Geodesic Conjugacy Rigidity of Nonpositively Curved Surfaces

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Certificate of Examination

This is to certify that the dissertation titled **Geodesic Conjugacy Rigidity of Non positively Curved Surfaces** submitted by **Mr. Jithin Paul M** (Reg. No. MS08026) for the partial fulfillment of BS-MS dual degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr.Krishnendu Gangopadhyay at Indian Institute of Science Education and Research Mohali.
This work has not been submitted in part or in full for a degree, a diploma or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgment of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sourses listed within have been detailed in the bibliography.

Jithin Paul M
(Candidate)
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In my capacity as the supervisor of the candidate's project work I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Krishnendu Gangopadhyay
(Supervisor)

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Contents

Introduction

It is a fundamental problem in Riemannian geometry to try and capture the geometry of a Riemannian manifold by certain of its geometric invariants. In this thesis we consider closed (compact without boundary) Riemannian manifolds M and the action of the geodesic flow g_M^t on the unit tangent bundle SM.

It turns out that if M has negative sectional curvature then the geodesic flow g_M^t has significant influence on the geometry of M; for instance, it is a well known fact that a typical geodesic in M is dense. This is in sharp contrast with the case of geodesics on the unit sphere in $\mathbb{R}^3$, where every geodesic is a great circle; in particular none of the geodesics is dense. The classification theorem for surfaces says that a closed surface M in $\mathbb{R}^3$ is homeomorphic to either a sphere or a torus or a surface of higher genus. The genus of a surface determines its Euler characteristic, which is a topological invariant; more precisely, the Euler characteristic $\chi(M)$ of a surface M of genus g is $2-2g$. The celebrated Gauss Bonnet theorem relates the Euler characteristic of a surface M to its Gaussian curvature K by the formula

$$\int_M K dA = 2\pi\chi(M)$$

where dA is the area form in M. A consequence of the Gauss Bonnet formula is that the sign of curvature on a given closed surface M, if the same sign holds at all points of M, is restricted to a single choice. For example on a sphere S^2, whose Euler characteristic is 2, a negative sign on the curvature at all of its point is not possible, whereas such a thing is possible on a surface of genus ≥ 2. The classical uniformization theorem for surfaces precisely confirms this possibility. That is, a surface M of genus ≥ 2 admits a metric of constant negative curvature -1.

The main theorem discussed in this thesis concerns metrics of non positive curvature on a surface M of genus ≥ 2 and proves that such metrics are determined up to isometry by the action of the geodesic flow g_M^t on SM. More precisely, we will discuss a proof of the following theorem.

Theorem 0.0.1 (Croke, 1990)**.** *Let N be a closed surface of genus ≥ 2 with non positive sectional curvature and M be a compact surface whose geodesic flow is conjugate to N via F; i.e., $F : SM \mapsto SN$ is a C^1-diffeomorphism such that $F \circ g_M^t = g_N^t \circ F$ for all t then $F = g_N^K \circ df$, where f is an isometry from M to N and K is a fixed number.*

Chapter 1

Chapter 1

1.1 Preliminaries

1.1.1 Curvature

Let M be a smooth manifold and T_pM denote the tangent space at p. Suppose for each $p \in M$ we have an inner product g_p on T_pM which varies smoothly with respect to p as a 2-tensor, called the metric tensor; then M is called a Riemanian manifold with metric g. We will denote g by $\langle \, , \rangle$.

Length of a curve $\gamma : [a, b] \mapsto M$ is defined as $\int_a^b \langle \dot\gamma(t), \dot\gamma(t)\rangle^{\frac{1}{2}} dt$.

Definition (Levi Civita Connection): *The Levi Civita Connection ∇ is the unique map which takes any two smooth vector fields (X, Y) on (M, g) to another smooth vector field $\nabla_X Y$ and satisfies the following properties:*
1. $\nabla_X Y$ is $\mathbb{R}-$linear in both X and Y.
2. $\nabla_X Y$ is $C^\infty(M)-$linear in X but obeys the following product rule for all $f \in C^\infty(M)$:

$$\nabla_X fY = X(f)Y + f\nabla_X Y$$

3. $\nabla_X Y - \nabla_Y X = [X, Y]$.
4. $Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$ where Z is also a smooth vector field.

It turns out that $\nabla_X Y(p)$ depends only on $X(p)$ and value of Y along any curve α such that $\dot\alpha(p) = X(p)$.

Definition (Covariant derivative) : *Consider a smooth curve γ and a smooth vector field V along γ. We define the covariant derivative D_tV by*

$$D_tV(t_0) = \nabla_{\dot\gamma(t_0)}V$$

The definition is meaningful because of the above remark and the fact that $\dot{\gamma}(t_0)$ and V can be extended to the whole manifold smoothly. Thus D_t has the following properties.
1. D_tV is $\mathbb{R}-linear$ in Y as well as the velocity of γ
2. $D_tfV = fV + fD_tV$
3. $\frac{d}{dt}\langle X, Y\rangle = \langle D_tX, Y\rangle + \langle X, D_tY\rangle$ for any two vector fields X and Y along γ.

When there is no scope for confusion we will denote D_tV simply by V'.

Definition (Parallel field) : *We say that a vector field V along γ is parallel if $V' = 0$ everywhere on the curve. Given any $v \in T_pM$ where p is any point on the curve γ, there exists a unique parallel field V such that $V(p) = v$. V is called the parallel translate of v.*

It should be noted that if two vectors v and w are parallelly translated, then their length as well as the angle between them are preserved.

Definition (Geodesics) : *A smooth curve γ is said to be a geodesic in M if $D_t\dot{\gamma} = 0$ everywhere on the curve.*

Thus for a geodesic, its tangent vector field is a parallel translate and hence every geodesic is a constant speed curve. It is a very important theorem that **given any vector $v \in TM$, the tangent bundle of M, there exist a unique geodesic γ_v such that $\dot{\gamma}_v(0) = v$.**

Now we will move on to the notion of curvature. We define the curvature endomorphism R by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

It induces a 4- tensor Rm called the Curvature tensor

$$Rm(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle.$$

We will be dealing only with 2-dimensional Riemannian manifolds(surfaces) for which the notions of Gaussian curvature and sectional curvature coincide. Now onwards M is a surface. Let X and Y be smooth vector fields defined in a neighborhood of $p \in M$ such that they are linearly independent everywhere in the neighborhood. Then (X, Y) is called a local frame at p

Definition(Curvature):

$$K = \frac{Rm(X,Y,Y,X)}{|X|^2|Y|^2 - \langle X,Y\rangle^2}$$

where (X,Y) is any local frame on M.

Definition(Isometry) : *Let (M, g_M) and (N, g_N) be Riemannian manifolds and $f : M \mapsto N$ be a (local) diffeomorphism such that $f^\star g_N = g_M$ ie. $f^\star g_N(v,w)$, which is by definition, $g_N(f_\star v, f_\star w)$ is same as $g_M(v,w)$ for all $v, w \in TM$. Then f is called an (local) isometry.* Gauss's Remarkable Theorem states that K is an isometry invariant.

1.1.2 Jacobi Fields

We will first define Jacobi fields in terms of geodesic variation.

Let $\gamma : (-\infty,\infty) \mapsto M$ be a smooth curve on a Riemannian manifold M. In our cases, this will be a maximal geodesic. Consider a smooth map $\Gamma : (-\epsilon,\epsilon) \times (-\infty,\infty) \mapsto M$ such that $\Gamma(0,t) = \gamma(t)$. Then Γ is said to be a variation of γ. If for each value of s, the curve $\gamma_s(t) = \Gamma(s,t)$ is a geodesic, then it is called a geodesic variation. The curves $\gamma_s(t)$ where t is the parameter are called main curves and the curves $\gamma_t(s)$ where s is the parameter are called transverse curves. We will denote the variation also by $\gamma_s(t)$ and the situation will always make clear whether we are referring to the variation or a main curve. We set $S = \Gamma_*(\frac{\partial}{\partial s})$ and $T = \Gamma_*(\frac{\partial}{\partial t})$. Note that S is tangential to transverse curves and T is tangential to main curves. Using the cordinate description in a neighborhood around each point, it is not difficult to prove that $D_sT = D_tS$. This is called symmetry lemma for obvious reason. For any variation $\gamma_s(t)$, the smooth vector field $S(0,t) = \Gamma_*(\frac{\partial}{\partial s}|_{(s=0)}) = \frac{\partial}{\partial s}|_{(s=0)}\gamma_s(t)$ along γ is called the variation field. Note that the variation field is tangent vectors to transverse curves at points of γ. Now we are ready to introduce the term Jacobi field.

Definition: *The variation field of a geodesic variation is called a Jacobi field*

It can be proved that a vector field J along γ is a Jacobi field if and only if J satisfies the vector equation

$$D_t^2 J + R(J,\dot\gamma)\dot\gamma = 0$$

where R is the curvature endomorphism. Since the equation is linear, the set of all Jacobi fields along a given geodesic γ, which will be denoted by ψ is a vector space over $\mathbb{R}$.

As one may expect due to the presence of a second order differential equation, We have the following theorem about the existence and uniqueness of Jacobi fields.

Theorem 1.1.1. *Let γ be a geodesic in M and $p = \gamma(a)$. For any pair of vectors $X, Y \in T_pM$ there is a unique Jacobi field J along γ satisfying the initial conditions $J(a) = X$ and $J^{'}(a) = D_tJ(a) = Y$*

As a corollary, it follows that along any geodesic γ, the map from ψ to $T_pM \times T_pM$ which takes J to $(J(a), J^{'}(a))$ is an isomorphism and hence the vector space ψ is of dimension $2n$. A Jacobi field J is said to be tangential(normal), if J is a multiple of (perpendicular to) to $\dot{\gamma}$. Two tangential Jacobi fields important to us are $\dot{\gamma}(t)$ and $t\dot{\gamma}(t)$ and the one dimensional spaces spanned by them are denoted by ψ^t and ψ^b respectively. Clearly, the set of tangential Jacobi fields and the set of normal Jacobi fields (denoted by $\psi^{\perp}$) are subspaces of ψ. The following lemma will provide hints to the dimension of tangential and normal spaces.

