HODGE THEORY

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Certificate of Examination

This is to certify that the dissertation titled "HODGE THEORY" submitted by Mr. Debjit Basu (Reg. No. MP16004) for the partial fulfilment of MS degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Dated: April 25, 2019

Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Chetan Balwe at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Debjit Basu
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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

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Chapter 1

Introduction

The main aims of this thesis are to prove the Ködaira Embedding Theorem (Theorem-25) and the degeneration of the Frölicher Spectral sequence (Theorem-19) for Kähler manifolds through Hodge Theory.
The first one tells us that there is an embedding of every Kähler manifold with integral Kähler cohomology class (called Polarised Manifolds) in some complex projective space. Now, by Chow's Theorem, every complex projective manifold is a complex algebraic variety. So we get a $(1,1)$-correspondence between smooth complex projective (algebraic) varieties and polarised manifolds.
The second one is equivalent to having $\frac{F^pH^k(X,\mathbb{C})}{F^{p+1}H^k(X,\mathbb{C})} \cong H^q(X,\Omega^p_X)$ for $p+q=k$ where $H^q(X,\Omega^p_X)$ is the sheaf cohomology of the holomorphic p-forms and F is the filtration $F^pH^k(X,\mathbb{C}) = \bigoplus_{r\geq p, r+q=k} H^{r,q}(X)$. (see Section-6.1 and Theorem-19) It is also equivalent to having the decomposition of betti numbers $b_k = \sum_{p+q=k} h^{p,q}$, where $b_k = dim_{\mathbb{C}} H^k(X,\mathbb{C})$ and $h^{p,q} := dim_{\mathbb{C}}(H^q(X,\Omega^p_X))$. Thus it can be called a weak version of the Hodge decomposition theorem (Theorem-13).
Now if $(X,\mathcal{O}^{alg}_X)$ is a complete complex algebraic variety (has the properties: quasi-projective, compact with respect to analytic topology) with Zariski topology, we define the algebraic de Rham complex to be the complex of the sheaves of algebraic differential forms with exterior derivative. These are by nature locally free coherent sheaves of $\mathcal{O}^{alg}_X$-modules. We also have the sheaf of holomorphic functions $\mathcal{O}^{an}_X$ on X. The identity map

$$Id_X : X_{an} = (X,\mathcal{O}^{an}_X) \to X_{zar} = (X,\mathcal{O}^{alg}_X)$$

is a morphism of sheaves. Let $Coh_{\mathcal{O}_X^{alg}}$ and $Coh_{\mathcal{O}_X^{an}}$ be the categories of coherent sheaves of $\mathcal{O}_X^{alg}$-modules and $\mathcal{O}_X^{an}$-modules respectively. Then we have the functor

$$T : Coh_{\mathcal{O}_X^{alg}} \to Coh_{\mathcal{O}_X^{an}}$$

defined by pulling back any element of $Coh_{\mathcal{O}_X^{alg}}$ by Id_X and tensoring it with $\mathcal{O}_X^{an}$ over the pull-back of $\mathcal{O}_X^{alg}$ by Id_X (see [5]). The Serre's GAGA principle tells us that:

Theorem 1 ***(Serre 1956)***
If X is a complete algebraic variety, the functor T is an equivalence of categories $Coh_{\mathcal{O}_X^{alg}}$ and $Coh_{\mathcal{O}_X^{an}}$. Moreover, for every $\mathcal{F} \in Coh_{\mathcal{O}_X^{alg}}$ we have

$$H^q(X_{zar}, \mathcal{F}) \cong H^q(X_{an}, T(\mathcal{F})) \quad \forall q$$

By the GAGA theorem,

$$H^k(X, \mathbb{C}) \cong \mathbb{H}^k(X, \Omega^*_{X,an}) \cong \mathbb{H}^k(X, \Omega^*_{X,alg})$$

where the first term is the complex cohomology of X with analytic topology, the last two terms are Hypercohomology (see Definition-5.3.1), the first isomorphism is given by Theorem-18 and the second isomorphism is obtained by applying GAGA to hypercohomologies on the complexes of sheaves.
Again by the GAGA principle degeneracy at E_1 of the Frölicher spectral sequence associated to the holomorphic and the algebraic de Rham complex are equivalent; i.e. one degenerates at E_1 if the other does. So this weak version of the Hodge decomposition theorem can be generalised for complete algebraic varieties.
We assume the reader is familiar with some multivariable complex analysis like Cauchy integral formulas, Hartog's theorem and Riemann's Theorem(these can be found in [3] and [1]), some differential geometry like De Rham's Theorem and Poincare lemma and Poincaré-Dolbeault lemma (these state the exactness of the de Rham complex and the Dolbeault complex respectively), some sheaf cohomology and $\widehat{C}$ech cohomology (these can be found in [5], [3] and [1]) By a smooth function we will mean a $\mathcal{C}^\infty$-function.

Chapter 2

Complex Manifolds

Definition 2.0.1 ***(Complex Manifolds)***
*Let $n \in \mathbb{Z}_{\geq 0}$. A **complex manifold** X **of dimension** n, is a smooth ($= \mathcal{C}^\infty$) $2n$-dimensional manifold and if there is a smooth atlas $\mathcal{U} = (U_\alpha, \phi_\alpha : U_\alpha \to \mathbb{R}^{2n} \cong \mathbb{C}^n)_{\alpha \in I}$, such that the transition maps $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_{\alpha,\beta}) \to \phi_\alpha(U_{\alpha,\beta})$ ($U_{\alpha,\beta} := U_\alpha \cap U_\beta$) $\forall \alpha, \beta \in I$ are holomorphic maps on open subsets of $\mathbb{C}^n$.*
*In this case $\mathcal{U}$ is said to be a **complex structure** on X.*

(The ideas presented in this section are adapted from the book [1])
Let $\mathcal{O}_{\mathbb{C}^n,z}$ be the stalk of the sheaf of holomorphic functions at a point $z \in \mathbb{C}^n$. Let $\mathcal{O}[n]$ denote $\mathcal{O}_{\mathbb{C}^n,(0,\ldots,0)}$. We have that $\mathcal{O}[n]$ is an integral domain, which follows from the **Identity theorem**, and also a local ring with the unique maximal ideal

$$\mathfrak{m}[n] := \{f \in \mathcal{O}[n] : f(0, \ldots, 0) = 0\}$$

We prove that:

Proposition 2.0.2 *$\mathcal{O}[n]$ is a unique factorization domain (UFD for brevity).*

Proof Clearly, $\mathcal{O}[0] = \mathbb{C}$ which is indeed an integral domain. Suppose $\mathcal{O}[n-1]$ is a UFD and let $f \in \mathcal{O}[n] \setminus \{0\}$. Since f is not identically zero, we can assume that f is a holomorphic function $f : U \to \mathbb{C}$ in a neighbourhood U (say) of $(0, .., 0) \in \mathbb{C}^n$ such that

$$f(0, \ldots, 0, w) \neq 0 \quad \forall w \in V \setminus \{0\},$$

where $V = \{z \in \mathbb{C} : (0, \ldots, 0, z) \in U\}$.
We first prove the following Lemma which is called the Weierstrass preparation theorem:

Lemma 1 *Let z_j be the complex coordinate axes in $\mathbb{C}^n$. If f is a holomorphic function around $(0,..,0)$ in $\mathbb{C}^n$ with $f(0,...,0)=0$ and is not identically zero on the z_n-axis, then in some smaller neighbourhood around the origin, there is a Weierstrass polynomial g of degree d in $z_n = t$ and a non-zero holomorphic function h such that f can be uniquely expressed as*

$$f = g \cdot h.$$

Proof of the Lemma : Let $V(f) := \{f = 0\}$.

Since f does not vanish identically on the z_n-axis, we have that the power series expansion for f around the origin has a term $c \cdot t^k$ with $c \neq 0$, and $k \geq 1$. Choose a very small $\epsilon > 0$. Then there is an open n-disc $\mathbb{D}_r$ of radius $r > 0$, around the origin and a $\delta \geq 0$, such that $f(0,..,0,t) \geq \delta > 0$ on the boundary sphere $\partial\mathbb{D}_r$ and $f(z_1,...,z_{n-1},t) \geq \frac{\delta}{2}$ on the cylinder $\mathbb{D}_\epsilon^{n-1} \times \partial\mathbb{D}_r$. Let $b_1(z_1,...,z_{n-1}),...,b_d(z_1,...,z_{n-1})$ be the solutions of the equation

$$f(z_1,...,z_{n-1},t) = 0$$

in $\mathbb{D}_\epsilon^{n-1} \times \mathbb{D}_r$. Then, by the **residue theorem**, we have the equality of holomorphic function

$$b_1^q(z) + \cdots + b_d^q(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\mathbb{D}_r} (\frac{t^d}{f(z,t)}) \frac{\partial f(z,t)}{\partial t} dt \quad \forall (z,t) \in \mathbb{D}_\epsilon^{n-1} \times \mathbb{D}_r$$

Now the elementary symmetric polynomials $E_1(z),...,E_d(z)$ are polynomials in $b^{(q)} := b_1^q + \cdots + b_d^q$, and $g(z,t) := \sum_{j=1}^d (-1)^j E_j(z) t^{d-j} \;\; \forall (z,t) \in \mathbb{D}_\epsilon^{n-1} \times \mathbb{D}_r$, and $f = 0 \Leftrightarrow g = 0$. Then $h = \dfrac{f}{g}$ is holomorphic on $(\mathbb{D}_\epsilon^{n-1} \times \mathbb{D}_r) \setminus V(f)$, and has only removable t-singularities for fixed z along $V(f)$. Then $h(z,t)$ extends to a function which is holomorphic in $t \in \mathbb{D}_r$ for each fixed $z \in \mathbb{D}_\epsilon$. But it also extends to a holomorphic function $\tilde{h}(z,t) = \dfrac{1}{2\pi\sqrt{-1}} \displaystyle\int_{\partial\mathbb{D}_r} \frac{f(z,v)}{v-t} dv \;\; \forall (z,t) \in \mathbb{D}_\epsilon^{n-1} \times \mathbb{D}_r$ This proves the lemma.
By this lemma we can write $f = g \cdot u$,

where u is invertible in $\mathcal{O}[n]$ and g is a Weierstrass polynomial with coefficients in $\mathcal{O}[n-1]$. Now by Gauss' Lemma, the ring of all Weierstrass polynomials W with coefficients in $\mathcal{O}[n-1]$ is a UFD. So there are irreducible polynomials $g_1,...,g_r \in W$ such that

$$f = u \cdot \prod_{j=1}^r g_j$$

and clearly $g_1, ..., g_r \in W$ are unique upto multiplication by units. The rest of the uniqueness part follows from the uniqueness in the Weierstrass preparation theorem.
□

2.1 Almost Complex Manifolds

Definition 2.1.1

1. ***(Complex Vector Bundles)*** *A* ***complex vector bundle*** *of rank r on a real manifold X is a real vector bundle (E, p) with $p^{-1}(X) \cong \mathbb{C}^r$ and a vector bundle endomorphism $J^2 = 1$.*

2. ***(Holomorphic Vector Bundles)*** *A* ***holomorphic vector bundle*** *of rank r on a complex manifold is a real vector bundle of rank $2r$ such that there is a trivialization $\mathcal{U} = (U_\alpha, \psi_\alpha)_{\alpha \in I}$ such that the transition maps $g_{\alpha,\beta} := \psi_\alpha \circ \psi_\beta^{-1} : U_{\alpha,\beta} \times \mathbb{C}^r \to U_{\alpha,\beta} \times \mathbb{C}^r$ are holomorphic (where $U_{\alpha,\beta} := U_\alpha \cap U_\beta$) $\forall \alpha, \beta \in I$.*

A complex structure $\mathcal{U} = (U_\alpha, \phi_\alpha)_{\alpha \in I}$ on a complex manifold X, induces an endomorphism

$$J :=: T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$$

with $J^2 = -1$ defined by $J|_{U_\alpha} := Id_{U_\alpha} \times (\sqrt{-1} \cdot Id_{\mathbb{C}^n}) : T_{X,\mathbb{R}}|_{U_\alpha} \to U_\alpha \times \mathbb{C}^n$. Now every vector bundle endomorphism $I :=: T_{X,\mathbb{R}} \to T_{X,\mathbb{R}}$ with $I^2 = -1$ need not come from a complex structure on a even dimensional manifold

Definition 2.1.2 ***(Almost complex structures and Integrable almost complex structures)*** *An* ***almost complex structure*** *on a manifold X, is a smooth vector bundle endomorphism J of the tangent bundle of X, such that $J^2 = -1$; and the pair (X, J) is called an* ***almost complex manifold****. Such an almost complex structure is said to be* ***integrable*** *if it comes from a complex structure on the manifold.*

Note: *Here by the "Newlander-Nirenberg question" we will mean the question that asks the conditions on the pair (X, J) which makes J integrable Let $T_{X,\mathbb{R}}$ be the real* *tangent bundle of an almost complex manifold (X, J) of real dimension $2n$ and define:*

$$T_{X,\mathbb{C}} := T_{X,\mathbb{R}} \underset{\mathbb{R}}{\otimes} \mathbb{C}$$
$$T_X^{1,0} := T_{X,\mathbb{R}} - iJT_{X,\mathbb{R}}$$
$$T_X^{0,1} := T_{X,\mathbb{R}} + iJT_{X,\mathbb{R}}$$

Clearly, these three above are complex vector bundles on X. Moreover, if J is integrable, $T_X^{1,0}$ is a holomorphic vector bundle on the complex manifold X. Let $\mathcal{A}^1_{X,\mathbb{C}} := T^*_{X,\mathbb{C}}$, $\mathcal{A}_X^{1,0} := (T_X^{1,0})^*$, $\mathcal{A}_X^{0,1} := (T_X^{0,1})^*$ be the complex duals of these bundles respectively. Let $\mathcal{A}^1_{X,\mathbb{R}}$ be the real dual of $T_{X,\mathbb{R}}$.
*Also define $\mathcal{A}^k_{X,\mathbb{R}} := \bigwedge^k(\mathcal{A}^1_{X,\mathbb{R}})$ and $\mathcal{A}^k_{X,\mathbb{C}} := \bigwedge^k(\mathcal{A}^1_{X,\mathbb{C}})$ and $\mathcal{A}_X^{p,q} := \bigwedge^p(\mathcal{A}_X^{1,0}) \otimes_{\mathbb{C}} \bigwedge^q(\mathcal{A}_X^{0,1})$, and $\Omega_X^p := \mathcal{A}_X^{p,0}$. The first two are called the bundle of **real differential** k-**forms**, **complex differential** k-**forms** on X and when J is integrable on X, the third one is called the bundle of **holomorphic** p- **forms** on X.*
*We have $\mathcal{A}^k_{X,\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{A}_X^{p,q}$. Let $\bigwedge(\mathcal{A}_{X,\mathbb{C}}) := \mathbb{C} \oplus \wedge^1(\mathcal{A}_{X,\mathbb{C}}) \oplus \wedge^2(\mathcal{A}_{X,\mathbb{C}}) \oplus \cdots$, $\bigwedge(\mathcal{A}_{X,\mathbb{R}}) := \mathbb{R} \oplus \wedge^1(\mathcal{A}_{X,\mathbb{R}}) \oplus \wedge^2(\mathcal{A}_{X,\mathbb{R}}) \oplus \cdots$ and $\wedge^k(\mathcal{A}_{X,\mathbb{C}})$ and $\wedge^k(\mathcal{A}_{X,\mathbb{R}})$ are homogeneous part of these of degree k respectively. Let $\mathrm{d}^0\alpha$ be the degree of a differential form (real or complex) alpha (or we sometimes say a homogeneous differential form of degree k, when we want to call all the elements of $\bigwedge(\mathcal{A}_{X,\mathbb{C}})$ and $\bigwedge(\mathcal{A}_{X,\mathbb{R}})$ as differential forms.) $\bigwedge(\mathcal{A}_{X,\mathbb{C}})$ and $\bigwedge(\mathcal{A}_{X,\mathbb{R}})$ are what we call **graded commutative algebras** meaning $\alpha \wedge \beta = (-1)^{d^0\alpha \cdot d^0\beta} \beta \wedge \alpha$ for any two homogeneous differential forms.*
Then for any chart $(U, x_1, y_1, ..., x_n, y_n)$ of X, with $J(x_j) = y_j$, and $J(y_j) = -x_j$, for all $j \in \mathbb{N}_n$. Let $z_j := x_j + i \cdot y_j$, $\overline{z}_j := x_j - i \cdot y_j$, $dz_j := dx_j + i \cdot dy_j$ and $d\overline{z}_j := dx_j - i \cdot dy_j$, $\forall j \in \mathbb{N}_n$.
For every $j \in \mathbb{N}_n$ and every smooth function f on U we define:

$$\frac{\partial}{\partial z_j} := \frac{1}{2} \cdot (\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$$
$$\frac{\partial}{\partial \overline{z}_j} := \frac{1}{2} \cdot (\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$$
$$\partial f := \sum_{k=1}^{n} \frac{\partial f}{\partial z_k} dz_k$$
$$\overline{\partial} f := \sum_{k=1}^{n} \frac{\partial f}{\partial \overline{z}_k} d\overline{z}_k$$

We have the maps $Re, Im : T_{X,\mathbb{C}} \to T_{X,\mathbb{R}}$ defined by $Re(v + i \cdot w) := v$ and $Im(v + i \cdot w) := w$, for real vector fields v, w on X. Then $Re : T_X^{1,0} \to T_{X,\mathbb{R}}$ and $Id_{T_{X,\mathbb{R}}} - \sqrt{-1} \cdot J$ are inverse to each other and hence $T_X^{1,0} \cong T_{X,\mathbb{R}}$ are isomorphic as real vector bundles. Then $(\frac{\partial}{\partial z_j})_{1 \leq j \leq n}$ is a basis of $T_X^{1,0}$, and $(\frac{\partial}{\partial \overline{z}_j})_{1 \leq j \leq n}$ is a basis of $T_X^{0,1}$; and $(dz_j)_{1 \leq j \leq n}$ and $(d\overline{z}_j)_{1 \leq j \leq n}$ are their dual bases, respectively. We extend d, ∂ and $\overline{\partial}$ on $\bigwedge(\mathcal{A}_{X,\mathbb{C}})$ by

$$\mu(\alpha \wedge \beta) = \mu(\alpha) \wedge \beta + (-1)^{d^0 \alpha} \mu(\beta) \wedge \alpha$$

where α and β are homogeneous differential forms $\forall \mu = d, \partial, \overline{\partial}$.
Then we have the sequences of sheaves:

$$\cdots \xrightarrow{d} \mathcal{A}_X^k \xrightarrow{d} \mathcal{A}_X^{k+1} \xrightarrow{d} \cdots \tag{2.1}$$

$$\cdots \xrightarrow{\overline{\partial}} \mathcal{A}_X^{p,q} \xrightarrow{\overline{\partial}} \mathcal{A}_X^{p,q+1} \xrightarrow{\overline{\partial}} \cdots \tag{2.2}$$

*are exact by the **Poincaré Lemma** and the **Poincaré-Dolbeault Lemma** respectively, which we will assume to be true here. For the proofs the reader may see [3]. Let us denote by $\mathcal{A}^k(X), \mathcal{A}^{p,q}(X), \mathcal{A}_{\mathbb{C}}^k(X)$ the space of global sections of $\mathcal{A}_{X,\mathbb{R}}^k, \mathcal{A}_X^{p,q}$ and $\mathcal{A}_{X,\mathbb{C}}^k$ respectively, and let $\mathcal{A}_{\mathbb{R},x}^k, \mathcal{A}_{X,x}^{p,q}$ and $\mathcal{A}_{\mathbb{C},x}^k$ be the stalks of the sheaves at x associated to these bundles respectively.*
Then the cohomologies of the complexes:

$$\cdots \xrightarrow{d} \mathcal{A}^k(X) \xrightarrow{d} \mathcal{A}^{k+1}(X) \xrightarrow{d} \cdots \tag{2.3}$$

$$\cdots \xrightarrow{\overline{\partial}} \mathcal{A}^{p,q}(X) \xrightarrow{\overline{\partial}} \mathcal{A}^{p,q+1}(X) \xrightarrow{\overline{\partial}} \cdots \tag{2.4}$$

*are called the **de Rham cohomology** and **Dolbeault cohomology** respectively and we denote the k^{th}-cohomology of 2.3 by $H_{DR}^k(X, \mathbb{R})$ and that of the complexification of 2.3 by $H_{DR}^k(X, \mathbb{C})$ and the q^{th}-cohomology of 2.4 by $H_{\overline{\partial}}^{p,q}(X)$, or $H_{\overline{\partial}}^{p,q}(X)$.*
Note: *The de Rham theorem tells us that $H_{DR}^k(X, \mathbb{K}) \cong H^k(X, \mathbb{K})$ where the second one is singular cohomology; for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We also have that $H^k(X, \mathbb{K})$ is isomorphic to the k^{th}-sheaf cohomology of the constant sheaf $\mathbb{K}$ on X, for $\mathbb{K} = \mathbb{R}, \mathbb{C}$.*

Definition 2.1.3

1. *Let $p : E \to X$ be a $\mathcal{C}^\infty$-vector bundle on a smooth ($= \mathcal{C}^\infty$) manifold X, of rank r. Then a smooth vector subbundle $\mathcal{V} \subseteq E$ of E of rank $k \leq r$, is called a k-**distribution** on (E, X).*

*A k-distribution on $(T_{X,\mathbb{R}}, X)$ is called a **real k-distribution** on X.*

*Similarly a **complex k-distribution** on X is a smooth subbundle of rank k of $(T_{\mathbb{C},X}, X)$.*

*A **holomorphic k-distribution** on a complex manifold is a holomorphic subbundle of $T_X^{1,0}$ of complex rank r.*

2. *A real (resp. complex) distribution $\mathcal{V}$ is said to be **integrable** if X has an open cover $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ such that $\exists$ a $\mathcal{C}^\infty$ map $\phi_U : U \to \mathbb{R}^{dim_{\mathbb{R}}X-k}$(resp. $\mathbb{C}^{2 \cdot dim_{\mathbb{R}}X-k}$), such that $\mathcal{V}|_U = ker(d\phi_U)$ $\forall U \in \mathcal{U}$*

3. *A holomorphic k-distribution $\mathcal{V}$ is one for which there is an open cover $\mathcal{U}$ and holomorphic maps $\phi_U : U \to \mathbb{C}^{dim_{\mathbb{C}}X-k}$ for each $u \in \mathcal{U}$ such that $\mathcal{V}|_U = ker(d\phi_U)$ $\forall U \in \mathcal{U}$.*

We use the following theorem that characterizes real distributions to answer the Newlander -Nirenberg Question:

Theorem 2 ***(Frobenius Theorem)***
A real distribution $\mathcal{V}$ is integrable if and only if $[\mathcal{V}, \mathcal{V}] \subseteq \mathcal{V}$. □

Frobenius theorem gives us the following answer to this question known as the famous Newlander-Nirenberg theorem:

Theorem 3 ***(Newlander-Nirenberg theorem)*** *An almost complex structure J on a manifold X is integrable iff $[T_X^{0,1}, T_X^{0,1}] \subseteq T_X^{0,1}$ (which is same as saying $[T_X^{1,0}, T_X^{1,0}] \subseteq T_X^{1,0}$ by taking conjugates)* □

Proof We prove this theorem by the following lemma which is a more stronger version of the Frobenius theorem:

Lemma 2 *Let E be a holomorphic k-distribution on a complex manifold X. Then E is integrable if and only if $[E, E] \subseteq E$* □

Proof of the Lemma : We see that since $E \subseteq T_X^{1,0}$ and $[E, E] \subseteq E \Rightarrow [Re(E), Re(E)] \subseteq Re(E)$ and thus by the Frobenius integrability theorem in real case, $Re(E)$ is integrable. So there is an open cover $\mathcal{U}$ of X and real smooth submersive maps $\phi_U : U \to V(U)$ where $V(U)$ is an open subset of $\mathbb{R}^{2n-2k}$ such that

$$Re(E)|_U = ker(d\phi_U)$$

$\forall U \in \mathcal{U}$. Clearly, $T_{V(U),\mathbb{R}} \cong (T_{X,\mathbb{R}}|_U)/Re(E)|_U$ $\forall U \in \mathcal{U}$, and since $Re(E)$ is invariant under the action of J, J induces a complex structure on $V(U)$ Let $M(U)$ be a submanifold of U, which is transverse to the fibers of ϕ_U, (such a manifold exists if we refine $\mathcal{U}$ sufficiently). Then ϕ_U is a diffeomorphism of $M(U)$ onto $V(U)$; as $Re(E)|_U \oplus T_{M(U),\mathbb{R}}$ (after shrinking U if necessary), the complex structure on $(T_{X,\mathbb{R}}|_U)/Re(E)|_U$ by J induces a complex structure on $T_{M(U),\mathbb{R}}$. As $[T_{M(U),\mathbb{R}}, T_{M(U),\mathbb{R}}] \subseteq T_{M(U),\mathbb{R}}$ (for $Re(E)|_U \oplus T_{M(U),\mathbb{R}}$) $\forall U \in \mathcal{U}$. we get that the complex structure on every $V(U)$ is integrable. This makes ϕ_U is a holomorphic map. This proves the lemma.

Since the statement is local, we assume X to be an open subset of $\mathbb{R}^{2n}$ and that J is real analytic and can be given by a convergent power series. Clearly, J extends to a holomorphic map $U \to \mathbb{C}^n$ where U is a neighbourhood of X in $\mathbb{C}^n$. J gives a holomorphic distribution E on U defined by the space corresponding to the eigenvalue $\sqrt{-1}$ of J. We have $E|_X = T_X^{1,0}$.

We see that $E = T_{U,\mathbb{C}} - i \cdot JT_{U,\mathbb{C}}$ and $E|_X = T_{X,\mathbb{R}} - i \cdot JT_{X,\mathbb{R}}$. So we see that $[E, E] \subseteq E$. As E is integrable, locally there are holomorphic submersions $\phi_V : V \to \mathbb{C}^n$ on U. Again as the statement we want to show is local, we assume there is a holomorphic map $\phi : U \to \mathbb{C}^n$ Now, $\phi|_X : X \to \phi(X)$ is a local diffeomorphism where $\phi(X)$ is open in $\mathbb{C}^n$; as $T_X \subseteq T_U \cong \mathbb{C}^{2n}$ can be identified with $\mathbb{R}^{2n} \subseteq \mathbb{C}^{2n}$ which is transversal to $Re(E) \cong T_X^{1,0}$. It follows that $d\phi|_{T_U}$ is an isomorphism and so ϕ is a local diffeomorphism.

We want to show that the pullback of the complex structure by ϕ and the the associated complex structure on X are equal. Clearly, J gives a complex structure on $T_{X,x}$, and we have a complex structure on $T_{U,x}$ given by the isomorphism $T_{U,x} \cong \mathbb{C}^{2n}$. We see that $Re(E)_x \subseteq T_{U,x} \cong T_{X,x} \otimes \mathbb{C}$ is generated by $-i \cdot J\alpha$ so that $\alpha = iJ\alpha$ in $T_{U,x}/Re(E)_x$ for all $\alpha \in T_{X,x}$ and we see that the composition

$$T_{X,x} \hookrightarrow T_{U,x} \to T_{U,x}/Re(E)_x$$

for every $x \in X$and it follows that the local isomorphism $d\phi : T_{X,\mathbb{R}} \to T_{\mathbb{C}^n,\mathbb{R}}$ identifies J with the complex structure on $\mathbb{C}^n$

Chapter 3

Kähler Manifolds

Let (X, J) be an almost complex manifold. A **Hermitian metric** $h : T_{X,\mathbb{R}} \times T_{X,\mathbb{R}} \to \mathbb{C}$, on (X, J) is a collection of Hermitian metrics $h_x : T_{\mathbb{R},x} \times T_{\mathbb{R},x} \to \mathbb{C}$, (of complex vector spaces) taking $T_{\mathbb{R},x}$ as a complex vector space, where multiplication by $\sqrt{-1}$ is given by the endomorphism J_x $\forall x \in X$, such that all the functions $h_{i,j} : x \to h_x(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ on X are smooth for all chosen chart $(U, (x_k)_{1\leq k\leq n})$, where $n = dim_{\mathbb{R}}X$. Since h takes values in $\mathbb{C}$, we can express h as $u + i \cdot v$ where g and h are tensor fields on X. We denote $Re(h) = u$ and $Im(h) = v$.
We have the following correspondence:

$$\{\text{The set of all smooth Hermitian metrics on X}\}$$

$$\leftrightarrow \{\text{The set of all smooth real (1, 1)-differential forms on X}\}$$

given by

$$h \mapsto -Im(h)$$

and

$$(\omega \circ (Id_{T_{X,\mathbb{R}}} \otimes J) - \sqrt{-1}) \cdot \omega \leftarrow\!\shortmid \omega$$

Definition 3.0.1

1. *We say that a $(1, 1)$ differential form ω on X is positive if $\omega(u, Jv) > 0$ for any two vector fields u and v*
We see that a $(1, 1)$ differential form is positive if and only if the associated Hermitian metric is positive definite.

2. *We say that the smooth Hermitian metric on X is* ***Kähler*** *and the associated form ω is* ***Kähler*** *and the manifold X is* ***Kähler*** *if the almost complex structure on X is integrable and the 2-form ω is closed.*

We also have the following proposition which is easy to show and so we omit the proof:

Proposition 3.0.2 *Let h be a Hermitian metric and ω be the associated $(1,1)$-differential form on an almost complex manifold (X, J). Then the volume form of X is equal to $\frac{\omega^n}{n!}$ where $2n = dim_{\mathbb{R}} X$.*

3.1 Example of Kähler Manifolds

Example 1 : (Riemann surfaces)

A Riemann surface X is a complex manifold. Any Hermitian metric on X is Kähler since X is a 2-dimensional real manifold implies that the symplectic real 2-form associated to the Hermitian metric has to be closed.

Example 2 : (Complex tori)

Let n be a positive integer and Γ be a lattice (i.e. a discrete subgroup) in $\mathbb{C}^n$, and consider the complex n-tori $\mathbb{T}^n := \frac{\mathbb{C}^n}{\Gamma}$. If we take Hermitian metrics with constant coefficients on $\mathbb{C}^n$, then that metric is invariant under translations and therefore induces a metric on $\mathbb{T}^n$.

Example 3 : (Kähler manifolds with Curvature form of a Line Bundle)

Let X be a complex manifold and let (L, h) be a holomorphic Hermitian line bundle on X. Let $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ is a complex atlas on X that trivializes L, via. the maps $\phi_\alpha : L|_{U_\alpha} \to U_\alpha \times \mathbb{C}^n$. This means that we have $\sigma_\alpha \in L(U_\alpha)$ such that $\sigma_\alpha(x) = 0 \ \ \forall x \in U_\alpha$ $\forall \alpha \in I$. Let $g_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathbb{C}$ be the transition function $\phi_\alpha \circ \phi_\beta^{-1}$. Then

$$\sigma_\alpha = g_{\alpha,\beta}\sigma_\beta : U_\alpha \cap U_\beta \to \mathbb{C}$$

Let $h_\alpha := h(\sigma_\alpha, \sigma_\alpha) \ \ \forall \alpha \in I$, then $h_\alpha = |g_{\alpha,\beta}|^2 \cdot h_\beta : U_\alpha \cap U_\beta \to \mathbb{C}$. The 2-forms

$$\omega_\alpha := \frac{1}{2\pi\sqrt{-1}}\overline{\partial}\partial(log(h_\alpha)) : U_\alpha \to \mathbb{C}$$

are such that ω_α and ω_β coincide on $U_\alpha \cap U_\beta$, as

$$\omega_\alpha - \omega_\beta = \frac{1}{2\pi\sqrt{-1}}\overline{\partial}\partial(log|g_{\alpha,\beta}|^2) = 0$$

Therefore the local 2-forms ω_α glue together to give a global 2-form $\Theta(L,h)$ and it is called the **curvature form** of (L,h). This form is closed for it is locally exact. Moreover it is of type $(1,1)$ and therefore it is a Kähler form on X.

Example 4 : (Fubini-Study metric on $\mathbb{CP}^n$)

Let l be any line through origin (by a line through origin in this section we will mean a complex vector subspace of complex dimension 1) on $\mathbb{C}^{n+1}$ and let $p : \mathbb{C}^{n+1} \to \mathbb{CP}^n$ be the quotient map. Let $U_i := p(\{(z_0, \cdots, z_n) \in \mathbb{CP}^n : z_i \neq 0\})$, for all $i \in \mathbb{N}_n$.

Then $\{p(l)\} \times l \subseteq \mathbb{CP}^n \times \mathbb{C}^{n+1}$. Let $\mathcal{O}_{\mathbb{CP}^n}(-1)$ be the union of all such $\{p(l)\} \times l$.

For every $i \in \mathbb{N}_n$, let $S(U_i)$ be the set of all lines through the origin l that intersects U_i, and for every $l \in S(U_i)$ choose an element $\sigma_i(p(l)) = [z_0, \cdots, z_n] \in l$ such that $z_i = 1$. Then $\sigma_i = \frac{z_i}{z_j} \cdot \sigma_j$ on $U_i \cap U_j$, where $\frac{z_i}{z_j}$ and $\frac{z_j}{z_i}$ are meromorphic functions on $\mathbb{CP}^n$ that are holomorphic on $U_i \cap U_j$. Thus, the maps $\phi_i : \mathcal{O}(-1)(U_i) = U_i \times S(U_i) \to U_i \times \mathbb{C} : (p(l), c \cdot \sigma_i(p(l))) \mapsto (p(l), c \cdot 1)\ \ \forall c \in \mathbb{C}$ and $\forall$ line through the origin l in $\mathbb{C}^{n+1}$, gives a line bundle structure on $\mathcal{O}_{\mathbb{CP}^n}(-1)$ by being the trivialization maps of it. This line bundle on $\mathbb{CP}^n$ is called the **tautological line bundle**.

