Some Topics in Algebraic Topology

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MP14010

A dissertation submitted for the partial fulfilment of

MS degree in Mathematics

Indian Institute of Science Education and Research Mohali

November 2017

Certificate of Examination

This is to certify that the dissertation titled “Some Topics in Algebraic Topology” submitted by Mr. Manoj Upreti (Reg. No. MP14010) for the partial fulfilment of MS degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: December 04, 2017

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Pranab Sardar at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within it have been detailed in the bibliography.

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(Candidate)

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Pranab Sardar

(Supervisor)

Acknowledgement

I Would like to thank Dr. Pranab Sardar for his guidance and for giving me enough freedom so that I could explore the subject according to my interest.
Many thanks are due to my parents for their strong emotional support.
I am also grateful to my friends at IISER Mohali, especially Damanvir Singh Binner and Manpreet Singh.
Many thank to the Mathematics StackExchange community for its discussions which supplemented my reading material and helped in clearing my doubts.

Contents

Preface

Algebraic topology is a branch of mathematics where algebra is uses to study topological spaces. The basic goal is to find algebraic invariants that classify topological spaces up to homeomorphism, though usually most classify up to homotopy equivalence.
The material in the thesis based on my undestanding mainly from the text: 'Algebraic topology by Allen Hatcher'. I tried my best to provide direct reference to any other sources I have used in bibliography.
Here are the brief introdution to the chapters.
In chapter 1, I have given some basic definition which I use in later chapters. Brouwer fixed point theorem, Homotopy extension and at the end Cell Complexes.
Chapter 2, it starts with the definition of path and path homotopy and contains very standard material must be learn to develope basics in algebraic topology. It contains some classical Theorems like 'Fundamental theorem of algebra' where we using algebraic tools to proof a algebraic result, Brouwer fixed poin theorem, and very well known theorem in algebraic topology 'The van Kampen Theorem'.
Chapter 3, this chapter in more about how to use algebra for proving results in topology, using tools from this chapter we can proof Brouwer fixed point theorem for general case in a very elegant way.

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Notation

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$: The integers, rational, real, and complex

$\mathbb{Z}_n$: The integers mod n

$\mathbb{R}^n$: n-dimensional Euclidean space

$\mathbb{C}^n$: Complex n-space

$I{=}[1, 0]$: The unit interval

S^n: The unit sphere in $\mathbb{R}^{n+1}$, all vectors of length 1

D^n: The unit disk or ball in $\mathbb{R}^n$, all vectors of length $\leqslant 1$

∂D^n: S^{n-1}: The boundary of the n-disk

$\mathbb{1}$: The identity function from the set to itself

$\sqcup$: Disjoint union of sets or spaces

x,$\sqcap$: Product of sets, group, or spaces

$A \subset B$ or $B \supset A$: Set-theoretic containment, not necessarily proper

iff: if and only if

Chapter 1

Some Basic Notions

1.1 Homotopy and Homotopy Type

Definition. If f and f' are continuous maps of the space X into the space Y, we say that f is **homotopic** to f' if there is a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for each x. The map F is called a **homotopy** between f and f' and we write $f \simeq f'$.

Remark. If $f \simeq f'$ and f' is a constant map, we say that f is **nulhomotopic**.

Definition. A **retraction** of X onto A is a map $r : X \to X$ such that $r(X) = A$ and $r|A{=}\mathbb{1}$.

Definition. A **deformation retraction** of a space X onto a subspace A is a family of maps $f_t : X \to X$, $t \in I$ such that $f_0 = \mathbb{1}$, $f_1(X) = A$ and $f_t|A = \mathbb{1}$ for all t. The family f_t should be continuous in the sense that the associated map $X \times I \to X$, $(x, t) \to f_t(x)$ is continuous.

Definition. A map $f : X \to Y$ is called a **homotopy equivalence** if there is a map $g : Y \to X$ such that $fg \simeq \mathbb{1}$ and $gf \simeq \mathbb{1}$. The space X and Y are said to be **homopopy equivalent** or to have the same **homotopy type** denoted as $X \simeq Y$.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

Definition. A space having the homotopy type of a point is called **contractible**.

1.2 Homotopy extension property

Suppose one is given a map $f_0 : X \to Y$, and on a subspace $A \subset X$ one is also given a homotopy $f_t : A \to Y$ of $f_0|A$ that one would like to extend to a homotopy $f_t : X \to Y$ of the given f_0. If the pair (X,A) is such that this extension problem can always be solved, one say that (X, A) has the homotopy extension property.

Definition. (X, A) has the **homotopy extension property** if every map $X \times \{0\} \cup A \times I \to Y$ can be extended to a map $X \times I \to Y$.

Example. The pair (D^n, S^{n-1}) has the Homotopy extension property suppose given a homotopy $f_t : S^{n-1} \to Y$ and a map $g_0 : D^n \to Y$ such that $g_0|S^{n-1} = f_0$. We assemble these to form a map $h : D^2 \to Y$ from the double size disk $D_2 = x \in R^n : ||x|| \leqslant 2$ by setting

$$\begin{cases} g_0(x) & ||x|| < 1 \\ f_{||x||-1}\left(\frac{x}{||x||}\right) for & 1 \leqslant ||x|| \leqslant 2 \end{cases}.$$

The desire homotopy $g_t : D^n \to Y$ is then just $g_t(x) = h((1 + t)x)$.

Example. (A closed subspace that does not have the homotopy extension property) (I, A) where $A = \{0\} \cup \{\frac{1}{n}|n = 1, 2, \cdots\}$ does not have the homotopy extension property since $I \times \{0\} \cup A \times I$ is not a retract of $I \times I$.

Proposition. If the pair (X, A) satisfy the homotopy extension property and A is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence.

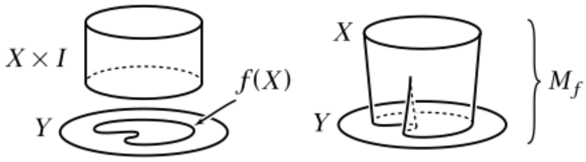
Another application of the homotopy extension property, is the following.

We finish this chapter with a technical result whose proof will involve several applications of the homotopy extension property.

Proposition. Suppose (X, A) and (Y, A) satisfy the homotopy extension property and $f : X \to Y$ is a homotopy equivalence with $f|A = \mathbb{1}$. Then F is a homotopy equivalence rel A.

Corollary. If (X, A) satisfy the homotopy extension property and the inclusion $A \hookrightarrow X$ is a homotopy equivalence, then A is a deformation retract of X.

Definition. For a map $f : X \to Y$, the **mapping cylinder** M_f is the quotient space of the disjoint union $(X \times I) \cup Y$ obtained by identifying each $(x, 1) \in X \times Y$ with $f(x) \in Y$.



Corollary. A map $f : X \to Y$ is a homotopy equivalence iff X is a deformation retract of the mapping cylinder M_f. Hence two spaces X and Y are homotopy equivalent iff there is a third space containing both X and Y as deformation retract.

1.3 Cell Complexes

CW complex is a space X constructed in the following way:
(1) Start with a discrete set X^0, where points are regarded as 0-cells.
(2) Inductively form the n-skeleton X^n from X^{n-1} by attaching n-cells ${e^n}_\alpha$ via maps

$$\varphi_\alpha : S^{n-1} \to X^{n-1}$$

.
This means that X^n is the quotient space of the disjoint union $X^{n-1} \sqcup_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identification $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus as a set $X^n = X^{n-1} \sqcup_\alpha e^n_\alpha$ where each e^n_α is an open n-disk.
(3) One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continoue indefinitely, setting $X = \cup_n X^n$. In the latter case X is given the weak topology: A set $A \in X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.
A space X constructed in this way is called a **cell complex** or **CW complex**. If $X = X^n$ for some n, then X is said to be finite dimensional, and the smallest such n is the dimension of X.

Definition. Each cell e^n_α in a cell complex X has a **characteristic map** $\Phi_\alpha : D^n_\alpha \to X$ which extends the attaching map φ_α and is a homeomorphism from the interior of D^n_α onto e^n_α, namely we can take Φ_α to be the composition $D^n_\alpha \hookrightarrow X^{n-1} \cup_\alpha D^n_\alpha \to X^n \hookrightarrow X$ where the middle map is the quotient map defining X^n.

Example. The space S^n has the structure of a cell complex with just two cells, e^0 and e^n, the n-cell being attached by the constant map $S^{n-1} \to e^0$. This is equivalent to regarding S^n as the quotient space $D^n/\partial D^n$.

In the canonical cell structure on S^n described in above example, a characteristic map for the n-cell is the quotient map $D^n \to S^n$ collapsing ∂D^n to a point.

Definition. A **subcomplex** of a cell complex X is a closed subspace $A \in X$ that is a union of cells of X. Since A is closed, the characteristic map of each cell in A has image contain in A, and in particular the image of the attaching map of each cell in A is contained in A so A is a cell complex in its own right.

