Superconducting Qubit

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CERTIFICATE OF EXAMINATION

This is to certify that the dissertation titled **Superconducting Qubit** submitted by Mr. Ankit Kumar Pandey (Registration Number: MS13137) for the partial fulfillment of BS-MS dual degree program of **Indian Institute of Science Education and Research, Mohali** has been examined by the thesis committee duly appointed by the institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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DECLARATION

The work presented in this dissertation has been carried out by me under the guidance of Dr. Sandeep K. Goyal at the Indian Institute of Science Education and Research, Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussion. This thesis is a bonafide record of work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidates project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Sandeep K. Goyal
(Supervisor)

to my family

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Abstract

Superconducting qubits are solid state electrical circuits manufactured using conventional techniques. Their main building block is Josephson tunnel junction, the only non-dissipative, strongly non-linear electrical circuit element works at ultra low temperature. Here we discuss the Josephson junction and it's characteristics which result in three basic electrical circuits using Josephson junction in three different combination, each of which act as a qubit i.e. Josephson junction qubit, Flux qubit, Charge qubit. Also we discuss the quantization of energy levels of each of the them and how their first two energy levels are used as qubit.

1 Introduction

Quantum computing is necessarily equipped and utilizing the laws of quantum mechanics in information processing. A conventional computer based on strings of bits encode with either 0 or 1, but a quantum computer uses quantum bits also called qubits[1]. Qubit is a quantum system that build two separable quantum states encoded with 0 and 1. As qubits are quantum in nature they exhibit quantum properties i.e. superposition, entanglement[2]. Because of these properties quantum computer can performs a huge number of calculations simultaneously. Other way of realization is that when classical computer performs an operation by a string encoded with either 0 or 1, on the other hand quantum computer have the benefit of using 0, 1 and their super-positions.

Certain complicated problems which have been thought impossible lately for classical computers will be solved by a quantum computer in a quick and efficient way. For example factorization of very large numbers(say 500 digits) is not possible for classical computer in reasonable time. In 1994, Peter Shor showed that if a quantum computer was available, it could factorize large numbers easily in polynomial time[3]. Other than that there are many other quantum algorithms like DeutschJozsa algorithm, Simon's algorithm, Grover's algorithm etc. Factorization of large numbers is the fundamental of modern day cryptography, RSA encryption, method to solve hard factorization. But Quantum Key Distribution (QKD), a quantum encryption method, works on completely random keys at a distance provides much higher secure cryptography[4]. Also there are physical limitations of modern classical computer such as, conventional computers are based on microprocessors made up of transistors. The size of transistors is becoming smaller every year. Right now transistors are as small as 7 nanometers, size lower than that could be really tough. So quantum computer could be a solution for that too. There are many more things that a quantum computer can do like Simulate quantum systems, Simulate quantum algorithms, Sort, filter, search through, and process Big Data, Decrypt encrypted messages to prevent criminal activities.[5]

The most basic step for the beginning of any quantum computation is State preparation. To process a quantum computation correctly the qubits need to be in a superposition state. In most simple way we can say that to build a quantum computer we need to build qubits which work in the way we want them to. These qubits could be constructed using photons, transition dipoles of ions, atoms in vacuum, spin of nuclei, electrons[$6-9$] or may be something else such as Josephson junction[8]. But qubits are extremely difficult to manage, because any external disturbance can make them to fall out of their quantum state, known as decoherence. It is the major weakness of quantum computing, but by time new experiments and quantum error correction examines how to fend off decoherence and other errors.

Superconducting qubit is one of the ways to build qubits using Josephson junction. In the following sections we will learn about Josephson junction, characteristics of electrical circuits equipped with Josephson junction and Josephson effects.Then we will see there are three different arrangements of Josephson junction associated with three different types of qubits i.e. Current biased Josephson junction qubit, Flux qubit and Charge qubit. After that we will discuss quantization of energy levels. Then Lagrangian and Hamiltonian corresponding to each of the qubit. Finally we will learn about three basic electrical circuits with different arrangement of Josephson junction and how they can be used as a qubit.

2 Josephson Junction

In superconductor electronics it is the junction consist two superconductors under suitable conditions like low voltage, low frequencies and sufficiently low temperatures, it allow electron pairs to tunnel(also called Josephson tunnel junction) one by one from one superconducting island to other also the wave functions of two superconductor slightly overlap. For the formation of quantum bits Josephson tunnel junction are good choice as it acts as a pure non-linear inductance[10]. When connected in parallel with the capacitance generated by overlapping of two superconducting metals it also fulfills the needs described in previous sections i.e. low dissipation, low noise, long coherence time etc. Linear circuit elements(inductors, capacitors) can also be used to build a resonators with low dissipation but we can't use them as qubit due to degeneracy in their energy levels. That's why Josephson element is fundamental element for the construction of qubit because its non-linearity breaks the degeneracy of the energy levels. Building a tight two-level dynamic system with negligible dissipation and relatively very large non-linearity which can be use as qubit.

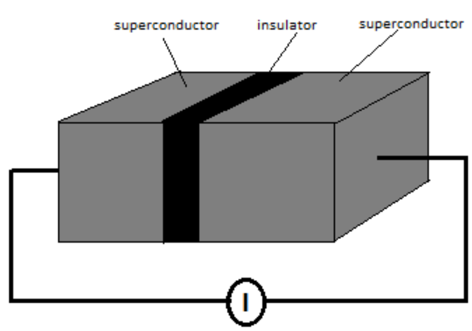

Figure 1: **Josephson Junction**: Superconducting materials can be two different or same connected via a insulating layer, also current flowing through circuit.

2.1 Josephson Effect

A macroscopic quantum phenomenon named after the British physicist Brian David Josephson, In 1962, who observed the effect that if two superconductors are separated by an insulating barrier and when insulating layer is sufficiently thin (of order of ten atoms thick) a pair of electron starts penetrating through the barrier resulting a flow of current in the circuit without any external power source. These electron pairs are called 'Cooper pair'[11] and the process known as 'Josephson tunneling'. If there is tunneling in the absence of any external electromagnetic field a DC current flows through the circuit between valuse $-I_c$ and I_c called critical current(maximum current a junction can hold) then this is known as *DC Josephson effect*[12]. If a constant power source is connected in circuit then a sinusoidal current with $-I_c$ as minimum and I_c as its maximum value then it is called *AC Josephson effect*.