Lemma 1.1.2. *Let $\gamma : I \mapsto M$ be a geodesic and $a \in I$.*

1 . *A Jacobi field along γ is normal if and only if $J(a) \perp \dot{\gamma}(a)$ and $J^{'}(a) \perp \dot{\gamma}(a)$.*

2 . *Any Jacobi field orthogonal to γ at two points is normal.*

Now, it follows easily that $\psi^{\perp}$ has dimension $2n - 2$ and the tangential space is $\psi^t + \psi^b$. Moreover the unique decomposition of a Jacobi fields into the sum of normal and tangential Jacobi fields can be obtained just by decomposing its initial value and initial derivative, and taking the unique Jacobi field determined by the perpendicular components and the one determined by the tangential components.

1.1.3 The geometry of tangent bundle

Our reference for this section is [P]. Let TM denote the tangent bundle of M and TTM be its bundle. For $\theta = (x, v) \in TM$, consider the set $V(\theta) = \{\zeta \in T_\theta TM : \zeta$ is the initial velocity of a curve $\sigma : (-\epsilon, \epsilon) \mapsto TM$ of the form $\sigma(t) = (x, v + tw)\}$. It can be shown that $V(\theta) = ker(d\pi(\theta))$ where π is the canonical projection from TM to M.

Now for each $\theta \in TM$, we define a map $K_\theta : T_\theta TM \mapsto T_xM$ called the connection map as follows.

Definition(Connection Map): *Given $\zeta \in T_\theta TM$, take a curve $z : (-\epsilon, \epsilon) \mapsto TM$ whose initial tangent vector is ζ. We can write $z(t) = (\alpha(t), Z(t))$ where $\alpha : (-\epsilon, \epsilon) \mapsto$*

M is a smooth curve and Z is a smooth vector field on α. Define $K_\theta(\zeta) = Z'(0)$, the co variant derivative of Z along α at $t = 0$.

We set $H(\theta) = ker(K_\theta)$. It is not difficult to show that K_θ is well defined(ie. independent of the curve z) and it is linear. We will prove the validity of the definition below.

Proposition 1.1.3. *K_θ is well defined.*

Proof. Let $\partial_1, \partial_2, ..., \partial_n$ be a basis of coordinate vectors in a neighborhood of x. Let $z_1(t) = (\alpha_1(t), Z_1(t))$ and $z_2(t) = (\alpha_2(t), Z_2(t))$ be two curves defining ζ. Let

$$Z_1 = a_1^i \partial_i$$
$$Z_2 = a_2^i \partial_i$$

We need to show that $\nabla_{\dot{\alpha_1}} Z_1(0) = \nabla_{\dot{\alpha_2}} Z_2(0)$. We have $a_1^i(0) = a_2^i(0)$ and $(a_1^i)'(0) = (a_2^i)'(0)$ since $\dot{z}_1(0) = \dot{z}_2(0)$.

Now $\dot{\alpha}(0) = (\pi \circ z_1)'(0) = d\pi(z_1(0))(z_1)'(0) = d\pi(\theta)(\zeta) = d\pi(z_2(0))(z_2)'(0) = (\pi \circ z_2)'(0) = \dot{\alpha}(0)$ and

$$\begin{aligned}
Z_1'(0) &= \nabla_{\dot{\alpha_1}} Z_1(0) \\
&= \nabla_{\dot{\alpha_1}} (a_1^i \partial_i)(0) \\
&= (a_1^i)'(0)\partial_i(0) + a_1^i(0)\nabla_{\dot{\alpha_1}} \partial_i(0) \\
&= (a_2^i)'(0)\partial_i(0) + a_2^i(0)\nabla_{\dot{\alpha_2}} \partial_i(0) \\
&= \nabla_{\dot{\alpha_2}} (a_2^i \partial_i)(0) \\
&= \nabla_{\dot{\alpha_2}} Z_2(0) \\
&= Z_2'(0)
\end{aligned}$$

Hence the proof is complete. □

Another equivalent way of constructing $H(\theta)$ is by means of the horizontal lift.

Definition (Horizontal Lift): *For $\theta = (x, v) \in TM$, we define the horizontal lift $L_\theta : T_xM \mapsto T_\theta TM$ as follows. Given $v' \in T_xM$, take a curve $\alpha : (-\epsilon, \epsilon) \mapsto M$ corresponding to v'. Let $Z(t)$ be the parallel transport of v along α. Let $\sigma : (-\epsilon, \epsilon) \mapsto TM$ be the curve $\sigma(t) = (\alpha(t), Z(t))$. Define $L_\theta(v') = \dot{\sigma}(0)$.*

It is clear that $K_\theta(L_\theta(v')) = 0$ for all $v' \in T_xM$. We take a curve σ corresponding to $L_\theta(v')$. Let $\sigma(t) = (\alpha(t), Z(t))$. Then $K_\theta(L_\theta(v')) = Z'(0)$,the covariant derivative of Z along α at $t = 0$. This is zero since Z is parallel along α by definition.

L_θ has the following properties.

1 . L_θ is well defined.

2 . L_θ is linear.

3 . $ker(K_\theta) = image(L_\theta)$.

4 . $d\pi(\theta) \circ L_\theta = Id_{T_xM}$, the identity map.

5 . The maps $d\pi(\theta)|_{H(\theta)} : H(\theta) \mapsto T_xM$ and $K_\theta|_{V(\theta)} \mapsto T_xM$ are linear isomophisms.

These results establish that $T_\theta TM = H(\theta)+V(\theta)$ as follows. from 5, $dim(H(\theta)) = dim(H(\theta)) = dim(T_xM) = n$. Also $H(\theta) \cap V(\theta) = 0$. This is because if $\zeta \in H(\theta) \cap V(\theta)$, then we can take $\zeta = L_\theta(v^{'})$ for some $v^{'} \in T_xM$. Then $v^{'} = Id_{T_xM}(v^{'}) = d\pi(\theta) \circ L_\theta(v^{'}) = d\pi(\theta)(\zeta) = 0$. Hence $\zeta = L(\theta)(v^{'}) = 0$.

Now it is quite easy to see that the map $j_\theta : T_\theta TM \mapsto T_xM \times T_xM$ defined by $j_\theta(\zeta) = (d\pi(\theta)(\zeta), K_\theta(\zeta))$ is linear and injective and hence an isomorphism (since the dimensions of domain space and codomine space are equal). Now onwards we can write ζ as (ζ_h, ζ_v) using this isomorphism. $\zeta_h = d\pi(\theta)(\zeta)$ is called the horizontal component of ζ and $\zeta_v = K_\theta(\zeta)$ is called the vertical component of ζ. Using the decomposition $T_\theta TM = H(\theta) + V(\theta) = T_xM \times T_xM$, we can define a Riemannian metric on TM such that $H(\theta)$ and $V(\theta)$ are orthogonal. This metric is called Sassaki metric.

Definition(Sassaki metric):

$$\langle\!\langle \zeta, \eta \rangle\!\rangle_\theta = \langle d\pi(\theta)(\zeta), d\pi(\theta)(\eta)\rangle_x + \langle K_\theta(\zeta), K_\theta(\eta)\rangle_x$$

1.1.4 Jacobi field correspondence

Proposition 1.1.4. *Let $\theta = (x, v) \in SM$. A vector $\xi \in T_\theta TM$ will lie in $T_\theta SM$ if and only if $\langle K(\xi), v\rangle = 0$.*

Proof. Let $\theta = (x, v) \in SM \subseteq TM$ and $\xi \in T_\theta TM$. Let $z(t)$ be the curve in TM corresponding to ξ. We can write $z(t) = (\alpha(t), Z(t))$ where $Z(t)$ is a smooth vector field along α.

If $\xi \in T_\theta SM$, then we can assume $z(t) \in SM$ for all t. ie , $\langle Z(t), Z(t)\rangle = 1$ for all t. Then

$$\frac{d}{dt}\langle Z(t), Z(t)\rangle = 2\langle \nabla_{\dot\alpha} Z(0), v\rangle = 0$$

But $\nabla_{\dot\alpha} Z(0) = K(\xi)$ by definition. This shows that $\langle K(\xi), v\rangle = 0$ if $\xi \in T_\theta SM$.

Conversely, let us assume that $\langle \nabla_{\dot\alpha} Z(0), v\rangle = 0$. ie $\frac{d}{dt}\langle Z(t), Z(t)\rangle = 0$. Since $\langle Z(0), Z(0)\rangle = \langle v, v\rangle = 1$, $\langle Z(t), Z(t)\rangle$ is away from zero at $t = 0$. So by replacing $z(t) = (\alpha(t), Z(t))$ by $(\alpha(t), \frac{Z(t)}{|Z(t)|})$, if necessary, we can assume that $z(t) \in SM$ in a neighborhood of $t = 0$. Then $\xi \in T_\theta SM$. □

The dimension of $T_\theta TM$ is $2n$. The condition $\langle K(\xi), v\rangle = 0$ brings down the dimension of $T_\theta SM$ to $2n - 1$.

Given a Jacobi Field J along a geodesic γ such that $J'(0) \perp v = \gamma'(0)$, we can identify J with $\xi = (J(0), J'(0)) \in T_\theta TM$. Then $\langle K(\xi), v)\rangle = \langle J'(0), v\rangle = 0$ and the above proposition $\xi \in T_\theta SM$. It is clear that this correspondence is an isomorphism. We will denote the Jacobi field corresponding to $\xi \in T_\theta SM$ by J_ξ.

1.2 Geodesic Conjugacy

Definition(Geodesic flow): *Let M be a complete manifold and SM denote its unit tangent bundle. For each $t \in \mathbb{R}$ we have a map $g^t : SM \mapsto SM$ defined by*

$$v \mapsto \gamma_v'(t).$$

The 1-parameter collection g^t is called the Geodesic Flow.

It can be easily verified that
1.$g^t \circ g^s = g^{s+t}$
2. $g^0 = Identity$
Thus we see that $\mathbb{R}$ acts on SM via the geodesic flow.