Definition. A pair (X, A) consisting of a cell complex X and a subcomplex A will be called a **CW pair**.

Example. Each skeleton X^n of a cell complex X is a subcomplex.

Now we state a proposition about compact subspace of a CW complex.

Proposition. A compact subspace of a CW complex is contained in a finite subcomplex.

Proposition. CW complexes are normal, and in particular, Hausdorff.

Proposition. Each point in a CW complex has arbitrarily small contractible open neighborhoods, so CW complexes are locally contractible.

Remark. In particular CW complexes are locally path connected. So a CW complex is path connected iff it is connected.

Product of CW complexes

Let X be a topological space and A_α are collection of subspaces such that $X = \cup A_\alpha$ these subspaces generate a possibly finer topology on X by defining a set $A \in X$ to be open iff $A \cap A_\alpha$ is open in A_α $\forall \alpha$.

In case $\{A_\alpha\}$ is the collection of compact subsets of X, we write X_c for this new compactly generated topology. If X is compact, or even locally compact, then $X = X_c$, that is, X is compactly generated.

Theorem. For CW complexes X and Y which chracteristic maps Φ_α and Ψ_β the product maps $\Phi_\alpha \times \Psi_\beta$ are the characteristic maps for a CW complex structure on $(X \times Y)_c$. If either X or Y is compact or more generally locally compact, then $(X \times Y)_c = X \times Y$. Also, $(X \times Y)_c = X \times Y$ if both X and Y have countably many cells.

Proposition. If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$, hence (X, A) has the homotopy extension property.

Remark. Most application of the homotopy extension property can be seen easily using above proposition.

Proposition. If (X, A) is a CW pair and we have a attaching maps $f, g : A \to X_0$ that are homotopic, then $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$ rel X_0.

Proposition. Collapsing a contractible subcomplex is a homotopy equivalence.

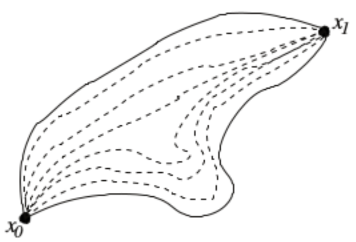
Chapter 2

Fundamental group

2.1 Path and Homotopy

Definition. A **path** in a space X is a continuous function $f : I \to X$, where $I = [0, 1]$.

Definition. If in a space X we have two paths say f_0 and f_1 we say that f_0 and f_1 are **path homotopic** if they have the same initial point x_0 and the same final point x_1



and if there exist a continuous map $F : I \times I \to X$ such that $F(s, 0) = f(s)$ and $F(s, 1) = f'(s)$ $F(0, t) = x_0$ and $F(1, t) = x_1$ for each $s \in I$ and $t \in I$. If f_1 is path homopotic to f_0, we write $f_0 \simeq_p f_1$.

Example. Any two paths say f_0 and f_1 in $\mathbb{R}^n$ with same endpoints x_0 and x_1 are path homotopic since we can give path homotopy between them as follows: $F : I \times I \to \mathbb{R}^n$ $F(s, t) = (1 - t)f_0(S) + tf_1(s)$.

Remark. More generally for any convex subspace $X \in \mathbb{R}^n$ result holds good.

Proposition. The relation of homotopy on paths with fixed endpoints in any space X is an equivalence relation.

Definition. For two paths say $f, g : I \to X$ such that $f(1) = g(0)$ then we can define a composition or **product path** $f \cdot g$ as follows:

$$f \cdot g(s) = \begin{cases} f(2s) & 0 \leqslant s \leqslant \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

Definition. Let f be a path in X such that $f(1) = f(0)$ where $f : I \to X$ is a path then f is called a **loop**.

Notation $\pi_1(X, x_0)$ =\{The set of all homotopy classes $[f]$ of loops $f : I \to X$ at the basepoint x_0\}.

Proposition. $\pi_1(X, x_0)$ is a group with respect to the product $[f][g] = [f \cdot g]$.

Proof. We will only check for the well definedness of the operation defined above i.e., we claim that if $f \simeq f'$ and $g \simeq g'$ then $f \cdot g \simeq f' \cdot g'$ where f, f', g, and g' are all loops based at x_0 in X.

Let $F : I \times I :\to X$ be a path homotopy between f and f' and $G : I \times I \to X$ is the homotopy between g and g'.

Define $H : I \times I \to X$ such that

$$H(s,t) = \begin{cases} F(2s, t) & s \in [1, 1/2] \\ G(2s-1, t) & s \in [1/2, 1] \end{cases}$$

Now since $F(1,t) = x_1 = G(0,t)\ \forall\, t \implies$ the map H is well-defined, and H is continuous by using the following result.

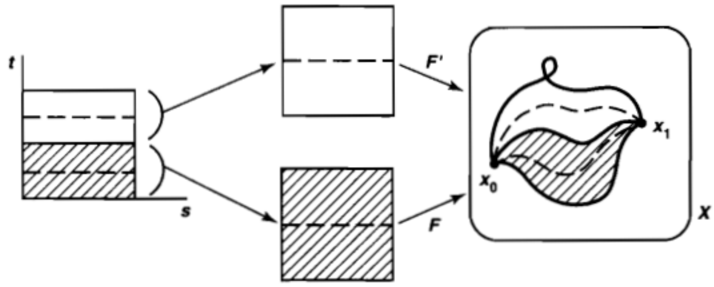
Proposition. (Pasting lemma) Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = g(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \in B$.

Now $H(s, 0) = f \cdot g$ and $H(s, 1) = f' \cdot g'$ and $H(0, t) = F(0, t) = x_0$ $H(1, t) = G(1, t) = x_0$

$\implies$ H is the required homotopy between $f \cdot g$ and $f' \cdot g'$.

□

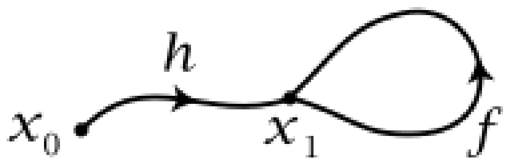
Remark. The identity element for the group $\pi_1(X, x_0)$ is the homopoty class of constant loop based at x_0.



Example. For a convex subspace X of $\mathbb{R}^n$ we have $\pi_1(X, x_0) = 0$ since any two loop f_0 and f_1 based at x_0 are homotopic with the homotopy $f_t(s) = (1-t)f_0(s) + tf_1(s)$.

Dependence of $\pi_1(X, x_0)$ on the choice of basepoint

As by definition $\pi_1(X, x_0)$ involves only the path-component of X containing x_0, so we find a relation between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ for two base points x_0 and x_1 only if x_0 and x_1 are in same path-component of X.
Let $h : I \to X$ be a path from x_0 to x_1 then we define inverse of path h by $\bar{h}(s) = h(1-s)$ which is a path from x_1 to x_0. Then corresponding to each loop f based at x_1 there is a loop $h.f.\bar{h}$ based at x_0.



Proposition. The map $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ defined by $\beta_h[f] = [h.f.\bar{h}]$ is an isomorphism.

Remark. It follows that for a path connected space X, the fundamental group $\pi_1(X, x_0)$ is, up to isomorphism, independent of the choice of basepoint x_0.

Definition. A space X is called simply connected if

a) It is path connected.

b) It has trivial fundamental group.

The following is a nice characterization of simply connected spaces.

Proposition. A space X is simply connected iff there is a unique homotopy class of paths connecting any two point in X.

Now we are going to discuss some applications of fundamental group.

Proposition. $\pi_1(S^n) = \{0\}$ if $n \geqslant 2$. Hence S^n is simply connected for $n \geqslant 2$.

Will discuss a proof using van Kampen theorem later.

Example. For a point x in $\mathbb{R}^n$, the compliment $\mathbb{R}^n - x$ is homeomorphic to $S^{n-1} \times \mathbb{R}$
$\implies \pi_1(\mathbb{R}^n - \{x\}) \approx \pi_1(S^{n-1}) \times \pi_1(\mathbb{R})$
$\implies \pi_1(\mathbb{R}^n - \{x\}) = \mathbb{Z}$ for $n = 2$ and 0 for $n > 2$.

Corollary. $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.