2.2 Equations

Any electrical element can be defined by its characteristic equation. Here defining branch flux of element

$$\Phi_{(t)} = \int_{\infty}^{t} V_{(t_1)} dt_1$$

although general magnetic flux is only defined for loop. Where

$$V_{(t)} = \int E dx$$

along the current line. According to above definition current flowing through an inductor is

$$I_{(t)} = \frac{\Phi_{(t)}}{L}$$

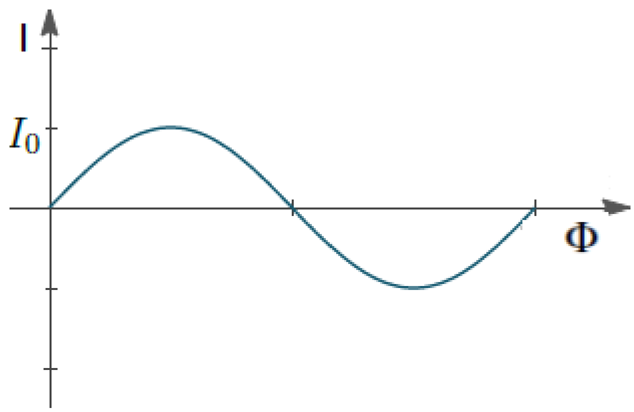

Figure 2: **Josephson tunnel** Junction current vs flux variation.

where $\Phi_{(t)}$ is branch flux and L is inductance of linear inductor. This branch can be generalized and described for every electrical element i.e. Josephson junction[10]. Josephson junction acts as a non-linear inductor and its behavior is like

$$I_J = I_0 \sin(\phi)$$

$$V = \frac{\Phi_0}{2\pi}\frac{d\phi}{dt}$$

where I_J is super-current flowing through Josephson element, I_0 is critical current that an element can hold, V is voltage difference between two ends of junction, $\Phi_0 = h/2e$ is flux quanta and $\phi = \phi_1 - \phi_2$ is phase difference between the wave-functions of two superconducting materials shown in Fig. 1. To understand the dynamical behavior of element differentiate first equation and substituting in second we get

$$\frac{dI_J}{dt} = I_0 \cos(\phi)\frac{2\pi}{\Phi_0}V$$

see dI_J/dt is proportional to V, which is the property of inductor, here defining inductance of Josephson element L_J also by $V = L_J dI_J/dt$ we get

$$L_J = \frac{\Phi_0}{2\pi I_0} \times \frac{1}{\cos(\phi)}.$$

$1/\cos(\phi)$ term indicates that it is non-linear inductance, as $\phi \to \pi/2$ inductance $\to \infty$ also when $\pi/2 < \phi < 3\pi/2$ inductance is negative. Inductance at zero bias is $L_{J0} = \frac{\Phi_0}{2\pi I_0}$. Now an inductor does have some energy stored in it, so to find inductive energy stored in Josephson element

$$U_J = \int\int I_J V dt$$

$$U_J = \int\int I_0 \sin{(\phi)} \frac{\Phi_0}{2\pi} \frac{d\phi}{dt} dt$$

$$U_J = \frac{I_0 \Phi_0}{2\pi} \int\int \sin{(\phi)} d\phi$$

$$\boxed{U_J = \frac{I_0 \Phi_0}{2\pi} (1 - \cos\phi)}$$

here $E_J = \frac{\Phi_0 I_c}{2\pi}$ sets the scale of energy an the term $(1 - \cos\phi)$ shows the dependence of energy on phase ϕ. Note for ground state U_J has minimum equal to zero.

Now to use Josephson junction in electric circuits to behave as qubit, they must show some characteristics which are necessary to form a qubit. There are some properties discussed below

2.3 Low Dissipation

First thing is that dissipation must be really low for an electrical circuit to behave quantum mechanically. More precisely all elements which are being used in circuit must have zero resistance at qubit transition frequency and at superconducting temperature. This is needed to carry electronic signals from one part of the circuit to another without any loss of energy i.e. not sufficient but necessary condition to keep quantum coherence. For this task superconductors(below their critical temperature) like aluminum, niobium are ideal[13].

2.4 Low noise

The degrees of freedom of the circuit must be cool down to temperatures where $\hbar\omega_{01}$, energy related with the transition between first two consecutive states $|0\rangle, |1\rangle$ is greater than kT, energy of thermal fluctuations. Reasons for last assertion will become clear in following subsections, to satisfy this frequency range is $5 - 20GHz$ and the operating temperature T should be $20mK$(Recall that $1K$ corresponds to about $20GHz$). The techniques [14] and requirements [15] for ultra-low noise filtering have been known for about 20 years. From the requirements $kT << \hbar\omega_{01}$ and $\hbar\omega_{01} << \Delta$, where Δ is the energy gap of the superconducting material.

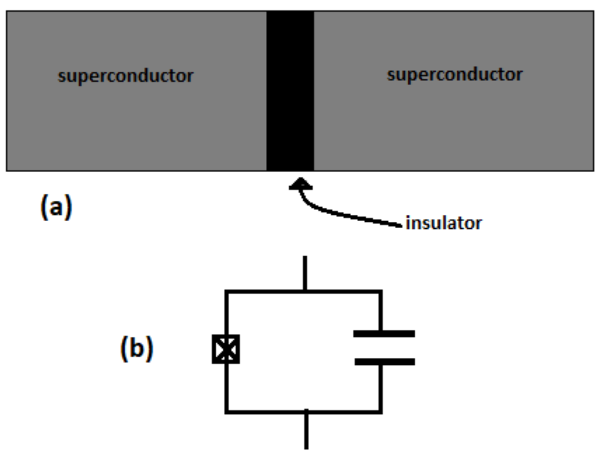

Figure 3: (a) Josephson junction element consists of a sandwich of two superconductors separated by a thin insulator, (b) Representation of Josephson junction and equivalent electric representation of junction.

2.5 Non-linear, Non-dissipative Elements: Tunnel Junctions

Quantum information can't be executed using just linear components, reason for that will be discussed in next subsection. For a quantum circuit, only presence of non-linear element is not sufficient but that element should also be non-dissipative in nature. Therefore even if PIN diode, CMOS transistors well operates at ultra-low temperatures, their use is not prescribed for non-dissipative operations. Till now only one electronic element is known which satisfy both the conditions i.e. non-dissipative and non-linear at very low temperature, the superconducting tunnel junction also called Josephson junction[10].

As shown in Fig.3 Josephson tunnel junction consist of two superconductors separated by a thin layer of insulator(of the order of $10^{-9}m$), which is thin enough to allow tunneling and its electrical representation. Generally insulating layer made up of oxidation of the superconducting metal.[16] Materialistic characteristics of amorphous aluminum oxide, alumina, forms fair tunnel insulating layer. Josephson effect is a complex natural effect associating tunneling and superconductivity but junction construction process is quite simple. Many means and perceptions of typical semiconductor electronics can be straightaway applied to quantum circuits. Still, there are many more important differences between traditional semiconductor electronics and quantum circuits at conceptual level. Now why the non-linearity is so important for quantum circuit also why not simply a harmonic oscillator could be used as a qubit, as its potential too has energy levels.

