epsilon_3} = y_2$, where $y_2 \in B$ (or $y_2 \in A$) and $\epsilon_{2k-2} = \epsilon_3$. Substituting this in (3.6),

$$g_0t^{\epsilon_1}g_1 \dots t^{\epsilon_{2k-3}}g_{2k-3}y_2g_3^{-1}t^{-\epsilon_4} \dots t^{-\epsilon_{2k}}g_{2k}^{-1} = 1. \tag{3.7}$$

In general, we get $g_{2k-i}x_{i-1}g_i^{-1} = x_i$, $x_{i-1}, x_i \in A$ (or B) such that $t^{\epsilon_{2k-i}}x_it^{-\epsilon_{i+1}} = y_i$, where $y_i \in B$ (or A) and $\epsilon_{2k-i} = \epsilon_{i+1}$, where $1 \leq i \leq k$.

In $g = g_0t^{\epsilon_1}g_1 \dots t^{\epsilon_{2k}}g_{2k}$, we put $g_{2k} = x_0g_0$ and for $1 \leq i \leq k$, $g_{2k-i} = x_ig_ix_{i-1}^{-1}$ and $\epsilon_{2k-i} = \epsilon_{i+1}$. This gives

$$g = g_0t^{\epsilon_1} \dots t^{\epsilon_k}x_kg_kx_{k-1}^{-1}t^{\epsilon_k}y_{k-1}g_{k-1}y_{k-2}^{-1}t^{\epsilon_{k-1}} \dots g_2x_1^{-1}t^{\epsilon_2}y_1g_2y_0^{-1}t^{\epsilon_1}x_0g_0.$$

Putting $t^{\epsilon_{i+1}}x_i = y_it^{\epsilon_{i+1}}$ for $-1 \leq i \leq k-1$, we get

$$g = g_0 \dots t^{\epsilon_k}x_kg_kx_{k-1}^{-1}x_{k-1}t^{\epsilon_k}g_{k-1}y_{k-2}^{-1}y_{k-2}t^{\epsilon_{k-1}} \dots x_1^{-1}x_1t^{\epsilon_2}g_2y_0^{-1}y_0t^{\epsilon_1}g_0.$$

This implies, $g = g_0t^{\epsilon_1}g_1 \dots t^{\epsilon_k}x_kg_kt^{\epsilon_k}g_{k-1}t^{\epsilon_{k-1}} \dots t^{\epsilon_3}g_2t^{\epsilon_2}g_2t^{\epsilon_1}g_0$. Thus, $g = g_0 \dots g_k't^{\epsilon_{k-1}}g_{k-1} \dots g_1t^{\epsilon_1}g_0$, where $g_k' = x_kg_k$. □

Lemma 3.2.4. *Let $g \in G_*$ be a product of k group-palindromes, say $g = p_1 p_2 \ldots p_k$. Then, $\Delta(g) \leq 7k - 6$.*

Proof. Let p be a group-palindrome in G_* of non-zero length. Then p can be represented as $p = uv\bar{u}$, where v is the maximal homogeneous palindromic sub-word in p and $\bar{u}$ is u written in reverse.

For example, if $p = g_0 t^{\epsilon_1} g_1 \ldots t^{-1} g_i t g_{i+1} t \ldots t \bar{g}_{i+1} t \bar{g}_i t^{-1} \bar{g}_{i-1} \ldots \bar{g}_1 t^{\epsilon_1} \bar{g}_0$, then $u = g_0 t^{\epsilon_1} g_1 \ldots t^{-1}, v = g_i t g_{i+1} t \ldots t \bar{g}_{i+1} t \bar{g}_i, \bar{u} = t^{-1} \bar{g}_{i-1} \ldots \bar{g}_1 t^{\epsilon_1} \bar{g}_0$.

Then for every k, $d_k(u) = d_k(\bar{u})$. As v is homogeneous, if k' is the length of $sqn(v)$, then $p_{k'}(p) = 2p_{k'}(u) + p_{k'}(v)$, or $m_{k'}(p) = 2m_{k'}(u) + m_{k'}(v)$. For all other k, $p_k(p) = 2p_k(u)$ and $m_k(p) = 2m_k(u)$. Therefore, $r_{k'}(p) = 1$, and $r_k(p) = 0$ for all other k. Thus, $\Delta(p) = 1$. If $p \in G$, then $\Delta(p) = 0$. So, $\Delta(p) \leq 1$. If $g \in G_*$ is a product of k group-palindromes, say $g = p_1 p_2 \ldots p_k$, then

$$\Delta(g) = \Delta(p_1 p_2 \ldots p_k) \leq \Delta(p_1) + \Delta(p_2) + \cdots + \Delta(p_k) + 6(k-1) \leq 7k - 6. \quad (3.8)$$

This completes the proof. □

3.2.1 Proof of Theorem 3.0.1

Now we prove that Δ is not bounded from above. For that purpose, we produce the following sequence of reduced words $\{a_i\}$, for which $\Delta(a_i)$ is increasing. Let $a_1 = g_0 t g_1 t^{-1} g_2 t g_3$. Then $d_1(a_1) = 1$, so $\Delta(a_1) = 1$.

For $a_2 = g_0 t g_1 t^{-1} g_2 t g_3 t^{-1} g_4 t^{-1} g_5 t g_6 t g_7 t^{-1} g_8 t^{-1} g_9$, $d_1(a_2) = 1$, $d_2(a_2) = -1$, so, $\Delta(a_2) = 2$.

$$a_3 = g_0 t g_1 t^{-1} g_2 t g_3 t^{-1} g_4 t^{-1} g_5 t g_6 t g_7 t^{-1} \cdots t^{-1} g_{13} t^{-1} g_{14} t^{-1} g_{15} t g_{16} t g_{17} t g_{18}.$$

Then, $d_1(a_3) = 1, d_2(a_3) = -1, d_3(a_3) = 1$, so, $\Delta(a_3) = 3$.

For each $a_i = g_0 t g_1 t^{-1} g_2 t \ldots$, we have $g_j \in G$ and since a_i is reduced, for subwords of the form $t^\epsilon g_i t^{-\epsilon}$, $g_i \notin A$ if $\epsilon = -1$ and $g_i \notin B$ if $\epsilon = 1$.

Given a_i, we construct a_{i+1} by attaching a segment with signature of length $3(i+1)$. In general,

$a_n = g_0 t g_1 t^{-1} g_2 \cdots g_{N-2n} t^{\mp 1} \cdots g_{N-n-1} t^{\mp 1} g_{N-n} t^{\pm 1} g_{N-n+1} t^{\pm 1} \cdots g_{N-1} t^{\pm 1} g_N$;

where $N = \frac{3n(n+1)}{2}$ and

$$sqn(a_n) = (1, -1, 1, -1, -1, 1, 1, 1, -1, -1, \ldots, \underbrace{\pm 1, \ldots, \pm 1}_{n \text{ times}}, \underbrace{\mp 1, \ldots, \mp 1}_{n \text{ times}}, \underbrace{\pm 1, \ldots, \pm 1}_{n \text{ times}})$$

Here, $\Delta(a_n) = n$. Then, by (3.8), we get that the palindromic width of G_* is infinite. This proves Theorem 3.0.1.

3.3 Palindromic width for amalgamated free products

We shall divide the proof of Theorem 3.0.2 into two cases.

3.3.1 Case 1

For a non-trivial $a \in A \cup B$ such that $CaC \neq Ca^{-1}C$, we shall prove the following:

Lemma 3.3.1. *Let $G = A *_C B$ be the free product of two groups A and B with amalgamated subgroup C. Let $|A : C| \geq 3$, $|B : C| \geq 2$ and there exists an element $a \in A \cup B$ for which $CaC \neq Ca^{-1}C$. Then $pw(G, \{A \cup B\})$ is infinite.*

To prove this, we shall use the quasi-homomorphism constructed in [21, 22]. We recall the construction here.

Quasi-homomorphisms

We recall the definition of a special form from [21].

Definition 3.3.2. ***Special form:*** *Let $a \in A$ such that $CaC \neq Ca^{-1}C$. Let $g \in G$, and $g = x_1x_2 \dots x_n$ be a reduced word representing it. Then the* special form *of g associated to this reduced word is obtained by replacing x_i by $ua^{\epsilon}u'$, whenever $x_i = ua^{\epsilon}u'$ for some $u, u' \in C$ and $\epsilon \in \{+1, -1\}$, in the following way:*

1. When $i = 1$, $x_1 = ua^{\epsilon}u'$, we write $g = ua^{\epsilon}x_2' \dots x_n$, where $x_2' = u'x_2$.

2. When $2 \leq i \leq n-1$, $x_i = ua^{\epsilon}u'$, we write $g = x_1x_2 \dots x_{i-1}'a^{\epsilon}x_{i+1}' \dots x_n$, where $x_{i-1}' = x_{i-1}u$ and $x_{i+1}' = u'x_{i+1}$.

3. When $i = n$ and $x_n = ua^{\epsilon}u'$, where $\epsilon \in \{+1, -1\}$ and $u, u' \in C$, we replace x_n by $g = x_1x_2 \dots x_{n-1}'a^{\epsilon}u'$, where $x_{n-1}' = x_{n-1}u$.

An *a-segment* of length $2k - 1$ is a segment of the reduced word of the following form

$$ax_1 \dots x_{2k-1}a,$$

where $x_j \neq a$ for $j = 1, \dots, 2k - 1$ such that the length of $x_1 \dots x_{2k-1}$ is $2k - 1$.

Similarly, an a^{-1}*-segment* of length $2k-1$ is a segment of the reduced word of the following form $a^{-1}x_1 \dots x_{2k-1}a^{-1}$, where $x_j \neq a^{-1}$ for $j = 1, \dots, 2k-1$ such that the length of $x_1 \dots x_{2k-1}$ is $2k - 1$.

For $g \in G$ expressed in special form, we define

$p_k(g)$ = number of a-segments of length $2k - 1$,

$m_k(g)$ = number of a^{-1}-segments of length $2k - 1$,

$d_k(g)$= $p_k(g) - m_k(g)$,

$r_k(g)$= remainder of $d_k(g)$ divided by 2, and

$$\Delta(g) = \sum_{k=1}^{\infty} r_k(g) \tag{3.9}$$

Clearly, $p_k(g^{-1}) = m_k(g)$ and so, $d_k(g^{-1}) + d_k(g) = 0$ for all $g \in$ G.

Lemma 3.3.3. Δ *is well-defined on special forms.*

Proof. Let $g \in G$ and $x_1 x_2 \dots x_n$ and $y_1 y_2 \dots y_n$ be two reduced forms of g.

Now, $x_1 x_2 \dots x_n = y_1 y_2 \dots y_n$ implies $x_1 \dots x_n y_n^{-1} \dots y_1^{-1} = 1$. Then $x_n y_n^{-1} = c_n \in C$. Further $x_{n-1} c_n y_{n-1}^{-1} = c_{n-1} \in C$ and so on. In general, for $1 \leq i \leq n$, $x_i c_{i+1} y_i^{-1} = c_i \in C$.

So, if $x_i = u a^{\epsilon} u'$ for some $u, u' \in C$ and $\epsilon \in \{+1, -1\}$, $u a^{\epsilon} u' c_{i+1} y_i^{-1} = c_i$. This gives $v a^{\epsilon} v' = y_i$, where $v = c_i^{-1} u \in C$ and $v' = u' c_{i+1} \in C$.

Thus, for any $k \in \mathbb{N}$, number of a^{ϵ} segments of length $2k-1$, for $\epsilon \in \{+1, -1\}$ is independent of the special form of $g \in G$. Thus, Δ is well-defined on special forms. □

We now prove that Δ is a quasi-homomorphism. Though this is same as the proof of [21, Lemma 1], we include it here for the sake of completion.

Lemma 3.3.4. *For any elements* $g, h \in G$, $\Delta(gh) \leq \Delta(g) + \Delta(h) + 9$, *i.e.* Δ *is a quasi-homomorphism.*

Proof. If g or h is in C or if g, h are in the same factor, i.e, in A or B, then this trivially holds. So we consider other possible cases. Let $g = g_1 g_2 \dots g_n$ and $h = h_1 h_2 \dots h_m$ be given in special forms.

Case 1: g_n and h_1 are in different factors. Then, $p_k(gh) =_1 p_k(g) + p_k(h)$ and $m_k(gh) =_1 m_k(g) + m_k(h)$. So, $d_k(gh) =_2 d_k(g) + d_k(h)$.

Case 2: $g_n h_1 \in A$ or B and $g_n h_1 \notin C$. Let $g_n h_1 \in A$ and $g_n h_1 = c_1 a^{\epsilon} c_2$, where $c_1, c_2 \in C$ and $\epsilon \in \{1, -1\}$. Then, $gh = g_1 \dots g'_{n-1} a^{\epsilon} h'_2 h_3 \dots h_m$, where $g'_{n-1} = g_{n-1} c_1$, $t = a^{\epsilon}$ and $h'_2 = c_2 h_2$. Suppose $g_n h_1$ is not of such form. Then, put $g'_{n-1} = g_{n-1}$, $t = g_n h_1$ and $h'_2 = h_2$. Let $\bar{g} = g_1 \dots g'_{n-1} t$ and $\bar{h} = h'_2 \dots h_m$. Clearly, t and h'_2 are different factors. So, this reduces to the Case 1, i.e, $d_k(\bar{g}\bar{h}) =_2 d_k(\bar{g}) + d_k(\bar{h})$. Suppose $d_k(\bar{g}) =_m d_k(g)$ and $d_k(\bar{h}) =_n d_k(h)$. Then, $d_k(gh) = d_k(\bar{g}\bar{h}) =_2 d_k(\bar{g}) + d_k(\bar{h}) =_{2+m+n} d_k(g) + d_k(h)$.

We consider the following possibilities: Let $h_1 = ca$, where $c \in C$. Then, $p_k(\bar{h}) =_1 p_k(h)$ and $m_k(\bar{h}) = m_k(h)$ and $n = 1$.

1. $g_n = ac', t = a$, for $c' \in C$; $p_k(\bar{g}) = p_k(g)$, $m_k(\bar{g}) = m_k(g)$, $m = 0$.
2. $g_n = ac', t = a^{-1}$, for $c' \in C$; $p_k(\bar{g}) =_1 p_k(g)$, $m_k(\bar{g}) =_1 m_k(g)$, $m = 2$.
3. $g_n = ac', t \neq a, a^{-1}$, for $c' \in C$; $p_k(\bar{g}) =_1 p_k(g)$, $m_k(\bar{g}) = m_k(g)$, $m = 1$.
4. $g_n = a^{-1}c', t = a$, for $c' \in C$; $p_k(\bar{g}) =_1 p_k(g)$, $m_k(\bar{g}) =_1 m_k(g)$, $m = 2$.
5. $g_n = a^{-1}c', t = a^{-1}$, for $c' \in C$; $p_k(\bar{g}) = p_k(g)$, $m_k(\bar{g}) = m_k(g)$, $m = 0$.
6. $g_n = a^{-1}c', t \neq a, a^{-1}$, for $c' \in C$; $p_k(\bar{g}) = p_k(g)$, $m_k(\bar{g}) =_1 m_k(g)$, $m = 1$.
7. $g_n \neq ac', a^{-1}c', t = a$, for any $c' \in C$; $p_k(\bar{g}) =_1 p_k(g)$, $m_k(\bar{g}) = m_k(g)$, $m = 1$.
8. $g_n \neq ac', a^{-1}c', t = a^{-1}$, for any $c' \in C$; $p_k(\bar{g}) = p_k(g)$, $m_k(\bar{g}) =_1 m_k(g)$, $m = 1$.
9. $g_n \neq ac', a^{-1}c', t \neq a, a^{-1}$, for any $c' \in C$; $p_k(\bar{g}) = p_k(g)$, $m_k(\bar{g}) = m_k(g)$, $m = 0$.

Let $h_1 = ca^{-1}$, for some $c \in C$. Then, $p_k(\bar{h}) = p_k(h)$ and $m_k(\bar{h}) = m_k(h)$. So, $n = 1$.Value of m is same as the above (1) to (9).

Let $h_1 \neq ca, ca^{-1}$, for any $c \in C$. Then, $p_k(\bar{h}) = p_k(h)$ and $m_k(\bar{h}) = m_k(h)$. Here, $n = 0$. Again, we have a repetition of values of m from above (1) to (9).

Case 3: $g_n h_1 = c \in C$. Then, $gh = g_1 g_2 \dots g_{n-1} c h_2 \dots h_m$. Put $\bar{g} = g_1 g_2 \dots g'_{n-1}$ and $\bar{h} = h_2 \dots h_m$, where $g'_{n-1} = g_{n-1}c$. Then, as in Case 1, we have $p_k(\bar{g}\bar{h}) =_1 p_k(\bar{g}) + p_k(\bar{h})$ and $m_k(\bar{g}\bar{h}) =_1 m_k(\bar{g}) + m_k(\bar{h})$ and so, $d_k(\bar{g}\bar{h}) =_2 d_k(\bar{g}) + d_k(\bar{h})$.

Now, if $gh = c$ for $c \in C$, then, $g = ch^{-1}$. In this case, we have $d_k(gh) = d_k(g) + d_k(h)$.

Suppose $gh = g_1 \dots g_{n-i} c h_{i+1} \dots h_m$, where $g_{n-i+1} \dots g_n h_1 \dots h_i = c$. Put $\bar{g} = g_1 \dots g_{n-i} c$ and $\bar{h} = h_{i+1} \dots h_m$.

Let $z = g_{n-i+1} \dots g_n$. Then, $h_1 \dots h_i = z^{-1}$. So, $d_k(gh) = d_k(\bar{g}\bar{h}) =_2 d_k(\bar{g}) + d_k(\bar{h})$. This reduces to Case 2. So, $d_k(gh) =_5 d_k(\bar{g}) + d_k(\bar{h})$.

Now, $d_k(g) =_2 d_k(\bar{g}) + d_k(t)$ and $d_k(h) = d_k(t^{-1}) + d_k(\bar{h})$. Then, $d_k(g) + d_k(h) =_4 d_k(\bar{g}) + d_k(\bar{h})$. Thus, we have $d_k(gh) =_9 d_k(g) + d_k(h)$. □

Normal form of palindromes

Definition 3.3.5. ***Group palindrome:*** *Let $g = x_1 \dots x_n$ be a reduced word of $g \in G$. Let $\bar{g}$ be the word obtained by writing g in the reverse order, i.e. $\bar{g} = x_n \dots x_1$. This is a non-trivial element of G. We say g is a* group-palindrome *if $\bar{g} = g$.*

Lemma 3.3.6. *A group-palindrome $g \in G$ has the form*

$$g = x_1 x_2 \dots x_k x'_{k+1} x_k x_{k-1} \dots x_1$$

where $x'_{k+1} = x_{k+1}c$ with $c \in C$.

Proof. Let g is a group-palindrome in G.

CASE 1: $l(g) = 2k + 1$.

Let $g = x_1 x_2 \dots x_k x_{k+1} \dots x_{2k} x_{2k+1}$. We know $g = \bar{g}$,

i.e., $x_1 x_2 \dots x_k x_{k+1} \dots x_{2k} x_{2k+1} = x_{2k+1} x_{2k} \dots x_2 x_1$.

This implies,

$$x_1 x_2 \dots x_{2k} x_{2k+1} {x_1}^{-1} {x_2}^{-1} \dots {x_{2k}}^{-1} {x_{2k+1}}^{-1} = 1. \tag{3.10}$$

Since the expression on the left side is reducible, $x_{2k+1} {x_1}^{-1} = c_1$; for $c_1 \in C$. This implies, $x_{2k+1} = c_1 x_1$. Thus,

$$x_1 x_2 \dots x_{2k} c_1 {x_2}^{-1} \dots {x_{2k-1}}^{-1} {x_{2k}}^{-1} = 1. \tag{3.11}$$

Further $x_{2k} c_1 {x_2}^{-1} = c_2$; for $c_2 \in C$. So, $x_{2k} = c_2 x_2 c_1^{-1}$. Substituting this in (3.11),

$$x_1 x_2 \dots x_{2k-1} c_2 {x_3}^{-1} \dots {x_{2k-1}}^{-1} {x_{2k}}^{-1} = 1. \tag{3.12}$$

In general we get $x_{2k-i} = c_{i+2} x_{i+2} c_{i+1}^{-1}$ for $0 \le i \le k-2$.

Then, $g = x_1 x_2 \dots x_k x_{k+1} \dots x_{2k-1} x_{2k}$, and so,

$g = x_1 x_2 \dots x_k x_{k+1} c_k x_k c_{k-1}^{-1} c_{k-1} x_{k-1} c_{k-2}^{-1} \dots c_3 x_3 {c_2}^{-1} c_2 x_2 {c_1}^{-1} c_1 x_1$,

which implies, $g = x_1 x_2 \dots x_k x_{k+1} c_k x_k \dots x_3 x_2 x_1$.

Therefore, $g = x_1 x_2 \dots x'_{k+1} x_k \dots x_3 x_2 x_1$, where $x'_{k+1} = x_{k+1} c_k$.

CASE 2: $l(g) = 2k$.

Let $g = x_1 x_2 \dots x_k x_{k+1} \dots x_{2k-1} x_{2k}$. We know $g = \bar{g}$, i.e.,

$x_1 x_2 \dots x_k x_{k+1} \dots x_{2k-1} x_{2k} = x_{2k} x_{2k-1} \dots x_2 x_1$. This implies,

$$x_1 x_2 \dots x_{2k-1} x_{2k} {x_1}^{-1} {x_2}^{-1} \dots {x_{2k-1}}^{-1} {x_{2k}}^{-1} = 1. \tag{3.13}$$

The left side of the equation is reducible. So, $x_{2k} {x_1}^{-1} = c_1$; for $c_1 \in C$, i.e,

$x_{2k} = c_1x_1$. Substituting this in (3.13),

$$x_1x_2 \dots x_{2k-1}c_1{x_2}^{-1} \dots {x_{2k-1}}^{-1}{x_{2k}}^{-1} = 1. \tag{3.14}$$

Further $x_{2k-1}c_1{x_2}^{-1} = c_2$; for $c_2 \in C$, i.e, $x_{2k-1} = c_2x_2c_1^{-1}$. Substituting this in (3.14), we have, $x_1x_2 \dots x_{2k-2}c_2{x_3}^{-1} \dots {x_{2k-1}}^{-1}{x_{2k}}^{-1} = 1$.

In general we get $x_{2k-i} = c_{i+1}x_{i+1}c_i^{-1}$ for $1 \le i \le k-1$. In particular, for $i = k$, $x_k = c_{k+1}x_{k+1}c_k^{-1}$. This is a contradiction as the consecutive syllables lie in different factors in a reduced word.

Thus, for $g \in G$ with $l(g) = 2k$, g cannot be a group-palindrome. □

Lemma 3.3.7. *Let $g \in G$ be a product of k group-palindromes, say $g = p_1p_2 \dots p_k$. Then, $\Delta(g) \le 12k - 9$.*

Proof. Let p be a group-palindrome in G of non-zero length.

Then p can be expressed as $p = hu\bar{h}$, where $\bar{h}$ is h written in reverse and u is of the form x_i' from Lemma 3.3.6. Then, for every k, $d_k(h) = d_k(\bar{h})$.

If $u = a$, we have

$$p_k(p) =_2 2p_k(h),$$

$$m_k(p) =_1 2m_k(h).$$

Then we get $d_k(p) =_3 2d_k(h)$.

If $u = a^{-1}$, as above, we get $d_k(p) =_3 2d_k(h)$.

If $u \ne a, a^{-1}$, we get $d_k(p) =_2 2d_k(h)$.

In general we have, $d_k(p) =_3 2d_k(h)$. Thus, $r_k(p) =_3 0$ and $\Delta(p) \le 3$.

So, if $g \in G$ is a product of k group-palindromes, say $g = p_1p_2 \dots p_k$, then

$$\Delta(g) = \Delta(p_1p_2 \cdots p_k) \le \Delta(p_1) + \Delta(p_2) + \cdots + \Delta(p_k) + 9(k-1) \le 12k - 9. \tag{3.15}$$

This completes the proof. □

Proof of Lemma 3.3.1

Now we prove that Δ is not bounded from above. For that purpose, we produce the following sequence $\{g_i\}$ for which $\Delta(g_i)$ is increasing.

Let $b \in B$ but not in C.

Let $g_1 = baba^{-1}ba$. Then, $p_1(g_1) = 0, p_2(g_1) = 1$ and $p_k(g_1) = 0$ for all other k; $m_k(g_1) = 0$ for all k; $d_2(g_1) = 1$ and $d_k(g_1) = 0$ for all other k. So, $\Delta(g_1) = 1$.

Let $g_2 = baba^{-1}baba^{-1}ba^{-1}ba$. Then $p_1(g_2) = 0, p_2(g_2) = p_3(g_2) = 1$ and $p_k(g_2) = 0$ for all other k, and, $m_1(g_2) = m_2(g_2) = 1$ and $m_k(g_2) = 0$ for all other k. So, $\Delta(g_2) = 2$.

Let $g_3 = baba^{-1}baba^{-1}ba^{-1}baba^{-1}ba^{-1}ba^{-1}ba$. Then $p_1(g_3) = 0, p_2(g_3) = p_3(g_3) = p_4(g_3) = 1$ and $p_k(g_3) = 0$ for all other k; $m_1(g_3) = 3, m_2(g_3) = 2$ and $m_k(g_3) = 0$ for all other k. So, $\Delta(g_3) = 4$.

In general, for $g_n = baba^{-1}ba(ba^{-1})^2 \dots ba(ba^{-1})^{n-1}ba(ba^{-1})^n ba$,

$p_1(g_n) = 0,\ p_k(g_n) = 1$ for $1 < k < n+1$; $m_1(g_n) = \frac{n(n-1)}{2},\ m_2(g_n) = n-1$ and for $k \neq 1, 2,\ m_k(g_n) = 0$. Thus we have $\Delta(g_n) = r_1 + r_2 + (n-1)$, where r_1 is the remainder of $\frac{n(n-1)}{2}$ divided by 2 and r_2 is that of n divided by 2. So, $\Delta(g_n) \geq n - 1$.

Then, by (3.15), we get that the palindromic width of G is infinite. This proves Lemma 3.3.1.

3.3.2 Case 2

For a non-trivial $a \in A \cup B$ such that $CaC = Ca^{-1}C$, we prove the following:

Lemma 3.3.8. *Let $G = A *_C B$ be the free product of two groups A and B with amalgamated subgroup C. Let $|A : C| \geq 3$, $|B : C| \geq 2$ and there exists an element $a \in A \cup B$ for which $CaC = Ca^{-1}C$. Then $pw(G, \{A \cup B\})$ is* *infinite.*

This lemma follows using similar methods as in Lemma 3.3.1. But there is a slight modification in the definition of special forms.

Definition 3.3.9. [22] ***Special form:*** *Let $a \in A$ such that $CaC = Ca^{-1}C$. Let $g \in G$, and $g = x_1x_2 \dots x_n$ be a reduced word representing it. If for any i, $x_i = c_1ac_1' = c_2a^{-1}c_2'$ for $c_1, c_2, c_1', c_2' \in C$, then we fix one such representation and denote it by $ca^{\epsilon}c'$. We define the* special form *of g associated to this reduced word to be the word obtained by replacing x_i by $ca^{\epsilon}c'$. It is done in the following way:*

1. When $i = 1$, $x_1 = ca^{\epsilon}c'$, we write $g = ca^{\epsilon}x_2' \dots x_n$, where $x_2' = c'x_2$.

2. When $2 \leq i \leq n-1$, $x_i = ca^{\epsilon}c'$, we write $g = x_1x_2 \dots x_{i-1}'a^{\epsilon}x_{i+1}' \dots x_n$, where $x_{i-1}' = x_{i-1}c$ and $x_{i+1}' = c'x_{i+1}$.

3. When $i = n$ and $x_n = ca^{\epsilon}c'$,we replace x_n by $g = x_1x_2 \dots x_{n-1}'a^{\epsilon}c'$, where $x_{n-1}' = x_{n-1}c$.

We now recall the definition of Δ from Equation 3.9. For $g \in G$ expressed in special form,

$p_k(g)$ = number of a-segments of length $2k-1$,

$m_k(g)$ = number of a^{-1}-segments of length $2k-1$,

$d_k(g)$= $p_k(g) - m_k(g)$,

$r_k(g)$= remainder of $d_k(g)$ divided by 2, and

$$\Delta(g) = \sum_{k=1}^{\infty} r_k(g).$$

Lemma 3.3.10. *For any elements $g, h \in G$, $\Delta(gh) \leq \Delta(g) + \Delta(h) + 9$.*

Proof of this lemma is exactly same as the proof of Lemma 3.3.4, except for the slight modification of the definition of special forms. So we omit this proof.