2.2 Fundamental theorem of algebra

Theorem. Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

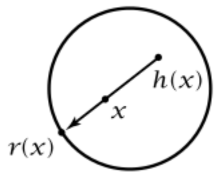
Proof. Suppose we have a polynomial $q(z) = z^n + a_1 z^{n-1} + ... + a_n$ let if possible say $q(z)$ has no roots in $\mathbb{C}$, then for each real number $r \geqslant 0$ the formula

$f_r(s)=\frac{q(re^{2\pi is}/q(r)}{|q(re^{2\pi is})/q(r)|}$ defines a loop in $S^1 \in \mathbb{C}$ based at 1. If we varies r, f_r will be homotopy of loos based at 1 as f_0 is trivial loop.
$\implies [f_r] \in \pi_1(S^1)$ is zero $\forall r$ now we fix some large value of r such that $r > |q_1| + |q_2| + ... + |q_n|$ and $r > 1$ then for $|z| = r$ we have
$|z^n| = r^n = r.r^{n-1} > (|a_1| + ... + |a_n|)|z^{n-1}| \geqslant |a_1 z^{n-1} + ... + a_n|$
$\implies |z^n| > |a_1 z^{n-1} + ... + a_n|$
$\implies$ polynomial $q_t(z) = z^n + t(a_1 z^{n-1} + ... + a_1)$ has no root on the circle $|z| = r$ when $0 \leqslant t \leqslant 1$.
Replacing p by p_t in the formula for f_r above and letting t go from 1 to 0, we get a homotopy from the loop f_r to the loop $w_n(s) = e^{2\pi ins}$ where w_n represents n times a generator of the infinite cyclic group $\pi_1(S^1)$ since we have shown that $[w_n] = [f_r] = 0$
$\implies n = 0$
$\implies q$ is a constant polynomial $\implies$ the only polynomial without roots in $\mathbb{C}$ are constants. $\square$

2.2.1 Brouwer fixed point theorem in 2-dimension

Theorem. Every constant map $h : D^2 \to D^2$ has a fixed point, that is, a point x with $h(x) = x$.

Proof. Let if possible $h(x) \neq x\ \forall x \in D^2$, then consider the map $r : D^2 \to S^1$ by letting $r(x)$ be the point of S^1 where the ray in $\mathbb{R}^2$ starting at $h(x)$ and passing through x leaves D^2.



Explicitly $r(x) = h(x) + k(x)(x - h(x))$ where $k(x) \geqslant 0$.

Claim: $r(x)$ is continuous, in order to show that $r(x)$ is continuous it would be enough to show $k(x)$ is continuous now consider, $r_1(x) = h(x)_1 + k(x_1 - h(x)_1)$ and $r_2(x) = h(x)_2 + k(x_2 - h(x)_2)$ where subscripts denote coordinates also as $r(x) \in S^1$

$\implies [r_1(x)]^2 + [r_2(x)]^2 - 1 = 0$

$\implies [h(x)_1 + k(x_1 - h(x)_1)]^2 + [h(x)_2 + k(x_2 - h(x)_2)]^2 - 1 = 0\ (*)$

Since the above polynomial is quadratic in k, it has two roots, which in view of the geometric context in which k arose, are both real and in fact one is positive and other is negative, or both are 0.

Let $k \geqslant 0$

$\implies$ $k(x)$ is well defined rational function in x_1 and x_2 by solving equation and calculating coefficient of k^2 we have $(x_1 - h(x)_1)^2 + (x_2 - h(x)_2)^2$ setting this coefficient equal to zero give $x_1 = h(x)_1$ and $x_2 = h(x)_2$ i.e, $x = h(x)$ but this is the contradiction to the hypothesis that the map h had no fixed point

$\implies$ leading coefficient of $(*)$ is not zero

$\implies$ k is continuous

$\implies$ $r(x)$ is continuous

Also note that $r(x) = x$ if $x \in S^1$

$\implies$ r is retraction of D^2 onto S^1.We will show that no such retraction can exist.

let f_0 be any loop in S^1.in D^2 there is a homotopy of f_0 to a constant loop, i.e,

$f_t(s) = (1-t)f_0(s) + tx_0$, where x_0 is the basepoint of f_0. Since the retraction r is the identity on S^1, the composition $r \circ f_t = f_0$ to the constant loop at x_0. $\square$

We will give a more general version of above theorem in Chapter 3.

2.2.2 Borsuk-Ulam theorem in 2-dimension

Theorem. For every continuous map $f : S^2 \to \mathbb{R}^2$ there exist a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$.

Corollary. Whenever S^2 is expressed as the union of three closed sets A_1, A_2 and A_3 then at least one of these sets must contain a pair of antipodal points $\{x, -x\}$.

Proof. Define d_1, d_2, $d_3 : S^2 \to \mathbb{R}^2$ as follows
$d_i(x) = inf_{y \in A_i}|x - y|$
this is continuous function, so using Borsuk-Ulam theorem to the map $S^2 \to \mathbb{R}^2$ given by $x \to (d_1(x), d_2(x))$

$\implies$ we will have a pair of antipodal points x and $-x$ with $d_1(x) = d_1(-x)$ and $d_2(x) = d_2(-x)$.
If either of these two distance is zero, then x, $-x \in A_1$ or x, $-x \in A_2$ since A_1 and A_2 are closed But if $d_1(x, A_1) > 0$ and $d_2(x, A_2) > 0$ then x, $-x \in A_3$. $\square$

2.2.3 Fundamental group of product of subspaces

Proposition. $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$ if X and Y are path connected.

Proof. First recall that a map $f : Z \to X \times Y$ is continuous iff the maps $g : Z \to X$ and $h : Z \to Y$ given as $f(z) = (g(z), h(z))$ are both continuous
$\implies$ a loop f in $X \times Y$ based at (x_0, y_0) is equivalent to a pair of loops g in X and h in Y based at x_0 and y_0 respectively. Similarly, a homotopy f_t of a loop in $X \times Y$ is equivalent to a pair of homotopies g_t and h_t of the corresponding loops in X and Y
$\implies$ we have a bijection and $[f] \to ([g], [h])$ is a group homomorphism
$\implies$ $\pi_1(X \times Y) \approx \pi_1(X) \times \pi_1(Y)$.

$\square$

Example. Using above result we can compute fundamental group of torus.
$\pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$.

2.3 Induced homomorphism

Consider the mpa $\varphi : X \to Y$ such that $\varphi(x_0) = y_0$ or write $\varphi(X, x_0) \to (Y, y_0)$ then define $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ as follows $\varphi_*([f]) = [\varphi f]$ then φ_* is well define since a hamotopy f_t of loops based at x_0 yields a composed homotopy φf_t of loops based at y_0 so $\varphi_*[f_0] = [\varphi f_0] = [\varphi f_1] = \varphi_*[f_1]$ also, φ_* is a homomorphism since $\varphi(f \cdot g) = (\varphi f) \cdot (\varphi g)$, both function have value $\varphi f(2s)$ for $0 \leqslant s \leqslant \frac{1}{2}$ and the value $\varphi g(2s-1)$ for $\frac{1}{2} \leqslant s \leqslant 1$. Two basic properties of induced homomorphism are :
a) $(\varphi\psi)_* = \varphi_*\psi_*$ for a composition $(X, x_0) \xrightarrow{\psi} (Y, y_0) \xrightarrow{\varphi} (Z, z_0)$.
b)$\mathbb{1}_* = \mathbb{1}$.

Proposition. If a space X retract onto a subspace A, then the homomorphism $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $i : A \hookrightarrow X$ is injective. If A is a deformation retract of X, then i_* is an isomorphism.

Proposition. If $\varphi : X \to Y$ is a homotopy equivalence, then the induced homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorphism $\forall x_0 \in X$.

Remark. Thus fundamental group is not a complete invariant of topological spaces.

2.4 Free product of groups

Let $\{G_\alpha\}_{\alpha\in\lambda}$ be a collection of groups then we construct a group $*_\alpha G_\alpha$ will be called the free product of the groups $\{G_\alpha\}$. $*_\alpha G_\alpha$ which as a set consist of all words $g_1 g_2 \cdots g_m$ of arbitrary finite length $m \geqslant 0$ where $g_i \in G_{\alpha_i}$ for some $\alpha_i \in \lambda$ and $g_i \neq e_{\alpha_i}$ (identity element of G_{α_i}) and adjacent letters g_i and g_{i+1} belong to different groups i.e., if $g_i \in G_{\alpha_i}$ and $g_{i+1} \in G_{\alpha_{i+1}}$ then $\alpha_i \neq \alpha_{i+1}$ words satisfying these conditions are called reduced.

Remark. a) Unreduced words can always be simplified to reduced words by writing adjacent letter that lie in the same group G_{α_i} as a single letter and by canceling trivial letters.

b)The empty word is allowed, and will serve as identity element of $*_\alpha G_\alpha$.

Now we define group operation on $*_\alpha G_\alpha$

Group operation in $*_\alpha G_\alpha$ is juxtaposition i.e., take any two reduced words say $(g_1 g_2 \cdot \cdots g_m)$ and $(h_1 h_2 \cdots h_n)$ then $g \cdot h$ is defined as $(g_1 g_2 \cdots g_m)(h_1 h_2 \cdots h_n) = g_1 \cdots g_m h_1 \cdots h_n$.

Remark. This product may not be reduced. If g_m and h_1 belong to the same G_α, they should be combined into a single letter $(g_m h_1)$ according to the multiplication in G_α and if this new letter happens to be the identity of G_α, it should be canceled from the product. This may allow g_{n-1} and h_2 to be combined and possibly canceled too. Repetition of this process eventually produces a reduced word.

