Neutrino Oscillations

Pratibha

A dissertation submitted for the partial fulfillment of MS degree in Science

Indian Institute of Science Education and Research Mohali

April 2019

Certificate of Examination

This is to certify that the dissertation titled "Neutrino Oscillations" submitted by Ms. Pratibha (Reg. No. MP16009) for the partial fulfilment of MS degree programme of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Vishal Bhardwaj

Dr. Satyajit Jena

Dr. Kinjalk Lochan

(Local-Guide)

Dr. Ketan Patel

(Supervisor)

Dated: April 25, 2019

Declaration

The work in this dissertation has been carried out by me under the guidance of Dr. Ketan Patel at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Pratibha

(Candidate)

Dated: April 25, 2019

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Ketan Patel

(Supervisor)

Acknowledgement

I would like to thank my thesis guide *Dr. Ketan Patel* for his valuable guidance, support & encouragement regarding this research project and for patiently resolving all my doubts. I also want to thank my thesis committee members *Dr. Kinjalk Lochan*, *Dr. Satyajit Jena* & *Dr. Vishal Bhardwaj* for their time to time support for my work.

I sincerely appreciate the moral support and encouragement of my parents throughout my work.

Atlast, I express my gratitude towards IISER Mohali for providing me this outstanding opportunity to do research projects like this along with all possible infrastructural facilities.

Pratibha

List of Figures

Notation

e	electron
μ	muon
τ	tau
ν_e	electron neutrino
ν_μ	muon neutino
m	mass of neutino
E	energy of neutino
p	momentum of neutrino
L	length
σ	width of wave-packet
v_g	group velocity
G	gravitational constant

Contents

Abstract

In this project, the main idea is to study the phenomenon of neutrino oscillations in flat & curved space-times and to reach to a common standard way to explain the flavour-oscillation probability. Here, calculation of the oscillation probability in plane-wave and wave-packet with assumptions like "same energy" & "same momentum" is done. Also the S-matrix formalism, importance of quantum-mechanical uncertainty relations, dependence of the sizes of production & detection regions, coherence and kinetic entanglement are discussed regarding neutrino oscillations.

Chapter 1

Theory

1.1 Introduction

In recent years, Neutrino Physics has emerged as one of the most active fields of research. Standard Model in particle physics describes neutrinos as massless and chargeless elementary particles that come in three different flavours ν_e, ν_μ & ν_τ. But many recent experiments indicate that neutrinos not only have mass but also have multiple mass eigenstates which are not identical to their flavour states. The existence of mass eigenstates indicates mixing, due to which neutrinos change flavour during their propagation. This phenomenon of changing flavour of neutrino during propagation is called neutrino oscillation.

The phenomenon of neutrino oscillation was first proposed by Bruno Pontecorvo in 1969 as an analogy with K^0 and antiK^0 oscillation. After that many neutrino experiments were performed and finally in 2004 the Super-Kamiokande experiment showed the first compelling evidence of neutrino oscillations[1.]. Also the values of the parameters affecting the probabilities of neutrino oscillation have been experimentally determined in most of the cases.

In this project, I have studied the phenomenon of neutrinos flavour change and derived the probabilities for the same in vacuum in flat space-time and in curved space-times.

1.2 Properties of Neutrinos

Neutrinos are elementary particles belonging to the Lepton-family of Standard Model and have following properties:-

- Exist in three different flavors ν_e, ν_μ & ν_τ, where each flavour corresponds to a lepton.
- Electrically neutral.
- Carry half-integer spin.
- Almost massless, i.e., they have very small masses compared to the other fermions.
- Interact only via the weak and gravitational interactions.
- Most abundant particles in the Universe.
- Each type of neutrino has antiparticle known as anti-neutrino.
- They are supposed to have three different mass eigen-states.

1.3 Sources of Neutrinos

Neutrinos are very abundant particles having sources of origin as:-

- Radioactive decays like beta decay of atomic nuclei or hadron.
- Nuclear reactions in the core of stars, artificial nuclear reactors, nuclear bombs and particle accelerators.
- Spin-down state of a neutron star.
- Collision of Cosmic rays with atoms in the our atmosphere.
- Nuclear reactions in the sun (a major source of solar neutrinos in the vicinity of earth i.e. about 65 billion solar neutrinos per second pass through every square centimeter perpendicular to the direction of sun).