Let $\mathcal{O}_{\mathbb{CP}^n}(+1)$ be the dual of the tautological line bundle, which we call the **Twisted bundle of Serre**. Let h be the standard Hermitian metric (usually denoted by $h_{standard}$) on $\mathbb{C}^{n+1}$. This defines a Hermitian metric on the subbundle

$$\mathcal{O}_{\mathbb{CP}^n}(-1) \subseteq \mathbb{CP}^n \times \mathbb{C}^{n+1}.$$

Let h^* be the metric on $\mathcal{O}_{\mathbb{CP}^n}(+1)$ dual to the metric h on the tautological bundle. Let ω be the curvature form $\Theta(\mathcal{O}_{\mathbb{CP}^n}(+1), h^*)$. Then

$$\omega_i := \omega|_{U_i} = \frac{1}{2\pi\sqrt{-1}} \partial\overline{\partial}(log(h^*(\sigma_i))$$

with notations as Example 3. Now $h^*(\sigma_i) = \dfrac{1}{h(\sigma_i)} : U_i \cong \mathbb{C}^n \to \mathbb{C}$ and

$h(\sigma_i)(z_1, \cdots, z_n) := 1 + \sum_{i=1}^n \|z\|^2$ and $\omega_i(z_1, \cdots, z_n) = \dfrac{1}{2\pi\sqrt{-1}} \partial\overline{\partial} log(\dfrac{1}{1+\sum_{j=1}^n \|z_j\|^2})$

We also get that this $(1,1)$-form is positive:

Lemma 3 ***(Positivity lemma of Fubini-Study metric)*** *The form ω on $\mathbb{CP}^n$ defined above is positive* □

Proof $\overline{\partial} log(\dfrac{1}{1+\sum_{j=1}^n \|z_j\|^2}) = -\dfrac{\overline{\partial}(1+\sum_{i=1}^n \|z_j\|^2)}{1+\sum_{j=1}^n \|z_j\|^2} = -\dfrac{\sum_{j=1}^n z_j d\overline{z_j}}{1+\sum_{j=1}^n \|z_j\|^2}$ implies,

$$\omega_i(z_1, \cdots, z_n) = \frac{\sqrt{-1}}{2\pi} \cdot \frac{(1+\sum_{j=1}^n \|z_j\|^2)(\sum dz_j \wedge d\overline{z}_j) + (\sum z_j d\overline{z}_j) \wedge (\sum(\overline{z}_j dz_j))}{(1+\sum_{j=1}^n \|z_j\|^2)^2}$$

and so $\omega_i(0,...,0) = \frac{\sqrt{-1}}{2\pi} \cdot \sum dz_j \wedge d\overline{z}_j$, which is positive. Now, from definition of the Fubini-Study metric, we see that it is invariant under biholomorphic maps $\mathbb{C}^{n+1}$ onto itself which preserve the standard metric, and clearly, $SU(n+1)$ acts on $\mathbb{C}^{n+1}$ by these maps, and this action keeps ω invariant. Thus ω is positive everywhere.

This construction generalizes in the case of projective bundles over a compact Kähler manifold, which is described below:

Example 5 : (Fubini-Study metric on Projective Bundles over a compact Kähler manifold)

Let $\pi : E \to X$ be a holomorphic vector bundle of rank $r+1$, on a complex manifold X. Let $E^* := E \setminus 0_E$ be the complement of the zero section 0_E of the bundle E. The constant sheaf $\mathbb{C}^*_X$ of stalks $\mathbb{C}^*$ is a group in itself and acts on E^* in the natural way. Let us denote the quotient of this action by $\mathbb{P}(E)$. Let $p_E : E^* \to \mathbb{P}(E)$

Let $\mathcal{U} := (U_\alpha, \phi_\alpha)_{\alpha \in I}$ be a trivializing open cover of E, by connected open sets. Then $\phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{C}^{r+1}$ are biholomorphic maps and by passing through the quotients of the action of $\mathbb{C}^*_X$ we get the maps ϕ_α satisfying the commutative diagram:

$$\begin{CD} \pi^{-1}(U_\alpha) \setminus 0_E @>{\phi_\alpha}>> U_\alpha \times \mathbb{C}^{r+1} \\ @V{p_E}VV @VV{Id_{U_\alpha} \times p_{\mathbb{C}^{r+1}}}V \\ \mathbb{P}(\pi^{-1}(U_\alpha)) @>{\mathbb{P}(\phi_\alpha)}>> U_\alpha \times \mathbb{CP}^r \end{CD}$$

The transition maps $g_{\alpha,\beta} := \phi_\alpha \circ \phi_\beta^{-1} : U_\alpha \cap U_\beta \to GL(r+1, \mathbb{C})$, send lines through origin to lines through origin and we have the commutative triangle of biholomorphic maps:

$$\begin{array}{ccc} (U_\alpha \cap U_\beta) \times \mathbb{P}^r & \xrightarrow{\mathbb{P}(\phi_\alpha) \circ \mathbb{P}(\phi_\beta)^{-1}} & (U_\alpha \cap U_\beta) \times \mathbb{P}^r \\ & {\scriptstyle \mathbb{P}(\phi_\beta)} \nwarrow \quad \nearrow {\scriptstyle \mathbb{P}(\phi_\alpha)} & \\ & \mathbb{P}(\pi^{-1}(U_\alpha \cap U_\beta)) & \end{array}$$

and so we have a well-defined complex manifold structure on $\mathbb{P}(E)$ which makes the map $\mathbb{P}(\pi)$ defined by the diagram:

$$\begin{array}{ccc} E & \xrightarrow{\pi} & X \\ & {\scriptstyle p_E} \searrow \quad \nearrow {\scriptstyle \mathbb{P}(\pi)} & \\ & \mathbb{P}(E) & \end{array}$$

holomorphic.

Definition 3.1.1 *The pair* $(\mathbb{P}(E), \mathbb{P}(\pi))$ *is called the* ***projective bundle*** *on* X, *associated to the vector bundle* (E, π).

We can generalize the tautological bundle on $\mathbb{CP}^r$ to $\mathbb{P}(E)$. Let $V := \mathbb{P}(\pi)^* E$ be the pull-back of the line bundle E to $\mathbb{P}(E)$. Let $q : V \to \mathbb{P}(E)$ be the pull-back $\mathbb{P}(\pi)^*(\pi)$. For every $x \in X$, define $\mathcal{O}_{\mathbb{P}(E)}(-1)_x$ to be the union of all pairs $q(l) \times l \subseteq \mathbb{P}(E)_x \times V_x$ where l is a line through origin in the fiber V_x over x and $\mathbb{P}(E)_x$ is the fiber of $\mathbb{P}(E)$ over x, and let $\mathcal{O}_{\mathbb{P}(E)}(-1) := \amalg_{x \in X} \mathcal{O}_{\mathbb{P}(E)}(-1)_x$., and we define the complex manifold structure on $\mathcal{O}_{\mathbb{P}(E)}(-1)$ in the natural way and it becomes a holomorphic bundle on $\mathbb{P}(E)$.

Definition 3.1.2 *The bundle* $\mathcal{O}_{\mathbb{P}(E)}(-1)$ *on* $\mathbb{P}(E)$ *is called the* ***tautological bundle****. The dual of the bundle* $\mathcal{O}_{\mathbb{P}(E)}(-1)$ *is denoted by* $\mathcal{O}_{\mathbb{P}(E)}(+1)$, *and we call it the* ***twisted bundle of Serre***

Proposition 3.1.3 *Let* (X, ω_X) *be a compact Kähler manifold and E be a holomorphic bundle on X, then the manifold* $\mathbb{P}(E)$ *is Kähler.*

Proof Let h be a Hermitian metric on E. Then h induces a Hermitian metric on both V and therefore induces a Hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(-1)$ by restriction. The curvature form ω_E of the induced metric on the dual $\mathcal{O}_{\mathbb{P}(E)}(+1)$ is a form on $\mathbb{P}(E)$ whose restriction on each fiber $\mathbb{P}(\pi)^{-1}(x)$ is positive (for this restriction is the Fubini-Study metric on $\mathbb{P}(E_x)$ coming from $(E_x, h_x) \cong (\mathbb{C}^{r+1}, h_{standard})$), but it may not be positive everywhere. Consider the unit spheres $S(E_x) := \{v \in E_x : h_x(v, v) = 1\}$ of each fibers E_x. Consider only the (biholomorphic) transition maps that take values in $U(r+1, \mathbb{C})$ (these maps actually take values in $SU(r+1)$); i.e. the transition maps preserve the Hermitian metric, (and therefore map unit vectors of one fiber to the unit vectors of the other fiber of E). Then by these transition maps the spaces $S(E_x)$ glue over X, to give a set $S(E)$, a map $s : S(E) \to X$ and trivializing charts $\psi_\alpha : s^{-1}(U_\alpha) \to U_\alpha \times \mathbb{S}^r_{\mathbb{C}}$, where $(\mathbb{S}^r_C := \{z \in \mathbb{C}^{r+1} : h_{standard}(z, z) = 1\})$ giving $S(E)$ the structure of a manifold and making s smooth. $S(E)$ is called the **sphere bundle** over X, associated to E, and if X is compact, $S(E)$ is also compact and $\mathbb{P}(E)$ is a quotient space of $S(E)$ in the natural way, so $\mathbb{P}(E)$ is compact. Being a form on a compact manifold ω_E is bounded; (i.e. bounded on the sphere bundle $S(\bigwedge^2 T^*_{\mathbb{P}(E)})$; where $T^*_{\mathbb{P}(E)})$ is the cotangent bundle on $\mathbb{P}(E)$). Thus, for sufficiently large $\lambda >> 0$, the form

$$\omega := \omega_E + \lambda \cdot \mathbb{P}(\pi)^* \omega_X$$

is a real closed form of type $(1, 1)$, which is positive on $\mathbb{P}(E)$. □

3.2 Blowups

Let Y be a complex submanifold of codimension $k$of a complex manifold of dimension n. For every $y \in Y$, there is a complex chart U around y and a collection of k-holomorphic functions $(f_1, ..., f_k)$ from U to $\mathbb{C}$ with independent differentials such that $U \cap Y = \{z \in U : f_1(z) = \cdots = f_k(z) = 0\}$. This set of functions is called the **system of local equations of Y around y**. We have the following proposition:

Proposition 3.2.1 *Let $y \in Y$. Let $f := (f_1, ..., f_k)$ and $g := (g_1, ..., g_k)$ be the system of local equations around the open sets U and V respectively around y, then there exist a matrix $M := (M_{i,j})_{1 \le i,j \le k}$ of functions holomorphic on $U \cap V$ and invertible along $U \cap V \cap Y$. Moreover, the restriction of the entries of this matrix to Y is uniquely determined by the local equations f and g.*

Proof Without loss of generality, consider the elements of f to be the first k complex coordinates $(z_1, ..., z_k)$ in $U \cap V$. Consider the power series of the functions g_i given by

$$g_i(z_1, ..., z_n) = c_0^{i,0} + (c_1^{i,1} z_1 + \cdots + c_n^{i,1} z_n) + (c_{1,1}^{i,2} z_1^2 + c_{1,2}^{i,2} z_1 z_2 \cdots + c_{n,n}^{i,n} z_n^2) + \cdots$$

Then

$$0 = g(z) = g(0, ..., 0, z_{k+1}, ..., z_n) =$$

$$= c_0^{i,0} + (c_{k+1}^{i,1} z_{k+1} + \cdots + c_n^{i,1} z_n) + (c_{k+1,k+1}^{i,2} z_{k+1}^2 + c_{k+1,k+2}^{i,2} z_{k+1} z_{k+2} \cdots + c_{n,n}^{i,n} z_n^2) + \cdots$$

$\forall z \in \{w \in U \cap V \cap Y : w_1 = \cdots = w_k = 0\}$. Thus all the coefficients of the above equation are zero and therefore there exist holomorphic functions $N_{i,j}(z_1, ..., z_n)$ $i, j \in \mathbb{N}_k$ such that

$$g_i(z_1, ..., z_n) = \sum_{j=1}^{k} N_{j,i}(z_1, ..., z_n) z_j.$$

Thus there are holomorphic functions $M_{i,j} : U \cap V \to \mathbb{C}$ for $i, j \in \mathbb{N}_k$ such that

$$g_i = \sum_{j=1}^{k} M_{j,i} f_j$$

Taking differentials,

$$dg_i = \sum_{j=1}^{k} M_{j,i} df_j + \sum_{j=1}^{k} f_j \cdot dM_{j,i}$$

and so,

$$dg_i|_Y = \sum_{j=1}^{k} (M_{j,i}|_Y \cdot df_j|Y)$$

The uniqueness follows from the fact that $(df_j|_Y)_j$ and $(dg_i|_Y)_i$ are independent on $U \cap V \cap Y$. □

Definition 3.2.2 ***(Conormal bundle)***

*Choose an open cover $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of X by open sets U_α of X, such that for every $\alpha \in I$, $\exists$ a system of local equations $f^\alpha = (f_1^\alpha, ..., f_k^\alpha)$ of Y, on U_α. Then by the above Proposition $\exists$ matrices $M^{\alpha,\beta} := (M_{i,j}^{\alpha,\beta})_{i,j}$ of holomorphic functions defined uniquely and invertible over Y, satisfying $df_i^\alpha|_Y = \sum_{j=1}^{k} M_{j,i}^{\alpha,\beta}|_Y \cdot df_j^\beta|_Y$. So, we see that the spaces $U_\alpha \times span_{\mathbb{C}}\{f_1^\alpha, ..., f_k^\alpha\}$ glue together to give a vector bundle $N^*_{Y/X}$ via. the transition functions $M^{\alpha,\beta}|_Y = (M_{i,j}^{\alpha,\beta}|_Y)_{i,j}$. The fibers $N^*_{Y/X,y}$ of this vector bundle consists of all the complex linear forms on $T_{X,y}$ which is zero on $T_{Y,y}$, for every $y \in Y$. This vector bundle is called the **conormal bundle** of Y in X.*

In the case of the above definition and $U \in \mathcal{U}$, with $f = (f_1, ..., f_k)$, being the system of local equations on U of $Y \hookrightarrow X$. Define

$$\tilde{U}_Y := \left\{(Z, z) \in \mathbb{CP}^{k-1} \times U : Z_i f_j(z) = Z_j f_i(z) \;\; \forall (i, j) \in \mathbb{N}_k\right\}$$

where $Z = [Z_1 : \cdots : Z_k]$ (i.e. Z is the line in $\mathbb{CP}^k$ that passes through origin and $(Z_1, ..., Z_k)$) and $Z_j \neq 0$ for some $j \in \mathbb{N}_k$. Define $\tau_U : \tilde{U}_Y \to U$ given by the restriction of the projection map $\mathbb{CP}^{k-1} \times U \to U$. Clearly, $\tilde{U}_Y$ is a complex submanifold of $\mathbb{CP}^{k-1} \times U$, and the map τ_U is holomorphic. By defining the map $\psi_U : U \setminus Y \to \tau_U^{-1}(U \setminus Y)$ defined by $\psi_U(z) = ([f_1(z) : \cdots : f_k(z)], z) \;\; \forall z \in U \setminus Y$, we see that $\tau_U|_{\tau_U^{-1}(U \setminus Y)}$ and ψ_U are holomorphic maps inverse to each other. We also see that $\tau_U^{-1}(y) = \mathbb{CP}^{k-1} \times \{y\}$, for every $y \in Y \cap U$.

Now let $U, V \in \mathcal{U}$, and let $f = (f_1, ..., f_k)$ and $g = (g_1, ..., g_k)$ be the system of local equations on U, and V respectively. Then by the last proposition there is a matrix $M^{U,V} = (M^{U,V}_{i,j})_{i,j}$ of holomorphic functions on $U \cap V \to \mathbb{C}$ that are uniquely determined and invertible on $U \cap V \cap Y$, such that

$$f_i^U = \sum_{j=1}^{k} M^{U,V}_{j,i} \cdot f_j^V \tag{3.1}$$

on $U \cap V \cap Y$. Then the equation 3.1 above defines $P^{U,V} : \tau_V^{-1}((U \cap V) \setminus Y) \to \tau_U^{-1}((U \cap V) \setminus Y)$ by

$$\begin{aligned} P^{U,V}([f_1^V(z) : \cdots f_k^V(z)], z) &= ([\sum_{j=1}^{k} M^{U,V}_{j,1} \cdot f_j^V : \cdots : \sum_{j=1}^{k} M^{U,V}_{j,k} \cdot f_j^V], z) \\ &= ([f_1^U(z) : \cdots f_k^U(z)], z) \end{aligned}$$

and we have that $P^{U,V}$ satisfies the commutative diagram,

$$\begin{array}{ccc} \tau_U^{-1}((U \cap V) \setminus Y) & \xrightarrow{P^{U,V}} & \tau_V^{-1}((U \cap V) \setminus Y) \\ & \searrow^{\tau_U} \quad \swarrow_{\tau_V} & \\ & (U \cap V) \setminus Y & \end{array}$$

and $P^{U,V}$ is nothing but $\mathbb{P}({}^tM^{U,V})$ and therefore $P^{U,V-1} = \mathbb{P}({}^tM^{U,V})^{-1}$ defines biholomorphic maps $\tau_U^{-1}((U \cap V) \setminus Y) \to \tau_V^{-1}((U \cap V) \setminus Y)$ which extend to a biholomorphic maps

$$\tau_U^{-1}(U \cap V) \to \tau_V^{-1}(U \cap V)$$

by continuity, which is again biholomorphic by Hartog's theorem.

Definition 3.2.3 *The spaces $\tilde{U}_Y$ and the maps $\tau_U : \tilde{U}_Y \to U$ defined above glue together over X, via. the transition maps $\mathbb{P}({}^tM^{U,V})^{-1}$, for $U, V \in \mathcal{U}$ to give a complex manifold $\tilde{X}_Y$ and a holomorphic map $\tau : \tilde{X}_Y \to X$. The pair $(\tilde{X}_Y, \tau)$ is called the* ***Blowup*** *of X along Y.*

Note:

1. The transition maps $\mathbb{P}({}^tM^{U,V})^{-1} = \mathbb{P}(({}^tM^{U,V})^{-1})$ are also the transition maps of the projective bundle corresponding to the **Normal Bundle** $N_{Y/X}$, where the normal bundle is defined to be the dual of the conormal bundle $N^*_{Y/X}$ of Y in X. So $\tau^{-1}(Y) \cong \mathbb{P}(N_{Y/X})$.

2. $\tau^{-1}(Y)$ is a complex submanifold of $\tilde{X}_Y$ of codimension 1, (a complex submanifold of codimension 1 is called a **Hypersurface** or a **Holomorphic divisor**). Because with notations as in Definition 3.2.2 if $U \in \mathcal{U}$ and $f^U = (f_1^U, ..., f_k^U)$ is the system of local equation of Y in X, then there is some $i \in \mathbb{N}_k$ such that $f_i \circ \tau(u) = 0 \Rightarrow f_j \circ \tau(u)$ for all $j \in \mathbb{N}_k$. We now prove the following lemma:

Lemma 4 *$\exists$ a holomorphic line bundle L on $\tilde{X}_Y$ which restricts to the twisted bundle $\mathcal{O}_{\mathbb{P}(N_{Y/X})}(+1)$ on $\tau^{-1}(Y)$, and it is trivial outside $\tau^{-1}(Y)$.* □

PROOF Let D be a holomorphic divisor of a complex manifold X. Let $\mathcal{V}$ be a covering of X by open sets, such that for every $U \in \mathcal{V}$, there is a holomorphic function $f^V : V \to \mathbb{C}$, such that $V \cap Y = \ f^V = 0$. We set $X \setminus D \in \mathcal{V}$ and $f^{X \setminus D} = 1$ on $X \setminus D$. By Equality 3.1 and Proposition 3.2.1 we get that the transition functions $g^{U,V} : U \cap V \to \mathbb{C}$ of the conormal bundle of $D \hookrightarrow X$, are given by $g^{U,V} = f_U/f_V$ and are invertible on $U \cap V$. Let $\mathcal{O}_X(-D)$ be the holomorphic line bundle with the transition functions $g^{U,V} \quad U, V \in \mathcal{V}$ then we call $\mathcal{O}_X(-D)$ the **Line Bundle associated to the divisor** D. Then this line bundle is trivial outside X. Now, by 3.1 and Proposition 3.2.1 we have $df^U = g^{U,V} df^V$ along D, so $\mathcal{O}_X(-D)|_D$ is isomorphic to the conormal bundle of $D \hookrightarrow X$.

We now prove the following claim:

Claim: The line bundle $\mathcal{O}_{X_Y}(-\tau^{-1}(Y))$ restricts to the twisted bundle $\mathcal{O}_{\mathbb{P}(N_{Y/X})}(+1)$ on $\tau^{-1}(Y) \cong \mathbb{P}(N_{Y/X})$.

Proof of the claim: By the arguments above we are reduced to showing that $N_{\tau^{-1}(Y)/\tilde{X}_Y} \cong \mathcal{O}_{\mathbb{P}(N_{Y/X})}(-1)$. The differential $\tau_* : T_{\tilde{X}_Y} \to \tau^* T_X$ induces a map $\tau_* : N_{\tau^{-1}(Y)/\tilde{X}_Y} \to \tau^* T_X$ and we check that this map is injective and gives the required isomorphism onto $\mathcal{O}_{\mathbb{P}(N_{Y/X})}(-1) \subseteq \tau^* T_X$ This follows from the identification $\tau^{-1}(Y) \cong \mathbb{P}(N_{Y/X})$. □We now show the next theorem:

Theorem 4 *The manifold $\tilde{X}_Y$ of the blowup $(\tilde{X}_Y, \tau)$ of a Kähler manifold (X, ω_X) by a compact complex submanifold Y is Kähler, and it is compact if X is compact.* □

PROOF The pull-back $\tau^*(\omega_X)$ of the Kähler form is positive outside $\tau^{-1}(Y)$, but only semi-positive along $\tau^{-1}(Y)$, for

$$ker(\tau^*(\omega_X)|_{\tau^{-1}(Y)}) = \amalg_{y \in Y} T_{\tau^{-1}(y)}$$

Now by the last lemma, $\exists$ a holomorphic line bundle L on $\tilde{X}_Y$ which restricts to the twisted bundle $\mathcal{O}_{\mathbb{P}(N_{Y/X})}(+1)$ on $\tau^{-1}(Y)$, and it is trivial outside $\tau^{-1}(Y)$. By partition of unity we can extend the Fubini-Study metric h on $\mathcal{O}_{\mathbb{P}(N_{Y/X})}(+1)$ to a Hermitian metric on L, which is the pull-back of the standard metric on $\mathbb{C}$ by the trivialization over $\tilde{X}_Y \setminus \tau^{-1}(Y)$, outside a compact neighbourhood of $\tau^{-1}(Y)$. The curvature form $\omega_L := \Theta(L, h_L)$ is the symplectic form of the Fubini-Study metric on $\tau^{-1}(Y) \cong \mathbb{P}(N_{Y/X})(+1)$ and zero outside a compact neighbourhood of $\tau^{-1}(Y)$ Moreover, ω_L is strictly positive on the bundle $ker(\tau_*)$ over the blow-up, and

$$ker(\tau^*(\omega_X)|_{\tau^{-1}(Y)}) = \amalg_{y \in Y} T_{\tau^{-1}(y)}$$

, we get that $supp(\omega_L)$ is compact and therefore there is some $\lambda >> 0$, such that

$$\omega := \lambda \cdot \tau^*(\omega_X) + \omega_L$$

is positive, and real closed of type $(1,1)$ and $(\tilde{X}_Y, \omega)$ is a Kähler manifold.
Since $\tilde{X}_Y$ is Hausdorff and X is locally compact Hausdorff and τ is closed and has compact fibers, we see that τ is a proper map; i.e. inverse image of compact subspaces is compact under τ. Thus $\tilde{X}_Y$ is compact if and only if X is compact. $\square$

Chapter 4

Harmonic Forms and Hodge Decomposition

Convention: In this chapter we will consider only smooth oriented manifolds. So whenever we say that M is a manifold, it means it is a smooth oriented manifold. In case it is a complex manifold we will mention it.

4.1 Hodge star operators and Laplacians

Convention: In this chapter we will consider only oriented manifolds. So whenever we say that M is a manifold, it means it is a oriented manifold.

4.1.1 The Hodge star operator

Let X be a differentiable Riemannian manifold with Riemann metric g. Let $e_1, ..e_n$ be an orthonormal basis of $T_{\mathbb{R},x}$ Then for this metric on $\mathcal{A}^k_{X,\mathbb{R}}$, and $e_1, ..e_n$ be an orthonormal basis of $T_{\mathbb{R},x}$, $e_1^* \wedge ... \wedge e_n^*$ is an orthonormal basis of $\mathcal{A}^k_{\mathbb{R},x}$. Now assume that X is oriented, $[X]$ be the orientation class of X and let Vol be the volume form of X relative to g. The L^2-metric on the space $A^k_{cs}(X)$ of C^∞-differential forms with compact support on X is defined by

$$\langle \alpha, \beta \rangle_{g,L^2} := \int_{[X]} g(\alpha, \beta)\ Vol \quad \forall \alpha, \beta \in A^k_{cs}(X)$$

. The pairing $\mathcal{A}^k_{\mathbb{R},x} \otimes_{\mathbb{R}} \mathcal{A}^{n-k}_{\mathbb{R},x} \to \mathcal{A}^n_{\mathbb{R},x} \cong \mathbb{R}$ given by the wedge product, gives an isomorphism $p : \mathcal{A}^{n-k}_{\mathbb{R},x} \overset{\cong}{\to} Hom(\mathcal{A}^k_{\mathbb{R},x}, \mathbb{R})$. We also have the $\mathbb{R}$-isomorphism of $\mathcal{A}^k_{\mathbb{R},x}$

with its dual, so we get an isomorphism of sections

$$*_x : \mathcal{A}^k_{\mathbb{R},x} \overset{\cong}{\to} \mathcal{A}^{n-k}_{\mathbb{R},x}.$$

which glues as a C^∞-map giving an isomorphism of bundles. This operator is called the **Hodge star operator**. We can consider this operator on the sheaves of sections of its domain bundle and codomain bundle.
This operator has the following essential property:

Proposition 4.1.1 $\langle \alpha, \beta \rangle_{g,L^2} = \int_{[X]} \alpha \wedge *(\beta) \quad \forall \alpha, \beta \in A^k_{cs}(X)$

But now if X is also a complex manifold, the metric g and the almost complex structure J on X together give a hermitian metric, say h on X and we have a L^2-metric (is this L^2 ?)

$$\langle \alpha, \beta \rangle_{h,L^2} := \int_{[X]} h(\alpha, \beta)\ Vol \quad \forall \alpha, \beta \in (A_{\mathbb{C}})^k_{cs}(X)$$

Now this h gives a $\mathbb{C}$-conjugate linear isomorphism of $\mathcal{A}^k_{\mathbb{C},x}$ with its dual. Now if we extend $*$ by $\mathbb{C}$-linearity, we get a $\mathbb{C}$-isomorphism

$$* : \mathcal{A}^k_{X,\mathbb{C}} \overset{\cong}{\to} \mathcal{A}^{n-k}_{X,\mathbb{C}}.$$

but not a $\mathbb{C}$-antilinear isomorphism. So vaguely saying "$*$ does not capture the structure provided by h, it only captures the structure provided by g". But if we define $\overline{*}(\beta) := \overline{*(\beta)}\ \forall \beta \in (A_{\mathbb{C}})^k_{cs}(X)$, then the operator $\overline{*}$ "does capture the structure given by h" and we get the following proposition:

Proposition 4.1.2 $\langle \alpha, \beta \rangle_{h,L^2} = \int_{[X]} \alpha \wedge \overline{*}(\beta) \quad \forall \alpha, \beta \in (A_{\mathbb{C}})^k_{cs}(X)$

This operator $\overline{*}$ is also considered as the *Hodge star operator* but here we will call it the **Conjugate star operator**
We have seen that for a complex manifold X, the conjugate star operator is more closely related to its complex structure than the (real) star operator. Having this in mind we define the star operator for holomorphic hermitian vector bundles on a complex manifold followingly:
Let (E, h_E) be a holomorphic hermitian vector bundle on a complex manifold X, with the Dolbeault operator $\overline{\partial}_E$. Then this also defines a $\mathrm{L}^2 - metric$:

$\langle \alpha, \beta \rangle_{h_E, L^2} := \int_{[X]} h_E(\alpha, \beta)\ Vol \quad \forall \alpha, \beta \in (A_{E,\mathbb{C}})^{0,q}_{cs}$
Then the pairing:

$$(\mathcal{A}^{0,q}_{\mathbb{X}} \underset{\mathbb{C}}{\otimes} E) \underset{\mathbb{C}}{\otimes} (\mathcal{A}^{n,n-q}_X \underset{\mathbb{C}}{\otimes} E^*) \to \mathcal{A}^{n,n}_X \cong \mathbb{C}$$

makes $\mathcal{A}^{n,n-q}_X \otimes_{\mathbb{C}} E^*$ dual to $\mathcal{A}^{0,q}_{\mathbb{X}} \otimes_{\mathbb{C}} E$ as $\mathbb{C}$-vector spaces. Now by the $\mathbb{C}$-conjugate linear isomorphism of $\Omega^{0,q}_{\mathbb{X}} \otimes_{\mathbb{C}} E$ with its dual. We define the **Hodge star operator** $*_E$, of the bundle E by the conjugate isomorphism:

$$*_E : \mathcal{A}^{0,q}_{\mathbb{X}} \underset{\mathbb{C}}{\otimes} E \to \mathcal{A}^{n,n-q}_X \underset{\mathbb{C}}{\otimes} E^*.$$

Consider the canonical bundle $K_X := \mathcal{A}^{n,0}_X$ on X. Then this is a holomorphic line bundle on X. Moreover,

$$\mathcal{A}^{n,n-q}_X \underset{\mathbb{C}}{\otimes} E^* \cong \mathcal{A}^{0,n-q}_X \underset{\mathbb{C}}{\otimes} (K_X \underset{\mathbb{C}}{\otimes} E^*)$$

Now since $\mathcal{A}^{p,q}_X = \mathcal{A}^{o,q}_X \otimes \mathcal{A}^{p,0}_X$. We can take $E = \Omega^{p,0}_X$. Then we get $\overline{\partial_E} = (-1)^p \overline{\partial}$. This also satisfies the similar proposition:

Proposition 4.1.3 $\langle \alpha, \beta \rangle_{h_E, L^2} = \int_{[X]} \alpha \wedge *_E(\beta) \quad \forall \alpha, \beta \in (\mathcal{A}_X)^{0,q}_{cs}$
Moreover, if we define $\overline{\partial_E}^* := (-1)^q {*_E}^{-1} \circ \partial_{K_X \otimes E^*} \circ *_E$. *Then* $\overline{\partial_E}^*$ *is the (formal) dual of* $\overline{\partial_E}$ *w.r.t. the* L^2 *metric induced by* h_E

Moreover, we have $\overline{\partial}^* = (-1)^p \frac{1}{2} \overline{\partial_E}^*$, where the coefficient 2 comes from the relation $2^k h^{(k)} = \Sigma h^{p,q}$. Now just replace E by $\mathcal{A}^{p,0}_X \otimes_{\mathbb{C}} E$

One of the great things the Hodge star operators do is to provide us with the formal duals of operators for which come paired with another operator, which we call its Quasi-adjoint here, such that these two together satisfy a certain condition which we will mention below. We are lucky that most of the operators we study in this Chapter have a Quasi-adjoint (mostly they are Quasi-adjoint to themselves):

4.1.2 Operators with a Quasi-adjoint and their Formal Duals

Let X be a Riemannian manifold of dimension n and let $V = T^*_{\mathbb{R},X}$ be the real cotangent bundle of X, and let $W = V \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified cotangent bundle. If X is a complex manifold we have the decomposition $\wedge^r W = \bigoplus_{p+q=r} W^{p,q}$, where $W^{p,q}$ is the set of all (p,q)-forms.

Let $\bigwedge(V) := \mathbb{R} \oplus \wedge^1 V \oplus \wedge^2 V \oplus \cdots$. and $\bigwedge(W) := \bigwedge(W) \otimes_{\mathbb{R}} \mathbb{C}$ Then under wedge product of forms $\bigwedge(V)$ and $\bigwedge(W)$ form what we call a **graded commutative algebra** (not to be confused with a commutative algebra which is graded) which means for $\alpha \in \wedge^k W$ and $\beta \in \wedge^l W$ we have $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$. for all $k, l \geq 0$.

Every element of $\wedge^k W$ can be called **homogeneous (differential) form** of degree k, for all non-negative k.

Let $\sqcap_r : \bigwedge(V) \to \wedge^r V$ and $\sqcap_{p,q} : \bigwedge(W) \to \wedge^{p,q} W$ be projection maps. We denote the complexification of $\sqcap_r$ by $\sqcap_r$ too. We define the following maps:

(a) Define $\kappa : \bigwedge(V) \to \bigwedge(V)$ by $\kappa := \sum_{r \geq 0} (-1)^{nr+r} \sqcap_r$ and we see at once that $*^2 = \kappa$ and $\kappa^2 = 1$

(b) Any homogeneous form η of degree r ($r \geq 0$) defines an operator $e(\eta) := \eta \wedge (*) : \bigwedge(W) \to \bigwedge(W)$

(c) When X is a complex manifold, define $J : \bigwedge(W) \to \bigwedge(W)$ by $J := \sum_{p \geq 0, q \geq 0} i^{p-q} \sqcap_{p,q}$ and we see that $J^2 = \kappa$.

Definition 4.1.4 *Let $r, s \in \mathbb{Z}$ and $T : \mathcal{A}^k(X) \to \mathcal{A}^{k+r}(X)$ be a $\mathbb{R}$-linear map. Then here we call, another $\mathbb{R}$-linear operator $S : \mathcal{A}^k(X) \to \mathcal{A}^{k+r}(X)$ to be the* ***Quasi-adjoint*** *to T if $\exists c : \mathbb{Z} \to \mathbb{R}$ such that*

$$\int_X T\alpha \wedge \beta = c(d^0 \beta) \int_X \alpha \wedge S\beta$$

for all α and β which are differential forms of appropriate orders and compact support. Then we define the (formal) dual of T, denoted by $T^* : \mathcal{A}^k(X) \to \mathcal{A}^{k-r}(X)$ as:

$$T^* := c(k) \cdot *^{-1} \circ S \circ * = c(k) \cdot \kappa \circ (*S*).$$

When X is a complex Hermitian manifold, a $\mathbb{C}$-linear map $T : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+r,q+s}(X)$ is ***Quasi- adjoint*** *to another $\mathbb{C}$-linear map $S : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+r,q+s}(X)$ if it satisfies a relation similar to above where c can be a complex number and it has its dual defined as above except the fact that we use the conjugate star operator $\overline{*}$ in place of $*$.*

The formal duals in the case of maps between differential forms with E coefficients where E is a holomorphic Hermitian vector bundle on a complex hermitian manifold is defined similarly using the Hodge star operator associated to the bundle and in this case also c is a complex number.