Next we varyfy associativity of above defined multiplication. Let W be the set of reduced words $g_1 \cdots \cdot g_m$ as above including the empty word. To each $g \in G_\alpha$ we associate the function $L_g : W \to W$ as $L_g(g_1 \cdots \cdot g_m) = g g_1 \cdots \cdot g_m$ where we combine g with g_1 if $g_1 \in G_\alpha$ to make $g g_1 ... g_m$ a reduced word. A key property of the association $g \to L_g$ is that $L_{gg'} = L_g L'_g$ for $g, g' \in G_\alpha$ i.e., $g(g'(g_1 ... g_m)) = (gg')(g_1 ... g_m)$ this special case of associating follows rather trivially from associativity in G_α. The formula $L_{gg'} = L_g L'_g$
implies that L_g is invertible with $(L_g)^{-1} = L_{g^{-1}}$ therefore the association $g \to L_g$ defines a homomorphism from G_α to the group $P(W)$ of all permutation of W. More generally, define $L : W \to P(W)$ by $L(g_1 \cdots \cdot g_m) = L_{g_1} \cdots \cdot L_{g_m}$ for each reduced word $g_1 \cdots \cdot g_m$. This function L is injective since the permutation $L(g_1 \cdots \cdot g_m)$ send the empty word to $g_1 \cdots \cdot g_m$. The product operation in W corresponds under L to composition in $P(W)$, because $L_{gg'} = L_g L'_g$. Since composition in $P(W)$ is associative, we conclude that the product in W is associative.

Remark. a)$G_{\alpha_i} \hookrightarrow *_\alpha G_\alpha \ \forall \alpha_i$.

b) Association implies that any two sequences of reduction operation performed on the same unreduced word always yield the same reduced word.

c)Suppose we have any collection of homomorphism $\varphi_\alpha : G_\alpha \to H$ then we can extend uniquely to a homomorphism $\varphi : *_\alpha G_\alpha \to H$ by the following process

the value of φ on a word $g_1 \cdots \cdot g_n$ with $g_i \in G_{\alpha_i}$ must be $\varphi_{\alpha_1}(g_1) \cdots \cdot \varphi_{\alpha_n}(g_n)$ since the process of reducing an unreduced product in $*_\alpha G_\alpha$ does not affect its image under φ, so φ is well defined. One can easily check that φ is a homomorphism.

2.5 The van Kampen Theorem

Suppose a space X is decomposed as the union of a collection of path-connected open subsets A_α, and $x_0 \in A_\alpha$ $\forall\alpha$ by the above remark c the homomorphism $j_\alpha : \pi_1(A_\alpha) \to \pi_1(x)$ induced by the inclusion $A_\alpha \hookrightarrow X$ extends to a homomorphism $\Phi : *_\alpha\pi_1(A_\alpha) \to \pi_1(X)$.

Remark. a) The van kampen theorem will say that Φ is very often surjective.

b)But we can expect Φ to have a nontrivial kernel in general.

c)For if $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ is the isomorphism induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ then $j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}$, both these composition being induced by the inclusion $A_\alpha \cap A_\beta \hookrightarrow X$, so the kernel of Φ contains all the elements of the form $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$ for $w \in \pi_1(A_\alpha \cap A_\beta)$.

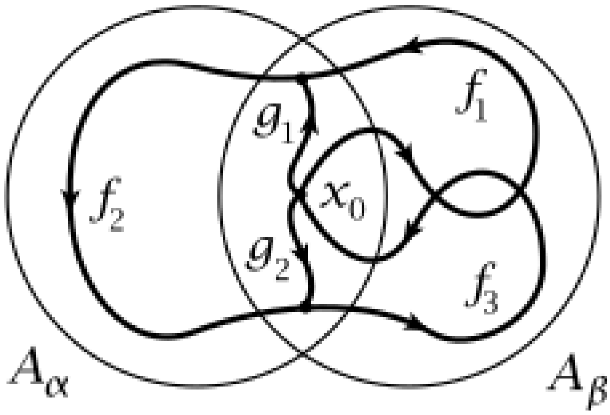
Theorem. (van Kampen theorem) If X is the union of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path connected, then the homomorphism $\Phi : *_\alpha\pi_1(A_\alpha) \to \pi_1(X)$ is surjective. If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$, and so Φ induces an isomophism $\pi_1(X) \simeq *_\alpha\pi_1(A_\alpha)/N$.

Proof. Let we are given a loop $f : I \to X$ at the basepoin x_0 since by definition f is continuous

$\Longrightarrow$ for each $s \in I$ there exist open neighborhood V_s in I mapped by f to some A_α.

We may infact take V_s to be an interval whose closure is mapped to a single A_α. Since I is compact so finite number of these intervals cover I, so the endpoint of this finite set of intervals then define a partition $0 = s_1 < s_1 < \cdots\cdot < s_m = 1$ of I such that each subinterval $[s_{i-1}, s_i]$ is mapped by f to a single A_α.

Let denote A_α containing $f([s_{i-1}, s_i])$ by A_i, and let f_i be a path obtained by restricting f to $[s_{i-1}, s_i]$. Then f is composition $f_1 \cdots\cdot f_m$ with f_i a path in A_i since we assume $A_i \cap A_{i+1}$ is path connected we may choose a path g_i in $A_i \cap A_{i+1}$ from x_0 to the point $f(s_i) \in A_i \cap A_{i+1}$ consider the loop $(f_1 \cdot \bar{g_1}) \cdot (g_1 \cdot f_2 \cdot \bar{g_2}) \cdot (g_2 \cdot f_3 \cdot \bar{g_3}) \cdots (g_{m-1} \cdot f_m)$ which is homotopic to f This loop is a composition of loops each lying in a single A_i, the loop indicated by the parentheses. Hence $[f]$ is in the image of Φ, and Φ is surjective.



Now before proceeding further we will introduce some terminology by a factorization of an element $[f] \in \pi_1(X)$ we shall mean a formal product $[f_1] \cdots [f_k]$ where
a) Each f_i is a loop in some A_α at the basepoin x_0, and $[f_i] \in \pi_1(A_\alpha)$ is the homotopy class.
b)The loop f is homotopic to $f_1 \cdots f_k$ in X.
A factorization of $[f]$ is thus a word in $*_\alpha \pi_1(A_\alpha)$ possibly unreduced, that is mapped to $[f]$ by Φ. Surjectivity of Φ implies that every $[f] \in \pi_1(X)$ has a fatorization now we move our focus toward the uniqueness of factorizations. Call two factorization of $[f]$ equivalent if they are related by a sequence of the following two sorts of moves or their inverses:
a') Combine adjacent terms $[f_i][f_{i+1}]$ into a single term $[f_i \cdot f_{i+1}]$ if $[f_i]$ and $[f_{i+1}]$ lie in the same group $\pi_1(A_\alpha)$.
b') Regard the term $[f_i] \in \pi_1(A_\alpha)$ as lying in the group $\pi_1(A_\beta)$ rather that $\pi_1(A_\alpha)$ if f_i is a loop in $A_\alpha \cap A_\beta$.
The first move does not change the element of $*_\alpha \pi_1(A_\alpha)$ defined by the factorization since all elements in $*_\alpha \pi_1(A_\alpha)$ are already reduced.
The second move does not change the image of this element in the quotient group $\mathbb{Q} = *_\alpha \pi_1(A_\alpha)/N$, by the definition of N. So equivalent factorization give the same element of $\mathbb{Q}$.
Consider the map $\Psi : \mathbb{Q} \to \pi_1(X)$ induced by Φ. i.e., $\Psi([f_1] \cdots [f_k]N) = \Phi([f_1] \cdots [f_k])$

Claim: This map Ψ is injective under the hypothesis that any two factorization of $[f] \in \pi_1(X)$ in $*_\alpha \pi_1(A_\alpha)$ are equivalent which we will prove

let $\Psi([f_1]\cdots\cdot[f_k]N) = \Psi([g_1]\cdots\cdot[g_t])N$

$\implies \Phi([f_1]\cdots\cdot[f_k]) = \Phi([g_1]\cdots\cdot[g_t]) = [f] \in \pi_1(X)$

$\implies [f_1]\cdots\cdot[f_k]$ and $[g_1]\cdots\cdot[g_t]$ are factorization of $[f]$ so by our hypothesis $[f_1]\cdots\cdot[f_k]$ and $[g_1]\cdots\cdot[g_t]$ are equivelent hence Ψ is injective.

$\implies ker\Psi = N$

Next we claim: $ker\Phi = N$

since $\Phi(i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}) = j_\alpha(i_{\alpha\beta}(w))j_\beta(i_{\beta\alpha}(w)^{-1}) = j_\alpha i_{\alpha\beta}(ww^{-1}) = [x_0]$ the second equality follow since $j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}$ and $j_\alpha i_{\alpha\beta}$ is a homomorphism.