2.6 Need for Non-linear Elements

Non-linear element is essential in circuit for quantum information. In harmonic oscillator potential is parabolic in shape and energy difference between all energy levels are equal. But for the formation of a qubit energy difference between ground state $|0\rangle$ and first excited state$|1\rangle$ must be significantly lower than all other energy differences or in other words frequency of transition between these two states much be much higher than other higher levels. Therefore Josephson junction are so important for quantum integrated circuits as their inductive potential is strong non-parabolic in shape.

Next we will study in detail about the three basic circuits using Josephson junction and their classical and quantum properties and how we can use them as a qubit.

3 Superconducting Qubit

In the following sections we will see mainly three type of circuit fabricated with superconducting elements which are building blocks in qubit application i.e. Josephson's junction connected in three different combinations according to which their names are stated Single Josephson Junction Qubit(also known as phase qubit), Flux Qubit and Charge Qubit. After that we will discuss about classical superconducting circuit and in section following that we sill see the quantum analog of these circuits.

3.1 Classical Superconducting Circuits

Three basic circuits are: single current biased Josephson junction; single Josephson junction (JJ) included in a superconducting loop (rf SQUID), two Josephson junctions included in a superconducting loop (dc SQUID), and an ultra-small superconducting island connected to a massive superconducting electrode via tunnel Josephson junction (Single Cooper pair Box, SCB).

3.1.1 Circuit of Current-biased Josephson Junction Qubit

In this circuit a constant current source is connected to a Josephson junction as shown in Fig.4. This is considered as the simplest superconducting quantum circuit. In terms of electrical elements we can represent a Josephson junction as a combination of a capacitor C, a resistance R and a Josephson element J as shown in Fig.4. Josephson element is an electrical element related to Josephson junction it's properties and electrical behavior we will see further.

Generally total resistance (R_N) of the circuit is different from value of re-

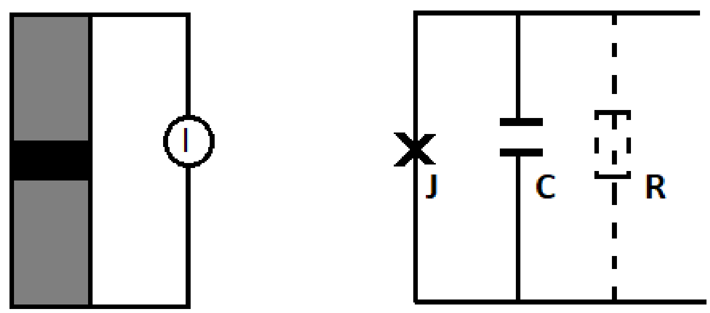

Figure 4: a. Current Biased Josephson Junction b. An equivalent electrical representation of Current Biased Josephson Junction.

sistance(R) shown above, that depends on material of Josephaon junction, temperature and external constant voltage source. Relation between current and voltage for trivial electrical elements capacitance and resistance are $I_C = C(dV/dt)$ and $I_R = V/R$ respectively. Now, for electrical relation of Josephson junction here we define new term superconducting phase difference($\phi(t)$). Superconducting phase difference is measured between both ends of the junction and which also depends on the potential drop between the two ends of the junction[17].

$$\phi(t) = \frac{2e}{\hbar} \int V dt + \phi$$

where ϕ is time independent part of phase difference. Expression of phase difference can also be written in form of magnetic flux and magnetic flux quanta($\Phi_0 = h/2e$). Through a superconducting loop quantity of magnetic flux is quantized and that quantity is called magnetic flux quantum.

$$\phi = \frac{2e}{\hbar}\Phi = 2\pi\frac{\Phi}{\Phi_0}$$

Behavior of current through a Josephson junction is a sinusoidal function of phase difference

$$I_J = I_c \sin\phi$$

where I_c is critical current through a Josephson junction which a junction can hold i.e. an amount of current that can flow through a junction without dissipation. From microscopic theory of superconductivity relation of critical current with known quantities is derived given as

$$I_c = \frac{\pi\Delta}{2eR_N}\tanh\frac{\Delta}{2T}$$

where Δ and T are superconducting order parameter and temperature.

Now, let total current flowing through external current source is I_e, called bias current and distributing to all branches of circuit in their respective way. By applying Kirchhoff's law over the circuit, equation of the above circuit is

$$C\frac{dV}{dt} + \frac{V}{R} + I_J = I_e$$
$$\frac{\hbar}{2e}C\ddot{\phi} + \frac{\hbar}{2eR}\dot{\phi} + I_c \sin\phi = I_e$$

where I_e is bias current. If we look above equation closely it looks like damped non-linear oscillator and it is describing dynamics of the phase[18]. Here second term of the equation is playing the role of dissipative term which regulates the life time of the qubit. But as here we are talking about the superconducting qubit working at superconducting temperature, we can neglect the resistive term. Then final equation will be

$$\frac{\hbar}{2e}C\ddot{\phi} + I_c \sin\phi = I_e$$

Now, we can write using known concepts from classical mechanics we can write Lagrangian and Hamiltonian for the system. As for writing Lagrangian we need Kinetic and Potential energies of dynamics. By comparing terms with harmonic oscillator equation $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$, we can deduce that current through Josephson element(I_J) behave as potential term and capacitive energy will be as kinetic energy.

Kinetic energy term associated with above equation

$$K(\dot{\phi}) = \left(\frac{\hbar}{2e}\right)^2 \frac{C\dot{\phi}^2}{2}$$

which is derived from electrostatic energy term of capacitor $CV^2/2$. Here defining charging energy of capacitor

$$E_C = \frac{(2e)^2}{2C}$$

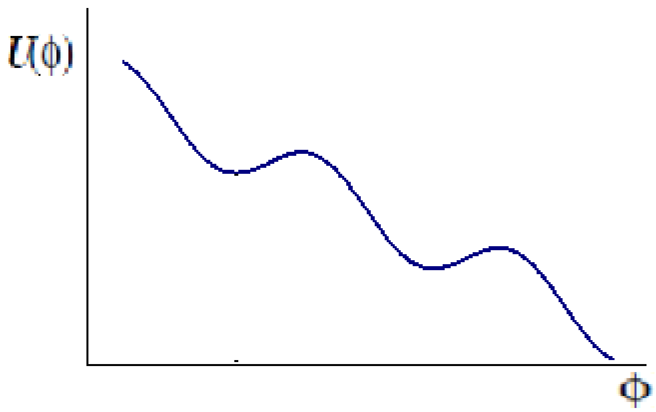

Figure 5: **Potential Energy of current biased Josephson junction** (Tilted washboard like potential)