The name dual comes from the following proposition:

Proposition 4.1.5 *Let X be a complex hermitian manifold. Let $T : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+r,q+s}(X)$ be Quasi-adjoint to the $\mathbb{C}$-linear map $S : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+r,q+s}(X)$. Then*

$$\langle T\alpha, \beta\rangle = \langle \alpha, T^*\beta\rangle$$

, for all α and β which are differential forms of appropriate type and compact support. The inner product above is the L^2-metric defined by the Hermitian metric h. The other (formal) duals also satisfy similar equalities. The only thing is that for real operators, the L^2-metric is defined by the Riemannian metric and for operators on vector valued differential forms on a complex Hermitian manifold, the L^2-metric comes from the Hermitian metric of the holomorphic vector bundle and the base space.

Proof $\langle T\alpha, \beta\rangle Vol. = \int_X T\alpha \wedge \overline{*}\beta = c \cdot \int_X \alpha \wedge S\ \overline{*}\beta = c \cdot \int_X \alpha \wedge \overline{*}^{-1} S\ \overline{*}\beta = \langle \alpha, T^*\beta\rangle Vol. \Rightarrow \langle T\alpha, \beta\rangle = \langle \alpha, T^*\beta\rangle$, for differential forms α and β of appropriate type.

Let X be a compact complex manifold and let ω be a Kähler form on X. Let

$$L := \omega \wedge (\cdot) : \mathcal{A}_X^k \to \mathcal{A}_X^{k+2} \quad : \quad \alpha \longmapsto \omega \wedge \alpha$$

We call this operator the **Lefschetz operator**.

Clearly, after complexifying L, $L = e(\omega)$.

The following corollary of the above proposition tell us what are the duals of certain useful operators.

Corollary 4.1.6

(a) Let X be a real Riemannian manifold with Riemannian metric g. The dual d^* of the exterior derivative $d : \mathcal{A}_X^k \to \mathcal{A}_X^{k+1}$ is given by

$$d^* = (-1)^k *^{-1} d*$$

(b) Let η be a homogeneous form of degree r ($r \geq 0$). The dual $e(\eta)^*$ of the operator $e(\eta) := \eta \wedge (*) : \wedge^k W \to \wedge^{k+r} W$ is given by

$$e(\eta)^* = (-1)^{r(dim_{\mathbb{R}} X - k - r)} (\overline{*}^{-1} e(\eta) \overline{*}).$$

(c) Let X be a complex Hermitian manifold with Hermitian metric h. The dual ∂^* *of the operator* $\partial : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p+1,q}$ *is given by*

$$\partial^* = (-1)^k \overline{*}^{-1} \partial \overline{*}$$

(d) Let X be a complex Hermitian manifold with Hermitian metric h. The dual $\overline{\partial}^*$ *of the operator* $\overline{\partial} : \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p,q+1}$ *is given by*

$$\overline{\partial}^* = (-1)^k \overline{*}^{-1} \overline{\partial} \overline{*}$$

(e) Let X be a complex Hermitian manifold with Hermitian metric h and let (E, h_E) *be the Hermitian holomorphic vector bundle X. The dual* $\overline{\partial}_E^*$ *of the operator* $\overline{\partial}_E$: $\mathcal{A}_E^{p,q} \to \mathcal{A}_E^{p,q+1}$ *is given by*

$$\overline{\partial}_E^* = (-1)^k \overline{*}_E^{-1} \overline{\partial}_E \overline{*}_E$$

(f) Let X be a compact complex manifold and let ω *be a Kähler form on X. The dual* L^* *of the Lefschetz operator* $L = \omega \wedge (\cdot) : \mathcal{A}_X^k \to \mathcal{A}_X^{k+2}$ *is given by* $L^* = \overline{*}^{-1} L \overline{*}$

Proof We will only prove 1 and 2. Clearly, 5. follows from 2. The proof of others are similar to that of 1.

Proof of 1 : Clearly, $\int_X d\alpha \wedge \beta = (-1)^{d^0\alpha+1} \int_X \alpha \wedge d\beta$ for all α and β which are differential forms of appropriate orders and compact support. This follows from Leibnitz rule for d and the Stokes' Theorem: Leibnitz rule gives us, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{d^0\alpha} \alpha \wedge d\beta$. Now by Stokes' Theorem $\int_X d(\alpha \wedge \beta) = 0$. Now, $d^0\beta = d^0\alpha + 1$. This proves 1.

Proof of 2 : Clearly,
$e(\eta)\alpha \wedge \beta = \eta \wedge \alpha \wedge \beta = (-1)^{d^0\alpha \cdot d^0\eta}(\alpha \wedge \eta \wedge \beta) = (-1)^{d^0\alpha \cdot d^0\eta}(\alpha \wedge e(\eta)\beta$ and $d^0\alpha \cdot d^0\eta = r(dim_{\mathbb{R}} X - k - r)$ therefore $\int_X e(\eta)\alpha \wedge \beta = (-1)^{r(dim_{\mathbb{R}} X - k - r)} \int_X \alpha \wedge e(\eta)\beta$
for any two complex differential forms α and β such that $d^0\alpha + d^0\beta + r = dim_{\mathbb{R}} X$. This proves 2. □

4.1.3 Computing the Hodge star operator locally on a Manifold

Let n be a positive integer. For every $p \in \mathbb{N}_n$, let

$$I_{p,n} := \{\mu = (\mu_1, ...\mu_p) \in \mathbb{N}^p \mid \mu_1 < \cdots < \mu_p\}$$

$$I_n := \amalg_{p \in \mathbb{N}_n} I_{p,n}$$

The set I_n (resp. $I_{p,n}$) is called the set of all strictly increasing multinidices (resp. multiindices of length p) chosen from 1,..,n. Suppose $\mu = (\mu_1, ...\mu_p) \in I_n$. Then we say that the integer p is the length of μ and write $|\mu| = p$. We also write $\mu_i \in \mu$ to mean that μ_i is a component of μ; i.e. we treat the multiindices as "ordered sets".

Real Manifolds with respect to the real co-ordinate chart

Let (X, g) be a Riemannian manifold of dimension n. Let $(U, \phi = (x_1, ..., x_n))$ be a chart of (X, g) and let V be the cotangent bundle over U with ϕ giving an oriented frame $dx = (dx_1, ..., dx_n)$ of V on U. For any $\mu \in I_p$, define $dx_\mu := dx_{\mu_1} \wedge \cdots \wedge dx_{\mu_p}$
Let $M(g) = (g_{i,j})$ be the matrix of g w.r.t. the basis x. Let $M(g)^{-1} = (g^{i,j})$
For any $\alpha = \sum_\mu \alpha_\mu dx_\mu$, define

$$\alpha^\mu = \alpha^{\mu_1,...,\mu_p} = g^{\mu_1,\nu_1} \cdots g^{\mu_p,\nu_p} \alpha_{\nu_1,..,\nu_p}$$

, where we follow the Einstein summation convention. Then if we declare, $*\alpha = \sum_{\eta=(\eta_1,...,\eta_{n-p})} (*\alpha_\eta) dx_\eta$ (where (μ, η) is a permutation of $(1, 2, .., n)$), we can find the coefficients $(*\alpha_\eta)$ by the following formula which is easy to show:

Lemma 5 $(*\alpha_\eta) = (sgn(\mu, \eta))\sqrt{detM(g)}\alpha^\mu$ □

Complex manifolds with respect to the complex co-ordinate chart

We now introduce a way to compute $*\zeta$ for every complex differential form ζ on some open subset of a complex manifold X of complex dimension n. Let U be a holomorphic chart of X. Let V be the real cotangent bundle on U and $W = V \otimes_\mathbb{R} \mathbb{C}$, $x_1, y_1, ..., x_n, y_n$ be a basis of V and $z_1, ..., z_n$ be the basis of $W^{1,0}$ given by $z_i = x_i + \sqrt{-1}y_i \quad \forall i = 1, ..., n$ and let $\overline{z}_i = x_i - \sqrt{-1}y_i \quad \forall i = 1, ..., n$.
For any $p \in \mathbb{N}_n$ and $\mu = (\mu_1, ..\mu_p) \in I_{p,n}$, define:

$$z_\mu := z_{\mu_1} \wedge ... \wedge z_{\mu_p}$$

$$x_\mu := x_{\mu_1} \wedge ... \wedge x_{\mu_p}$$

$$...etc. \quad and$$

$$\omega_\mu := \bigwedge_{1 \leqslant i \leqslant p} (z_{\mu_i} \wedge \overline{z}_{\mu_i})$$

We see that any form in $\bigwedge^* W = \mathbb{R} \oplus \wedge^1 W \oplus \cdots$ is a complex linear combination of the terms $z_A \wedge \overline{z}_B \wedge \omega_M$ for $(A, B, M) \in I_n^3$ such that A, B and M are mutually disjoint. With the help of this we can compute the Hodge star operator:

Lemma 6 $*(z_A \wedge \overline{z}_B \wedge \omega_M) = \lambda(a, b, m) z_A \wedge \overline{z}_B \wedge \omega_{M'}$,
where A, B, M are mutually disjoint multinidices in I_n, $a = |A|$, $b = |B|$, $m = |M|$, and $M' = \mathbb{N}_n \setminus (A \cup B \cup M)$ and

$$\lambda(a, b, m) = i^{a-b}(-1)^{\frac{p(p+1)}{2}+m}(-2i)^{p-n}$$

where $p = a + b + 2n$ is the total degree of $z_A \wedge \overline{z}_B \wedge \omega_M$. □

4.1.4 Laplacians

Definition 4.1.7 ***(Laplacians and Harmonic Forms)***

- ***Laplacian and Harmonic forms associated to*** d :

  Let X be a smooth Riemannian manifold. We define the ***Laplacian associated to d*** *by,* $\Delta_d := d \circ d^* + d^* \circ d : \mathcal{A}^k_{X,cs} \to \mathcal{A}^k_{X,cs}$.
  A form $\alpha \in \mathcal{A}^k_{X,cs}$ *is* Δ_d-***Harmonic*** *if* $\Delta_d \alpha = 0$.
  The space of all Δ_d-***Harmonic forms*** *of order k on X is denoted by* $\mathcal{H}^k_d$ *or* $\mathcal{H}^k_X$

- ***Laplacian associated to*** ∂ :

  Let X be a complex manifold equipped with a Hermitian metric h. We define the ***Laplacian associated to*** ∂ *by:* $\Delta_\partial := \partial \circ \partial^* + \partial^* \circ \partial : \mathcal{A}^{p,q}_{X,cs} \to \mathcal{A}^{p,q}_{X,cs}$.
  A form $\alpha \in \mathcal{A}^{p,q}_{X,cs}$ *is* Δ_∂-***Harmonic*** *if* $\Delta_\partial \alpha = 0$.
  The space of all Δ_∂-***Harmonic forms*** *of type (p,q) on X is denoted by* $\mathcal{H}^{p,q}_\partial$

- ***Laplacian associated to*** $\overline{\partial}$:

  Let X be a complex manifold equipped with a Hermitian metric h. We define the ***Laplacian associated to*** $\overline{\partial}$ *by:* $\Delta_{\overline{\partial}} := \overline{\partial} \circ \overline{\partial}^* + \overline{\partial}^* \circ \overline{\partial} : \mathcal{A}^{p,q}_{X,cs} \to \mathcal{A}^{p,q}_{X,cs}$.
  A form $\alpha \in \mathcal{A}^{p,q}_{X,cs}$ *is* $\Delta_{\overline{\partial}}$-***Harmonic*** *if* $\Delta_\partial \alpha = 0$.
  The space of all $\Delta_{\overline{\partial}}$-***Harmonic forms*** *of type (p,q) on X is denoted by* $\mathcal{H}^{p,q}_{\overline{\partial}}$

- ***Laplacian associated to*** $\overline{\partial}_E$:

  Let X be a complex manifold equipped with a Hermitian metric h and let (E, h_E) be a holomorphic Hermitian vector bundle on X. We define the ***Laplacian associated to*** $\overline{\partial}_E$ *by:* $\Delta_{\overline{\partial}_E} := \overline{\partial}_E \circ \overline{\partial}^*_E + \overline{\partial}^*_E \circ \overline{\partial}_E : (\mathcal{A}_{E,X})^{p,q}_{cs} \to (\mathcal{A}_{E,X})^{p,q}_{cs}$.

*A form $\alpha \in (\mathcal{A}_{X,E})^{p,q}_{cs}$ is $\Delta_{\overline{\partial}_E}$-**Harmonic** if $\Delta_{\overline{\partial}_E}\alpha = 0$.*
*The space of all $\Delta_{\overline{\partial}}$-**Harmonic forms** of type (p,q) on X is denoted by $\mathcal{H}^{p,q}_E$*

Then following lemma shows us that harmonic forms are closed.

Lemma 7 *If X is compact smooth Riemannian manifold. Then*

$$\langle \alpha, \Delta_d \alpha \rangle = \langle d\alpha, d\alpha \rangle + \langle d^*\alpha, d^*\alpha \rangle$$

Corollary 4.1.8 *In the case of the above lemma, $ker\Delta_d = ker(d) \cap ker(d^*)$*

4.2 Elliptic Differential Operators

4.2.1 Differential Operators

Let X be a smooth manifold and $E \xrightarrow{\tau} X$ and $F \xrightarrow{\sigma} X$ are two complex vector bundles of ranks r and s respectively, on X. A $\mathbb{C}$-linear map of sheaves (this is not in general a map of C^∞_X-modules !) $L : \varepsilon_E \to \varepsilon_F$ is called a **Differential operator of order k** if $\exists$ an open cover by charts $(\mathcal{U}, \phi) = (U_\alpha, \phi_\alpha = (x_{1,\alpha}, .., x_{n,\alpha}))_\alpha$ of X that trivialize both E and F by the local frames $\zeta^\alpha := (\zeta^\alpha_1, ..., \zeta^\alpha_r)$ and $\eta^\alpha = (\eta^\alpha_1, ..., \eta^\alpha_s)$ respectively on U_α and $\mathbb{C}$-linear maps $C^\infty_X(U_\alpha)^r \xrightarrow{\widetilde{L}_\alpha} C^\infty_X(U_\alpha)^s$ such that the following diagram commutes:

$$\begin{array}{ccc} \varepsilon_E(U_\alpha) & \xrightarrow{L|_{U_\alpha}} & \varepsilon_F(U_\alpha) \\ \downarrow \cong & & \downarrow \cong \\ C^\infty_X(U_\alpha)^r & \xrightarrow{\widetilde{L}_\alpha} & C^\infty_X(U_\alpha)^s \end{array} \qquad (4.1)$$

and the maps L_α are given by

$$L_\alpha(f_1, ..., f_r) = (\sum_{|\sigma|\ \leq k,\ 1\leq j\leq r} a^{j,i}_{\alpha,\sigma} D^\sigma_\alpha(f_j)\ )_{1\leq i\leq s}$$

where D^σ_α are just some monomials of degree at most k in the polynomial ring (multiplication is given by composition of operators) $\mathbb{R}[\ \frac{\partial}{\partial x_{1,\alpha}}, \cdots, \frac{\partial}{\partial x_{n,\alpha}}\ ]$ that acts as an operator on $C^\infty_X(U_\alpha)$ and there is some σ with $|\sigma| = k$ such that some $a^{j,i}_{\alpha,\sigma} \neq 0$. Fix some indices α and β for the open sets. Let us denote $U = U_\alpha \cap U_\beta$, restrict the frames and charts on U_α to U and denote these without the index α. Let us explain what happens to the Differential operators when we change the chart and the frames of E and F on U. Let

$$g : U \to GL(n, \mathbb{R}),\ u : U \to GL(r, \mathbb{C}),\ v : U \to GL(s, \mathbb{C})$$

be the change of frame maps of the tangent bundle, E and F respectively. Then,

(a) if $y_1, .., y_n$ be the new coordinates on U, the linear operator g defined on the tangent bundle of U, gives a graded homomorphism of commutative graded $\mathbb{R}$-algebra homomorphism of the polynomial rings (with the obvious grading) $\mathbb{R}[\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}] \to \mathbb{R}[\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n}]$, this maps $D^\sigma \in \mathbb{R}[\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}]$ to a homogeneous polynomial of the same degree in $\mathbb{R}[\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n}]$. Thus all linear sums of D^σ with $|\sigma| = m$ give all the sections of the symmetric product $S^m(T_{\mathbb{R},X})\ \ \forall m \leqslant k$.

(b) The action of v on frames changes L in the following way : The coefficients $a^{i,j}_\sigma$ of L change according to the formula:

$$v \star (a^{i,j}_\sigma)_i = {}^t v \cdot (a^{i,j}_\sigma)_i$$

, where $v \star (a^{i,j}_\sigma)_i$ are the new coefficients and the dot on the right hand side of the above is matrix multiplication. So, it behaves like a section of F.

(c) But because of the Leibnitz rule, u changes L in a co-ordinate dependent way, therefore L is not a C^∞_X-module homomorphism of sheaves or a morphism of vector bundles. But we do get a section of $E^* \otimes F = Hom(E, F)$ in the following way:

For each point $(x, v) \in T^*_{\mathbb{R},x}$ with $v \neq 0$ choose a function $g \in C^\infty_X(V)$ for some open set V of X such that $v = dg$.

Define $\quad \sigma_L(x, v)(f) := L(\frac{i^k}{k!}(g - g(x))^k f)(x) \in F_x \quad \forall f \in E_x$

and call it the **principal symbol** of L. This does define a section of the vector bundle $Hom(E, F)$ over some open set V of X if v is a differential 1-form on V and g is defined all over V. In this construction of σ_L, only the principal part of L is important; i.e. if two differential operators have the same principal part then they have the same principal symbol. But this gives us more: let θ be the 1-form on X which is zero everywhere the principal symbol gives a homogeneous morphism

$$\Omega^1_X \setminus \theta \to Hom(E, F).$$

meaning, $\sigma_L(x, \rho v) = \rho^k \sigma_L(x, v) \quad \forall v \in T^*_{\mathbb{R},x} \setminus 0,\ \forall x \in X$ and $\forall \rho > 0$.

Note: The **principal part** of the differential operator L is:

$$L_{prin} = (\sum_{|\sigma| \ =k,\ 1\leq j\leq r} a^{j,i}_{\alpha,\sigma} D^{\sigma}_{\alpha}(f_j) \ )_{1\leq i\leq s}$$

We are now ready to define and study elliptic differential operators.

4.2.2 The Fundamental Theorem of Elliptic Differential Operators

Definition 4.2.1 *We say that a differential operator $P : \varepsilon_E \to \varepsilon_F$ is* ***elliptic*** *if $\sigma_P(x, v)$ is injective for every v and x such that v is a non-zero cotangent vector at x and x is a point in X.*

Suppose (X, g) is a compact Riemannian manifold and let (E, g$_E$)$and(F, g_F)betwo$C$^\infty$-bundles over X, with Riemannian metrics g_E and g_F on them, respectively. Let differential operator $P : \varepsilon_E \to \varepsilon_F$. Let $f_1 : E \to E^*$ and $f_2 : F \to F^*$ be the isomorphisms of E and F with their dual given by the metrics on them.

We define the **formal adjoint** of P by $P^* : \varepsilon_{F^*} \to \varepsilon_{E^*}$ that satisfies $\langle \alpha, P\beta \rangle_{g_{F^*},L^2} = \langle P^*\alpha, \beta \rangle_{g_E,L^2}$. We will construct such an adjoint later but for now assume that it exists (**?**)...... ***Note:***

The proof of the existence tells something about the symbol of the adjoint. We see that:

Proposition 4.2.2 *The symbol of P^* is equal to the adjoint of the symbol of P; i.e. $\sigma_P(x, v) = (\sigma_{P^*}(x, v))^* \quad \forall v \in T^*_{\mathbb{R},x} \setminus 0 \ \ \forall x \in X$. particularly, P^* is an elliptic operator if and only if P is the same.*

Theorem 5 ***The Fundamental Theorem of Elliptic Differential Operators*** *Let $P : E \to F$ be an elliptic differntial operator on a compact Riemannian manifold X between two vector bundles E, F of the same rank with equipped Riemannian metrics on them. Then the kernel and the image of P are finite dimensional $\mathbb{R}$-vector spaces, and we have the decomposition $\varepsilon_E = kerP \oplus ImP^*$.* □

4.2.3 Laplacian as an Elliptic Differential Operator and the Hodge Isomorphism

By computing the symbol of the Laplacians we can see that they are all Elliptic Differential Operators:

Lemma 8 ***Symbols of Laplacians***

(a) ***Symbol of*** Δ_d

Let $x \in (X, g)$*, where* (X, g) *is a Riemannian manifold. On a sufficiently small chart around* x*, the symbol* σ_{Δ_d} *is given by*

$$\sigma_{\Delta_d}(v)(\omega) = -\|v\|^2 \omega$$

where v *is a non-zero section of the cotangent bundle and* ω *is a non-zero section of some* $\mathcal{A}_X^k$ *around the point* $x \in X$*, where* $\|\cdot\|$ *is the norm induced by* g*.*

(b) ***Symbol of*** $\Delta_{\overline{\partial}}$

If $x \in (X, h)$ *is a point on a complex manifold* X *with Hermitian metric* h*. Then on a sufficiently small chart of* X*, the symbol* $\sigma_{\Delta_{\overline{\partial}}}$ *is given by*

$$\sigma_{\Delta_{\overline{\partial}}} = -\frac{1}{2}\|v\|^2 Id.$$

on $\mathcal{A}_X^{p,q}$ *around* $x \in X$*, where* $\|\cdot\|$ *is the norm given by* h

(c) ***Symbol of*** $\Delta_{\overline{\partial}_E}$

If $x \in (X, h)$ *is a point on a complex manifold* X *with Hermitian metric* h *and* (E, h_E) *be a holomorphic Hermitian vector bundle on* X*. Then on a sufficiently small chart of* X*, the symbol* $\sigma_{\Delta_{\overline{\partial}_E}}$ *is given by*

$$\sigma_{\Delta_{\overline{\partial}_E}} = \|v\|^2 Id.$$

on $\mathcal{A}_E^{p,q}$ *around* $x \in X$*, where* $\|\cdot\|$ *is the norm induced by* h*.*

Thus all the Laplacians above are Elliptic Differential Operators. □

PROOF $2a$: Let $\alpha = \sum_\mu \alpha_\mu dx_\mu$ be any differential form. Let η be the unique element in $I_{n-p,n}$, where $|\mu| = p$, such that (μ, η) is a permutation of $(1, ..., n)$.
Then

$$*\alpha = \sum_\eta (sgn(\mu, \eta))\sqrt{det(M(g))}(\sum_\nu g^{\mu_1,\nu_1} \cdots g^{\mu_p,\nu_p}\alpha_\nu)dx_\eta$$

and define, $G^{\mu}_{\nu,\eta} := \sqrt{det(M(g))}(g^{\mu_1,\nu_1}\cdots g^{\mu_p,\nu_p})$. The G's are smooth functions on U. I would like to make a note here that the index η in G has not much use except that it is helpful to keep note of what happens when we apply the Hodge star operator. Then

$$*\alpha = \sum_{\nu,\eta}(sgn(\mu,\eta))G^{\mu}_{\nu,\eta}\alpha_\nu dx_\eta$$

and

$$d(*\alpha) = \sum_{\nu,\eta,i}(sgn(\mu,\eta))(\frac{\partial G^{\mu}_{\nu,\eta}}{\partial x_i}\alpha_\nu dx_i \wedge dx_\eta) + \sum_{\nu,\eta}(sgn(\mu,\eta))G^{\mu}_{\nu,\eta}(\sum_i \frac{\partial \alpha_\nu}{\partial x_i}dx_i \wedge dx_\eta)$$

Let $H_{i,\eta}$ be the coefficients of $dx_i \wedge dx_\eta$ in the right hand side of the last equality above. Though the multiindex (i,η) is not strictly positive we will work with these. Now the complement of the underlying set of (i,η) is $\{\mu_1,..,\mu_p\}\setminus\{i\}$, call this $\mu - i$ Now if $i = \mu_r$ then $sgn(\mu,\eta) = (-1)^{p-r}sgn(\mu - i,(i,\eta)) = (-1)^{p-r}(-1)^{(p-1)(n-p+1)}sgn((i,\eta),\mu - i)$. Now as we defined $G^{\mu}_{\nu,\eta}$ for α, we define $G^{(i,\eta)}_{(j,\tau),\mu-j}$ in the similar way and we get:

$$(*d*)(\alpha) = \sum_{\mu,\nu,i,j,\tau}(-1)^{(p-r)+(p-1)(n-p+1)}G^{(i,\eta)}_{(j,\tau),\mu-i}(\frac{\partial G^{\mu}_{\nu,\tau}}{\partial x_j}\alpha_\nu + G^{\mu}_{\nu,\tau}\frac{\partial \alpha_\nu}{\partial x_j})dx_{\mu-i}$$

The formula above is complicated because of the choice of the chart. If we change the chart and $dx_1,...,dx_n$ is an orthonormal basis for the metric g, then the above formula becomes simple; e.g. $\sum_\nu G^{\mu}_{\nu,\eta}\alpha_\nu$ becomes α_μ. So we get,

$$(*d*)(\alpha) = \sum_{\mu,i}(-1)^{(p-r)+(p-1)(n-p+1)}\frac{\partial \alpha_\mu}{\partial x_i}dx_{\mu-i}$$

If we apply d to the above formula we get $d(*d*)$ and if we replace α by $d\alpha$ we get $(*d*)(d\alpha)$. Then adding them with appropriate signs we can see that

$$\Delta_d(\alpha) = -\sum_{\mu,i}(\frac{\partial^2 \alpha_\mu}{\partial x_i^2}dx_\mu)$$

. Thus the symbol of Δ_d is given by:

$$\sigma_{\Delta_d}(v)(\alpha) = -(v_1^2 + ... + v_i^2 + ... + v_n^2)\alpha = -\|v\|^2\alpha$$

The proofs of 2b and 2c are similar to 2a.

Now we can apply the Fundamental Theorem of Elliptic Operators on the Laplacians and get a similar decomposition.

Suppose that (X, g) be a compact Riemannian manifold. Note that Laplacians are self-adjoint, so we have

$$\mathcal{A}^k(X) = \mathcal{H}^k_X \oplus \Delta_d(\mathcal{A}^k(X)).$$

This gives us the projection map $\mathcal{A}^k(X) \to \mathcal{H}^k_X$. Let $Z^k(X)$ be the space of all d-closed differential forms of degree k on X and $B^k(X) = d(A^{k-1}(X))$, then we get the induced map $\mu : Z^k(X) \to \mathcal{H}^k_X$.

Let $\beta \in B^k(X)$ and let $\beta = \alpha + \Delta_d(\gamma)$, where $\alpha \in \mathcal{H}^k_X$ and $\gamma \in \mathcal{A}^k(X)$. So, $\alpha + d^*d(\gamma)$ is exact and so in particular it is closed. Thus $d^{ast}d(\gamma)$ is closed. and therefore it lies in $ker(d) \cap Im(d^*) = 0$, so α is exact. Thus, α is in $ker(d^*) \cap Im(d) = 0$, and this shows that $Z^k(X) \to \mathcal{H}^k_X$ induces a map,

$$\phi : H^k_{DR}(X, \mathbb{R}) \to \mathcal{H}^k_X$$

We also have the projection map $\tau : \mathcal{Z}^k(X) \to H^k_{DR}(X, \mathbb{R})$, which when restricted to Harmonic form gives the map:

$$\psi : \mathcal{H}^k_X \to H^k_{DR}(X, \mathbb{R})$$

which sends every harmonic form to its cohomology class.

The following theorem is of a significant importance to us and I call it the Hodge Isomorphism Theorem:

Theorem 6 ***(Hodge Isomorphism Theorem)***

The maps ϕ and ψ above are inverse to each other. □

PROOF Let $\beta \in Z^k(X)$ and let $\beta = \alpha + \Delta_d(\gamma)$, where $\alpha \in \mathcal{H}^k_X$ and $\gamma \in \mathcal{A}^k(X)$. So, $\alpha + d^*d(\gamma)$ is closed and so is in $ker(d) \cap Im(d^*) = 0$, and hence β is cohomologous to α. So we see that the diagram below commutes:

$$\begin{array}{ccc} Z^k(X) & \xrightarrow{\mu} & \mathcal{H}^k_X \\ & {\scriptstyle\tau}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & H^k_{DR}(X, \mathbb{R}) & \end{array} \tag{4.2}$$

Since both μ and τ are zero on exact forms, the following diagram commutes:

$$\begin{array}{ccc} H^k_{DR}(X, \mathbb{R}) & \xrightarrow{\phi} & \mathcal{H}^k_X \\ & {\scriptstyle Id_{H^k_{DR}(X,\mathbb{R})}}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & H^k_{DR}(X, \mathbb{R}) & \end{array} \tag{4.3}$$

Now if we show that ψ is injective, we will get $\phi \circ \psi = Id_{\mathcal{H}_X^k}$. But that follows from $\mathcal{H}_X^k \cap B^k(X) \subseteq ker(d^*) \cap Imd = 0$ ■

The similar Isomorphism theorems hold for Complex De Rham cohomology of X, Dolbeault cohomology of a compact complex Hermitian manifold and the Dolbeault cohomology of a Hermitian Holomorphic vector bundle on a compact complex vector bundle.

There is this amazing corollary of the Hodge Isomorphism theorem :

Corollary 4.2.3 *The cohomology groups $H^k(X, \mathbb{R})$ are all finite dimensional. Moreover, the same holds for the complex cohomologies of X, the Dolbeault cohomologies of a compact complex manifold and the Dolbeault cohomologies of a holomorphic hermitian vector bundle on a compact complex manifold.*

4.3 Lefschetz Representation of the Lie algebra sl(2, C)

Let X be a compact complex manifold of complex dimension n and let ω be a Kähler form on X. Let

$$L := \omega \wedge (\cdot) : \mathcal{A}^*_{X,\mathbb{R}} \to \mathcal{A}^{*+2}_{X,\mathbb{R}} \;\; (or, \mathcal{A}^{p,q}_X \to \mathcal{A}^{p+1,q+1}_X) \quad : \quad \alpha \longmapsto \omega \wedge \alpha$$

be the Lefschetz operator and let L^* denote the formal dual of L obtained using the Hodge star operator defined by the metric h associated to the Kähler form ω. Then recall that we have seen that $L^* = \overline{*}^{-1} L \overline{*} = \kappa \;\overline{*}\; L \;\overline{*}$. Where

$$\kappa : \bigwedge \mathcal{A}_{X,\mathbb{R}} \to \bigwedge \mathcal{A}_{X,\mathbb{R}} \;\; \alpha \mapsto \sum_{r \geq 0} (-1)^r \sqcap_r \alpha$$

and $\bigwedge \mathcal{A}_{X,\mathbb{R}} = \mathbb{R} \oplus \mathcal{A}^0_{X,\mathbb{R}} \oplus \mathcal{A}^1_{X,\mathbb{R}} \oplus \cdots$. and $\sqcap_r : \bigwedge \mathcal{A}_{X,\mathbb{R}} \to \mathcal{A}^r_{X,\mathbb{R}}$ are the projection maps for all non-negative r. But what really is κ when we take $L : \mathcal{A}^k_{X,\mathbb{R}} \to \mathcal{A}^{k+2}_{X,\mathbb{R}}$? Clearly it is $(-1)^{k-2} = (-1)^k$. Also recall that we were introduced to the map $J := \sum_{p \geq 0, q \geq 0} i^{p-q} \sqcap_{p,q} : \bigwedge \mathcal{A}_{X,\mathbb{C}} \to \bigwedge \mathcal{A}_{X,\mathbb{C}}$, where $\sqcap_{p,q} : \bigwedge \mathcal{A}_{X,\mathbb{C}} \to \mathcal{A}^{p,q}_X$ and

$$\bigwedge \mathcal{A}_{X,\mathbb{C}} = (\bigwedge \mathcal{A}_{X,\mathbb{R}}) \underset{\mathbb{R}}{\otimes} \mathbb{C}.$$

Consider the non-commutative algebra $\mathcal{G}$ of all $\mathbb{C}$-linear morphism of sheaves $\mathcal{A}_{X,\mathbb{C}} \to \mathcal{A}_{X,\mathbb{C}}$ where multiplication is given by composition of maps, and the $\mathbb{C}$-vector space structure is pointwise. Then the operator $[\cdot,\cdot] : \mathcal{G} \times \mathcal{G} \to \mathcal{G} \quad : \quad [A, B] := A \circ B - B \circ A, \quad \forall A, B \in \mathcal{G}$. This operator is called the **commutator** because if $[A, B] = 0$ then $A \circ B = B \circ A$, i.e A and B commute. Clearly $(\mathcal{G}, [\cdot,\cdot])$ forms a Lie algebra. We get the following proposition at once:

Proposition 4.3.1

(a) $[L^*, L] = \sum_{0 \le r \le 2n} (n - r)\sqcap_r,$

(b) $\overline{*} \circ \sqcap_{p,q} = \sqcap_{n-q,n-p} \circ \overline{*},$

(c) $[L, J] = [L^*, J] = [L, \omega] = [L^*, \omega] = 0.$

By $\mathbb{K}$ we will mean either one of the fields $\mathbb{R}$ and $\mathbb{C}$.

Definition 4.3.2 *(Lie Groups and Lie Algebras)*

(a) ***(Lie Group)*** *Let G be a smooth manifold (real or complex) such that the multiplication map $G \times G \to G$ is also smooth (if G is a complex manifold we require this map to be homogeneous).*

(b) ***(Lie Algebra)*** *Let $\mathbb{F}$ be any field. A* ***Lie algebra*** *over $\mathbb{F}$ is a pair $(V, [[\cdot,\cdot]])$ where V is a vector space over $\mathbb{F}$ and $[[\cdot,\cdot]] : V \times V \to V$ is a $\mathbb{F}$-bilinear map that satisfies the following two axioms:*

i. ***(Anticommutativity)*** $[[v, v]] = 0 \;\; \forall v \in V$

This implies $[[v, w]] = -[[w, v]] \;\; \forall v, w \in V$

ii. ***(Jacobi identity)*** $[[v, w]] + [[w, u]] + [[u, v]] = 0 \;\; \forall u, v, w \in V$

Example 4.3.3

(a) Clearly $(\mathcal{G}, [\cdot,\cdot])$ forms a Lie algebra.