$\implies N \subseteq ker\Phi$

Next let $[g_1]\cdots\cdot[g_n] \in ker\Phi$

$\implies \Psi([g_1]\cdots\cdot[g_n]N) = \Phi([g_1]\cdots\cdot[g_n]) = [x_0]$

$\implies [g_1]\cdots\cdot[g_n]N \in ker\Psi$

$\implies [g_1]\cdots\cdot[g_n] \in N$

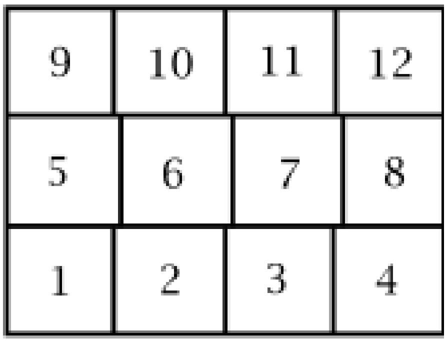
implies $ker\Phi \subseteq N$

hence $ker\Phi = N$

and the proof will be completed so now we prove our hypothesis that any two factorization of $[f]$ are equivalent.

Let say $[g_1]\cdot\cdots\cdot[g_k]$ and $[g_1']\cdot\cdots\cdot[g_l']$ be two factorization of $[f]$. Then the composed paths $g_1\cdot\cdots\cdot g_k$ and $g_1'\cdot\cdots\cdot g_l'$ are homotopic so let $F : I \times I \to X$ be a homotopy from $g_1\cdot\cdots\cdot g_k$ to $g_1'\cdot\cdots\cdot g_l'$. There exist partitions $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 < t_0 < t_1 < \cdots < t_n = 1$ such that each ractangle $[s_{i-1}, s_i] \times [t_{j-i}, t_j]$ is mapped by F into a single A_α, which we label A_{ij}.

Note: These partitions may be obtained by covering $I \times I$ by finitely many rectangles $[a, b] \times [c, d]$ each mapping to a single A_α using a compectness argument, then partitioning $I \times I$ by the union of all the horizontal and vertical lines containing edges of these rectangles we may assume the s-partition subdivides the partitions giving the products $g_1\cdot\cdots\cdot g_k$ and $g_1'\cdot\cdots\cdot g_l'$. Since F maps a neighborhood of R_{ij} to A_{ij}, we may perturb the vertical sides of the rectangles R_{ij} so that each point of $I \times I$ lies in atmost three $R_{ij}'s$ we may assume there are at least three row of rectangles, so we may do this perturbation just on the rectangles in the intermediate rows, leaving the top and bottom rows unchanged let us relabel the new rectangles $R_1, R_2, \cdots \cdot R_{mn}$, ordering them as in the figure.



If γ is a path in $I \times I$ from the left edge to the right edge, then the restriction $F|\gamma$ is a loop at the basepoint x_0 since F maps both the left and the right edges of $I \times I$ to x_0 let γ_r be the path separating the first r rectangles $R_1, \cdots, R_r$ from the remaining rectangles. Thus γ_0 is the bottom edge of $I \times I$ and Γ_{mn} is the top edge. We pass from γ_r to γ_{r+1} by pushing across the rectangle R_{r+1} let us call the corners of the $R_r's$ vertices. For each vertex v with $F(v) \neq x_0$, let g_v be a path from x_0 to $F(v)$. We can choose g_v to lie in the intersection of the two or three $A'_{ij}s$ corresponding to the R'_rs containing v since we assume the intersection of any two or three $A'_{ij}s$ is path connected. If we insert into $F|\gamma_r$ the appropriate paths $\bar{g}_v g_v$ at successive vertices, as in the proof of surjectivity of Φ, then we obtain a factorization of $[F|\gamma_r]$ by regarding the loop corresponding to a horizontal or vertical segment between adjacent vertices as lying in the A_{ij} for either of the R'_ss containing the segment. Different choice of these containing R'_ss change the factorization of $[F|\gamma_r]$ to an equvalent factorization. Furthermore the factorization associated to successive paths γ_r and γ_{r+1} are equivalent since pushing γ_r across R_{r_1} to γ_{r+1} changes $F|\gamma_r$ to $F|\gamma_{r+1}$ by a homotopy within the A_{ij} corresponding to R_{r+1}, and we choose this A_{ij} for all the segments of γ_r and γ_{r+1} in R_{r+1}. We can arrange the factorization associated to γ_0 is equivalent to the factorization $[g_1] \cdots \cdot [g_k]$ by choosing the path g_v for each vertex v along the lower edge of $I \times I$ to lie not just in the two $A'_{ij}s$corresponding to the R'_ss containing v but also to lie in the A_α for the f_i containing v in its domain. In case v is the common end point of the domain of two consecutive f'_is we have $F(v) = x_0$ so there is no need to choose g_v. In similar way we may assume that the factorization associated to the final

γ_{mn} is equivalent to $[g'_1]\cdots[g'_l]$. Since the factorizations associated to all the $\gamma'_r s$ are equivalent, we conclude that the factorization $[g_1]\cdots[g_k]$ and $[g'_1]\cdots[g'_l]$ are equivalent.

$\square$

Example. Let (X, x_0) and (Y, y_0) be two topological spaces we define their wedge sum as $X \vee Y := X \sqcup Y / x_0 \sim y_0$ let V_{x_0}, V_{y_0} be disc neighbourhoods of x_0, y_0 so that $X \vee Y = U_1 \sqcup U_2$ with $U_1 = [X \sqcup V_{y_0}]$ and $U_2 = [V_{x_0} \sqcup Y]$ we conclude that $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$. In particular $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = \mathbb{F}_2$.

Example. We will show that S^n for $n > 1$ are simply-connected. Let decompose S^n as two hemispheres H_1 H_2, then $H_1 \cap H_2 = S^{n-1}$ so all these are connected
$\implies \pi_1(S^n) = \pi_1(H_1) *_{\pi_1(S^{n-1})} \pi_1(H_2)$ is the trivial group since both H_1 and H_2 are contractible.

Example. Let $\{X_i\}_{i=1}^n$ such that each X_i is a connected space with a basepoint which has contractible neighborhood we get $\pi_1(\vee_{i=1}^n X_i) = *_{i=1}^n \pi_1(X_i)$. For example, the fundamental group of a bouquet of n-circles is the free group on n-generators.

Example. (Infinite mug) We will calculate fundamental group of topological cylinder $S^1 \times \mathbb{R}$ with a handle attached (the handle can be thought as a segment of a curve). Let call this space X. Note X retract into $S^1 \cup Y$ by deformation, where Y is a segment of curve, therefore up to homotopy this space is $S^1 \vee S^1$ so $\pi_1(X) \simeq \pi_1(S^1 \vee S^1) \simeq \mathbb{Z} * \mathbb{Z}$.

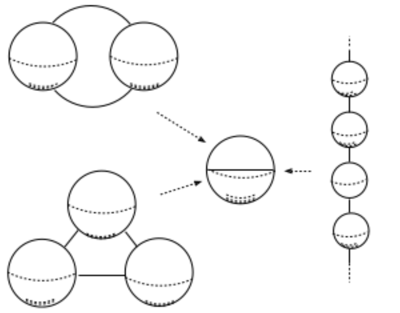
2.6 Covering spaces

Definition. A covering space of a space X is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ satisfying the following condition:
There exist an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by p homeomorphically onto U_α.

Remark. We do not require $p^{-1}(U_\alpha)$ to be nonempty, so p need not be surjective.

2.6.1 Fundamental group of S^1

Definition. Let $p : E \to B$ be a covering map; let $b_0 \in B$ choose e_0 so that $p(e_0) = b_0$. Given an element $[f]$ of $\pi_1(B, b_0)$, let $\tilde{f}$ be the lifting of f to a path in E that be-



Three different coverings of a 2-sphere with a diameter attached

gins at e_0. Let $\phi([f])$ denotes the end point $\tilde{f(1)}$ of $\tilde{f}$. Then ϕ is a well-defined set map

$$\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$

we call ϕ the **lifting correspondence** derived from the covering map p. It depends of course on the choice of the pint e_0.

Theorem. Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$ if E is path connected, then the lifting correspondence $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is surjective. If E is simply connected, it is bijective.

Proof. If E is path connected, then given $e_1 \in p^{-1}(b_0)$ there is a path $\tilde{f}$ in E from e_0 to e_1. Then $f = p \circ \tilde{f}$ is a loop in B at b_0, and $\phi([f]) = e_1$

$\implies$ ϕ is surjective.

Next we show that if E is simply connected then ϕ is bijective. Since simply connected space is also path connected by definition so ϕ is surjective so it only remains to show ϕ is injective.

Let $[f]$ and $[g] \in \pi_1(B, b_0)$ be any such that $\phi([f]) = \phi([g])$

$\implies$ $\tilde{f(1)} = \tilde{g(1)}$

where $\tilde{f}$ and $\tilde{g}$ be lifting of f and g, respectively to paths in E that begin at e_0 now, as E is simply connected

$\implies$ there is a path homotopy $\tilde{F}$ in E between $\tilde{f}$ and $\tilde{g}$. Then $p \circ \tilde{F}$ is a path homotopy in B between f and g. □

Theorem. The fundamental group of S^1 is isomorphic to the additive group of integers.