putting it back in previous equation then we get kinetic energy term as

$$K(\dot{\phi}) = \frac{\hbar^2 \dot{\phi}^2}{4E_C}$$

From the remaining two terms deriving potential energy for the dynamics of phase which is in the form of Josephson energy and bias current

$$U(\phi) = E_J(1 - \cos\phi) - \frac{\hbar}{2e} I_e \phi$$

where E_J is Josephson energy, can be written as

$$E_J = \frac{\hbar}{2e} I_c$$

Now, plotting the potential energy v/s phase to see the dynamics of phase. As shown in Fig.5 potential energy graph is in the form of washboard but tilted. Comparing this potential term with the energy variation of a simple pendulum performing simple harmonic motion with small amplitude, from that we found an expression of frequency

$$\omega_J = \sqrt{\frac{2eI_c}{\hbar C}}$$

which called as plasma frequency of the junction. By analyzing potential energy graph we see that if there would not be any biased current the graph would have been exactly like washboard but due to application of biased current(I_e), graph get tilted. More we increase value of biased current more minima(s) of graph will become shallow and will get completely vanish when bias current becomes equal to critical current $I_e = I_C$. When $I_e = I_C$ plasma oscillation will become unstable and the whole system will start to dissipate energy. Now, writing Lagrangian of the phase dynamics of above circuit using expression of kinetic and potential energies

$$L(\phi, \dot{\phi}) = K(\dot{\phi}) - U(\phi)$$

$$L(\phi, \dot{\phi}) = \frac{\hbar^2 \dot{\phi}^2}{4E_C} - E_J(1 - \cos\phi) + \frac{\hbar}{2e} I_e \phi$$

We can crosscheck that Kirchhoff law represents dynamics of the phase using Lagrangian

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0$$

Now first we will see the similar kinetic energy, potential energy and Lagrangian for the other two circuits i.e. Flux qubit and Charge qubit then wee will their quantum behaviors.

3.1.2 Circuit of Flux Qubit

In this circuit we have a loop of superconducting material and both connected through a Josephson junction as shown Fig.6. We control the amount and the direction of the magnetic field lines through that superconducting loop[20]. Due to change in magnetic flux through loop by varying magnetic field there will be a current flow in loop by Lenz's law, that current will flow through Josephson junction. In the same Fig. 6 electrical representation of whole arrangement is shown. Same as previous circuit Josephson junction will act

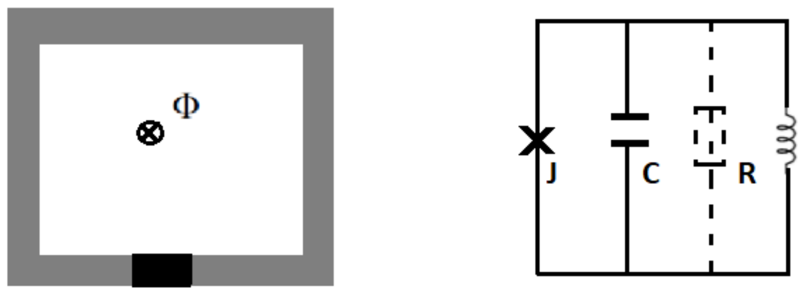

Figure 6: **Flux Qubit:** Mechanical circuit and it's equivalent electrical circuit

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as a combination of a Josephson element, a capacitor and a resistance(we will neglect the resistance term as we are talking about superconducting temperature) attached to it in parallel, in addition here superconducting loop will behave as an inductor(L)[21]. Now, before we write Kirchhoff equation and other dynamics equation first we introduce current and energy terms for the new inductor(L)

$$I_L = \frac{\hbar}{2eL}(\phi - \phi_e)$$

$$\phi_e = \frac{2e}{\hbar}\Phi_e$$

where Φ_e is magnetic flux through superconducting loop which we are controlling externally. Writing Kirchhoff law for circuit in Fig. 6,

$$\frac{\hbar}{2e}C\ddot{\phi} + \frac{\hbar}{2eR}\dot{\phi} + I_c \sin\phi + \frac{\hbar}{2eL}(\phi - \phi_e) = 0$$

When we see above equation closely, if there is no third term it would be a equation of dapmed linear LC oscillator, resonant frequency

$$\omega_{LC} = \frac{1}{\sqrt{LC}}$$

with damping $\gamma = 1/RC$ Now, for superconducting temperature neglecting resistance term

$$\frac{\hbar}{2e}C\ddot{\phi} + I_c \sin\phi + \frac{\hbar}{2eL}(\phi - \phi_e) = 0$$

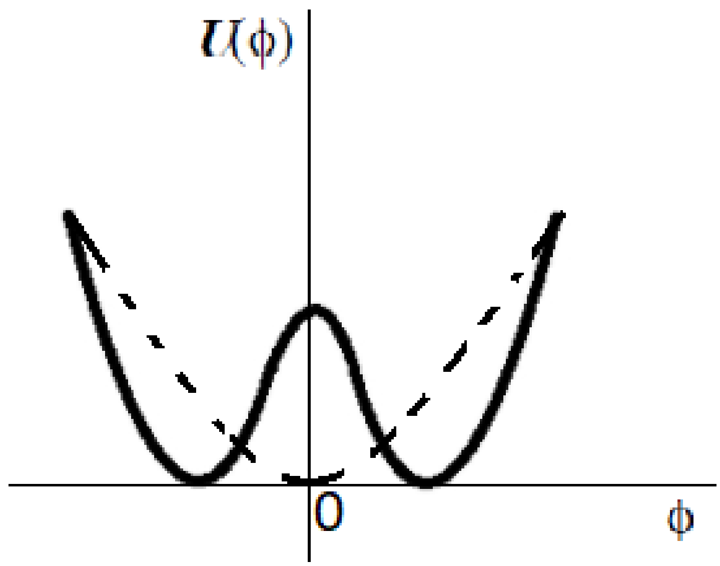

Figure 7: **Potential energy of Flux Qubit**

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By dropping resistance term there is no loss, so in this case equation shown below will be the kinetic and potential energy terms for flux qubit.

$$K(\dot{\phi}) = \frac{\hbar^2 \dot{\phi}^2}{4E_C}$$

$$U(\phi) = E_J(1 - \cos\phi) + E_L \frac{(\phi - \phi_e)^2}{2}$$

and corresponding Lagrangian will be

$$L(\phi, \dot{\phi}) = \frac{\hbar^2 \dot{\phi}^2}{4E_C} - E_J(1 - \cos\phi) - E_L \frac{(\phi - \phi_e)^2}{2}$$

where E_C, E_J, E_L are energies stored in capacitor, Josephson element and inductor respectively. Expressions of E_C, E_J are same as in the previous section and E_L is

$$E_L = \frac{{\Phi_0}^2}{4\pi^2 L}.$$

Fig. 7 shows potential energy variation with phase corresponding to flux qubit. Now the shape of graph depends on the amount of flux penetrating

through superconducting loop i.e. if the quantity of flux is an integral multiple of magnetic flux quantum(as defined in section 3.1.1) then there will only be single minima at $\phi = \phi_e$ and if it's multiple of half integer to magnetic flux quantum there will be two minima on either side of $\phi = \phi_e$. These two minima correspond to two current states flowing through the superconducting loop.
Now the third circuit, Charge qubit, its kinetic energy, potential energy and Lagrangian will be discussed next.