If $g \in G$ is a product of k group-palindromes, say $g = p_1p_2 \dots p_k$, then as in Lemma 3.3.7, we have,

$$\Delta(g) = \Delta(p_1p_2 \dots p_k) \leq \Delta(p_1) + \Delta(p_2) + \cdots + \Delta(p_k) + 9(k-1) \leq 12k - 9. \tag{3.16}$$

And finally, for the same sequence used in Section 3.3.1, using (3.16), we get that the palindromic width of G is infinite.

3.3.3 Proof of Theorem 3.0.2

The result follows by combining Lemma 3.3.1 and Lemma 3.3.8.

3.3.4 The index two case

So far we have shown that the palindromic width of $G = A*_C B$, when $|A : C| \geq 3$, $|B : C| \geq 2$, is infinite. Let's now consider the case when $|A : C| \leq 2$, $|B : C| \leq 2$.

Proposition 3.3.11. *Let $G = A*_C B$ be the free product of two groups A and B with amalgamated subgroup C and $|A : C| \leq 2$, $|B : C| \leq 2$. Let S and T be the generating sets of A and B respectively. If $pw(C, \{S, T\})$ is finite, then $pw(G, \{A \cup B\})$ is finite.*

Proof. We only need to consider the case of $|A : C| = 2$, $|B : C| = 2$. Then C is a normal subgroup of both A and B. Let T_A and T_B be the sets of right coset representatives of C in A and C in B respectively. Here, $T_A \cong \mathbb{Z}_2$ and $T_B \cong \mathbb{Z}_2$.

Any $g \in G$ can be expressed uniquely as a C-normal form. A C-normal form of g is a sequence $(x_0, x_1, \dots, x_n)$, where $g = x_0x_1 \cdots x_n$ with $x_0 \in C$, $x_i \in T_A \setminus \{1\} \sqcup T_B \setminus \{1\}$ and consecutive x_i, x_{i+1} lie in distinct sets. Existence and uniqueness of such a C-normal form follows from [15, Theorem 11.3].

So, for $g = x_0 x_1 \cdots x_n$, where $(x_0, x_1, \ldots, x_n)$ is the C-normal form of g, clearly, $x_1 x_2 \cdots x_n \in T_A * T_B \cong \mathbb{Z}_2 * \mathbb{Z}_2$. This implies that $pw(x_1 x_2 \cdots x_n) \leq 2$. Therefore, $pw(g) \leq pw(x_0) + pw(x_1 x_2 \cdots x_n) \leq 3$. $\square$

3.4 Palindromic width of graph of groups

Fundamental group of any graph of groups has a representation which is an amalgamated free product or a HNN extension (see [23]). Let Y be a finite graph and $G = \pi_1(\mathcal{G}, Y)$ be the fundamental group. There are two cases to consider.

1. Suppose there exists an edge $e = [v_1, v_2]$ such that removing e, while retaining v_1 and v_2 results in a union of disjoint connected subgraphs, $Y_1 \sqcup Y_2$. Let $v_i \in V(Y_i)$, $i = 1, 2$. Let $G_i := \pi_1(\mathcal{G}, Y_i)$. Then we have $G \cong G_1 *_{G_e} G_2$, where G_e is the group associated to the edge e.

2. Let $e = [v_1, v_2]$ be an edge in Y such that removing e does not separate Y. Let Y' be the graph obtained by removing e while retaining the vertices v_1 and v_2. Let $G' = \pi_1(\mathcal{G}, Y')$ and for the edge group G_e, the embeddings $\phi_i : G_e \to G_{v_i}$, for $i = 1, 2$, induce embeddings $G_e \to G'$, with H_1 and H_2 as the images of $\phi_1(G_e)$ and $\phi_2(G_e)$ respectively, in G'. Then, $G \cong G_*$, where the isomorphism $\phi : H_1 \to H_2$ is given by $H_1 \to G_e \to H_2$.

We have the following consequence of Theorem 3.0.1 and Theorem 3.0.2.

Corollary 3.4.1. *Let Y be a non-empty, connected graph. Let $\pi_1(\mathcal{G}, Y)$ be the fundamental group of the graph of groups of Y with the standard generating set S. Then the palindromic width of $\pi_1(\mathcal{G}, Y)$ is infinite if one of the following holds:*

1. *Y is a loop with a vertex P and edge e; and the image of G_e is a proper subgroup of G_P.*

2. *Y is a tree and has an oriented edge $e = [P_1, P_2]$ such that removing e, while retaining P_1 and P_2, gives two disjoint graphs Y_1 and Y_2 with $P_i \in \text{vert}\, Y_i$ satisfying the following: extending $G_e \to G_{P_i}$ to $\phi_i : G_e \to \pi_1(\mathcal{G}, Y_i)$, $i = 1, 2$, we get $[\pi_1(\mathcal{G}, Y_1) : \phi_1(G_e)] \geq 3$ and $[\pi_1(\mathcal{G}, Y_2) : \phi_2(G_e)] \geq 2$.*

3. *Y has an oriented edge $e = [P_1, P_2]$ such that removing the edge, while retaining P_1 and P_2 does not separate Y and gives a new graph Y' satisfying the following: extending $G_e \to G_{P_i}$ to $\phi_i : G_e \to \pi_1(\mathcal{G}, Y')$, $i = 1, 2$, we have $\phi_i(G_e) = H_i$ and H_1, H_2 are proper subgroups of $\pi_1(\mathcal{G}, Y')$.*

The fundamental group in (1) is an HNN extension of G_P and so, (1) follows from Theorem 3.0.1. In (2), the fundamental group is an amalgamated free product of $\pi_1(\mathcal{G}, Y_1)$ and $\pi_1(\mathcal{G}, Y_2)$ with proper amalgamated subgroups $\phi_1(G_e) \cong \phi_2(G_e)$. The result follows from Theorem 3.0.2. Finally, the fundamental group in (3) is an HNN extension of G', with G' being the fundamental group of the graph of groups corresponding to Y'. Hence, this also follows from Theorem 3.0.1.

Chapter 4

A limit set intersection theorem for graph of relatively hyperbolic groups

In this chapter, we prove the following theorem:

Theorem 4.0.1. *Let G be a group admitting a decomposition into a finite graph of relatively hyperbolic groups $(\mathcal{G}, Y)$ satisfying the qi-embedded condition. Further, suppose the monomorphisms from edge groups to vertex groups is strictly type-preserving, and that induced tree of coned-off spaces also satisfy the qi-embedded condition. If G is hyperbolic relative to the family $\mathcal{C}$ of maximal parabolic subgroups, then the set of conjugates of vertex and edge groups of G satisfy a limit set intersection property for conical limit points.*

In Section 4.1, we quickly recall definitions and basic results pertaining to limit sets and the limit set intersection theorem. In Section 4.2, we construct a tree of relatively hyperbolic metric spaces associated to a given graph of relatively hyperbolic groups. In Section 4.3, we give a slightly modified ladder construction. Finally, in Section 4.4, we prove Theorem 4.0.1.

4.1 Preliminaries on limit sets

Let (X, d) be a metric space. For a subset $Y \subset X$ and $R > 0$, $N_R(Y) = \{x \in X \mid \exists\, y \in Y \text{ with } d(x, y) \leq R\}$.

Definition 4.1.1. ***Conical limit point:***

(1) Let X be a proper hyperbolic metric space and $Y \subset X$. Then $\xi \in \partial X$ is called a conical limit point *of Y if for any geodesic ray γ in X asymptotic to ξ, there is a constant $R < \infty$ such that, there exists sequence $\{y_n\}$ in $Y \cap N_R(\gamma)$ with $\lim y_n \to \xi$.*

(2) For a group H acting on X by isometries, $\xi \in \Lambda(H)$ is a conical limit point *of H if ξ is a conical limit point of the orbit $H \cdot x_0$ for any $x_0 \in X$.*

(3) The set of all conical limit points of H is called the conical limit set *and it is denoted by $\Lambda_c(H)$.*

The first two parts of Definition 4.1.1 also make sense for an infinite subgroup or subset H of a hyperbolic group G. In that case, we may take X to be a Cayley graph of G and the action of H on X. We state two results on the conical limit set of any such H.

Lemma 4.1.2. *Suppose G is a hyperbolic group and let H be a subset of G. Then for every $g \in G$,*

(1) $\Lambda_c(gHg^{-1}) = \Lambda_c(gH)$;

(2) $\Lambda_c(gH) = g\Lambda_c(H)$.

Proof. Let $g \in G$.

(1) For any $\{h_n\}$ in H, we have $d(gh_ng^{-1}, gh_n) = l(g)$. Then (1) follows from Lemma 2.2.29.

(2) G acts on Γ_G by isometries and, by Lemma 2.2.30, this induces an action of G on ∂G by homeomorphisms. So, for $\xi \in \Lambda_c(gH)$, if for $\{h_n\}$ in H, $\{gh_n\}$ converges to ξ, then $\{h_n\}$ converges to $g^{-1}\xi \in \Lambda_c(H)$. Thus, ξ lies in $g\Lambda_c(H)$. Similarly, if ξ is in $\Lambda_c(H)$, then there exists a sequence $\{h_n\}$

in H such that h_n converges to ξ. Then, $g\xi \in g\Lambda_c(H)$ and clearly, $\{gh_n\}$ converges to $g\xi$. Therefore, $g\xi \in \Lambda_c(gH)$. □

This lemma also holds for general limit set $\Lambda(H)$ (see [49, Lemma 2.9]) and the set of non-conical limit points. Let $\Lambda_{nc}(H) := \Lambda(H) \setminus \Lambda_c(H)$ denote the set of non-conical limit points. Then, for every $g \in G$, we have

(1) $\Lambda(gHg^{-1}) = \Lambda(gH)$ and $\Lambda(gH) = g\Lambda(H)$.

(2) $\Lambda_{nc}(gHg^{-1}) = \Lambda_{nc}(gH)$ and $\Lambda_{nc}(gH) = g\Lambda_{nc}(H)$.

Definition 4.1.3. ***Limit set intersection property:*** *Suppose G is a (relatively) hyperbolic group. Let $\mathcal{S}$ be a collection of subgroups of G. Then $\mathcal{S}$ is said to have the* limit set intersection property *if for every $H,\ K \in \mathcal{S}$, $\Lambda(H) \cap \Lambda(K) = \Lambda(H \cap K)$.*

It is known that if G is hyperbolic and $\mathcal{S}$ is a collection of quasiconvex subgroups of G, then limit intersection property holds for $\mathcal{S}$, i.e., for every $H,\ K \in \mathcal{S}$, $\Lambda(H) \cap \Lambda(K) = \Lambda(H \cap K)$. If G is relatively hyperbolic and $\mathcal{S}$ is a collection of relatively quasiconvex subgroups of G, then we have the following theorem by Yang:

Theorem 4.1.4. [53, Theorem 1.1] *Let H and J be two relatively quasiconvex subgroups of a relatively hyperbolic group G. Then,*

$$\Lambda(H) \cap \Lambda(J) = \Lambda(H \cap J) \sqcup E$$

where the exceptional set E consists of the limit points isolated in $\Lambda(H) \cap \Lambda(J)$.

We prove the following version of limit set intersection property.

Definition 4.1.5. ***Conical limit intersection property:*** *A collection $\mathcal{S}$ of subgroups of a relatively hyperbolic group G is said to have the* conical limit intersection property *if for every $H,\ K \in \mathcal{S}$, $\Lambda_c(H) \cap \Lambda_c(K) = \Lambda_c(H \cap K)$.*

4.2 Trees of spaces

In [14], Bestvina and Feighn introduced graph of spaces for a finite graph. In [41], Mj defined a closely related notion, namely tree of metric spaces for infinite trees.

Definition 4.2.1. [41] ***Tree of hyperbolic metric spaces:*** *A tree* (T) *of metric spaces satisfying the quasiisometrically (qi) embedded condition is a metric space* (X, d) *admitting a map* $p : X \to T$ *onto a simplicial tree* T*, such that there exists* $\epsilon \geq 0$ *and* $K > 0$ *satisfying the following:*

1. *For each vertex* $v \in V(T)$*,* $X_v := p^{-1}(v) \subset X$ *with the induced path metric* d_v *is a metric space. Further, the inclusion map* $i_v : X_v \to X$ *is uniformly proper.*

2. *For each edge* $e \in E(T)$*, let* X_e *be the pre-image of the midpoint of* e *under* p*. Then with the induced path metric* d_e*,* X_e *is a metric space.*

3. *For each* $e \in E(T)$ *with initial and terminal vertices* v_1 *and* v_2 *respectively, there exists a map* $f_e : X_e \times [0, 1] \to X$ *such that* $f_e|_{X_e \times (0,1)}$ *is an isometry onto the pre-image of the interior of* e *equipped with the path metric.*

4. *The maps* $f_{e,v_1} := f_e|_{X_e \times \{0\}}$ *and* $f_{e,v_2} := f_e|_{X_e \times \{1\}}$ *are* (K, ϵ)*-qi embeddings. This is called the* ***qi-embedded condition****.*

If there exists $\delta > 0$ such that all the vertex spaces and edge spaces are δ-hyperbolic, then X is a *tree of hyperbolic metric spaces.*

This was generalised to the case of relatively hyperbolic metric spaces by Mj and Pal in [43].

Definition 4.2.2. [43] ***Tree of relatively hyperbolic metric spaces:*** *A tree of metric spaces* (X, d) *is a tree* (T) *of relatively hyperbolic metric spaces if the following conditions are also satisfied:*

1. *Each vertex space X_v is strongly hyperbolic relative to a collection of subsets $\mathcal{A}_v$ and each edge space X_e is strongly hyperbolic relative to a collection of subsets $\mathcal{A}_e$. Further, for every $v \in V(T)$, $\mathcal{E}(X_v, \mathcal{A}_v)$ is uniformly δ-hyperbolic for some $\delta > 0$.*

2. *The maps f_{e,v_i}, for $i = 1, 2$, are **strictly type-preserving**, i.e, for any $A_{v_i,\alpha} \in \mathcal{A}_{v_i}$, $f^{-1}_{e,v_i}(A_{v_i,\alpha})$ is either empty or some $A_{e,\beta}$ in $\mathcal{A}_e$. Also for every $A_{e,\beta} \in \mathcal{A}_e$, $f_{e,v_i}(A_{e,\beta})$ lies in some $A_{v_i,\alpha}$ in $\mathcal{A}_{v_i}$.*

3. *For $i = 1, 2$, the induced maps $\hat{f}_{e,v_i} : \mathcal{E}(X_e, \mathcal{A}_e) \to \mathcal{E}(X_{v_i}, \mathcal{A}_{v_i})$ are uniform qi-embeddings. This is called the **qi-preserving electrocution condition**.*

Now we give a construction of trees of relatively hyperbolic metric spaces associated to a graph of relatively hyperbolic groups. Let Y be a finite graph with vertex set $V(Y)$ and edge set $E(Y)$.

Definition 4.2.3. ***Graph of relatively hyperbolic groups:*** *A graph of groups* $(\mathcal{G}, Y)$ *is a* graph of relatively hyperbolic group*s if for each $v \in V(Y)$, G_v is hyperbolic relative to a collection of subgroups $\{H_{v,\alpha}\}_\alpha$ and for each $e \in E(Y)$, G_e is hyperbolic relative to a collection of subgroups $\{H_{e,\alpha}\}_\alpha$.*

4.2.1 Trees of relatively hyperbolic metric spaces from a graph of relatively hyperbolic graph of groups

Let Y be a finite graph and $(\mathcal{G}, Y)$ be a graph of relatively hyperbolic groups. Let T be a maximal subtree of Y and $G = \pi_1(\mathcal{G}, Y, T)$ be the fundamental group of $(\mathcal{G}, Y)$. For each $v \in V(Y)$, let G_v be the vertex group hyperbolic relative to $\mathcal{H}_v = \{H_{v,\alpha}\}_\alpha$ and for each $e \in E(Y)$, let G_e be the edge group hyperbolic relative to $\mathcal{H}_e = \{H_{e,\alpha}\}_\alpha$. For $v \in V(Y)$, we fix the generating set of G_v to be S_v and $e \in E(Y)$, we fix the generating set of G_e to be

S_e satisfying $f_{e,t(e)}(S_e) \subset S_{t(e)}$. Then $S = \bigcup_{v \in V(Y)} S_v \bigcup (E(Y) \setminus E(T))$ is a generating set of G. Let $\Gamma(G, S)$ denote the Cayley graph of G with respect to S. A tree of relatively hyperbolic metric spaces X for $(\mathcal{G}, Y)$ is a metric space admitting a map $p : X \to \mathcal{T}$ and satisfying the following:

1. For every vertex $\tilde{v} = gG_v \in V(\mathcal{T})$, $X_{\tilde{v}} = p^{-1}(\tilde{v})$ is a subgraph of $\Gamma(G, S)$ with $V(X_{\tilde{v}}) = gG_v$ and $gx, gy \in X_{\tilde{v}}$ are connected by an edge if $x^{-1}y \in S_v$. With the induced path metric $d_{\tilde{v}}$, $X_{\tilde{v}}$ is a geodesic metric space hyperbolic relative to $\mathcal{H}_{\tilde{v}} = \{gg_{v,\alpha}H_{v,\alpha} |\, g_{v,\alpha}H_{v,\alpha} \text{ is a left coset of } H_{v,\alpha} \text{ in } G_v\}$.

2. For every edge $\tilde{e} = gG_e^e \in E(\mathcal{T})$, $X_{\tilde{e}}$ is the pre-image of the midpoint of $\tilde{e}$ and it is a subgraph of $\Gamma(G, S)$ with $V(X_{\tilde{e}}) = geG_e^e$ and $gex, gey \in X_{\tilde{e}}$ are connected by an edge if $x^{-1}y \in \phi_{t(e)}(S_e)$. With the induced path metric $d_{\tilde{e}}$, $X_{\tilde{e}}$ is a geodesic metric space hyperbolic relative to $\mathcal{H}_{\tilde{e}} = \{gg_{e,\alpha}H_{e,\alpha} |\, g_{e,\alpha}H_{e,\alpha} \text{ is a left coset of } H_{e,\alpha} \text{ in } G_e\}$.

3. For an edge $\tilde{e} = gG_e^e$ connecting vertices $\tilde{u} = gG_{o(e)}$ and $\tilde{v} = geG_{t(e)}$, if $x \in G_e^e$, we join $gex \in X_{\tilde{e}}$ to $gexe^{-1} \in X_{\tilde{u}}$ and $gex \in X_{\tilde{v}}$ by edges of length $\frac{1}{2}$. These extra edges give us maps $f_{\tilde{e},\tilde{u}} : X_{\tilde{e}} \to X_{\tilde{u}}$ and $f_{\tilde{e},\tilde{v}} : X_{\tilde{e}} \to X_{\tilde{v}}$ with $f_{\tilde{e},\tilde{u}}(gex) = gexe^{-1}$ and $f_{\tilde{e},\tilde{v}}(gex) = gex$.

4. There exists a $\delta > 0$ such that $\mathcal{E}(X_{\tilde{v}}, \mathcal{H}_v)$ and $\mathcal{E}(X_{\tilde{e}}, \mathcal{H}_e)$ are δ-hyperbolic metric spaces.

For a tree of relatively hyperbolic metric spaces with vertex spaces $X_{\tilde{v}}$ and edge spaces $X_{\tilde{e}}$, we can associate a tree of coned-off metric spaces with vertex spaces $\mathcal{E}(X_{\tilde{v}}, \mathcal{H}_v)$ and edge spaces $\mathcal{E}(X_{\tilde{e}}, \mathcal{H}_e)$. This is called the **induced tree of coned-off spaces**. We denote it by $\mathcal{TC}(X)$. The maps $f_{\tilde{e},\tilde{u}} : X_{\tilde{e}} \to X_{\tilde{u}}$ and $f_{\tilde{e},\tilde{v}} : X_{\tilde{e}} \to X_{\tilde{v}}$ induce $\hat{f}_{\tilde{e},\tilde{u}} : \mathcal{E}(X_{\tilde{e}}, \mathcal{H}_{\tilde{e}}) \to \mathcal{E}(X_{\tilde{u}}, \mathcal{H}_{\tilde{u}})$ and $\hat{f}_{\tilde{e},\tilde{v}} : \mathcal{E}(X_{\tilde{e}}, \mathcal{H}_{\tilde{e}}) \to \mathcal{E}(X_{\tilde{v}}, \mathcal{H}_{\tilde{v}})$. If these induced maps are qi-embeddings, then this tree of spaces satisfies qi-preserving electrocution condition.

Fix $v_0 \in V(Y)$. Then, $\tilde{v}_0 = G_{v_0} \in V(\mathcal{T})$. Let $x_0 \in X_{v_0}$ denote the identity element of G_{v_0}. By Milnor-Schwarz lemma, the orbit map $\Theta : G \to X$ given by $g \mapsto gx_0$ is a quasiisometry.

Lemma 4.2.4. [49, Lemma 3.5] *There exists a constant D_0 such that for every vertex space $gG_v \subset X$, $\mathrm{Hd}(\Theta(gG_v), gG_v) \leq D_0$.*

Proof. For any $gg' \in gG_v$, $\Theta(gg') = gg'x_0$. Let x denote the identity element in G_v. Suppose γ_v be a geodesic joining x_0 to x in X. Then $gg'\gamma_v$ is a path joining $gg'x_0$ to $gg'x$ in X, for every $g' \in G_v$. We choose $D_0 = \max\{l(\gamma_v) \mid v \in V(Y)\}$. □

Let $\tilde{v} = gG_v \in V(\mathcal{T})$. Θ induces a quasiisometry $\Theta_{g,v} : gG_v \to X_{\tilde{v}}$. For each $x \in gG_v$, we map x to $y \in X_{\tilde{v}}$ such that $d_X(\Theta(x), y) \leq D_0$. This map is coarsely well-defined. Θ induces a quasiisometry $\Theta^h : G^h \to X^h$ and $\Theta_{g,v}$ induces a quasiisometry $\Theta^h_{g,v} : gG^h_v \to X^h_{\tilde{v}}$.

Definition 4.2.5. [44] ***Cone locus:*** *The cone locus of $\mathcal{TC}(X)$ is defined as a graph with the vertex set consisting of cone points in the vertex spaces, $\{c_v \mid v \in V(\mathcal{T})\}$ and the edge set consists of the cone points in the edge spaces, $\{c_e \mid e \in E(\mathcal{T})\}$. For $u, v \in V(\mathcal{T})$, c_u and c_v are joined by an edge c_e, for $e \in E(\mathcal{T})$ if $o(e) = u$, $t(e) = v$ in $\mathcal{T}$, c_u, c_v and c_e are cone vertices attached to horosphere-like sets H_u in $\widehat{X}_u$, H_v in $\widehat{X}_v$ and H_e in $\widehat{X}_e$ respectively, and $f_{e,u}(H_e) \subset H_u$ and $f_{e,v}(H_e) \subset H_v$. Then the edge $c_e \times [0,1]$ joins c_u and c_v by identifying $c_e \times \{0\}$ to c_u and $c_e \times \{1\}$ to c_v.*

The connected components of a cone locus are trees, each of which can be naturally identified with a subtree of $\mathcal{T}$. Corresponding to each such connected component, we get a tree of horosphere-like subsets in X. We denote the collection of such tree of horosphere-like sets by $\mathcal{C} = \{C_\alpha\}$, where C_α's are the tree of horosphere-like sets.

Denote by X^h, the quotient space $\mathcal{G}(X, \mathcal{C})$ obtained by attaching hyperbolic cones C^h_α to $C_\alpha \in \mathcal{C}$ by identifying $(x, 0)$ to x for all $x \in C_\alpha$. By Theorem 2.3.15 due to Bowditch, $\mathcal{G}(X, \mathcal{C})$ is a δ-hyperbolic metric space for some $\delta > 0$.

Lemma 4.2.6. [43] *For each $v \in V(\mathcal{T})$, the inclusion $i_v : (X_v, \mathcal{H}_v) \to (X, \mathcal{C})$ induces a uniform proper embedding $\hat{i}_v : \widehat{X}_v \to \mathcal{TC}(X)$.*

Proof. Let $x, y \in \widehat{X}_v$ such that $d_{\mathcal{TC}(X)}(x, y) \leq M$, for some $M > 0$. Let $\widehat{\alpha}$ be a geodesic in $\mathcal{TC}(X)$ joining x to y. Then, $\widehat{\alpha}$ passes through at most M many horosphere-like subsets C_α's. Suppose $\widehat{\alpha}$ passes through the following vertex spaces $\widehat{X}_{v_1}, \widehat{X}_{v_2}, \ldots, \widehat{X}_{v_n}$, where $v_1 = v$. For $1 \leq i \leq n-1$, let e_i be an edge joining v_i to v_{i+1} in $\mathcal{T}$.

Let α^b be the disjoint union of the portions of $\widehat{\alpha}$ lying outside the horosphere-like sets. So we have, $l_{\mathcal{TC}(X)}(\alpha^b) = l_X(\alpha^b) \leq M$.

Let β_n be the maximal portion of α^b lying in $\widehat{X}_{v_n}$ with end points in $f_{e_{n-1},v_n}(X_{e_{n-1}})$. Suppose β_n joins $f_{e_{n-1},v_n}(x_n)$ and $f_{e_{n-1},v_n}(y_n)$, for $x_n, y_n \in X_{e_{n-1}}$. Then, $l_{\mathcal{TC}(X)}(\beta_n) = l_X(\beta_n) \leq M$. Since $\hat{f}_{e_{n-1},v_n}(\widehat{X}_{e_{n-1}})$ is a quasiconvex subset of $\widehat{X}_{v_n}$, without loss of generality, we assume β_n to be lying in $f_{e_{n-1},v_n}(X_{e_{n-1}})$. And since X_{v_n} is properly embedded in X, there exists $M_1 > 0$ depending on M such that $l_{X_{v_n}}(\beta_n) \leq M_1$. Now $\widehat{\alpha}|_{[f_{e_{n-1},v_n}(x_n), f_{e_{n-1},v_n}(y_n)]}$ passes through at most M horosphere-like subsets. So,

$d_{\widehat{X}_{v_n}}(f_{e_{n-1},v_n}(x_n), f_{e_{n-1},v_n}(y_n)) \leq M_1 + M$.

Also, $d_{\mathcal{TC}(X)}(f_{e_{n-1},v_{n-1}}(x_n), f_{e_{n-1},v_{n-1}}(y_n)) \leq M_1 + M + 2$.

Now, let β_{n-1} be the union all the portions of α^b lying in $\widehat{X}_{v_{n-1}}$ and the image of β_n under the map $\phi_{v_n,v_{n-1}}$ in $\widehat{X}_{v_{n-1}}$ joining $f_{e_{n-1},v_{n-1}}(x_n)$ and $f_{e_{n-1},v_{n-1}}(y_n)$ such that the endpoints of β_{n-1} are $f_{e_{n-2},v_{n-1}}(x_{n-1})$ and $f_{e_{n-2},v_{n-1}}(y_{n-1})$, for $x_{n-1}, y_{n-1} \in X_{e_{n-2}}$. So, $l_{\mathcal{TC}(X)}(\beta_{n-1}) = l_X(\beta_{n-1}) \leq M + d_{\mathcal{TC}(X)}(f_{e_{n-1},v_{n-1}}(x_n), f_{e_{n-1},v_{n-1}}(y_n)) \leq 2M + M_1 + 2$. Again, assuming β_{n-1} lies in $f_{e_{n-2},v_{n-1}}(X_{e_{n-2}})$, we have, $l_{X_{v_{n-1}}}(\beta_{n-1}) \leq M_2$, where M_2

depends on $M_1 + M + 2$, i.e., M. So, as above,

$$d_{\widehat{X}_{v_{n-1}}}(f_{e_{n-2},v_{n-1}}(x_{n-1}), f_{e_{n-2},v_{n-1}}(y_{n-1})) \leq M_2 + M.$$

Further, $d_{\mathcal{TC}(X)}(f_{e_{n-2},v_{n-2}}(x_{n-1}), f_{e_{n-2},v_{n-2}}(y_{n-1})) \leq M_2 + M + 2$. Continuing this till $v = v_1$, we have $d_{\widehat{X}_{v_1}}(f_{e_1,v_1}(x_2), f_{e_1,v_1}(y_2)) \leq M'$, for some $x_2, y_2 \in X_{e_1}$, where $M' > 0$ depends on M. Then,

$$\begin{aligned} d_{\widehat{X}_{v_1}}(x,y) &\leq d_{\widehat{X}_{v_1}}(x, f_{e_1,v_1}(x_2)) + d_{\widehat{X}_{v_1}}(f_{e_1,v_1}(x_2), f_{e_1,v_1}(y_2)) + d_{\widehat{X}_{v_1}}(f_{e_1,v_1}(y_2), y) \\ &\leq 2M + M'. \end{aligned}$$

Thus, for $N = 2M + M'$, $d_{\widehat{X}_v}(x,y) \leq N$. □

The following lemma, due to Mj and Pal (cf. [43, Lemma 1.20], [44, Lemma 2.11]), shows that quasigeodesics in X^h and $\mathcal{TC}(X)$, joining the same pair of points, track each other.