Definition (Geodesic Conjugacy): *Let M and N be two complete manifolds with geodesic flows g_M^t and g_N^t respectively. M and N are said to have conjugate geodesic flows via F if there exists a C^1- diffeomorphism $F : SM \mapsto SN$ such that*

$$g_N^t \circ F = F \circ g_M^t$$

for all $t \in \mathbb{R}$

Proposition 1.2.1. *F takes a geodesics γ to a geodesic $F(\gamma)$*

Proof. By this we mean that F takes the tangent field of a geodesic in M to the tangent field of a geodesic in N. Let $v = \gamma^{'}(0)$.

$$F(\gamma^{'}(t)) = F \circ g^t_M(v) = g^t_N \circ F(v) = \gamma^{'}_{F(v)}(t)$$

Thus F takes γ to $\gamma_{F(v)}$ which we will call as $F(\gamma)$ □

Definition(Geodesic Vector Field) : *Let G be the vector field on SM which generate the geodesic flow. i.e.*

$$G(v) = \frac{d}{dt}|_{t=0} g^t_M(v).$$

We call G as a geodesic (flow) vector field.

Proposition 1.2.2. *Let F be a geodesic conjugacy from M to N. Then $F_\star(G_M(v)) = G_N(F(v))$.*

Proof.

$$\begin{aligned} F_\star(G_M(v)) &= (F \circ g^t_M(v))^{'}(0) \\ &= (g^t_N \circ F(v))^{'}(0) \\ &= G_N(F(v)) \end{aligned}$$

□

Proposition 1.2.3. *If $\tilde{M}$ is the universal covering space of M , then $S\tilde{M}$ is a cover of SM.*

Proof. Let $P : \tilde{M} \mapsto M$ be the projection under which $\tilde{M}$ is the universal cover. Then by the definition of the metric in $\tilde{M}$, P is a local isometry. Define a map $SP : S\tilde{M} \mapsto SM$ by $SP(\tilde{x}, \tilde{v}) = (P(\tilde{x}), dP(\tilde{v}))$. Let $(x, v) \in SM$. Consider a neighborhood $U \times S^1$ where U is an evenly covered neighborhood of x. Let $V_1, V_2, ..., V_n$ be the disjoint homeomorphic copies of U in $\tilde{M}$. We will show that $U \times S^1$ is evenly covered by $V_1 \times S^1, V_2 \times S^1, ..., V_n \times S^1$ under the map SP.

Clearly $V_1 \times S^1, V_2 \times S^1, ..V_n \times S^1$ are mutually disjoint since $V_1, V_2, ...Vn$ are so.

To show that $SP^{-1}(U \times S^1) \subseteq \sqcup_{i=1}^n V_i \times S^1$ let us assume that $(\tilde{y}, \tilde{w}) \in SP^{-1}(U \times S^1)$. This means $\tilde{y} \in V_i$ for some i and $\tilde{w} \in S^{'}$ since U is evenly covered and dP is a local isometry. Then $(\tilde{y}, \tilde{w}) \in V_i \times S^1$

Conversely let $(\tilde{y}, \tilde{w}) \in V_i \times S^1$. Then again $P(\tilde{y}) \in U$ and $dP(\tilde{w}) \in S^1$. So that $SP(\tilde{y}, \tilde{w}) \in U \times S^1$. Thus we get $SP^{-1}(U \times S^1) = \sqcup_{i=1}^{n} V_i \times S^1$.

It remains to show that SP is surjective. This is clear since P is surjective and dP is a local isometry □

Now onwards in this chapter we will concentrate on surfaces of genus ≥ 2.

Let K_M be the subgroup of $\pi_1(SM)$ generated by the fiber, where M is a surface of genus ≥ 2. If M is orientable, then K_M is the center of $\pi_1(SM)$. If M is non orientable then $K_M = \{a \in \pi_1(SM) |\ bab^{-1} = a \text{ or } a^{-1} \text{ for all } b \in \pi_1(SM)\}[B - K]$. Since $F_\star$ is an isomorphism it is easy to see that $F_\star(K_M) = K_N$.

Lemma 1.2.4. *F lifts to a map $\tilde{F} : S\tilde{M} \mapsto S\tilde{N}$.*

Proof. The condition for the lift to exist is that

$$(F \circ SP_M)_\star \pi_1(S\tilde{M}) \subseteq (SP_N) \star \pi_1(S\tilde{N})$$

. Let $\tilde{K_M}(resp. \tilde{K_N})$ be the subgroup of $\pi_1(S\tilde{M})$ ($\pi_1(S\tilde{N})$ respectively) generated by fiber. We will soon show that $\tilde{K_M} = \pi_1(S\tilde{M})$ ($\tilde{K_N} = \pi_1(S\tilde{N})$ respectively). Hence our lifting condition is translated to

$$F_\star(SP_M)_\star(\tilde{K_M}) \subseteq SP_{N\star}(\tilde{K_N})$$

ie.

$$F_\star(K_M) \subseteq K_N$$

which is true by the above remark. □

Proposition 1.2.5. $\tilde{K_M} = \pi_1(S\tilde{M}), (also\ \tilde{K_N} = \pi_1(S\tilde{N})$.

Proof. Consider a loop $(\tilde{x}(t), \tilde{v}(t))$ in $S\tilde{M}$. We will homotope it to a loop $(\tilde{x}(0), \tilde{w}(t))$ in the fiber.

Consider $H : [0, 1] \times [0, 1] \mapsto S\tilde{M}$ such that

$$(0, t) \mapsto (\tilde{x}(t), \tilde{v}(t))$$
$$(1, t) \mapsto (\tilde{x}(0), \tilde{w}(t))$$

We will define $H(s, t)$ as follows. Join $\tilde{x}(t)$ to $\tilde{x}(0)$ by a geodesic parametrised by s which varies from 0 to 1. Call it γ_t. Define $H(s, t) = (\gamma_t(s), P_s\tilde{v}(t))$ where $P_s\tilde{v}(t)$ is the parallel translate of $\tilde{v}(t)$ along γ_t for time s. The speed of γ_t varies continuously with $dist(\gamma(t), \tilde{x}(0))$ which in turn varies continuously w.r.t t. So H is continuous and we get required homotopy. □

Lemma 1.2.6. *$\tilde{F} : S\tilde{M} \mapsto S\tilde{N}$ is a geodesic conjugacy.*

Proof. We need to show that $\tilde{F} \circ g^t_{\tilde{M}} = g^t_{\tilde{N}} \circ \tilde{F}$ for all t. For a given $\tilde{v} \in S\tilde{M}$, consider the geodesic $\gamma_{\tilde{v}}$ in $\tilde{M}$. Since dP_M is a local isometry this geodesic is taken to a geodesic γ_v in M by the map dP_M. Clearly $g^t_{S\tilde{M}}(v)$ is taken to $g^t_M(v)$ because of the same reason. F takes $g^t_M(v)$ to $g^t_N(F(v))$. Now again, since dP_N is an isometry, $\gamma_{F(v)}$ must come from a geodesic above, whose two tangent vectors are $\tilde{F}(\tilde{v})$ and $\tilde{F} \circ g^t_{\tilde{M}}(\tilde{v})$. Thus we get that

$$\tilde{F} \circ g^t_{S\tilde{M}}(v) = g^t_{\tilde{N}} \circ \tilde{F}(v).$$

□

Proposition 1.2.7. *F induces an isomorphism from $\pi_1(M)$ to $\pi_1(N)$.*

Proof. Let α be a nontrivial element in $\pi_1(M)$. Since $(\pi_M)_\star$ is a surjection, α comes from a loop $\tilde{\alpha}$ in SM. $\tilde{\alpha}$ cannot be in K_1 since α is nontrivial. Since $F_\star(K_1) = K_2$, $F_\star(\tilde{\alpha})$ cannot be in K_2. Hence $(\pi_N)_\star \circ F_\star(\tilde{\alpha})$ is nontrivial. Define it to be $F(\alpha)$. By taking $F_\star^{-1}$ instead of $F_\star$, we can show that F is invertible. F is clearly a homomorphism. □

Thus the map F on closed geodesics induces an isomorphism of free homotopy classes. It is known that on a manifold of non positive curvature, every free homotopy class contains a geodesic as the shortest curve and if the class has more than one geodesic, all of them must have the same length. Now by F, the same is true or M also. ie. two freely homotopic geodesics has same length in M.

Lemma 1.2.8. *M has no conjugate points.*

Proof. By [B], closed geodesics are dense in N meaning the set of periodic vectors $\{v \in SN : \gamma_v \text{ is a closed geodesic}\}$ is dense in SN. By F^{-1} closed geodesics are dense in M also. So if we show that there are no pair of conjugate points along any closed geodesic, by the denseness it holds for all geodesics in M. By the remark following previous proposition, a closed geodesic $\gamma : [0, L] \mapsto M$ is the shortest curve in its free homotopy class. Hence the segment of the lift $\tilde{\gamma}$ from 0 to L is also minimizing(the projection map P is isometry), It applies to iterates of γ that represent the elements in the fundamental group which are powers of the element represented by γ as well; hence $\tilde{\gamma}$ is minimizing for all time. Hence there are no conjugate points along $\tilde{\gamma}$. Let

$\gamma_v(0) = p$ and. If possible let q be a point conjugate to p along γ. Reparameterise γ such that $q = \gamma_v(1)$. p is conjugate to q along a geodesic γ in M if and only if the exponential map $E_M : TM \mapsto M$ fails to be a local diffeomorphism at v. By inverse function theorem , this is if and only if $E_{M\star}$ at v is not invertible. But $dP \circ E_M = E_M \circ P$ where dP is the projection from $T\tilde{M}$ to TM. Taking derivative and using the fact that $\tilde{\gamma}$ has no conjugate point , it follows that γ has no conjugate point. □

1.3 The Contact structure on SM

Definition (Contact Manifold): *Let M be a $2n-1$ dimensional manifold. A 1-form α on M is called a contact form if $\alpha \wedge (d\alpha)^{n-1}$ is non vanishing. Then the pair (M, α) is called a contact manifold. A flow ψ_t on M which preserves α, that is for which $\psi_t^*\alpha = \alpha$, is called a* **contact flow**.

Every contact manifold (M, α) comes with a unique vector field X such that

$$i_X\alpha = 1$$
$$i_X d\alpha = 0$$

such an X is called the **characteristics vector field**. The flow of X is called the **characteristic flow**. It can be shown that the characteristic flow preserves α.