3.1.3 Circuit of Charge Qubit

In this circuit Josephson junction is a bit different from previous two circuits, here one end of Josephson junction is same superconducting material with same dimensions but at the other end a massive(with respect to usual thickness) superconducting electrode[20]. In other words we can say a Josephson junction is connected with superconducting island. Other end of Josephson junction is connected to a DC voltage source($V_g$called gate voltage) via a external capacitor called gate capacitor. Superconducting island is also connected with same voltage source as shown in Fig. 8. This is considered most important circuit of all, generally known as Single Cooper Pair Box[$22 - 23$].

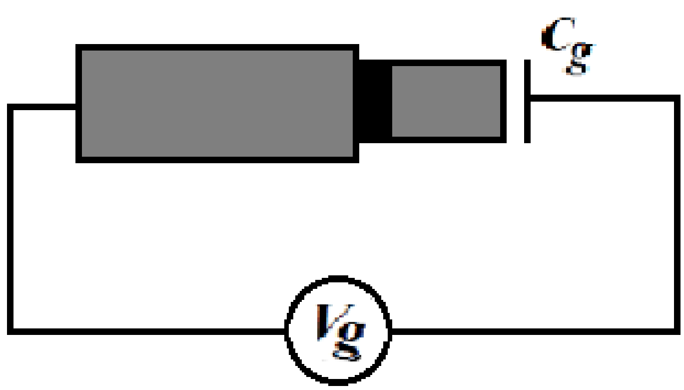

Figure 8: **Charge Qubit**

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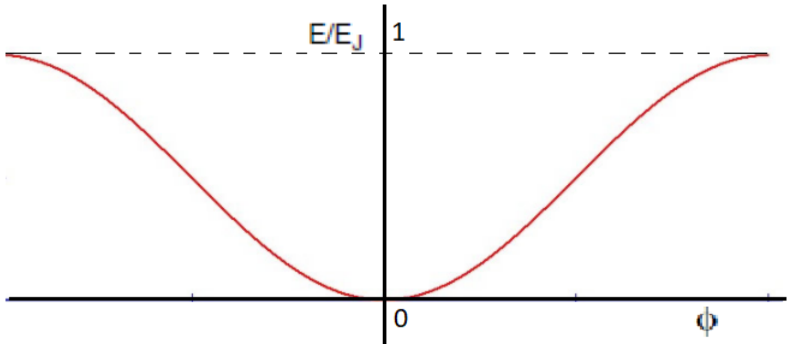

Figure 9: **Potential energy**

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Similarly a previous sections kinetic and potential energies terms are as

$$K(\dot{\phi}) = \frac{C_\Sigma}{2}\left(\frac{\hbar}{2e}\dot{\phi} - \frac{C_g}{C_\Sigma}V_g\right)^2$$

$$U(\phi) = E_J\,(1 - \cos\phi)$$

and resulting Lagrangian we can get by $L = K - U$.

$$L(\phi, \dot{\phi}) = \frac{C_\Sigma}{2}\left(\frac{\hbar}{2e}\dot{\phi} - \frac{C_g}{C_\Sigma}V_g\right)^2 - E_J(1 - \cos\phi)$$

where C_Σ total capacitance. Above graph(Fig.9) shows a term potential energy divided by Josephson energy as a function of phase(ϕ).

Now in section 3.2 we will see the quantum behaviors of three circuits discussed above i.e. circuit of Current biased Josephson junction, circuit of Flux qubit, circuit of Charge qubit and details about their energy graphs. In subsequent section we will see that how these circuits can be used as qubit.

3.2 Quantum Superconducting Circuits

All the equations written previously whatever may it was defining dynamics of phase contained electric current in one way or another. That gives rise to electromagnetic field term which is a quantum in nature. So further we will look for the quantization of Lagrangian and Hamiltonian. For the quantization, we will use traditional process that is proposing Hamiltonian and replace the momentum term with momentum operator of quantum mechanics. Now,

$$H(p, \phi) = p\dot{\phi} - L$$

Above is the well known relation between Hamiltonian and Lagrangian, where p is canonical momentum conjugated to coordinate ϕ. In comparison with conventional Hamiltonian q is replaced with ϕ.

$$p = \frac{\partial L}{\partial \dot{\phi}}$$

From Lagrangian of phase qubit the momentum operator comes out to be

$$p = \left(\frac{\hbar}{2e}\right)^2 C\dot{\phi}$$

Now what would the momentum represent in this scenario? For that observe that momentum is proportional to the charge on the capacitor $q = CV$, expanding

$$p = \frac{\hbar}{2e}q$$

and $q/2e$ is n number of cooper pairs present on capacitor.

$$p = \hbar n$$

Now writing all the operators in their differential form, momentum operator can be written as

$$\hat{p} = -\iota\hbar\frac{\partial}{\partial\phi}$$

then by relation $p = (\hbar/2e)q$ charge operator would be

$$\hat{q} = -2e\iota\frac{\partial}{\partial\phi}$$

and using $p = \hbar n$ pair number operator would be

$$\hat{n} = -\iota \frac{\partial}{\partial \phi}.$$

Interpretation of whole quantization process we followed above would be that '*the phase and charge dynamical variables can not be exactly determined by means of physical measurements.*' So commutation relation of phase and cooper pair number operator would be

$$[\phi, \hat{n}] = \iota.$$

3.2.1 Hamiltonian of three circuits

For circuit of Current-biased Josephson junction

Classical

$$H = E_C n^2 - E_J \cos\phi - \frac{\hbar}{2e} I_e \phi$$

Quantum

$$\hat{H} = E_C \hat{n}^2 - E_J \cos\phi - \frac{\hbar}{2e} I_e \phi$$

For circuit of Flux Qubit

Classical

$$H = H(n, \phi) = E_C n^2 - E_J \cos\phi + E_L \frac{(\phi - \phi_e)^2}{2}$$

Quantum

$$\hat{H} = E_C \hat{n}^2 - E_J \cos\phi + E_L \frac{(\phi - \phi_e)^2}{2}$$

For circuit of Charge Qubit

Classical

$$H = E_C(n - n_g)^2 - E_J \cos\phi$$

Quantum

$$\hat{H} = E_C(\hat{n} - n_g)^2 - E_J \cos\phi$$

where $E_C = (2e)^2/2C_\Sigma$ and $n_g = -C_g V_g/2e$ represents the number of cooper pairs on superconducting island.