Lemma 4.2.7. [43, Lemma 1.20], [44, Lemma 2.11] *Given $k, \epsilon \geq 0$, there exists $K > 0$ such that if α and β denote respectively a (k,ϵ)- quasigeodesic in $\mathcal{TC}(X)$ and a (k,ϵ)- quasigeodesic in X^h joining a and b, then $\beta \cap X$ lies in a K-neighbourhood of (any representative of) α in (X, d). Here, d denotes the original metric on X.*

4.3 Cannon-Thurston maps for a tree of relatively hyperbolic spaces

4.3.1 Ladder construction

Recall that for any edge $e \in V(\mathcal{T})$ joining vertices u and v, the maps $f_{e,u} : X_e \to X_u$ and $f_{e,v} : X_e \to X_v$ are qi-embeddings and these induce qi-embeddings $f^h_{e,u} : X^h_e \to X^h_u$ and $f^h_{e,v} : X^h_e \to X^h_v$ respectively. Let $C_2 > 0$ such that $f^h_{e,u}(X^h_e)$ and $f^h_{e,v}(X^h_e)$ are C_2-quasiconvex subset of X^h_u and X^h_v respectively. Let $C = C_{2.2.21} + C_2$, with $C_{2.2.21}$ from Lemma 2.2.21. Let $D_{2.2.21}$ be the constant from Lemma 2.2.21. Further, $f_{e,u}$ and $f_{e,v}$ give a partially defined map from X_u to X_v with the domain restricted to $f_{e,u}(X_e)$. However, we denote the map simply by $\phi_{u,v} : X_u \to X_v$, i.e., $\phi_{u,v}(f_{e,u}(x)) = f_{e,v}(x)$.

We construct the ladder for geodesic rays. Recall that $p : \mathcal{TC}(X) \to \mathcal{T}$ is an induced tree of coned-off metric spaces. Fix the vertex v_0 as the base point. Let $v \neq v_0$ be a vertex of $\mathcal{T}$.

Let $\widehat{\alpha}_v \subset \widehat{X_v}$ be a geodesic ray starting at a point outside the horosphere-like sets. Let α_v be the corresponding electro-ambient quasigeodesic ray. Consider the set of all edges incident on v except for the edge lying in the geodesic joining v_0 to v in $\mathcal{T}$. Among them, choose the collection of all edges $\{e_k\}_{k\in I}$ such that diameter of the subset $N^h_C(\alpha_v) \cap f_{e_k,v}(X_{e_k})$ is greater than $D_{2.2.21}$. Suppose each e_k joins v to $v_k \in V(\mathcal{T})$. For each $k \in I$, we have the following two cases:

- **Case 1:** Diameter of $N^h_C(\alpha_v) \cap f_{e_k,v}(X_{e_k})$ is infinite in X^h_v.

  Let p_k be a nearest point projection of $\alpha_v(0)$ in $N^h_C(\alpha_v) \cap f_{e_k,v}(X_{e_k})$. Let $\widehat{\mu}_k$ be an electric geodesic in $\widehat{X_v}$ starting at p_k such that, for its electro-ambient quasigeodesic μ, we have $\mu(\infty) = \alpha_v(\infty)$ in ∂X^h_v. Let $\widehat{\Phi}(\widehat{\mu}_k)$ denote the electric geodesic ray in $\widehat{X_{v_k}}$, starting at $\phi_{v,v_k}(p_k)$ such that its electro-ambient quasigeodesic ray denoted by $\Phi(\widehat{\mu}_k)$ and the quasigeodesic ray $\phi^h_{v,v_k}(\mu_k)$ are asymptotic to the same point in $\partial X^h_{v_k}$.

- **Case 2:** Diameter of $N^h_C(\alpha_v) \cap f_{e_k,v}(X_{e_k})$ is finite in X^h_v.

  In this case, choose $p_k, q_k \in N^h_C(\alpha_v) \cap f_{e_k,v}(X_{e_k})$ such that $d_v(p_k, q_k)$ is maximal. Let $\widehat{\mu}_k$ be an electric geodesic in $\widehat{X_v}$ joining p_k and q_k. Let $\widehat{\Phi}(\widehat{\mu}_k)$ denote the electric geodesic in $\widehat{X_{v_k}}$, joining $\phi_{v,v_k}(p_k)$ and $\phi_{v,v_k}(q_k)$.

Define $B_1(\widehat{\alpha}) = \hat{i}_v(\widehat{\alpha}) \cup \bigcup_k \widehat{\Phi}(\widehat{\mu}_k)$.

Now, suppose we have constructed $B_m(\widehat{\alpha})$. Let $w_k \in p(B_m(\widehat{\alpha})) \setminus p(B_{m-1}(\widehat{\alpha}))$ and let $\hat{i}_{w_k}(\widehat{\alpha_k}) = p^{-1}(w_k) \cap B_m(\widehat{\alpha})$, where $\widehat{\alpha}_k$ is a geodesic (ray) in $\widehat{X}_{w_k}$. So $B_{m+1}(\widehat{\alpha}) = B_m(\widehat{\alpha}) \cup \bigcup_k B_1(\widehat{\alpha}_k)$. The ladder $B_{\widehat{\alpha}} = \cup_{m\geq 1} B_m(\widehat{\alpha})$.

Convex hull of $p(B_{\widehat{\alpha}})$ is a subtree of $\mathcal{T}$ and we denote it by $\mathcal{T}_1$.

Lemma 4.3.1. [43, Lemma 2.4] *Let $\hat{\mu}_1 \subset \widehat{X}_v$ be an electric geodesic with endpoints lying outside horosphere-like sets. Let μ_1 be the corresponding electro-ambient quasigeodesic in X_v^h. Let $p, q \in N_C^h(\mu_1) \cap f_{e,v}^h(X_e^h)$ be such that $d_{X_v^h}(p,q)$ is maximal. Let $\hat{\mu}_2$ be a geodesic in $\widehat{X}_v$ joining p and q and μ_2 be its electro-ambient representative. If $z \in f_{e,v}(X_e)$, then $d_{\widehat{X}_v^h}(\hat{\pi}_{\hat{\mu}_1}(z), \hat{\pi}_{\hat{\mu}_2}(z)) \leq D_{4.3.1}$ for some $D_{4.3.1}$.*

Retraction map

Definition 4.3.2. ***Retraction map:*** *For each $v \in V(\mathcal{T}_1)$, let $\hat{\pi}_{\widehat{\alpha}_v} : \widehat{X}_v \to \widehat{\alpha}_v$ be the electric projection of $\widehat{X}_v$ onto $\widehat{\alpha}_v$.*

The retraction map $\widehat{\Pi}_{\widehat{\alpha}} : \mathcal{TC}(X) \to B_{\widehat{\alpha}}$ is defined by:

$$\widehat{\Pi}_{\widehat{\alpha}}(x) = \hat{i}_v(\hat{\pi}_{\widehat{\alpha}_v}(x)) \quad \text{if } x \in \widehat{X}_v \text{ for } v \in V(\mathcal{T}_1).$$

If $x \in p^{-1}(V(\mathcal{T}) \setminus V(\mathcal{T}_1))$, we choose $x_1 \in p^{-1}(V(\mathcal{T}_1))$ such that $d(x, x_1) = d(x, p^{-1}(V(\mathcal{T}_1)))$. Then, $\widehat{\Pi}_{\widehat{\alpha}}(x) = \widehat{\Pi}_{\widehat{\alpha}}(x_1)$.

Now we prove the following theorem.

Theorem 4.3.3. *If $\mathcal{TC}(X)$ is hyperbolic, then $B_{\widehat{\alpha}}$ is uniformly quasiconvex (independent of $\widehat{\alpha}$).*

This is done in two steps. In the first step, we show that the retraction map is coarsely Lipschitz. While the proof of this result is identical to the proof of [43, Theorem 2.2], it is included for the sake of completion. In the second step, we show that first step implies the quasiconvexity of the ladder.

Lemma 4.3.4. [43, Theorem 2.2] *There exists* $C_{4.3.4} \geq 0$ *such that*

$$d_{\mathcal{TC}(X)}(\widehat{\Pi}_{\widehat{\alpha}}(x), \widehat{\Pi}_{\widehat{\alpha}}(y)) \leq C_{4.3.4} d_{\mathcal{TC}(X)}(x, y) + C_{4.3.4}$$

for $x, y \in \mathcal{TC}(X)$.

Proof. It is enough to prove this for $x, y \in \mathcal{TC}(X)$ such that $d_{\mathcal{TC}(X)}(x, y) \leq 1$. So let $d_{\mathcal{TC}(X)}(x, y) \leq 1$.

Case 1: Let $x, y \in p^{-1}(v)$ for some $v \in V(\mathcal{T}_1)$. Then by Lemma 2.3.13,

$$d_{\mathcal{TC}(X)}(\widehat{\Pi}_{\widehat{\alpha}}(x), \widehat{\Pi}_{\widehat{\alpha}}(y)) \leq d_{\hat{X}_v}(\hat{\pi}_{\widehat{\alpha}_v}(x), \hat{\pi}_{\widehat{\alpha}_v}(y)) \leq 2D_{2.3.13}.$$

Case 2: Let $x \in p^{-1}(v)$ and $y \in p^{-1}(w)$ for distinct $v, w \in \mathcal{T}_1$ satisfying $d_{\mathcal{T}}(v, w) = 1$. Let e be the edge joining v to w. Let $\widehat{\alpha}_v = B_{\widehat{\alpha}} \cap p^{-1}(v)$ and $\widehat{\alpha}_w = B_{\widehat{\alpha}} \cap p^{-1}(w)$. Recall that $\widehat{\alpha}_w = \widehat{\Phi}(\widehat{\mu})$, where

1. if diameter of $N^h_C(\alpha_v) \cap f_{e,v}(X_e)$ is infinite, then $\widehat{\mu}$ is an electric geodesic ray in $\widehat{X_v}$ starting at a point $p \in N^h_C(\alpha_v) \cap f_{e,v}(X_e)$ such that its electro-ambient quasigeodesic μ satisfies $\mu(\infty) = \alpha_v(\infty)$, where α_v is the electro-ambient quasigeodesic of $\widehat{\alpha}_v$ and $\widehat{\Phi}(\widehat{\mu})$ is an electric geodesic ray in $\widehat{X_w}$ starting at $\phi_{v,w}(p)$

2. otherwise, $\widehat{\mu}$ is an electric geodesic joining two points p and q at maximal distance from each other in X^h_v and $\widehat{\Phi}(\widehat{\mu})$ is an electric geodesic in $\widehat{X_w}$ joining $\phi_{v,w}(p)$ and $\phi_{v,w}(q)$.

Now, by Lemma 4.3.1, $d_{\mathcal{TC}(X)}(\hat{\pi}_{\widehat{\alpha}_v}(x), \hat{\pi}_{\widehat{\mu}}(x)) \leq d_{\widehat{X}_v}(\hat{\pi}_{\widehat{\alpha}_v}(x), \hat{\pi}_{\widehat{\mu}}(x)) \leq D_{4.3.1}$.

Then,

$$d_{\mathcal{TC}(X)}(\hat{\pi}_{\widehat{\alpha}_v}(x), \hat{\phi}_{v,w}(\hat{\pi}_{\widehat{\mu}}(x))) \leq D_{4.3.1} + 1. \tag{4.1}$$

By Lemma 2.3.14,

$$d_{\mathcal{TC}(X)}(\hat{\phi}_{v,w}(\hat{\pi}_{\widehat{\mu}}(x)), \hat{\pi}_{\widehat{\alpha}_w}(\hat{\phi}_{v,w}(x))) \leq d_{\widehat{X}_w}(\hat{\phi}_{v,w}(\hat{\pi}_{\widehat{\mu}}(x)), \hat{\pi}_{\widehat{\alpha}_w}(\hat{\phi}_{v,w}(x))) \leq D_{2.3.14}. \tag{4.2}$$

$\widehat{X_w}$ is properly embedded in $\mathcal{TC}(X)$. So there exists $N > 0$ such that if

$$d_{\mathcal{TC}(X)}(\hat{\phi}_{v,w}(x), y) \leq d_{\mathcal{TC}(X)}(\hat{\phi}_{v,w}(x), x) + d_{\mathcal{TC}(X)}(x, y) \leq 2,$$

then we have, $d_{\widehat{X}_w}(\hat{\phi}_{v,w}(x), y) \leq N$. Again using Lemma 2.3.13,

$$d_{\mathcal{TC}(X)}(\hat{\pi}_{\widehat{\alpha}_w}(\hat{\phi}_{v,w}(x)), \hat{\pi}_{\widehat{\alpha}_w}(y)) \leq D_{2.3.13} d_{\widehat{X}_w}(\hat{\phi}_{v,w}(x), y) + D_{2.3.13} \tag{4.3}$$

$$\leq D_{2.3.13} N + D_{2.3.13}. \tag{4.4}$$

Then from (4.1), (4.2) and (4.4), we have

$$\begin{aligned} d_{\mathcal{TC}(X)}(\widehat{\Pi}_{\widehat{\alpha}}(x), \widehat{\Pi}_{\widehat{\alpha}}(y)) &= d_{\mathcal{TC}(X)}(\hat{\pi}_{\widehat{\alpha}_v}(x), \hat{\pi}_{\widehat{\alpha}_w}(y)) \\ &\leq D_{4.3.1} + 1 + D_{2.3.14} + D_{2.3.13} N + D_{2.3.13}. \end{aligned}$$

Case 3: $p([x, y])$ is not contained in $\mathcal{T}_1$. Let $x_1, y_1 \in p^{-1}(V(\mathcal{T}_1))$ such that $d(x, x_1) = d(x, p^{-1}(V(\mathcal{T}_1)))$ and $d(y, y_1) = d(y, p^{-1}(V(\mathcal{T}_1)))$. Since $d_{\mathcal{TC}(X)}(x, y) = 1$, $p(x_1) = v = p(y_1)$, where $v \in V(\mathcal{T}_1)$. In fact, $x_1, y_1 \in f_{e,v}(X_e)$ where e is an edge with initial vertex v. Then $\widehat{\Pi}_{\widehat{\alpha}}(x) = \widehat{\Pi}_{\widehat{\alpha}}(x_1) = \hat{\pi}_{\widehat{\alpha}_v}(x_1)$ and $\widehat{\Pi}_{\widehat{\alpha}}(y) = \widehat{\Pi}_{\widehat{\alpha}}(y_1) = \hat{\pi}_{\widehat{\alpha}_v}(y_1)$.

If $d_{X_v^h}(\pi_{\alpha_v}(x_1), \pi_{\alpha_v}(y_1)) \leq D_{2.2.21}$, we have $d_{\mathcal{TC}(X)}(\widehat{\Pi}_{\widehat{\alpha}}(x), \widehat{\Pi}_{\widehat{\alpha}}(y)) \leq d_{\hat{X}_v}(\hat{\pi}_{\widehat{\alpha}_v}(x_1), \hat{\pi}_{\widehat{\alpha}_v}(y_1)) \leq D_{2.2.21}$.

So assume $d_{X_v^h}(\pi_{\alpha_v}(x_1), \pi_{\alpha_v}(y_1)) > D_{2.2.21}$. Then, by Lemma 2.2.21, $[x_1, \pi_{\alpha_v}(x_1)] \cup [\pi_{\alpha_v}(x_1), \pi_{\alpha_v}(y_1)] \cup [\pi_{\alpha_v}(y_1), y_1]$ is a quasigeodesic lying in a $C_{2.2.21}$-neighbourhood of a geodesic $[x_1, y_1]$ in X_v^h. Since $f_{e,v}^h(X_e^h)$ is C_2-quasiconvex in X_v^h, there exists $x_2, y_2 \in f_{e,v}^h(X_e^h)$ such that $d_{X_v^h}(\pi_{\alpha_v}(x_1), x_2) \leq C_{2.2.21} + C_2 = C$ and $d_{X_v^h}(\pi_{\alpha_v}(y_1), y_1) \leq C_{2.2.21} + C_2 = C$. Identifying $\widehat{X^h}_w$ and $\widehat{X}_w$, there exists $x_3, y_3 \in N_C^h(\alpha_v) \cap f_{e,v}(X_e)$ such that $d_{\hat{X}_v}(x_2, x_3) \leq 1$ and $d_{\hat{X}_v}(y_2, y_3) \leq 1$.

Let $D_1 > D_{2.3.13} D_{2.2.21} + D_{2.3.13}$. Then if $d_{\hat{X}_v}(\hat{\pi}_{\widehat{\alpha}_v}(x_3), \hat{\pi}_{\widehat{\alpha}_v}(y_3)) > D_1$, we have, $d_{\hat{X}_v}(x_3, y_3) > D_{2.2.21}$. This is a contradiction, as the edge e joins v to a vertex outside $\mathcal{T}_1$. So $d_{\hat{X}_v}(\hat{\pi}_{\widehat{\alpha}_v}(x_3), \hat{\pi}_{\widehat{\alpha}_v}(y_3)) \leq D_1$.

Then, $d_{\mathcal{TC}(X)}(\widehat{\Pi}_{\widehat{\alpha}}(x), \widehat{\Pi}_{\widehat{\alpha}}(y)) \leq d_{\hat{X}_v}(\hat{\pi}_{\widehat{\alpha}_v}(x_1), \hat{\pi}_{\widehat{\alpha}_v}(y_1))$

$\leq d_{X_v^h}(\pi_{\alpha_v}(x_1), x_2) + d_{X_v^h}(\pi_{\alpha_v}(y_1), y_2) + d_{\hat{X}_v}(x_2, x_3) + d_{\hat{X}_v}(y_2, y_3) + d_{\hat{X}_v}(x_3, y_3)$

$\leq 2C + 2 + d_{\hat{X}_v}(\hat{\pi}_{\widehat{\alpha}_v}(x_3), x_3) + d_{\hat{X}_v}(\hat{\pi}_{\widehat{\alpha}_v}(y_3), y_3) + d_{\hat{X}_v}(\hat{\pi}_{\widehat{\alpha}_v}(x_3), \hat{\pi}_{\widehat{\alpha}_v}(y_3))$

$\leq 4C + D_1 + 2.$

Take $C_{4.3.4} = \max\{D_{2.3.13}, D_{4.3.1} + 1 + D_{2.3.14} + D_{2.3.13}N + D_{2.3.13}, 4C + D_1 + 2\}$. □

Now, for the second step, we have the following theorem by Bowditch:

Lemma 4.3.5. [17, Lemma 4.2] *Suppose (Y, d) is a hyperbolic geodesic metric space. Let $Q \subset Y$ and $\psi : Y \to Q$ be a map which restricts to inclusion on Q and with the property that $d(\psi(x), \psi(y))$ is bounded above by a fixed linear function of $d(x, y)$. Then Q is quasiconvex.*

4.3.2 Vertical quasigeodesic rays

Let $\widehat{\alpha}_v$ be an electric geodesic ray in $\widehat{X_v}$ starting at a point outside horospheres. Let α_v be its electro-ambient quasigeodesic. We have the ladder $B_{\widehat{\alpha}_v} = \bigcup_{u \in V(\mathcal{T}_1)} \hat{i}_u(\widehat{\alpha}_u)$. Let $B^b_{\alpha_v} = \bigcup_{u \in V(\mathcal{T}_1)} \hat{i}_u(\alpha^b_u) \subset B_{\widehat{\alpha}_v}$. For any $x \in B^b_{\alpha_v}$, there exists $u \in V(\mathcal{T}_1)$ such that $x \in \alpha^b_u$. Let $\sigma = [u_n, u_{n-1}] \cup \cdots \cup [u_1, u_0]$ be the geodesic in $\mathcal{T}_1$ with $u_0 = v$ and $u_n = u$.

Definition 4.3.6. [43] ***Vertical quasigeodesic ray:*** *A vertical quasigeodesic ray starting at x is a map $r_x : \sigma \to B^b_{\alpha_v}$ satisfying the following for a constant $C' \geq 0$:*

$d_\sigma(u, w) \leq d(r_x(u), r_x(w)) \leq C' d_\sigma(u, w)$, for all $u, w \in \sigma$.

Note: $r_x(u_i) \in X_{u_i}$ and $r_x(u_n) = x$.

We end this section with one of the most important results we use.

Theorem 4.3.7. [43] *For each $v \in V(\mathcal{T})$, CT map exists for the inclusion map $i_v : (X_v, \mathcal{H}_v) \to (X, \mathcal{C})$.*

4.4 Limit set intersection theorem

Let u, v be vertices connected by an edge e. Recall that $\phi_{u,v} : X_u \to X_v$ is a partially defined qi-embedding. By Lemma 2.3.10, we know that the induced map $\phi^h_{u,v} : f^h_{e,u}(X^h_e) \to f^h_{e,v}(X^h_e)$ is a qi-embedding and it induces the embedding $\partial\phi^h_{u,v} : \partial f^h_{e,u}(\partial X^h_e) \to \partial f^h_{e,v}(\partial X^h_e)$ defined by $\partial\phi^h_{u,v}(\partial f^h_{e,u}(x)) = \partial f^h_{e,v}(x)$.

Definition 4.4.1. ***Flow of a boundary point:*** *Let $\xi \in \partial X^h_u$ and $\partial\phi^h_{u,v}(\xi) = \eta \in \partial X_v$. Then we say η is a* flow *of ξ and that ξ can be flowed into ∂X^h_v.*

If $u_0 \neq u_n$ and $u_0, u_1, ..., u_n$ is the sequence of consecutive vertices in the geodesic $[u_0, u_n]$ in $\mathcal{T}$ then we say $\xi \in \partial X^h_{u_0}$ can be flowed into $\partial X^h_{u_n}$ if there exists $\xi_i \in \partial X^h_{u_i}$ such that $\xi_0 = \xi$ and $\xi_{i+1} = \partial\phi^h_{u_i,u_{i+1}}(\xi_i)$ for $0 \leq i \leq n-1$. And ξ_n is called a flow of ξ.

Lemma 4.4.2. *Suppose $\xi_1 \in \partial X^h_{v_1}$ can be flowed to $\partial X^h_{v_2}$ and let ξ_2 be the flow. Then ξ_1 and ξ_2 map to the same limit point in ∂X^h under the respective CT maps.*

Proof. It is enough to prove the case when v_1 and v_2 are adjacent vertices. Let e be the edge in $\mathcal{T}$ joining v_1 to v_2.

By the definition of flow, $\xi_2 = \partial\phi^h_{v_1,v_2}(\xi_1)$. There exists $\xi_e \in \partial {X_e}^h$ such that $\partial f^h_{e,v_i}(\xi_e) = \xi_i$, for $i = 1, 2$. Let $\{x_n\}$ be a sequence in X^h_e with $x_n \to \xi_e$ as $n \to \infty$. Then, for $i = 1, 2$, $\{f^h_{e,v_i}(x_n)\}$ is a sequence in $X^h_{v_i}$ with $f^h_{e,v_i}(x_n) \to \xi_i$ as $n \to \infty$ and $d_{X^h}(\hat{i}_e(x_n), \hat{i}_{v_i}(f^h_{e,v_i}(x_n))) = \frac{1}{2}$. This implies that $d_{X^h}(\hat{i}_{v_1}(f^h_{e,v_1}(x_n)), \hat{i}_{v_2}(f^h_{e,v_2}(x_n))) = 1$. So, under CT map, both ξ_1 and ξ_2 map to the same element of ∂X^h. $\square$

The converse of this lemma is false. However, we have the following:

Proposition 4.4.3. *Let $v_1 \neq v_2 \in \mathcal{T}$. Suppose $\xi_i \in \partial X^h_{v_i}$, $i = 1, 2$, map to the same point, say ξ, under the CT maps $\partial X^h_{v_i} \to \partial X^h$ such that ξ is a limit point of both X_{v_1} and X_{v_2}. Then there exists $w \in [v_1, v_2] \subset \mathcal{T}$ such that ξ_1 and ξ_2 can be flowed to ∂X^h_w.*

Proof. We assume the contrary. Suppose there exists no $w \in [v_1, v_2] \subset \mathcal{T}$ such that ξ_1 and ξ_2 can be flowed to ∂X^h_w. Then there exists $v'_1, v'_2 \in [v_1, v_2]$ such that ξ_1 can be flowed only till $\partial X^h_{v'_1}$ in the direction of $X^h_{v_2}$ and ξ_2 can be flowed only till $\partial X^h_{v'_2}$ in the direction of $X^h_{v_1}$. Then, there are two possibilities.

Case 1: Suppose $v'_1 \in [v'_2, v_2]$.

In this case, we are done by taking w to be v'_1.

Case 2: Suppose $v'_1 \notin [v'_2, v_2]$.

We will show that this is not possible. We prove by contradiction. Using Lemma 4.4.2, without loss of generality, assume $v_1 = v'_1$ and $v_2 = v'_2$. For $i = 1, 2$, let $\widehat{\alpha}_i \subset \widehat{X}_{v_i}$ be an electric geodesic ray with corresponding electro-ambient quasigeodesic ray α_i such that $\alpha_i(\infty) = \xi_i$. Let B_i denote the ladder $B_{\widehat{\alpha}_i}$. Let $u_i \in V(\mathcal{T})$ be the vertex adjacent to v_i in $[v_1, v_2]$, and let the edge connecting v_i and u_i be e_i. Since ξ_i cannot be flowed into ∂X_{u_i}, for $i = 1, 2$, $N^h_C(\alpha_i) \cap f_{e_i}(X_{e_i})$ has finite diameter in $X^h_{v_i}$.

Let $\{x_n\}$ be a sequence of elements in α^b_1 such that $\lim x_n = \xi$ in ∂X^h and γ be a geodesic ray in X^h with $\gamma(0) = x_1$ and $\gamma(\infty) = \xi$. For each $n > 0$, let $y_n \in \gamma$ be a nearest point projection of x_n in γ. By Lemma 2.2.38, the path $\gamma|_{[x_1,y_n]} * [y_n, x_n]$, denoted by γ_n, is a quasigeodesic . Similarly we choose $\{x'_n\}$ in α^b_2 with $\lim x'_n = \xi$ in ∂X^h. Let γ' be a geodesic ray in X^h with $\gamma'(0) = x'_1$ and $\gamma'(\infty) = \xi$. As above, for a nearest point projection $y'_n \in \gamma'$ of x'_n in γ', we get a sequence of quasigeodesics $\gamma'_n = \gamma'|_{[x'_1,y'_n]} * [y'_n, x'_n]$. We will show that if Case 2 holds, then $\mathrm{Hd}_{X^h}(\gamma, \gamma') = \infty$, which is a contradiction.

Claim: $\mathrm{Hd}(\gamma \cap X, \gamma' \cap X) = \infty$.

Proof of the claim: Suppose not. Suppose there exists some $M > 0$

such that $\mathrm{Hd}(\gamma \cap X, \gamma' \cap X) = M$. Let $\{z_k\} \subset \gamma \cap X$ and $\{z'_k\} \subset \gamma' \cap X$ such that $z_k \to \xi$, $z'_k \to \xi$ in X^h and $d(z_k, z'_k) \leq M$. For each $k > 0$, there exists n_k such that $z_k \in \gamma_{n_k}$ and $z'_k \in \gamma'_{n_k}$. Let β_{n_k} and β'_{n_k} denote geodesics joining x_1 to x_{n_k} and x'_1 to x'_{n_k} in B_1 and B_2 respectively. By Theorem 4.3.3, these are quasigeodesics in $\mathcal{TC}(X)$. By Lemma 4.2.7, there exists $K > 0$ such that $\gamma_{n_k} \bigcap X$ and $\gamma'_{n_k} \bigcap X$ lie in K-neighbourhood of β_{n_k} and β'_{n_k} respectively. So there exists $w_k \in \beta^b_{n_k}$ and $w'_k \in \beta'^b_{n_k}$ such that $d(z_k, w_k) \leq K$ and $d(z'_k, w'_k) \leq K$. Then, $d(w_k, w'_k) \leq M + 2K = B$, say.

Let Y_1 and Y_2 be the connected components obtained by removing X_{e_1} from X, with Y_1 containing X_{v_1} and Y_2 containing X_{v_2}. Since $N^h_C(\alpha_1) \cap f_{e_1}(X_{e_1})$ has finite diameter, only finitely many β_{n_k} pass through it. So for infinitely many k, $w_k \in Y_1$. Since, for all such k, $w'_k \in Y_2$ and $d(w_k, w'_k) \leq B$, there is a sequence $\{t_k\}$ in $f_{e_1}(X_{e_1})$, and hence in $f^h_{e_1}(X^h_{e_1})$, satisfying $d(w_k, t_k) \leq B$. Thus, there exists a flow of ξ_1 into $X^h_{u_1}$, which contradicts our assumption. This proves the claim and also that $\mathrm{Hd}_{X^h}(\gamma, \gamma') = \infty$. $\square$

Now we show that the flow of a conical limit point is a conical limit point.