(b) More generally, let V be any vector space over a field $\mathbb{F}$. Let $End_{\mathbb{F}}(V)$ the non-commutative algebra of all $\mathbb{F}$-linear endomorphisms on V with pointwise vector space structure and composition of functions. Then $(End_{\mathbb{F}}(V), [\cdot,\cdot])$ is a Lie algebra, where $[\cdot,\cdot] : V \times V \to V \quad [a, b] := a \circ b - b \circ a \;\; \forall a, b \in End_{\mathbb{F}}(V)$. This particular Lie algebra is denoted by $\mathfrak{gl}(V)$ and this operator $[\cdot,\cdot]$ is also called a commutator for the same reason. When $V = \mathbb{F}^n$ we denote $\mathfrak{gl}(V)$ by $\mathfrak{gl}(n, \mathbb{F})$

(c) The elements of $\mathfrak{gl}(V)$ which have trace zero form a Lie algebra with the commutator of $\mathfrak{gl}(V)$. This Lie algebra is denoted by $\mathfrak{sl}(V)$. When $V = \mathbb{F}^n$ we denote $\mathfrak{sl}(V)$ by $\mathfrak{sl}(n, \mathbb{F})$

Definition 4.3.4

*(a) A **represerntation** (V, ρ) of a Lie group G consists of a vector space V over $\mathbb{K}$ and a smooth group homomorphism*

$$\rho : G \to GL(V).$$

*(b) A **representation** (V, τ) of a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ is a vector space homomorphism*

$$\tau : \mathfrak{g} \to \mathfrak{gl}(V)$$

over ⅂ such that

$$\tau(\ [[v, w]]\ ) = [\tau(v), \tau(w)] \quad \forall v, w \in V,$$

for any field $\mathbb{F}$.

*(c) A **Lie group homomorphism** $\rho : G \to H$ is a group homomorphism between Lie groups which is also a smooth map.*
We can similarly define real analytic Lie group homomorphism of real analytic Lie groups and holomorphic Lie group homomorphisms of holomorphic Lie groups.

*(d) A **Lie algebra homomorphism** $\rho : (L, [\cdot, \cdot]_L) \to (M, [\cdot, \cdot]_M)$ is a vector space map $\phi : L \to M$ between Lie algebras $(L, [\cdot, \cdot]_L)$ and $(M, [\cdot, \cdot]_M)$ such that*

$$[v, w]_L = [\tau(v), \tau(w)] \quad \forall v, w \in L$$

Example 4.3.5 *Let G be any Lie group and for any $g \in G$, let $L_g : G \to G \quad a \mapsto g \cdot a$ and $R_g : G \to G \quad a \mapsto a \cdot g$ be the left and right multiplication action of G on itself. By definition of Lie group these maps are smooth. A vector field X on G is said to be **left-invariant** (resp. **right-invariant**) if*

$$dL_g(X_a) = X_{g \cdot a} \quad \forall g, a \in G.$$

(resp. $dR_g(X_a) = X_{g \cdot a} \quad \forall g, a \in G$. We state the following propositions without proof:

Proposition 4.3.6 *All the left invariant sections of the tangent bundle $T_{\mathbb{K},G}$, forms a subbundle $L[G]$ of $T_{\mathbb{K},G}$ and becomes a Lie algebra under the Lie bracket $[\cdot, \cdot]$*

of vector fields defined by $[X,Y]f := X(Yf) - Y(Xf)$ $\forall$ *sections* X, Y *of* $T_{\mathbb{F},G}$ *and a* $\mathbb{F}$*-valued* $\mathcal{C}^\infty$*-function* f *(viewing vector fields as derivations of germs of* $\mathcal{C}^\infty$ *functions.). Moreover the* $\mathbb{K}$*-vector spaces* $L[G]$ *and* $\mathfrak{h}$ *are isomorphic, where* $\mathfrak{g}$ *is the* $\mathbb{K}$*-tangent space at the identity element of* G*. Thus,* $\mathfrak{g}$ *becomes a Lie algebra in a natural way and the cotangent bundle* $\mathfrak{g}^*$ *at identity also becomes a Lie algebra by the natural isomorphism between a vector space and its dual. (We can replace the left invariant vector fields by right invariant vector fields in this proposition. The bundle of all right invariant vector fields is denoted by* $R[G]$*.) We say that* $\mathfrak{g}$ *is the Lie algebra associated to the Lie group* G*, and we express this statement by saying* $Lie(G) = \mathfrak{g}$

For any Lie group homomorphism $\rho : G \to H$ *the differential* $d\rho : Lie(G) \to Lie(H)$ *is a Lie algebra homomorphism.*

Definition 4.3.7

(a) A morphism of two Lie group representations $\rho_j : G \to GL(V_j)$ $j = 1, 2$*, is a map of vector spaces* $\phi : V_1 \to V_2$ *such that*

$$\rho_2(g) \circ \phi = \phi \circ \rho_1(g) \quad \forall g \in G.$$

(b) A morphism of two Lie algebra representations $\tau_j : \mathfrak{g} \to \mathfrak{gl}(V_j)$ $j = 1, 2$*, is a map of vector spaces* $\phi : V_1 \to V_2$ *such that*

$$\tau_2(v) \circ \phi = \phi \circ \tau_1(v) \quad \forall v \in \mathfrak{g}.$$

(c) A representation (V, ρ) *(Lie algebra or Lie group representation) is said to be a* ***subrepresentation*** *of another representation* (W, p) *(of the same type) if there is an injective morphism of representations* $\phi : (V, \rho) \to (W, p)$*.*

(d) A representation (V, ρ) *(Lie algebra or Lie group representation) is said to be* ***irreducible*** *if it has no proper subrepresentation.*

(e) A representation (V, ρ) *(Lie algebra or Lie group representation) is said to be* ***completely reducible*** *if it can be expressed as a direct sum of irreducible subrepresentations.*

Definition 4.3.8 *Define* $B := \sum_{0 \le r \le 2n} (n - r) \sqcap_r$ *.*

Consider the elements $e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ *and* $h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *of* $\mathfrak{sl}(2,\mathbb{C})$ *and we get* $[e,f] = h$, $[h,e] = 2e$ *and* $[h,f] = -2f$. *The elements* h, e, f *are the basis of the Lie algebra* $\mathfrak{sl}(2,\mathbb{C})$ *and the map*

$$\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathcal{G}$$

defined by

$$h \mapsto B, \; e \mapsto L^*, \; f \mapsto L$$

gives a representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. *We call this representation of* $\mathfrak{sl}(2,\mathbb{C})$ *the* ***Lefschetz representation***.

Definition 4.3.9

(a) Let V *be a complex vector space. For any Lie algebra representation* $\tau : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V)$, *an element* $v \in V$ *is said to have* ***weight*** $\lambda \in \mathbb{C}$ *if* $\tau(h)(v) = \lambda \cdot v$; *i.e.* λ *is an eigenvalue of* $\tau(h)$ *with eigenvector* v. *Under this representation* ρ *of* $\mathfrak{sl}(2,\mathbb{C})$ *every differential form of degree p, has weight n-p, as* $\rho(h) = B = \sum_{j=0}^{\infty}(n-j)\sqcap_j$. *The subspace of all all vectors of weight* λ *is called the* ***space of weight***- λ.

(b) In the above case of the representation (V, τ) *of* $\mathfrak{sl}(2,\mathbb{C})$, *an eigen-vector* $v \in V$ *of* $\tau(h)$ *is said to be* ***primitive*** *if* $\tau(e)(v) = 0$.

We have the following theorem for $\mathfrak{sl}(2,\mathbb{C})$ Lie algebras:

Theorem 7 *Every finite dimensional* $\mathfrak{sl}(2,\mathbb{C})$ *is completely reducible.* □

Theorem 8 *Every finite dimensional representation of* $\mathfrak{sl}(2,\mathbb{C})$ *has a primitive vector. Let* v_0 *be a primitive vector of weight* λ *in an irreducible representation* (V, ρ) *Then defining*

$$v_n = \begin{cases} 0 & \text{for } n = -1 \\ \frac{1}{n!}\rho(f)^n v_0 & \text{for } n = 0, 1, 2, \cdots \end{cases}$$

then we have:

$$\rho(h)v_n = (\lambda - 2n)v_n \tag{4.4}$$

$$\rho(f)v_n = (n+1)v_{n+1} \tag{4.5}$$

$$\rho(e)v_n = (\lambda - n + 1)v_{n-1} \tag{4.6}$$

Moreover, $\lambda \in \mathbb{Z}$ *and* $\lambda + 1 = dim_{\mathbb{C}}V$, *and* $\rho(f^n)v_0 = 0 \;\; \forall n > m$. □

Theorem 9 ***(Classification of finite dimensional irreducible*** $\mathfrak{sl}(2,\mathbb{C})$ ***representations)*** *Let V be a vector space of dimension $m \in \mathbb{Z}^{\geq 0}$ with the basis $(v_0, ..., v_m)$. Then we have a $\mathfrak{sl}(2,\mathbb{C})$ representation on V, defined by:*

$$\rho(h)v_n = (m-2n)v_n \tag{4.7}$$

$$\rho(f)v_n = (n+1)v_{n+1} \tag{4.8}$$

$$\rho(e)v_n = (m-n+1)v_{n-1} \tag{4.9}$$

and with the convention $v_{-1} = v_{m+1} = 0.$, which is irreducible. Conversely, every complex representation of $\mathfrak{sl}(2,\mathbb{C})$ of dimension $m+1$ is equivalent this representation.□

For the proofs of these theorems see [2] pg. $171-178$.

Corollary 4.3.10 *Suppose (V,ρ) be an irreducible complex representation of $\mathfrak{sl}(2,\mathbb{C})$ of dimension $m+1$, where $m \in \mathbb{Z}_{\geq 0}$. For every vector $u \in V$ of weight λ, we have an integer $r \geq 0$ and a primitive vector u_0 of weight $\lambda + 2r$, such that*

$$u = \rho(f^r)u_0$$

and

$$u_0 = \frac{(m-r)!}{m!r!}\rho(e)^r u$$

Proof Let $B = (v_0, ..., v_m)$ be a basis of V such that the pair (ρ, B) satisfies the condition of Theorem 9. From 4.9, we get

$$\rho(e)^r v_r = (m-r+1)(m-r+2)\cdots(m-1)m\cdot v_0 = \frac{m!}{(m-r)!}v_0.$$

and by 4.8 we get

$$\rho(f)^r\rho(e)^r v_r = \frac{m!}{(m-r)!}v_r$$

$0 \leq \forall r \leq m$. Since every irreducible subrepresentation of a $\mathfrak{sl}(2,\mathbb{C})$ representation intersects every space of fixed weight in a subspace of dimension 1 (a complex line through origin), we get that u is a scalar multiple of some v_r, say $u = c\cdot v_r$, for fixed $c \in \mathbb{C}$ and fixed r, $(0 \leq r \leq m)$. Then

$$u = \frac{(m-r)!}{m!r!}\rho(f)^r\rho(e)^r u$$

and hence $\frac{(m-r)!}{m!r!}\rho(e)^r u$ is primitive. □

We will now consider an action of $SL(2,\mathbb{C})$ and $\mathfrak{sl}(2,\mathbb{C})$ on the symmetric power

$S^m(\mathbb{C}^2), \quad \forall m \in \mathbb{Z}_{>0}$. Let $v_{1,0} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_{1,1} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $v_{m,k} := v_{1,0}^{m-k} \cdot v_{1,1}^k \quad 0 \leq \forall k \leq m$. Then $(v_{m,k})_{0\leq k\leq m}$ is a basis of $S^m(\mathbb{C}^2)$. Let $\tau_1 : SL(2,\mathbb{C}) \to GL(S^1(\mathbb{C}^2)) = GL(2,\mathbb{C})$ is the inclusion map $SL(2,\mathbb{C}) \hookrightarrow GL(2,\mathbb{C})$, and $\tau_m = \tau_1^{\otimes m}$. Let $\rho_m := d\tau_m \quad \forall m \in \mathbb{Z}_{>0}$. Then we get the formulas:

$$\begin{aligned} \rho_m(h)v_{m,k} = (m-2k)v_{m,k} \quad 0 \leq k \leq m \\ \rho_m(e)v_{m,0} = 0 \\ \rho_m(f)v_{m,m} = 0 \\ \rho_m(e)v_{m,k} = k \cdot v_{m,k-1} \quad 0 < k \leq m \\ \rho_m(f)v_{m,k} = (m-k)v_{m,k+1} \quad 0 \leq k \leq m-1. \end{aligned}$$

Let $u_k := \rho_m(f)^k v_{m,0} = \frac{m!}{(m-k)!} v_{m,k}$. Then u_0 is primitive, and the pair $(\rho_m, (u_0, ..., u_m))$ satisfies the condition of Theorem 9 and therefore $(S^m(\mathbb{C}^2), \rho_m)$ is irreducible. Let $\gamma := i(e+f) = exp(i(e+f)) \in SL(2,\mathbb{C}) \cap \mathfrak{sl}(2,\mathbb{C})$. Then,

$$\begin{aligned} \tau_m(\gamma)u_k = \frac{m!}{(m-k)!}\tau_m(\gamma)v_{m,k} &= \frac{m!}{(m-k)!}(\tau_1(\gamma)(v_{1,0}))^{m-k} \cdot (\tau_1(\gamma)(v_{1,1}))^k \\ &= \frac{i^m k!}{(m-k)!} u_{m-k} \end{aligned}$$

as $(\tau_1(\gamma)v_{1,\epsilon} = iv_{1,1-\epsilon}, \quad \epsilon = 0,1)$. Hence,

$$\tau_m \rho_m(f)^k u_0 = \frac{i^m k!}{(m-k)!} \rho_m(f)^{m-k} u_0 \tag{4.10}$$

4.4 Lefschetz Decomposition and important identities on Kähler Manifolds

4.4.1 The Lefschetz Decomposition of Diffrerential Forms

Definition 4.4.1 *A complex differential p-form α is said to be* ***primitive*** *if*

$$\rho(e)\alpha = L^*\alpha = 0$$

We have the following proposition about primitiveness:

Proposition 4.4.2

(a) *Any complex p-form α is primitive then $L^k\alpha = 0 \forall k \geq n-p+1$*

(b) *All the primitive forms have degree less than n (the complex dimension of X).*

PROOF Let $\mathfrak{sl}(2,\mathbb{C}) \cdot \alpha$ vector space spanned by the orbit of α under the action of $\mathfrak{sl}(2,\mathbb{C})$. Then by the theory of Lie algebra representation of $\mathfrak{sl}(2,\mathbb{C})$, we have:

$$dim_{\mathbb{C}}(\mathfrak{sl}(2,\mathbb{C}) \cdot \alpha) = n-p+1.$$

with $\alpha, \rho(f)\alpha, ..., \rho^{n-p}(f)\alpha$ being a basis of $\mathfrak{sl}(2,\mathbb{C}) \cdot \alpha$ and $\rho^{n-p+1}(f)\alpha = L^{n-p+1}\alpha = 0$.
2. is true because of dimension count and $\alpha \neq 0$. □

Let $(\cdot)^+ : \mathbb{R} \to \mathbb{R}$ be defined by $(x)^+ := maxx, 0 \ \ \forall x \in \mathbb{R}$.

Theorem 10 *(Lefschetz decomposition on differential forms)*

(a) *Let α be a complex differential form of degree p. Then there is an unique representation,*

$$\alpha = \sum_{2r \leq p} L^r \alpha_r$$

where for each r, α_r is a primitive p-form of degree $(p-2r)$. Moreover,

$$\alpha_r = \sum_{r,s} a_{r,s} L^s (L^*)^{r+s} \alpha$$

for some fixed, $a_{r,s} \in \mathbb{Q}$.

(b) *If $L^m\alpha = 0$, then $\alpha_r = 0$ for all $r \geq (n-p+m)^+$.*

In particular, if $p \leq n$ and $L^{n-p}\alpha = 0$, then $\alpha = 0$. □

PROOF Let W be the complex vector space $\bigwedge(\mathcal{A}_{X,\mathbb{C}})$. Since every finite dimensional $\mathfrak{sl}(2,\mathbb{C})$-representation can be decomposed into irreducible subrepresentations, we have:

$$W = W_1 \oplus \cdots \oplus W_l.$$

Now if α is any p-form in W and it $\alpha = \sum_j \beta_j$ for $\beta_j \in W_j$, then $d(\beta_j) = d(\beta) = p$ for all $j = 1, 2, ..., l$.
Now, since each β_j has weight $n-p$, the sub-representation generated by β_j has dimension $n-p+1$ and there is some $r_j \in \mathbb{N}$ such that

$$\beta_j = L^{r_j}\eta_j$$

where η_j is primitive form of degree $(p-2r_j)$ for all j, and we have

$$\eta_j = c_j \cdot (L^*)^{r_j}\beta_j$$

where $c_j \in \mathbb{Q}$ for all j.

For every r, let $\alpha_r := \sum_{r_j=r} \beta_j$, then we get

$$\alpha = \sum_{2r \leq p} L^r \alpha_r$$

Clearly, proving the expression for α_r will prove the uniqueness of the decomposition of α, which we will now do:

If $\alpha = \alpha_0 + L\alpha_1 + \cdots + L^m\alpha_m$, then $(L^*)^m\alpha = (L^*)^m\alpha_0 + (L^*)^{m-1}(L^* \circ L)\alpha_1 \cdots + (L^*)^m \circ L^m\alpha_m$

Since α_r are primitive and $(\rho(e)^r \circ \rho(f)^r)(\alpha_r) = b_r\alpha_r$ for $r = 1, ..., m$ (from 4.10), where $b_r \in \mathbb{Q}$ we get $\alpha_m = \frac{1}{b_m}\alpha$. The formula for α_r in (a) is then proved using induction on r.

Part (b) then follows from the uniqueness in part (a) and the previous proposition. $\square$

4.4.2 Important Identities

Let η be any differential form (possibly complex) of degree r, ($r \geq 0$), and recall the operator $e(\eta)$ and $e(\eta)^*$ that we were introduced to earlier this Chapter. For any two homogeneous forms α and β we have $e(\alpha \wedge \beta) = e(\alpha) \circ e(\beta)$. We also have the following proposition about formal duals:

Proposition 4.4.3 *Suppose* $T_1 : \wedge^*\mathcal{A}_{X,\mathbb{C}} \to \wedge^{*+k_1}\mathcal{A}_{X,\mathbb{C}}$ *and* $T_2 : \wedge^*\mathcal{A}_{X,\mathbb{C}} \to \wedge^{*+k_2}\mathcal{A}_{X,\mathbb{C}}$ *be two* $\mathbb{C}$*-linear maps of sheaves that have Quasi-adjoints* S_1 *and* S_2 *respectively and let* $c_1, c_2 : \mathbb{Z} \to \mathbb{C}$ *be the constants that give us* $\int_X T_j\alpha_j \wedge \beta_j = c_j(d^0\beta_j) \cdot \int_X \alpha_j \wedge S_j\beta_j$ *for any two homogeneous forms* α_j *and* β_j *satisfying* $k_j + d^0\alpha_j + d^0\beta_j = dim_{\mathbb{R}}X$*, for* $j = 1, 2$.

(a) If c_1 *and* c_2 *satisfy*

$$c_2(d^0(S_1\beta)) = c_2(d^0((\overline{*}^{-1}S_1\overline{*})\beta))$$

for any homogeneous form β*, then:* $(T_1 \circ T_2)^* = T_2^* \circ T_1^*$*. The converse is also true.*

(b) Moreover, if $(T_1 \circ T_2)^* = T_2^* \circ T_1^*$ *and* $(T_2 \circ T_1)^* = T_1^* \circ T_2^*$ *and* c_1*,* c_2 *satisfy*

$$c_2(d^0(S_1\beta)) \cdot c_1(d^0\beta) = c_1(d^0(S_2\beta)) \cdot c_2(d^0\beta)$$

for any homogeneous form β*, then*

$$[T_1, T_2]^* = [T_2^*, T_1^*]$$

PROOF We have

$$\int_X (T_1 \circ T_2)(\alpha) \wedge \beta = c_2(d^0(S_1\beta)) \cdot c_1(d^0\beta) \cdot \int_X \alpha \wedge (S_2 \circ S_1)(\beta)$$

$$\int_X (T_2 \circ T_1)(\alpha) \wedge \beta = c_1(d^0(S_2\beta)) \cdot c_2(d^0\beta) \cdot \int_X \alpha \wedge (S_1 \circ S_2)(\beta)$$

for any two homogeneous forms α and β satisfying $k_1 + k_2 + d^0\alpha + d^0\beta = dim_{\mathbb{R}}X$.

We also have

$$T_2^* \circ T_1^* = [(c_2 \circ d^0) \cdot (\overline{*}^{-1} S_2 \overline{*})] \circ [(c_1 \circ d^0) \cdot (\overline{*}^{-1} S_1 \overline{*})]$$

$$= (c_1 \circ d^0) \cdot (c_2 \circ d^0 \circ (\overline{*}^{-1} S_1 \overline{*})) \cdot (\overline{*}^{-1} (S_2 \circ S_1) \overline{*})$$

The rest of the proof is trivial. □

Note:

(a) If $c_1, c_2 : \mathbb{Z} \to \mathbb{C}^\times$ are group homomorphisms from the additive group of integers and the multiplicative group of non-zero complex numbers, with $c_1^2 = c_2^2 = 1$ (multiplication comes from the multiplication of $\mathbb{C}^\times$) then 1. is true and if moreover $c_2(k_1) = c_1(k_2)$, then 2. is true.

(b) If η_1 and η_2 are homogeneous differential forms of degrees of the same parity, then the operators $T_1 = e(\eta_1)$ and $T_2 = e(\eta_2)$ satisfy the above proposition. Moreover if η_1 and η_2 are homogeneous differential forms of degrees of opposite parity, then one of the η's have even degree and therefore the commutator $[e(\eta_1), e(\eta_2)] = 0$, and $[e(\eta_2)^*, e(\eta_1)^*] = 0$.

Let $\{x_1, y_1, ..., x_n, y_n\} = \{v_1, ..., v_{2n}\}$ be the orthonormal basis for the complex cotangent bundle of X w.r.t. the Hermitian metric given by the Kähler form ω. Then, $\omega = \sum_{1 \le j \le n} x_j \wedge y_j$ and from what we noted above we have

$$L = e(\omega) = \sum_{1 \le j \le n} e(x_j) \circ e(y_j)$$

which implies,

$$L^* = e(\omega)^* = \sum_{1 \le j \le n} e(y_j)^* \circ e(x_j)^*$$

Now, for 1-forms η we have $e(\eta)^* = *(e(\eta))*$ and we see that

$$e(v_{j_1})^*(v_{j_1} \wedge \cdots \wedge v_{j_k}) = v_{j_2} \wedge \cdots \wedge v_{j_k}$$

if $j_1 \notin \{j_2, \cdots j_k\}$ and zero otherwise. Next, we prove the following lemma:

Lemma 9

(a) $[L^*, e(x_j)] = e(y_j)^*$ *and* $[L^*, e(y_j)] = -e(x_j)^* \quad \forall j = 1, 2, \cdots n.$

(b) $[L^*, e(\zeta)] = -ie(\overline{\zeta})^*$ *and* $[L^*, e(\zeta)] = ie(\overline{\zeta})^*$ *for every form* ζ *of type* $(1,0)$.

(c) $[L^*, e(\alpha)] = -Je(\alpha)^* J^{-1}$ *for every real 1-form* α. □

Proof Fix $j \in \mathbb{N}_n$

$$[L^*, e(x_j)] = (\sum_{1 \leq k \leq n} e(y_k)^* e(x_k)^*)(e(x_j)) - (e(x_j))(\sum_{1 \leq k \leq n} e(y_k)^* e(x_k)^*)$$

$$= e(y_j)^* e(x_j)^* e(x_j) - e(x_j) e(y_j)^* e(x_j)^*$$

(from the last equality above we have that $e(x_j)$ commutes with $e(v_s)$ for all $v_s \neq x_j$). Suppose η is any homogeneous form. Then $\eta = \eta_0 + x_j \wedge \eta_1 + y_j \wedge \eta_2 + x_j \wedge y_j \wedge \eta_3$ where the forms $\eta_0, \eta_1, \eta_2, \eta_3$ are homogeneous forms where neither x_j nor y_j occur in them. Now, $[L^*, e(x_j)]\eta = \eta_2 - x_j \wedge \eta_3$ and $e(y_j)^*(\eta) = \eta_2 - x_j \wedge \eta_3$.
Now,

$$[L^*, e(y_j)] = [L^*, e(Jx_j)] = [L^*, J \circ e(x_j) \circ J^{-1}] = J \circ [L^*, e(x_j)] \circ J^{-1}$$

$$= Je(y_j)^* J^{-1} = J\overline{*}e(y_j)\overline{*}J^{-1} (since\ e(\alpha)^* = \overline{*}e(\alpha)\overline{*}\ \ \forall\ real\ 1 - form\ \alpha)$$

$$= \overline{*}Je(y_j)J^{-1}\overline{*} = \overline{*}e(-x_j)\overline{*}$$

This proves 1. and everything else follows from 1. □

Consider the exponential map

$$exp : \mathfrak{gl}(n, \mathbb{C}) \to GL(n, \mathbb{C}) \ : \quad A \mapsto e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

We use the fact that $e^{ad(A)} \circ P = e^A P e^{-A}$ for all $A \in \mathfrak{gl}(n, \mathbb{C})$ and for all $P \in End(n, \mathbb{C})$, where $ad(A)(B) := [A, B]$ for all $A, B \in \mathfrak{gl}(n, \mathbb{C})$ The exponential map can be defined in a similar way for any Lie Group G and its associated Lie algebra $\mathfrak{g}$. For the representation $\rho : \mathfrak{sl}(2, \mathbb{C}) \to \mathcal{G}$ we define

$$\chi := exp(\frac{i}{2}\pi\rho(e + f)) = exp(\frac{i}{2}\pi(L^* + L)).$$

Define $t \mapsto e_t(\alpha) := exp(it\rho(e + f)) \circ e(\alpha) \circ exp(-it\rho(e + f))$ from $\mathbb{C}$ to $\mathcal{G}$.

Lemma 10 *Let* α *be a real 1-form. Then* $\chi e(\alpha)\chi^{-1} = -iJe(\alpha)^* J^{-1}$ □

PROOF Clearly, $e_0(\alpha) = e(\alpha)$ and $e_{\frac{\pi}{2}}(\alpha) = \chi e(\alpha)\chi^{-1}$. The idea of the proof is to construct a initial value problem, i.e. to construct a differential equation in independent variable t and dependent variable $f(t)$ taking values in $\mathcal{G}$ such that $f(0) = e(\alpha)$.

Clearly $e_t(\alpha) = \sum_{k=0}^{\infty} \frac{1}{k!}[ad(it(L^* + L))]^k e(\alpha)$ and every $[ad(L^* + L)]^k$ is a sum of monomials in $ad(L^*)$ and $ad(L)$ and since we have:

(a) $B = [L^*, L] = L^* \circ L - L \circ L^*$

(b) $ad(L) \circ e(\alpha) = 0$ and $ad(-B) \circ e(\alpha) = e(\alpha)$

we get that

$$e_t(\alpha) = \sum_{k=0}^{\infty} c_k(t) ad(L^*)^k$$

where $c_k(t)$ are given by power series in t with complex coefficients.

Now, by the last lemma $[ad(L^*)]^k e(\alpha) = 0 \quad \forall k \geq 2$. So we get that

$$e_t(\alpha) = c_0(t)e(\alpha) + c_1(t)ad(L^*)e(\alpha)$$

Now differentiating $e_t(\alpha) = \sum_{k=0}^{\infty} \frac{1}{k!}[ad(it(L^* + L))]^k e(\alpha)$ and

$$e_t(\alpha) = c_0(t)e(\alpha) + c_1(t)ad(L^*)e(\alpha)$$

and comparing the coefficients, we get the system of differential equations: $c_0'(t) = ic_1(t)$ and $c_1'(t) = -c_0(t)$ now the unique solution of the above system that satisfies the initial condition is given by: $c_o(t) = \cos t$ and $c_1(t) = i \sin t$. So,

$$e_t(\alpha) = (\cos t)e(\alpha) + (i \sin t)ad(L^*)e(\alpha)$$

and the proof of the lemma is complete by putting $t = \frac{\pi}{2}$ above. $\square$

Lemma 11 *Let β be any complex differential p-form. Then*

$$\overline{*}(\beta) = i^{p^2 - n} J^{-1} \chi(\beta)$$

PROOF Let $\# := i^{p^2-n}J^{-1}\chi$. Then from equation-4.10 applied to the Lefschetz decomposition we get

$$\chi \cdot \rho(f)^k u_0 = \frac{i^m r!}{(m-r)!} \rho(f)^{m-r} u_0 \tag{4.11}$$

for every primitive form of weight m. Now, since $u_0 = 1$ is a primitive 0-form of weight $n = dim_{\mathbb{C}}X$, from equation-4.11 above we get

$$\chi(1) = \frac{i^n}{n!}L^n(1).$$

Hence $\#(1) = \dfrac{1}{n!}L^n(1) = Vol.$
Now, $\# \, e(\zeta)\beta = i^{(p+1)^2-n}J^{-1}\chi e(\zeta)\beta$

$$= i^{(p+1)^2-n}J^{-1}(\chi e(\zeta)\chi^{-1})\chi\beta = (-1)^p e(\zeta)^*(i^{p^2-n}J^{-1}\chi\beta) = (-1)^p e(\zeta)^* \# \, \beta$$

for every 1-form ζ. But, $\overline{*}$ also satisfies:

$$\overline{*}(1) = Vol.$$

$$\overline{*} \circ e(\zeta)\beta = (-1)^p e(\zeta) \circ \overline{*}\beta \quad \forall p - \text{form}\beta.$$

Now, as 1-forms generate the whole space of all forms of all degrees (denoted previously by W) on X, $\overline{*}$ and $\#$ are equal. □

Proposition 4.4.4 *Let β be any primitive complex differential p-form. Then*

$$\overline{*}L^r\beta = (-1)^{\frac{p(p+1)}{2}}\frac{r!}{(n-p-r)!}L^{n-p-r}J\beta$$

for all r such that $0 \le r \le n-p$

Proof Applying Theorem-8 for Lefschetz representation we see that the vector space generated by $\{\beta, \rho(f)\beta, ..., \rho(f)^{n-p}\beta\}$ is an irreducible sub-representation of the Lefschetz representation. Again applying equation-4.10 to the Lefschetz representation, we get:

$$\chi L\beta = \frac{i^{n-p} \cdot r!}{(n-p-r)!}L^{n-p-r}\beta$$

and by the last lemma we get

$$\overline{*}L^r\beta = i^{(p+2r)^2-n}J^{-1}\chi L^r\beta = i^{p^2-n}J^{-1}\frac{i^{n-p} \cdot r!}{(n-p-r)!}L^{n-p-r}\beta$$

$$= i^{p^2-p}J^{-2}\frac{r!}{(n-p-r)!}L^{n-p-r}J\beta$$

(as J commutes with L.) But this is exactly equal to the right hand side of the equality in the proposition. Notice that, $J^{-2}(\phi) = (-1)^p\phi$ for any p-form ϕ and we can put $\phi = J\beta$ □

4.5 Hodge Decomposition Theorem

Let d_c and d_c^* be defind by: $d_c := J^{-1} \circ d \circ J$ and $d_c^* = J^{-1} \circ d^* \circ J$ We will now prove the following lemma:

Lemma 12 ***(More Identities)***

(a) $d_c\phi = -i(\partial - \overline{\partial})\phi$ *and* $dd_c = 2\sqrt{-1}\partial\overline{\partial}$

(b) For a compact Kähler manifold X : $[L, d] = [L^*, d^*] = 0$ *and* $[L, d^*] = d_c$ *and* $[L^*, d] = -d_c$

(c) $[L, d_c] = [L^*, d_c^*] = 0$ *and* $[L, d_c^*] = -d$ *and* $[L^*, d_c] = d^*$

(d) $[L, \partial] = [L, \overline{\partial}] = [L^*, \partial^*] = [L^*, \overline{\partial}^*] = 0$, $[L, \partial^*] = i\overline{\partial}$, $[L, \overline{\partial}^*] = -i\partial$, $[L^*, \partial] = i\overline{\partial}^*$ *and* $[L, \overline{\partial}] = -i\partial^*$

(e) $d^*d_c = -d_cd^* = d^*Ld^* = -d_cL^*d_c$, $dd_c^* = -d_c^*d = d_c^*Ld_c^* = -dL^*d$

(f) $\partial\overline{\partial}^* = -\overline{\partial}^*\partial = -i\overline{\partial}^*L\overline{\partial}^* = -i\partial L\partial$ *and* $\overline{\partial}\partial^* = -\partial^*\overline{\partial} = i\overline{\partial}L\overline{\partial} = i\partial^*L\partial^*$ □

PROOF For every complex smooth function f,

(a)

$$Jd(Jf) = Jdf = J[\sum_{j=1}^{n}(\frac{\partial f}{\partial z_j}dz_j) + \sum_{j=1}^{n}(\frac{\partial f}{\partial \overline{z}_j}d\overline{z}_j)] =$$

$$= \sum_{j=1}^{n}(\frac{\partial f}{\partial z_j}Jdz_j) + \sum_{j=1}^{n}(\frac{\partial f}{\partial \overline{z}_j}Jd\overline{z}_j) = \sum_{j=1}^{n}(\frac{\partial f}{\partial z_j}\sqrt{-1}dz_j) + \sum_{j=1}^{n}(\frac{\partial f}{\partial \overline{z}_j}(-\sqrt{-1})d\overline{z}_j) = \sqrt{-1}(\partial - \overline{\partial})f$$

So, $dd_cf = (\partial + \overline{\partial})[\sqrt{-1}(\partial - \overline{\partial})]f = 2\sqrt{-1}\partial\overline{\partial}f$.

(b) The first part of (b) follows from the Kähler hypothesis on X; i.e. we have $[L, d] = 0$ and $[L^*, d^*] = 0$ by taking adjoints.