The geodesic vector field G gives a contact structure on SM. We define α by

$$\alpha_v(\xi) = \langle\langle \xi, G(v) \rangle\rangle$$

We see that for $\xi \in T_vSM$.

$$\begin{aligned} \alpha_v(\xi) &= \langle\langle \xi, G(v) \rangle\rangle \\ &= \langle d\pi(\xi), v \rangle + \langle K(\xi), v \rangle \\ &= \langle d\pi(\xi), v \rangle. \end{aligned}$$

As shown in detailed in [P] , α is a contact form whose characteristic flow is g_M^t. Hence G is the characteristic vector field of α. g_M^t preserves α.

Now it is easy to see that $ker\alpha = G^{\perp}$, the $2n-2$ subspace of T_vSM orthogonal to the span of $G(v)$. And also

$$d\theta(\xi,\eta) = \langle \xi_h, \eta_v\rangle - \langle \xi_v, \eta_h\rangle$$

It is also shown in [P] that the volume form induced by the Sassaki metric is $(n-1)!\alpha \wedge (d\alpha)^{n-1}$. Thus for a surface the volume form on SM is $\alpha \wedge d\alpha$.

Proposition 1.3.1. *Geodesic conjugacy preserves contact and volume forms.*

Proof. Let F be a geodesic conjugacy from SM to SN. We will denote the contact forms on SM and SN by α_1 and α_2 respectively. Then we need to show that

$$F^{\star}\alpha_2 = \alpha_1$$

For $\xi \in T_vSM$,

$$\begin{aligned} F^{\star}\alpha_2(\xi) &= \alpha_2(F_{\star}\xi) \\ &= \langle F(v), (F_{\star}\xi)\rangle \\ &= \langle F(v), (F_{\star}(aG_M(v) + \xi^{\perp}))_h\rangle \end{aligned}$$

Where we have decomposed ξ into $aG_M(v)$ and $\xi^{\perp} \in G_M^{\perp}$, $a \in \mathbb{R}$.

$$RHS = \langle F(v), (F_{\star}(aG_M(v)))_h\rangle + \langle F(v), F_{\star}(\xi^{\perp})_h\rangle$$

Now

$$\begin{aligned} F_{\star}(aG(v)) &= aF_{\star}G(v) \\ &= aF_{\star}(\frac{d}{dt}|_{(t=0)} g^t_M(v)) \\ &= a\frac{d}{dt}|_{(t=0)}(F \circ g^t_M(v) \\ &= a\frac{d}{dt}|_{(t=0)}(g^t_N(F(v)) \\ &= aG_N(F(v)) \end{aligned}$$

Thus $F_{\star}$ takes span of G_M to span of G_N. Since $F_{\star}$ is an isomorphism and $T_vSM = \langle G_M\rangle + G_M^{\perp}$ and $T_{F(v)}SN = \langle G_N\rangle + G_N^{\perp}$, $F_{\star}$ must take $G_M^{\perp}$ to $G_N^{\perp}$. Thus $(F_{\star}(\xi^{\perp}))_h = 0$

and we get

$$F_\star\alpha_2(\xi) = a$$

On the other hand

$$\begin{aligned}\alpha_1(\xi) &= \langle\, v, \xi_n\rangle \\ &= \langle\, v, (aG_M(v) + \xi^\perp)_h\rangle \\ &= \langle\, v, av\rangle = a.\end{aligned}$$

Now we will show that

$$F_\star d\alpha_2 = d\alpha_1$$

$$\begin{aligned}F_\star d\alpha_2(\xi,\eta) &= d\alpha_2(F_\star\xi, F_\star\eta) \\ &= F_\star(\xi)\alpha_2(F_\star\eta) - F_\star(\eta)\alpha_2(F_\star\xi) - \alpha_2([F_\star(\xi), F_\star(\eta)]\end{aligned}$$

by the definition of $d\alpha$.

$$\begin{aligned}F_\star(\xi)\alpha_2|_{F(v)}(F_\star\eta) &= F_\star(\xi)F^\star\alpha_2|_{F^{-1}F(v)}(\eta) \\ &= \xi\alpha_1(\eta) \\ \alpha_2|_{F(v)}([F_\star(\xi), F_\star\eta]) &= \alpha_2|_{F(v)}F_\star[\xi,\eta] \\ &= F^\star\alpha_2(\xi,\eta) \\ &= \alpha_1[\xi,\eta]\end{aligned}$$

Thus $LHS = \xi\alpha_1(\eta) - \eta\alpha_1(\xi) - \alpha_1[\xi,\eta]$ which is, by definition, $d\alpha_1(\xi,\eta)$.

Since $F^\star(\alpha\wedge\beta) = F^\star\alpha\wedge F^\star\beta$ for all forms of α and β, we get $F^\star(\alpha_2\wedge d\alpha_2) = \alpha_1\wedge d\alpha_1$.

Hence F is volume and orientation preserving. □

1.4 F-induced correspondences of Jacobi fields

We have already seen that the space of Jacobi fields ψ along a geodesic γ splits naturally as $\psi = \psi^\perp + \psi^t + \psi^b$ where $\psi^\perp$ consists of those Jacobi fields that are perpendicular to γ, ψ^t is spanned by $\gamma^{'}$, and ψ^b is spanned by $t\gamma^{'}$. Although all Jacobi fields arises from variations of geodesics, we have :

Proposition 1.4.1. *Only the Jacobi fields in $\psi^{\perp} + \psi^{t}$ comes from variations of geodesics γ_s which are all parametrized by arc length.*

Proof. Let $\Gamma(s,t)$ denotes the smooth map corresponding to the variation γ_s. Let $S = \Gamma_{\star}(\frac{\partial}{\partial s})$ and $T = \Gamma_{\star}(\frac{\partial}{\partial t})$. Then by symmetry lemma, we have $D_s T = D_t S$. Now suppose to the contrary that $J \in \psi^{b}$ arises from a variation γ_s through unit speed geodesics. Then $\langle T, T\rangle = 1$ identically. Differentiating with respect to s, this gives $\langle D_s T, T\rangle = 0$. By symmetry lemma, $\langle D_t S, T\rangle = 0$ identically. At $s = 0$, this implies $\langle D_t J, \gamma^{'}\rangle = 0$. We can take $J = at\gamma^{'}$ where $a \in \mathbb{R}$ and then $0 = \langle D_t J, \gamma^{'}\rangle = a\langle \gamma^{'} + tD_t\gamma^{'}, \gamma^{'}\rangle = a\langle \gamma^{'}, \gamma^{'}\rangle$ which is clearly a contradiction. □

All of our geodesics will be parametrized by arc length (unit speed) unless otherwise stated so that we can restrict ourself to $\psi^{\perp} + \psi^{t}$.

Note that $\psi^{\perp} + \psi^{t}$ is exactly the subspace consisting of Jacobi fields J for which $J^{'}$ is perpendicular to γ. Hence under the correspondence $\xi \mapsto J_{\xi}$, $\psi^{\perp} + \psi^{t}$ is isomorphic to $\{\xi \in T_v TM | \langle K(\xi), v\rangle = 0\}$. By proposition 0.1.1, this subspace is $T_v SM$. Thus **every element J of $\psi^{\perp} + \psi^{t}$ can be represented by a unique $\xi \in T_v SM$ isomorphically. We will denote J by J_{ξ}.**

Definition($F_{\star}$): *Since $F_{\star}$ is an isomorphism from $T_v SM$ to $T_{F(v)}SN$, we readily get an isomorphism from $\psi_M^{\perp} + \psi_M^{t}$ to $\psi_N^{\perp} + \psi_N^{t}$. We will denote this map also by $F_{\star}$. Thus*

$$F_{\star}(J_{\xi}) = J_{F_{\star}(\xi)}$$

There is another way to go from a Jacobi field in M to a Jacobi field in N. Corresponding to each J there is a geodesic variation γ_s. F takes γ_s to a geodesic variation $F(\gamma_s)$ in N. We define $\phi(J)$ to be the variation field of $F(\gamma_s)$.

Is ϕ same as $F_{\star}$? We will explore now.

Let $T\gamma$ denote the tangent field of γ. It is a smooth curve in SM. $T\gamma_s$ is a variation of $T\gamma$. Define TJ to be the variation field of $T\gamma$.

$$\begin{aligned}
d\pi(TJ) &= d\pi(\frac{d}{ds}|_{(s=0)} T\gamma_s(t)) \\
&= \frac{d}{ds}|_{(s=0)}(\pi \circ \gamma_s^{'}(t)) \\
&= \frac{d}{ds}|_{(s=0)}\gamma_s(t) \\
&= J
\end{aligned}$$

Now $K(TJ) = \nabla_{\dot{\gamma}_s}\gamma_s^{'}(t)(0)$ where $\dot{\gamma}_s$ is the derivative w.r.t s and $\gamma_s^{'}(t)$ is that w.r.t t. $RHS = D_sT$ at $s = 0$, which is, by symmetry lemma, same as D_tS at $t = 0$. This is nothing but $J^{'}$.

Thus we get $TJ_\xi(0) = \xi$.

Now, if we show that $F_\star(TJ) = T\phi J$, it will establish that $F_\star$ is same as ϕ.

$$\begin{aligned} T\phi J &= \frac{d}{ds}|_{(s=0)}F(\gamma_s)^{'}(t) \\ &= \frac{d}{ds}|_{(s=0)}F(\gamma_s^{'}(t)) \\ &= F_\star(\frac{d}{ds}|_{(s=0)}\gamma_s^{'}(t)) \\ &= F_\star(TJ) \end{aligned}$$

What we have achieved is the knowledge that we can go from J to $F_\star(J)$ also by taking variations to variations.

Definition(Stable and Unstable Jacobi fields):

We denote by ψ^s (resp. ψ^u) the subspace of ψ which consists of the Jacobi fields J for which $|J(t)|^2 + |J^{'}(t)|^2 \mapsto 0$ as $t \mapsto \infty$ (resp. $t \mapsto -\infty$). It is easy to see that ψ^s (resp. ψ^u) $\subseteq \psi^\perp$.

Definition(Weakly stable and weakly unstable Jacobi fields):

We will denote by ψ^{ws} (resp. ψ^{wu}) the subspace of $\psi^\perp$ consisting of Jacobi fields J for which $|J(t)|^2 + |J^{'}(t)|^2$ remains bounded as $t \mapsto \infty$ (resp. $t \mapsto -\infty$).