Lemma 4.4.4. *Let $v \in V(\mathcal{T})$ and let $\xi_v \in \partial X^h_v$ such that its image under the CT map, say ξ, is a conical limit point of X_v. Suppose ξ_v can be flowed into ∂X^h_u and let ξ_u be the flow. Then ξ_u also maps to a conical limit point of X^h_u under the CT map.*

Proof. It is enough to check the case when v and u are adjacent. Rest follows by induction. So without loss of generality, assume that $d_{\mathcal{T}}(v, u) = 1$. Let e be the edge in $\mathcal{T}$ joining u to v. Let $\widehat{\alpha}_u$ be an electric geodesic ray in $\widehat{X_u}$ with an electro-ambient quasigeodesic ray α_u satisfying $\alpha_u(\infty) = \xi_u$. Let $B_{\widehat{\alpha}_u}$ be a ladder. Since ξ_u is a flow of ξ_v, we have $\xi_u \in \partial f^h_{e,u}(\partial X^h_e)$. So $f^h_{e,u}(X^h_e)$ is an unbounded subset of X^h_u. Let $p \in f^h_{e,u}(X^h_e)$ be a nearest point projection of $\alpha_u(0)$ on $f^h_{e,u}(X^h_e)$ and let μ be a geodesic ray in X^h_u starting at p with

$\mu(\infty) = \xi_u$. Then, α_u and μ are finite Hausdorff distance apart in X^h_u. By quasiconvexity of $f^h_{e,u}(X^h_e)$, $\mu \subset N^h_{C_2}(f^h_{e,u}(X^h_e))$. Let $x \in \alpha_u$ such that for a nearest point projection $y \in \mu$ of x on μ, y satisfies $d_{X^h_u}(p, y) > D_{2.2.21}$. Then by Lemma 2.2.21, $[\alpha_u(0), p] \cup \mu|_{[p,y]} \cup [y, x] \subset N^h_{C_{2.2.21}}(\alpha_u)$. Doing this for all such x we have, $\mu \subset N^h_{C_{2.2.21}}(\alpha_u)$. Therefore, for $C = C_{2.2.21} + C_2$, $N^h_C(\alpha_u) \cap f_{e,u}(X_e)$ has infinite diameter in X^h_u. Hence, by the construction of $B_{\widehat{\alpha}_u}$, the ladder extends to $\widehat{X}_v$ and $\widehat{\alpha}_v = B_{\widehat{\alpha}_u} \cap p^{-1}(v)$ is a geodesic ray in $\widehat{X}_u$ and for its electro-ambient quasigeodesic ray, $\alpha_v(\infty) = \xi_v$.

Let $\{x_n\}$ be a sequence of elements in α^b_v such that $\lim x_n = \xi$ in ∂X^h and let γ be a geodesic ray with $\gamma(0) = x_1$ and $\gamma(\infty) = \xi$. For each $n > 0$, let $y_n \in \gamma$ be a nearest point projection of x_n in γ. By Lemma 2.2.38, $\gamma_n = \gamma|_{[x_1,y_n]} * [y_n, x_n]$ is a quasigeodesic ray in X^h. Since ξ is a conical limit point of X_v, by the definition of conical limit points, there exists a real number $R \geq 0$ and an infinite sequence of elements $\{w_k\}$ in X_v such that $\lim w_k = \xi$ and $w_k \in N^h_R(\gamma)$. Using Lemma 2.3.9, there is $R_1 = R_1(R)$ such that, for each k, there exists $w'_k \in \gamma \cap X$ satisfying $d_{X^h}(w_k, w'_k) \leq d(w_k, w'_k) \leq R_1$. Let $n_k > 0$ such that $w'_k \in \gamma_{n_k}$. For each $k > 0$, let β_{n_k} be a geodesic in $B_{\widehat{\alpha}_u}$ joining x_1 to x_{n_k}. This is quasigeodesic in $\mathcal{TC}(X)$. By Lemma 4.2.7, for each k, there exists $z_k \in \beta^b_{n_k}$ such that $d(z_k, w'_k) \leq K$, where K is the constant from Lemma 4.2.7. This implies that $d(z_k, w_k) \leq K + R_1$. Since $w_k \in X_v$, we have $d_{\mathcal{T}}(v, p(z_k)) \leq K + R_1$ and $d_{\mathcal{T}}(u, p(z_k)) \leq K + R_1 + 1$. Then using the vertical quasigeodesic ray starting at z_k, we get a sequence $\{t_k\} \subset \alpha^b_u \subset X_u$ satisfying $d(t_k, z_k) \leq C'(K + R_1 + 1)$. Then

$d_{X^h}(t_k, w'_k) \leq d_{X^h}(t_k, z_k) + d_{X^h}(z_k, w'_k) \leq d(t_k, z_k) + d(z_k, w'_k) \leq C'(K + R_1 + 1) + K = L$, say.

Thus, we have an infinite sequence $\{t_k\}$ in X_u such that $\lim t_k = \xi$ in X^h and $t_k \in N^h_L(\gamma)$. Hence, ξ is a conical limit point for X_u. □

This is the last lemma required to prove Theorem 4.0.1.

Lemma 4.4.5. *Let $v \in V(\mathcal{T})$ and $\partial i_v : \partial X_v^h \to \partial X^h$ be the CT map. If $\xi \in \partial i_v(\partial X_v^h)$ is a conical limit point of X_v, then $|\partial i_v^{-1}(\xi)| = 1$.*

Proof. We prove by contradiction. Let $\xi_1, \xi_2 \in \partial X_v^h$ such that $\partial i_v(\xi_1) = \partial i_v(\xi_2) = \xi \in \partial X^h$. For $i = 1, 2$, let $\widehat{\alpha}_i$ be a geodesic in $\widehat{X_v}$ with its electro-ambient quasigeodesic α_i satisfying $\alpha_i(\infty) = \xi_i$. We follow the steps of the proof of Lemma 4.4.4 with respect to $\widehat{\alpha}_1$ and $\widehat{\alpha}_2$ to get a pair of sequences of elements that are bounded distance apart but converge to two different boundary points in X_v^h.

Let $\{x_n\}$ and $\{x'_n\}$ be sequences of elements in α_1^b and α_2^b such that $\lim x_n = \lim x'_n = \xi$ in ∂X^h. Let γ and γ' be geodesic rays with $\gamma(0) = x_1$, $\gamma'(0) = x'_1$ and $\gamma(\infty) = \gamma'(\infty) = \xi$. So there exists $K' > 0$ such that $\mathrm{Hd}_{X^h}(\gamma, \gamma') \leq K'$. For each $n > 0$, let $y_n \in \gamma$ and $y'_n \in \gamma'$ be nearest point projection of x_n on γ and x'_n on γ' respectively. By Lemma 2.2.38, $\gamma_n = \gamma|_{[x_1, y_n]} * [y_n, x_n]$ and $\gamma'_n = \gamma'|_{[x'_1, y'_n]} * [y'_n, x'_n]$ are quasigeodesics in X^h. Since ξ is a conical limit point of X_v, by the definition of conical limit points, there exists a real number $R \geq 0$ and an infinite sequence of elements $\{w_k\}$ in X_v such that $\lim w_k = \xi$ and $w_k \in N_R^h(\gamma)$. Using Lemma 2.3.9, there is $R_1 = R_1(R)$ and $R_2 = R_2(R + K')$ such that, for each k, there exists $z_k \in \gamma \cap X$ and $z'_k \in \gamma' \cap X$ satisfying $d(w_k, z_k) \leq R_1$ and $d(w_k, z'_k) \leq R_2$. For each $k > 0$, there exists $n_k > 0$ such that $z_k \in \gamma_{n_k}$ and $z'_k \in \gamma'_{n_k}$. For $i = 1, 2$, let $B_i = B_{\widehat{\alpha}_i}$. For each $k > 0$, let β_{n_k} denote a geodesic in B_1 joining x_1 to x_{n_k} and λ_{n_k} denote a geodesic in B_2 joining x'_1 to x'_{n_k}. These are quasigeodesics in $\mathcal{TC}(X)$.

By Lemma 4.2.7, there exists a constant $K > 0$ such that $\gamma_{n_k} \bigcap X$ lies in K-neighbourhood of β_{n_k} and $\gamma'_{n_k} \bigcap X$ lies in K-neighbourhood of λ_{n_k} in X. So there exists $t_k \in \beta_{n_k}^b$ and $t'_k \in \lambda_{n_k}^b$ such that $d(z_k, t_k) \leq K$ and $d(z'_k, t'_k) \leq K$. Thus, $d(w_k, t_k) \leq R_1 + K$ and $d(w_k, t'_k) \leq R_2 + K$. Since $w_k \in X_v$, for each k, $d_{\mathcal{T}}(v, p(t_k))) \leq R_1 + K$ and $d_{\mathcal{T}}(v, p(t'_k))) \leq R_2 + K$.

Using vertical quasigeodesic rays, we get sequences $\{s_k\}$ and $\{s'_k\}$ in α_1^b and α_2^b respectively, such that $d(s_k, w_k) \leq C'(R_1+K)$ and $d(s'_k, w_k) \leq C'(R_2+K)$. Then $d(s_k, s'_k) \leq C'(R_1+R_2)+2C'K$. Since $X_v \to X$ is a proper embedding, $d_{X_v}(s_k, s'_k)$ is uniformly bounded in X_v and $\lim s_k = \lim s'_k$. Hence, $\xi_1 = \xi_2$. □

Corollary 4.4.6. *Let $v_1 \neq v_2 \in \mathcal{T}$. Suppose $\xi_i \in \partial X^h_{v_i}$, $i = 1, 2$, map to the same point, say ξ, under the CT maps $\partial X^h_{v_i} \to \partial X^h$, such that it is a conical limit point for both X_{v_1} and X_{v_2}, then ξ_1 can be flowed into ∂X_{v_2} and ξ_2 can be flowed into ∂X_{v_1}.*

4.4.1 Proof of Theorem 4.0.1

Proof. For $i = 1, 2$, let $w_i = g_i G_{v_i} \in V(\mathcal{T})$, $g_i \in G$ and $v_i \in V(Y)$.

Then $G_{w_i} = \mathrm{Stab}_G(w_i) = g_i G_{v_i} g_i^{-1}$.

$\Lambda_c(g_i G^h_{v_i} g_i^{-1}) = \Lambda_c(g_i G^h_{v_i}) = g_i \Lambda_c(G^h_{v_i})$.

So, it is enough to show that $\Lambda_c(G^h_{v_1}) \cap \Lambda_c(gG^h_{v_2}) = \Lambda_c(G^h_{v_1} \cap gG^h_{v_2} g^{-1})$.

It is clear that $\Lambda_c(G^h_{v_1} \cap gG^h_{v_2} g^{-1}) \subset \Lambda_c(G^h_{v_1}) \cap \Lambda_c(gG^h_{v_2})$ and we only need to prove

$$\Lambda_c(G^h_{v_1}) \cap \Lambda_c(gG^h_{v_2}) \subset \Lambda_c(G^h_{v_1} \cap gG^h_{v_2} g^{-1}).$$

Let $\xi \in \Lambda_c(G^h_{v_1}) \cap \Lambda_c(gG^h_{v_2})$. Then there exists $\xi_1 \in \Lambda_c(G^h_{v_1})$ and $\xi_2 \in \Lambda_c(gG^h_{v_2})$ such that under the CT maps, $\xi_1, \xi_2 \mapsto \xi$ in ∂G^h.

$\Theta^h_{1,v_1} : G^h_{v_1} \to X^h_{w_1}$ and $\Theta^h_{g,v_2} : gG^h_{v_2} \to X^h_{w_2}$ are quasiisometries, so there exists $\xi'_1 \in \partial X^h_{w_1}$ and $\xi'_2 \in \partial X^h_{w_2}$ such that $\partial\Theta^h_{1,v_1}(\xi_1) = \xi'_1$ and $\partial\Theta^h_{g,v_2}(\xi_2) = \xi'_2$. For $i = 1, 2$, let λ_i be a geodesic ray in $X^h_{w_i}$ with $\lambda_i(\infty) = \xi'_i$. By Corollary 4.4.6, there is a flow of ξ'_1 into $\partial X^h_{w_2}$ and ξ'_2 is the flow. It also follows from the proof of Lemma 4.4.4 that the ladder B_{λ_1} extends to $\widehat{X}_{w_2}$ and without loss of generality, take $\lambda_2 = B_{\lambda_1} \cap \widehat{X}_{w_2}$. Let $\{p_k\}$ be a sequence of points on λ_1 lying outside horoball-like sets such that $\lim p_k = \xi'_1$. Let

$d_{\mathfrak{T}}(w_1, w_2) = N$. Then using vertical quasigeodesic rays, there exists $C' \geq 0$ and a sequence $\{q_k\}$ in λ_2, lying outside horoball-like sets, such that $\lim q_k = \xi'_2$ and $d(p_k, q_k) \leq C'N$. For each $k > 0$, let $(\Theta^h_{1,v_1})^{-1}(p_k) = a_k \in G^h_{v_1}$ and $(\Theta^h_{g,v_2})^{-1}(q_k) = b_k \in gG^h_{v_2}$. Then $d(a_k, b_k) \leq d(a_k, p_k) + d(p_k, q_k) + d(q_k, b_k) \leq D_0 + C'N + D_0 = D'$. So we have sequence of points $\{a_k\}$ in G_{v_1} and $\{b_k\}$ in gG_{v_2} such that $d(a_k, b_k) \leq D'$ for all $k > 0$, and lim $a_k = \xi_1$ and lim $b_k = \xi_2$. Let $\{\omega_k\}$ be a sequence of geodesics in the Cayley graph $\Gamma(G, S)$ joining a_k to b_k and let W_k be a word labeling ω_k. Since there are only finitely many such words, there exists a constant subsequence $\{W_{k_l}\}$ of $\{W_k\}$. Let $h_l = a^{-1}_{k_1} a_{k+l}$ and $h'_l = b^{-1}_{k_1} b_{k_l}$. Let $h \in G$ be the element represented by W_{k_l}. Then $a_{n_1} h_l h = a_{n_1} h h'_l$, i.e., $h_l = h h'_l h^{-1}$. Since h'_l connects two elements of gG_{v_2}, $h'_l \in G_{v_2}$. This implies that $h_l \in G_{v_1} \cap hG_{v_2} h^{-1}$
Then $a_{k_1} h_l a^{-1}_{k_1} \in a_{k_1} G_{v_1} a^{-1}_{k_1} \cap a_{k_1} h G_{v_2} h^{-1} a^{-1}_{k_1} = G_{v_1} \cap gG_{v_2} g^{-1}$.

Since $d(a_{k_1} h_l a^{-1}_{k_1}, a_{k_1} h_l) = d(a_{k_1} h_l a^{-1}_{k_1}, a_{k_l}) = d(1, a_{k_1})$ for all $l \in \mathbb{N}$, $\lim_{l \to \infty} a_{k_1} h_l a^{-1}_{k_1} = \lim_{l \to \infty} a_{k_l} = \xi_1$. This completes the proof. □

While we are far from understanding a limit intersection theorem for general limit points of vertex and edge groups of a graph of relatively hyperbolic groups satisfying the conditions of Theorem 4.0.1, the following proposition sheds some light into the case of bounded parabolic limit points. For a finitely generated relatively hyperbolic group G, under the action of G on ∂G^h, $g \in G$ is a *parabolic element* if it has infinite order and fixes exactly one point in ∂G^h. A subgroup containing only parabolic elements is a *parabolic subgroup* and it has a unique fixed point in the boundary. This point is called a *parabolic limit point*. And a parabolic limit point p is *bounded parabolic* if its stabiliser G_p in G acts cocompactly on $\partial G^h \setminus \{p\}$.

Proposition 4.4.7. [53, Proposition 3.3] *Let H, J be infinite subgroups of a relatively hyperbolic group G. If $\xi \in \Lambda(H) \cap \Lambda(J)$ is a bounded parabolic point of H and J, then ξ is either a bounded parabolic point of $H \cap J$, or an* *isolated point in* $\Lambda(H) \cap \Lambda(J)$ *and does not lie in* $\Lambda(H \cap J)$.

Chapter 5

Pullbacks of metric bundles

In this chapter we prove the following:

Theorem 5.0.1. *Suppose $G(\mathcal{Y})$ is a complex of groups, where $\mathcal{Y}$ is a finite complex. Let T be a maximal tree in the 1-skeleton of the first barycentric subdivision of $\mathcal{Y}$. Suppose $G = \pi_1(G(\mathcal{Y}), T)$ is hyperbolic. Further, suppose the following conditions hold:*

1. $G(\mathcal{Y})$ is developable with development B'.

2. All groups G_σ, for $\sigma \in V(\mathcal{Y})$, and G_e, for $e \in E(\mathcal{Y})$, are hyperbolic and the injective homomorphisms $G_e \to G_{o(e)}$ and $G_e \to G_{t(e)}$ have finite index images in the target groups.

Let $\mathcal{Z}$ be any connected subcomplex of $\mathcal{Y}$ with maximal subtree $T_1 \subset T$ in the 1-skeleton of the first barycentric subdivision of $\mathcal{Z}$, $H = \pi_1(G(\mathcal{Z}), T_1)$ and A' be a development of $G(\mathcal{Z})$. If the natural homomorphism $i : H \to G$ is injective and the natural map $A' \to B'$ is a qi-embedding, then H is also hyperbolic and i admits a Cannon-Thurston map $\partial i : \partial H \to \partial G$.

Here, $\mathcal{Y}, \mathcal{Z}$ are in fact, the scwols associated to the finite polyhedral complexes under consideration.

In Section 5.1, we quickly recall a characterization of hyperbolicity and some results related to it. In Section 5.2, we recall metric (graph) bundles and define metric (graph) bundle morphisms, as well as, pullback bundles. In Section 5.3, we recall some required tools from [45]. In Section 5.4, we prove the existence of CT maps, followed by few corollaries in Section 5.5.

5.1 Preliminaries

Throughout this chapter, for a geodesic metric space X and a geodesic subspace Y, a geodesic joining a pair of points $x, y \in X$, is denoted by $[x, y]$. If $x, y \in Y$, we denote the geodesic in Y joining them by $[x, y]_A$. To prove the main theorem we have the following weaker analogous to a result due to Hamenstadt (cf. [31, Lemma 3.5]).

Suppose X is a δ-hyperbolic metric graph and I is an interval in $\mathbb{R}$ whose both end points are in the set $\mathbb{Z} \cup \{\infty, -\infty\}$. Suppose $Y \subset X$ is a K-quasiconvex subset which admits a surjective map $\pi : Y \to I$. Let $Y_i = \pi^{-1}(i)$ for all $i \in I \cap \mathbb{Z}$ and $Y_{ij} = \pi^{-1}([i, j])$ for all $i, j \in I \cap \mathbb{Z}$ with $i < j$. Suppose we also have the following.

(1) All the sets Y_i, Y_{ij}, $i < j, i \in I$ are K-quasiconvex in X.

(2) Y_i uniformly coarsely bisects Y into $Y_i^- = \pi^{-1}((-\infty, i] \cap I)$ and $Y_i^+ = \pi^{-1}([i, \infty) \cap I)$ for all $i \in \mathbb{Z}$ in the interior of I.

(3) $d(Y_{ii+1}, Y_{jj+1}) > 2K + 1$ for all $i, j \in I$ if $j + 1 \in I$ and $i + 1 < j$.

Proposition 5.1.1. *Given $\delta, D \geq 0, \lambda \geq 1$ and $\epsilon \geq 1$, there exists $\lambda' = \lambda_{5.1.1}(\delta, K, D, \lambda, \epsilon) \geq 1, \mu = \mu_{5.1.1}(\delta, K, D, \lambda, \epsilon) \geq 0$ such that the following holds.*

Let $m, n \in I \cap \mathbb{Z}, m < n$. Suppose Y_i, Y_j are D-cobounded in X for $m \leq i < j \leq n - 1$ for some D independent of i, j. Let $y \in Y_m, y' \in Y_n$ and let $\{y_i\}$, $m \leq i \leq n$ be a sequence points in Y defined as follows: $y_m = y, y_{i+1}$ is an ϵ-approximate nearest point projection of y_i on Y_{i+1} for $m \leq i \leq n - 1$. Let $\alpha_i \subset Y_{ii+1}$ be a λ-quasigeodesic in X joining y_i, y_{i+1}, $m \leq i \leq n - 1$

and $\beta \subset Y_n$ be a λ-quasigeodesic in X joining y_n, y'. Then the concatenation of all the α_i's and β, denoted by α is a λ'-quasigeodesic in X joining y, y'. Moreover, for $m+2 \leq i \leq n$, y_i is a μ-approximate nearest point projection of y on Y_i.

Proof. The proof is broken down into the following three claims.

Claim 1: Let $x \in Y_i^-$ and $\bar{x} \in Y_i$ be an ϵ-approximate nearest point projection of x on Y_i. Then there exists ϵ' depending on ϵ and other constants from the hypothesis of the proposition such that $\bar{x}$ is an ϵ'-approximate nearest point projection of x on Y_i^+.

Proof of Claim 1: Let x' be a 1-approximate nearest point projection of x on Y_i^+. Then by Lemma 2.2.38, $[x, x'] * [x', \bar{x}]$ is a $k = K_{2.2.38}(\delta, K)$-quasigeodesic in X. Then by stability of quasigeodesics, there exists $z \in [x, \bar{x}]$ such that $d(z, x') \leq D_{2.2.18}(\delta, k) =: D_1$. Now, since Y_i^- is K-quasiconvex, there exists $w \in Y_i^-$ such that $d(z, w) \leq K$. We have $w \in Y_i^-$ and $x' \in Y_i^+$. As Y_i uniformly coarsely bisects Y into Y_i^+ and Y_i^-, there exists $D' \geq 0$ such that $[w, x'] \cap N_{D'}(Y_i) \neq \emptyset$ and so, there exists $z_1 \in [w, x']$ such that $d(z_1, Y_i) \leq D'$. Then, $d(z_1, w) \leq d(w, x') \leq d(w, z) + d(z, x') \leq K + D_1$. Now, by Lemma 2.2.13, $\bar{x}$ is an $\epsilon + 3$-approximate nearest point projection of z on Y_i, i.e, $d(z, \bar{x}) \leq d(z, Y_i) + \epsilon + 3$ and $d(z, Y_i) \leq d(z, w) + d(w, z_1) + d(z_1, Y_i) \leq 2K + D_1 + D'$. So, $d(\bar{x}, x') \leq d(\bar{x}, z) + d(z, x') = 2K + 2D_1 + D' + \epsilon + 3$. So, for $\epsilon' = 2K + 2D_1 + D' + \epsilon + 3$, Claim 1 is proved.

Claim 2: For $m+2 \leq i \leq n-1$, there is a uniformly bounded set $A_i \subset Y_i$ such that the ϵ-approximate nearest point projection of any point in Y_j^-, $j < i$, on Y_i is contained in A_i.

Proof of Claim 2: Let $B_i \subset Y_i$ be the set of 1-approximate nearest point projections of points of Y_{i-1} on Y_i in X. Then diameter of B_i is at most D. Let $x \in Y_j^-, j < i$. Let x_1, x_2 be the ϵ-approximate nearest point projections of x on Y_{i-1}, Y_i respectively and let x_3 be the ϵ-approximate nearest point projection of x_1 on Y_i. Then by Claim 1, x_1 is an ϵ'-approximate nearest point projection of x on Y_{i-1}^+ and x_2, x_3 are ϵ'-approximate nearest point projections of x, x_1 on Y_i^+ respectively. Then by Corollary 2.2.40, $d(x_2, x_3) \leq D_{2.2.40}(\delta, K, \epsilon')$. Now let x_1' be a 1-approximate nearest point projection of x_1 on Y_i. Then $x_1' \in B_i$ and since $\epsilon \geq 1$, again by Corollary 2.2.40, $d(x_3, B_i) \leq d(x_3, x_1') \leq D_{2.2.40}(\delta, K, \epsilon')$. So, $d(x_2, B_i) \leq 2D_{2.2.40}(\delta, K, \epsilon')$. Therefore, we take $A_i = N_{2D_{2.2.40}(\delta,K,\epsilon')}(B_i) \cap Y_i$. Clearly, since B_i has bounded diameter, so does A_i.

Now by Lemma 2.2.4, it is enough to show that α is contained in a uniformly small neighbourhood of a geodesic in X joining y, y' and α is uniformly properly embedded in X.

Claim 3: α is contained in a uniformly small neighbourhood of a geodesic in X joining y, y'.

Proof of Claim 3: Suppose γ is any geodesic in X joining y, y'. Note that by Corollary 2.2.39, $\alpha_{n-1} * \beta$ is a $K_{proj-qj}(\delta, \lambda, \epsilon)$-quasigeodesic in X. We show that the points y_i, for $m+1 \leq i \leq n-1$, are uniformly close to γ. Now, y_{m+1} is an ϵ-approximate nearest point projection of $y = y_m$ on Y_{m+1}. Then by Claim 1, y_{m+1} is an ϵ'-approximate nearest point projection of y on Y_{m+1}^+. Let γ_{m+1} be a geodesic in X joining y_{m+1} and y'. Then by Lemma 2.2.18, the Hausdorff distance between $\alpha_m * \gamma_{m+1}$ and γ is unifromly small and so, there exists $x_{m+1} \in \gamma$ such that y_{m+1} and x_{m+1} are uniformly close. Now for any $m+2 \leq i \leq n-1$, y_i is an ϵ-approximate nearest point projection of y_{i_1} on Y_i. Let r denote the supremum of diameter of A_i, $m+1 \leq i \leq n-1$. Then, by the proof of Claim 2, y_i is an $\epsilon + r$-approximate nearest point projection of y on Y_i and again by Claim 1, it is an $(\epsilon + r)'$-approximate nearest point projection of y on Y_i^+. Again, as above we have $x_i \in \gamma$ such that $d(y_i, x_i)$ is uniformly small.

Claim 4: α is uniformly properly embedded in X.

Proof of Claim 4: Define $L = \sup\{d(y_i, x_i) \mid m+1 \leq i \leq n-1\}$. Let $x, x' \in \alpha$ with $d(x, x') \leq N$ and $\pi(x) = k, \pi(x') = l$. Without loss of generality, we assume that $k < l$. We claim that $l \leq k + N$. This is true because for any adjacent vertices u, v in γ with $u \in N_K(Y_{ss+1})$ and $v \in N_K(Y_{tt+1})$, by condition (3), we have $s \leq t \leq s+1$. Now, let $\alpha(s_k) = x, \alpha(s_i) = y_i$, for $k+1 \leq i \leq l-1$ and $\alpha(s_l) = x'$. Now, $d(\alpha(s_i), \alpha(s_{i+1}) \leq 2L + N$ for $l \leq i \leq l-1$. Since $l - k \leq N$ and the segments of α joining $\alpha(s_i), \alpha(s_{i+1})$ are uniform quasigeodesics, we are done.

For the second part of the proposition, we already know that y_i is an $(\epsilon + r)$-approximate nearest point projection of any point in Y_j^- on Y_i, $j < i$, $m+2 \leq i \leq n-1$. In particular, y_i is an $(\epsilon + r)$-approximate nearest point projection of y on Y_i. Now, y_{n-1} is an $(\epsilon + r)'$-approximate nearest point projection of y on Y_{n-1}^+. Let y_n' be a 1-approximate nearest point projection of y on Y_n. Then, by Lemma 2.2.40, $d(y_n', y_n) \leq D_{2.2.40}(\delta, K, (\epsilon + r)')$. Thus, y_n is an $(1 + D_{2.2.40}(\delta, K, (\epsilon + r)'))$-approximate nearest point projection of y on Y_n. Take $\mu = \max\{(\epsilon + r)', 1 + D_{2.2.40}(\delta, K, (\epsilon + r)')\}$.

5.2 Metric bundles and metric graph bundles

In this section, we recall the definitions and some elementary properties of the primary objects of study in this chapter, namely metric bundles and metric graph bundles, from [45].

3

Definition 5.2.1. [45] ***Metric bundle (Definition 1):*** *Suppose* (X, d) *and* (B, d_B) *are geodesic metric spaces; let* $c \geq 1$ *and let* $f_0 : \mathbb{R}^+ \to \mathbb{R}^+$ *be a function.* X *is an* (f_0, c)*-* metric bundle *over* B *if there is a surjective* 1*-Lipschitz map* $\pi : X \to B$ *such that the following conditions hold:*

1. *For each point* $b \in B$, $F_b := \pi^{-1}(b)$ *is a geodesic metric space*

with respect to the path metric d_b *induced from* X. *The inclusion maps* $i_b : (F_b, d_b) \to X$ *are uniformly metrically proper as measured by* f_0.