1.4 Lepton-Mixing

- Neutrino flavour eigenstates differ from it's mass eigenstates and a specific neutrino flavour eigenstate is quantum superposition of the three neutrino mass eigenstates. This is possible due to the uncertainty principle because the three masses differ so little that they can not be experimentally distinguished within any practical flight path.

The proportion of each mass eigenstate in the produced pure flavour state has been found to depend strongly on that flavour. It is not known which one out of the three neutrinos (ν_e, ν_μ & ν_τ) is the heaviest.
The relationship between the flavour and mass eigenstates is encoded in the PMNS (Pontecorvo–Maki–
Nakagawa–Sakata) matrix or we can say:-

$$\mid \nu_\alpha\rangle = \sum_{i=1}^{3} U_{\alpha i} \mid \nu_i\rangle \tag{1.1}$$

where,

$\mid \nu_i\rangle$ =ith mass eigenstate of neutrino.

$\mid \nu_\alpha\rangle$ =αth flavour eigenstate of neutrino where, $\alpha = e, \mu, \tau$

$U_{\alpha i}$: Unitary lepton mixing matrix (also known as Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix &

$$U = \left\{ \begin{matrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{matrix} \right\}$$

where, U$U^\dagger$ = $U^\dagger$U = I & each $U_{\alpha i}$ denotes the composition of ith mass eigenstate in αth flavour eigenstate.

It is because of this mixing that neutrinos can change their flavours after travelling some distance.

Chapter 2

Analysis

2.1 Neutrino Oscillations in vacuum (flat space-time)

Plane-wave approach

We assume that neutrinos have mass. Thus there is a spectrum of neutrino mass eigenstates, ν_i, i = 1, 2 & 3, each with mass m_i and we define three flavour states, ν_α, α = e, μ, τ as known today. To understand this leptonic-mixing, let us consider the a leptonic decay as follows[8.]:-

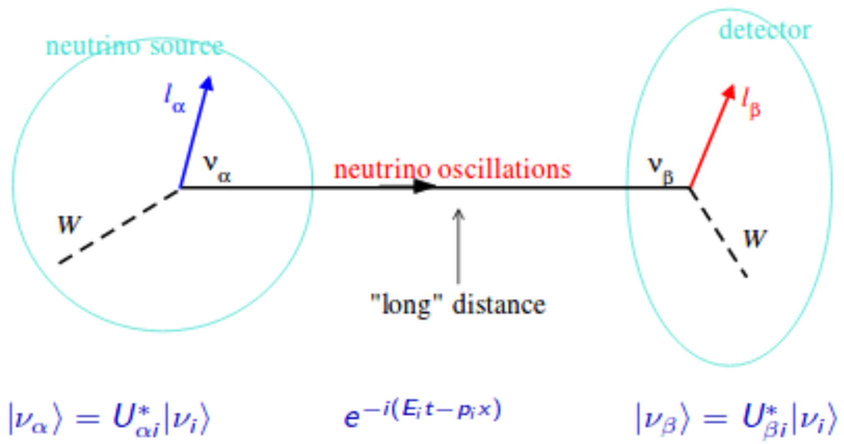

Figure 2.1: Neutrino Oscillation

Here, $W^+ \rightarrow l_\alpha + \nu_\alpha$

where l_α is a charged lepton of flavour α. Mixing suggests that every time the above decay produces a particular anti l_α, the neutrino flavour eigenstate ν_α is accompanied a mass eigenstate same ν_i which may or may not be the same for each decay even if the lepton has a fixed flavour. Thus, we assume that each ν_α is actually a superposition of several mass eigenstates ν_i's out of which only one state can be distinguished during a single decay. So, we can write a flavour state ν_α as

$$\mid \nu_\alpha\rangle = \sum_{i=1}^{3} U_{\alpha i} \mid \nu_i\rangle \tag{2.1}$$

where ,

$\mid \nu_i\rangle$: i$^t h$ mass eigen-state of neutrino.