A Jacobi field J_ξ in $\psi^\perp + \psi^t$ corresponds to a curve $\xi(t)$ in TSM where $\xi(0) = \xi$. Since $|J_\xi(t)|^2 + |J_\xi(t)^{'}|^2 = |\xi(t)|^2$ the above definitions can be restated as $J_\xi \in \psi^s$ (resp. ψ^u) if $|\xi(t)| \mapsto 0$ and $J_\xi \in \psi^{ws}$ (resp. ψ^{wu}) if $|\xi(t)|$ remains bounded as $t \mapsto \infty$ (resp. $t \mapsto -\infty$).

As $|\xi(t) + c\eta(t)| \leq |\xi(t)| + |c||\eta(t)|$ for any constant c, it is clear that the sets defined above are actually subspaces of ψ.

Lemma 1.4.2. *Along every geodesic γ of M, $F_\star$ takes the sets $\psi_M^\perp, \psi_M^s, \psi_M^u, \psi_M^{ws}, \psi_M^{wu}$ to the corresponding sets $\psi_N^\perp, \psi_N^s, \psi_N^u, \psi_N^{ws}$ and ψ_N^{wu} along $F(\gamma)$.*

Proof. Clearly $\psi_M^\perp$ corresponds to $G_M^\perp(v) \subseteq T_vSM$ where $v = \gamma^{'}(0)$. We have already seen that $F_\star(G_M^\perp(v)) = G_N^\perp(F(v))$. Hence $F_\star(\psi_M^\perp) = \psi_N^\perp$. Since $F : SM \mapsto SN$ is a C^1 map between compact manifolds, there is a number $a > 1$ such that $\frac{1}{a}|\xi| < |F_\star(\xi)| < a|\xi|$, for every $\xi \in TSM$. Hence $|F_\star\xi(t)| \mapsto 0$ (respectively remains bounded) if and only if $|\xi(t)| \mapsto 0$ (respectively remains bounded). This implies $F_\star(\psi_M^s) = \psi_N^s$ and

$F_\star(\psi_M^u) = \psi_N^u$ directly and along with the fact $F_\star(\psi_M^\perp) = \psi_N^\perp$, this implies $F_\star(\psi_M^{ws}) = \psi_N^{ws}$ and $F_\star(\psi_M^{wu}) = \psi_N^{wu}$. $\square$

We have $|F_\star\xi| \leq |F_\star||\xi| < (|F_\star| + 1|)|\xi|$ and

$$\begin{aligned} |\xi| &= |F_\star^{-1}(F_\star(\xi))| \\ &\leq |F_\star^{-1}||F_\star\xi| \\ &< (|F_\star^{-1}| + 1)|F_\star\xi| \end{aligned}$$

where $|F_\star|$ and $|F_\star^{-1}|$ are the operator norms of $F_\star$ and $F_\star^{-1}$ respectively. These are continuous functions on SM and SN respectively. Hence the function $f(v) = max\{|F_\star|_v, |F_\star^{-1}|_{F(v)}\} + 1$ is continuous and attains a maximum a on the compact space SM. Then $\frac{1}{a}|\xi| < |F_\star\xi| < a|\xi|$ for all $\xi \in TSM$.

Chapter 2

2.1 Vanishing Fields

Now onwards our manifolds are surfaces, ie. $n = 2$.

In this section we will show that a normal Jacobi field vanishing in M has its image also vanishing in N. The difference of parameters at which they vanish is of special interest to us.

For a geodesic on a surface we can choose a parallel unit field X normal to γ' along γ by parallel translating a unit vector v normal to γ at $\gamma(0)$. (The other choice of X would be the parallel translate of $-v$). Every Jacobi field $J(t)$ in $\psi^{\perp}$ can (and will) be written as $J(t) = j(t)X(t)$ where $j(t)$ is function. We will sometimes confuse the Jacobi field with the function j.

By lemma 2.2 of [I-H] , for any geodesic γ on N and for any $v \in T_pN$ where $p = \gamma(0)$, there exists a unique weakly stable Jacobi field Y_1 and a unique weakly unstable Jacobi field Y_2 such that

$$Y_1(0) = Y_2(0) = v.$$

Note that if these two Jacobi fields are linearly dependent, then both of them are bounded in both the directions. On a manifold of non positive curvature, a bounded Jacobi field is necessarily parallel.] and hence can be realized as the variation field of a flat strip. Hence, if the geodesic γ passes through a region of negative curvature, these two Jacobi fields are necessarily independent. Otherwise they may be the same.

Via F_*^{-1} we thus see there are nontrivial elements of ψ_M^{ws} and ψ_M^{wu}. By scaling, if necessary, we can choose a Jacobi field $J_M^s \in \psi_M^{ws}$ with $J_1^s(0) = X(0)$. Thus, for a fixed geodesic γ of M, we will from now on denote by J_M^s, the element of ψ_M^{ws} with $j_M^s(0) = 1$. Similarly we define $J_M^u \in \psi_M^{wu}$ by demanding $j_M^u(0) = 1$. J_M^s may coincide with J^u_M as explained above.

We have already shown that like N, M also has no conjugate points. Along a geodesic γ, where no pair of points on γ are conjugate, it is natural look at $\psi^\perp = \psi^n \cup \psi^z$ where ψ^n consists of Jacobi fields that never vanish and ψ^z those that do. By [Gre] along a geodesic without conjugate points a Jacobi field that vanishes must be unbounded at ∞ and $-\infty$ and hence ψ^{ws} and ψ^{wu} are contained in ψ^n.

By the above J^s_M never vanishes and so we can define a new Jacobi field $J^z_M \in \psi^z_M$ by

$$j^z_M(t) = j^s_M(t) \int_0^t \frac{dy}{j^s_M(y)^2} \tag{2.1}$$

clearly J^s_M and J^z_M are linearly independent and since $\psi^\perp$ is of dimension 2, any Jacobi field in $\psi^\perp_M$ is a linear combination of J^s_M and J^z_M.

For $v = \gamma^{'}(x)$, we let J^v_M be the Jacobi field along γ such that

$$j^v_M(x) = 0 \text{ and } j^{v'}_M(x) = 1$$

That is to say $J^v_M(x) = 0$ and $J^{v'}_M(x) = X(x)$. By the existence and uniqueness theorem for Jacobi fields, we know that such a J^v_M exists uniquely.

We see that

$$J^v_M(t) = j^z_M(x).J^z_M(t) + j^s_M(x) \int_x^0 \frac{dy}{j^s_M(y)^2} J^s_M(t) \tag{2.2}$$

One can easily verify that $j^v_M(x) = 0$ and $j^{v'}_M(x) = 1$. Along the geodesic $F(\gamma)$, we will let $J^s_N = F_\star(J^s_M)$. By lemma (1.4.2), we know that $J^s_N \in \psi^{ws}_N \subseteq \psi^n_N$. We define J^z_N from J^s_N in the same way that J^z_M was defined from J^s_M. We know there are constants c_1 and c_2 such that

$$F_\star(J^z_M) = c_1 J^s_N + c_2 J^z_N \tag{2.3}$$

because $F_\star(J^z_M) \in \psi^\perp$ which is spanned by J^s_N and J^z_N.

Lemma 2.1.1. *In the above $c_2 = 1$.*

Proof. Let $J^s_M = J_\xi$ and $J^z_M = J_\eta$.

F preserves $d\alpha$

$$
\begin{aligned}
&\Rightarrow F^* d\alpha_2(\xi,\eta) = d\alpha_1(\xi,\eta)\\
&\Rightarrow d\alpha_2(F_*\xi, F_*\eta) = d\alpha_1(\xi,\eta)\\
&\Rightarrow\ \langle\, (F_*\xi)_h, (F_*\eta)_v\rangle - \langle\, (F_*\xi)_v, (F_*\eta)_h\rangle = \langle\, \xi_h, \eta_v\rangle - \langle\, \xi_v, \eta_h\rangle\\
&\Rightarrow\ \langle\, F_*(J_M^s)(0), F_*(J_M^z)'(0)\rangle - \langle\, F_*(J_M^s)'(0), F_*(J_M^z)(0)\rangle\\
&=\ \langle\, J_N^s(0), c_1 {J_N^s}'(0) + {J_N^z}'(0)\rangle - \langle\, {J_N^s}'(0), c_1 J_N^s(0) + c_2 J_N^z(0)\rangle\\
&= j_M^s(0).{j_M^z}'(0) - {j_M^s}'(0).j_M^z(0)
\end{aligned}
$$

In the last equality,

$$
\begin{aligned}
RHS &= j_M^s(0).{j_M^z}'(0) - {j_M^s}'(0).j_M^z(0)\\
&= j_M^s(0).\frac{j_M^s(0)}{j_M^s(0)^2} - 0\\
&= 1\\
LHS &= j_N^s(0)(c_1 {j_N^s}'(0) + c_2 {j_N^z}'(0)) - {j_N^s}'(0)(c_1 j_N^s(0) + c_2 j_N^z(0))\\
&= c_2(j_N^s(0){j_N^z}'(0) - {j_N^s}'(0) j_N^z(0))\\
&= c_2(j_N^s(0).\frac{j_N^s(0)}{j_N^s(0)^2} - 0)\\
&= c_2
\end{aligned}
$$

Thus $c_2 = 1$. □

Definition(g): *Let γ be geodesic in M with $\gamma'(x) = v$. We have defined J_M^v by $j_M^v(x) = 0$ and ${j_M^v}'(x) = 1$. Thus J_M^v vanishes once and since there are no conjugate points in M, J_M^v cannot vanish again. If $F_\star$ takes ψ_M^z to ψ_N^z, then $F_\star(J_M^v)$ will also vanish exactly once say at $F(\gamma)(t_0)$. We will set $g(v) = t_0 - x$.*

The following lemma validates our definition of the function $g : SM \mapsto \mathbb{R}$ and proves that g is bounded.