2. Suppose $b_1, b_2 \in B$, $d_B(b_1, b_2) \leq 1$ *and let* γ *be a geodesic in* B *joining them. Then for any point* $x \in F_b$, $b \in \gamma$, *there is a path in* $\pi^{-1}(\gamma)$ *of length at most* c *joining* x *to both* F_{b_1} *and* F_{b_2}.

Given geodesic metric spaces X *and* B, X *is a* **metric bundle** *over* B *if* X *is an* (f_0, c)*-metric bundle over* B *in the above sense for some function* $f_0 : \mathbb{R}^+ \to \mathbb{R}^+$ *and some constant* $c \geq 1$.

If X is a metric bundle over B in the above sense, then we refer to it as a **geodesic metric bundle**. But as the above definition seems a little restrictive, we make a minor modification to this definition. However, this new definition implies the original definition of metric bundle.

Definition 5.2.2. ***Metric bundle (Definition 2):*** *Suppose* (X, d) *and* (B, d_B) *are* length spaces*; let* $c \geq 1$ *and let* $f_0 : \mathbb{R}^+ \to \mathbb{R}^+$ *be a function. We say that* X *is an* (f_0, c)*-* length metric bundle *over* B *if there is a surjective* 1*-Lipschitz map* $\pi : X \to B$ *such that the following conditions hold:*

1. For each point $b \in B$, $F_b := \pi^{-1}(b)$ *is a length space with respect to the path metric* d_b *induced from* X. *The inclusion maps* $i_b : (F_b, d_b) \to X$ *are uniformly metrically proper as measured by* f_0.

2. Suppose $b_1, b_2 \in B$, *and let* γ *be a* path *of length at most* 1 *in* B *joining them. Then for any point* $x \in F_b$, $b \in \gamma$, *there is a path in* $\pi^{-1}(\gamma)$ *of length at most* c *joining* x *to both* F_{b_1} *and* F_{b_2}.

From now on, by metric bundle, we mean a length metric bundle, unless otherwise specified. The following proposition is an exact replica of [45, Proposition 1.4]. Since the proofs are very similar, we omit it.

Proposition 5.2.3. *Let* X *be an* (f_0, c)*-metric bundle over* B. *Then there exists* $K_{5.2.3} = K_{5.2.3}(f_0, c) \geq 1$, *such that the following holds.*

Suppose $b_1, b_2 \in B$ and let γ be a path in B of length at most 1 joining them. Let $\phi : F_{b_1} \to F_{b_2}$, be any map such that for every $x_1 \in F_{b_1}$ there is a path of length at most c in $\pi^{-1}(\gamma)$ joining x_1 to $\phi(x_1)$. Then ϕ is a $K_{5.2.3}$-quasiisometry.

We refer to an (f_0, c)- metric bundle as an (f_0, c, K)**- metric bundle**.

Definition 5.2.4. ***Metric graph bundle:*** *Suppose X and B are metric graphs. Let $f_0 : \mathbb{N} \to \mathbb{N}$ be a function. Then X is an f_0-*metric graph bundle *over B if there exists a surjective simplicial map $\pi : X \to B$ such that:*

(1) For each $b \in V(B)$, $F_b := \pi^{-1}(b)$ is a connected subgraph of X and the inclusion maps $i_b : V(F_b) \to X$ are uniformly metrically proper (as measured by f_0) for the path metric d_b induced on F_b.

(2) Suppose $b_1, b_2 \in V(B)$ are adjacent vertices.Then each vertex x_1 of F_{b_1} is connected by an edge to a vertex in F_{b_2}.

Remark 2. *Since the map p is simplicial, it follows that it is 1-Lipschitz.*

Now, we have the following analog of Proposition 5.2.3.

Proposition 5.2.5. *Suppose X is an f_0-metric graph bundle over B. Then there exists $K_{5.2.5} = K_{5.2.5}(f_0) \geq 1$ such that the following holds.*

Suppose $b_1, b_2 \in V(B)$ are adjacent vertices. Let $\phi : F_{b_1} \to F_{b_2}$ be any map such that each $x_1 \in V(F_{b_1})$ is connected to $\phi(x_1) \in V(F_{b_2})$ by an edge, and any interior point on an edge of F_{b_1} is sent to the image of one of the vertices on which the edge is incident. Then any such ϕ is a $K_{5.2.5}$-quasiisometry.

We refer to an f_0-metric graph bundle as an (f_0, K)**-metric graph bundle** (with $K = K_{5.2.5}(f_0)$), or simply as a **metric graph bundle** when f_0, K are understood.

Terminology:

(1) For a metric (graph) bundle, the spaces (F_b, d_b), $b \in B$ or $b \in V(B)$, will be referred to as **horizontal spaces** or **fibers** and the distance between

two points in F_b will be referred to as their **horizontal distance**. A geodesic in F_b will be called a **horizontal geodesic**.

(2) The spaces X and B will be referred to as the *total space* and the *base space* respectively.

(3) By X being a metric bundle (resp. metric graph bundle) we mean that it is the total space of a metric bundle (resp. metric graph bundle).

(4) For metric (graph) bundle X and any $x, y \in X$, $d(x, y)$ denotes the distance between x, y in X.

(5) For $b_1, b_2 \in B$, $d_B(b_1, b_2)$ denotes the distance between b_1, b_2 in B.

Corollary 5.2.6 (**Bounded flaring condition for metric graph bundles**). [45, Corollary 1.14] *For all $k \in \mathbb{R}$, $k \geq 1$ there is a function $\mu_k : \mathbb{N} \to [1, \infty)$ such that the following holds:*

Suppose X is an (f_0, K)-metric graph bundle with base space B. Let $\gamma \subset B$ be a geodesic joining $b_1, b_2 \in V(B)$, and let $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ be two k-qi lifts of γ in X which join x_1 to x_2 and y_1 to y_2 respectively, so that $p(x_i) = p(y_i) = b_i$, $i = 1, 2$. For all $N \in \mathbb{N}$, if $d_B(b_1, b_2) \leq N$ then

$$d_{b_1}(x_1, y_1) \leq \mu_k(N) max\{d_{b_2}(x_2, y_2), 1\}.$$

In the rest of the chapter, we summarise the conclusion of Corollary 5.2.6 by saying that a metric graph bundle satisfies a **bounded flaring condition**.

Definition 5.2.7. ***Quasiisometric sections:*** *Suppose $A \subset B$ and $K \geq 1$. A K-*qi section *over A is a K-qi embedding $s : V(A) \to X$ such that $\pi \circ s = Id_A$. A K-qi section over a geodesic α is also called a K-*qi lift *of α.*

Remark 3. *We will refer to the image of a qi section as qi section most of the time. Clearly, the image determines the qi section. Also, it is clear that given a qi section $s : V(A) \to X$ we can always extend the definition to all of A. However, we generally do not need that, and so use the restricted definition.*

The following lemma is immediate from the definition of a metric (graph) bundle.

Lemma 5.2.8. *Suppose $\pi : X \to B$ is an (f_0, c)-metric bundle or f_0-metric graph bundle.*
(1) Suppose $b_1, b_2 \in B$. Let $\gamma : [0, L] \to B$ be a continuous, rectifiable, arc length parametrized path (resp. an edge path) in B joining b_1, b_2. Given any $x \in F_{b_1}$, there is a path $\tilde{\gamma}$ in $\pi^{-1}(\gamma)$ such that $l(\gamma) \leq Lc$ (resp. $l(\gamma) = L$) joining x to some point in F_{b_2}.

In particular, when X is a metric graph bundle over B any geodesic γ of B can be lifted to a geodesic starting from any given point of $\pi^{-1}(\gamma)$.

(2) In the case of a metric bundle given any $k \geq 1$, $\epsilon \geq 0$, any dotted (k, ϵ)-quasigeodesic $\beta : [m, n] \to B$ has a lift $\tilde{\beta}$ starting from any point of $F_{\beta(m)}$ such that for all $i, j \in [m, n]$, we have

$$-\epsilon + \frac{1}{k}|i - j| \leq d_X(\tilde{\beta}(i), \tilde{\beta}(j)) \leq c \cdot (k + \epsilon + 1)|i - j|.$$

In particular, it is a $c(k + \epsilon + 1)$-qi lift of β. Also, $l(\tilde{\beta}) \leq ck(k + \epsilon + 1)(\epsilon + d_B(b_1, b_2))$.

Proof. Let $0 = t_0, t_1, \ldots, t_n = L$ be a sequence of points in $[0, L]$ such that the length of $l(\gamma)|_{[t_i, t_{i+1}]} = 1$ for $0 \leq i \leq n - 1$ and $l(\gamma)|_{[t_{n-1}, t_n]} \leq 1$ in the metric bundle case. For metric graph bundle, we have $t_i = i, 0 \leq i \leq L = n$. Let $x = x_0$. Then there exists a path $\tilde{\gamma}_0$ in $\pi^{-1}(\gamma[0, t_1])$ with length at most c joining x to some point in F_{t_1}. We denote this point by x_1. By induction, given $x_i \in F_{t_i}$, there exists a path $\tilde{\gamma}_i$ in $\pi^{-1}(\gamma[t_i, t_{i+1}])$ with length at most c joining x_i to some point in $F_{t_{i+1}}$, denoted by x_{i+1}. For metric graph bundles, we have $c = 1$. Then $\tilde{\gamma}$ is the concatenation of the paths $\tilde{\gamma}_i$ and clearly, $l(\tilde{\gamma}) = l(\tilde{\gamma}_0) + \ldots + l(\tilde{\gamma}_{n-1}) \leq Lc$. Clearly, in the case of metric graph bundles, we have $l(\tilde{\gamma}) = L$.

In the case of metric graph bundles, suppose γ is a geodesic. Then $d_B(b_1, b_2) = L$. Now, if $\tilde{\gamma}$ joins x to $x_n \in F_{t_n}$, then $d(x, x_n) \geq L$. Also, $L = d(x, x_n) \leq l(\tilde{\gamma}) = l(\tilde{\gamma}_0) + \ldots + l(\tilde{\gamma}_{n-1}) = L$. Thus, $\tilde{\gamma}$ is a geodesic in X.
(2) Let $x \in F_{\beta(m)}$. We know that, for $m \leq i \leq n-1$, $d_B(\beta(i), \beta(i+1)) \leq k+\epsilon$. Then there exists a path β_i in B joining $\beta(i), \beta(i+1)$ with length at most $k+\epsilon+1$. Then, there exists a path in $\pi^{-1}(\beta_i)$ in X joining any point of $F_{\beta(i)}$ to a point of $F_{\beta(i+1)}$ with length at most $(k+\epsilon+1)c$. Thus, we can inductively construct a sequence of points $x_i \in F_{\beta(i)}$, $m \leq i \leq n$ with $x_m = x$. Let $\tilde{\beta}_i$ be the path in $\pi^{-1}(\beta_i)$ joining x_i, x_{i+1} for $m \leq i \leq n-1$. Finally, $\tilde{\beta}$ is defined by setting $\tilde{\beta}(i) = x_i$, $m \leq i \leq n$.

Clearly, $d(\tilde{\beta}(i), \tilde{\beta}(j)) \leq c.(k+\epsilon+1)|i-j|$. Also, $d_B(\beta(i), \beta(j)) = d_B(\pi \circ \tilde{\beta}(i), \pi \circ \tilde{\beta}(j)) \leq d(\tilde{\beta}(i), \tilde{\beta}(j))$. Then using the fact that β is a (k, ϵ)-dotted quasigeodesic in B, we have

$$-\epsilon + \frac{1}{k}|i-j| \leq d_X(\tilde{\beta}(i), \tilde{\beta}(j)) \leq c \cdot (k+\epsilon+1)|i-j|.$$

Also, $l(\tilde{\beta}) = \sum_{i=m}^{n-1} d(\tilde{\beta}(i), \tilde{\beta}(i+1)) \leq \sum_{i=m}^{n-1} c \cdot (k+\epsilon+1) = (n-m)c \cdot (k+\epsilon+1)$.

Since $-\epsilon + \frac{1}{k}(n-m) \leq d_B(b_1, b_2)$, we have the required inequality. □

The proof of the following corollary appears in [45, Proposition 2.10]. We include a proof for the sake of completeness.

Corollary 5.2.9. *Let $d_B(b_1, b_2) = l$. Then we can define a map $\phi : F_{b_1} \to F_{b_2}$ such that for all $x \in F_{b_1}$, $d(x, \phi(x)) \leq 3c + 3cl$ (in the case of metric graph bundles, $d(x, \phi(x)) = l$).*

Proof. The proof is trivially true for the case of metric graph bundles by the definition. In the case of metric bundles, let γ be a dotted $(1,1)$-quasigeodesic in B joining b_1, b_2. For any $x \in F_{b_1}$, let $\tilde{\gamma}$ be a dotted lift starting x as constructed in Lemma 5.2.8 (2). Define $\phi(x)$ to be the endpoint of $\tilde{\gamma}$. Then the statement follows from Lemma 5.2.8 (2). □

Remark 4. *For any $b_1, b_2 \in B$ any map $\phi : F_{b_1} \to F_{b_2}$ with $d(x, \phi(x)) \leq D$, for all $x \in F_{b_1}$ and some constant D independent of x, will be referred to as a fiber identification map.*

Lemma 5.2.10. *Let $\pi : X \to B$ be an (f_0, c)-metric bundle (f_0-metric graph bundle). For any $b_1, b_2 \in B$ (or $b_1, b_2 \in V(B)$), let $\phi_{b_1b_2} : F_{b_1} \to F_{b_2}$ be a map such that for all $x \in F_{b_1}$, $d(x, \phi_{b_1b_2}(x)) \leq R$ (in the case of metric graph bundles, we have $\phi : V(F_{b_1}) \to V(F_{b_2})$). Then ϕ is a $D_{5.2.10}$-surjective $K_{5.2.10} = K_{5.2.10}(R)$-qi.*

Proof. We will prove this only for the case of metric bundles as the metric graph bundle case is similar. First we will show that ϕ_{b_1,b_2} is coarsely Lipschitz. Let $x, y \in F_{b_1}$ such that $d_{b_1}(x, y) \leq 1$. Then, $d(x, y) \leq 1$ and $d(\phi_{b_1,b_2}(x), \phi_{b_1,b_2}(y)) \leq d(\phi_{b_1,b_2}(x), x) + d(x, y) + d(y, \phi_{b_1,b_2}(y)) \leq 2R + 1$. Then, $d_{b_2}(\phi_{b_1,b_2}(x), \phi_{b_1,b_2}(y)) \leq f_0(2R + 1)$. Now, $d_B(b_1, b_2) \leq R$. By Lemma 5.2.9, we can also define $\phi_{b_2b_1} : F_{b_2} \to F_{b_1}$ such that for every $x' \in F_{b_2}$, $d(x', \phi_{b_2b_1}(x')) \leq 3c(R + 1)$. Then as above, $\phi_{b_2b_1}$ is coarsely $f_0(6c(R+1)+1)$-Lipschitz. Further, for any $x \in F_{b_1}$, $d(x, \phi_{b_2b_1} \circ \phi_{b_1b_2}(x)) \leq d(x, \phi_{b_1b_2}(x)) + d(\phi_{b_1b_2}(x), \phi_{b_2b_1} \circ \phi_{b_1b_2}(x)) \leq R + (3c+1)R$ and so, $d_{b_1}(x, \phi_{b_2b_1} \circ \phi_{b_1b_2}(x)) \leq f_0(R+(3c+1)R)$. Similarly, for $x' \in F_{b_2}$, $d_{b_2}(x', \phi_{b_1b_2} \circ \phi_{b_2b_1}(x')) \leq f_0(R + (3c + 1)R)$. Thus, for $D_{5.2.10} = f_0(R + (3c + 1)R)$ and $K_{5.2.10} = K_{2.2.1}(f_0(2R + 1), f_0(6c(R + 1) + 1), D_{5.2.10})$, we have the result. □

Clearly, these maps are coarsely well-defined. The next lemma easily follows using 5.2.10 and the definition of metric (graph) bundles.

Lemma 5.2.11. *Let $\pi : X \to B$ be a metric bundle (metric graph bundle) and let $b_1, b_2 \in B$ (or $b_1, b_2 \in V(B)$) with $d_B(b_1, b_2) \leq R$. Suppose $\phi_{b_1b_2} : F_{b_1} \to F_{b_2}$ is a fiber identification map as constructed in the proof of Corollary 5.2.9. Then, ϕ_{b_1,b_2} is a $K_{5.2.11} = K_{5.2.11}(R)$-qi.*

Example 5.2.12 (Tangent bundle of a manifold). *Suppose M is a (complete) Riemannian manifold. Consider the Sasaki metric on the tangent bundle TM of M. We claim that (TM, M, π) is a metric bundle where $\pi : TM \to M$ is the natural foot-point projection map. Given $p \in M$, the fiber of π is the tangent space T_pM. We know that the inclusion maps $T_pM \to TM$, $p \in M$ are isometric embeddings in the Riemannian sense and hence in our sense too. In particular, the fibers of π are uniformly properly embedded in TM. On the other hand, given $p, q \in M$ $v \in T_pM$, and a piecewise smooth path $\gamma \subset M$ joining p, q we can consider the parallel transport of v along γ. This gives a lift $\tilde{\gamma}(t) := (\gamma(t), v(t))$ of γ in TM joining $(p, v) \in T_pM$ to a point of T_qM. For the Sasaki metric, $l(\tilde{\gamma}) = l(\gamma)$. This checks all the hypotheses of a metric bundle.*

Definition 5.2.13. ***(1) Metric bundle morphisms:*** *Let (X_i, B_i, π_i), $i = 1, 2$ be (f_0, c)- metric bundles. A* morphism *from (X_1, B_1, π_1) to (X_2, B_2, π_2) (or simply from X_1 to X_2 when there is no possibility of confusion) consists of a pair of maps (f, g), where $f : X_1 \to X_2$ and $g : B_1 \to B_2$ are such that there is a constant $D > 0$ with the following conditions:*

(i) The maps f, g are coarsely D-Lipschitz.
(ii) $\pi_2 \circ f = g \circ \pi_1$, i.e., for all $b \in B_1$ we have $f(\pi_1^{-1}(b)) \subset \pi_2^{-1}(g(b))$. Let $f_b : \pi_1^{-1}(b) \to \pi_2^{-1}(g(b))$ denote the restriction of f to $\pi_1^{-1}(b)$.

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow[g]{} & B_2 \end{array}$$

Figure 5.1

(iii) Let $a, b \in B_1$ be any points joined by a path of length at most 1,

and let $p = g(a), q = g(b) \in B_2$. *Let* $\phi^1_{ab} : \pi_1^{-1}(a) \to \pi_1^{-1}(b)$ *and* $\phi^2_{pq} : \pi_2^{-1}(p) \to \pi_2^{-1}(q)$ *be the fiber identification maps for the two bundles* X_1 *and* X_2 *respectively. Then the maps* $\phi^2_{pq} \circ f_a$ *and* $f_b \circ \phi^1_{ab}$ *are uniformly close to each other. This is automatic since* f *is Lipschitz.*

(2) Metric graph bundle morphisms: *In the case of* (f_0*-) metric graph bundles, we further require that the maps* f, g *send vertices to vertices and edges linearly to edge paths (so that the maps* f, g *are Lipschitz).*

(3) Isomorphisms: *A morphism* (f, g) *from a metric (graph) bundle* (X_1, B_1, π_1) *to a metric (graph) bundle* (X_2, B_2, π_2) *is called an* isomorphism *if there is a morphism* (f', g') *from* (X_2, B_2, π_2) *to* (X_1, B_1, π_1) *such that* $f \circ f'$ *and* $f' \circ f$ *are both at a finite distance from the corresponding identity maps.*

We refer to the maps f_b in the definition of bundle morphisms as *fiber maps*. We note that the map f determines the map g, and hence we will generally say that 'f is a bundle morphism from X_1 to X_2'. Basic properties of morphisms are recorded as a proposition below.

Proposition 5.2.14. *Suppose* (f, g) *is a morphism of metric (graph) bundles as in the definition above. Then the following hold:*

(1) *The restrictions of* f *on the fibers are uniformly coarsely Lipschitz.*

(2) *Suppose* $\gamma \subset B_1$ *is a* $(1, 1)$*-quasigeodesic (or a geodesic in the case of a metric graph bundle) and suppose* $\tilde{\gamma}$ *is an qi lift of* γ, $L \geq 1$. *If* g *is coarsely Lipschitz, then* $f \circ \tilde{\gamma}$ *is a uniform qi lift over* $g \circ \gamma$.

In particular, in case of metric graph bundles, the edges in X_1 *connecting points of different fibers are mapped under* f *to an edge path with no edge in any fiber of* X_2.

(3) *If* (f, g) *is an isomorphism of metric (graph) bundles then the maps* f, g *are quasiisometries and all the fiber maps are uniform quasiisometries.*

Conversely, if the map g is a qi and the fiber maps are uniform qi then (f,g) is an isomorphism.

Proof. (1) Let $b \in B_1$ and $x, y \in \pi_1^{-1}(b)$ be such that $d_b(x,y) \leq 1$. Since f is coarsely D-Lipschitz, $d_{X_2}(f(x), f(y)) \leq D d_{X_1}(x,y) + D$. Now, $d_{X_1}(x,y) \leq d_b(x,y)$ since d_b is the induced length metric on $\pi_1^{-1}(b)$ from X_1. Thus $d_{X_2}(f(x), f(y)) \leq 2D$. Now, the fibers of π_2 are uniformly properly embedded in X_2. Hence, $d_{g(b)}(f(x), f(y))$ is uniformly bounded. Therefore, we are done by Lemma 2.2.5.

(2) First, observe that for $b_1, b_2, b_3 \in B_i$, where $i = 1, 2$, and $z \in F_{b_1}$, $d(\phi_{b_1b_3}(z), \phi_{b_2b_3} \circ \phi_{b_1b_2}(z)) \leq d(\phi_{b_1b_3}(z), z) + d(z, \phi_{b_1b_2}(z)) + d(\phi_{b_1b_2}(z), \phi_{b_2b_3} \circ \phi_{b_1b_2}(z)) \leq c(d_{B_1}(b_1, b_3) + d_{B_1}(b_1, b_2) + d_B(b_2, b_3))$. Since the fibers are uniformly properly embedded as measured by f_0, $d_{b_3}(\phi_{b_1b_3}(z), \phi_{b_2b_3} \circ \phi_{b_1b_2}(z)) \leq f_0(c(d_{B_i}(b_1, b_3) + d_{B_i}(b_1, b_2) + d_{B_i}(b_2, b_3))) =: D(c, d_{B_i}(b_1, b_3))$.

Let $\gamma : [0, l] \to B_1$ be a $(1,1)$-quasigeodesic in B_1 (geodesic in the case of metric graph bundles) with $\gamma(0) = b_1, \gamma(l) = b_2$. Let $\tilde{\gamma}$ join $x \in F_{b_1}$ to $y \in F_{b_2}$. By Corollary 5.2.10, there exists a $K_{5.2.10}(l)$-quasiisometry $\phi_{\gamma(t_1)\gamma(t_2)} : F_{\gamma(t_1)} \to F_{\gamma(t_2)}$ with $\phi_{\gamma(t_1)\gamma(t_2)}(\tilde{\gamma}(t_1)) = \tilde{\gamma}(t_2)$, for $t_1, t_2 \in [0, l]$. Now, let $\widetilde{g \circ \gamma}$ be a lift of the path $g \circ \gamma$, starting at $f(x)$. We have, $f(\tilde{\gamma}(1)) = f \circ \phi_{b_1\gamma(1)}(x) = f_{\gamma(1)} \circ \phi_{b_1\gamma(1)}(x)$. By (iii) of Definition 5.2.13, $\phi_{g(b_1)g(\gamma(1))}(f(x)) = \phi_{g(b_1)g(\gamma(1))} \circ f_{b_1}(x)$ is uniformly close to $f_{\gamma(1)} \circ \phi_{b_1\gamma(1)}(x)$. Then, $f(\tilde{\gamma}(1)) = \phi_{g(b_1)g(\gamma(1))}(f(x))$ and $f(\tilde{\gamma}|_{[0,1]})$ is a lift of $g \circ \gamma([0,1])$ with $l(f(\tilde{\gamma}|_{[0,1]})) \leq D l(\tilde{\gamma}|_{[0,1]}) + D =: D'$. We continue this inductively. By the above observation, for any $t_1, t_2, t_3 \in [0, l]$, $\phi_{\gamma(t_1)\gamma(t_3)}(z) = \phi_{\gamma(t_2)\gamma(t_3)} \circ \phi_{\gamma(t_1)\gamma(t_2)}(z)$. Also, by (iii) of Definition 5.2.13, $\phi_{g(b_1)g(b_2)}(f(x)) = f_{b_2} \circ \phi_{b_1b_2}(x) = f(y)$. So, $f \circ \tilde{\gamma}$ is a lift of $\widetilde{g \circ \gamma}$. Moreover, for any $s, t \in [0, l]$, $l(f(\tilde{\gamma}|_{[s,t]})) \leq D' l_{B_2}(g \circ \gamma([s,t]))$. Thus, $f \circ \tilde{\gamma}$ is a qi-lift of $g \circ \gamma$.

(3) We shall prove this only in the case of a metric bundle. The proof for a metric graph bundle is very similar and hence we skip it.

If (f, g) is an isomorphism then f, g are qi by Lemma 2.2.1(1). We need to show that the fiber maps are quasiisometries.

Suppose (f', g') is a coarse inverse of (f, g) such that $d_{X_2}(f \circ f'(x_2), x_2) \leq R$ and $d_{X_1}(f' \circ f(x_1), x_1) \leq R$ for all $x_1 \in X_1$ and $x_2 \in X_2$. It follows that for all $b_1 \in B_1, b_2 \in B_2$, we have $d_{B_1}(b_1, g' \circ g(b_1)) \leq R$ and $d_{B_2}(b_2, g \circ g'(b_2)) \leq R$. Suppose f', g' are coarsely D'-Lipschitz. Let $D_1 = f_0(2D)$ and $D_2 = f_0(2D')$. Then for all $u \in B_1$, $f_u : \pi_1^{-1}(u) \to \pi_2^{-1}(g(u))$ is coarsely D_1-Lipschitz and for all $v \in B_2$, $f'_v : \pi_2^{-1}(v) \to \pi_1^{-1}(g'(v))$ is coarsely D_2-Lipschitz by (1).

Let $b \in B_1$. To show that $f_b : \pi_1^{-1}(b) \to \pi_2^{-1}(g(b))$ is a uniform quasiisometry, it is enough, by Lemma 2.2.1(1), to find a uniformly coarsely Lipschitz map $\pi_2^{-1}(g(b)) \to \pi_1^{-1}(b)$ which is a uniform coarse inverse of f_b. We already know that $f'_{g(b)}$ is D_2-coarsely Lipschitz. Let $b_1 = g' \circ g(b)$. We also note that $d_{B_1}(b, b_1) \leq R$. Hence, it follows by Corollary 5.2.9 and Corollary 5.2.11 that we have a $K_{5.2.10}(R)$-qi $\phi_{b_1 b} : \pi_1^{-1}(b_1) \to \pi_1^{-1}(b)$ such that $d_{X_1}(x, \phi_{b_1 b}(x)) \leq 3c + 3cR$ for all $x \in \pi_1^{-1}(b_1)$. Let $h = \phi_{b_1 b} \circ f'_{g(b)}$. We claim that h is a uniformly coarsely Lipschitz, uniform coarse inverse of f_b. Since $f'_{g(b)}$ is D_2-coarsely Lipschitz and clearly $\phi_{b_1 b}$ is $K_{5.2.10}(R)$-coarsely Lipschitz, it follows by Lemma 2.2.3(1) that h is $(D_2 K_{5.2.10}(R) + K_{5.2.10}(R))$-coarsely Lipschitz. Moreover, for all $x \in \pi_1^{-1}(b)$ we have $d_{X_1}(x, h \circ f_b(x)) \leq d_{X_1}(x, f'_{g(b)} \circ f_b(x)) + d_{X_1}(f'_{g(b)} \circ f_b(x), h \circ f_b(x)) \leq R + 3c + 3cR$. Hence, $d_b(x, h \circ f_b(x)) \leq f_0(R + 3c + 3cR)$. Let $y \in \pi_2^{-1}(g(b))$. Then

$$\begin{aligned} d_{X_2}(y, f_b \circ h(y)) &= d_{X_2}(y, f \circ \phi_{b_1 b} \circ f'(y)) \\ &\leq d_{X_2}(y, f \circ f'(y)) + d_{X_2}(f \circ f'(y), f \circ \phi_{b_1 b} \circ f'(y)) \\ &\leq R + D(3c + 3cR) + D. \end{aligned}$$

since $d_{X_1}(f'(y), \phi_{b_1 b} \circ f'(y)) \leq 3c + 3cR$. Hence, $d_{g(b)}(y, f_b \circ h(y)) \leq f_0(R + D(3c + 3cR) + D)$. Hence by Lemma 2.2.1(1) f_b is a uniform qi.