Proof. Let $p : R \to S^1$ be the covering map given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ let $e_0 = 0$ and let $b_0 = p(e_0)$ then $p^{-1}(b_0) = \mathbb{Z}$. Since R is simply connected the lifting correspondence $\phi : \pi_1(S^1, b_0) \to Z$ is bijective. We will show ϕ is a homomorphism then it will follows that $\pi_1(S^1, b_0) \approx \mathbb{Z}$ let $[f]$, $[g] \in \pi_1(B, b_0)$, let $\tilde{f}$ and $\tilde{g}$ be their respective lifting to paths on $\mathbb{R}$ beginning at 0 let $n = f\tilde{(1)}$
$m = g\tilde{(1)}$
then $\phi([f]) = n$ and $\phi([g]) = m$, let $\bar{\bar{g}}$ be the path $\bar{\bar{g}}(s) = n + g\bar{(s)}$ on $\mathbb{R}$ since we have $p(n + x) = p(x) \forall x \in \mathbb{R}$ the path $\bar{\bar{g}}$ is a lifting of g; it begins at n. Then the product $\bar{f} * \bar{\bar{g}}$ is defined and it is lifting of $f * g$ that begins at 0. The end point of this path is $g\bar{\bar{(1)}} = n + m$
$\implies \phi([f] * [g]) = n + m = \phi([f]) + \phi([g])$
$\implies \phi$ is bijective and isomorphism
$\implies \pi_1(S^1, b_0) \simeq \mathbb{Z}$. □

Definition. A lift of a map $f : Y \to X$ is a map $\tilde{f} : Y \to \tilde{X}$ such that the following diagram commute.

$$\begin{tikzcd}
Y \arrow[r, "\tilde{f}"] \arrow[dr, "f"'] & \tilde{X} \arrow[d, "p"] \\
 & X
\end{tikzcd}$$

i.e., $p \circ \tilde{f} = f$

2.7 Lifting Properties

2.7.1 Homotopy lifting property

Proposition. Given a covering space $p : \tilde{X} \to X$, a homotopy $f_t : Y \to X$ and a map $\tilde{f}_0 : Y \to \tilde{X}$ lifting f_0, then there exists a unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ of $\tilde{f}_0$ that lifts f_t.

Remark. Taking Y to be a point gives the path lifting property for a covering $p : \tilde{X} \to X$, which says that for each path $f : I \to X$ and each lift $\tilde{x_0}$ of the starting point $f(0) = x_0$ there is a uniqe path $\tilde{f} : I \to \tilde{X}$ lifting f starting at $\tilde{x_0}$.

Remark. Taking Y to be I we will get path lifting homotopy.

The following proposition is an application of homotopy lifting.

Proposition. The map $p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$ induced by a covering space $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ is injective. The image subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0})$ in $\pi_1(X, x_0)$ consists of the homotopy classes of loops in X based at x_0 whose lifts to $\tilde{X}$ starting at $\tilde{x_0}$ are loops.

Definition. If $p : \tilde{X} \to X$ is a covering space, then the cardinality of the set $p^{-1}(x_0)$ is locally constant over X. Hence if X is connected, this cardinality is constant as x ranges over all of X. It is called number of **sheets** of the covering.

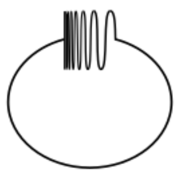
Proposition. The number of sheets of a covering space $p : (\tilde{x}, \tilde{x_0}) \to (X, x_0)$ with X and $\tilde{X}$ path-connected equals the index of $p_*(\pi_1(\tilde{X}, \tilde{x_0})$ in $\pi_1(X, x_0)$.

2.7.2 Lifting criterion

Proposition. Suppose given a covering space $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ and a map $f : (Y, y_0) \to (X, x_0)$ with Y path connected and locally path-connected. Then a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$ of f exists iff $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x_0}))$.

The importance of Y to be locally path-connectd can be made clear by following example.

Example. Let Y be the quasi-circle shown in the figure, a closed subspace of $\mathbb{R}^2$ consisting of a portion of the graph of $y = \sin(1/x)$, the segment $[-1, 1]$ in the $y - axis$, and an arc connecting these two pieces.



Collapsing the segment of Y in the $y - axis$ to a point gives a quotient map $f : Y \to S^1$. Show that f does not lift to the covering space $\mathbb{R} \to S^1$, even though $\pi_1(Y) = 0$. Thus local path–connectedness of Y is a necessary hypothesis in the lifting criterion.

Let $p : \mathbb{R} \to S^1$ be given by $p(t) = (\cos 2\pi t, \sin 2\pi t$ let l be the segment on the $y - axis$ without loss of generality assume that $f(l) = \{1\}$ and let $\tilde{f} : Y \to \mathbb{R}$ is a lifting of f Now as Y/l is connected
$\implies \tilde{f}(Y/l)$ is connected
$\implies \tilde{f}(Y/l)$ must be the component of $p^{-1}(f(Y/l)) = \mathbb{R}/2\pi\mathbb{Z}$ say $(0, 2\pi)$ since f is surjective.
$\implies \tilde{f}(Y/l)$=$(0, 2\pi)$ since Y is compact
$\implies \tilde{f}(Y) \supset [0, 2\pi]$
$\implies \{0, 2\pi\} \subset \tilde{f}(L)$.
Now since L is connected $\implies \tilde{f}(L)$ should be connected but $\tilde{f}(L) \subset 2\pi\mathbb{Z}$ is a discrete set containing at least two points.

2.7.3 Unique lifting property

Proposition. Given a covering space $p : \tilde{X} \to X$ and a map $f : Y \to X$ with two lifts $\tilde{f}_1$, $\tilde{f}_2 : Y \to \tilde{X}$ that agree at one point of Y, then if Y is connected, these two maps must agree on all of Y.

The proof of above proposition is very easy. Now we move towards classification of covering spaces.

2.8 The classification of covering spaces

Definition. A space X is **semilocally simply-connected** if for each $x \in X$ has a neighborhood U such that the inclusion-induced map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

Definition. A covring space $p : X \to B$ is called a universal cover of B if X is simply connected.

Theorem. Suppose B is path connected and locally path connected space; then a universal cover of B exists iff B is semilocally simply connected.

Remark. In all the results of the remaining part or this section we will assume that all of the spaces have the following properties; path connected, locally path connected, and semi locally simply connected. Because this gives a good classification of covering spaces.

Proposition. Suppose X is path-connexted, locally path-connected, and semilocally simply-connected. Then for every subgroup $H \subset \pi_1(X, x_0)$ there is a covering space $p : X_H \to X$ such that $p*(\pi_1(X_H, \tilde{x}_0) = H$ for a suitably chosen basepoint $\tilde{x}_0 \in X_H$.

Having taken care of the existence of covering spaces of X corresponding to all subgroups of $\pi_1(X)$, we turn now to the question of uniqueness. More specifically, we are interested in uniqueness up to isomorphism, where an isomorphism between covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ such that $p_1{=}p_2 f$.

Proposition. If X is path-connected and locally path connected, then two path-connected covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ are isomorphic via an isomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ taking a basepoint $\tilde{x}_1 \in p_1^{-1}$ to a base point $\tilde{x}_2 \in p_2^{-1}$ iff $p_1 * (\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_2 * (\pi_1(\tilde{X}_2, \tilde{x}_2))$.

We can conclude the first half of the following classification theorem:

Theorem. Let X be path-connected, locally path-connected, and semilocally simply-connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and the set of subgroups of $\pi_(X, x_0)$, obtained by associating the subgroup $p * (\pi_1(\tilde{X}, \tilde{x}_0))$ to the covering space $(\tilde{X}, \tilde{x}_0)$. If basepoint are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p : \tilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

2.9 Deck Transformation and group action

Definition. For a covering space $p : \tilde{X} \to X$ the isomorphism $\tilde{X} \to \tilde{X}$ are called **deck transformation** or **covering transformation**.

Definition. A covering space $p : \tilde{X} \to X$ is called **normal** if for each $x \in X$ and each pair of lifts $\tilde{x}$, $\tilde{x}'$ of x there is a deck transformation taking $\tilde{x}$ to $\tilde{x}'$.

Proposition. Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a path-connected covering space of the path connected, locally path-connected space X, and let H be the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0) \subset \pi_1(X, x_0)$. Then:

a) This covering space is normal iff H is a normal subgroup of $\pi_1(X, x_0)$.

b)$G(\tilde{X})$ is isomorphic to the quotient $N(H)/H$ where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$ in particular, $G(\tilde{X})$ is isomorphic to $\pi_1(x, x_0)/H$ if $\tilde{X}$ is a normal covering. Hence for the universal cover $\tilde{X} \to X$ we have $G(\tilde{X}) \approx \pi_1(x)$.