$d^* = (-1)^{p+1} * d*^{-1}$ on p-forms. Now from Lemma 11 we get

$$\chi^{-1}f = i^{(2n-p)^2}\overline{*}^{-1}J^{-1}f$$

Therefore,

$$\chi d\chi^{-1}f = i^{-(2n-p+1)^2+n}i^{p^2-n}J\overline{*}d\overline{*}^{-1}J^{-1}f = iJd^*J^{-1}f$$

Define $d_t := exp(it(L^* + L)) \circ d \circ exp(-it(L^* + L))$. Then following the proof of Lemma 10, we get

$$d_t = \sum_{k=0}^{\infty}\frac{1}{k!}[ad(it(L^* + L))]^k d$$

and by the Kähler hypothesis we get $[L, d] = 0$, and therefore, we have:

$$d_t = \sum_{k=0}^{\infty} c_k(t)[ad(L^*)]^k d$$

where $c_k(t)$ are analytic functions. We see that $d_{\frac{\pi}{2}} f = \chi \circ d \circ \chi^{-1} = iJd^*J^{-1}f$ which is an operator of degree -1, and therefore we have

$$d_{\frac{\pi}{2}} = c_1(\frac{\pi}{2})ad(L^*)d$$

and $c_1(\frac{\pi}{2})ad(L^*)d = iJd^*J^{-1}f$ we have $[ad(L^*)]^k f = 0 \quad \forall k \geq 2$. Thus,

$$d_t = c_0(t) + c_1(t)ad(L^*)d$$

and following analogous arguments of the proof of Lemma 10 we get

$$d_t = \cos(t) + i\sin(t)ad(L^*)d$$

So, just by letting $t = \frac{\pi}{2}$ we are done.

(c) This follows from (a) and (b).

By, $[L^*, d_c] = d^*$, we have $d^*d_c = -d_cL^*d_c + L^*d_cd_c = - - d_cL^*d_c$ and $-d_cd^* = d_cd_cL^* - d_cL^*d_c = -d_cL^*d_c$ This proves, $d^*d_c = -d_cd^*$.
The rest of the proof is similar. □

Theorem 11 ***(Kähler Identity)*** *Let X be a compact Kähler manifold. Then the Laplacian Δ_d commutes with $\overline{*}, d$ and L, and we have $\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$. So as a consequence we get that $\Delta_d(\mathcal{A}^{p,q}) \subseteq \mathcal{A}^{p,q} \quad \forall p, q$ that are non-negative integers, and that Δ_∂ and $\Delta_{\overline{\partial}}$ are real operators. Moreover, this implies*

$$[\Delta, \sigma] = 0,$$

for $\sigma = L^*, d, d^*, \partial, \overline{\partial}, \partial^*$ □

PROOF $[\Delta_d, L] = -d[L, d^*] - [L, d^*]d = -dd_c - d_cd = 0$, by the last lemma. Now

$$\Delta_d = dd^* + d^*d = d[L^*, d_c] + [L^*, d_c]d$$

$J\Delta_dJ^{-1} = -d_cL^*d + d_cdL^* - L^*dd_c + dL^*d_c = \Delta_d$ as $dd_c = -d_cd$
From the last lemma we get, $2\partial = d + id_c$ and $2\partial^* = d^* - id_c^*$. Therefore,

$$4\Delta_\partial = (dd^* + d^*d) + (d_cd_c^* + d_c^*d_c) + i(d_cd^* + d^*d_c) - i(dd_c^* + d_c^*d)$$

We also have

$$\Delta_d = J^{-1}\Delta_d J = d_c d_c^* + d_c^* d_c$$

and we are done ☐

Note: As $[L^*, \Delta_d] = [L^*, \Delta_d] = [L^*, \Delta_\partial] = [L^*, \Delta_{\overline{\partial}}] = 0$, the Lefschetz decomposition theorem gives a decomposition of cohomology. Let $\mathcal{H}_X^r$ be the set of all Δ_d-harmonic forms of degree r, $\mathcal{H}_X^{p,q}$ be the set of all $\Delta_{\overline{\partial}}$-harmonic forms of type (p, q) and let $\mathcal{H}_{X,prim}^r$ be the set of all **primitive harmonic forms** of degree r and let $\mathcal{H}_{X,prim}^{p,q}$ be the set of all **primitive harmonic forms** type (p, q).

Theorem 12 ***(Lefschetz decomposition theorem on cohomology)*** *For a compact Kähler manifold X, we have the decomposition of cohomology:*

$$H^k(X, \mathbb{C}) = \bigoplus_{2r \le k} L^r H_{prim}^{k-2r}(X, \mathbb{C})$$

and

$$H^{p,q}(X, \mathbb{C}) = \bigoplus_{2r \le k} L^r H_{prim}^{p-r,q-r}(X, \mathbb{C})$$

where $H_{prim}^l(X, \mathbb{C})$ is the cohomology of primitive l-forms. ☐

Proof We have the Lefschetz decomposition theorem of Δ_d-Harmonic forms

$$\mathcal{H}_X^k = \bigoplus_{2r \le k} L^r \mathcal{H}_{X,prim}^k$$

as $[L, \Delta_d] = 0$ and the first part follows from the Hodge isomorphism theorem, and the second is similar.

Theorem 13 ***(Hodge decomposition theorem on cohomology)*** *Let X be a Kähler manifold. Then we have the decomposition of cohomologies*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H_{\overline{\partial}}^k(X)$$

Proof We have the decomposition of Harmonic forms :

$$\mathcal{H}_X^k = \bigoplus_{p+q=k} \mathcal{H}_X^{p,q}$$

and the theorem follows from the Hodge Isomorphism Theorem. Indeed, if for any differential k-form α on X, if $\alpha = \sum_{p+q=k} \alpha_{p,q}$ where $\alpha_{p,q}$ is a differential form of type (p,q) on X. Then by the Kähler Identity, $\Delta_d(\alpha) = \sum_{p+q=k} \Delta_{\overline{\partial}}(\alpha_{p,q})$ is the corresponding decomposition of $\Delta_d(\alpha)$. So, if $\alpha \in \mathcal{H}_X^k$, then $\alpha_{p,q} \in \mathcal{H}_X^{p,q}$. So there is a map $\pi : \mathcal{H}_X^k \to \bigoplus_{p+q=k} \mathcal{H}_X^{p,q}$ defined by $\pi(\alpha) = (\alpha_{p,q})_{p+q=k}$. This map is injective by construction and it is also surjective since for any $\beta \in \mathcal{H}_X^{p,q}$,

$$\Delta_d \alpha = 2\Delta_{\overline{\partial}} \alpha = 0$$

and therefore $\beta \in \mathcal{H}_X^k$ and $\pi(\beta) = \beta$.

4.6 Applications

Lemma 13 ***$\partial\overline{\partial}$-Lemma*** *Let xi be a differential form on a Kähler manifold X, which is ∂ and $\overline{\partial}$-closed. Then if xi is exact under one of the operators $d, \partial, \overline{\partial}$, then xi is $\partial \circ \overline{\partial}$ exact; i.e. there is some differential form β on X such that $xi = \partial\overline{\partial}\beta$* □

PROOF Suppose that $\xi = \partial\phi$, for some differential form ϕ on X. Then by the Hodge isomorphism theorem, $\phi = \phi_1 + \Delta_d \phi_2$ where $\Delta_d \phi_1 = 0$. Now, ϕ_1 is ∂ closed since $\Delta = 2 \cdot \Delta_\partial$

From Lemma 12, we get $\partial\overline{\partial}^* = -\overline{\partial}^* \partial$. But then,

$$\xi = \partial\phi_1 + 2\partial\Delta_{\overline{\partial}}\phi_2 = 2\partial(\overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*)\phi_2$$

$$= -2\overline{\partial}^*(\partial\overline{\partial}\phi_2) + 2\partial(\overline{\partial}\partial)\phi_2$$

Now $-2\overline{\partial}^*(\partial\overline{\partial}\phi_2)$ is ∂-closed, as $-2\overline{\partial}^*(\partial\overline{\partial}\phi_2) = \xi - 2\partial(\overline{\partial}\partial)\phi_2$ and it is in the image of $\overline{\partial}^*$, and so, $-2\overline{\partial}^*(\partial\overline{\partial}\phi_2) = 0$, and $\xi = 2\partial(\overline{\partial}\partial)\phi_2$ □

Lemma 14 *For a Kähler manifold (X, ω), $H^{p,q}(X)$ is canonically isomorphic to $H^q(X, \Omega_X^p)$ and $H_{\overline{\partial}}^{p,q}(X)$* □

PROOF Now, $H^k(X, \mathbb{C}) \cong \mathcal{H}_d^k$, and $H^{p,q}(X) \cong \mathcal{H}_d^{p,q}$. But $\Delta_d = \Delta_{\overline{\partial}} \Rightarrow \mathcal{H}_d^{p,q} = \mathcal{H}_{\overline{\partial}}^{p,q}$

But by the analogue of Hodge isomorphism theorem $\mathcal{H}_{\overline{\partial}}^{p,q} \cong H^q(X, \Omega_X^p)$.

Clearly the Dolbeault complex

$$\cdots \xrightarrow{\overline{\partial}} \mathcal{A}^{p,q}(X) \xrightarrow{\overline{\partial}} \mathcal{A}^{p,q+1}(X) \xrightarrow{\overline{\partial}} \cdots$$

is a resolution of Ω_X^p by acyclic (fine actually) sheaves $\mathcal{A}_X^{p,q}$ and therefore, $H^q(X, \Omega_X^p) \cong H_{\overline{\partial}}^{p,q}(X)$ □

Definition 4.6.1 *Let* (X,ω) *be a compact Kähler manifold and consider the pairing*

$$Q^{(k)} : H^k(X,\mathbb{R}) \times H^k(X,\mathbb{R}) \to \mathbb{R}$$

$$Q^{(k)}([\alpha],[\beta]) := \int_X \omega^{n-k} \wedge \alpha \wedge \beta$$

Also define $H_Q^{(k)}([\alpha],[\beta]) = i^k Q^{(k)}([\alpha],[\overline{\beta}])$
for any two k-forms α *and* β *on* X; *where* $[\alpha],[\beta] \in H^k(X,\mathbb{R})$ *are cohomology classes of these forms* $\forall k \in \mathbb{Z}_{\geq 0}$.

Then we have the following proposition which is easy to see so we omit the proof:

Proposition 4.6.2

(a) $H_Q^{(k)}$ *are Hermitian and so* $Q^{(k)}$ *is alternating when* k *is odd and symmetric when* k *is even.*

(b) The Lefschetz decomposition

$$H^k(X,\mathbb{C}) = \bigoplus_{2r\leq k} L^r H^{k-2r}_{prim}(X,\mathbb{C})$$

is orthogonal with respect to $H_Q^{(k)}$; *and*

$$L^*(H_Q^{(k)}|_{L^r H^{k-2r}_{prim}(X,\mathbb{C})}) = (-1)^r H_Q^{(k-2r)}$$

(c) The Hodge decomposition

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

is orthogonal with respect to $H_Q^{(k)}$; *and* $(-1)^{\frac{k(k+1)}{2}} i^{p-q-k} H_Q^{(k)}$ *is a positive definite on the complex subspace* $H^{p,q}_{prim}(X,\mathbb{C})$

(d) The signature $sign(Q^{(n)})$ *of* $Q^{(n)}$ *is* $\sum_{p,q}(-1)^p h^{p,q}(X)$, *where* $h^{p,q} := dim_{\mathbb{C}} H^{p,q}(X)$

Definition 4.6.3 *Define* $K^p\mathcal{A}^k_X := \bigoplus_{r\geq p, r+q=k} \mathcal{A}^{r,q}_X$ *and* $F^pH^k(X,\mathbb{C}) := \bigoplus_{r\geq p, r+q=k} H^{r,q}(X)$

Then we have the following proposition:

Proposition 4.6.4 $F^pH^k(X,\mathbb{C}) = \dfrac{ker(d|_{K^pA^k})}{d(K^pA^{k-1})}$
In particular $H^{p,0}(X)$ *is isomorphic to the space of all holomorphic forms of degree* p *on* X.

Chapter 5

Spectral Sequences and Hypercohomology

5.1 Derived Category and Derived Functors

Let $\mathcal{A}$ be an abelian category that has enough injectives.

Definition 5.1.1

(a) ***(Complexes, Left Bounded Complexes and Injective complexes)***

i. *A complex (of increasing type)*

$$\cdots \to M^{k-1} \stackrel{d^{k-1}}{\to} M^k \stackrel{d^k}{\to} M^{k+1} \stackrel{d^{k+1}}{\to} \cdots$$

*in $\mathcal{A}$ (i.e. $d^{k+1} \circ d^k = 0 \quad \forall k \in \mathbb{Z}$) is said to be **left-bounded** if $\exists n \in \mathbb{Z}$ such that $M^k = 0, \forall k \leq n$. We denote any complex as above by (M^*, d) or by (M, d).*

ii. *A **non-negative complex** is a left-bounded complex (M, d_M) such that $M^k = 0, \forall k < 0$.*

iii. ***(Cochain Maps)***

*Let (M^*, d_M) and (N^*, d_N) be two complexes (or, left bounded complexes). A sequence of maps $f = (f^k : M^k \to N^k)_{k \in \mathbb{Z}}$ is said to be a **cochain map** $f : (M, d_M) \to (N, d_N)$ if*

$$f \circ d_M = d_N \circ f.$$

(with proper indices),i.e. the following diagram commutes:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & M^{k-1} & \overset{d_M^{k-1}}{\longrightarrow} & M^k & \overset{d_M^k}{\longrightarrow} & M^{k+1} & \overset{d_M^{k+1}}{\longrightarrow} \cdots \\
& \Big\downarrow{\scriptstyle f^{k-1}} & & \Big\downarrow{\scriptstyle f^k} & & \Big\downarrow{\scriptstyle f^{k+1}} & \\
\cdots \longrightarrow & N^{k-1} & \overset{d_N^{k-1}}{\longrightarrow} & N^k & \overset{d_N^k}{\longrightarrow} & N^{k+1} & \overset{d_N^{k+1}}{\longrightarrow} \cdots
\end{array} \tag{5.1}$$

iv. *Let $Kom(\mathcal{A})$ be the category of all complexes(as objects) and cochain maps(as morphisms) of $\mathcal{A}$, let ${}_{lb}Kom(\mathcal{A})$ be the subcategory of $Kom(\mathcal{A})$ consisting all left-bounded complexes in $\mathcal{A}$ and let ${}_0Kom(\mathcal{A})$ be the sub-category of all non-negative complexes of $\mathcal{A}$.*

v. *Here we call a complex $(I, d_I) \in Kom(\mathcal{A})$ to be* ***injective*** *if every I^k is an injective object in $\mathcal{A}$.*

(b) *Two cochain maps $(M, d_M) \underset{g}{\overset{f}{\rightrightarrows}} (N, d_N)$ are said to be* ***homotopic*** *(or* ***cochain homotopic****) if $\exists h = (h^k : M^k \to N^{k-1})_{k\in\mathbb{Z}}$ such that $f - g = d \circ h + h \circ d$ (with appropriate indices).*

This map h is called a ***cochain homotopy*** *or a* ***homotopy****.*

(c) *For any $n \in \mathbb{N}$, the n^{th}-cohomology of any complex $(M, d_M) \in Kom(\mathcal{A})$ is defined by $H^n(M, d_M) := \dfrac{ker(d_M^n)}{Im(d_M^{n-1})} \in \mathcal{A}$ (since $\mathcal{A}$ is closed under kernel, Image and quotients).*

A cochain map $f = (f^k : M^k \to N^k)_{k\in\mathbb{Z}}$ gives a morphism of of n^{th}-cohomologies

$$H^n(f) : H^n(M, d_M) \to H^n(N, d_N) \quad : \quad \alpha + Im(d_M^{n-1}) \longmapsto f^n(\alpha) + Im(d_N^{n-1})$$

This makes taking n^{th}-cohomology a covariant functor

$$\begin{array}{rcl}
H^n : \qquad Kom(\mathcal{A}) & \longrightarrow & \mathcal{A} \\
(M, d_M) & \longrightarrow & H^n(M, d_M) \\
((M, d_M) \overset{f}{\to} (N, d_N)) & \longrightarrow & (H^n(M, d_M) \overset{H^n(f)}{\to} H^n(N, d_N))
\end{array} \tag{5.2}$$

It can be seen easily that this functor maps homotopic cochain maps to the same map.

(d) *A chain map $(M, d_M) \overset{f}{\to} (N, d_N)$ is said to be a* ***quasi-isomorphism*** *if all the morphisms $H^n(M, d_M) \overset{H^n(f)}{\to} H^n(N, d_N)$ are isomorphisms in $\mathcal{A}$.*

Let $S(\mathcal{A})$ be the full sub-category of $Kom(\mathcal{A})$ in which morphisms are all quasi-isomorphisms.

Let ${}_{lb}S(\mathcal{A})$ be the full subcategory of ${}_{lb}Kom(\mathcal{A})$ whose morphisms are all quasi-isomorphisms.

Let ${}_0S(\mathcal{A})$ be the full subcategory of ${}_0Kom(\mathcal{A})$ whose morphisms are all quasi-isomorphisms.

(e) *For every* $n \in \mathbb{Z}$ *and any complex* $(M, d_M) \in Kom(\mathcal{A})$ *define the* n*-shift of* M *as another complex* $(M[n], d) \in Kom(\mathcal{A})$ *such that* $M[n]^k := M^{k+n}$ *and* $d^k := (-1)^n d_M^{k+n} \quad \forall k \in \mathbb{Z}$. *So we can make any left-bounded complex a non-zero complex by shifting it.*

(f) ***(Homotopy Categories)***

For any (M, d_M) (N, d_N) *in* $Kom(\mathcal{A})$, *let* $K^n(M, N) := Kom(\mathcal{A})(M, N[n]) \in Ab.$, *i.e. the set of all cochain maps* $((M, d_M) \xrightarrow{f} (N[n], d_{N[n]}))$ *in* $\mathcal{A}$, *(where Ab. is the category of abelian groups).*

Define $D^n : K^n(M, N) \to K^{n+1}(M, N)$ *by*

$$(D^n(f))^k := (-1)^{n+1} d_N^{k+n} \circ f^k + f^{k+1} \circ d_M^k = -d_{N[n]} \circ f^k + f^{k+1} \circ d_M^k \quad \forall k \in \mathbb{Z}.$$

Then we see that $(K(M, N), D)$ *is a cochain-complex in Ab.*

We define the ***Homotopy category*** $K(\mathcal{A})$ *whose objects are the objects of* $Kom(\mathcal{A})$ *but for any two* (M, d_M) (N, d_N) *in* $Kom(\mathcal{A})$, *the morphisms from* M *to* N *are defined by* $K(\mathcal{A})(M, N) := H^0(K(M, N), D)$; *i.e. the* 0^{th}*-cohomology of the complex* $(K(M, N), D)$ *in Ab.*

It can be easily checked that the morphisms in $K(\mathcal{A})(M, N)$ *are the cochain maps* $((M, d_M) \xrightarrow{f} (N, d_N))$ *in* $\mathcal{A}$ *upto homotopy.*

Consider the (covariant) functor $\kappa : Kom(\mathcal{A}) \to K(\mathcal{A})$ *from chain complexes in* $\mathcal{A}$ *to the homotopy category of* $\mathcal{A}$, *that maps every object to itself and every cochain map to its homotopy class.*

Define ${}_{lb}K(\mathcal{A}) := \kappa({}_{lb}Kom(\mathcal{A}))$ *and* ${}_0K(\mathcal{A}) := \kappa({}_0Kom(\mathcal{A}))$.

(g) ***(Localization of a Category)***

Let K *be a category and* S *be a subcategory of* K *whose objects are same as the objects of* K *but whose morphisms are some morphisms in* K, *which is closed under composition (and has the identity element corresponding to every object in* K*). A (covariant) functor* $T : K \to C$ *is said to be a* ***localizer*** *of the pair* (K, S) *if* $T(s)$ *is an isomorphism in* C *for every morphism* s *of* S. *Let* L *be the category of all localizers of the pair* (K, S) *with morphisms being natural transformations. The initial object* $F : K \to K_S$ *of* L *(if it exists) is called the* ***localization*** *of* K *at* S. *Most of the time we will call the category* K_S *the* ***localization*** *of the pair* (K, S).

Note that if K *is an abelian category we would want all the functors* $T : K \to C$ *to* *be additive functors and C be an abeliam category and $F : K \to K_S$ is additive. But mostly K_S is an additive category but not abelian category.*

(h) ***(Derived Categories)***

The category $S(\mathcal{A})$ of quasi-isomorphisms naturally gives a subcategory of $K(\mathcal{A})$ which we also denote by $S(\mathcal{A})$. Then the ***derived category*** *of $\mathcal{A}$ is defined by $K(\mathcal{A})_{S(\mathcal{A})}$, (if it exists) and is denoted by $\mathcal{D}(\mathcal{A})$.*

Similarly we define the derived categories : ${}_{lb}\mathcal{D}(\mathcal{A}) := {}_{lb}K(\mathcal{A})_{{}_{lb}S(\mathcal{A})}$ and ${}_0\mathcal{D}(\mathcal{A}) := {}_0K(\mathcal{A})_{{}_0S(\mathcal{A})}$

If we assume that $\mathcal{D}(\mathcal{A})$ exists, then taking n^{th}-cohomology is a functor $H^n : D(\mathcal{A}) \to Ab$.

(i) ***(Cone of a morphism of complexes)***

Let $(M, d_M) \xrightarrow{f} (N, d_N)$ be a morphism of complexes in $\mathcal{A}$. We define the **cone** *of f by the element $C_f := (C^k, d_C^k)_{k\in\mathbb{Z}}$ defined by :*

$C^k := M^k \oplus N^{k-1}$ and $d_C^k := \begin{pmatrix} d_M^k & 0 \\ f^k & -d_N^{k-1} \end{pmatrix}$,

treating elements of C_f^k as column vectors $\begin{pmatrix} v \\ w \end{pmatrix}$ such that $v \in M^k$, $w \in N^{k-1}$

We now state the following theorem without proof:

Theorem 14 *Let $0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} P \to 0$ be a short exact sequence in the category $Kom(\mathcal{A})$. Then, taking cohomology gives an exact sequence:*

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H^{k-1}(M, d_M) & \xrightarrow{H^{k-1}(\alpha)} & H^0(N, d_N) & \xrightarrow{H^{k-1}(\beta)} & H^{k-1}(P, d_P) \\
 & & & & & \swarrow{\scriptstyle \delta^{k-1}} & \\
H^k(P, d_P) & \xleftarrow[H^k(\beta)]{} & H^k(N, d_N) & \xleftarrow[H^k(\alpha)]{} & H^k(M, d_M) & & \\
\downarrow{\scriptstyle \delta^k} & & & & & & \\
H^{k+1}(M, d_M) & \xrightarrow{H^{k+1}(\alpha)} & H^{k+1}(N, d_N) & \xrightarrow{H^{k+1}(\beta)} & H^{k+1}(P, d_P) & \xrightarrow{\delta^{k+1}} & \ldots
\end{array} \tag{5.3}$$

□

where the map (called the connecting maps) $\delta^k : H^k(P, d_P) \to H^{k+1}(M, d_M)$ is defined by

$$\delta^k(p + Im(d_P^{k-1})) := (\alpha^{-1} \circ d_N \circ \beta^{-1})(p) + Im(d_M^k).$$

From, the definition above of the mapping cone of a morphism $(M, d_M) \xrightarrow{f} (N, d_N)$ we get the split exact sequence:

$$0 \to N[-1] \to C_f \to M \to 0$$

in $\mathcal{A}$. Then by the above theorem we get the long exact sequence of cohomology and the connecting maps

$$\delta^k : H^k(M, d_M) \to H^{k+1}(N[-1], d_{N[-1]}) = H^k(N, d_N)$$

are given by $H^k(f)$.

Thus we get the following proposition:

Proposition 5.1.2

(a) The cochain map f is a quasi-isomorphism iff the mapping cone (C_f, d_{C_f}) is acyclic.

(b) The objects M and N are injective if and only if the mapping cone C_f is injective.

Proof The first part is clear from the discussion above.

The second part is clear since the cone is constructed using direct sum. We now prove the following lemma for left-bounded complexes:

Lemma 15 *Let $(M, d_M) \in {}_{lb}Kom(\mathcal{A})$. Then there is an injective complex $(I, d_I) \in {}_{lb}Kom(\mathcal{A})$ and a cochain map $i : (M, d_M) \to (I, d_I)$ such that:*

(a) Each i^k is monic.

(b) i is a quasi-isomorphism. □

Note: For now suppose that $(M, d_M) \in {}_0Kom(\mathcal{A})$, otherwise we will shift the indices of M, and make it happen.

Since $\mathcal{A}$ has enough injectives, for each M^k, there is an injective resolution

$$0 \to M^k \to J^{k,0} \underset{j^{k,0}}{\to} J^{k,1} \underset{j^{k,1}}{\to} J^{k,2} \underset{j^{k,2}}{\to} \cdots$$

So we immediately get a double sequence $(J^{k,l})_{k,l \in \mathbb{Z}}$ coming from these injective resolutions. So if we take $J^n := \bigoplus_{k+l=n} J^{k,l}$ and

$$d_J^n := \bigoplus_{k+l=n} j^{k,l} : J^n \to J^{n+1}$$

then it is true that (J, d_J) is an injective complex and the obvious inclusion $M^n \to J^n$ (j^n say) is monic.

However it is not clear whether the map $j := (j^n)_{n\in\mathbb{Z}} : (M, d_M) \to (J, d_J)$ is a cochain map or not. But we need a cochain map like that because otherwise we will not get a map between cohomologies. So we follow the following lines of proof. But first we need the next definition:

Definition 5.1.3 ***(Double Complexes and their morphisms and the associated Total Complex)***

A ***double complex*** (M, d_1, d_2) *consists of double-sequence* $M := (M^{k,l})_{k,l\in\mathbb{Z}}$ *in* $\mathcal{A}$ *with maps* $d_1^{k,l} : M^{k,l} \to M^{k+1,l}$ *and* $d_2^{k,l} : M^{k,l} \to M^{k,l+1}$ *such that* $d_1 \circ d_2 = d_2 \circ d_1$ *with appropriate indices.*

The ***total complex*** *associated to this double complex is a complex* $(Tot(M), D)$ *where* $Tot(M)^n := \bigoplus_{k+l=n} M^{k,l}$ *and* $D^n := \bigoplus_{k+l=n}(d_1^{k,l} + (-1)^k d_2^{k,l})$ *for all integer* $n \geq 0$

A ***morphism of double complexes*** $f : (M, d_1, d_2) \to (N, \partial_1, \partial_2)$ *is a collection of maps* $f = (f^{k,l} : M^{k,l} \to N^{k,l})_{k,l\in\mathbb{Z}}$ *such that* $f^{*,l} : (M^{*,l}, d_1^{*,l}) \to (N^{*,l}, \partial_1^{*,l})$ *and* $f^{k,*} : (M^{k,*}, d_2^{k,*}) \to (N^{k,*}, \partial_2^{k,*})$ *are cochain maps* $\forall k, l \in \mathbb{Z}$.

Let $f : (M, d_1, d_2) \to (N, \partial_1, \partial_2)$ *be a morphism of double complexes. Then this map induces a map of the assocaited total complexes* $Tot(f) : (Tot(M), d) \to (Tot(N), \partial)$*, defined by:*

$$Tot(f)^n := \bigoplus_{k+l=n} f^{k,l} : Tot(M)^n \to Tot(N)^n \quad \forall n \in \mathbb{Z}$$

then $Tot(f)$ *is a cochain map, called the* ***total map associated to*** f.

Proof First we construct an injective complex $(I^{*,0}, d_{1,I}^*)$ and a co-chain map of monics $i^{*,0} : (M^*, d_M^*) \to (I^{*,0}, d_{1,I}^{*,0})$, which is done in three steps below. Then if we apply the same construction to $Coker(i^{*,0})$ in place of (M, d_M), we get another injective complex $(I^{*,1}, d_{1,I}^*)$ and cochain maps $d_{2,I}^{*,0} : (I^{*,0}, d_{1,I}^*) \to (I^{*,1}, d_{1,I}^*)$ and we keep repeating to get a double complex $((I^{k,l})_{k,l\geq -1}, d_{1,I}, d_{2,I})$, where $(I^{*,-1}, d_{1,I}^{*,-1}) := (M, d_M)$, $d_{2,I}^{*,-1} := i^{*,-1}$ and $i^{*,0} : (M^*, d_M^*) \to (I^{*,0}, d_{1,I}^*)$ is a chain map of monics and $(I^{k,*}, d_{2,I})$ is a resolution of M^k for all integers $k \geq 0$ (i.e.

$$0 \to M^k \to I^{k,0} \to I^{k,1} \to \cdots$$

is an exact sequence.)

Step 1: Let $I^{0,0}$ be an injective object in $\mathcal{A}$ such that $M^0 \overset{i^{0,0}}{\to} I^{0,0}$ is monic.

We will now construct the push-out of the adjacent diagram.

Clearly, this push-out is $Coker(i^{0,0}, -d_M)$.

$$\begin{array}{ccc} M^0 & \xrightarrow{i^{0,0}} & I^{0,0} \\ \big\downarrow{\scriptstyle d_M^0} & & \\ M^1 & & \end{array}$$

Now let $I^{1,0}$ be an injective object in $\mathcal{A}$ and $p^1 : Coker(i^{0,0}, -d_M) \to I^{1,0}$ is monic. Let $i^{1,0} : M^1 \to I^{1,0}$ be the composition of $M^1 \to Coker(i^{0,0}, -d_M)$ with p^1 and let $d_{1,I}^{0,0} : I^{0,0} \to I^{1,0}$ be the composition of $I^{0,0} \to Coker(i^{0,0}, -d_M)$ with p^1.

Then since $i^{0,0}$ is monic, so is $i^{1,0}$ and moreover, $d_{1,I}^{0,0} \circ i^{0,0} = i^{1,0} \circ d_M$. So till this step the adjacent diagram commutes:

$$\begin{array}{ccc} M^0 & \xrightarrow{i^{0,0}} & I^{0,0} \\ \big\downarrow{\scriptstyle d_M^0} & & \big\downarrow{\scriptstyle d_{1,I}^{0,0}} \\ M^1 & \xrightarrow{i^{1,0}} & I^{1,0} \end{array}$$

Step-2 : Now in the next step, not only we want to extend the last box diagram to the next diagram:

$$\begin{array}{ccc} M^0 & \xrightarrow{i^{0,0}} & I^{0,0} \\ \big\downarrow{\scriptstyle d_M^0} & & \big\downarrow{\scriptstyle d_{1,I}^{0,0}} \\ M^1 & \xrightarrow{i^{1,0}} & I^{1,0} \\ \big\downarrow{\scriptstyle d_M^1} & & \big\downarrow{\scriptstyle d_{1,I}^{1,0}} \\ M^2 & \xrightarrow{i^{2,0}} & I^{2,0} \end{array}$$

But we also want that the two vertical arrows on the right; i.e. $d_{1,I}^0$ and $d_{1,I}^1$ compose to zero so that $(I, d_{1,I})$ becomes a complex. So we don't just construct the pushout of

$$\begin{array}{ccc} M^1 & \xrightarrow{i^{1,0}} & I^{1,0} \\ \big\downarrow{\scriptstyle d_M^1} & & \\ M^2 & & \end{array}$$

but we construct the push-out, say P of the diagram:

$$\begin{array}{ccc} M^1 & \longrightarrow & Coker(d_{1,I}^0) \\ \big\downarrow{\scriptstyle d_M^1} & & \\ M^2 & & \end{array}$$

in a similar way as before (where the horizontal arrow is the map induced by $i^{1,0}$.) and find an injective object $I^{2,0}$ and a map $p^2 : P \to I^{2,0}$ which is monic. Let $d_{1,I}^2 : I^{1,0} \to I^{2,0}$ be the composition of $I^{0,0} \to Coker(d_{1,I}^0)$ with $Coker(d_{1,I}^0) \to P$ composed by p^2. Let $i^{2,0}$ be the composition of $M^2 \to P$ with p^2. Then we get what we want in the diagram:

$$\begin{array}{ccc} M^0 & \xrightarrow{i^{0,0}} & I^{0,0} \\ \downarrow d_M^0 & & \downarrow d_{1,I}^{0,0} \\ M^1 & \xrightarrow{i^{1,0}} & I^{1,0} \\ \downarrow d_M^1 & & \downarrow d_{1,I}^{1,0} \\ M^2 & \xrightarrow{i^{2,0}} & I^{2,0} \end{array}$$

Now for every integer $k \geq 2$, having constructed $I^{k,0}$ the way we want, we construct $I^{k+1,0}$ by applying Step-2 to the box diagram:

$$\begin{array}{ccc} M^{k-1} & \xrightarrow{i^{k-1,0}} & I^{k-1,0} \\ \downarrow d_M^{k-1} & & \downarrow d_{1,I}^{k-1,0} \\ M^k & \xrightarrow{i^{k,0}} & I^{k,0} \end{array}$$

(end of step-2)

Now take the total complex (I, d_I) associated to the double complex $((I^{k,l})_{k,l\geq 0}, d_{1,I}, d_{2,I})$ and for every integer $n \geq 0$,

let i^n be the map $i^{n,0}$ composed with the inclusion $I^{n,0} \to I^n$. The proof is almost complete except for the fact that the cochain map i^n is a quasi-isomorphism, which is the next step.