Conversely, suppose all the fiber maps of the morphism (f, g) are (λ, ϵ)-qi which are coarsely R-surjective and g is a (λ_1, ϵ_1)-qi which is R_1-surjective. Let g' be a coarsely (K, C)-quasiisometric, L-coarse inverse of g where $K = K_{2.2.1}(\lambda_1, \epsilon_1, R_1)$, $C = C_{2.2.1}(\lambda_1, \epsilon_1, R_1)$ and $L = L_{2.2.1}(\lambda_1, \epsilon_1, R_1)$. For all $u \in B_1$ let $\bar{f}_u$ be a L_1-coarse inverse of $f_u : F_u \to F_{g(u)}$. We will define a map $f' : X_2 \to X_1$ such that (f', g') is morphism from X_2 to X_1 and f' is a coarse inverse of f as follows.

For all $u \in B_2$ we define $f^{'}_u : F_u \to F_{g'(u)}$ as the composition $\bar{f}_{g'(u)} \circ \phi_{ug(g'(u))}$ where $\phi_{ug(g'(u))}$ is a fiber identification map as constructed in the proof of Corollary 5.2.9. Collectively this defines $f^{'}$. Now we shall check that f' satisfies the desired properties.

(i) We first check that (f', g') is a morphism. It is clear from the definition that $\pi_1 \circ f' = g' \circ \pi_2$. Hence we will be done by showing that $f^{'}$ is coarsely Lipschitz. By Lemma 2.2.5 it is enough to show that for all $u_2, v_2 \in B_2$ and $x \in F_{u_2}, y \in F_{v_2}$ with $d(x, y) \leq 1$, $d_{X_1}(f^{'}(x), f^{'}(y))$ is uniformly small. Note that it follows that $d_{B_2}(u_2, v_2) \leq 1$. Let $u_1 = g'(u_2)$ and $v_1 = g'(v_2)$. Then $d_{B_1}(u_1, v_1) \leq K + C$, $d_{B_2}(u_2, g(u_1)) \leq L$ and $d_{B_2}(v_2, g(v_1)) \leq L$. This means $d_{X_2}(x, \phi_{u_2 g(u_1)}(x)) \leq 3(Lc + c)$ and $d_{X_2}(y, \phi_{v_2 g(v_1)}) \leq 3(Lc + c)$ by Lemma 5.2.8 and Corollary 5.2.9. Hence, $d_{X_2}(\phi_{u_2 g(u_1)}(x), \phi_{v_2 g(v_1)}(y)) \leq 1 + 6(Lc + c)$. Let $x_2 = \phi_{u_2 g(u_1)}(x)$, $y_2 = \phi_{v_2 g(v_1)}(y)$, $x_1 = f'(x) = \bar{f}_{g(u_1)}(x_2)$ and $y_1 = f'(y) = \bar{f}_{g(v_1)}(y_2)$. Therefore, $d_{X_2}(x_2, y_2) \leq 1 + 6(Lc + c) = R_2$, say and we want to show that $d_{X_1}(x_1, y_1)$ is uniformly small. Let $x'_2 = f(x_1) = f_{u_1}(x_1), y'_2 = f(y_1) = f_{v_1}(y_1)$. Then $d_{X_2}(x_2, x'_2) \leq L_1$ and $d_{X_2}(y_2, y'_2) \leq L_1$. Hence, $d_{X_2}(x'_2, y'_2) \leq 2L_1 + R_2$. Since $d_{B_1}(u_1, v_1) \leq K + C$ there is a point $y'_1 \in F_{u_1}$ such that $d_{X_1}(x_1, y'_1) \leq (K + C)c + c$. Hence, $d_{X_2}(x'_2, f(y'_1)) \leq ((K + C)c + c).D + D$. Hence, $d_{X_2}(f(y'_1), y'_2) \leq d_{X_2}(f(y'_1), x'_2) + d_{X_2}(x'_2, y'_2) \leq ((K + C)c + c).D + D + 2L_1 + R_2$. This implies that $d_{v_2}(f(y'_1), f(y_1)) \leq f_0(((K + C)c + c).D + D + 2L_1 + R_2) = L_2$, say. Since f_{v_1} is a (λ, ϵ)-qi we have $-\epsilon + \frac{1}{\lambda} d_{v_1}(y_1, y'_1) \leq L_2$. Hence, $d_{v_1}(y_1, y'_1) \leq (\epsilon + L_2)\lambda$. Thus,

$d_{X_1}(x_1, y_1) \leq d_{X_1}(x_1, y_1') + d_{X_1}(y_1', y_1) \leq (K+C)c + c + (\epsilon + L_2)\lambda$.

(ii) We already know that g' is a coarse inverse of g. Hence we will be done by checking that f' is a coarse inverse of f. Let $x \in X_1$ and $\pi_1(x) = u$. Let $g' \circ g(u) = u'$. First we need to show that $d_{X_1}(x, f' \circ f(x))$ is uniformly bounded. $d_{X_1}(x, f' \circ f(x)) \leq d_{X_1}(x, \phi_{u'u} \circ f' \circ f(x)) + d_{X_1}(\phi_{u'u} \circ f' \circ f(x), f' \circ f(x))$. Now, $\phi_{u'u} \circ f' \circ f(x) = \phi_{u'u} \circ f'_{g(u)} \circ f_u(x)$, where $\phi_{u'u} \circ f'_{g(u)} = \bar{f}_u$. So, $d_{X_1}(x, f' \circ f(x)) \leq d_{X_1}(x, \bar{f}_u \circ f_u(x)) + d_{X_1}(\phi_{u'u} \circ f'_{g(u)} \circ f_u(x), f'_{g(u)} \circ f_u(x)) \leq L_1 + 3c + 3cd_{B_1}(u', u) \leq L_1 + 3c + 3cL$.

To show that $d_{X_1}(x, f \circ f'(x))$ is uniformly bounded, let $y \in X_2$, $\pi_2(y) = v$ and $g \circ g'(v) = v'$. Now, $f'(y) = f'_v(y) = \bar{f}_{g'(v)} \circ \phi_{vv'}$. So, $d_{X_2}(y, f \circ f'(y)) = d_{X_2}(y, f_{g'(v)} \circ \bar{f}_{g'(v)} \circ \phi_{vv'}(y)) \leq d_{X_2}(y, \phi_{vv'}(y)) + d_{X_2}(\phi_{vv'}(y), f_{g'(v)} \circ \bar{f}_{g'(v)} \circ \phi_{vv'}(y)) \leq 3c + 3cd_{B_2}(v, v') + L_1 \leq 3c + 3cL + L_1$. □

Definition 5.2.15. ***Subbundle:*** *Suppose (X_i, B, π_i), $i = 1, 2$ are metric (graph) bundles with the same base space B. Then (X_1, B, π_1) is* subbundle *of (X_2, B, π_2) or simply X_1 is a subbundle of X_2 if there is a metric (graph) bundle morphism (f, g) from (X_1, B, π_1) to (X_2, B, π_2) such that $g = Id_B$ and the fiber maps f_b, $b \in B$ are uniform qi embeddings.*

The most important example of a subbundle that concerns us is that of ladders which we discuss in a later section.

Definition 5.2.16. ***Pullback bundle:*** *Given a metric (graph) bundle (X, B, π) and a map $g : B_1 \to B$, the* pullback *of g is a metric (graph) bundle (X_1, B_1, π_1) together with a morphism $(f : X_1 \to X, g : B_1 \to B)$ such that the following universal property holds: if (X_2, B_1, π_2) is a metric (graph) bundle together with a morphism $(f' : X_2 \to X, Id_{B_1} : B_1 \to B_1)$ such that the whole diagram commutes, then there exists a coarsely unique $f'' : X_2 \to X_1$ such that for any $x \in X_2$, $d_{X_2}(f'(x), f \circ f''(x))$ is uniformly bounded and $\pi_2 = \pi_1 \circ f''$.*

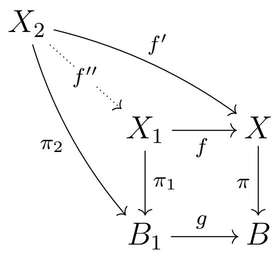

Figure 5.2

The following proposition and its proof shows that our definition of pullback bundle is natural.

Proposition 5.2.17 (Pullbacks of metric bundles). *Suppose* (X, B, π) *is a metric bundle and* $g : B_1 \to B$ *is a coarsely Lipschitz map. Then there is a pullback.*

Proof. We will work with the set theoretic pullback $\{(x, t) \in X \times B_1 \mid g(t) = \pi(x)\}$. We put on it the induced length metric from $X \times B_1$. We denote the metric space thus obtained by X_1. Let $\pi_1 : X_1 \to B_1$ the restriction of the projection map $X \times B_1 \to B_1$ to X_1. We first show that X_1 is a length space. Suppose g is coarsely L-Lipschitz. Given $(x, s), (y, t) \in X_1$. We join s, t by a rectifiable path say α. Then $g \circ \alpha$ is a rectifiable path in B of length at most $l(\alpha)L + L$.Then this path can be lifted to a rectifiable path in X starting from x and ending at some point say z in F_t. The length of such a path is at most $c(Ll(\alpha) + L)$. By construction this lift is contained in X_1. Finally, we can join $(y, t), (z, t)$ by a $(1, 1)$-quasigeodesic in F_t. This shows that (x, s) and (y, t) can be joined in X_1 by a rectifiable path and so, X_1 is a length space. Now, since $\pi_1^{-1}(t) = \pi^{-1}(g(t))$ is uniformly properly embedded in X and X is properly embedded in $X \times B_1$, $\pi_1^{-1}(t)$ is uniformly properly embedded in X_1 for all $t \in B_1$. The same argument also shows that any path in B_1 of length at most L can be lifted to a path of length at most $c(L + 1)$, verifying the condition 2(i) of metric bundles.

Hence (X_1, B_1, π_1) is a metric bundle. Let $f : X_1 \to X$ be the restriction of the projection map $X \times B_1 \to X$ to X_1. Clearly, $f : X_1 \to X$ is a morphism of metric bundles. Finally, the universal property is immediate since we are working with the set theoretic pullback. □

Example: Suppose (X, B, π) is a metric bundle and $B_1 \subset B$ which is path connected and with respect to the induced path metric from B, it is a length space. Let $X_1 = \pi^{-1}(B_1)$ be endowed with induced path metric from X. Let $\pi_1 : X_1 \to B_1$ be the restriction of π to X_1. Let $g : B_1 \to B$ and $f : X_1 \to X$ be the inclusion maps. It is clear that (X_1, B_1, π_1) is a metric bundle and also, X_1 is the pullback of g.

Proposition 5.2.18 (Pullback for metric graph bundles). *Suppose (X, B, π) is a metric graph bundle, B_1 is a metric graph and $g : B_1 \to B$ is a map that sends vertices to vertices and edges to edge paths. We assume that g is coarsely L-Lipschitz for some constant $L \geq 1$. Then there is a pullback $\pi_1 : X_1 \to B_1$ of g.*

Proof. We first construct a metric graph X_1, a candidate for the total space of the pullback bundle. The vertex set of X_1 is the disjoint union of vertex sets of $\pi^{-1}(g(b))$, $b \in V(B_1)$. There are two types of edges. First, for all $b \in V(B_1)$, we take all the edges appearing in $\pi^{-1}(g(b))$. In other words, the full subgraph $\pi^{-1}(g(b))$ is contained in X_1. Let us denote that by F_b. For adjacent vertices $s, t \in B_1$, we introduce some other edges with one end point in F_s and the other in F_t. We note that $F_s, F_t \subset X_1$ are identical copies of $F_{g(s)}$ and $F_{g(t)}$ respectively. Let $f_s : F_s \to F_{g(s)}$ denote this identification. Let e be an edge joining s, t and let α be the image of e under g. Now for each $v \in F_s$ we lift the path α isometrically to $\tilde{\alpha}$. For each such lift we join v by an edge to $w \in V(F_t)$ if and only if $f_t(w) = \tilde{\alpha}(g(t))$. This completes the construction of X_1. We notice that there is a natural map

$f : X_1 \to X$ which are identity maps on the fibers of π_1 and the other edges are mapped to edge paths. It follows that this map is coarsely Lipschitz, say coarsely L_1-Lipschitz. It is also clear that $\pi \circ f = \pi_1 \circ g$. We need to verify that the fibers are uniformly properly embedded in X_1 to check that it is a metric graph bundle. Suppose $x, y \in F_s$ and $d_{X_1}(x, y) \leq D$. Let α be a geodesic in X_1 joining x, y. Then $f \circ \gamma$ is a path of length at most $L_1 D + L_1$. Thus, $d_X(f(x), f(y)) \leq L_1 D + L_1$. Since the fiber $F_{g(s)}$ is uniformly properly embedded in X we know that $d_{g(s)}(x, y) \leq D_1$ for some D_1 depending on D. Since f sets an isometry from F_s to $F_{g(s)}$ we have $d_s(x, y) \leq D_1$. The last condition of morphism is immediate.

Now we check that X_1 is a pullback of X under g. Suppose $\pi_2 : Y \to B_1$ is a metric graph bundle and $f^Y : Y \to X$ is a graph morphism such that (f^Y, g) is a morphism from Y to X. We need to find a coarsely Lipschitz map $f' : Y \to X_1$ such that (f', Id_{B_1}) is a morphism from Y to X_1 and the whole diagram commutes.

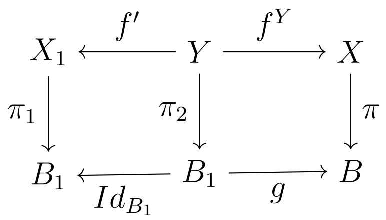

Figure 5.3

We define f' on each fiber $\pi_2^{-1}(s)$ as the composition $f_s^{-1} \circ f_s^Y$. Suppose $s, t \in B_1$ are adjacent vertices and suppose $x \in \pi_2^{-1}(s)$ and $y \in \pi_2^{-1}(t)$ are connected by an edge e in Y. Then $f^Y(e)$ is a path in X starting from $f^Y(x)$ and it is a lift of $g(\pi_2(e))$. By the construction of X_1, this corresponds to an edge in X_1 starting from $f'(x)$. The map $f'(e)$ is defined to be that edge. It is now clear that $f \circ f' = f^Y$. We need to show that f' is coarsely unique. Suppose for $f'' : Y \to X_1$, (f'', Id_{B_1}) also be a morphism from Y

to X such that $\pi_2 = \pi_1 \circ f''$ and there exists $k > 0$ satisfying the following: for any $y \in Y$, $d_X(f^Y(y), f \circ f''(y) \leq k$. Since $\pi_2^{-1}(s)$ is properly embedded in Y, there exists k_1 depending on k such that $d_{g(s)}((f^Y)_s(y), f_s \circ f''_s(y)) \leq k_1$, i.e., $d_{g(s)}(f_s \circ f'_s(y), f_s \circ f''_s(y)) \leq k_1$. Since f_s is an isometry, we have $d_s(f'_s(y), f''_s(y)) \leq k_1$ and $d_{X_1}(f'(y), f''(y)) \leq k_1$. Thus, f' is coarsely unique.

Finally, it is enough to check that the pullback is coarsely unique. Suppose (X_2, B_1, π_2) is also a pullback bundle of g with graph morphisms $f' : X_2 \to X_1$ and $f_2 : X_2 \to X$ such that (f', Id_{B_1}) and $(f_2, g \circ Id_{B_1})$ are morphisms from X_2 to X_1 and X_2 to X respectively.

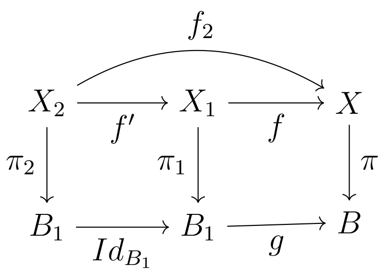

Figure 5.4

By definition of f' we have, for each $s \in B_1$, $f'_s = f_s^{-1} \circ (f_2)_s$ on the fiber $\pi_2^{-1}(s)$. The maps f and f_2 are identity maps on fibers. So $f'_s : \pi_2^{-1}(s) \to \pi_1^{-1}(s)$ is an isometry for each $s \in B_1$. Then, by Proposition 5.2.14, the morphism (f', Id_{B_1}) is an isomorphism. Thus, the pullback bundle is unique up to metric bundle isomorphism. This completes the proof. □

Lemma 5.2.19. *Suppose $\pi : X \to B$ is a metric (graph) bundle. Let $g : B_1 \to B$ be a Lipschitz qi embedding. Let $A = g(B_1), X_A = \pi^{-1}(A)$ and let π_A be the restriction of π. If $\pi_1 : X_1 \to B_1$ is the pullback of X and suppose we have the following pullback diagram:*

Then X_1 is the pullback of X_A. Moreover, X_A is quasiisometric to X_1.

Main example: Suppose $1 \to N \to G \xrightarrow{\pi} Q \to 1$ be a short exact sequence of finitely generated groups. We fix a generating set S_N of N, a

$$\begin{array}{ccccc} X_1 & \xrightarrow{f} & X_A & \xrightarrow{i_{X_A}} & X \\ \pi_1\big\downarrow & & \pi_A\big\downarrow & & \big\downarrow\pi \\ B_1 & \xrightarrow[g]{} & A & \xrightarrow[i_A]{} & B \end{array}$$

Figure 5.5

generating set $S_G \supseteq S_N$ of G and the generating set $S_Q = \pi(S_G) \setminus \{1\}$ of Q. Then we have a metric graph bundle $\pi : \Gamma(G, S_G) \to \Gamma(Q, S_Q)$. Suppose $H < Q$ is a finitely generated subgroup. Let S_H be a finite generating set of H. Then we can define a Lipschitz map $g : \Gamma(H, S_H) \to \Gamma(Q, S_Q)$. Let $G_1 = \pi^{-1}(H)$. Let $\pi_1 : X_1 \to \Gamma(H, S_H)$ be the pullback of the metric graph bundle under g. Then by the above lemma, X_1 is quasiisometric to G_1.

5.3 Geometry of metric (graph) bundles

In this section, we recall some results from [45] and also include some new results to be used for the proof of the main theorem.

5.3.1 Metric graph bundles from metric bundles

An analogue of the following result is proved in [45] (see Lemma 1.17 through Lemma 1.21 in [45]). We give an independent and relatively simpler proof here. We also construct an approximating metric graph bundle morphism from a given metric bundle morphism. However, one disadvantage of this construction is that the resulting metric graphs are never proper.

Proposition 5.3.1. *Suppose $p : X' \to B'$ is an (f_0, c, K)-metric bundle. Then there is a metric graph bundle $\pi : X \to B$, along with quasiisometries $\psi_B : B' \to B$ and $\psi_X : X' \to X$ such that*

(1) $\pi \circ \psi_X = \psi_B \circ p$, and

(2) for all $b \in B'$, ψ_X restricted to $p^{-1}(b)$ is a uniform quasiisometry onto $\pi^{-1}(\psi_B(b))$.

Moreover, the maps ψ_X, ψ_B have coarse inverses ϕ_X, ϕ_B respectively, making the following diagram commutative:

$$\begin{array}{ccc} X' & \underset{\phi_X}{\overset{\psi_X}{\rightleftarrows}} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \underset{\phi_X}{\overset{\psi_B}{\rightleftarrows}} & B \end{array}$$

Figure 5.6

Proof. For the proof we use the construction of Lemma 2.2.7. We briefly recall the construction of the spaces. We define $V(B) = B'$ and $s, t \in V(B)$ are connected by an edge if and only if $s \neq t$ and $d_{B'}(s,t) \leq 1$. This defines the graph. Then the natural map $\psi_B : B' \to B$, which is the inclusion map when B' is identified with the vertex set of B, is a $(1,1)$-quasiisometry. To define X, we take $V(X) = X'$. Edges are of two types.

Type 1 edges: For all $s \in B'$, $x, y \in p^{-1}(s)$ are connected by an edge in X if and only if $d_s(x,y) \leq 1$.

Type 2 edges: For $s \neq t \in B'$, $x \in p^{-1}(s)$ and $y \in p^{-1}(t)$, x, y are connected by an edge in X if and only if $d_{B'}(s,t) \leq 1$ and $d(x,y) \leq c$.

The map $\psi_X : X' \to X$ is also defined to be the inclusion map. Also, $\pi \circ \psi_X = \psi_B \circ p$. First we need to verify that ψ_X is a quasiisometry. Clearly, ψ_X is 1-Lipschitz. So it is enough to produce Lipschitz coarse inverses ϕ_X, ϕ_B as claimed in the second part of the proposition and then apply Lemma 2.2.1. We first choose a coarse inverse ϕ_B of ψ_B as follows. On $V(B)$, it is simply the identity map. The interior of each edge is then sent to one of its end points. The map ϕ_X on $V(X)$ is also defined as the identity map. The interior of a type 1 edge is sent to one of its end points. The interior of each type 2 edge $e = [x, y]$ is sent to x or y depending on the image of $\pi(e)$ under ϕ_B. It follows that the diagram in Figure 5.6 commutes. Now for $x, y \in X$ such that $d_X(x, y) \leq 1$, we have $d(\phi_X(x), \phi_X(y)) \leq c$. So, by Lemma 2.2.5, ϕ_X is coarsely Lipschitz. Moreover, for any $x \in X'$ and $s \in B'$, we have $d(\phi_X \circ \psi_X(x), x) = 0$ and $d_{B'}(\phi_B \circ \psi_B(s), s) = 0$. Also, for $y \in X$ and $t \in B$, we have $d(\psi_X \circ \phi_X(y), y) \leq 1$ and $d_B(\psi_B \circ \phi_B(t), t) \leq 1$. So, ϕ_X and ϕ_B are coarse inverses of ψ_X and ψ_B respectively.

Let $s \in B'$ and $\psi_B(s) = t \in B$. By the construction of X, ψ_X restricted to $p^{-1}(s)$ is mapped to $\pi^{-1}(t)$ such that the vertex set of $\pi^{-1}(t)$ is $p^{-1}(s)$ and there exists an edge joining any pair of elements $x, y \in \pi^{-1}(t)$ if and only if $d_s(x, y) \leq 1$. Then, by the construction in Lemma 2.2.7, ψ_X restricted to $p^{-1}(s)$ is a $(1, 1)$-quasiisometry.

Finally, we need to check that (X, B, π) is a metric graph bundle. Let $s \in B$ and $x, y \in \pi^{-1}(s)$ such that $d_X(x, y) \leq M$ for some $M > 0$. Since ϕ_X is a quasiisometry, $d(x, y) \leq M'$, where $M' > 0$ depends on M and ϕ_X. Since $p^{-1}(\phi_B(s))$ is properly embedded in X' as measured by f_0, we have $d_{\phi_B(s)}(x, y) \leq f_0(M')$. Now, using the above fact that $p^{-1}(\phi_B(s))$ is $(1, 1)$-quasiisometric to $\pi^{-1}(s)$, we have $d_s(x, y) \leq f_0(M') + 1$. Hence, $\pi^{-1}(b)$ is uniformly properly embedded in X.

Now we check condition (2) of Definition 5.2.4. Suppose $s, t \in V(B)$ are adjacent vertices. Then, $d_{B'}(s, t) \leq 1$. Let α be a path in B' joining s, t with $l_{B'}(\alpha) \leq 1$. Then, for any $x \in p^{-1}(s)$, α can be lifted to a path of length at most c, joining x to some $y \in p^{-1}(t)$. Then there exists an edge joining x and y in X, which is a lift of the edge joining s and t in B. □

Approximating a metric bundle morphism

Suppose $p : X' \to B'$ is a metric bundle and $g : A' \to B'$ is a (bi-Lipschitz) qi embedding. Suppose Y' is the pullback of the bundle under the map g. Let $g^*p : Y' \to A'$ is the corresponding bundle projection map and $f : Y' \to X'$ is the pullback map. Suppose, using the above proposition, we construct metric graph bundles $\pi : X \to B$, $\pi_Y : Y \to A$ with quasiisometries $\psi_A : A' \to A$, $\psi_B : B' \to B$, $\psi_Y : Y' \to Y$ and $\psi_X : X' \to X$ such that $\pi_Y \circ \psi_Y = \psi_A \circ g^*p$ and $\pi \circ \psi_X = \psi_B \circ p$.

Suppose $\phi_X, \phi_B, \phi_Y, \phi_A$ are the coarse inverses (as constructed in the Proposition 5.3.1, above) of ψ_X, ψ_B, ψ_Y, and ψ_A respectively. We then have a commutative diagram: (see Figure 5.7)

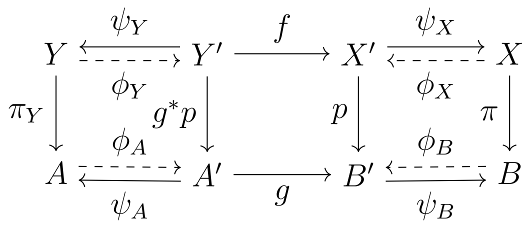

Figure 5.7

Lemma 5.3.2. *(1) The pair of maps $(\psi_X \circ f \circ \phi_Y, \psi_B \circ g \circ \phi_A)$ can be redefined on edges so that the modified pair gives a pullback diagram for metric graph bundle morphism, i.e., Y is a pullback of X under $\psi_B \circ g \circ \phi_A$.*

(2) In case, X', Y' are hyperbolic then f admits a CT map if and only if so does $\psi_X \circ f \circ \phi_Y$.

Proof. (1) Let $F = \psi_X \circ f \circ \phi_Y$ and $G = \psi_B \circ g \circ \phi_A$. By Proposition 5.2.18, X has a pullback bundle (Y_1, A, π_1) over G. Let $F_1 : Y_1 \to X$. Then, by the universal property, there exists a coarsely unique $F' : Y \to Y_1$ such that the images of F and $F_1 \circ F'$ are uniformly close in X and $\pi_Y = \pi_1 \circ F'$. Recall from the construction of a pullback in Proposition 5.2.18 that for any $a \in A$, $\pi_1^{-1}(a)$ is isometric to $\pi^{-1}(G(a))$. And by (2) of Proposition 5.3.1, $\pi^{-1}(G(a))$ is uniformly quasiisometric to $p^{-1}(\phi_B \circ G(a))$. Again, since Y' is a pullback of X' under g, $(g^*p)^{-1}(\phi(a))$ is isometric to $p^{-1}(g \circ \phi(a))$. Now, $\phi_B \circ G = \phi_B \circ \psi_B \circ g \circ \phi_A$ and since ϕ_B is a coarse inverse of ψ_B, the maps $\phi_B \circ G(a)$ and $g \circ \phi(a)$ are uniformly close. Then by Corollary 5.2.10, $p^{-1}(\phi_B \circ$

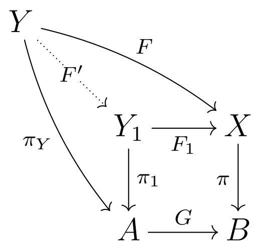

Figure 5.8

$G(a))$ is quasiisometric to $p^{-1}(g \circ \phi(a))$. Thus, $\pi_1^{-1}(a)$ is quasiisometric to $(g^*p)^{-1}(\phi(a))$.

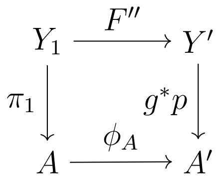

Figure 5.9

We already have that ϕ_A is a quasiisometry. Therefore, by Proposition 5.2.14, (F'', ϕ_A) is an isomorphism. Then, again by Proposition 5.2.14, F'' is a quasiisometry and so, $\psi_Y \circ F'' : Y_1 \to Y$ is a quasiisometry. Therefore, Y_1 is isomorphic to Y and thus, Y is a pullback of X under G.