Lemma 2.1.2. *We have $F_\star(\psi_M^n) = \psi_N^n$ and $F_\star(\psi_M^z) = \psi_N^z$. Further the map g is continuous and hence bounded.(say $|g(v)| \leq g_0$)*

Proof. Let $J(t) = j(t)X(t) \in \psi_M^z$ along γ be vanishing at $\gamma(x)$. Then by the uniqueness theorem of Jacobi fields

$$J = j'(x) J_M^v$$

where $v = \gamma^{'}(x)$.

Hence to show $F_{\star}(\psi^z_M) \subseteq \psi^z_N$ along any geodesic, it is enough to show $F_{\star}(J^v_M)$ vanishes for all $v \in SM$. Using equations (2.1),(2.2),(2.3) and lemma[3.5] we see that

$$F_{\star}(J^v_M)(t) = j^s_M(x)\{c_1 j^s_N(t) + j^s_N(t)\int_0^t \frac{dy}{{j^s_N}^2}\} + j^s_M(x).\int_x^0 \frac{dy}{j^s_M(y)^2}.j^s_N(t)$$

. Hence

$$F_{\star}(J^v_M)(t) = j^s_M(x) j^s_N(t)\{c_1 + \int_0^t \frac{dy}{j^s_N(y)^2} - \int_0^x \frac{dy}{j^s_M(y)^2}\} \tag{2.4}$$

Thus $F_{\star}(J^v_M)$ will vanish somewhere if and only if there is a t_x such that

$$c_1 + \int_0^{t_x} \frac{dy}{j^s_N(y)^2} = \int_0^x \frac{dy}{j^s_M(y)^2} \tag{2.5}$$

Since J^s_M and J^s_N are in ψ^{ws}, they are bounded at ∞, and hence both sides of the above equation are monotonically increasing. So if such a t_{x_0} exists for some x_0, then it must exists for all $x \geq x_0$. Since we can also pick x so that the right hand side is $\geq c_1$, we see that t_x exists for all large x, say $x \geq x_0$. In particular we can find x_0 such that $t_{x_0} = 0$. Now we could have gone through the whole process above (starting just before equation (2.1) starting with j^u_M instead of j^s_M to derive the equation, corresponding to (2.4) and (2.5) only with j^s_M and j^s_N replaced with J^u_M and J^u_N; where c_1 may be different, because the only property of J^s_M that we used was that it never vanished). In this case since j^u_M and j^u_N are bounded at $-\infty$, such t_x exists for all $x \leq x_0$.(t_{x_0} must exist in this case also because we have already shown that $F_{\star}(J^v_M)$ where $v = \gamma^{'}(x_0)$ vanishes. Hence $F_{\star}(J^u_M)$ vanishes for all v on γ and since γ was arbitrary $F_{\star}(\psi^z_M) \subseteq \psi^z_N$ along any geodesic.

Now we will prove that g is continuous.

We have the Jacobi equation,

$$D_t^2 J + R(J, \dot{\gamma})\dot{\gamma} = 0$$

For $J(t) = j(t)X(t)$

$$j^{''}(t)X(t) + R(j(t)X(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0$$

Since $J \perp \dot{\gamma}$, they span the tangent space.

Hence sectional curvature is given by

$$
\begin{aligned}
K &= \frac{R_m(J,\dot\gamma,\dot\gamma,J)}{|\dot\gamma|^2|J|^2 - \langle\,\dot\gamma, J\rangle} \\
&= \frac{R_m(J,\dot\gamma,\dot\gamma,J)}{|J|^2} \\
&= R_m(\frac{J}{|J|},\dot\gamma,\dot\gamma.\frac{J}{|J|}) \\
&= R_m(X,\dot\gamma,\dot\gamma,X) \\
&= \langle\, R(X,\dot\gamma)\dot\gamma, X\rangle
\end{aligned}
$$

Now by the symmetries of R_m, $R_m(X,\dot\gamma,\dot\gamma,\dot\gamma)=0$. That is $\langle\, R(X,\dot\gamma)\dot\gamma,\dot\gamma\rangle = 0$ which means $R(X,\dot\gamma)\dot\gamma$ is along X. . Thus $R(X,\dot\gamma)\dot\gamma = \langle\, R(X,\dot\gamma)X\rangle X = KX$.

Let $K_v(t)$ represent the curvature of the surface M_0 at $F(\gamma_v)(t)$. Then by the above,the Jacobi equation can be transformed as follows.

$$D_t^2 J + R(J,\dot\gamma)\gamma = 0$$

ie.

$$j^{''}(t)X(t) + j(t)R(X,\dot\gamma)\dot\gamma = 0$$

ie.

$$j^{''}(t)X(t) + j(t)K_v(t)X(t) = 0$$

ie.

$$j^{''}(t) + j(t)K_v(t) = 0$$

As v varies continuously, K_v will vary continuously. Also, since $J^v(0)$ and $(J^v)^{'}(0)$ are continuous, $F_\star(J^v)(0)$ and $F_\star(J^v)^{'}(0)$ varies continuously with respect to v. Since the coefficient as well as the initial conditions are varying continuously, by the theory of differential equations, the solution $F_\star(J^v)(t)$ is continuous in both v and t. On a surface without conjugate points Jacobi field $F_\star(J^v)$ vanish exactly once. Hence $F_\star(J^v)(t)$ as a real valued function of t crosses t axis transversely and hence the zero varies continuously with v. Thus $g(v)$ is continuous.

Now we will show that $\psi_N^z \subseteq F_*(\psi_M^z)$. Let $J \in \psi_N^z$ be vanishing at t_0 on a given geodesic. We can take this geodesic to be $F(\gamma)$, where γ is a geodesic on M because F and F^{-1} takes geodesics to geodesics. Since g is bounded and continuous, as t varies from $-\infty$ to ∞, so does $g(\gamma^{'}(t)) + t$ taking every value; in particular taking t_0.

Let $g(\gamma'(x)) + x = t_0$. This means $F_\star(J_M^v)$, where $v = \gamma'(x)$, vanishes at $F(\gamma)(t_0)$. Since the space of Jacobi fields along a geodesic vanishing at a given point is one dimensional,

$$J = aF_\star(J_M^v) = F_\star(aJ_M^v)$$

for some $a \in \mathbb{R}$. This completes the proof. □

Lemma 2.1.3. *There is a number $R > 0$ such that if $\tilde{\gamma}$ and $\tilde{\sigma}$ are geodesics in $\tilde{M}$ such that $\tilde{\gamma}(0) = \tilde{\sigma}(0)$ any $\tilde{\gamma} \neq \tilde{\sigma}$, then $\tilde{F}(\tilde{\sigma})(R) \notin \tilde{F}(\tilde{\gamma})(-g_0, \infty)$*

Proof. Recall that $\tilde{F} : S\tilde{M} \mapsto S\tilde{N}$ is a geodesic conjugacy and hence we can define $\tilde{F}(\tilde{\sigma})$ and $\tilde{F}(\tilde{\gamma})$.

$p \in N$ and $v \in S_pN$. For any $\tilde{v} \in \tilde{N}$ which projects to v and any $w \in S_pN$ we let $\tilde{\gamma_v}$ and $\tilde{\gamma_w}$ be the geodesics in $\tilde{N}$ starting at $\tilde{p}$ with initial tangent vectors that project to v and w respectively. We can find a Jacobi field $J \in \psi_{\tilde{N}}^z$ which arises from a geodesic variation taking $\tilde{\gamma_v}$ to $\tilde{\gamma_w}$. Then J must vanish at 0. Since $\tilde{F}_\star(\psi_{\tilde{M}}^z) = \psi_{\tilde{N}}^z$, $\tilde{F_\star}^{-1}(J)$ must vanish somewhere. That is to say $\tilde{F}^{-1}(\tilde{\gamma_v})$ and $\tilde{F}^{-1}(\tilde{\gamma_w})$ must intercept at some $\tilde{F^{-1}}(\tilde{\gamma_w})(t)$. We know that $t < g_0$. By the continuity of g, t will be less that $g_0 + 1$ if $\tilde{\gamma}_v'(0)$ and $\tilde{\gamma}_w'(0)$ are sufficiently close. Since the projection $dP_N : S\tilde{N} \mapsto SN$ is a local isometry, this will happen if v and w are close say within an angle $\theta(v)$. $\theta(v)$ must be a continuous function of v if we take $\theta(v)$ to be the supremum of such angles. We can find a single angle θ_v for all $v \in SN$ by taking the minimum. Choose θ such that $0 < \theta < \theta_v$. Since J can vanish only once, $\tilde{F^{-1}}(\tilde{\gamma_v})$ and $\tilde{F^{-1}}(\tilde{\gamma_w})$ do not intercept for $t > g_0 + 1$.

Now let $\tilde{\gamma}$ and $\tilde{\sigma}$ be as in the statement of the theorem. Let R be greater than $max\{g_0 + 1, \frac{\pi a}{sin\theta}, g_0 + \pi a\}$ where a as in the proof of lemma 0.3.1. Assume $\tilde{F}(\tilde{\gamma})(t_0) = \tilde{F}(\tilde{\sigma})(R)$. By the above, since the difference between the parameters of $\tilde{\sigma}$ and $\tilde{F}(\tilde{\sigma})$ where they vanish is $R\rangle g_0 + 1$, $\tilde{F}(\tilde{\sigma})'(R)$ make an angle greater than θ with $\tilde{F}(\tilde{\gamma})'(t_0)$. If $t_0 \geq 0$, then $d(\tilde{F}(\tilde{\sigma})(0), \tilde{F}(\tilde{\gamma})(0)) \geq Rsin\theta \geq \pi a sin\theta$, since $\tilde{N}$ has non positive sectional curvature. If $t_0\langle 0$ for $t_0 \geq -g_0$, the triangle inequality again gives $d(\tilde{F}(\tilde{\sigma})(0), \tilde{F}(\tilde{\gamma})(0)) \geq R - g_0\rangle\pi a$. On the other hand there is a path in $S\tilde{M}$ from $\tilde{\gamma}'(0)$ to $\tilde{\sigma}'(0)$ of length $\leq \pi$, namely arc of the unit speed circle. By the definition of a, its image under F in $S\tilde{N}$ is a curve of length $\leq \pi a$ which when projected to $\tilde{N}$ become a curve of length $\leq \pi a$ from $\tilde{F}(\tilde{\gamma})(0)$ to $\tilde{F}(\tilde{\sigma})(0)$. This contradiction yields the lemma.