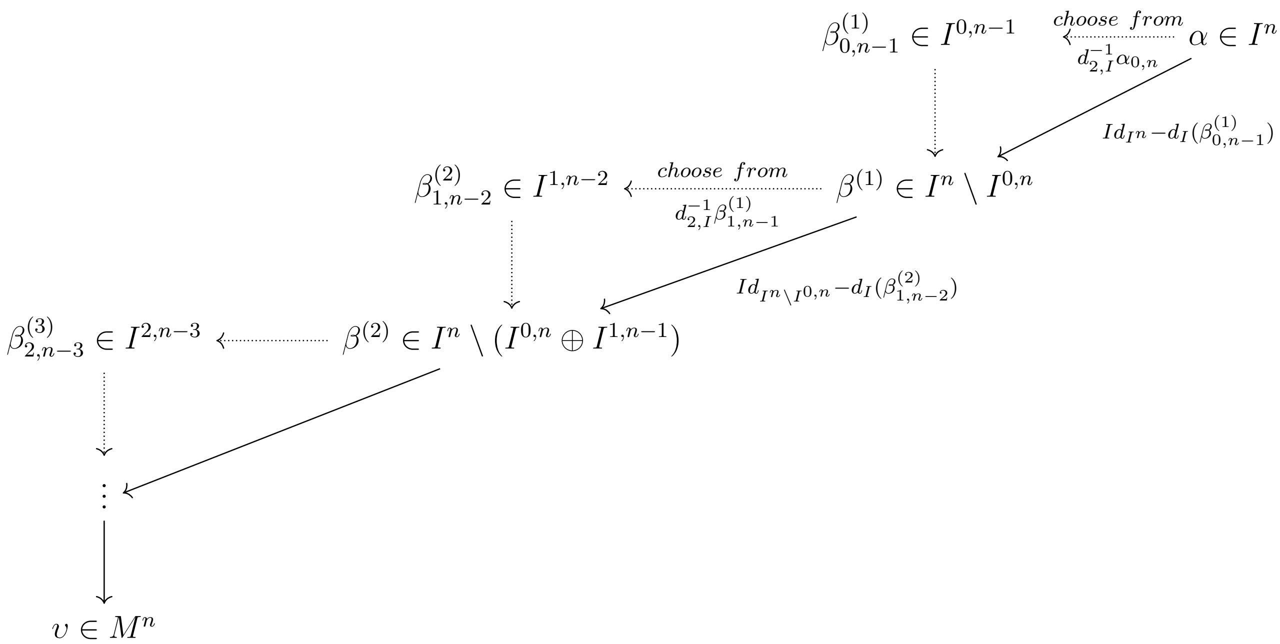
Step-3 : The **Mitchell's Full Imbedding theorem** tells us that : **Every small abelian category admits a full and faithful exact functor from itself to the category of abelian groups** (See Mitchell's, **Theory of Categories**, page no. 151, and [7]) Following this we will prove this lemma only in the category of abelian groups:

Surjectivity of $H^n(i)$:

Suppose that $\alpha = \sum_{p+q=n} \alpha_{p,q} \in ker d_I^n$ and then

$$d^n(\alpha) = \sum_{p+q=n} (d_{1,I}^{p,q} + (-1)^p d_{2,I}^{p,q})\alpha_{p,q} = d_{2,I}^{0,n}\alpha^{0,n} + \sum_{p+q=n, p\neq 0} (d_{1,I}^{p,q} + (-1)^p d_{2,I}^{p,q})\alpha_{p,q}$$

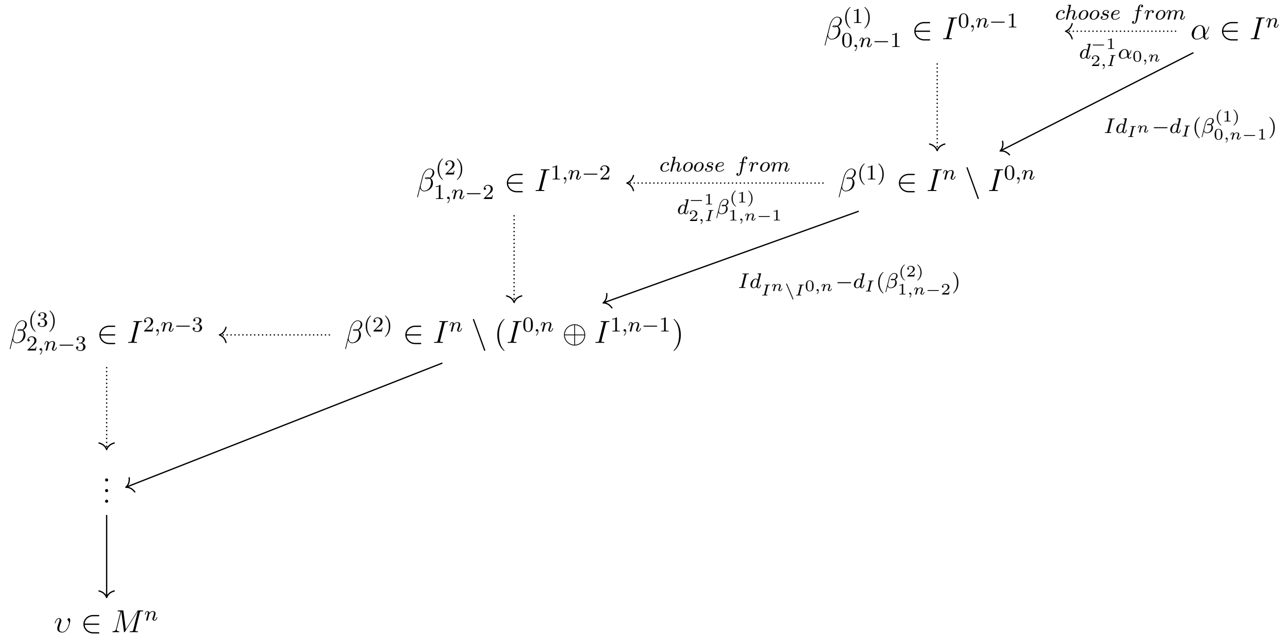
(see the double-complex:
$$\begin{array}{ccccccc}
M^0 & \xrightarrow{i^{0,0}} & I^{0,0} & \xrightarrow{d_{2,I}^{0,0}} & I^{0,1} & \xrightarrow{d_{2,I}^{0,1}} & \cdots \\
\downarrow{\scriptstyle d_M^0} & & \downarrow{\scriptstyle d_{1,I}^{0,0}} & & \downarrow{\scriptstyle d_{1,I}^{0,1}} & & \\
M^1 & \xrightarrow{i^{1,0}} & I^{1,0} & \xrightarrow{d_{2,I}^{1,0}} & I^{1,1} & \xrightarrow{d_{2,I}^{1,1}} & \cdots \\
\downarrow{\scriptstyle d_M^1} & & \downarrow{\scriptstyle d_{1,I}^{1,0}} & & \downarrow{\scriptstyle d_{1,I}^{1,1}} & & \\
M^2 & \xrightarrow{i^{2,0}} & I^{2,0} & \xrightarrow{d_{2,I}^{2,0}} & I^{2,1} & \xrightarrow{d_{2,I}^{2,1}} & \cdots \\
\downarrow{\scriptstyle d_M^2} & & \downarrow{\scriptstyle d_{1,I}^{2,0}} & & \downarrow{\scriptstyle d_{1,I}^{2,1}} & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$
) and therefore, $d_{2,I}\alpha_{0,n} = 0 \in I^{0,n+1}$ and $d_1\alpha_{p,q} + (-1)^{p+1}d_2\alpha_{p+1,q-1} = 0 \in I^{p+1,q}$ $\forall q \geq 1$. The exactness of $M^k \to I^{k,0} \to I^{k,1} \to I^{k,2} \to \cdots$ for all integers $k \geq 0$ implies that there is some $\beta_{0,n-1} \in I^{0,n-1}$ such that $\alpha_{0,n} = d_{2,I}\beta^{(1)}_{0,n-1}$. We get, $\beta^{(1)} := \alpha - d_I\beta^{(1)}_{0,n-1} \in \bigoplus_{p+q=n, p\neq 0} I^{p,q}$ is cohomologous to α in the total complex and $d_{2,I}\beta^{(1)}_{1,n-1} = 0$. Now we apply similar arguments on $\beta^{(1)}$ and keep doing this inductively unless we obtain an element $\gamma \in I^{n,0}$. Now, $d_I\gamma = 0$ implies $d_{1,I}\gamma = 0$ and $d_{2,I}\gamma = 0$. $d_{2,I}\gamma = 0$ means that $\exists \upsilon \in M^n$ such that $\gamma = i^n(\upsilon)$ and therefore we see that $d_{1,I}\gamma = 0$ implies $\upsilon \in ker(d_M)$. The process of finding successive β's from the *alpha*'s can be described by the following downward staircase:



where $Id_S : S \to S$ is the identity map of S for every set S.

Injectivity of $H^n(i)$:

Suppose that $\mu \in M^{n+1}$ is such that $i^n(\mu) = d_I\alpha$ for some $\alpha \in I^n$, then using almost similar arguments (since $i^n(\mu) \in I^{n,0}$) we find an absolutely same downward staircase of successive β's as before:

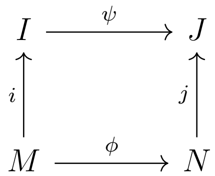


where $Id_S : S \to S$ is the identity map of S for every set S. and since each new term β is cohomlogous to α we get that $\mu = d_M(\upsilon)$. $\square$

Let $T : \mathcal{A} \to \mathcal{B}$ be a left-exact additive functor from an abelian category $\mathcal{A}$, with enough injectives, into an abelian category $\mathcal{B}$. Let $(M, d_M) \in {}_0Kom(\mathcal{A})$. The above lemma guarantees the existence of an injective complex (I, d_I) in ${}_0Kom(\mathcal{A})$ and an injective quasi-isomorphism $i : (M, d_M) \to (I, d_I)$, let us denote this pair by (I, i), and call it an **injective embedding** of (M, d_M).

Proposition 5.1.4 ***(Functoriality)*** *Let $(M, d_M), (N, d_N) \in {}_0Kom(\mathcal{A})$, $i : (M, d_M) \to (I, d_I)$ is a co-chain map and $j : (N, d_N) \to (J, d_J)$ is any cochain map to an injective complex (J, d_J).*

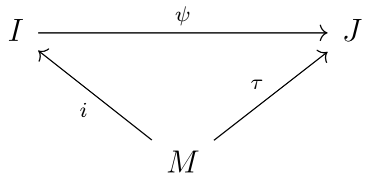
Every morphism $\phi \in {}_0Kom(\mathcal{A})(M, N)$ induces a morphism $\psi \in {}_0Kom(\mathcal{A})(I, J)$ such that the adjacent diagram commutes:



Moreover, $T(\psi)$ is a quasi-isomorphism in the category $\mathcal{B}$, if ϕ and j are quasi-isomorphisms and (I, i) is an injective embedding of (M, d_M). Lastly, ψ is unique upto homotopy if ϕ is monic.

Proof Consider the composed map $\tau := j \circ \phi$ and we prove the following lemma:

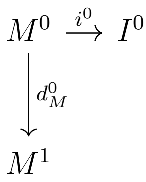
Lemma 16 *Suppose that (I, i) is an injective embedding of (M, d_M) and $\tau : (M, d_M) \to (J, d_J)$ is a cochain map to an injetive complex (J, d_J) Then $\exists$ a cochain map $\psi : (I, d_I) \to (J, d_J)$ such that the adjacent diagram commutes:*



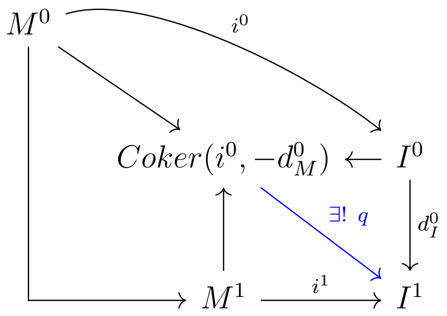
such that $T(\psi)$ is a quasi-isomorphism in the category $\mathcal{B}$, if i ,ϕ and τ are quasi-isomorphisms.$\square$

Proof of the Lemma : Let $\psi^0 : I^0 \to J^0$ be a map such that $\psi^0 \circ i^0 = \tau^0$ which exists since i^0 is monic and J^0 is injective.

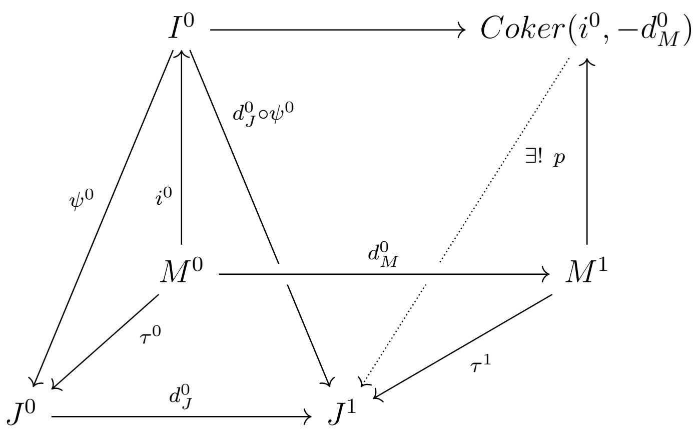
Clearly, $Coker(i^0, -d_M^0)$ is the



push-out of the adjacent diagram and so $\exists! q$ (shown in the diagram below as a blue arrow) commuting the adjacent diagram



and therefore $Im(q) \subseteq Im(i^1 + d_I^0)$ and $\exists! p$ (shown in the diagram below)



Now if q is monic, by injectivity of J^1, $\exists!$ ψ^1 (shown in the diagram below as a red arrow) such that the following diagram commutes:

$$\begin{array}{ccc} Coker(i^0, -d_M^0) & \xrightarrow{q} & I^1 \\ \uparrow & \overset{p}{\searrow} & \downarrow {\scriptstyle \exists!\ \psi^1} \\ M^1 & \xrightarrow{\tau^1} & J^1 \end{array}$$

but q is injective if the chain complex $M^0 \overset{i^0+d_M^0}{\rightarrow} I^0 \oplus M^1 \overset{d_I^0-i^1}{\rightarrow} I^1$ is an exact sequence. We get $\psi^1 \circ d_I^0 = d_J^0 \circ \psi^0$. Now having defined $\psi^k : I^k \to J^k$ such that $\psi^k \circ d_I^{k-1} = d_J^{k-1} \circ \psi^{k-1}$ and $\tau^1 = \psi^1 \circ i^1$, we define $\psi^{k+1} : I^{k+1} \to J^{k+1}$ by replacing the indices 0 by k everywhere in the above argument except for the first line.
(**prove the exactness of** $(M^0 \overset{i^0+d_M^0}{\rightarrow} I^0 \oplus M^1 \overset{d_I^0-i^1}{\rightarrow} I^1)$ **or find another argument.**
Moreover, it is clear that ψ is a quasi-isomorphism if i, ϕ, j are quasi-isomorphisms. So what is left in the proof of this lemma is to prove the following claim:

Claim: If $\psi : (I, d_I) \to (J, d_J)$ is a quasi-isomorphism, so is

$$T(\psi) : (T(I), d_{T(I)}) \to (T(J), d_{T(J)}).$$

Proof of the claim : This claim follows from Proposition 5.1.2 and the fact that $T(I^*)$ is exact if T is left-exact and I^* is injective.
Now since ψ is a quasi-isomorphism, the cone C_ψ of ψ is acyclic and injective and therefore $T(C_\psi)$ is exact, and we have the long exact sequence:

$$\begin{array}{l} \cdots \longrightarrow H^{k-1}(T(J^*)[-1]) \longrightarrow H^{k-1}(T(C_\psi^*)) \longrightarrow H^{k-1}(T(I^*)) \\ \xrightarrow{H^{k-1}(T(\psi))} H^k(T(J^*)[-1]) \longrightarrow H^k(T(C_\psi^*)) \longrightarrow H^k(T(J^*)) \\ \xrightarrow{H^k(T(\psi))} H^{k+1}(T(J^*)[-1]) \longrightarrow H^{k+1}(T(C_\psi^*)) \longrightarrow H^{k+1}(T(I^*)) \longrightarrow \ldots \end{array}$$

and thus $T(\psi)$ is a quasi-isomorphism. 16 and the lemma proves the proposition except the uniqueness (upto homotopy) part which we will show now: Let μ be another morphism satisfying all the properties of ψ proved so far. Consider the morphism $\psi - \mu : (I, d_I) \to (J, d_J)$. Clearly, this is a morphism satisfies the commutative daigram:

$$\begin{array}{ccc} (I, d_I) & \xrightarrow{\psi-\mu} & (J, d_J) \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle j} \\ (M, d_M) & \xrightarrow{0} & (N, d_N) \end{array}$$

□

The last proposition allows us to define the following:

Definition 5.1.5 *Let $T : \mathcal{A} \to \mathcal{B}$ is an left-exact additive functor of abelian categories where the domain category has enough injectives.*

(a) ***(Injective resolution)***

Let $Q_A : {}_{lb}Kom(\mathcal{A}) \to {}_{lb}\mathcal{D}(\mathcal{A})$ and $Q_B : {}_{lb}Kom(\mathcal{B}) \to {}_{lb}\mathcal{D}(\mathcal{B})$ be the obvious (covariant) functors from the categories $\mathcal{A}$ and $\mathcal{B}$ to their derived categories. The above proposition shows us that for every $(M, d_M) \in {}_{lb}Kom(\mathcal{A})$ we have an element $I(M) \in \mathcal{D}(\mathcal{A})$ (namely $I(M) = Q_A((I, d_I))$ where (I, i) is an injective embedding of (M, d_M)) which is unique upto unique isomorphism, such that $I(M)$ is Q_A-image of an injective complex and the homologies of $I(M)$ and M are same. This $I(M)$ is called the ***injective resolution*** *of (M, d_M). The last proposition shows that every morphism $\phi : (M, d_M) \to (N, d_N)$ defines a morphism $I(\phi) : I(M) \to I(N)$, which (comes from ψ) is unique upto unique homotopy.*

(b) ***(Derived Functor)***

Define $RT : {}_{lb}\mathcal{D}(\mathcal{A}) \to {}_{lb}\mathcal{D}(\mathcal{B})$, by: $RT((M, d_M)) := T(I(M))$, then RT satisfies $RT \circ Q_A = Q_B \circ T$. We call RT the (right-) ***derived functor*** *of T.*

For all integer $k \geq 0$, define $R^kT := H^k \circ T$, then R^kT is called the k^{th}-(right) ***derived functor*** *of T.*

Define $RT(\phi) := T(I(\phi))$. Then we see that RT is actually a functor.

Theorem 15 ***(Derived Functors through Acyclic resolutions)*** *Let $(M, d_M) \xrightarrow{\phi} (N, d_N)$ be a quasi-isomorphism and N^k be an acyclic complex $\forall k$ for the functor $T : \mathcal{A} \to \mathcal{B}$. Then ϕ induces an isomorphism*

$$R^kT(M, d_M) \cong H^k(T(N), T(d_N)) \quad \forall k.$$

5.2 Filtered complexes and Spectral Sequences

5.2.1 Filtered Complexes

Definition 5.2.1

(a) ***(Filtrations of Chain Complexes)*** *Let $\mathcal{A}$ be an abelian category. An **increasing filtration** W_*A on $A \in \mathcal{A}$ is a sequence $(W_pA)_{p\in\mathbb{Z}}$ such that*

$$\cdots \subseteq W_pA \subseteq W_{p+1}A \subseteq \cdots \subseteq A,$$

*where $\subseteq$ means "sub-object or equal to." Similarly a **decreasing filtration** F^*A on $A \in \mathcal{A}$ is a sequence $(F^pA)_{p\in\mathbb{Z}}$ such that*

$$\cdots \supseteq F^pA \supseteq F^{p+1}A \supseteq \cdots$$

*An **increasing filtered complex** is a triple (A, W, d) where $(A, d) \in Kom(\mathcal{A})$, and for every $p \in \mathbb{Z}$, $W_pA^k \subseteq A^k$, $d(W_pA^k) \subseteq W_pA^{k+1}$ such that $(W_pA^*, d|_{W_pA^*}) \hookrightarrow (A, d)$ is a cochain map (notice that we need the last condition $(W_pA^*, d|_{W_pA^*}) \hookrightarrow (A, d)$ because by $\subseteq$ we mean "sub-object or equal to," which may not be the same as subset). Similarly, we define a **decreasing filtered complex**.*

(b) ***(Filtration of Cohomologies and Derived Functors)***

i. ***(Filtration of Cohomologies)***

Let $T : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories and let the domain category have enough injectives.

Let (A, F, d) be a decreasing filtered complex in category $\mathcal{A}$, and for every $p \in \mathbb{Z}$, let $i_{F^p} : (F^pA^*, d|_{F^pA^*}) \hookrightarrow (A, d)$ be the inclusion map. Then this defines a morphism of cohomologies:

$$H^k(i_{F^p}) : H^k(F^pA, d|_{F^pA}) \to H^k(A, d)$$

Define $F^pH^k(A, d) := Im(H^k(i_{F^p}))$. Then $F^*H^k(A, d)$ defines a filtration on $H^k(A, d)$.

ii. ***(Filtration on Derived Functors)***

The inclusions $i_{F^p} : (F^pA^*, d|_{F^pA^*}) \hookrightarrow (A, d)$ induce morphisms of injective resolutions $I(F^pA^*) \overset{I(i_{F^p})}{\to} I(A^*)$, and we have the morphism

$$RT(i_{F^p}) = T(I(i_{F^p})) : RT(F^pA^*) = Q_B(T(I(F^pA^*))) \to Q_B(T(I(A^*))) = RT(A^*).$$

Define $F^pRT(A) := Im(RT(i_{F^p}))$. *Then* $F^*RT(A)$ *defines a filtration of* $RT(A^*) \in \mathcal{B}$.

Everything above can be defined similarly for increasing filtrations.

(c) ***(Shift of Indices)***

Let W_*A *be an increasing filtration on* A. *Then the* n-***shift*** *of* W *is the filtration* $W_*[n]A$ *of* A, *defined by* $W_k[n]A := W_{n+k}A \ \forall k \in \mathbb{Z}$.

Let (A, W, d) *be any increasing filtered complex in the category* $\mathcal{A}$, *and* $n \in \mathbb{Z}$. *Then we define the* n-***shift*** $(A, W[n], d)$ *of* W *as*

$$W_k[n](A^l) = (W[n])_k(A^l) := W_{k+n}(A^l)$$

for all $l \in \mathbb{Z}$ *and* $k \in \mathbb{Z}$.

Similarly, we can define shifts of decereasing filtrations and filtered complexes.

(d) ***(Filtered Double Complexes)***

An ***decreasing filtered double complex*** *is a quadraple* (A, F, d_1, d_2) *such that :* $(A^{*,*}, d_1, d_2)$ *be a double complex in the category* $\mathcal{A}$, *and* $F^*(A^{k,l})$ *is an decreasing filtration on* $A^{k,l}$, *such that* $d_1^{k,l}(F^pA^{k,l}) \subseteq F^pA^{k+1,l}$ *and* $d_2^{k,l}(F^pA^{k,l}) \subseteq F^pA^{k,l+1}$, *and the inclusion* $(F^pA^{*,*}, d_1|_{F^pA^{*,*}}, d_2|_{F^pA^{*,*}}) \hookrightarrow (A^{*,*}, d_1, d_2)$ *is a morphism of double complexes.*

Define $F^pTot(A)^n := \bigoplus_{k+l=n} F^pA^{k,l} \ \ \forall k, l, p \in \mathbb{Z}$.

Then $(Tot(A), F^*(Tot(A)), d^* = d_1^* + (-1)^* d_2^*)$ *is a filtered complex, called the* ***total filtered complex*** *associated to the double filtered complex* (A, F, d_1, d_2).

Similarly, we can define increasing filtered double complexes.

(e) ***(Filtration coming from Truncations)***

Let $(A, d) \in {}_{lb}Kom(\mathcal{A})$, *such that*

$$(A, d) = A^m \overset{d_A^m}{\to} A^{m+1} \to \cdots$$

Then define

$$(T^p(A), d|_{T^p(A)}) := \underbrace{0 \to \cdots \to 0}_{p-zeroes} \to A^{m+p} \overset{d_A^{m+p}}{\to} A^{m+p+1} \to \cdots$$

$\forall p \in \mathbb{Z}$ *(defined in the obvious way.) (where we use the convention that* $A^l = 0 \ \ \forall$ *integers* $l < m$*)*

Clearly, (A, T, d) is a decreasing filtered complex (where the inclusion maps are defined in the obvious way), and it is called the ***filtration coming from truncation****.*

(f) ***(Gradations assocaited to Filtrations)***

*Let F^*A be a decreasing filtration(resp. W_*A be an increasing filtration) on $A \in \mathcal{A}$. Then the p^{th}-**gradation** associated to F^*A (resp. W_*A) is denoted by $Gr_F^p(A)$ (resp. $Gr_p^W(A)$) and is defined by :*

$Gr_F^p(A) := \dfrac{F^pA}{F^{p+1}A}$ *(resp.* $Gr_p^W(A) := \dfrac{W_pA}{W_{p-1}A}$ *)*

5.2.2 Spectral Sequences

We will work in the category of Abelian groups here because most of our work in this section is through diagram chasing and by Mitchel's theorem it will also be true in any small abelian category.

Let (A, F, d) be a decreasing filtered complex with filtrations that are **upper bounded on components and uniformly lower bounded**; i.e.

(a) **(upper bounded on components)**

for every $k \in \mathbb{Z}$, $\exists l \in \mathbb{Z}$ such that

$$F^lA^k = 0$$

(b) **(uniformly lower bounded)**

$\exists m \in \mathbb{Z}$ such that $F^lA^k = 0$, $\forall\ k \in \mathbb{Z}$ and all integers $l \geq m$.

We assume for now that the uniform lower bound m is zero for otherwise we will shift the indices. We call the filtered complexes with these properties to be "**nicely bounded filtered complexes**".

Consider the differential $\overline{d} : Gr_F^p(A^n) \to Gr_F^p(A^{n+1})$ satisfying the commutative diagram:

$$\begin{array}{ccc} Gr_F^p(A^n) & \xrightarrow{\overline{d}} & Gr_F^p(A^{n+1}) \\ {\scriptstyle \pi_F^{p,n}}\uparrow & & \uparrow{\scriptstyle \pi_F^{p,n+1}} \\ F^pA^n & \xrightarrow{d} & F^pA^{n+1} \end{array}$$

(where the vertical maps are quotient maps)

Then we see that $(Gr_F^pA, \overline{d})$ is a complex and we can take the n^{th}-cohomology group

$H^n(Gr_F^p A,\ \overline{d})$ $\forall$ integer $n \geq 0$. We also have the gradations of homologies

$$Gr_F^p H^n(A) = \frac{F^p H^n(A)}{F^{p+1} H^n(A)}.$$

Theorem 16 ***(Spectral Sequence of nicely bounded filtered complexes)***

There is a triple sequence $(E_r^{p,q}, d_r^{p,q})_{p,r \in \mathbb{Z}^{\geq 0}, q \in \mathbb{Z}}$ *with* $E_r^{p,q} \in Ab.$ *and* $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ *and* $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$*, such that*

(a) $E_0^{p,q} = Gr_F^p A^{p+q}$, $d_0^{p,q} = (\overline{d})^{p+q}$,

(b) $(E_1^{p,q}, d_1^{p,q}) \cong (H^{p+q}(Gr_F^p(A)), \delta^{p,q})$, *(isomorphism analogous to the isomorphism of chain complexes) where* $\delta^{p,q} : H^{p+q}(Gr_F^p(A)) \to H^{p+q+1}(Gr_F^{p+1}(A))$ *is the connecting map of homologies associated to the short exact sequence*

$$0 \to Gr_F^{p+1}(A) \to \frac{F^p(A)}{F^{p+2}(A)} \to Gr_F^p(A) \to 0$$

(c) $E_{r+1}^{p,q} \cong H^r(E^{p,q}, d^{p,q})$, *the* r^{th}*-cohomology of* $(E^{p,q}, d^{p,q})$*, defined by*

$$\frac{ker(d_r^{p,q})}{Im(d_r^{p-r,q+r-1})},$$

(d) For $p+q$ *fixed,* $\exists N \in \mathbb{Z}^{\geq 0}$ *such that*

$$E_r^{p,q} = Gr_F^p(H^{p+q}(A))$$

$\forall r \geq N.$ □

Proof We define: $Z_r^{p,q} := \{x \in F^p A^{p+q} : dx \in F^{p+r} A^{p+q+1}\}$ and $B_r^{p,q} := Z_{r-1}^{p+1,q-1} + d(Z_{r-1}^{p-r+1,q+r-2})$, then $B_r^{p,q} \subseteq Z_r^{p,q}$. By definition of Z's and B's, $d(Z_r^{p,q}) \subseteq Z_r^{p+r,q-r+1}$ and $d^{p+q}(B_r^{p,q}) \subseteq B_r^{p+r,q-r+1}$. Define $E_r^{p,q} := \dfrac{Z_r^{p,q}}{B_r^{p,q}}$ and $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ be the map induced by d. Then we already have $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ (since $d^{p+q+1} \circ d^{p+q} = 0$).
First we prove 3. **:** Since $d(Z_{r+1}^{p,q}) \subseteq Z_{r-1}^{p+r+1,q-r} \subseteq B_r^{p+r,q-r+1}$, the quotient map $\pi : Z_r^{p,q} \to E_r^{p,q}$ restricts to $Z_{r+1}^{p,q} \to ker(d_r^{p,q})$. On the other hand, $B_{r+1}^{p,q} = Z_r^{p+1,q-1} + d(Z_r^{p-r,q+r-1}) \equiv d(Z_r^{p-r,q+r-1})$ (mod $B_r^{p,q}$) and so π restricts to $B_{r+1}^{p,q} \to Im(d_r^{p-r,q+r-1})$ and therefore we have the map $\phi : E_r^{p,q} \to \dfrac{ker(d_r^{p,q})}{Im(d_r^{p-r,q+r-1})}$ given by π.
We prove that ϕ is bijective. It is surjective since from $\pi^{-1}(ker(d_r^{p,q})) = Z_r^{p,q} \cap d^{-1}(B_r^{p+r,p-r+1})$ and $z \in \pi^{-1}(ker(d_r^{p,q})) \Rightarrow dz = dz_1 + z_2 \in d(Z_{r-1}^{p+1,q-1}) + Z_{r-1}^{p+r+1,q-r}$

where, $z_1 \in Z_{r-1}^{p+1,q-1}$ and $z_2 \in Z_{r-1}^{p+r+1,q-r} \subseteq F^{p+r+1}A^{p+q+1}$, we get $z - z_1 \in Z_{r+1}^{p,q}$ and $\pi(z_1) = 0$, as $B_r^{p,q} \supseteq Z_{r-1}^{p+r+1,q-r}$, and therefore $\pi^{-1}(ker(d_r^{p,q})) \subseteq Z_{r+1}^{p,q}$.

It is injective since $\pi^{-1}(Im(d_r^{p-r,q+r-1})) = d(Z_r^{p-r,q+r-1}) + B_r^{p,q} \equiv B_{r+1}^{p,q}$ (as we mentioned earlier).

We now prove 1. **:** Clearly $Z_0^{p,q} = F^pA^{p+q}$ and $B_r^{p,q} = F^{p+1}A^{p+q}$. So 1. is trivial.

We now prove 2. **:** The fact $(E_1^{p,q}, d_1^{p,q}) \cong (H^{p+q}(Gr_F^p(A)), \delta^{p,q})$ follows from 1. and 3. To prove 2. we only need to show that d_1 comes from the connecting maps. Let $\pi_F^{p,n} : F^pA^n \to Gr_F^p(A^n)$ be the quotient maps. Then, $(\pi_F^{p,p+q})^{-1}(ker(\overline{d})) = Z_1^{p,q}$ and the isomorphism $(E_1^{p,q}, d_1^{p,q}) \cong (H^{p+q}(Gr_F^p(A)), \delta^{p,q})$ is induced by $\pi_F^{p,p+q}$, (by our previous arguments). Now $d(Z_r^{p,q}) \subseteq Z_r^{p+r,q-r+1}$ and therefore we see that δ of an element is obtained by taking a representative z of an element $[z]$ of $H^{p+q}(Gr_F^p(A))$ in $ker(\overline{d})$, taking a preimage y of this representative in F^pA^{p+q} and y is in fact in $Z_1^{p,q}$, taking $dy \in F^{p+1}A^{p+q+2}$ (since y is in $Z_1^{p,q}$) and taking its $\pi_F^{p+1,p+q+1}$. But this is same as d_1.

We now prove 4. **:** For any fixed $m \in \mathbb{Z}$, we can take large enough non-negative integer l such that $F^lA^k = 0$ for $k = m-1, m, m+1$. As F is decreasing $F^{p+l+1}A^{m+1} = 0$ and therefore $Z_{l+1}^{p,q} = ker(d_A^m) \cap F^pA^m$ and $d(Z_l^{p-l,q+l-1}) = F^pA^m \cap Im(d_A^{m-1})$ as $l - q \geq 0\ \forall p+q = m$ with p being a positive integer. Hence, ${}_FE_{l+1}^{p,q} \cong Gr_F^p(H^m(A^*))$

□

Theorem 17 *Let (A, d) be the total complex associated to the double complex (B, D_1, D_2). Then if $K^pA^n := \bigoplus_{k\geq p, k+l=n} A^{k,l} \subseteq A^n$, then (A, K, D) defines a filtered complex, which we call the **Hodge filtered complex**, and K is called the **Hodge filtration**.If ${}_KE_r^{p,q}$ is the Spectral Sequence associated to the Hodge filtration, then:*

(a) ${}_KE_0^{p,q} = A^{p,q}$ *and* $d_0^{p,q} = (-1)^pD_2^{p,q}$

(b) $({}_KE_1^{p,q}, d_1^{p,q}) \cong (H_{D_2}^q(K^{p,*}), H_{D_2}^q(D_1^{p,*}))$, *where* $(H_{D_2}^q(K^{p,*}))$ *is the cohomology of* $(K^{p,*}, D_2^{p,*})$ *and* $H_{D_2}^q(D_1)$ *is the morphism* $H_{D_2}^q(K^{p,*}) \to H_{D_2}^q(K^{p+1,*})$ *is induced by* $D_1^{p,*}$

□

Proof Clearly 1. follows from 1. of Theorem 16, as $d_0^{p,q} = \overline{d} = (-1)^pD_2^{p,q} : Gr_K^p(A^{p+q}) = B^{p,q} \to Gr_K^p(A^{p+q+1}) = B^{p,q+1}$

Now 2. follows from 2. of Theorem 16 as the short exact sequence $0 \to Gr_K^{p+1}(A) \to$

$\dfrac{K^p(A)}{K^{p+2}(A)} \to Gr_K^p(A) \to 0$ is the same as

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B^{p+1,*-1} & \longrightarrow & B^{p,*} \oplus B^{p+1,*-1} & \longrightarrow & B^{p,*} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (-1)^{p+1} D_2^{p+1,*}} & & \downarrow{\scriptstyle (-1)^p D_2^{p,*} \oplus (-1)^{p+1} D_2^{p+1,*}} & & \downarrow{\scriptstyle (-1)^p D_2^{p,*}} & & \\
0 & \longrightarrow & B^{p+1,*} & \longrightarrow & B^{p,*+1} \oplus B^{p+1,*} & \longrightarrow & B^{p,*+1} & \longrightarrow & 0
\end{array}$$

and d_1 is same as the connecting map of the long exact sequence of homology of this complex. □

5.3 Hypercohomology

Let (X, τ_X) be a topological space and let $Sh(X, \mathcal{A})$ be the category of $\mathcal{A}$-sheaves (i.e. $\mathcal{F} \in Sh(X, \mathcal{A})$, $\Leftrightarrow \mathcal{F} : \tau_X^{op} \to \mathcal{A}$; i.e. a contravariant functor $\tau_X \to \mathcal{A}$). We have the global section functor $\Gamma_X : Sh(X) \to \mathcal{A}$. such that $\Gamma_X(\mathcal{F}) := \mathcal{F}(X)$ for all $\mathcal{F} \in Sh(X, \mathcal{A})$. Now, Γ_X is left-exact and if $\mathcal{A}$ has enough injectives, so does $Sh(X, \mathcal{A})$.