Previously as I was saying charge qubit is an important case of qubit because if instead of one massive island Josephson junction is connected to gate capacitor through both large island there would not be any boundary condition and no discreteness in charge, but charge would be a continuous variable. It will be a whole new story in case of charge qubit, tunneling is not allowed electrons will be trapped on single island if other island is comparably really thin to show charging effect. Due to quantization of charge number of electrons trapped will be an integer. Now there be two cases of corresponding energy when the number of electrons trapped are even or odd in number, for even number there will be an extra energy E_C and for odd number $E_C/2$. E_C is energy associated with a cooper pair and $E_C/2$ is associated with single electron. Flow of charge will be unidirectional, from Josephson junction to island as transfer of only pair of electrons is possible. To satisfy condition that transfer of only cooper pairs is allowed there will be constraint on wave function

$$\psi(\phi) = \psi(\phi + 2\pi)$$

Energy spectrum of linear oscillator is already studied in great details in quantum mechanics

$$E_n = \hbar\omega_{LC}\left(n + \frac{1}{2}\right) \qquad n = 1, 2, 3...$$

Next we will see basic qubit structure of three circuits discussed and their energy spectrum. As we have seen above that number of energy levels corresponding to all three superconducting circuits is large, which fulfill the first requirement to be used as qubit that energy levels must be quantized. Second condition for energy spectrum to be use as qubit is that first two energy levels must be well separated from higher energy levels. So there could be transition only between these two levels or the probability of transition to any other level should be very very less. Now we will see energy levels and Hamiltonian of each of the three circuits.

3.3 Current Biased Josephson Junction Qubit

Josephson Junction Qubit is the simplest case of qubit, it works at zero voltage state $I < I_0$ and charging energy smaller than Josephson energy $E_C < E_J$ that means Josephson behavior dominant over charge behavior and charge number has large quantum fluctuation. Energy levels becomes noticeable in local minima of potential energy washboard with small capacitance and high resistance. For zero voltage state($\dot{\bar{\phi}} = 0$) particle oscillate in local potential well and for non-zero voltage state($\dot{\bar{\phi}} \neq 0$) or higher excited states particle will cross local potential well and then whole system will not be non-dissipative. Quantum Hamiltonian as stated previously

$$\hat{H} = E_C \hat{n}^2 - E_J \cos\phi + \frac{\hbar}{2e} I_e \phi$$

In the expression expanding the $\cos\phi$ term upto ϕ^4

$$\hat{H} = E_C \hat{n}^2 - E_J \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!}\right) + \frac{\hbar}{2e} I_e \phi$$

$$\hat{H} = E_C \hat{n}^2 - E_J + E_J \frac{\phi^2}{2!} + \frac{\hbar}{2e} I_e \phi - E_J \frac{\phi^4}{4!}$$

Now first four terms will act as primary Hamiltonian(H_0) and last term will be perturbation(H') in the Hamiltonian

$$H_0 = E_C \hat{n}^2 - E_J + E_J \frac{\phi^2}{2!} + \frac{\hbar}{2e} I_e \phi$$

$$H' = -E_J \frac{\phi^4}{4!}$$

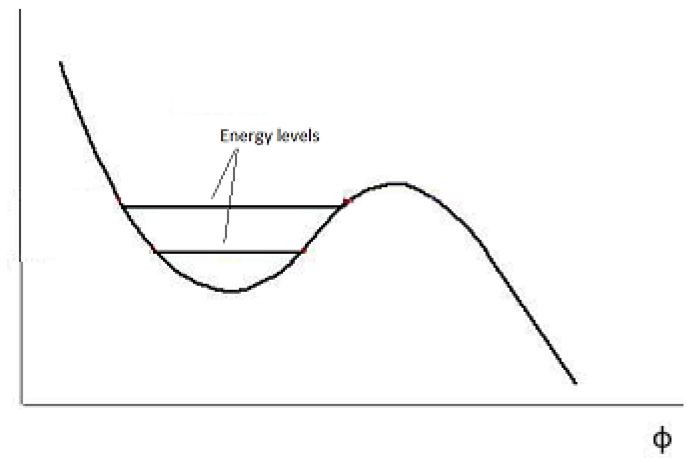

Figure 10: Energy level quantization in local potential well of potential washboard.

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then introducing creation and annihilation operators as

$$\hat{a} = \alpha\hat{\phi} + \iota\beta\hat{n}$$

$$\hat{a}^{\dagger} = \alpha\hat{\phi} - \iota\beta\hat{n}$$

putting values of $\hat{\phi}$ and $\hat{n}$ in form of creation and annihilation operators back in Hamiltonian and calculating we get difference between energies of first excited state and ground state as $12E_0$ whereas difference between second and first excited states as $24E_0$. Where E_0 is energy in terms of E_J and E_C. From her we can clearly see that first two levels are well separated from second excited energy level. So it can be used as a tight two level qubit.[19] There are many different ways to truncate the whole Hilbert space to two levels above shown is one of the ways. In qubit ground state is represented by $|0\rangle$ and first excited level by $|1\rangle$, also Hamiltonian can also be written in form

$$H = -\frac{1}{2}\epsilon\sigma_z$$

where ϵ is energy difference between the two levels. $\epsilon = E_1 - E_0$. The actual energy difference between two energy levels is managed by bias current.

3.4 Flux Qubit

A basic flux qubit can be made up from circuit described in section 3.1.2. Let's consider Hamiltonian from section 3.2.1.

$$H = H(n, \phi) = E_C n^2 - E_J \cos\phi + E_L \frac{(\phi - \phi_e)^2}{2}$$

this qubit works in function at $E_J >> E_C$. When external flux is multiple of half integer bias flux, mathematically $\phi_e = \pi$ minima of potential energy will degenerate and there will be two identical potential wells on either side of $\phi = \phi_e$ as shown in Fig. 11.