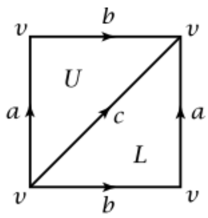
Chapter 3

Homology

3.1 Simplicial and singular homology

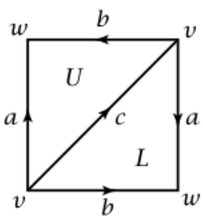
We start with the following motivating examples:

a) **Torus** can be obtained from a square by identifying opposite edges in the way indicated by the arrows in the figure below:

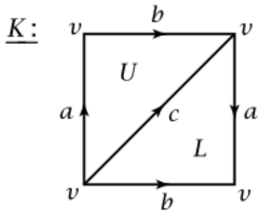


b) Similarly projective plane $\mathbb{R}P^2$, and the klein bottle K can be obtained.

$\underline{\mathbb{R}\mathrm{P}^2}$:



If we cut the square along a diagonal, we get two triangles so each of these surfaces can also be constructed from two triangles by identifying certain pair of edges. In the same way, a polygon with any number of sides can be cut along diagonals into



triangle, In fact all closed surfaces can be constructed from triangles by identifying edges.

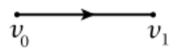
That is we have a single building block, the triangle, from which all surfaces can be constructed using only triangles we could also construct a large class of $2-$ *dimensional* spaces that are not surfaces in the strict sense, by allowing more than two edges to be identified together at a time.

Remark. $\triangle$-Complexes are a generalization of this idea, using the $n - dimensional$ analog of the triangle.

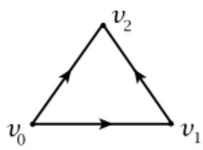
Definition. $n-simplex$, this is the smallest convex set in $\mathbb{R}^{n+1}$ containig $n+1$ points $v_0, v_1, \cdots, v_n$ that do not lie in a hyperplane of dimension less than n.

Example. $0 - simplex$ is a point.

Example. $1 - simplex$ is just a line.

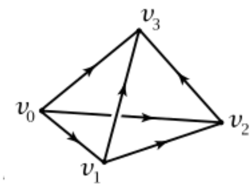


Example. $2 - simplex$ is a triangle.



Example. $3 - simplex$ is a Tetrahedron.

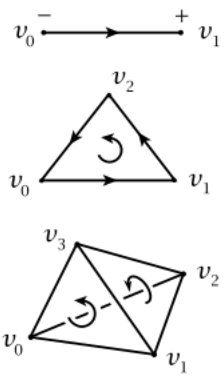
Definition. A **face** of a simplex $[v_0, \cdots, v_n]$ is the subsimplex with vertices any nonempty subset of the $v_i's$. The subset need not be a proper subset, $[v_0, \cdots, v_n]$ is regarded as a face of itself.



Definition. $\triangle-$ *complex* is a quotient space of a collection of disjoint simplices obtained by identifying certain of their faces via the canonical linear homeomorphisms that preserve the ordering of vertices.

3.2 Simplicial Homology

Let $\triangle_n(X)$ be the free abelian group with basis the open $n-$ *simplices* e_α^n of X elements of $\triangle_n(X)$ called $n-$ *chains*. $n-$ *chains* can be written as finite formal sums $\sum_\alpha n_\alpha \sigma_\alpha$ where $\sigma_\alpha : \triangle^n \to X$ is the characteristic map of e_α^n. The boundary of the $n-$ *simplex* $[v_0, \cdots, v_n]$ consists of the various $(n-1)-$ *dimensional* simplices $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$, where the 'hat' symbol ^ over v_i indicates that this vertex is deleted from the sequence $v_0, \cdots v_n$.



$$\partial[v_0, v_1] = [v_1] - [v_0]$$

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

$$\partial[v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$$

Definition. For a general $\triangle-$ *complex* X, a **boundary homomorphism** given as follows:

$\partial_n : \triangle_n(X) \to \triangle_{n-1}(x)$

$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha | [v_0, \cdots, \hat{v}_i, \cdots, v_n]$.

Proposition. The composition $\triangle_n(X) \xrightarrow{\partial_n} \triangle_{n-1}(X) \xrightarrow{\partial_{n-1}} \triangle_{n-2}(X)$ is zero.

$$\cdots \to \triangle_n(X) \xrightarrow{\partial} \triangle_{n-1}(X) \to \cdots$$

So define simplicial homology groups to be $H_n^{\triangle} = Ker\partial_n/Im\partial_{n+1}$.

Example. $X = T$, the torus with the $\triangle - complex$ struture pictured earlier, having one vertex, three edges a, b, and c, and two 2-simplices U and L. As in the figure give at the starting of chapter, $\partial_1 = 0$ so $H_0^{\triangle} \approx \mathbb{Z}$. Since $\partial_2 U = a + b - c = \partial_2 L$ and $\{a, b, a + b - c\}$ is a basis for $\triangle_1(T)$, it follows that $H_1^{\triangle} \approx \mathbb{Z} \oplus \mathbb{Z}$. Since there are no 3-simplices, $H_2^{\triangle}$ is equal to $Ker\partial_2$, which is infinite cyclic generated by $U - L$ since $\partial(pU + qL) = (p + q)(a + b - c) = 0$ only if $p = -q$.
Thus

$$H_n^{\triangle}(T) \approx \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 0, 2 \\ 0 & n \geqslant 3 \end{cases}$$

3.3 Singular Homology

A singular $n - simplex$ in a space X is by defintion just a map $\sigma : \triangle^n \to X$. Let $C_n(X)$ be the free abelian group with basis the set of singular $n - simplex$ in X. Elements of $C_n(X)$, called $n - chains$, or more precisely singular $n - chains$, are finite formal sums $\sum_i n_i \sigma_i$ for $n_i \in \mathbb{Z}$ and $\sigma_i : \triangle^n \to X$ and boundary map is defined as
$\partial_n : C_n(X) \to C_{n-1}(X)$
$\partial_n(\sigma) = \sum_i (-1)^i \sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_n]$.
The proof of the lemma also hold good here i.e., $\partial_n \partial_{n+1} = 0$ so we can define singular homology group $H_n(X) = Ker\partial_n/Im\partial_{n+1}$.

Proposition. Corresponding to the decomposition of a space X into its path-components X_α there is an isomorphism of $H_n(X)$ with the direct sum $\bigoplus_\alpha H_n(X_\alpha)$.

Proposition. A If X is nonempty and path-connected, then $H_0(X) \approx \mathbb{Z}$ hence for any space X, $H_0(X)$ is a direct sum of $\mathbb{Z}'s$, one for each path-component of X.

Corollary. Thus $H_0(X)$ is a free abelian group and rank$(H_0(X))$ is equal to the number of path component of X.

Proposition. If X is a point, then $H_n(X) = 0$ for $n > 0$ and $H_0(X) \approx \mathbb{Z}$. Sine $C_0(X) \approx \mathbb{Z}$ and $C_n(X) = 0$ for $n \geqslant 1$.

3.4 Reduced homology groups

Reduced homology groups $\tilde{H}_n(X)$ is defined to be the homology groups of the augmented chain complex

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $\varepsilon(\sum_i n_i\sigma_i) = \sum_i n_i$

since $\varepsilon\partial_1 = 0$, ε vanishes on $Im\partial_1$, and hence induces map $H_0(X) \to \mathbb{Z}$ with $Kernel = \tilde{H}_0(X)$

so, $H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}$. And $H_n(X) \approx \tilde{H}_n(X)$ for $n > 0$.

3.5 Homotopy Invariance

For a map $f : X \to Y$, and induced homomorphism $f_\#(\sigma) = f\sigma : \triangle^n \to Y$, then extending $f_\#$ linearly via $f_\#(\sum_i n_i\sigma_i) = \sum_i n_i f_\#(\Sigma_i) = \sum_i n_i f\sigma_i$. Also note
$f_\#\partial(\sigma) = f_\#(\sum_i(-1)^i\sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_n]) = \sum_i(-1)^i f\sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_n] = \partial f_\#(\sigma)$.
Thus we have diagram on next page.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\sharp"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

The fact that the maps $f_\# : C_n(X) \to C_n(X)$ satisfy $f_\#\partial = \partial f_\#$ is also express by saying that the $f'_\#s$ define a chain map from the singular chian complex of X to that of Y also $f_\#\partial = \partial f_\#$

$\implies$ $f_\#$ takes cycles to cycles since $\partial\alpha = 0$

$\implies$ $\partial(f_\#\alpha) = f_\#(\partial\alpha) = 0$ and $f_\#$ takes boundaries to boundaries since $f_\#(\partial\beta) = \partial(f_\#\beta)$.

Hence $f_\#$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$

Thus we have proved the following proposition.

Proposition. A chain map between chain complexes induces homomorphisms between the homology groups of the two complexes.

Properties:

a)$(fg)_* = f_*g_*$.

b)$\mathbb{1}_* = \mathbb{1}$.

Theorem. B If two maps $f,\ g : X \to Y$are homotopic, then they induce the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$.