Definition 5.3.1 ***(Hypercohomologies)*** *Let $(\mathcal{F}, d)$ be a complex of sheaves. Then the derived functors $R^k\Gamma_X$ is called the k^{th}-**hypercohomology functor** and is denoted by $\mathbb{H}_X^k$ and $R^k\Gamma_X(\mathcal{F}, d)$ is the k^{th}-**hypercohomology** of $(\mathcal{F}, d)$, denoted by $\mathbb{H}_X^k(\mathcal{F}, d)$*

Theorem 15 says that hypercohomologies can be computed from a chain complex $(\mathcal{G}, d_{\mathcal{G}})$ which is quasi-isomorphic to $(\mathcal{F}, d)$, and all whose terms are Γ_X-acyclic.

Definition 5.3.2 *Let $\phi : (X, \tau_X) \to (Y, \tau_Y)$ be a continuous map. Then ϕ defines a (covariant functor) $\phi_* : Sh(X, \mathcal{A}) \to Sh(Y, \mathcal{A})$, called the **push-forward**, defined by*

$$\phi_*(\mathcal{F})(U) := \mathcal{F}(\phi^{-1}(U)) \quad \forall U \in \tau_Y$$

Chapter 6

Hodge Theory Revisited

Definition 6.0.1 *Let* $\mathcal{A}$ *be an abelian category with enough injectives and let* $\mathcal{F} \in Sh(X, \mathcal{A})$*. Consider the inclusion functor*

$$L : Sh(X, \mathcal{A}) \to {}_0Kom(Sh(X, \mathcal{A}))$$

defined by

$$L(\mathcal{F}) := (\mathcal{F} \to 0 \to 0 \to \cdots \to 0)$$

A complex $(\mathcal{G}, d) \in {}_0Kom(Sh(X, \mathcal{A}))$ *is called a* ***resolution*** *of* $\mathcal{F}$ *if there is a monic* $i : \mathcal{F} \to \mathcal{G}^0$ *such that* $L(i)$ *is a quasi-isomorphism and each* $\mathcal{G}^k$ *is* Γ_X*-acyclic* $\forall k \in \mathbb{Z}^{\geq 0}$*. This does mean that*

$$H^k(X, \mathcal{F}) \cong \mathbb{H}^k_X(\mathcal{G}, d)$$

Definition 6.0.2 ***(Holomorphic de Rham complex)***
Let X *be a complex manifold of dimension* n *and let* Ω^k_X *be the sheaf of holomorphic forms of degree* k *on* X *and let* $\mathcal{O}_X$ *be the structure sheaf (i.e. the sheaf of holomorphic functions on* X*). Then the exterior derivative* d *and the* $(1,0)$*-part* ∂ *on* Ω^k_X *coincide and we have the finite complex*

$$0 \to \mathcal{O}_X \xrightarrow{\partial} \Omega^1_X \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^n_X \to 0$$

This complex is called the ***holomorphic de Rham complex*** *of* X*, denoted by* Ω^*_X *or* (Ω_X, ∂) *and we use the convention* $\Omega^0_X = \mathcal{O}_X$*.*

Let $\mathbb{C}_X$ be the constant sheaf on X, with fibers (stalks) $\mathbb{C}$ over each point. Consider the inclusion of sheaves $i : \mathbb{C}_X \hookrightarrow \mathcal{O}_X$. Then we have the following resolution theorem:

Theorem 18 *The holomorphic de Rham complex is a resolution of the constant sheaf* $\mathbb{C}_X$, *via.* i. *Thus,* $H^k(X, \mathbb{C}) = H^k(X, \mathbb{C}_X) = \mathbb{H}^k_X(\Omega_X, \partial)$ □

PROOF We wish to show that all the terms of the holomorphic de Rham complex are injective sheaves, which means that the sheaves of cohomology $\mathcal{H}^l(\Omega^k_X) = 0 \; \forall l > 0$ and $\forall k \geq 0$, where every sheaf of abelian groups, $\mathcal{F}$, $\mathcal{H}^l(\mathcal{F})$ is the sheafification of the presheaf of the sheaf cohomologies $H^l(U, \mathcal{F})$ for all integers $l \geq 0$.
To show this, first we want the following lemma:

Lemma 17 *Let* (A, d) *be the total complex associated to the double complex* (B, D_1, D_2) *and let* $i : (M, d_M) \to (B^{*,0}, D_1)$ *be a cochain map such that*

$$0 \to M^p \xrightarrow{i^p} I^{p,0} \xrightarrow{D_2} I^{p,1} \xrightarrow{D_2} \cdots$$

is acyclic, meaning that the cohomology of this complex is zero in positive degrees, then

$$H^k(M, d_M) = H^k(A, d) \;\; \forall k \geq 0$$

The proof of the above lemma is similar to Step-3 of Lemma 15. Now, we have the inclusion of the holomorphic de Rham complex into the de Rham complex $(\Omega_X, \partial) \hookrightarrow (\mathcal{A}^*_X, d)$, and the de Rham complex is the total complex associated to the double complex $(\mathcal{A}^{p,q}_X, \partial, (-1)^p\overline{\partial})$, where the complex $(\mathcal{A}^{p,*}_X, (-1)^p\overline{\partial})$ is acyclic, by the Poincaré - Dolbeault Lemma. Thus by the last lemma, the holomorphic de Rham complex is quasi-isomorphic to the de Rham complex.

6.1 Frölicher Spectral Sequence

Let X be a complex manifold of dimension n. Consider the filtration from truncations

$$T^p\Omega^*_X := \Omega^p \xrightarrow{\partial} \Omega^{p+1} \xrightarrow{\partial} \cdots$$

on the holomorphic de Rham complex and the Hodge filtration

$$K^p\mathcal{A}^k := \bigoplus_{r \geq p, r+s=k} \mathcal{A}^{p,q}_X$$

of the de Rham complex. Then for every p, $(K^p\mathcal{A}^*_X, d)$ is the total complex associated to the double complex $(\mathcal{A}^{k,l}_X, \partial, (-1)^k\overline{\partial})_{k \geq p, l \geq 0}$ and clearly, $(\mathcal{A}^{k,0}_X, \partial)_{k \geq p} = T^p\Omega^*_X$.

Thus by arguments similar to Theorem 18, we see that $(K^p\mathcal{A}, d)$ and $(T^p\Omega_X, \partial)$ are quasi-isomorphic. Now each one of the sheaves $K^p\mathcal{A}_X^k$ are fine sheaves and therefore they are acyclic w.r.t. the global section functor. Hence by Theorem 15 we get that

$$H^q(K^pA^k(X), d) = \mathbb{H}_X^q(T^p\Omega_X, \partial)$$

where $(K^pA^k(X), d)$ is the complex of global sections of $(K^p\mathcal{A}_X^k, d)$.

Definition 6.1.1 ***(Frölicher Spectral Sequence)*** *The spectral sequence ${}_KE_r^{p,q}$ associated to the Hodge filtration on the de Rham complex $(A(X)d)$ is called the* ***frölicher Spectral Sequence*** *or* ***FSS*** *for brevity.*

By Theorem 17 we have:

$$({}_KE_1^{p,q}, d_1^{p,q}) \cong (H^q(A^{p,*}(X), (-1)^p\overline{\partial}) = (H_{\overline{\partial}}^{p,q}(X), \partial) \cong (H^q(X, \Omega_X^p), \partial)$$

where the last isomorphism follows from Serre duality.

6.1.1 Degeneration of FSS at E_1 for a Kähler Manifold

For a compact Kähler manifold, the Hodge decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H_{\overline{\partial}}^{p,q}(X)$$

gives us a filtration

$$F^pH^k(X, \mathbb{C}) = \bigoplus_{r\geq p, r+q=k} H^{r,q}(X)$$

of $H^k(X, \mathbb{C})$ for which we have $Gr_F^pH^k(X, \mathbb{C}) \cong H_{\overline{\partial}}^{p,k-p}(X) \cong H^{k-p}(X, \Omega_X^p)$. So we have the following theorem:

Theorem 19 *The Frölicher spectral sequence K of a compact Kähler manifold degenerates at E_1*

Proof By, 16 we have ${}_KE_\infty^{p,q} \cong Gr_K^pH_{DR}^{p+q}(X, \mathbb{C})$ and ${}_KE_{r+1}^{p,q} = H^r({}_KE_r^{p,q}, d_r^{p,q}) \Rightarrow$

$$dim_{\mathbb{C}}({}_KE_{r+1}^{p,q}) \leq dim_{\mathbb{C}}({}_KE_r^{p,q}) \tag{6.1}$$

and the Hodge decomposition gives, ${}_K E_1^{p,q} \cong Gr_K^p H_{DR}^{p+q}(X,\mathbb{C})$. So $dim_{\mathbb{C}}({}_K E_r^{p,q}) \leq dim_{\mathbb{C}}({}_K E_1^{p,q}) \ \ \forall r \geq 1$.

But by the Hodge decomposition, $Gr_K^p H_{DR}^{p+q}(X,\mathbb{C}) \cong^* Gr_F^p H^{p+q}(X,\mathbb{C})$ and the equality in 6.1 occurs for every, (p,q) iff $d_r^{p,q} = 0 \ \forall p,q$. ($\cong^*$ is true since the Hodge decomposition implies

$F^p H^k(X,\mathbb{C}) = ker(d|_{K^p A^k})/d(F^p A^{k-1}(X)))$

6.2 Normal Crossing Divisors and Open Manifolds

Definition 6.2.1 *(a) **(Normal Crossing Divisors)** A hypersurface D in a complex manifold X is called a **normal crossing divisor** or **NCD** (for brevity) if there is a locally constant function $r : X \to \mathbb{N}_n$ such that for each point $x \in X$, there is a neighbourhood V of x and there are n-coordinate functions $z_1, ..., z_n$ on V, such that that where r is constant on V and $D \cap U = \left\{ y \in U : \prod_{j=1}^{r(U)} z_j(y) = 0 \right\}$. We say that the complex chart $(V, (z_1, .., z_n))$ **expresses** D. We say that D is **fully normal crossing** if we can take $V = X$.*

*(b) **Open Manifold associated to a NCD** The **open manifold** associated to a NCD D in a complex manifold X, is its complement $X \setminus D$ in X*

In this chapter, we will assume X is a complex manifold of complex dimension n, D is a NCD on X, and $U = X \setminus D$ is the associated open manifold, and k a non-negative integer.

Definition 6.2.2 *Let $\Omega_{X,D}^k$ be the sheaf of meromorphic k-forms on X, that are holomorphic on U. We say that a section ξ of $\Omega_{X,D}^k$ over some complex chart V has **logarithmic singularities/ logarithmic poles** if it has only poles of order at most one along $V \cap D$, and the same holds for $d\alpha$. Then the sections of $\Omega_{X,D}^k$ with logarithmic singularities forms a subsheaf of $\Omega_{X,D}^k$, denoted by*

$$\Omega_X^k(logD)$$

*, called the **sheaf of** k- **forms with logarithmic poles***

Proposition 6.2.3 *Let $(V, z_1, ..., z_n)$ be a complex chart expressing D, then $\Omega^k_X(logD)|_V$ is a sheaf of free $\mathcal{O}_X|_V$-modules, with basis*

$$(\frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_p}}{z_{i_p}} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_q} : i_s \leq r(V), j_t \geq r(V), p+q=k)$$

In particular, $\Omega^k_X(logD)$ is a sheaf of free $\mathcal{O}_X$-modules.

Proof Let $\xi \in \Gamma(V, \Omega^k_X(logD))$. Let $f := \prod_{j=1}^{r(V)} z_j$Then ξ has a pole of order at most 1 along $\{f=0\}$. So there is a holomorphic k-form σ on V, such that $f \cdot \xi = \sigma$. Then $df \wedge \xi + f \wedge d\sigma$ and therefore $df \wedge \xi$ vanishes along D. Putting $\xi = \sum_{A,B} \xi_{A,B} dz_A \wedge dz_B$ for $A \subseteq \mathbb{N}_{r(V)}$ and $B \subseteq \mathbb{N}_n \setminus \mathbb{N}_{r(V)}$ we get that $\xi_{A,B}$ vanishes on $\left\{g_A := \prod_{j \in \mathbb{N}_r \setminus A} z_j = 0\right\}$ and as g_A is a product of primes in the unique factorization domain $\mathcal{O}(V)$, (taking V sufficiently small) we see that g_A divides $\xi_{A,B}$. The fact that $\mathcal{O}(V)$, is a UFD follows from 2.0.2. □

Definition 6.2.4 ***(Logarithmic de Rham complex)*** *As, $d(\Omega^k_X(logD)) \subseteq \Omega^{k+1}_X(logD)$ we see that $(\Omega^*_X(logD), d)$ forms a complex of sheaves on X, called the* ***logarithmic de Rham complex***

Let $j : U \hookrightarrow X$ be the inclusion of the assocaited open manifold of D in X, then, j induces the following inclusions of chain complexes

$$\Omega^{\cdot}(logD) \hookrightarrow j_* \Omega^{\cdot}_X|_U \hookrightarrow j_* \mathcal{A}^{\cdot}_X|_U$$

Let ϕ be the composition of these inclusions. We have the following theorem:

Theorem 20 *ϕ induces a quasi-isomorphism.* □

Proof We prove this statement for fully normal crossing divisor D on a compact manifold X.

So, we assume that X is the n-polydisc $D_1 \times \cdots \times D_n$ and $D = \left\{(z_1, ..., z_n) \in D_1 \times D_n : \prod_{j=1}^r z_j = 0\right\}$ $U = X \setminus D = D_1^\times \times \cdots \times D_r^\times \times D_{r+1} \times D_n$ and there is a deformation retraction of U to the torus $\mathbb{T} := \prod_{j=1}^r \partial D_j$. Now, as $H^1(\mathbb{T}, \mathbb{Z}) = \mathbb{Z}^r$, we have :

$$H^1(\mathbb{T}, \mathbb{C}) \cong \mathbb{C}^r$$

$$\bigwedge^k H^1(\mathbb{T}, \mathbb{C}) \cong H^k(\mathbb{T}, \mathbb{C})$$

where the second ismorphism is given by the cup-product and follows from the Künneth formula. The homology class of the cycles $_j$ generates $H_1(\mathbb{T},\mathbb{Z}) = \mathbb{Z}^r$ and we have $\frac{1}{2\pi\sqrt{-1}} \cdot \int_{\partial D_j} \frac{dz_i}{z_i} = \delta_{i,j}$, where $\delta_{i,j}$ is the Krönecker delta, and $\frac{dz_i}{z_i} \in \Gamma(U, \Omega^k_X(logD))\ \ \forall i \in \mathbb{N}_r$. So, by the Poincaré duality, we get that $(\frac{dz_j}{z_j})_{1\leq j\leq r}$ is a basis of $H^1(\mathbb{T},\mathbb{C})$.

Let $\frac{dz_I}{z_I} := \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_k}}{z_{i_k}}$ for any multiindex $I = (i_1, ..., i_k) \in \mathbb{N}^k_r\ 1 \leq \forall k \leq r$

Then the map

$$H^k(\Gamma(U, \Omega^k_X(logD))) \to H^k(\Gamma(U, j_*\mathcal{A}_X|_U)) = H^k(U,\mathbb{C})$$

$$\frac{dz_I}{z_I} \wedge dz_J \longmapsto \prod_{j\in I} \int_* (\frac{dz_j}{z_j})$$

is surjective.

To prove injectivity, we show that any section α of $\Gamma(U, \Omega^k_X(logD))$ is cohomologous in $H^k(\Gamma(U, \Omega^k_X(logD)))$ to some $\frac{dz_I}{z_I}$. The approach we take to show this is induction on r.

For $r = 0$, the statement is same as the statement of Lemma holresolutionthm. Assuming the statement to be true for $r-1$, and $\forall k$, we see that any section α of $\Gamma(U, \Omega^k_X(logD))$ can be expressed as $\alpha = \frac{dz_r}{z_r} \wedge \beta + \gamma$, where γ is holomorphic in z_r, and β is independent of z_r. Now if α is exact, then $\frac{dz_r}{z_r} \wedge \beta$ is holomorphic on $\{z_r = 0\}$, so this implies, $d\beta = 0$.

Now, β and γ are elements of $\Gamma(U, \Omega^{k-1}_X(logD'))$ and $\Gamma(U, \Omega^k_X(logD'))$, where $D' = \left\{\prod_{j=1}^{r-1} z_j = 0\right\}$. We now apply the induction hypothesis on β and γ and we get the result for α. □

Corollary 6.2.5 *The above map induces the isomorphism of cohomologies:*

$$H^k(U,\mathbb{C}) \cong \mathbb{H}^k(X, \Omega^*_X(logD))$$

Proof ϕ gives an isomorphism of hypercohomology,

$$\mathbb{H}^k(X, \Omega^k_X(logD)) \cong \mathbb{H}^k(X, j_*\mathcal{A}_X|^{\cdot}_U) = \mathbb{H}^k(X, j_*\mathcal{A}_X|^{\cdot}_U) \cong H^k(\Gamma(X, j_*\mathcal{A}_X|^{\cdot}_U))$$

where the last isomorphism follows because the sheaves considered are acyclic for the global section functor. Now, $\Gamma(X, j_*\mathcal{A}_X|^{\cdot}_U) = \Gamma(U, \mathcal{A}_X|^{\cdot}_U)$. Then $H^k(\Gamma(U, \mathcal{A}_X|^{\cdot}_U)) \cong H^k(U,\mathbb{C})$.

6.2.1 Filtrations on the log-deRham complex

Definition 6.2.6 *Let D is a fully normal crossing divisor and adapt the notations of the last theorem. Let l be a non-negative integer $\leq r$, and let $M_X^* := \Omega_X^*(logD)$. Then we define an increasing filtration W, by $W_l M_X^* := \bigwedge^l M_X^1 \wedge \Omega_X^{*-l}$; i.e. $W_l M_X^n$ has the basis $\frac{dz_I}{z_I} \wedge dz_J$ where $I \subseteq \mathbb{N}_r, |I| \leq l$ and $I \cap J$ is empty, $|I| + |J| = n$.*
Now, for any normal crossing divisor, we define the filtration W locally, by the above paragraph.

Definition 6.2.7 *A divisor D is said to be a* ***globally normal crossing divisor****, if $D = \cup_{i\in I} D_i$, where all the D_i's are smooth hypersurfaces and the intersections $D_{i_1} \cap \cdots \cap D_{i_l}$ are transversal for every multiindex $(i_1, ..., i_l)$ and for all l.*

Let us give a total order on I, and define $D^{(0)} = X$, $D^{(k)} := \amalg_{K\subseteq I, |K|=k}(\cap_{i\in K} D_i)$. Then either $\cap_{i\in K} D_i$ is empty or it is a complex submanifold of X of codimension k, as the intersections are transversal. Let $j_k : D^{(k)} \to X$ be the morphism induced by the inclusions $j_K : \cap_{i\in K} D_i \hookrightarrow X$.

Theorem 21 *There is an isomorphism $Gr_k^W M_X \cong (j_k)_* \Omega_{D^{(k)}}^{\cdot - k}$* □

Proof Let $(V, (z_1, ..., z_n))$ be a complex chart that expresses D, then $V \cap D$, is a fully normal crossing divisor of V. Define the map

$$Res^k : \Gamma(V, W_k M_X^\cdot) \to \Gamma(V, j_* \Omega_{D^{(k)}}^{\cdot - k}) = \oplus_{|K|=k} [\Gamma(V, j_* \Omega_{\cap_{i\in K} D_i}^{\cdot - k})]$$

by

$$Res^k(\frac{dz_A}{z_A} \wedge dz_B)_K = \begin{cases} (2\pi\sqrt{-1})^k dz_B|_{(\cap_{i\in K} D_i)\cap V} & \text{for } A \subseteq \mathbb{N}_r \subseteq I \text{ with } |A| = k \\ 0 & \text{otherwise} \end{cases}$$

Then $\Gamma(V, W_{k-1} M_X) \subseteq ker(Res^k)$, and Res gives a map $Gr_k^W \Gamma(V, M_X^\cdot) \to \Gamma(V, j_* \Omega_{D^{(k)}}^{\cdot - k})$. Let $(z_1', ..., z_n')$ be another set of local coordinates on V, such that $(V, (u_1, ..., u_n))$ expresses D. Then there are non-zero holomorphic functions $f_j : V \to \mathbb{C}^\times \ \ \forall j \in \mathbb{N}_n$, such that $u_j = f_j z_j \ \ \forall j \in \mathbb{N}_n$; then

$$\frac{du_j}{u_j} = \frac{df_j}{f_j} + \frac{dz_j}{z_j} \quad \forall j \in \mathbb{N}_r$$

Then $\frac{dz_A}{z_A}$ and $\frac{du_A}{u_A}$ have the same image in $Gr_k^W(\Gamma(U, M_X^*))$ as the forms $\frac{df_j}{f_j}$ are holomorphic on V. So Res is independent of the chosen coordinates that express

D, because it anyway does not depend on the last $n-r$ coordinates of any chart expressing D chosen on V.

Res is injective: If $\alpha = \sum_{A,B} \alpha_{A,B} \dfrac{dz_A}{z_A} \wedge dz_B \in \Gamma(V, W_k M_X^n)$ be such that $Res(\alpha)_K = 0, \ \forall K$, with $|K| = k$; and $|A| \leq r$, then $\alpha_{A,B} = 0 \ \ \forall A$, with $|A| = k$, and hence $\alpha \in \Gamma(V, W_{k-1} M_X^n)$.

Res is surjective: Let $(\alpha_K)_K \in \oplus_{|K|=k}[\Gamma(V, j_* \Omega^{\cdot - k}_{\cap_{i\in K} D_i})]$, then, each α_K extend as a holomorphic form to a neighbourhood of $\cap_{i\in K} D_i$ to β_K, say. Shrink V to be contained in the intersection of all the domain of definitions (assumed to be open), then $\beta := \sum_K \frac{1}{(2\pi\sqrt{-1})^k} \dfrac{dz_K}{z_K} \wedge \beta_K$ is defined on V (here the good news is that the β_K are finite and so the intersection of the domain of definitions is open). □Since we have considered only hypercohomology of increasing complex of sheaves we consider the hypercohomology of increasing complexes, let $M^k := W_{-k}$ be the associated increasing filtration to W_*. Then we have the following theorem:

Theorem 22 *We have* ${}_W E_1^{p,q} \cong H^{2p+q}(D^(- p), \mathbb{C})$. □

Proof From Theorem-16, ${}_W E_1^{p,q} \cong \mathbb{H}^{p+q}(X, Gr_M^p(\Omega_X^*(logD))$ and $Gr_M^p(\Omega_X^*(logD)) = Gr_{-p}^W \Omega_X^*(logD)) \cong (j_{-p})_*(\Omega^{*+p}_{D^{(-p)}})$ (by the last theorem). Moreover, for every complex $\mathcal{F}^*$ of sheaves on $D^{(-p)}$ we have $R^k(j_{-p})_* \mathcal{F}^l = 0 \ \ \forall k > 0$, as the maps j_{-p} are proper maps and have finite fibers, we have

$$\mathbb{H}^l(D^{(-p)}, \mathcal{F}^*) \cong \mathbb{H}^l(X, (j_{-p})_* \mathcal{F}^*)$$

And we note that $H^{2p+q}(D^{(-p)}, \mathbb{C}) \cong \mathbb{H}^{2p+q}(D^{-p}, \Omega_{D^{(-p)}})$ □

Chapter 7

Hodge Structures and Polarisation

7.1 Pure Hodge Structures

Let X be a compact manifold and let $\mathbb{K}$ be a field of characteristic zero. Consider the morphism of constant sheaves $\mathbb{Z} \to \mathbb{K}$. The sheaf cohomology group $H^k(X, \mathbb{K})$ can be identified with the *$\widehat{C}$ech cohomology* group $\widehat{H}^k(\mathcal{U}, \mathbb{K})$, where $\mathcal{U}$ is a *$\widehat{C}$ech cover* of X(and similarly $H^k(X, \mathbb{Z}) = \widehat{H}^k(\mathcal{U}, \mathbb{Z})$). Since $\mathbb{K}$ has characteristic zero and $C^*(\mathcal{U}, \mathbb{Z})$ is a complex of free abelian groups of finite rank , we have

$$C^*(\mathcal{U}, \mathbb{Z}) \bigotimes_{\mathbb{Z}} \mathbb{K} = C^*(\mathcal{U}, \mathbb{K})$$

and

$$\widehat{H}^k(\mathcal{U}, \mathbb{Z}) \bigotimes_{\mathbb{Z}} \mathbb{K} = \widehat{H}^k(\mathcal{U}, \mathbb{K}).$$

Since tensoring with $\mathbb{K}$ kills the torsion part, the integral cohomology modulo torsion is identified with its image $\widehat{H}^k(\mathcal{U}, \mathbb{Z}) \otimes_{\mathbb{Z}} 1$ in the cohomology with $\mathbb{K}$-scalars $\widehat{H}^k(\mathcal{U}, \mathbb{K})$. Suppose now that X is a compact Kähler manifold. Then it has the Hodge decomposition,

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

where $H^{p,q}(X)$ is a complex subspace and we have the Hodge symmetry

$$H^{p,q}(X) = \overline{H^{q,p}(X)}.$$

Let ω be the Kähler form on X, and $L : H^k(X, \mathbb{R}) \to H^{k+2}(X, \mathbb{R})$ be the cup product with the class $[\omega] \in H^2(X, \mathbb{R})$. Then we have the Lefschetz decomposition

$$H^k(X, \mathbb{R}) = \bigoplus_r L^r H^{k-2r}_{prim}$$

where each primitive component has a Hodge decomposition induced by the Hodge decomposition of $H^k(X,\mathbb{R})$. We also have the intersection form Q on $H^k(X,\mathbb{R})\ \forall k \leqslant n$ defined by,

$$Q(\alpha,\beta) := \int_X \omega^{n-k}\wedge\alpha\wedge\beta = \langle L^{n-k}\alpha,\beta\rangle.$$

Q is alternating if k is odd and symmetric otherwise. The induced Hermitian form

$$H_Q(\alpha,\beta) := i^k Q(\alpha,\overline{\beta})$$

on $H^k(X,\mathbb{C})$ satisfies Proposition-4.6.2.

Definition 7.1.1 ***(Category of Pure Hodge Structures)***

- ***(Integral pure Hodge structure of weight k)*** *An* ***integral pure Hodge structure of weight k*** $V = (V_{\mathbb{Z}}, (V^{p,q})_{p,q\in\mathbb{Z}})$ *is given by a free abelian group* $V_{\mathbb{Z}}$ *of finite type together with a decomposition*

$$V_{\mathbb{C}} := V_{\mathbb{Z}} \underset{\mathbb{Z}}{\otimes} \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$$

where $V^{p,q}$ *are* $\mathbb{C}$ *vector spaces satisfying*

$$V^{p,q} = \overline{V^{q,p}}.$$

and

$$V^{p,q} = 0,\ \forall p,q\in\mathbb{Z},\ p+q\neq k$$

- ***(Morphisms of Integral pure Hodge structures)*** *A morphism* $\phi : V \to W$ *of* ***integral pure Hodge structures*** $V = (V_{\mathbb{Z}}, (V^{p,q})_{p,q\in\mathbb{Z}})$ *and* $W = (W_{\mathbb{Z}}, (W^{p,q})_{p,q\in\mathbb{Z}})$ *of weights* n *and* $m = n+2r$ *is an abelian group homomorphism* $\phi : V_{\mathbb{Z}} \to W_{\mathbb{Z}}$ *such that the induced homomorphism* $\phi : V_{\mathbb{C}} \to W_{\mathbb{C}}$ *is a* $\mathbb{C}$*-linear map that satisfies* $\phi(V^{p,q}) \subseteqq W^{p+r,q+r}\ \forall p,q\in\mathbb{Z}$ *and* $\forall r\in\mathbb{Z}$

Definition 7.1.2 ***(Category of Pure Hodge Complexes)*** *Let A be a commutative ring with unity.*

- ***(Filtrations, Filtered Complexes and Conjugate Filtrations)*** *Let M be an A-module. A collection* $(F^pM)_{p\in\mathbb{Z}}$ *of submodules of M is said to be an* ***increasing filtration(resp. decreasing filtration)*** *of A if*

$$F^pM \subseteqq F^{p+1}M\ \forall p\in\mathbb{Z}$$

$$\textit{(resp.)}\ F^pM \supseteq F^{p+1}M\ \forall p\in\mathbb{Z}$$

and the ordered pair $(M,(F^pM)_{p\in\mathbb{Z}})$ is said to be a ***filtered complex*** *of the same type.*

A filtration $F = (F^pM)_{p\in\mathbb{Z}}$ of a complex vector space M gives another filtration $\overline{F} = (\overline{F}^pM)_{p\in\mathbb{Z}}$ called the ***conjugate filtration*** *defined by $\overline{F}^pM = \overline{F^pM}$.*

- ***(Morphism of Filtered complexes)*** *A morphism $\phi : (M,F) \to (N,G)$ of two filtered complexes $(M, F = (F^pM)_{p\in\mathbb{Z}})$ and $(N, G = (G^pN)_{p\in\mathbb{Z}})$ of degree $r \in \mathbb{Z}$ over A consists of a morphism $\phi : M \to N$ of A-modules such that*

$$\phi(F^pM) \subseteqq G^{p+r}N \ \forall p \in \mathbb{Z}.$$

- ***(n-Opposite Filtrations)***

  Let $n \in \mathbb{N}$, $F = (F^pM)_{p\in\mathbb{Z}}$ and $G = (G^pM)_{p\in\mathbb{Z}}$ be two filtrations of an A-module M. Then F and G are said to be ***n-opposite filtrations*** *of M iff*

$$\forall p,q,\ p+q = n+1 \Rightarrow F^pM \oplus G^qM \cong M.$$

- ***(Pure Hodge Filtrations and Pure Hodge Complexes)***

  Let $k \in \mathbb{N}$. and $V_\mathbb{Z}$ be an abelian group of finite type and $V_\mathbb{C} := V_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$ and filtration $F = (F^pV_\mathbb{C})_{p\in\mathbb{Z}}$ of a $V_\mathbb{C}$ is said to be a ***Pure Hodge Filtration of weight k*** *of $V_\mathbb{Z}$ if F is k-opposite to its conjugate $\overline{F}$.*

  A ***Pure Hodge Complex of weight k*** *is an ordered pair $(V_\mathbb{Z}, F)$ where F is a pure Hodge Filtration of weight k on $V_\mathbb{Z}$, where $V_\mathbb{Z}$ is an abelian group of finite type.*

- ***(Morphism of Pure Hodge Complexes)***

  A ***morphism of Pure Hodge Complexes*** *$(V_\mathbb{Z}, F^*V_\mathbb{C})$ and $(W_\mathbb{Z}, F^*W_\mathbb{C})$ of weights n and $m = n+2r, r \in \mathbb{Z}$, $-\frac{n}{2} \leq r$ respectively, is a morphism $\phi : V_\mathbb{Z} \to W_\mathbb{Z}$ such that the induced map $\phi : V_\mathbb{C} \to W_\mathbb{C}$ gives a morphism of filtered complexes $\phi : (V_\mathbb{C}, F^*V_\mathbb{C}) \to (W_\mathbb{C}, F^*W_\mathbb{C})$ of degree r.*

  Kernel and image of a morphism of Hodge structures has a natural Hodge structure.

The Pure Hodge Complexes with morphisms among them defined as above form the ***category of Pure Hodge Complexes****.*

Proposition 7.1.3 *Let $k \in \mathbb{N}$. The categories of Pure Hodge structures and Pure Hodge complexes of weight k are isomorphic.*

Definition 7.1.4 *An **integral polarised Hodge structure of weight**-k is a triple $(V_{\mathbb{Z}}, F, Q)$ such that $(V_{\mathbb{Z}}, F)$ is a pure Hodge structure of weight k, and a bilinear form Q which is symmetric if k is even and alternating if k is odd.*

Chapter 8

Chern Class and the Ködaira embedding Theorem

Definition 8.0.1 ***(Polarised Manifold)***
*A **polarised manifold** is a pair $(X, [\omega])$, consisting of a compact complex manifold X and an integral Kähler class $[\omega] \in H^k(X, \mathbb{R})$ on X.*

Definition 8.0.2 ***(Line Bundles with Special Propperties)***
Let X be a compact complex manifold of dimension n and $p : L \to X$ be a holomorphic line bundle over X.

- ***(Base Point Free ones)***
*L is said to be **base point free/spanned** if there exist a map $\phi : X \to H^0(X, L)$ such that if $s = \phi(x)$, then the germ $s_x \neq 0$.*
Suppose L is spanned. Let $P := \mathbb{P}(H^0(X, L))$, the projectivized space of global sections of L, and let P^* be the dual projective space of P. We can define a map $j_L : X \to P^*$ followingly: the collection of all sections of L that vanish at x forms a hyperplane, say H_x in P. Define $j_L(x) := H_x$.
Choose a basis $s := (s_0, s_1 ..., s_n) \in H^0(X, L)$, which exists because of compactness of X. Let $\mathcal{U} := (U)_{\alpha \in I}$ be an open cover of X. Then s can be given by its restrictions $s_\alpha := s_{i,\alpha}$ $\forall i \in \mathbb{N}_n$ and $\alpha \in I$ of L over the open sets in $\mathcal{U}$. Now if we define the local maps $i_{L,\alpha} : U_\alpha \to P^* \cong \mathbb{P}^n$ by $i_{L,\alpha}(x) := [s_0(x) : s_1(x) : ... : s_n(x)]$, then these local maps are well defined since L is spanned and glue together to give a global map $i_L : X \to P^* \cong \mathbb{P}^n$, since the local maps are compatibe with the transition functions

of L on intersections.