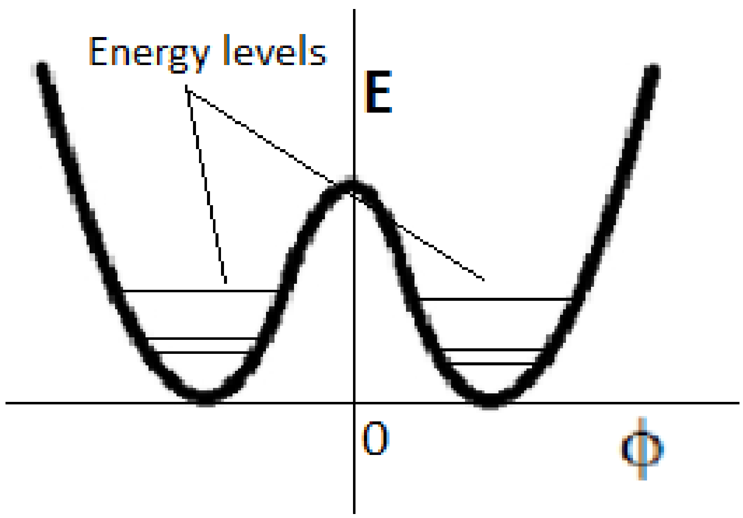

Figure 11: Energy levels in double well potential

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After truncating the whole Hilbert space to two levels Hamiltonian of flux can also be written as

$$\hat{H} = -\frac{1}{2}\left(\epsilon\sigma_z + \Delta\sigma_x\right)$$

where $\epsilon = \frac{E_J}{E_L} - 1 << 1$ and Δ is energy difference between ground and first excited level.

3.5 Charge Qubit

A basic Charge qubit can be made up from circuit described in section 6.1.3. In case of charge qubit over charge behavior dominant Josephson behavior i.e. $E_C >> E_J$. This expression also means that omitting Josephson behavior in circuit which give rise to isolation of the island. So number of cooper pairs will be some n, then eigenfunction

$$E_C\left(\hat{n}-n_g\right)^2|n\rangle = E_N|n\rangle$$

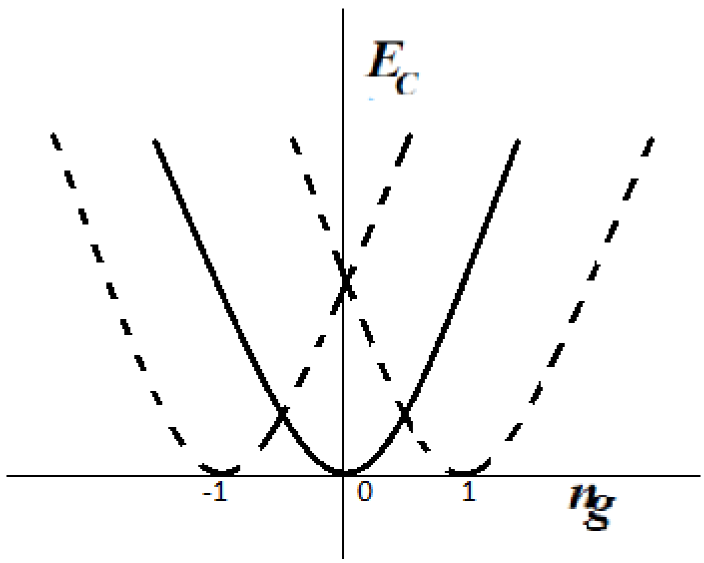

Figure 12: Energy spectrum of charge qubit

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The energy spectrum corresponding to different charge states($n = 1, 2, 3...$) will be $E_n = E_C(n-n_g)^2$, shown in Fig. 12. Now truncation of Hilbert space on two level system will result the Hamiltonian as

$$\hat{H} = -\frac{1}{2}\left(\epsilon\sigma_z - \Delta\sigma_x\right)$$

where $\epsilon = E_C(1-2n_g)$ and $\Delta = E_J$. Energy of two level of qubit is govern by

$$E_{1,2} = \mp\frac{1}{2}\sqrt{E_C^2(1-2n_g)^2 + E_J^2}$$

as in flux qubit inter-level distance is govern by external, here it is controlled gate voltage.

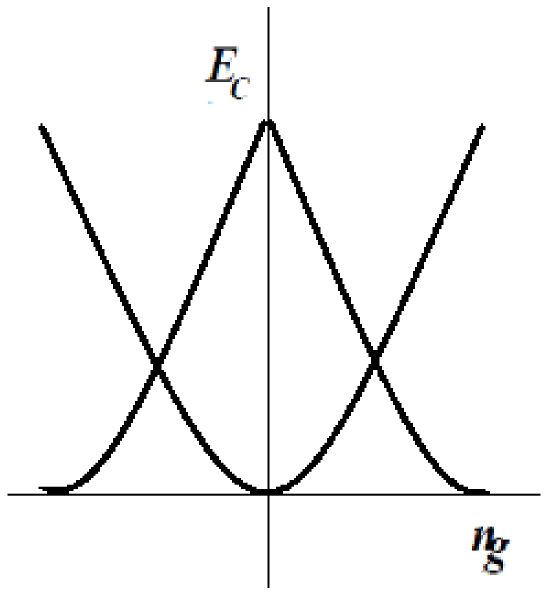

Figure 13: Charging energy as a series of parabolas centered at $n_g = N$ and $N + 1$

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To understand two level system of charge qubit, let's assume there are N number of cooper pairs on island then charging energy shows quadratic behavior with n_g. If n_g is close to N then $|N\rangle$ will be ground state and as we increase n_g charging energy will also increase quadratically till it is greator than the energy of $|N + 1\rangle$ state and it turns the new ground state. At that point of time if tunneling happens it would mean that charging energy is a periodic function of n_g. Fig. 13 shows the charging energy for each is a parabola centred on ng = N, and the parabolas cross at half-integer values of n_g, where the states $|N\rangle$ and $|N + 1\rangle$ are degenerate.

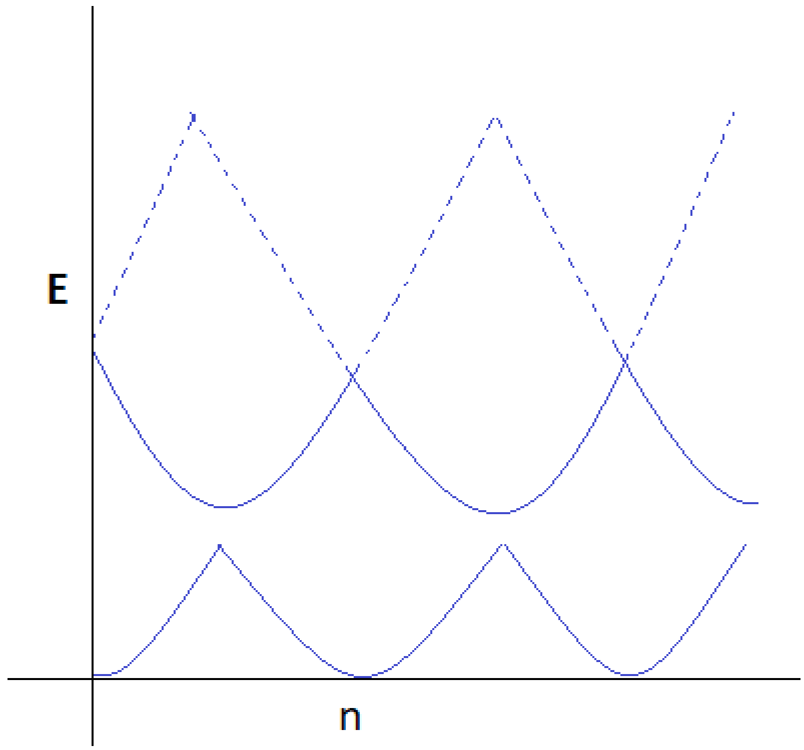

Figure 14: Degeneracy of the charging energies at half-integer values of n_g
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Here Josephson energy comes into account to break the degeneracy at points $n_g = N + 1/2$ as shown in Fig.14. At these points, the two lowest energy eigenstates will be the symmetric and anti-symmetric combinations of the N and N + 1 states and the other states will be separated in energy by E_C. Hence the new Hamiltonian in terms of number state can be written as