$\mid \nu_\alpha\rangle$: α flavour eigenstate of neutrino with $\alpha = e, \mu, \tau$

$U_{\alpha i}$: Unitary lepton mixing matrix, also known as PMNS (Pontecorvo-Maki-Nakagawa-Sakata) matrix and for $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$ matrix $U_{\alpha i}$ has form

$$U = \left\{ \begin{matrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{matrix} \right\}$$

where, U$U^\dagger$ = $U^\dagger$U = I

As inverse of eq.(2.1), we can write each mass eigenstate as a superposition of flavours as

$$\mid \nu_i\rangle = \sum_{i=1}^{3} U^*_{\alpha i} \mid \nu_\alpha\rangle \tag{2.2}$$

(Here, for the mass eigenstates the mixing matrix is $U^\dagger$.)

In neutrino oscillations, neutrinos change flavour during travelling due to time evolution, so let us assume that neutrino is detected with flavour ν_β (where, $\beta = e$, μ & τ) at the end of a path of length L. If $\alpha \neq \beta$, then the neutrino has changed it's flavour in its journey. This neutrino flavour change, ν_α to ν_β is a quantum mechanical phenomenon and we want to find out the probability, P($\nu_\alpha \rightarrow \nu_\beta$) of this oscillation.

2.1.1 Oscillation Probability

We are assuming that each ν_α is described by a plane wave with a certain energy and momentum which is a superposition of ν_i's, so we have to add all the individual contribution coming from each one of the travelling ν_i while calculating oscillation probability, P($\nu_\alpha \to \nu_\beta$). The amplitude of oscillation probability will depend on all the factors on which contribution of each ν_i depends on and all such factors are:-

- The amplitude for ν_i when a l_α is produced at their source of origin or by decay is given by $U_{\alpha i}$.
- The amplitude for ν_i to propagate from source to detector, say denoted as $A(\nu_i)$.
- The amplitude for ν_i when l_α is detected at the detector given by $U^*_{\beta i}$.

Then, the amplitude of flavour change from $\nu_\alpha \to \nu_\beta$ is,

$$A(\nu_\alpha \to \nu_\beta) = \sum_{i=1}^{3} U_{\alpha i} A(\nu_i) U^*_{\beta i} \tag{2.3}$$

To find out the value of $A(\nu_i)$, we consider the state vector of neutrino at time, t_0 in it's rest frame following the time dependent Schrödinger equation given by:-

$$i \frac{\partial}{\partial t_0} \mid \nu_i(t_0) \big\rangle = m_i \mid \nu_i(t_0) \big\rangle \tag{2.4}$$

having solution

$$\mid \nu_i(t_0) \big\rangle = e^{-m_i t_0} \mid \nu_i(0) \big\rangle \tag{2.5}$$

Then the probability amplitude for ν_i to travel from it's source to the detector in proper time t_i(in its rest frame) is:-

$$A(\nu_i) = \big\langle \nu_i(0) \mid \nu_i(t_0) \big\rangle = e^{-m_i t_0} \tag{2.6}$$

Now, we need $A(\nu_i)$ in the lab frame and for that we need to use a Lorentz transform to find the corresponding expression in the lab frame, where the lab frame variables are:

- distance between source and detector, L.

- laboratory-frame time, t.

- energy of mass eigenstate ν_i, E_i.

- momentum of mass eigenstate ν_i, p_i.

- By Lorentz invariance,

$$m_i t_0 = E_i t - p_i L \tag{2.7}$$

Approximation:- Every neutrino described by a plane wave has same energy for each mass eigenstate ν_i i.e. $E_i = E$.