Moreover,

(a) ***the map i_L is holomorphic*** *:*

for if $V_i := z = [z_0, .., z_n] \in \mathbb{P}^n : \ z_i \neq 0$ then $i_L^{-1}(V_i)$ is the open subset of X where the section s_i does not vanish anywhere. Now,

$$i_L^{-1}(V_i) \xrightarrow{\quad i_L \quad} V_i \cong \mathbb{C}^n \tag{8.1}$$

$$x \longmapsto [\frac{s_{0,\alpha}(x)}{s_{i,\alpha}(x)}, .., 1, ..., \frac{s_{n,\alpha}(x)}{s_{i,\alpha}(x)}] \longmapsto (\frac{s_{0,\alpha}(x)}{s_{i,\alpha}(x)}, .., \widehat{\frac{s_{i,\alpha}(x)}{s_{i,\alpha}(x)}}, ..., \frac{s_{n,\alpha}(x)}{s_{i,\alpha}(x)})$$

(where the 1 is at the i^{th}-position) is a holomorphic injection since the meromorphic sections $\frac{s_{j,\alpha}(x)}{s_{i,\alpha}(x)}$ are holomorphic on $i_L^{-1}(V_i)$ $\forall j \ \in \mathbb{N}_n$ only. Hence i_L is holomorphic.

(b) *The map i_L is independent of the basis element s upto a projective transformation.*

(c) *Let $D = \Sigma_{i=0}^n a_i z_i$ be a divisor, then $i_L^*(D) = \Sigma_{i=0}^n a_i s_i = \{x \in X : \Sigma_{i=0}^n a_i s_i(x) = 0\} \in H^0(X, L)$ is a divisor and therefore is the element $\prod_i s_i^{a_i} \in Div(X) = H^0(X, \frac{\mathcal{M}_X^*}{\mathcal{O}_X^*})$ and thus*

$$i_L^* : H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}^*) \to H^1(X, \mathcal{O}_X^*)$$

is such a map that $i_L^*(\mathcal{O}_{\mathbb{P}^n(1)}) = L$ and clearly,

$$i_L^* : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(X, L)$$

is surjective.

- ***(Very Ample Line Bundles)***

  L is said to be ***very ample*** *if i_L^* is an embedding.*

- ***(Ample Line Bundles and Ample Divisors)***

  L is said to be ***ample*** *if $L^{\otimes n}$ is a very ample line bundle for some $n \in \mathbb{N}$. A divisor D is* ***ample*** *if its associated line bundle $\mathcal{O}_X(D)$*

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- ***(Positive Line Bundles and Positive Divisors)*** *(a) Recall that a (1,1)-differential form ξ is positive iff $(T_{\mathbb{C}}X)^2 \ni (u,v) \to \xi(u, Jv)$ is a positive definite inner product; where J is the complex structure on X.*

(b) Let h be a Hermitian metric on L and let $\Theta(L,h) \in \mathcal{A}_X^{1,1}$ be the curvature form which is locally defined by $\Theta(L,h) = \overline{\partial}\partial(\log h_\alpha)$, where h_α is the square of the local norm defined by h on U_α, and $\mathcal{U} = (U)_{\alpha \in I}$ is a trivializing open cover of X.

*(c) L is said to be a **positive line bundle** if $\exists$ a Hermitian metric h on L such that the curvature form $\Theta(L,h)$ is a positive form.*

We now define the Chern class of isomorphism classes of Line Bundle using Sheaf Cohomology:

Consider the morphism of chain complexes of sheaves of $\mathbb{C}$-modules:

$$\begin{array}{ccccccccccccccc} 0 & \xrightarrow{0} & \mathbb{C} & \xrightarrow{i} & \mathcal{A}_X^0 & \xrightarrow{d} & \mathcal{A}_X^1 & \xrightarrow{d} & \mathcal{A}_X^2 & \xrightarrow{d} & .. & \xrightarrow{d} & \mathcal{A}_X^n & \xrightarrow{0} & 0 \\ & & \downarrow{\scriptstyle f=f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & \downarrow{\scriptstyle f^3} & & & & \downarrow{\scriptstyle f^{n+1}} & & \\ 0 & \xrightarrow{0} & \mathcal{O}_X & \xrightarrow{j} & \mathcal{A}_X^{0,0} & \xrightarrow{\overline{\partial}} & \mathcal{A}_X^{0,1} & \xrightarrow{\overline{\partial}} & \mathcal{A}_X^{0,2} & \xrightarrow{\overline{\partial}} & .. & \xrightarrow{\overline{\partial}} & \mathcal{A}_X^{0,n} & \xrightarrow{0} & 0 \end{array} \tag{8.2}$$

induced by the inclusion $f : \mathbb{C} \hookrightarrow \mathcal{O}_X$. Then f^{k+1} is just the projection map (upto homotopy) corresponding to the decomposition of C_X^∞-modules:

$$\mathcal{A}_X^k = \bigoplus_{p+q=k} \mathcal{A}_X^{p,q}$$

$\forall k \in \mathbb{N}_n$. Note that the maps i, j above are inclusions. The inclusion $f : \mathbb{C} \hookrightarrow \mathcal{O}_X$ defines a morphism of sheaf cohomology:

$$f^k : H^k(X, \mathbb{C}) \to H^k(X, \mathcal{O}_X)$$

The chain map also define maps of cohomology groups

$$f^k : H^k_{DR}(X, \mathbb{C}) \to H^{0,k}_{\overline{\partial}}(X)$$

and the last two maps of cohomology coincide by De Rham's Theorem and Dolbeault's Theorem.

Since the Laplacians Δ_d and $\Delta_{\overline{\partial}}$ coincide upto a factor of 2, f^k maps Δ_d-harmonic forms to $\Delta_{\overline{\partial}}$-harmonic forms, giving the chain map of global Harmonic forms:

$$\begin{array}{ccccccccccccccc} 0 & \xrightarrow{0} & \Gamma(X,\mathbb{C}) & \xrightarrow{i} & \mathcal{H}_X^0 & \xrightarrow{0} & \mathcal{H}_X^1 & \xrightarrow{0} & \mathcal{H}_X^2 & \xrightarrow{0} & .. & \xrightarrow{0} & \mathcal{H}_X^n & \to & 0 \\ & & \downarrow{\scriptstyle f=f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & \downarrow{\scriptstyle f^3} & & & & \downarrow{\scriptstyle f^{n+1}} & & \\ 0 & \xrightarrow{0} & \Gamma(X,\mathcal{O}_X) & \xrightarrow{j} & \mathcal{H}_X^{0,0} & \xrightarrow{0} & \mathcal{H}_X^{0,1} & \xrightarrow{0} & \mathcal{H}_X^{0,2} & \xrightarrow{0} & .. & \xrightarrow{0} & \mathcal{H}_X^{0,n} & \to & 0 \end{array} \tag{8.3}$$

where horizontal maps $d, \overline{\partial}$ are zero on Harmonic forms and f^{k+1} is the projection given by the Hodge decomposition:

$$\mathcal{H}_X^k = \bigoplus_{p+q=k} \mathcal{H}_X^{p,q}$$

$\forall k \in \mathbb{N}_n$. Now by the identifications $H^k(X,\mathbb{C}) \cong \mathcal{H}_X^k$ and $H^{p,q}(X,\mathbb{C}) \cong \mathcal{H}_X^{p,q}$ from the above chain complex we have the maps $f^k : H^k(X,\mathbb{C}) \to H^{0,k}(X)$ whcih is the same as the map $f^k : H^k_{DR}(X,\mathbb{C}) \to H^{0,k}_{\overline{\partial}}(X)$.

**Here I have the following question:*

The kernel of this projection map f^k is $F^1H^k(X)$ and the real forms in the kernel of f^3 are representable by real (1,1)-harmonic forms.

In the following diagram:

$$\begin{array}{ccccccccccc} 0 & \xrightarrow{0} & H^0(X) & \xrightarrow{\delta} & H^1(X) & \xrightarrow{\delta} & H^2(X) & \xrightarrow{\delta} .. \xrightarrow{\delta} & H^n(X) & \to & 0 \\ & & \downarrow f^1 & & \downarrow f^2 & & \downarrow f^3 & & \downarrow f^{n+1} & & \\ 0 & \xrightarrow{0} & H^{0,0}(X) & \xrightarrow{\delta} & H^{0,1}(X) & \xrightarrow{\delta} & H^{0,2}(X) & \xrightarrow{\delta} .. \xrightarrow{\delta} & H^{0,n}(X) & \to & 0 \end{array} \tag{8.4}$$

(where the horizontal maps are boundary maps), do the vertical maps together give a morphism of exact sequences?* *Now the exact sequence of sheaves of abelian groups:*

$$0 \xrightarrow{0} \mathbb{Z} \xrightarrow{h} \mathcal{O}_X \xrightarrow{exp 2\pi\sqrt{-1}} \mathcal{O}_X^* \xrightarrow{0} 0 \tag{8.5}$$

Consider the exact sequence of cohomology groups:

$$\begin{array}{l} 0 \xrightarrow{0} H^0(X,\mathbb{Z}) \xrightarrow{h} H^0(X,\mathcal{O}_X) \xrightarrow{exp2\pi\sqrt{-1}} H^0(X,\mathcal{O}_X^*) \xrightarrow{\delta} \\ H^1(X,\mathbb{Z}) \xrightarrow{h} H^1(X,\mathcal{O}_X) \xrightarrow{exp2\pi\sqrt{-1}} H^1(X,\mathcal{O}_X^*) \xrightarrow{\delta} \\ H^2(X,\mathbb{Z}) \xrightarrow{h} H^2(X,\mathcal{O}_X) \xrightarrow{exp2\pi\sqrt{-1}} H^2(X,\mathcal{O}_X^*) \xrightarrow{\delta} ... \end{array} \tag{8.6}$$

The commutativity of the inclusion of sheaves:

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{h} & \mathcal{O}_X \\ \downarrow i & \nearrow f & \\ \mathbb{C} & & \end{array} \tag{8.7}$$

gives the map of cohomologies:

$$
\begin{array}{ccc}
H^*(X,\mathbb{Z}) & \xrightarrow{h^*} & H^*(X,\mathcal{O}_X) \\
\downarrow{\scriptstyle i^*} & \nearrow{\scriptstyle f^*} & \\
H^*(X,\mathbb{C}) & &
\end{array}
\tag{8.8}
$$

The vertical map i^* in the last commuting triangle is inclusion upto torsion part. Now clearly, the kernels of i^* and h^* contain the torsion part and we have:

$$kerh^* = (kerf^* \cap H^*(X,\mathbb{Z})) \oplus (torsion\ part).$$

Now the L.H.S. is equal to $im\delta^*$ and clearly $im\delta^1 = (H^{1,1}(X) \cap H^2(X,\mathbb{Z})) \oplus (torsion\ part)$.

Definition 8.0.3 ***(Chern Classes of Line Bundles)*** *The Chern class associated to the line bundle L is defined by $c_1(L) := 2\pi\sqrt{-1}\cdot\delta^1([L])$ where $[L]$ is the isomorphism class of L in the Picard group $H^1(X,\mathcal{O}_X^*)$.*

Theorem 23 ***(Fundamental Theorem for (first) Chern class of a Line Bundle)*** *Let L be a holomorphic line bundle over X, and let h_L be a Hermitian metric on L. Then the class of the Curvature form $\Theta(L,h_L)$ is equal to the image of $c_1(L)$ in $H^2(X,\mathbb{Z})$. For every real form ξ of type (1,1) whose class is equal to the image of $c_1(L)$ in $H^2(X,\mathbb{Z})$, $\exists$ a metric h on L such that $\Theta(L,h_L) = \xi$. In particular, the cohomology class of the curvature form is the same for any Hermitian metric on L. Moreover, if L is positive line bundle with respect to one Hermitian metric on it, then L is positive with respect to every other Hermitian metric on it.* □

Proof Let $\mathcal{U} = (U)_{\alpha\in I}$ be a trivializing open covering of L by simply connected open sets, and let σ_α be the non-zero holomorphic section over U_α that trivializes $L|_{U_\alpha}$ $\forall\alpha \in I$. Let $h_\alpha = h_L(\sigma_\alpha,\sigma_\alpha)$ $\forall\alpha \in I$.
Let $\mathbb{Z}_X$ be the constant sheaf on X with stalks $\mathbb{Z}$. Since the open sets U_α are simply connected, the exponential exact sequences restricted to U_α splits on the right, followingly:

$$\text{(a)}\quad 0 \xrightarrow{0} \mathbb{Z}|_{U_\alpha} \xrightarrow{h} \mathcal{C}^\infty_X|_{U_\alpha} \underset{\frac{1}{2\pi\sqrt{-1}}log}{\overset{exp2\pi\sqrt{-1}}{\rightleftarrows}} (\mathcal{C}^\infty_X)^*|_{U_\alpha} \xrightarrow{0} 0$$

$$\text{(b)}\quad 0 \xrightarrow{0} \mathbb{Z}|_{U_\alpha} \xrightarrow{h} \mathcal{O}_X|_{U_\alpha} \underset{\frac{1}{2\pi\sqrt{-1}}log}{\overset{exp2\pi\sqrt{-1}}{\rightleftarrows}} \mathcal{O}^*_X|_{U_\alpha} \xrightarrow{0} 0$$

$\forall \alpha \in I$, let $\omega_\alpha := \Theta(L,h)|_{U_\alpha}$. Then $\omega_\alpha = \frac{1}{2\pi\sqrt{-1}}\partial\overline{\partial}(\log h_\alpha)$ and so $\omega_\alpha = d\xi_\alpha$ where $\xi_\alpha = \frac{1}{2\pi\sqrt{-1}}\overline{\partial}(\log h_\alpha)$

Let $(g_{\alpha,\beta} : U_{\alpha,\beta} \to \mathbb{C}^\times)_{\alpha,\beta\in I}$ be the transition functions of L over $\mathcal{U}$, where $U_{\alpha,\beta} = U_\alpha \cap U_\beta$. Then $\sigma_\alpha = g_{\alpha,\beta}\sigma_\beta$, then we have $\xi_\alpha - \xi_\beta = \dfrac{1}{2\pi\sqrt{-1}}\overline{\partial}\log|g_{\alpha,\beta}|^2$

Let $f_{\alpha,\beta} \in \mathcal{O}^*_X(U_{\alpha,\beta})$, such that $exp[2\pi\sqrt{-1}f_{\alpha,\beta}] = g_{\alpha,\beta}$. Then $exp(-2\pi\sqrt{-1}\cdot\overline{f}_{\alpha,\beta}) = \overline{g}_{\alpha,\beta}$ So $\frac{1}{2\pi\sqrt{-1}}\log|g_{\alpha,\beta}|^2 = f_{\alpha,\beta} - \overline{f}_{\alpha,\beta} \Rightarrow \xi_\alpha - \xi_\beta = -d\overline{f}_{\alpha,\beta}$.

Moreover, the cocycle codition $g_{\alpha,\beta}\cdot g_{\beta,\gamma}\cdot g_{\gamma,\alpha} = 1$ of transition maps gives us $f_{\alpha,\beta} + f_{\beta,\gamma} + f_{\gamma,\alpha} \in \mathbb{Z}$ We have,

$$g_{\alpha,\beta}\cdot g_{\beta,\gamma}\cdot g_{\gamma,\alpha} = 1 \tag{8.9}$$

$$\omega_\alpha = d\xi_\alpha \tag{8.10}$$

$$\xi_\alpha - \xi_\beta = -d(\overline{f}_{\alpha,\beta}) \tag{8.11}$$

Let $\mathcal{K}^{p,q,r} := \underline{\widehat{C}}^r(\mathcal{A}^{p,q}_X), \mathcal{K}^{k,r} := \underline{\widehat{C}}^r(\mathcal{A}^k_X), \mathcal{M}^r := \underline{\widehat{C}}^r(\mathcal{C}^\infty_X)^*)$ where for every sheaf $\mathcal{F}$ on X, $\underline{\widehat{C}}^*(\mathcal{F})$ denotes the sheaf of $\widehat{C}$ech complexes on X.

The exponential short exact sequence (a) gives a short exact sequence of sheaves of $\widehat{C}$ech complexes :

$$0 \to \underline{\widehat{C}}(\mathbb{Z}_X) \to \underline{\widehat{C}}(\mathcal{A}^0_X) \to \mathcal{M} \to 0$$

Take the long exact sequence of cohomology of the above complex on the open cover, $\mathcal{U}$. Then the connecting map $\delta^1 : \widehat{H}^1(\mathcal{U}, \mathcal{C}^\infty_X)^*) \to \widehat{H}^2(\mathcal{U}, \mathbb{Z}_X)$ is given by

$$\delta^1 = \frac{1}{2\pi\sqrt{-1}}\delta^1_{\mathcal{K}} \circ log$$

where $\delta^{k,*}_{\mathcal{K}}$ be the $\widehat{C}$ech differential of $K^{k,*}$. Then,

$$\delta^1(g_{\alpha,\beta})_{\alpha,\beta} = \frac{1}{2\pi\sqrt{-1}}\delta^1_{\mathcal{K}}(f_{\alpha,\beta})_{\alpha,\beta} \tag{8.12}$$

Let $(\mathcal{K}, D)$ be the total complex associated to the double complex $((\mathcal{K}^{k,r})_{k,r\geq 0}, d, \delta_{\mathcal{K}})$. Since $\mathcal{A}^k_X$ are fine sheaves, their sheaf cohomologies vanish on positive degree. So, we can refine $\mathcal{U}$ so that

$$\widehat{H}^r(\mathcal{U}, \mathcal{A}^k_X) = 0 \ \ \forall r \geq 1.$$

Then $(\mathcal{K}, D)$ is a resolution of the constant sheaf $\mathbb{C}_X$ on X, with stalks $\mathbb{C}$. Let (K, D) be the complex of global sections of $(\mathcal{K}, D)$, then there is an isomorphism,

$H^k(X,\mathbb{C}) \to H^k(K,D)$. By the inclusion of the de Rham complex in (K,D), we get the morphism of cohomologies $H^k_{DR}(X,\mathbb{C}) \to H^k(K,D)$ and the inclusion of sheaves $\mathbb{C}_X \hookrightarrow \mathcal{K}^0$, defines a morphism of sheaf cohomologies $\widehat{H}^k(\mathcal{U},\mathbb{C}_X) \to H^k(K,D)$. Moreover, all these morphisms of cohomology are isomorphisms.
Equations 8.10, 8.11 can be written in the complex $(\mathcal{K},D)|_{\mathcal{U}}$ as:

$$(\omega_\alpha)_\alpha + D(\overline{f}_{\alpha,\beta})_{\alpha,\beta} = \delta_{\mathcal{K}}(\xi_\alpha) \tag{8.13}$$

$$\delta_{\mathcal{K}}(\xi_\alpha) + D(\overline{f}_{\alpha,\beta})_{\alpha,\beta} = \delta_{\mathcal{K}}(\overline{f}_{\alpha,\beta})_{\alpha,\beta} = \delta_{\mathcal{K}}(f_{\alpha,\beta})_{\alpha,\beta} \text{ as coefficients of } \delta_{\mathcal{K}}(f_{\alpha,\beta})_{\alpha,\beta} \text{ are integers} \tag{8.14}$$

Then we get $(\omega_\alpha)_\alpha$ is cohomologous to $\delta_{\mathcal{K}}(f_{\alpha,\beta})_{\alpha,\beta}$ in (K,D). So, from equality 8.12 we get $2\pi\sqrt{-1}\cdot\delta^1(g_{\alpha,\beta})_{\alpha,\beta}$ is cohomologous to $(\omega_\alpha)_\alpha$, and $(g_{\alpha,\beta})_{\alpha,\beta} \in H^1(X,\mathcal{O}^*_X)$ represents the isomorphism class of L, and so $2\pi\sqrt{-1}\cdot\delta^1(g_{\alpha,\beta})_{\alpha,\beta}$ as a cohomology class is equal to $c_1(L)$. This completes the first part of the Theorem.
For the second part, let h be the Hermitian metric induced on L by ξ. Then $\xi-\Theta(L,h)$ is exact, and therefore it is ∂-closed and $\overline{\partial}$-closed. So, by $\partial\overline{\partial}$-lemma, we conclude that there is a smooth function μ on X, such that $\xi - \Theta(L,h) = \frac{1}{2\pi\sqrt{-1}}\cdot\partial\overline{\partial}\mu$. Then we set $h_L = e^\mu h$ and we are done with the second part.
The last part follows from observing that in the proof of the second part, h is positive definite if and only if h_L is positive definite. The following theorem is known as the Kodaira-Akizuki-Nakano Vanishing theorem (KAN Vanishing theorem for brevity) which we state without proof. The proof can be found in [2]:

Theorem 24 *Let L be a positive holomorphic line bundle over a compact complex manifold. Then for any $q>0$, we have $H^q(X,K_X\otimes L)=0$.* □

This helps us to get:

8.1 Ködaira embedding theorem

Theorem 25 ***(Ködaira embedding theorem)*** *Let X be a compact complex manifold and let ω be a positive Kähler form on X. Let (L,h) be a holomorphic Hermitian line bundle, such that the curvature form $\Theta(L,h)=\omega$. Then (L,h) is positive and there* *exists a holomorphic embedding $\phi : X \hookrightarrow \mathbb{P}^N$ for some sufficiently large integer $N > 0$.* □

PROOF Let $x \in X$ and let $\tau : X_x \to X$ be the blow-up of X at the point x, let $\tilde{X} = X_x$, and let $E = \tau^{-1}(x)$. Let $I_E \subseteq \mathcal{O}_{\tilde{X}}$ be the ideal sheaf corresponding to the hypersurface E of X; i.e. $I_E(U)$ is the set of all holomorphic maps $f : U \to \mathbb{C}$ such that f vanishes on $U \cap E$, for every complex chart U of X.
Let $\tilde{L}$ be the sheaf of holomorphic sections of the pull-back bundle $\pi^*(L)$. Then the restriction of $\tilde{L}$ on E is $\tilde{L}|_E$ defined by the short-exact sequence:

$$0 \to I_E\tilde{L} \to \tilde{L} \to \tilde{L}|_E \to 0 \tag{8.15}$$

and which is the sheaf of holomorphic sections of $\pi^{-1}L|_E$. We want to show the following claim:
Claim: $\tau^*(K_X) = K_{X_x} \otimes I_E^{\otimes(n-1)}$
Proof of the claim : Let us choose a complex chart (U, ϕ) around x such that $\phi : U \to \mathbb{D}^n$ (where $\mathbb{D}^n$ is the complex unit disc in $\mathbb{C}^n$), such that $\phi(x) = 0$. Let $\phi = (z_1, ..., z_n)$, then ϕ itself is the system local equations of $Y \hookrightarrow X$ on U, and

$$\tilde{U}_x = \left\{(Z, z) \in \mathbb{CP}^{n-1} \times U : Z_i z_j = z_i Z_j \quad \forall i, j \in 1, 2, ..., n\right\}$$

where $Z = [Z_1 : \cdots : Z_n] \in \mathbb{CP}^{n-1}$. We define $\tilde{U}_x^i = \left\{(Z, z) \in \tilde{U}_x : Z_i \neq 0\right\}$ and

$$\phi_j^{(i)}(Z, z) = \begin{cases} z_j/z_i = Z_j/Z_i & \text{for } j \neq i \\ z_i & \text{otherwise} \end{cases}$$

Let $\tau(\tilde{U}_x^i) = U_x^i$.
Then $\phi^{(i)} := (\phi_1^{(i)}, ..., \phi_n^{(i)})$ define local coordinates on

$$\tilde{U}_x^i = \left\{[\phi_1^{(i)}(Z, z) : \cdots : \phi_n^{(i)}(Z, z)] \times (z_i \cdot \phi_1^{(i)}(Z, z), ..., 1 \cdot \phi_i^{(i)}(Z, z), ..., z_i \cdot \phi_n^{(i)}(Z, z)) : z \in U_x^i\right\}$$

Let $\alpha = g \cdot dz_1 \wedge \cdots \wedge dz_n$ be a monomial section of K_X over U_x^i. Then

$$\tau^*(\alpha) = (\tau^*(g)) \cdot d(\tau^* z_1) \wedge \cdots \wedge d(\tau^* z_n) = (\tau^*(g)) \cdot d(z_i \cdot \phi_1^{(i)}) \wedge \cdots \wedge d(\phi_i^{(i)}) \wedge \cdots \wedge d(z_i \cdot \phi_n^{(i)})$$

$$= (\tau^*(g)) \cdot (-1)^{(n-1)} z_i^{n-1} \cdot d(\phi_1^{(i)}) \wedge \cdots \wedge d(\phi_n^{(i)})$$

This defines isomorphisms $\psi_i : \tau^*(K_X)|_{\tilde{U}_x^i} \to K_{X_x} \otimes I_E^{\otimes(n-1)}|_{\tilde{U}_x^i}$ by

$$\tau^*(\alpha) \mapsto [(\tau^*(g)) \cdot d(\phi_1^{(i)}) \wedge \cdots \wedge d(\phi_n^{(i)})] \otimes (-1)^{(n-1)} z_i^{n-1}$$

which glue together as we vary i to give an isomorphism $\psi : \tau^*(K_X)|_{\tilde{X}\setminus E} \to K_{X_x} \otimes I_E^{\otimes(n-1)}|_{\tilde{X}\setminus E}$ which extends to all of $\tilde{X}$ by continuity. This proves the claim. ■

Let N be any integer.

Now tensoring the equation of the above claim by I_E and $\tilde{L}^{\otimes N}$ we get :

$$\tau^*(K_X) \otimes I_E \otimes \tilde{L} \cong K_{X_x} \otimes I_E^{\otimes(n-1)} \otimes I_E \otimes \tilde{L}^{\otimes N} = K_{X_x} \otimes I_E^{\otimes n} \otimes \tilde{L}^{\otimes N}$$

which gives us

$$K_{X_x}^{-1} \otimes I_E L^{\otimes N} \cong I_E^{\otimes n} \otimes \tau^*(L^{\otimes N} \otimes K_X^{-1})$$

Consider the line bundle associated to the divisor E, denoted by $\mathcal{O}_{\ X}(-E)$. This is the line bundle corresponding to the sheaf I_E on $\tilde{X}$ and by the proof of 4 it restricts to $\mathcal{O}_{\mathbb{P}(N_{\{x\}/X})}(+1)$ where $E \cong \mathbb{P}(N_{\{x\}/X})$. So the curvature form ω_E of $\mathcal{O}_{\ X}(-E)$, with the metric induced from the Fubini-Study metric on $\mathcal{O}_{\mathbb{P}(N_{\{x\}/X})}(+1)$ through partition of unity is positive on E, and therefore ω_{nE} which is the curvature metric of the line bundle coming from the sheaf $I_E^{\otimes n}$ is positive on E.

Let h_X be the Hermitian metric on K_X induced by ω and $\omega_{K_X} := \Theta(K_X, h_X)$, then the curvature form of $(L^{\otimes N} \otimes K_X^{-1}, h^{\otimes n} \otimes h_X^{-1})$ is given by $N \cdot \omega - \omega_{K_X}$ for any integer N, and $\Theta(I_E^{\otimes n} \otimes \tau^*(L^{\otimes N} \otimes K_X^{-1})) = \omega_{nE} + \tau^*(N \cdot \omega + \omega_{K_X})$ Now, as X is compact, so is $\tilde{X}$ by Theorem 4, so all the terms of $\omega_{nE} + \tau^*(N \cdot \omega + \omega_{K_X})$ are bounded for fixed N. As ω is positive and ω_{nE} is positive on E, we see that $\omega_{nE} + \tau^*(N \cdot \omega + \omega_{K_X})$ is positive on the blow-up $\tilde{X}$ for $N >> 0$.

Let $N >> 0$. By the KAN vanishing theorem,

$$0 = H^1(\tilde{X}, K_{X_x} \otimes (K_{X_x}^{-1} \otimes I_E \tilde{L}^{\otimes N})) = H^1(\tilde{X}, I_E \tilde{L}^{\otimes N}) \tag{8.16}$$

Now consider the exact sequence of cohomology using the short exact sequence 8.15 with L replaced by $L^{\otimes N}$:

$$\begin{array}{ccccc} 0 \longrightarrow H^0(\tilde{X}, I_E \tilde{L}^{\otimes N}) & \longrightarrow & H^0(\tilde{X}, \tilde{L}^{\otimes N}) & \longrightarrow & H^0(\tilde{X}, \tilde{L}^{\otimes N}|_E) \\ & & & & \downarrow \delta \\ \cdots \longleftarrow H^1(\tilde{X}, \tilde{L}^{\otimes N}|_E) & \longleftarrow & H^1(\tilde{X}, \tilde{L}^{\otimes N}) & \longleftarrow & H^1(\tilde{X}, I_E \tilde{L}^{\otimes N}) \end{array}$$

Then, by 8.16, we get that the map $H^0(\tilde{X}, \tilde{L}^{\otimes N}) \to H^0(\tilde{X}, \tilde{L}^{\otimes N}|_E)$ is surjective. So there is a section of $\tilde{L}^{\otimes N}$ which does not vanish on E. This means that there is a holomorphic section of $L^{\otimes N}$ that does not vanish on x. Thus we see that the line bundle $L^{\otimes N}$ is spanned/ base point free. Consider the holomorphic map $i := i_{L^{\otimes N}} : X \to \mathbb{P}((H^0(X, L^{\otimes N}))^*) =: P$ as considered in Definition 8.0.2, which is well-defined since X is compact and therefore $H^0(X, L^{\otimes N})$ is finite-dimensional. All we need to show is that i and its derivative at each point is injective.

i **is injective :** First we notice that if we consider the blowup along a complex submanifold Y of X of codimension k and if $C = \tau^{-1}(Y)$, then the claim we proved before takes the form

$$\tau^*(K_X) \cong K_{\tilde{X}_Y} \otimes I_C^{\otimes k-1} \otimes \tau^*(K_Y)$$

So, if $Y = \{x, y\} \subseteq X$, then $K_Y = 1$ and $k = n$, so anyway we get

$$\tau^*(K_X) \cong K_{\tilde{X}_Y} \otimes I_C^{\otimes n-1},$$

and thus all the arguments made above hold if we replace $\{x\}$ by $Y = \{x, y\}$.

So fix the notation $Y := \{x, y\}$.

Now, $L^{\otimes N}|_Y = \mathcal{F}_Y(L_x^{\otimes N} \oplus L_y^{\otimes N})$, where $\mathcal{F}_Y(L_x^{\otimes N} \oplus L_y^{\otimes N})$ is the **skyscraper sheaf** corresponding to $L_x^{\otimes N} \oplus L_y^{\otimes N}$ around Y; i.e. the direct sum of skyscraper sheaves corresponding to L_x around x and that of L_y around y. We see that the map of global sections $H^0(X, L^{\otimes N}) \to H^0(X, \mathcal{F}_Y(L_x^{\otimes N} \oplus L_y^{\otimes N})) = L_x^{\otimes N} \oplus L_y^{\otimes N}$ is surjective, and so there is a global section of $L^{\otimes N}$ that vanishes on x (resp. x) but does not vanish on y (resp. x). Finally we show that for every, $x \in X$,

$d_x i$ **is injective :** Let $\mathfrak{a}_X \subseteq \mathcal{O}_X$ be the sheaf of holomorphic functions that vanish at x. Let $s_0, ..., s_p$ be a basis of $H^0(X, L^{\otimes N})$ such that $s_0(x) \neq 0$ and $s_1(x) = \cdots = s_p(x) = 0$, where the integer $p \geq 0$ is $dim(H^0(X, L^{\otimes N})) - 1$. Then $s_1, ..., s_n$ is the basis of $H^0(X, \mathfrak{a}_X \otimes L^{\otimes N})$. Then on a complex chart U around x, i is given by $i(z) = [s_0(z) : \cdots : s_p(z)]$ for all $z \in U$. Choose U small enough such that $s_0(z) \neq 0$ for all $z \in U$. Then, on U, i can be given by $i(z) = (\frac{s_1(z)}{s_0(z)}, ..., \frac{s_p(z)}{s_0(z)}) \in \mathbb{C}^p$.

Then $d_x(i)$ is injective if and only if its dual map from the holomorphic cotangent space of P at $i(x)$ to the holomorphic cotangent space of X at x is surjective; i.e. if $d_x(\frac{s_1(z)}{s_0(z)}), ..., d_x(\frac{s_p(z)}{s_0(z)})$ span the holomorphic cotangent space of X at x. Clearly, the holomorphic cotangent space of X at x is given by $\frac{\mathfrak{a}_{X,x}}{\mathfrak{a}_{X,x}^2}$, where $\mathfrak{a}_{X,x}$ is the stalk of $\mathfrak{a}_X$ at the point $x \in X$, and $\frac{s_1}{s_0}, ..., \frac{s_p}{s_0} \in H^0(U, \mathfrak{a}_X \otimes L^{\otimes N})$

Consider the short exact sequences of sheaves:

$$0 \to \mathfrak{a}_X^2 \otimes L^{\otimes N} \to \mathfrak{a}_X \otimes L^{\otimes N} \overset{d_x}{\to} \frac{\mathfrak{a}_X}{\mathfrak{a}_X^2} \otimes L^{\otimes N}|_x = \frac{\mathfrak{a}_{X,x}}{\mathfrak{a}_{X,x}^2} \otimes L_x^{\otimes N} \to 0 \tag{8.17}$$

$$0 \to I_E^2 \otimes \tau^* L^{\otimes N} \to I_E \otimes \tau^* L^{\otimes N} \overset{d_E}{\to} \frac{I_E}{I_E^2} \otimes \tau^* L^{\otimes N}|_E = \frac{I_E}{I_E^2} \otimes \tau^* L_E^{\otimes N} \to 0 \tag{8.18}$$

Then, by the KAN vanishing theorem, $H^1(X_x, I_E^2 \tau^* L^{\otimes N}) = 0$, and therefore

$$H^0(X_x, I_E \tau^* L^{\otimes N}) \to H^0(X_x, \frac{I_E}{I_E^2} \otimes \tau^* L^{\otimes N}|_E) = H^0(X_x, \tau^*(\frac{\mathfrak{a}_X}{\mathfrak{a}_X^2} \otimes L^{\otimes N})|_E) = H^0(X, (\frac{\mathfrak{a}_X}{\mathfrak{a}_X^2} \otimes L^{\otimes N})|_x)$$

is surjective and we have

$$H^0(X_x, I_E \tau^* L^{\otimes N}) = H^0(X_x, \tau^*(\mathfrak{a}_X L^{\otimes N})) = H^0(X, \mathfrak{a}_X \otimes L^{\otimes N})$$

Thus we see that the map

$$H^0(d_x) : H^0(X, \mathfrak{a}_X \otimes L^{\otimes N}) \to H^0(X, \frac{\mathfrak{a}_X}{\mathfrak{a}_X^2} \otimes L^{\otimes N}|_x) = \frac{\mathfrak{a}_{X,x}}{\mathfrak{a}_{X,x}^2} \otimes L_x^{\otimes N}$$

of the long exact sequence of cohomology is surjective. This completes the proof $\square$

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