$$\hat{H} = E_C \sum |N\rangle (N - n_g)^2 \langle N| - \frac{E_J}{2} \sum (|N\rangle\langle N+1| + |N+1\rangle\langle N|)$$

As we are discussing only about two level system then Hamiltonian can be modified as

$$\hat{H} = E_C \left(\frac{1}{2} - n_g\right) (|1\rangle\langle 1| - |0\rangle\langle 0|) + \frac{E_J}{2} (|1\rangle\langle 0| + |0\rangle\langle 1|)$$

4 Conclusion

The elementary superconducting qubit in this thesis represents an important step towards acquiring a quantum computer. Yet, moving forward, there is still a long way to go for quantum information processing through a quantum computer. The rapid progress of superconducting qubit based quantum computing has been due in large part to the already well-developed protocols for NMR, photon, and trapped-ion quantum computing systems. Here, some reflections on some basic structure of superconducting qubit presented in this thesis.

After an introduction part first we discussed formation of Josephson junction and essential requirements for that in section 2, then we talked about Josephson effect and basic equations of current flow through Josephson junction. After that some properties of Josephson junction are discussed like such as low dissipation, low noise and non-linearity of the junction. Then why there is need of a non-linear element in the circuit, for this circuit to be use as a qubit.

In section 3 we talked about the superconducting qubits. First we discussed about circuit formation using Josephson junction along with other conventional electrical elements. There are three different arrangements of Josephson junction associated with three different qubits i.e. Current biased Josephson junction qubit, Flux qubit and Charge qubit. After that we discussed quantum behavior of circuit and quantization of energy levels. Then we discussed Lagrangian and Hamiltonian corresponding to each of the qubit. There we saw that difference between first two energy levels were significantly less than that of second and third energy level in other words transition frequency between first two energy levels is much greater than other energy levels. That makes a tight two levels system resulting in the formation of a qubit.

Reference

[1] Hey, Tony. *"Quantum computing: an introduction". Computing and Control Engineering Journal* 10.3 (1999):105-112.

[2] Hill, Scott, and William K. Wootters. *"Entanglement of a pair of quantum bits." Physical review letters* 78.26 (1997): 5022.

[3] Shor, Peter W. *"Algorithms for quantum computation: Discrete logarithms and factoring." Foundations of Computer Science, 1994 Proceedings. 35th Annual Symposium on. Ieee*, 1994.

[4] Rivest, Ronald L., Adi Shamir, and Leonard Adleman. *"A method for obtaining digital signatures and public-key cryptosystems." Communications of the ACM* 21.2 (1978): 120-126.

[5] Shor, Peter W., and John Preskill. *"Simple proof of security of the BB84 quantum key distribution protocol." Physical review letters*85.2 (2000): 441.

[6] Wallraff, Andreas, et al. *"Strong coupling of a single photon to a superconducting qubit using circuit quantum electrodynamics."* Nature 431.7005 (2004): 162.

[7] Meier, Florian, Jeremy Levy, and Daniel Loss. *"Quantum computing with spin cluster qubits." Physical review letters* 90.4 (2003): 047901.

[8] Makhlin, Yuriy, Gerd Schn, and Alexander Shnirman. *"Quantum-state engineering with Josephson-junction devices." Reviews of modern physics* 73.2 (2001): 357.

[9]. Dykman, M. I., P. M. Platzman, and P. Seddighrad. *"Qubits with electrons on liquid helium." Physical Review* B 67.15 (2003): 155402.

[10] B. D. Josephson, in *Superconductivity*, R. D. Parks (ed.) (Marcel Dekker, New York, 1969).

[11] L. N. Cooper, *Phys. Rev.* 104, 1189 (1956).

[12] Barone, Antonio, and Gianfranco Paterno. *Physics and applications of the Josephson effect. Vol.* 1. New York: Wiley, 1982.

[13] M. Tinkham,*Introduction to Superconductivity* (Krieger, Malabar, 1985).

[14] J. M. Martinis, M. H. Devoret, J. Clarke,*Phys. Rev. Lett.* 55, 1543-1546 (1985); M. H. Devoret, J. M. Martinis, J. Clarke, *Phys. Rev. Lett.* 55, 1908- 1911 (1985); J. M. Martinis, M. H. Devoret and J. Clarke,*Phys. Rev.* 35, 4682 (1987).

[15] J. M. Martinis and M. Nahum,*Phys Rev.* B 48, 18316-19 (1993).

[16] I. Giaever, *Phys. Rev. Lett.* 5, 147, 464 (1960).

[17] Devoret, Michel H., John M. Martinis, and John Clarke. *"Measurements of macroscopic quantum tunneling out of the zero-voltage state of a current-biased Josephson junction." Physical review letters* 55.18 (1985): 1908.

[18] M. Steffen, J. Martinis and I. L. Chuang, *Phys. Rev.* B 68, 224518 (2003).

[19] Martinis, John M., Michel H. Devoret, and John Clarke. *"Energy-level quantization in the zero-voltage state of a current-biased Josephson junction." Physical review letters* 55.15 (1985): 1543.

[20] Wendin, Gran, and V. S. Shumeiko.*"Superconducting quantum circuits, qubits and computing." arXiv preprint cond-mat*/0508729 (2005).

[21] A. Barone and G. Patern'o, *Physics and Applications of the Josephson Effect* (Wiley, New York, 1992).

[22] M. Buttiker, *Phys. Rev.* B 36, 3548 (1987).

[23] V. Bouchiat, D. Vion, P. Joyez, D. Esteve, M. H. Devoret, *Physica Scripta* T76 (1998) p.165-70.

[24] *Superconducting Qubits: A Short Review* M. H. Devoret, A. Wallraff, J. M. Martinis *arXiv:cond-mat/0411174 [cond-mat.mes-hall].*