Then for $m_i^2 \ll \mathrm{E}^2$, we can write

$$p_i = \sqrt{E^2 - m_i^2} \approx E - \frac{m_i^2}{2E} \tag{2.8}$$

hence

$$m_i t_0 \approx Et - EL + \frac{m_i^2}{2E} \tag{2.9}$$

Here, the $E(t - L)$ term is common to every interfering mass eigenstate so we will consider only the i-dependent part
then using eq.(2.9) in eq.(2.6), we get:-

$$A(\nu_i) = \langle \nu_i(0) \mid \nu_i(t_0) \rangle = e^{-m_i t_0} = e^{-i\frac{m_i^2}{2E}L} \tag{2.10}$$

and now we can write eq.(2.3) as:-

$$A(\nu_\alpha \to \nu_\beta) = \sum_{i=1}^{3} U_{\alpha i} e^{-i\frac{m_i^2}{2E}L} U_{\beta i}^* \tag{2.11}$$

So, Oscillation Probability,

$$P(\nu_\alpha \to \nu_\beta) =\mid A(\nu_\alpha \to \nu_\beta) \mid^2$$
$$= \left(\sum_{i=1}^{3} U_{\alpha i} e^{-i\frac{m_i^2}{2E}L} U_{\beta i}^* \right)^* \left(\sum_{j=1}^{3} U_{\alpha j} e^{-i\frac{m_j^2}{2E}L} U_{\beta j}^* \right)$$
$$= \sum_{i=1}^{3} \sum_{j=1}^{3} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* e^{-i\frac{L}{2E}(m_i^2 - m_j^2)}$$
$$= \sum_{i=j}^{3} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* + \sum_{i \neq j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* e^{-i\frac{L}{2E}(\Delta m_{ji}^2)}$$

where,

$$\Delta m_{ji}^2 = (m_i^2 - m_j^2) \tag{2.12}$$

Now, using the identity

$$e^{i\theta} = \cos\theta + \sin\theta = 1 - 2\sin^2\frac{\theta}{2} + i\sin\theta \tag{2.13}$$

and after solving, we get:-

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i>j}^{3} R(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}) \sin^2\left(\Delta m_{ji}^2 \frac{L}{4E}\right) + 2\sum_{i>j}^{3} I(U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}) \sin\left(\Delta m_{ji}^2 \frac{L}{2E}\right) \tag{2.14}$$

2.1.2 2-Flavour Limit

Let us consider only two flavour states of neutrino ν_e and ν_μ each having two mass eigenstates ν_1 and ν_2. Then for mixing we have a 2 × 2 mixing matrix U which is unitary. We know that a 2 × 2 unitary matrix has 1 rotation angle and 3 phase factors. As the phase factors have no effect on neutrino oscillation so we can exclude them. Then, the possible unitary matrix with one angle parameter is:-

$$U = \left\{ \begin{matrix} U_{e1} & U_{e2} \\ U_{\mu 1} & U_{\mu 2} \end{matrix} \right\}$$

$$= \begin{Bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{Bmatrix}$$

$$\text{U}^{\dagger} = \begin{Bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{Bmatrix}$$

where, $\text{U}U^{\dagger} = U^{\dagger}\text{U} = I$

then, eq.(2.14) for $\alpha \neq \beta$ becomes as:-

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - \left(-\sin^2 2\theta\right)\sin^2\left(\Delta m_{ji}^2 \frac{L}{4E}\right) + 0 = \sin^2 2\theta \sin^2\left(\Delta m_{ji}^2 \frac{L}{4E}\right) \quad (2.15)$$

i.e. for a fix value of L, the probability will vary with E.

<u>Variation of Oscillation Probability, $P_{\alpha\to\beta}(L)$ with L</u>

Here, $L = \dfrac{10^{lgl}}{0.197 \times 10^{-12}}$

If we consider the values of the fixed parameters in eq.(2.15) as:-

- E, energy of each neutrino
- p, momentum of each ν & $E \approx$p$\approx 1MeV$
  m$_1$, mass of mass 1st eigenstate of ν =0.001eV
- m_2, mass of mass 1st eigenstate of ν=0.005eV
- θ, angle of mixing of flavours of $\nu = \frac{\pi}{4}$
- L, distance between the source & detector region of neutrinos