(2) Suppose X', Y' are hyperbolic. Then clearly, X and Y are also hyperbolic. Suppose f admits a CT map $\partial f : \partial Y' \to \partial X'$. Since ψ_X and ϕ_Y are quasiisometries, by (4) of Lemma 2.2.36, they admit homeomorphisms $\partial \psi_X : \partial X' \to \partial X$ and $\partial \phi_Y : \partial Y \to \partial Y'$ respectively. Then by Lemma 2.2.36, this gives a map $\partial \psi_X \circ \partial f \circ \partial \phi_Y = \partial(\psi_X \circ f \circ \phi_Y) : \partial Y \to \partial X$. Conversely, if $\psi_X \circ f \circ \phi_Y$ admits a CT map, since ϕ_X and ψ_Y are coarse inverses of ψ_X and ϕ_Y respectively and f is coarsely Lipschitz, $\phi_X \circ \psi_X \circ f \circ \phi_Y \circ \psi_Y$ and f are at finite distance. So, by (2) of Lemma 2.2.36, $\partial(\phi_X \circ \psi_X \circ f \circ \phi_Y \circ \psi_Y) = \partial f$. Thus, f admits a CT map $\partial f : \partial Y' \to \partial X'$. $\square$

5.3.2 Flaring condition

Motivated by the Bestvina-Feighn's hallway flaring condition [14], the following was defined in [45].

Definition 5.3.3. *Suppose $\pi : X \to B$ is a metric bundle or a metric graph bundle. Then it satisfies a* **flaring condition** *if for all $k \geq 1$, there exist $\lambda_k > 1$ and $n_k, M_k \in \mathbb{N}$ such that the following holds:*
Let $\gamma : [-n_k, n_k] \to B$ be a geodesic and let $\tilde{\gamma_1}$ and $\tilde{\gamma_2}$ be two k-qi lifts of γ in X. If $d_{\gamma(0)}(\tilde{\gamma_1}(0), \tilde{\gamma_2}(0)) \geq M_k$, then we have

$$\lambda_k.d_{\gamma(0)}(\tilde{\gamma_1}(0), \tilde{\gamma_2}(0)) \leq max\{d_{\gamma(n_k)}(\tilde{\gamma_1}(n_k), \tilde{\gamma_2}(n_k)), d_{\gamma(-n_k)}(\tilde{\gamma_1}(-n_k), \tilde{\gamma_2}(-n_k))\}.$$

5.3.3 QI sections and ladders

For the rest of this section, we assume that all our metric (graph) bundles have the following property:

Each of the fibers F_b , $b \in B$ (resp. $b \in V(B)$) is a δ'-hyperbolic metric space with respect to the path metric d_b induced from X.

Definition 5.3.4. [45] ***Barycenter:*** *For a δ-hyperbolic metric space (Y, d) with $|\partial Y| \geq 3$, for any distinct triple of points $\xi_1, \xi_2, \xi_3 \in \partial Y$, $x \in Y$ is a* D-barycenter *of the (quasi) geodesic triangle $\Delta = \Delta\xi_1\xi_2\xi_3$, for some $D \geq 0$, if x lies in a D-neighbourhood of all three sides of the triangle Δ.*

The following lemma says that any ideal triangle has a coarsely unique barycenter.

Lemma 5.3.5. [45, Lemma 2.9] *Let $k \geq 1$. If (Y, d) is as in Definition 5.3.4 and $x, x' \in Y$ are D-barycenters of an ideal k-quasigeodesic triangle Δ, then $d(x, x') \leq L_{5.3.5}$, where $L_{5.3.5} = L_{5.3.5}(\delta, k, D)$.*

A point $x \in Y$ is a *barycenter* of an ideal quasigeodesic triangle Δ if x is a D-barycenter of Δ for some $D \geq 0$.

Definition 5.3.6. ***Barycenter map:*** *Let (Y, d) be a δ-hyperbolic metric space with $|\partial Y| \geq 3$. A* barycenter map *$\phi : \partial^3 Y \to Y$ sends any distinct triple (ξ_1, ξ_2, ξ_3) to a barycenter of the ideal triangle $\Delta\xi_1\xi_2\xi_3$.*

By Lemma 5.3.5, a barycenter map is coarsely well-defined. The following was proved in [45].

Proposition 5.3.7. [45, Proposition 2.10] **Global qi sections for metric graph bundles:** *For all $\delta', N \geq 0$ and proper $f_0 : \mathbb{N} \to \mathbb{N}$ there exists $K_0 = K_0(f_0, \delta', N)$ such that the following holds.*

Suppose $p : X \to B$ is an (f_0, K)-metric graph bundle with (1) uniformly hyperbolic fibers and (2) the barycenter maps $\phi_b : \partial^3 F_b \to F_b$, $b \in V(B)$ are uniformly coarsely surjective. Then there is a K_0-qi section over B through each point of $V(X)$.

Since any length space is uniformly quasiisometric to a metric graph by Lemma 2.2.7, and by Lemma 2.2.36(4), quasiisometries induce bijection of the boundaries of hyperbolic spaces, it follows that the same could be done for any length space as well.

Proposition 5.3.8. **Global qi sections for metric bundles:** *Any metric bundle satisfies the following:*

1) fibers are uniformly hyperbolic, and

2) the barycenter maps of these spaces are uniformly coarsely surjective,

admits a uniform qi section through each point.

Definition 5.3.9. [45, Definition 2.13] *Suppose Σ_1 and Σ_2 are two K-qi sections of the metric graph bundle X. For each $b \in V(B)$ we join the points $\Sigma_1 \cap F_b$, $\Sigma_2 \cap F_b$ by a geodesic in F_b. We denote the union of these geodesics by $\mathcal{L}(\Sigma_1, \Sigma_2)$, and call it a K-***ladder*** *(formed by the sections Σ_1 and Σ_2).*

By Proposition 5.3.7([45, Proposition 2.10]), through every point of X there is a uniform global qi section. Hence there are plenty of ladders in X. Ladders as defined above are not connected. However, a uniformly small neighborhood of it is connected:

Lemma 5.3.10. [45, Lemma 3.3, Lemma 3.5, Remark 3.6] *The $2K$-neighborhood of a K-ladder is connected and with respect to the induced path metric, it is uniformly properly embedded.*

The following is the crucial motivation to define a ladder.

Proposition 5.3.11. *Given $K \geq 0$, there is $C = C_{5.3.11}(K) \geq 0$ such that the following holds:*

(1) (cf. [45, Theorem 3.2]) *Suppose Σ_1, Σ_2 are two K-qi sections in X and $\mathcal{L} = \mathcal{L}(\Sigma_1, \Sigma_2)$ is the ladder formed by them. Then there is a coarsely C-Lipschitz retraction $\pi_{\mathcal{L}} : X \to \mathcal{L}$.*

(2) A uniform neighborhood of $\mathcal{L}$ is uniformly qi embedded in X. In particular, when X is δ-hyperbolic, $\mathcal{L}$ is $K_{5.3.11}(\delta, K)$-quasiconvex in X.

Proof. (2) Let $Y = N_{2K}(\mathcal{L})$. By 5.3.10, Y is connected and uniformly properly embedded in X. Let $x, y \in Y$ such that $d(x, y) = n$. Let $x = x_0, x_1, \dots, x_n = y$ be consecutive vertices on $[x, y]$. For $0 \leq i \leq n$, let $y_i = \pi_{\mathcal{L}}(x_i)$ and let $\alpha = [x, y_0] \cup [y_0, y_1] \dots [y_n, y]$. Since $\pi_{\mathcal{L}}$ is C-coarsely Lipschitz and $d(x_i, x_{i+1}) = 1$, $d(y_i, y_{i+1}) \leq 2C$, for $0 \leq i \leq n-1$. Suppose Y is uniformly properly embedded as measured by $f' : [0, \infty) \to [0, \infty)$. Then, $d_Y(y_i, y_{i+1}) \leq f'(2C)$. Now, clearly, $d_Y(x, \pi_{\mathcal{L}}(x)) \leq f'(2K)$ and $d_Y(y, \pi_{\mathcal{L}}(y)) \leq f'(2K)$. So $l_Y(\alpha) \leq 2f'(2K) + nf'(2C) = f'(2C)d(x, y) + 2f'(2K)$. Also, $d(x, y) \leq d_Y(x, y) \leq l_Y(\alpha)$. Thus,

$$d(x, y) \leq l_Y(\alpha) \leq f'(2C)d(x, y) + 2f'(2K).$$

So, Y is qi-embedded in X.

Now let X be δ-hyperbolic. Then, Y is $K' = D_{2.2.18}(\delta, K)$-quasiconvex in X. Now, let $x, y \in \mathcal{L}$. Then, $x, y \in Y$ as well and $[x, y] \subset N_{K'}(Y) = N_{K'+2K}(\mathcal{L})$. So, $\mathcal{L}$ is a $K_{5.3.11} := K' + 2K$-quasiconvex. □

As a consequence, given a pair of points in a ladder, a geodesic in the ladder joining them is a uniform quasigeodesic in X. We recall the construction of uniform quasigeodesics in ladders from [45], in the next section. The description of these quasigeodesics is essential for the proof of the main result. However, we first recall some related concepts.

Definition 5.3.12. *(1)* ***Neck of Ladders:*** *Suppose X is a metric graph bundle over B and suppose Σ_1, Σ_2 are any two qi sections and $A \geq 0$. Then $U_A(\Sigma_1, \Sigma_2) = \{b \in V(B) \mid d_b(\Sigma_1 \cap F_b, \Sigma_2 \cap F_b) \leq A\}$ is called the A-**neck** of the ladder $\mathcal{L}(\Sigma_1, \Sigma_2)$.*

(2) ***Small girth ladders:*** *Given two K-qi sections Σ_1, Σ_2, the ladder $\mathcal{L}(\Sigma_1, \Sigma_2)$ is called a small girth ladder if $U_A(\Sigma_1, \Sigma_2) \neq \emptyset$, where $A = M_K$.*

We need the following two lemmas before describing the quasigeodesics.

Lemma 5.3.13 (Neck of a ladder is quasiconvex). [45, Lemma 2.18]

Let X be an (f_0, K)-metric graph bundle over B satisfying (M_k, λ_k, n_k)-flaring for all $k \geq 1$ (cf. Definition 5.3.3), and let μ_k be the bounded flaring function (cf. Corollary 5.2.6). Then for all $c_1 \geq 1$ and $R > 1$, there exists $D_{5.3.13} = D_{5.3.13}(c_1, R)$ and $K_{5.3.13} = K_{5.3.13}(c_1)$ such that the following holds: Let X_1, X_2 be two c_1-qi sections of B in X and $A \geq max\{M_{c_1}, d_h(X_1, X_2)\}$.

(1) Let $\gamma : [t_0, t_1] \to B$ be a geodesic, $t_0, t_1 \in \mathbb{Z}$, such that

a) $d_{\gamma(t_0)}(X_1 \cap F_{\gamma(t_0)}, X_2 \cap F_{\gamma(t_0)}) = AR$.

b) $\gamma(t_1) \in U_A := U_A(X_1, X_2)$ but for all $t \in [t_0, t_1) \cap \mathbb{Z}$, $\gamma(t) \notin U_A$.

Then the length of γ is at most $D_{5.3.13}(c_1, R)$.

(2) U_A is $K_{5.3.13}$-quasiconvex in B.

(3) If $d_h(X_1, X_2) \geq M_{c_1}$ then the diameter of the set U_A is at most $D'_{5.3.13} = D'_{5.3.13}(c_1, A)$.

Lemma 5.3.14 (QI sections in ladders). [45, Lemma 3.1] *For $K \geq 1$, there exists $C_{5.3.14} = C_{5.3.14}(K)$ such that if Σ_1 and Σ_2 are two K-qi sections, then through each $x \in \mathcal{L}(\Sigma_1, \Sigma_2)$, there exists a $C_{5.3.14}$-qi section contained in $\mathcal{L}(\Sigma_1, \Sigma_2)$.*

We fix the following notations: $K_1 := C_{5.3.14}(K)$, $K_i := C_{5.3.14}(K_{i-1})$, $i \geq 2$.

Next we find a relation between $d_h(\Sigma_1, \Sigma_2)$ and $d(\Sigma_1, \Sigma_2)$.

Lemma 5.3.15. *Given $D \geq 0, K \geq 1$ there is $R = R_{5.3.15}(D, K)$ such that the following holds.*

Suppose Σ is a K-qi section in X and $x \in X$. Let $b = \pi(x)$. Then $d(x, \Sigma) \geq D$ if $d_b(x, \Sigma \cap F_b) \geq D$.

Proof. Let $y \in \Sigma$ be a nearest point projection of x on Σ. Let γ be a lift of $[b, \pi(y)]$ on Σ. Since $d_B(b, \pi(y)) \leq d(x, y)$, we have $d(y, \Sigma \cap F_b) \leq K d(x, y) + K$. Then, $d(x, \Sigma \cap F_b) \leq d(x, y) + d(y, \Sigma \cap F_b) \leq (K+1) d(x, y) + K$. Thus, $d(x, y) \geq \frac{d(x, \Sigma \cap F_b)}{K+2}$. Then if, $d_b(x, \Sigma \cap F_b) \geq R := f_0(D(K+2))$, we have $d(x, y) \geq D$. □

The following corollary is immediate.

Corollary 5.3.16. *Given $D \geq 0, K \geq 1$ there is $R = R_{5.3.16}(D, K)$ such that the following holds.*

Suppose Σ, Σ' are K-qi section in X. Then $d(\Sigma, \Sigma') \geq D$ if $U_R(\Sigma, \Sigma') = \emptyset$.

5.4 Cannon-Thurston maps for pullback bundles

In this section, we prove the main result of the paper. Here is the setup. From now on, we assume the following hypotheses:

We suppose $\pi : X \to B$ is a metric graph bundle such that

(H1) B is a δ_0-hyperbolic metric space.

(H2) For $b \in V(B)$, the fiber F_b is a δ_0-hyperbolic geodesic space with respect to the path metric induced from X. And there exists $f_0 : \mathbb{N} \to \mathbb{N}$ such that $i_b : F_b \to X$ is uniformly metrically proper as measured by f_0.

(H3) The barycenter maps $\partial^3 F_b \to F_b$, $b \in V(B)$, are uniformly coarsely surjective.

(H4) The flaring condition is satisfied.

The following theorem is the main result of [45]:

Theorem 5.4.1. [45, Theorem 4.3 and Proposition 5.8] *If $\pi : X \to B$ is a geodesic metric bundle or a metric graph bundle satisfying $H1, H2, H3$ then, X is a hyperbolic metric space if and only if X satisfies $H4$.*

Remark 5. The sole purpose of $H3$ is to have global uniform qi sections through every point of X. Thus, $H3$ can be relaxed to the following.

(H3′) Through any point of X, there is a global K_0-qi section.

Corollary 5.4.2. Theorem 5.4.1 *holds for length metric bundles as well.*

We are now ready to state and prove the main theorem.

Theorem 5.4.3 (Main Theorem). [35] *Suppose $\pi : X \to B$ is a metric (graph) bundle satisfying the above hypotheses and X is δ-hyperbolic. Let $g : A \to B$ be a k-qi embedding and $p : Y \to A$ be the pullback bundle. Let $f : Y \to X$ be the pullback map. Then Y is a hyperbolic metric space and the CT map exists for $f : Y \to X$.*

By Propositions 5.3.1 and 5.3.2, it is enough to consider the case of metric graph bundles. Using Lemma 5.2.19, we are reduced to the case $A \subset B$ and $Y = \pi^{-1}(A) \subset X$. Since Y is an induced metric bundle over a qi embedded subset A of B, by Theorem 5.4.1, the space Y is hyperbolic (see [45, Remark 4.4]). We then use Lemma 2.2.33 and compare geodesics in Y and in X whose common end points are in Y. This is done in three steps. First, we describe a set of uniform quasigeodesics in X joining each pair of points $x, y \in X$. This is extracted from [45]. When $x, y \in Y$, we suitably modify these paths to obtain uniform quasigeodesics in Y. These modified paths are referred to as the 'cut-paste paths' below. Finally, we show that the quasigeodesic segments of X are far from a base point if the Y-quasigeodesic segments are far from the same point, thus verifying Lemma 2.2.33. To maintain modularity of the arguments, we state intermediate observations as lemmas and propositions. First, we show that Y is properly embedded in X.

Lemma 5.4.4. *Suppose $\pi : X \to B$ is a metric graph bundle satisfying the above hypotheses. Suppose $g : A \to B$ is a k-qi embedding and $p : Y \to A$ is the pullback bundle. Let $f : Y \to X$ be the pullback map. Then Y is properly embedded in X.*

Proof. Let $x, y \in Y$ such that $d_X(x, y) \leq M$. Let $\pi(x) = b_1$ and $\pi(y) = b_2$. Then, $d_B(b_1, b_2) \leq M$. Let $[b_1, b_2]_A$ be a geodesic joining b_1 and b_2 in A. This is a quasigeodesic in B. By Lemma 5.2.8, there exists an isometric section γ over $[b_1, b_2]_A$, through x in Y. Clearly, γ is a qi lift in X, say k'-qi lift. We have, $l_X(\gamma) \leq k'(kM + k) + k' =: D(M)$. The concatenation of γ and the fiber geodesic $[\gamma \cap F_{b_2}, y]_{F_{b_2}}$ is a path, denoted by α, joining x and y in X. So, $d_X(\gamma \cap F_{b_2}, y) \leq d_X(\gamma \cap F_{b_2}, x) + d_X(x, y) \leq l_X(\gamma) + d_X(x, y) \leq D(M) + M$.

Now, since F_{b_2} is uniformly properly embedded as measured by f_0, we have, $d_{b_2}(\gamma \cap F_{b_2}, y) \leq f_0(D(M) + M)$.

Now, α lies in Y. So, $l_Y(\gamma) \leq kM + k$. As in the case of X, the concatenation of γ and the fiber geodesic $[\gamma \cap F_{b_2}, y]_{F_{b_2}}$ is a path joining x and y in Y. Then, $d_Y(x, y) \leq l_Y(\alpha) \leq l_Y(\gamma) + d_Y(\gamma \cap F_{b_2}, y) \leq kM + k + d_{b_2}(\tilde{\gamma} \cap F_{b_2}, y)$. Therefore, $d_Y(x, y) \leq kM + k + f_0(D(M) + M)$. So, for $D_{5.4.4}(M) := kM + k + f_0(D(M) + M)$, we have $d_Y(x, y) \leq D_{5.4.4}(M)$. □

Now we recall the description of quasigeodesics in X from [45].

Step 1: Descriptions of uniform quasigeodesics in X.

Let $y, y; \in X$ and let Σ, Σ' be a pair of K_0-qi sections containing y, y' respectively. Let $\mathcal{L} = \mathcal{L}(\Sigma, \Sigma')$ be the ladder formed by them.

Step 1(a): Ladder decomposition

Lemma 5.4.5. *Given $K \geq 1$. Then there exists $D_{5.4.5} = D_{5.4.5}(K)$ such that the following holds.*

Suppose Σ, Σ' are two K-qi sections and $d_h(\Sigma, \Sigma') \geq M_K$. Then Σ, Σ' are uniformly $D_{5.4.5}$-cobounded in X.

Proof. The K-qi sections Σ, Σ' are $K' := D_{2.2.18}(\delta, K, K)$-quasiconvex. Let $P : X \to \Sigma$ be a 1-approximate nearest point projection map. Let the diameter of $P(\Sigma')$ be greater that $D = D_{2.2.41}(\delta, K', 1)$. Then by Corollary 2.2.41, $d(\Sigma, \Sigma') \leq R := R_{2.2.41}(\delta, K', 1)$. Then for $x \in \Sigma'$ such that $d(x, \Sigma) \leq R$, we have $d(x, \Sigma \cap F_{\pi(x)}) \leq R + KR + K =: \bar{R}$. Then $\pi(P(\Sigma')) \subset U_{\bar{R}}(\Sigma, \Sigma')$. By Lemma 5.3.13, the diameter of is at most $D_{5.3.13}(K, \bar{R})$. Thus, diameter of $P(\Sigma)$ is at most $K + KD_{5.3.13}(K, \bar{R})$ and for $D_{5.4.5} = \max\{D, K + KD_{5.3.13}(K, \bar{R})\}$, Σ, Σ' are uniformly $D_{5.4.5}$-cobounded in X. □

We fix the following notations:

(i) Each K_i-qi section is K'_i-quasiconvex in X, where we have

$K'_i = \max\{D_{2.2.18}(\delta, K), K_{5.3.11}(\delta, K)\}$.

(ii) If Σ, Σ' are K_i-qi sections and $d_h(\Sigma, \Sigma') \geq M_{K_i}$, then Σ, Σ' are $D_i = D_{5.4.5}(K_i)$.

(iii) If Σ, Σ' are K_i-qi sections with $d_h(\Sigma, \Sigma') > r_i := R5.3.16(2K'_i + 1, K_i)$, then $d(\Sigma, \Sigma') > 2K'_i + 1$.

We fix $b_0 \in A$. Let Σ, Σ' be K_0-qi sections in X. Suppose $\alpha : [0, l] \to F_b$ is an isometry onto $\mathcal{L}(\Sigma, \Sigma') \cap F_{b_0}$ such that $\alpha(0) = \Sigma \cap F_{b_0}, \alpha(l) = \Sigma' \cap F_{b_0}$. We restate [45, Proposition 3.14] as:

Proposition 5.4.6. *There is a constant L_0 such that for all $L \geq L_0$, there is a partition $0 = t_0 < t_1 < \cdots < t_n = l$ of $[0, l]$ and K_1-qi sections Σ_i, passing*

through $\alpha(t_i)$, $1 \leq i \leq n-1$, *inside* $\mathcal{L}(\Sigma, \Sigma')$ *such that the following holds.*
(1) $\Sigma_0 = \Sigma, \Sigma_n = \Sigma'$.
(2) Each Σ_i, $1 \leq i \leq n-1$ *coarsely separates* $\mathcal{L}(\Sigma, \Sigma')$.
(3) For $0 \leq i \leq n-2$, $\Sigma_{i+1} \subset \mathcal{L}(\Sigma_i, \Sigma')$.
(4) For $0 \leq i \leq n-2$ *either (I)* $d_h(\Sigma_i, \Sigma_{i+1}) = L$, *or (II)* $d_h(\Sigma_i, \Sigma_{i+1}) > L$ *and there is a* K_2*-qi section* Σ_i' *through* $\alpha(t_{i+1}-1)$ *inside* $\mathcal{L}(\Sigma_i, \Sigma_{i+1})$ *such that* $d_h(\Sigma_i, \Sigma_i') < C + CL$, *where* $C = C_{5.3.11}(K_1)$.
(5) $d_h(\Sigma_{n-1}, \Sigma_n) \leq L$.

Convention We fix $R_0 = L_0 + M_{K_3} + r_3$ and $R_1 = C + CL$, where $C = C_{5.3.11}(K_1)$.

Thus, we have the following corollary.

Corollary 5.4.7. *There is a partition* $0 = t_0 < t_1 < \cdots < t_n = l$ *of* $[0, l]$ *and* K_1*-qi sections* Σ_i, *passing through* $\alpha(t_i)$, $1 \leq i \leq n-1$, *inside* $\mathcal{L}(\Sigma, \Sigma')$ *such that the following holds.*
(1) $\Sigma_0 = \Sigma, \Sigma_n = \Sigma'$.
(2) Each Σ_i, $1 \leq i \leq n-1$ *coarsely separates* $\mathcal{L}(\Sigma, \Sigma')$.
(3) For $0 \leq i \leq n-2$, $\Sigma_{i+1} \subset \mathcal{L}(\Sigma_i, \Sigma')$.
(4) For $0 \leq i \leq n-2$ *either (I)* $d_h(\Sigma_i, \Sigma_{i+1}) = R_0$, *or (II)* $d_h(\Sigma_i, \Sigma_{i+1}) > R_0$ *and there is a* K_2*-qi section* Σ_i' *through* $\alpha(t_{i+1}-1)$ *inside* $\mathcal{L}(\Sigma_i, \Sigma_{i+1})$ *such that* $d_h(\Sigma_i, \Sigma_i') < R_1$.
(5) $d_h(\Sigma_{n-1}, \Sigma_n) \leq R_0$.

Remark 6. *(1) Note that* Σ_{n-1}, Σ_n *need not be mutually cobounded.*
(2) We call the sub ladders in (4)(I) of Corollary 5.4.7 as type (I) ladders and those in (4)(II) as type (II) ladders.

Now we have all the conditions of Proposition 5.1.1. Hence, we can find a uniform quasigeodesic joining x, y in $\mathcal{L} = \mathcal{L}(\Sigma, \Sigma')$. It suffices to find uniform approximate nearest point projection of points of Σ_i onto Σ_{i+1} and describe uniform quasigeodesics joining the successive points.

Step 1(b): Joining y, y' in the ladder. Let $y = 0$ and for $0 \leq i \leq n-1$, let $y_{i+1} \in \Sigma_{i+1}$ be a uniform approximate nearest point projection of y_i on Σ_{i+1} in X. Let $y' = y_{n+1}$. Let α_i denote the uniform quasigeodesic in X joining y_i to y_{i+1} such that $\alpha_i \subset \mathcal{L}(\Sigma_i, \Sigma_{i+1})$. Let $alpha_n$ denote the lift of $[\pi(y_n), \pi(y')]$ in Σ'. Next we recall how to find y_i and construct α_i, for $0 \leq i \leq n-1$.

Case I: $\mathcal{L}_i = \mathcal{L}(\Sigma_i, \Sigma_{i+1})$ is of type (I) or $i = n-1$. In this case, $U_{R_0}(\Sigma_i, \Sigma_{i+1}) \neq \emptyset$. Let u_i be a nearest point projection of $\pi(y_i)$ on $U_{R_0}(\Sigma_i, \Sigma_{i+1})$. We define $y_{i+1} = \Sigma_{i+1} \cap F_{u_i}$. Let γ_i be the lift of $[\pi(y_i), u_i]$ in Σ_i, let σ_i be the fiber geodesic of $F_{u_i} \cap \mathcal{L}_i$ joining $\gamma_i(u_i)$ and y_{i+1}. Then α_i is a concatenation of γ_i and σ_i.

The following lemma shows that y_{i+1} is a uniform approximate nearest point projection of y_i on Σ_{i+1}.

Lemma 5.4.8. *Given $K \geq 1, R \geq M_K$, there are constants $\epsilon_{5.4.8} = \epsilon_{5.4.8}(K, R), \epsilon'_{5.4.8} = \epsilon'_{5.4.8}(K, R)$ such that the following holds.*

Let $\mathcal{Q}_1, \mathcal{Q}_2$ be K-qi sections and $d_h(\mathcal{Q}_1, \mathcal{Q}_2) \leq R$. Let $x \in \mathcal{Q}_1$ and $U = U_R(\mathcal{Q}_1, \mathcal{Q}_2)$. Suppose b is a nearest point projection of $\pi(x)$ on U. Then $\mathcal{Q}_2 \cap F_b$ is an $\epsilon_{5.4.8}$-approximate nearest point projection of x on $\mathcal{Q}_2$.

If $d_h(\mathcal{Q}_1, \mathcal{Q}_2) > M_K$ then any $b' \in U$, the point $\mathcal{Q}_2 \cap F_{b'}$ is an $\epsilon'_{5.4.8}$-approximate nearest point projection of $\mathcal{Q}_1$ on $\mathcal{Q}_2$.

This lemma follows from Corollary 1.40 and Proposition 3.4 of [45].

Case II: $\mathcal{L}_i = \mathcal{L}(\Sigma_i, \Sigma_{i+1})$ is of type (II). Then $d_h(\Sigma_i, \Sigma_{i+1}) > R_0$ and there is a K_2-qi section Σ'_i inside $\mathcal{L}_i = \mathcal{L}(\Sigma_i, \Sigma_{i+1})$ passing through $\alpha(t_{i+1}-1)$ such that $d_h(\Sigma_i, \Sigma'_i) \leq R_1$. Let v_i be a nearest point projection of $\pi(y_i)$ on $U_{R_1}(\Sigma_i, \Sigma'_i)$ and let w_i be a nearest point projection of v_i on $U_{R_0}(\Sigma'_i, \Sigma_{i+1})$. Let $y'_i = \Sigma'_i \cap F_{v_i}, y_{i+1} = \Sigma_{i+1} \cap F_{w_i}$. Let γ_i, γ'_i be the lifts of $[\pi(y_i), v_i], [v_i, w_i]$ on Σ_i, Σ'_i respectively. Let σ_i, σ'_i be the fiber geodesics of $F_{v_i} \cap \mathcal{L}(\Sigma_i, \Sigma'_i), F_{w_i} \cap$

$\mathcal{L}(\Sigma_i', \Sigma_{i+1})$ respectively. then, $\alpha_i = \gamma_i * \sigma_i * \gamma_i' * \sigma_i'$.

Borrowing the notation from [45], we denote the uniform quasigeodesic obtained by concatenating $\alpha_0, \alpha_1, \ldots, \alpha_n$ by $c(y, y')$.

Step 2: Construction of cut-paste paths.