□

Since $\tilde{F}$ is the lift of F, we can assert the same lemma for N. ie. there is a number $R > 0$ such that if γ and σ are geodesic in N such that $\gamma(0) = \sigma(0)$ and $\gamma \neq \sigma$, then $F(\sigma)(R) \notin F(\gamma)[-g_0, \infty]$.

Proposition 2.1.4. *In the situation of the main theorem we have for every $p \in M$ (we parametrize geodesics γ_v so that $\gamma_v'(0) = v$ for all $v \in S_pM$)*

$$2\pi \geq \int_{S_p} F_\star(J^v)(g(v))dv$$

Proof. The inequality is an application of the lemma in Appendix. We parametrize S_p as usual by θ in $(0, 2\pi)$ then $a(\theta)$ will be $g(\theta)$ and R comes from the above lemma. We define $H(\theta, s) = F(\gamma_\theta)(s)$ into N. Note that the $\gamma_\theta(s)$ in the Appendix corresponds to $F(\gamma_\theta(s))$. To avoid any confusion, we will show that H has the required properties.

We have $\partial_1 = \{\theta, g(\theta)\}$. If $H(\theta_1, g(\theta_1)) = H(\theta_2, g(\theta_2))$ then $F(\gamma_{\theta_1})(g(\theta_1)) = F(\gamma_{\theta_2})(g(\theta_2))$. Since $g(\theta_2) \in [-g_0, \infty]$ this is contradictory to the above lemma. So H maps ∂_1 in a 1-1 fashion to an imbedded circle ∂ in N which will bound a disk. $\gamma_\theta(s)$ is a geodesic variation and if we take γ_θ as the central curve J^θ is the variation field along γ_θ for each θ.

Recall that we can obtain $F_\star(J^\theta)$ also by taking the variation field of $F(\gamma_\theta)$. $F_\star(J^\theta)(R)$ is the tangent vector to the transverse curve $F(\gamma_\theta)(R)$ parametrized by θ. But this curve is ∂. Thus $F_*(J^\theta)(R)$ is tangent to ∂. Since J^θ is normal to $F(\gamma_\theta)$. Thus $F(\gamma_\theta)$ is the geodesic normal to ∂. As s goes to ∞, $F(\gamma_\theta)(s)$ goes to ∞ and hence eventually lies outside D.

By virtue of the above lemma $F(\gamma_\theta)(R, \infty) \cap \partial = \phi$. Hence $F(\gamma_\theta)(R, \infty)$ lies outside D. Again by the above lemma $F(\gamma_\theta)(-g_0, R) \cap \partial = \phi$ and hence we have $F(\gamma_\theta)(-g_0, R)$ lies in D. In particular $H(\partial_0)$ lies in the interior of D and property(4) is satisfied.

For any $p \in D$, let τ be a minimizing geodesic from p to ∂. Then τ is perpendicular to γ so than $p = F(\gamma_\theta)(t)$ for some θ and t. We need to show that $g(\theta) \leq t \leq R$. By the previous paragraph $t \leq R$. Since $F_\star(J^\theta)(g(\theta)) = 0$ and $F_\star(J^\theta)$ is the variation field of the variations of normal geodesics the usual variation argument will say that, since τ is the shortest path from p to ∂, t cannot be $< g(\theta)$. Hence D is the image of H and property(3) is satisfied.

Now we can apply the lemma. Again recall that γ_θ in the lemma is our $F(\gamma_\theta)$, J is $F_\star(J^\theta)$.

$$\nabla_{\gamma_\theta(a(\theta))} J(\theta, s) = F_\star(J^\theta)'(g(\theta))$$

and $\gamma_\theta^\perp$ is our X, the unit normal field. Since $F_\star(J^\theta)$ is normal, $F_\star(J^\theta)'$ will be perpendicular to γ_θ so that the integral is just $F_\star(J^\theta)'(g(\theta))$(as a real valued function). Thus

$$2\pi \geq \int_{S_p} F_\star(J^v)(g(v))dv$$

Here dv is just the Lebesgue measure on S^1.

2.2 Proof of the main theorem

Finally, we have reached the proof of the main theorem.

Proof. Integrating the inequality of the previous lemma over M, we get

$$2\pi.Vol(M) \geq \int_{SM} F_\star(J^v)'(g(v))dv$$

From the invariance of the canonical measure under the geodesic flow we get for each $L > 0$:

$$2\pi L.Vol(M) \geq \int_{SM}\int_0^L F_\star(J^{g^t_M(v)})'(g(g^t_M(v))dtdv$$

For a fixed v, let $\gamma_v(t)$ be the geodesic with $\gamma_v'(0) = v$ so that $g^x_M(v) = \gamma_v'(x)$. Equation (2.4) says

$$F_\star(J^v_M)(t) = j^s_M(x).j^s_N(t)\{c_1 + \int_0^t \frac{dy}{j^s_N(y)^2} - \int_0^x \frac{dy}{j^s_M(y)^2}\}$$

Taking the covariant derivative along $F(\gamma)$,

$$F_\star(J_M^{g^x_{M(v)}})'(t) = j^s_M(x).{j^s_N}'(t)\{c_1 + \int_0^t \frac{dy}{j^s_N(y)^2} - \int_0^x \frac{dy}{j^s_M(y)^2}\} + \frac{j^s_M(x).j^s_N(t)}{{j^s_N}^2(t)}$$

By equation (2.5), at $t = t_x = g(v) + x$,

$$F_\star(J^v_M)'(g(v) + x) = \frac{j^s_M(x)}{j^s_N(g(v) + x)}$$

Plugging in $g^t(v)$ and noting that $F_\star(J^{g^t_M(v)})$ is a Jacobi field along $F(\gamma)$ with parameter shifted by t we get, for any $v \in SM$ and t,

$$F_\star(J^{g^t_M(v)})(g(g^t_M(v)) + x) = \frac{j^s_M(t)}{j^s_N(g(\gamma'(t) + t)}$$

Apply lemma in Appendix A with $f(t) = g(\gamma^{'}(t)) + t$, $j = j^s_M$ and $\bar{j} = j^s_N$. Equation (5 **this number is to renamed appropriately**) tells that these functions satisfy the condition of the lemma with $c_2 = 1$. We take $[a,b] = [0,L]$ so that

$$[\bar{a},\bar{b}] = [g(v), g(g^L_M(v)) + L].$$

Then

$$\int_0^L \frac{j^s_M(t)}{j^s_N(g(\gamma'(t)) + t)}dt \geq \frac{L^{\frac{3}{2}}}{[L + g(g^L(v)) - g(v)]^2}$$

Thus we find that

$$2\pi L.Vol(M) = L.Vol(SM) \geq \int_{SM} \frac{L^{\frac{3}{2}}}{(L + g(g^L(v)) - g(v))^{\frac{1}{2}}}dv$$

Rearranging the terms we see,

$$1 \geq \frac{1}{Vol(SM)}.\int_{SM} \frac{1}{(1 + \frac{g(g^L(v)) - g(v)}{L})^{\frac{1}{2}}}dv$$

Jensen's inequality says that on a measure space (Ω, μ) with $\mu(\Omega) = 1$, if g is a real valued μ-integrable function and ψ is a convex function on $\mathbb{R}$, then

$$\int_\Omega (g \circ \psi)d\mu \geq g(\int_\Omega \psi d\mu)$$

Take $g(x) = x^{\frac{-1}{2}}$ and $\psi(v) = 1 + \frac{g(g^L(v)) - g(v)}{L}$ and $d\mu = \frac{dv}{vol(SM)}$ so that we will get

$$1 \geq [\frac{1}{Vol(SM)}.\int_{SM} (1 + \frac{g(g^L(v)) - g(v)}{L})dv]^{\frac{-1}{2}}$$

with equality holds only if $g(g^L(v)) = g(v) + c(L)$, where $c(L)$ is a constant depending at most on L. On the other hand the invariance of dv under g^t says,

$$\int_{SM} g(g^L(v))dv = \int_{SM} g(v)dv$$

hence $\int_{SM} c(L)dv = 0$

Then since $c(L)$ is a constant it must be zero. Hence g is constant (say K) on the unit tangent vectors of a given geodesic. But there are dense geodesics on M, ie. there are geodesics γ such that $\{\gamma^{'}(t) : t \in \mathbb{R}\}$ is dense in SM. Hence we get that the function g, on which we were contemplating so far, is just a constant(say K).

By replacing F by $g_N^K \circ F$, we can assume that $g(v) = 0$ for all $v \in SM$.

Now let $x \in M$ and $c(\theta)$ be a curve in the fiber S_xM. For each θ, $c^{'}(\theta) \in TSM$ and it correspond to a Jacobi field $J_{c^{'}(\theta)}$ along $\gamma_{c(\theta)}$.

$$\begin{aligned} J_{c^{'}(\theta)}(0) &= d\pi(c^{'}(\theta)) \\ &= (\pi \circ c)^{'}(\theta) \\ &= 0 \end{aligned}$$

Since $g(v) = 0$ for all $v \in SM$, on particular $g(c^{'}(\theta)) = 0$, $F_*(J^{c(\theta)})$ will vanish at $F(\gamma^{'}_{c(\theta)})(0)$ for all θ

Thus $F_\star(J^{c(\theta)})(0) = d\pi(F_*(c^{'}(\theta)) = (\pi \circ F \circ c)^{'}(\theta) = 0$ for all θ. So $\pi \circ F(c(\theta))$ is independent of θ and hence for $x \in M$ we can define a function $f(x) = \pi \circ F(v)$ where v is any vector in the fiber S_xM.

To finish the proof we need only to note that $f : M \mapsto N$ is an isometry and $df = F$. Since F takes tangent vector field of γ to that of $F(\gamma) = \gamma_{F(v)}$, f takes γ to $F(\gamma)$. In particular if γ is a minimizing geodesic from p to q then $f(\gamma)$ is a minimizing geodesic of the same length from $f(p)$ to $f(q)$. This shows that f is an isometry.

Finally for $v \in SM$,

$$\begin{aligned} df(v) &= (f \circ \gamma_v)^{'}(0) \\ &= \frac{d}{dt}|_{(t=0)} \pi \circ F(\gamma^{'}(t)) \\ &= (\pi \circ \gamma_{F(v)})^{'}(0) \\ &= F(v) \end{aligned}$$

$\square$

Chapter 3

3.1 Appendix A

We will reproduce the lemma in [C].