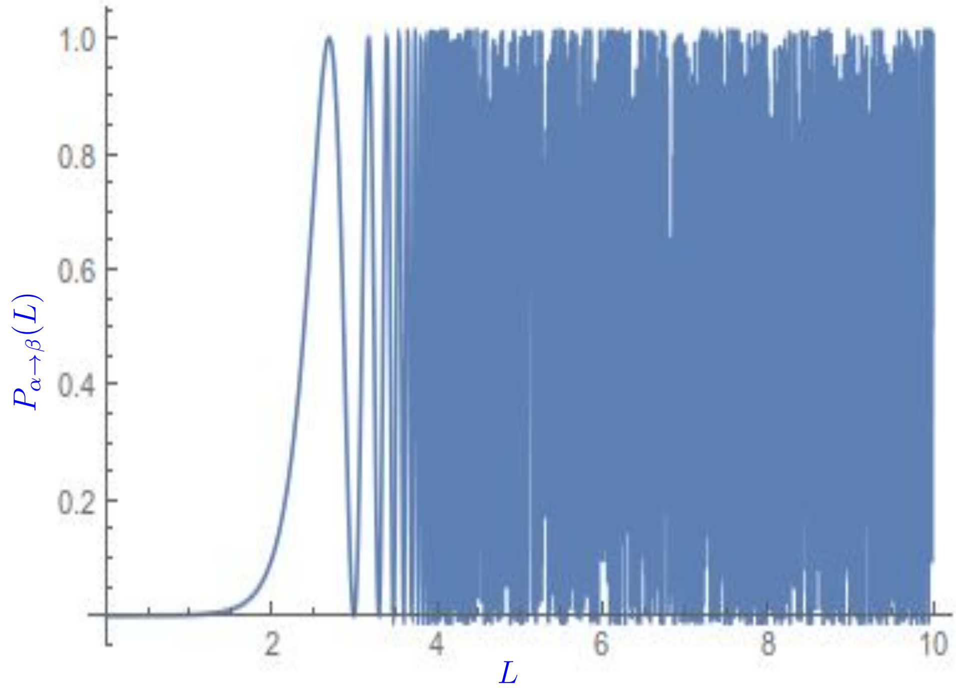

Figure 2.2: Variation of oscillation probability, $P_{\alpha\to\beta}(L)$ with L (Plane-wave approach)

This graph shows that as far as the neutrino-source is disturbed, fluctuations in their flavour-oscillation process goes on. Hence a proper choice for the range of E ensures proper sensitivity (Smaller values of E will cause very rapid fluctuations, while larger values will be monotonous).

2.1.3 Discussion

a.) In the above calculation we have considered neutrinos travelling in vacuum, so it is clear that the phenomenon of flavour change arises from the time evolution of a neutrino itself not from it's interactions with matter.

b.) As the probability of neutrino flavour change is a sum of sinusoidal and sine-squared functions oscillating with the value of $\frac{L}{E}$, so it justifies the term "Neutrino Oscillation".

c.) If there was no leptonic mixing then all the off-diagonal terms in $U_{\alpha i}$ would be zeroes. Then atleast one $U^*_{\alpha i}$ *or* $U_{\alpha j}$ out of $U^*_{\alpha i}U_{\beta i}U_{\alpha j}U_{\beta j}$ is zero for i>j, Which again reduces eq.(2.14) to

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} \tag{2.16}$$

which implies that if the neutrinos are changing flavour then that indicates the existence of leptonic mixing.

d.) If all neutrinos are massless, then $\Delta m_{ji}^2 = (m_i^2 - m_j^2)$=0 resulting eq.(2.14) into

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} = 0 \tag{2.17}$$

where, $\nu_\alpha \to \nu_\beta$ and $\alpha \neq \beta$.
This means that the observation of flavour change of neutrino in vacuum implies that they are not massless and their mass eigenstates are not degenerate.

e.) Equation (2.14) and (2.15) contain the term Δm_{ij}^2, but do not contain the mass of each mass eigenstate explicitly. Hence, although we can find out the squared-mass splitting from neutrino oscillation experiments, we cannot find out the mass of each eigenstate.

f.) Length scale for neutrino oscillations, also known as oscillation length, L_{osc} is given by:-

$$L_{osc} = \frac{4E}{\Delta m_{ji}^2} \tag{2.18}$$

But, generally neutrinos have neither equal energy nor equal momentum. Also in the plane wave approach neutrinos have well defined momentum and thus loses their locality and the source of plane waves should be undisturbed for infinite period of time.