The following is an immediate consequence.

Corollary. The maps $f_* : H_n(X) \to H_n(Y)$ induced by a homotopy equivalence $f : X \to Y$are isomorphisms for all n.

Definition. Given a homotopy $F : X \times I \to Y$ from f to g, we can define prism operators $P : C_n(X) \to C_{n+1}(Y)$ by $P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times \mathbb{1}) \mid [v_0, \cdots, v_i, w_i, \cdots, w_n]$ for $\sigma : \triangle^n \to X$, where $F \circ (\sigma \times \mathbb{1})$ is the composition $\triangle^n \times I \to X \times I \to Y$, this prism operators satisfy the relation $\partial P = g_\# - f_\# - P\partial$
i.e, $\partial P + P\partial = g_\# - f_\#$. This relationship is expressed by saying P is a chain homotopy between the chain maps $f_\#$ and $g_\#$. Theorem B then follows from:

Proposition. Chain homotopic chain maps induce the same homomorphism on homology.

3.6 Relative homology groups

Relative homology groups are defined in the following way. Given a space X and a subspace $A \subset X$, let $C_n(X, A)$ be the quotient group $C_n(X)/C_n(A)$. Since the boundary map $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, it induces the quotient boundary map $\partial : C_n(X, A) \to C_{n-1}(X, A)$. Letting n vary, we have a sequence of boundary maps

$$\cdots \to C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \to \cdots$$

The relation $\partial^2 = 0$ holds for these boundary maps since it holds before passing to quotient groups. so we have chain complex, and the homology groups $Ker\partial_n/Im\partial_{n+1}$ of this chain complex are by definition the **relative homology group** $H_n(X, A)$.

Theorem. The sequence of homology groups
$\cdots\cdot \to H_n(A) \xrightarrow{i*} H_n(B) \xrightarrow{j*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i*} H_{n-1}(B) \to \cdots\cdot$ is exact.

Example. In the long exact sequence of reduced homology groups for the pair $(D^n, \partial D^n)$ the maps $H_j(D^n, \partial D^n) \xrightarrow{\partial} \widetilde{H}_{j-1}(S^{n-1})$ are isomorphism for all $j>0$ since the remaining terms $\widetilde{H}_j(D^n)$ are zeros for all j.

Hence we have

$$H_j(D^n, \partial D^n) \approx \begin{cases} \mathbb{Z} \ for \ j = n \\ 0 \ otherwise \end{cases}$$

Example. Applying the long exact sequence of reduced homology groups to a pair (X, x_0) with $x_0 \in X$ yields isomorphisms $H_n(X, x_0) \approx \widetilde{H}_n(X)$ for all n since $\widetilde{H}_n(x_0) = 0 \ \forall n$.

Proposition. If two maps $f, g : (X, A) \to (Y, B)$ are homotopic through the maps of pair $(X, A) \to (Y, B)$, then $f_* = g_* : H_n(X, A) \to H_n(Y, B)$.

3.7 Long exact sequences in homology

Definition. A sequence of homorphisms

$\cdots \to \mathrm{A}_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} \cdots$ is said to be exact if $ker\alpha_n = Im\alpha_n$ for each n.

The following chain is called long exact sequence of homology groups:

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots \to H_0(X, A) \to 0$$

3.8 Excision theorem

Theorem. Given subspaces $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, then the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphism $H_n(X - Z, A - Z) \to H_n(X, A)$ for all n.

Equivalently, for subspaces A, B contained in X whose interiors cover X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphism and $H_n(B, A \cap B) \to H_n(X, A)$ for all n.

Proposition. For good pairs (X, A), the quotient map $q : (X, A) \to (X/A, A/A)$ induces isomprphism $q_* : H_n(X, A) \to H_n(X/A, A/A) \approx \widetilde{H}_n(X/A) \ \forall \ n$.

Theorem. If X is a space and A is nonempty closed subspace that is a deformation retract of some neighbourhood in X, then there is an exact sequence

$\cdots \to \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X/A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \xrightarrow{i_*} \widetilde{H}_{n-1}(X) \to \cdots \to \widetilde{H}_0(X/A) \to 0$ where i is the inclusion $A \hookrightarrow X$ and j is the quotient map $X \to X/A$.

Definition. Pair of spaces (X, A) satisfying the hypothesis of the theorem will be called **good pairs**.

Corollary. $\widetilde{H}_n(S^n) \approx \mathbb{Z}$ and $\widetilde{H}_j(S^n) = 0$ for $j \neq n$

Corollary. ∂D^n is not a retract of D^n. Hence every map $f : D^n \to D^n$ has a fixed point.

Proof. Let if possible $r : D^n \to \partial D^n$ is a retraction then using defination of retraction we have ri=$\mathbb{1}$ where i is the inclusion map $i : \partial D^n \hookrightarrow D^n$ now corresponding to map r and i we can induce maps $r_* : \tilde{H}_{n-1}(\partial D^n) \to \tilde{H}_{n-1}(\partial D^n)$ and $i_* : \tilde{H}_{n-1}(\partial D^n) \to \tilde{H}_{n-1}(\partial D^n)$ since by property we defined earlier we have $(ri)_* = r_* i_* = \mathbb{1}_* = \mathbb{1} = (ri)$ that means the composition $\tilde{H}_{n-1}(\partial D^n) \xrightarrow{i_*} \tilde{H}_{n-1}(D^n) \xrightarrow{r_*} \tilde{H}_{n-1}(\partial D^n)$ is then the identity map on $\tilde{H}_{n-1}(\partial D^n) \approx \mathbb{Z}$.But both i_* and r_* are 0 since $\tilde{H}_{n-1}(D^n) = 0$ and we have a contradiction. $\square$

Definition. The local homology groups of a space X at a point $x \in X$ are defined to be the groups $H_n(X, X - \{x\})$.

Theorem. If nonempty open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic then $m = n$.

The above theorem is an application of result of Brouwer known as 'invariance of dimension,' which tells us $\mathbb{R}^m$ is not homeomoephic to $\mathbb{R}^n$ if $m \neq n$.

3.9 The equivalence of simplicial and singular homology

There is a canonical homomorphism from $H_n^{\triangle}$ to $H_n(X, A)$ induced by the chain map $\triangle_n(X, A) \to C_n(X, A)$ ending each n-simplex of X to its characteristic map. The possibility $A = \varnothing$ is not excluded, in which case the relative group reduce to absolute groups.

Theorem. The homomorphisms $H_n^{\Delta}(X, A) \to H_n(X, A)$ are isomorphisms for all n and all Δ-complex pairs (X, A).

3.10 Computations and applications

Definition. For a map $f : S^n \to S^n$, the induced $f_* : \widetilde{H}_n(S^n) \to \widetilde{H}_n(S^n)$ is a homorphism from an infinite cyclic group to itself and so must be of the form $f_*(\alpha) = d\alpha$ for some integer d depending only on f. This integer is called the degree of f, with the notation $deg\ f$.

Now we first define action of a group on a space X then give a application of degree map. An action of a group G on a space X is a homomorphism from from G to the hroup Homeo(X) of homeomorphism $h : X \to X$, and action is free if the homeomorphism corresponding to each nontrivial element of G has no fixed points.

Example. $\mathbb{Z}_2$ is the only nontrivial group that can act freely on S^n if n is even. Since the degree of homeomorphism must be 1 or -1, an action of a group G on S^n determines a degree function $d : G \to \{+1, -1\}$. This is a homomorphism since degfg=deg$f$deg$g$. If the action is free, then d sends every nontrivial element of G to $(-1)^{n+1}$. Thus when n is even, d has trivial kernel, so $G \subset \mathbb{Z}_2$.

3.11 Cellular homology

Theorem. If X is a CW complex, then

(a) $H_k(X^n, X^{n-1})$ is zero for $k \neq n$ and is free abelian for $k = n$, with a basis in one-to-one correspondence with n-cells of X.

(b) $H_k(X^n) = 0$ for $k{>}n$. In particular, if X is finite-dimensional then $H_k(X) = 0$ for $k > dimX$

(c) the inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \to H_k(X)$ if $k{<}n$.

With the help of above result we define **cellular homology groups** of a CW complex X. Let X be a CW complex. Using above theorem, portions of the long exact sequences for the pair (X^{n+1}, X^n), (X^n, X^{n-1}), and (X^{n-1}, X^{n-2}) fit into a diagram

$$
\begin{tikzcd}
 & & & & 0 & & \\
 & 0 \arrow[dr] & & H_n(X^{n+1}) \approx H_n(X) \arrow[ur] & & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1},X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n,X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1},X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}
$$

where d_{n+1} and d_n are defined as $d_{n+1} = j_n\partial_{n+1}$ and $d_n = j_{n-1}\partial_n$. Since the composition d_nd_{n+1} includes two successive maps in one of the exact sequences, hence is zero. The horizontal row in the diagram is a chain complex, called the **cellular chain complex** of X. The homology group of this cellular chain complex are called the **cellular homology groups** of X.

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