In this step, we construct the cut-paste path $\tilde{c}(y, y')$ in Y corresponding to $c(y, y')$ in X. Using Proposition 5.1.1, we prove that $\tilde{c}(y, y')$ is a uniform quasigeodesic in Y. Note that since $\mathcal{L}$ satisfies all the hypotheses of Proposition 5.1.1, so does $\mathcal{L} \cap Y$. For $0 \leq i \leq n$, let b_i denote a nearest point projection of $\pi(y_i)$ on A. Since $y \in A$, $b_0 = \pi(y)$. Let $\tilde{y}_i = F_{b_i} \cap \Sigma_i$. For $0 \leq i \leq n-1$, define a path $\tilde{\alpha}_i \subset \mathcal{L}(\Sigma_i \cap \Sigma_{i+1} \cap Y)$ joining $\tilde{y}_i, \tilde{y}_{i+1}$. Let $\tilde{\alpha}_n$ be the lift of $[\pi(\tilde{y}_n), y']_A$ on Σ'.

Case 1: $\mathcal{L}_i = \mathcal{L}(\Sigma_i, \Sigma_{i+1})$ is of type (I) or $i = n-1$. Let $\tilde{\gamma}_i$ be the lift of $[b_i, b_{i+1}]_A$ in Σ_i starting at $\tilde{y}_i$ and let $\tilde{\sigma}_i$ be the fiber geodesic of $F_{b_{i+1}} \cap \mathcal{L}_i$ joining $\gamma_i(u_i)$ and y_{i+1}. Then $\tilde{\alpha}_i$ is a concatenation of $\tilde{\gamma}_i$ and $\tilde{\sigma}_i$.

Case 2: $\mathcal{L}_i = \mathcal{L}(\Sigma_i, \Sigma_{i+1})$ is of type (II). Let b_i' be a nearest point projection of $\pi(y_i')$ on A. Let $y_i' = \Sigma_i' \cap F_{v_i}, y_{i+1} = \Sigma_{i+1} \cap F_{w_i}$. Let $\tilde{\gamma}_i, \tilde{\gamma'}_i$ be the lifts of $[b_i, b_i'], [b_i', b_{i+1}]$ on Σ_i, Σ_i' respectively. Let $\tilde{\sigma}_i, \tilde{\sigma'}_i$ be the fiber geodesics of $F_{b_i'} \cap \mathcal{L}(\Sigma_i, \Sigma_i'), F_{b_{i+1}} \cap \mathcal{L}(\Sigma_i', \Sigma_{i+1})$ respectively. then, $\alpha_i = \tilde{\gamma}_i * \tilde{\sigma}_i * \tilde{\gamma'}_i * \tilde{\sigma'}_i$.

Step 3: Proving that $\tilde{c}(y, y')$ is a uniform quasigeodesic in Y.

By Proposition 5.1.1, it is enough to show that for $0 \leq i \leq n-1$, $\tilde{y}_{i+1}$ is a uniform approximate nearest point projection of $\tilde{y}_i$ on Σ_{i+1} and $\tilde{\alpha}_i$ is a uniform quasigeodesic. We prove this for each type of sub ladder $\mathcal{L}_i$. First we have the following lemma.

Lemma 5.4.9. *Suppose $b \in A$, $x, y \in F_b$ and suppose for all $K \geq K_0$ and $R \geq M_K$, there exists a constant $D = D_{5.4.9}(K, R$ such that for all $x', y' \in [x, y]_b$ and any two K-qi sections $\mathcal{Q}_1$ and $\mathcal{Q}_2$ passing through x', y' respectively, either $U_R(\mathcal{Q}_1, \mathcal{Q}_2) = \emptyset$ or $d_B(b, U_R(\mathcal{Q}_1, \mathcal{Q}_2)) \leq D$. Then $[x, y]_b$ is a $\lambda_{5.4.9}$-quasigeodesic in X, where $\lambda_{5.4.9}$ depends on D and the metric graph* *bundle parameters. In particular, if $\mathcal{Q}$ and $\mathcal{Q}'$ are two K-qi sections passing through x, y respectively then y (resp. x) is a uniform approximate nearest point projection of x (resp. y) on $\mathcal{Q}'$ (resp. $\mathcal{Q}$).*

Proof. (1) Since fibers are uniformly properly embedded in X, the arc length parametrization of $[x, y]_b$ is also uniformly properly embedded in X. So by Lemma 2.2.4, it is enough to show that $[x, y]_b$ lies uniformly close to a quasigeodesic in X joining x, y. Let Σ_x, Σ_y be K_0-qi sections in X passing through x, y respectively. Let $z \in [x, y]_b$ and Σ_z be a K_1-qi section through z inside $\mathcal{L}(\Sigma_x, \Sigma_y)$. Then $\mathcal{L}(\Sigma_x, \Sigma_y) = \mathcal{L}(\Sigma_x, \Sigma_z) \cup \mathcal{L}(\Sigma_z, \Sigma_y)$.

Claim: A nearest point projection of x on Σ_z lies uniformly close to z.

Proof of the claim: Suppose $U_{M_{K_1}}(\Sigma_x, \Sigma_z) \neq \emptyset$. Then by Step 1(b) and Lemma 5.4.8, it follows that z is uniformly close to an nearest point projection of x on Σ_z.

Now suppose $U_{M_{K_1}}(\Sigma_x, \Sigma_z) = \emptyset$. Let $\alpha_{zx} : [0, l] \to F_b$ be the unit speed parametrization of the fiber geodesic $F_b \cap \mathcal{L}(\Sigma_x, \Sigma_z)$ joining x, z. By Proposition 5.4.6, there exists a K_2-qi section $\Sigma_{z'}$ through $\alpha_{zx}(t) = z'$ such that $\mathcal{L}(\Sigma_z, \Sigma_{z'})$ is a sub ladder of type (I) or (II), where $t \in [0, l]$. Let x' be a nearest point projection of x on $\Sigma_{z'}$. We consider the following cases:

(I) Suppose $d_h(\Sigma_z, \Sigma_{z'}) = R_0$. Then by Lemma 5.4.8, for any $v \in U_{R_0}(\Sigma_z, \Sigma_{z'})$, $F_b \cap \Sigma_z$ is an $\epsilon'_{5.4.8}(K_2, R_0)$-approximate nearest point projection of x' on Σ_z. Since in this case, $d_B(b, U_{R_0}(\Sigma_{z'}, \Sigma_z)) \leq D$, and diameter of $U_{R_0}(\Sigma_z, \Sigma_{z'})$ is uniformly bounded, $d(z, F_v \cap \Sigma_z)$ is uniformly small.

(II) Suppose $d_h(\Sigma_z, \Sigma_{z'}) > R_0$. Then there is a K_3-qi section $\Sigma_{z''}$ in $\mathcal{L}(\Sigma_{z'}, \Sigma_z)$ passing through $z'' = \alpha_{zx}(t-1)$ such that $U_{R_0}(\Sigma_z, \Sigma_{z''}) \neq \emptyset$. Let w be a nearest point projection of b on $U_{R_0}(\Sigma_z, \Sigma_{z''})$. Then by Lemma 5.4.8, $F_w \cap \Sigma_z$ is a uniform approximate nearest point projection of z'' on Σ_z. Since $d(z', z'') \leq 1$, $F_w \cap \Sigma_z$ is a uniform approximate nearest point projection of z' on Σ_z as well. Since $\Sigma_z, \Sigma_{z'}$ are cobounded, $F_w \cap \Sigma_z$ is a uniform approximate nearest point projection of x' on Σ_z.

Since this is true for every $z \in [x, y]_b$, by the construction of $c(x, y)$ in X, each z lies uniformly close to some point in $c(x, y)$.

(2) Let $\mathcal{Q}, \mathcal{Q}'$ be K-qi sections through x, y respectively. We will only that x is a uniform approximate nearest point projection of y on $\mathcal{Q}$ as the other proof is similar. Let $x' \in \mathcal{Q}$ be a nearest point projection of y on $\mathcal{Q}$. Consider the lift β of $[\pi(x'), b]$ on $\mathcal{Q}$. Since $\mathcal{Q}$ is $K' = D_{2.2.18}(K, \delta)$-quasiconvex in X, by Lemma 2.2.38, the concatenation of the geodesic $[y, x']$ in X and β is a $K_{2.2.38}(\delta, K', K)$-quasigeodesic in X. By Lemma 2.2.18, $d(x', [x, y]_b) \leq 2D_{2.2.18}(\delta.k')$ where $k' = \max\{K_{2.2.38}(\delta, K', K), \lambda_{5.4.9}\}$. Then $d_B(\pi(x'), b) \leq 2D_{2.2.18}(\delta.k')$ and so, $d(x', F_b \cap \mathcal{Q}) = d(x', x) \leq 2D_{2.2.18}(\delta.k')K + K$. Thus, for $\epsilon = 2D_{2.2.18}(\delta.k')K + K$, x is an ϵ-approximate nearest point projection of y on $\mathcal{Q}$. $\square$

Let $k_0 := D_{2.2.18}(k, \delta_0)$. Then A is k_0-quasiconvex in B. Let $P_A : B \to A$ denote a nearest point projection map of B onto A.

Lemma 5.4.10. *Given $R \geq 0, K, K' \geq 1$ and $R' \geq M_{K'}$, there exists $R_{5.4.10} = R_{5.4.10}(R, R', K, K'), D_{5.4.10} = D_{5.4.10}(R, R', K, K')$ such that the following holds.*

Let $u \in B$ and $P_A(u) = b$. Let $x, y \in F_b$ and γ_1, γ_2 be K-qi lifts of $[u, b]$ in X through x, y respectively. Let $\mathcal{Q}_1, \mathcal{Q}_2$ be K'-qi sections over A in Y and let $U = U_{R'}(\mathcal{Q}_1, \mathcal{Q}_2)$. If $d_u(\gamma_1(u), \gamma_2(u)) \leq R$ and $U \neq \emptyset$, then $d_b(x, y) \leq R_{5.4.10}$ and $d_A(b, U) \leq D_{5.4.10}$.

Proof. Suppose $d_u(\gamma_1(u), \gamma_2(u)) \leq R$ and $U \neq \emptyset$. Let $b' \in U$. Then by Lemma 2.2.38, $\beta = [u, b] \cup [b, b']_A$ is a $K_{2.2.38}(\delta, k_0, k, 0)$-quasigeodesic in B. Clearly, the concatenation of γ_i and the lift of $[b, b']_A$ on $\mathcal{Q}_i$, for $i = 1, 2$ is a k'-qi lift of β in X through x, y respectively, where $k' = \max\{K, K'\}$. Let $k'' = K_{2.2.38}(\delta, k_0, k, 0)$. Then by Lemma 2.2.3, these qi lifts are $(k'k'', k'k'' + k')$-quasigeodesics in X. Since $d_u(\gamma_1(u), \gamma_2(u)) \leq R$ and $d(F_{b'} \cap \mathcal{Q}_1, F_{b'} \cap \mathcal{Q}_2) \leq R'$,

x lies in a $D' = 2\delta + R + R' + 2D_{2.2.18}(\delta, k'k'', k'k''+k')$ to the qi section through y. Then by Lemma 5.3.15 to the restrictions of the lifts to $[u, b]$ and $[b, b']_A$, $d_b(x, y) \leq R_{5.4.10}$, where $R_{5.4.10} = \max\{R_{5.3.15}(D', K), R_{5.3.15}(D', K')\}$. By Lemma 5.4.10, $d_A(b, U) \leq D_{5.4.10}$, where $D_{5.4.10} = D_{5.3.13}(K', R_{5.4.10}/M_{K'})$.

□

Lemma 5.4.11. *Given $K \geq K_0$ and $R \geq M_K$, there exists $K_{5.4.11} = K_{5.4.11}(K, R), D_{5.4.11} = D_{5.4.11}(K, R)$ such that the following holds.*

Suppose $\mathcal{Q}, \mathcal{Q}'$ are K-qi sections in X and $d_h(\mathcal{Q}, \mathcal{Q}') \leq R$ in X. Let $U = U_R(\mathcal{Q}, \mathcal{Q}')$. Suppose $d_h(\mathcal{Q} \cap Y, \mathcal{Q}' \cap Y) \geq R$. Then the following hold.

(1) Diameter of $P_A(U)$ is at most $D_{5.4.11}$.

(2) For $b \in P_A(U)$, $F_b \cap \mathcal{L}(\mathcal{Q}, \mathcal{Q}')$ is a $K_{5.4.11}$-quasigeodesic in Y.

(3) $F_b \cap \mathcal{Q}$ is an $\epsilon_{5.4.11}$-approximate nearest point projection of $\mathcal{Q}'$ on $\mathcal{Q}$ and vice versa.

Proof. Recall that A is k_0-quasiconvex in B and U is $K_{5.3.13}(K)$ quasiconvex in B. Let $\lambda' = \max\{k_0, K_{5.3.13}(K)\}$. Suppose there exists $u, v \in P_A(U)$ such that $d_B(u, v) \geq D_{2.2.41}(\delta, \lambda', 0)$. Then by Corollary 2.2.41, there exists $u', v' \in U$ such that $d_B(u, u') \leq R_{2.2.41}(\delta, \lambda', 0), d_B(v, v') \leq R_{2.2.41}(\delta, \lambda', 0)$. Let $D = R_{2.2.41}(\delta, \lambda', 0)$. Then by the bounded flaring condition, $d_u(\mathcal{Q} \cap F_u, \mathcal{Q}' \cap F_u) \leq \mu_K(D)R, d_v(\mathcal{Q} \cap F_v, \mathcal{Q}' \cap F_v) \leq \mu_K(D)R$. For $R_1 = \mu_K(D)R$, $u, v \in U_{R_1}(\mathcal{Q}, \mathcal{Q}')$. Since $R_1 \geq M_K$, the diameter of $U_{R_1}(\mathcal{Q}, \mathcal{Q}')$ is at most $D_{5.3.13}(K, R_1)$ and so, $d_B(u, v) \leq D_{5.3.13}(K, R_1)$. For $D_{5.4.11} = \max\{D_{5.3.13}(K, R_1), D_{2.2.41}(\delta, \lambda', 0)\}$, we have the proof of (1).

Let $u \in U$ such that $b = P_A(u)$. Let $x, y \in F_b \cap \mathcal{L}(\mathcal{Q}, \mathcal{Q}')$. Let $\mathcal{Q}_1, \mathcal{Q}_1'$ be K'-qi sections over A in Y passing through x, y respectively and let $U' = U_{M_{K'}}(\mathcal{Q}_1, \mathcal{Q}_1')$. Suppose $U' \neq \emptyset$. Let γ_1, γ_2 be K-qi lifts of $[u, b]$ in $\mathcal{Q}, \mathcal{Q}'$ respectively. Then $d(\gamma_1(u), \gamma_2(u)) \leq R$. By Lemma 5.4.10, $d_B(b, U')$ is small and then, by Lemma 5.4.9, (2) and (3) are proved. □

Lemma 5.4.12. *Given $D \geq 0, K \geq K_0$ and $R \geq M_K$, there exists $K_{5.4.12} = K_{5.4.12}(D, K, R), D_{5.4.12} = D_{5.4.12}(D, K, R)$ and $\epsilon_{5.4.12} = \epsilon_{5.4.12}(D, K, R)$ such that the following holds.*

Suppose $\mathcal{Q}, \mathcal{Q}'$ are K-qi sections in X and $d_h(\mathcal{Q}, \mathcal{Q}') \leq R$ in X. Let $U = U_R(\mathcal{Q}, \mathcal{Q}')$. Suppose $U \neq \emptyset$. Then the following hold.

(1) Diameter of $P_A(U)$ is at most $D_{5.4.12}$.

(2) For $b \in P_A(U)$, $F_b \cap \mathcal{L}(\mathcal{Q}, \mathcal{Q}')$ is a $K_{5.4.12}$-quasigeodesic in Y.

(3) $F_b \cap \mathcal{Q}$ is an $\epsilon_{5.4.12}$-approximate nearest point projection of $\mathcal{Q}'$ on $\mathcal{Q}$ and vice versa.

Proof. By Corollary 2.2.41, P_A is coarsely $L := L_{2.2.41}(\delta_0, k_0, 0)$-Lipschitz. So, the diameter of $P_A(U)$ is at most $LD + L =: D_{5.4.12}$. Proof of (2) and (3) are same as in the proof of Lemma 5.4.11, once we show that $d_h(\mathcal{Q} \cap Y, \mathcal{Q}' \cap Y) \geq R$. Suppose not, i.e., $d_h(\mathcal{Q} \cap Y, \mathcal{Q}' \cap Y) < R$. Then the diameter of $U_R(\mathcal{Q} \cap Y, \mathcal{Q}' \cap Y)$ is at most $k + kD$, since A is a k-qi embedded subset of B. Then the proof follows by the first part of Lemma 5.4.8. □

Lemma 5.4.13. *Given $K \geq K_0$ and $R \geq M_K$, there exists $D_{5.4.13} = D_{5.4.13}(D)$ such that the following holds.*

Suppose $\mathcal{Q}, \mathcal{Q}'$ are K-qi sections in X and $d_h(\mathcal{Q} \cap Y, \mathcal{Q}' \cap Y) \leq R$ in X. Let $U = U_R(\mathcal{Q} \cap F_b, \mathcal{Q}' \cap F_b) \leq D_{5.4.13}$.

Proof. Let $u \in U$ such that $P_A(u) = b$. If $u \in A$, then $b = u$ and $d_b(\mathcal{Q} \cap F_b, \mathcal{Q}' \cap F_b) \leq R$. So let $u \notin A$. Let $v \in U_R(\mathcal{Q} \cap Y, \mathcal{Q}' \cap Y)$. Then $[u, b] \cup [b, v]$ is $K_{2.2.38}(\delta_0, k_0, 1, 0)$-quasigeodesic in B. Since U is $K_{5.3.13}$-quasiconvex in B and $u, v \in U$, $d_B(b, U) \leq D_{2.2.18}(\delta_0, K_{2.2.38}(\delta_0, k_0, 1, 0)) + K_{5.3.13}$. Let $D := D_{2.2.18}(\delta_0, K_{2.2.38}(\delta_0, k_0, 1, 0)) + K_{5.3.13}$. Then by applying bounded flaring, $d_b(\mathcal{Q} \cap F_b, \mathcal{Q}' \cap F_b) \leq R \max\{\mu_K(D), 1\}$. Thus $D_{5.4.13} = R \max\{\mu_K(D), 1\}$. □

Now, finally we prove Step 3.

Lemma 5.4.14. *For $0 \leq i \leq n-1$, we have the following:*
(1) $\tilde{y}_{i+1}$ is a uniform approximate nearest point projection of $\tilde{y}_i$ on $\Sigma_{i+1} \cap Y$.
(2) $\tilde{\alpha}_i$ is a uniform quasigeodesic in Y.

Proof. **Case 1: $i \leq n-2$ and $\mathcal{L}_i$ is of type (I):** By Lemma 5.3.13, diameter of $U_{R_0}(\Sigma_i, \Sigma_{i+1})$ is uniformly small. By Lemma 5.4.12(3), $\tilde{y}_{i+1}$ is a uniform approximate nearest point projection of $\tilde{y}_i$ on $\Sigma_{i+1} \cap Y$. Also, $\Sigma_i \cap F_{b_{i+1}}$ is a uniform approximate nearest point projection of $\tilde{y}_{i+1}$ on $\Sigma_i \cap Y$ and by Lemma 5.4.12(2), $\tilde{\sigma}_i = [\Sigma_i \cap F_{b_{i+1}}, \Sigma_{i+1} \cap F_{b_{i+1}}]_{b_{i+1}}$ is a uniform quasigeodesic in Y. Using Lemma 2.2.38, $\tilde{\alpha}_i$ is a uniform quasigeodesic in Y.

Case 2: $\mathcal{L}_i$ is of type (II): We know that $d_h(\Sigma_i, \Sigma'_i) \leq R_1$. We consider the following sub cases:

Sub case 1: $d_h(Y \cap \Sigma_i, Y \cap \Sigma'_i) \leq R_1$. Then by Lemma 5.4.13, length of $\tilde{\sigma}_i = [\Sigma_i \cap F_{b'_i}, \Sigma'_i \cap F_{b'_i}]_{b'_i}$ has uniformly small length. Then, since $\tilde{\gamma}_i$ is a uniform quasigeodesic in Y, $\tilde{\gamma}_i * \tilde{\sigma}_i$ is a uniform quasigeodesic in Y.

Sub case 2: $d_h(Y \cap \Sigma_i, Y \cap \Sigma'_i) > R_1$. Then by Lemma 5.4.11, $[\Sigma_i \cap F_{b'_i}, \Sigma'_i \cap F_{b'_i}]_{b'_i}$ is a quasigeodesic in Y. Also, $\tilde{y}'_i$ is a uniform approximate nearest point projection of $\tilde{y}_i$ on $\Sigma'_i \cap Y$. Also, $\Sigma_i \cap F_{b'_i}$ is a uniform approximate nearest point projection of $\tilde{y}'_i$ on $\Sigma_i \cap Y$ and by Lemma 2.2.38, $\tilde{\gamma}_i * \tilde{\sigma}_i$ is a uniform quasigeodesic in Y.

Now, $d_h(\Sigma'_i \cap Y, \Sigma_{i+1} \cap Y) \leq 1$. Since $b_{i+1} \in P_A(U_{R_0}(\Sigma'_i, \Sigma_{i+1}))$, length of $\tilde{\sigma}'_i = [\Sigma'_i \cap F_{b_{i+1}}, \Sigma_{i+1} \cap F_{b_{i+1}}]_{b_{i+1}}$ has uniformly small length. Then, since $\tilde{\gamma}'_i$ is a uniform quasigeodesic in Y, $\tilde{\gamma}'_i * \tilde{\sigma}'_i$ is a uniform quasigeodesic in Y.

Case 3: $i = n-1$: We know that $d_h(\Sigma_{n-1}, \Sigma_n) \leq R_0$. Then we have the following possibilities : (i) $d_h(\Sigma_{n-1} \cap Y, \Sigma_n \cap Y) \leq R_0$, (ii) $d_h(\Sigma_{n-1} \cap Y, \Sigma_n \cap Y) > R_0$. Then proof of (i) and (ii) follow from Lemma 5.4.13 and Lemma 5.4.11 as in the proof of Sub case 1 and Sub case 2 respectively. □

Applying Proposition 5.1.1, we have proved the following result:

Proposition 5.4.15. *Let $x, y \in Y$ and Σ, Σ' be K_0 qi sections in X passing* *through x, y respectively. Let $c(x, y)$ be a geodesic in $\mathcal{L}(\Sigma, \Sigma')$ joining x, y. Then the corresponding cut-paste path $\tilde{c}(x, y)$ is a uniform quasigeodesic in Y.*

Step 4: Verification of the hypothesis of Lemma 2.2.33

Let $b_0 \in A$ and $y_0 \in F_{b_0}$. Let $y, y' \in Y$.

Proposition 5.4.16. *Given $D > 0$, there exists $D_{5.4.16} = D_{5.4.16}(D)$ such that the following holds.*

If $d(y_0, c(y, y')) \leq D$, then $d_Y(y_0, \tilde{c}(y, y')) \leq D_{5.4.16}$.

Proof. Let $x \in c(y, y')$ such that $d(x, y_0) \leq D$. Then $d_B(\pi(x), b_0) \leq D$. Let $i \in \{0, 1, \ldots, n\}$ such that $x \in \alpha_i$. Suppose $\mathcal{L}_i$ is of type (I). In that case, $\alpha_i = \gamma_i * \sigma_i$ joins y_i, y_{i+1}, where γ_i is a lift of $[y_i, y_{i+1}]$ on Σ_i. Clearly, $\pi(x) \in [y_i, y_{i+1}]$. Let w_1, w_2 be nearest point projections of $\pi(x)$ on $[b_i, b_{i+1}], A$ respectively. Then, $d_B(w_1, w_2) \leq D_{2.2.42}$. By Lemma 2.2.18, $d_B(w_1, [b_i, b_{i+1}]_A) \leq D_{2.2.18}(\delta_0, k)$ and $d_B(\pi(x), w_2) \leq d_B(\pi(x), b_0) \leq D$. Thus, there exists $w \in [b_i, b_{i+1}]_A$ such that $d_B(\pi(x), w) \leq D_{2.2.18}(\delta_0, k) + D =: D'$. Recall that $\pi(\tilde{\alpha}_i) = [b_i, b_{i+1}]_A$. Let Σ_x be a K_2-qi section in $\mathcal{L}_i$ through x. Then, $d(x, \Sigma_x \cap F_w) \leq K_2 D' + K_2$. Then, $d(y_0, \tilde{c}(x, y)) \leq K_2 D' + K_2 + D$. Then $d_Y(y_0, \tilde{c}(x, y)) \leq D_{5.4.4}(K_2 D' + K_2 + D)$.

Now, suppose $\mathcal{L}_i$ is of type (II). In that case, $\alpha_i = \gamma_i * \sigma_i * \gamma'_i * \sigma'_i$ joins y_i, y_{i+1}, where γ_i is a lift of $[y_i, y'_i]$ on Σ_i and γ'_i is a lift of $[y'_i, y_{i+1}]$ on Σ'_i. Then $x \in \gamma_i * \sigma_i \in \mathcal{L}(\Sigma_i, \Sigma'_i)$ or $x \in \gamma'_i * \sigma'_i \in \mathcal{L}(\Sigma'_i, \Sigma_{i+1})$. Repeating the above calculations, we have $D_{5.4.16} = \max\{D_{5.4.4}(K_2 D' + K_2 + D), D_{5.4.4}(K_3 D' + K_3 + D)\}$. ☐

This proposition verifies Lemma 2.2.33.

Proof of Theorem 5.0.1: It is given that, $G(\mathcal{Y})$ is a complex of groups, where $\mathcal{Y}$ is a finite complex. T is a maximal tree in the 1-skeleton of the first barycentric subdivision of $\mathcal{Y}$. Also, $G = \pi_1(G(\mathcal{Y}), T)$ is hyperbolic. We have,

$B' = D(\mathcal{Y}, i_T)$, where $i_T : G(\mathcal{Y}) \to G$ is the natural morphism.

We take B to be the 1-skeleton of the first barycentric subdivision of B'. Then we construct a graph X, similar to the construction in the subsection 4.2.1 on which G has a simplicial, proper and cocompact action and a G-equivariant map $\pi : X \to B$. Here, for each $\sigma \in V(\mathcal{Y})$, X_σ is a copy of the Cayley graph of G_σ. For $\tau \subset \sigma$, by 2 of the hypotheses of the theorem, the corresponding morphism $G_\tau \to G_\sigma$ is a quasiisometry. By the construction of X, (X, B, π) is a metric graph bundle.

Now, we take A to be the 1-skeleton of the first barycentric subdivision of A'. We similarly construct a metric graph bundle Y such that there exists a simplicial, proper and cocompact action of H on Y and also, an H-equivariant map $\pi_A : Y \to A$. Clearly, Y is a pullback X under $A' \to B'$. Then the proof follows by Theorem 5.4.3. □

5.5 Consequences

Given a short exact sequence of finitely generated groups, one can naturally associate a metric graph bundle to it ([45, Example 1.8]). Having said that, Theorem 5.4.3 gives the following as an immediate consequence.

Theorem 5.5.1. *Let* $1 \to N \to G \xrightarrow{\pi} Q \to 1$ *be a short exact sequence of hyperbolic groups. Suppose* Q_1 *is a finitely generated, qi embedded subgroup of* Q *and* $G_1 = \pi^{-1}(Q_1)$*. Then,* G_1 *is hyperbolic and the inclusion* $G_1 \to G$ *admits CT.*

Theorem 5.4.3 has the following analog for metric bundles.

Theorem 5.5.2. *Suppose* $\pi : X \to B$ *is a metric bundle satisfying the hypotheses of section 5. Suppose* $g : A \to B$ *is a* k*-qi embedding and suppose* $\pi : Y \to A$ *is the pullback bundle. Let* $f : Y \to X$ *be the pullback map. Then* Y *is a hyperbolic metric space and the CT map exists for* $f : Y \to X$*.*

Proof. By Proposition 5.3.1 and the first part of Lemma 5.3.2, we are reduced to considering the pullback of a metric graph bundle. Clearly, all the metric graphs in question are hyperbolic. Therefore, the result follows from Theorem 5.4.3 and the second part of Lemma 5.3.2. □

Chapter 6

Future Plan

The next natural course of action would be to investigate the analogue of Theorem 1.3.3 in the relative hyperbolic setting. Let $\pi : X \to B$ is a metric (graph) bundle such that B is a hyperbolic metric space and each fiber is hyperbolic relative to a collection of subsets such that the coned-off fibers will be uniformly hyperbolic. Let $g : A \to B$ be a k-qi embedding and $p : Y \to A$ be the pullback bundle. Let $f : Y \to X$ be the pullback map. We need to establish the sufficient conditions under which the metric (graph) bundle is relatively hyperbolic. Then show that Y is a hyperbolic metric space and the CT map exists for $f : Y \to X$. This would generalize the result by Pal in [46].

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