Lemma 3.1.1. *Let j and $\bar{j}$ be positive real valued continuous functions defined on intervals of $\mathbb{R}$. For constants C_1 and C_2 with $C_2 > 0$ define $f : [a, b] \mapsto [\bar{a}, \bar{b}]$ by:*

$$C_2. \int_a^{f(t)} \frac{ds}{\bar{j}^2(s)} + C_1 = \int_a^t \frac{ds}{j^2(s)} \tag{3.1}$$

where j is assumed to be defined at least on $[a, b]$ and $\bar{j}$ on $[a, b] \cup [\bar{a}, \bar{b}]$. Then we have

$$\int_a^b \frac{C_2.j(t)}{\bar{j}(f(t))} dt \geq [\frac{(b-a)^3.C_2}{(\bar{b}-\bar{a})}]^{\frac{1}{2}}$$

with equality if and only if

$$f(t) = \frac{\bar{b}-\bar{a}}{b-a}(t-a) + \bar{a} \text{ and } \frac{j(t)}{\bar{j}(f(t))} = [\frac{(b-a)}{C_2(\bar{b}-\bar{a})}]^{\frac{1}{2}}$$

Proof. Differentiating (3) with respect to t we see that

$$f^{'}(t) = \frac{\bar{j}^2(f(t))}{C_2.j^2(t)}$$

Hence using the substitution $u = f(t)$ gives

$$\int_a^b \frac{C_2.j(t)}{\bar{j}(f(t))} dt = \int_{\bar{a}}^{\bar{b}} \frac{C_2^2.j^3(f^{-1}(u))}{\bar{j}^3(u)} du$$

(Note that $C_2 > 0$ implies $f^{'}(t) > 0$ and hence that $f^{-1}(u)$ is well defined.)A Holder inequality applied to the right hand side, RHS, of the above yields:

$$[RHS]^{\frac{2}{3}}.[\bar{b}-\bar{a}]^{\frac{1}{3}} \geq \int_{\bar{a}}^{\bar{b}} \frac{C_2^{\frac{4}{3}}.j^2(f^{-1}(u))}{\bar{j}^2(u)} du = C_2^{\frac{1}{3}}.(b-a). \tag{3.2}$$

The equality above comes from the substitution $t = f^{-1}(u)$. The inequality in 3.2 will be equality if and only if $j(f^{-1}(u))/(\bar{j}(u))$ is a constant, say F. Rearranging 3.2 yields the inequality in the lemma. If equality holds then we see that $C_2.F.(b-a) = [C_2(b-a)^3/(\bar{b}-\bar{a})]^{\frac{1}{2}}$ and hence $F = [(b-a)/\{C_2(\bar{b}-\bar{a})\}]^{\frac{1}{2}}$. Further our computation of $f^{'}(t)$ yields in the equality case $f^{'}(t) = 1/(C_2.F^2) = (\bar{b}-\bar{a})/(b-a)$. These results plus the fact that $f(a) = \bar{a}$ yield the equality case in the lemma. $\square$

For $\theta \in S^1$, let $a(\theta) < R$ be a bounded continuous function, where R is a constant,

$$Q = \{(\theta, s) | a(\theta) \leq s \leq R\} \subseteq S^1 \times \mathbb{R}$$

Let $H : Q \mapsto M$ be a map into a two dimensional Riemannian manifold with the following properties;

1 Each curve $\alpha_\theta(s) = H(\theta, s)$ is a unit speed geodesic in M.

2 On the interior of Q, H is a C^1-immersion.

3 The image $H(Q)$ is a manifold whose boundary is the 1-1 image of ∂_1.

4 The image of ∂_0 lies in the interior of $H(Q)$.

We will let $J(\theta, s)$ be the variation field $H_\star(\frac{\partial}{\partial\theta})$. Hence for fixes θ, $J(\theta, s)$ is a Jacobi field along α_θ. We also choose a unit normal field $\alpha_\theta^\perp$ along each geodesic α_θ which we assume has $\langle J(\theta, s), \alpha_\theta^\perp \rangle > 0$ for all $a(\theta) < s < R$. This can be done by initially choosing $\alpha_\theta^\perp$ such that $\langle J(\theta, s), \alpha_\theta^\perp \rangle > 0$. Since H is immersion, $J(\theta, s) = H_\star(\frac{\partial}{\partial\theta})$ can not neither vanish nor become tangential. So that $\langle J(\theta, s), \alpha_\theta^\perp \rangle$ never changes sign.

Lemma 3.1.2. *If in the above M has non positive curvature then we have:*

$$2\pi = \int_{S^1} \langle \nabla_{\dot{\alpha}_\theta(a(\theta))} J(\theta, s), \alpha_\theta^\perp \rangle d\theta$$

Proof. For m in M, let $K(m)$ represent the curvature of M at m. Since H may be more than 1 to 1, and since $K(m) \leq 1$, we have

$$\int_{H(\theta)} K(m)dm \geq \int_0^{2\pi} \int_{a(\theta)}^{R} K(H(\theta, s))\langle J(\theta, s), \alpha_\theta^\perp \rangle ds d\theta$$

Jacobi equation along α_θ says

$$D_s^2 J + R(J, \dot{\alpha_\theta})\dot{\alpha_\theta} = 0$$

$$\begin{aligned} -\langle D_s^2 J, \alpha_\theta^\perp \rangle &= \langle R(J, \dot{\alpha_\theta})\dot{\alpha_\theta}, \alpha_\theta^\perp \rangle \\ &= R_m(\langle J, \alpha_\theta^\perp \rangle \alpha_\theta^\perp + \ \langle J, \dot{\alpha_\theta} \rangle \dot{\alpha_\theta}, \dot{\alpha_\theta}, \dot{\alpha_\theta}, \alpha_\theta^\perp) \end{aligned}$$

By the symmetries of the curvature tensor R_m,this become $R_m(\langle J, \alpha_\theta^\perp \rangle \alpha_\theta^\perp, \dot{\alpha_\theta}, \dot{\alpha_\theta}, \alpha_\theta^\perp \rangle)$ which is $\langle J, \alpha_\theta^\perp \rangle R_m(\alpha_\theta^\perp, \dot{\alpha_\theta}, \dot{\alpha_\theta}, \alpha_\theta^\perp)$ But by definition

$$\begin{aligned} K &= \frac{R_m(\alpha_\theta^\perp, \dot{\alpha_\theta}, \dot{\alpha_\theta}, \alpha_\theta^\perp)}{\frac{1}{|\dot{\alpha_\theta}|}^2 |\alpha_\theta|^2 - \langle\, \alpha_\theta^\perp, \dot{\alpha_\theta} \rangle^2} \\ &= R_m(\alpha_\theta^\perp, \dot{\alpha_\theta}, \dot{\alpha_\theta}, \alpha_\theta^\perp) \end{aligned}$$

Hence the integrand on the RHS become $-\langle\, D_s^2 J, \alpha_\theta^\perp \rangle$ which is $-\frac{d}{ds}\langle\, D_s J, \alpha_\theta^\perp \rangle$, because $\alpha_\theta^\perp$ is parallel along α_θ. Hence RHS become

$$\int_0^{2\pi} \langle\, \nabla_{\dot{\alpha}_\theta(a(\theta))} J(\theta, s), \alpha_\theta^\perp \rangle d\theta - \int_0^{2\pi} \langle\, \nabla_{\dot{\alpha}_\theta(R)} J, \alpha_\theta^\perp \rangle d\theta$$

Gauss Bonnet theorem of a surface with boundary says that

$$\int_M K dA + \int_{\partial M} K_g ds = 2\pi\chi(M)$$

where K_g is the geodesic curvature of ∂M. Since the boundary component of $H(Q)$ is a single circle, the Euler characteristics is ≤ 1 (in our applications $H(Q)$ will in fact always be a disk) and hence LHS is less than or equal to $2\pi-$ boundary term, $B\partial$, of Gauss Bonnet. Hence the lemma follows when we see that

$$B\partial = \int_0^{2\pi} \langle\, \nabla_{\dot{\alpha_\theta}(R)} J(\theta, s), \alpha_\theta^\perp \rangle d\theta$$

□

3.2 Appendix B

In case N has non positive sectional curvature and genus 1(ie. flat torus, by Gauss Bonnet theorem) M must be isometric to N but F need not be of the form $g^K \circ dI$.

We have the following theorem in [C]

Theorem 3.2.1. *If the geodesic flow of a closed surface M is conjugate to a flat torus N, then M is isometric to N*

We consider an example to exhibit that F need not to be of the form $g^K \circ dI$.

Example 3.2.2 Let N be a flat torus say $N = \mathbb{R}^2/\Gamma$ for a lattice Γ. Let (x, y) be standard coordinates of $\mathbb{R}^2$ and θ be the angle from x-axis. Then

$$SN = \{(x, y, \theta) \in \mathbb{R}^2/\Gamma \times \mathbb{R}^1/2\pi\}$$

Note that

$$g^t(x, y, \theta) = (x + tcos\theta, y + tsin\theta, \theta)$$

Hence the diffeomorphism $F : SN \mapsto SN$ defined by $F(x, y, \theta) = (x+a(\theta), y+b(\theta), \theta)$ where $(a(0), b(0)) = (0, 0)$ and $(a(2\pi), (2\pi)) \in \Gamma$ induce a geodesic conjugacy. It is easy to see that $g^t \circ F = F \circ g^t$. One can show that if $(a(2\pi), b(2\pi)) \in \Gamma - (0, 0)$; then F is not homotopic to a fiber preserving map so cannot be of the form $g^K \circ dI$. Even if $(a(2\pi), b(2\pi)) \in \Gamma - (0, 0)$ as long as a or b is not identically zero, F is not fiber preserving and(except for special choice $a(\theta) = (1 - cos\theta), b(\theta) = -tsin(\theta))$ cannot be made so by following a fixed amount. Hence again F is not $g^K \circ dI$.

It should be pointed out for general surfaces there is no theorem like the main theorem. In particular zoll surfaces have geodesic flow that are conjugate to the geodesic flow on the round sphere,(see[W]).

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