2.1.4 Drawbacks of Plane-Wave Approach

In the above calculation, we assumed that each neutrino is a superposition of it's different mass eigenstate described by a stationary plane wave of certain energy E_i and momentum p_i but this assumption is somewhat contradictory with the phenomenon of neutrino oscillations because each neutrino travels in space and time but plane waves picturizing them are stationary.

Also to detect flavour oscillation of neutrinos, there should be some finite distance between the source and the detector, but plane waves describing them have infinite extents so we can't differentiate between the source and detection regions which makes flavour oscillation detection difficult to observe. Moreover, it is practically impossible that all the neutrinos produced by a single source will have the same energy.

Hence, although we got an analytical idea for flavour oscillation but it's not correct and is not applicable for every source of neutrino production and detection.

2.2 Quantum field theory & Neutrino Oscillations (Wave-Packet Approach)

In the previous section[2.1], we saw that it's not the standard and correct approach to calculate the general expression of neutrino oscillations probability, so now we will try to explain this phenomenon by quantum field theory.

As we know that, in particle physics, any physical process like scattering of particles, particle decays, etc.. , can be described using quantum field theory & the S-matrix formalism just by making use of appropriate initial conditions. Similarly, we can have the S-matrix formalism of the phenomenon of neutrino oscillations in the regime of quantum field theory[3.].

2.2.1 S-matrix formalism of Neutrino Oscillations

Like any other process in quantum field theory, the S-matrix formalism of Neutrino Oscillations is

- Neutrinos are produced in a certain confined space-time region known as source.

- After production, neutrinos propagate and are detected in another confined space-time region known as detector.
- Detector and source have finite sizes and are separated by a finite distance L which is much larger than the sizes of the production and detection regions.
- Neutrino oscillation process contains two distinct interaction regions.
- Integration over the 4-coordinate of the two interaction points is performed over two finite & different space-time interval.
- Initial and final states of neutrinos are described by wave-packets.
- Pictorial representation of neutrino oscillation in S-matrix formalism is:-

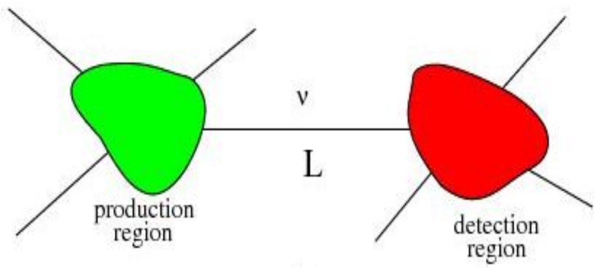

Figure 2.3: S-matrix representation of neutrino oscillation

2.2.2 Contradiction between Law of Energy-momentum conservation & Neutrino Oscillations

In case of Neutrino Oscillations, if we apply the exact energy and momentum conservation to the incoming and outgoing particles in the neutrino production and detection processes then that would make this phenomenon impossible, because

- If we know the exact energy and momentum of all the incoming particles taking part in the production process of the neutrino, then by exact energy-momentum conservation we can calculate the energy E_ν and momentum p_ν of neutrino from these incoming particles. Since neutrinos propagate macroscopic distances and are therefore on the mass shell, so we can calculate the mass of exact energy and momentum from the relation $E_\nu^2 = p_\nu^2 + m^2$. This would imply that the neutrino state is a specific mass eigenstate and not a coherent superposition of different mass eigenstates.

- Moreover according to exact energy-momentum conservation all the involved particles have sharp energies and momenta which means that they are described by plane waves. This makes localization of the neutrino source and detector impossible leading to the non-observability of the neutrino oscillations.

2.2.3 QFT and energy-momentum conservation

Now, we will discuss the flavour oscillation phenomenon quantum field theoretically and will check if this approach leads to some explanation which can resolve the inconsistency of oscillation and the energy-momentum conservation law. For that we will consider a process of neutrino production, propagation and detection in space-time given by:-

$$\pi \rightarrow \mu + v_\mu \tag{2.19}$$

and the electron neutrinos that appear as a result of the oscillations are detected via the process

$$v_e + n \rightarrow p + e \tag{2.20}$$

as shown in the image:-

where,

- $(T_S,\, X_S)$, 4-coordinates of the central point of the neutrino production region.
- $(T_D,\, X_D)$, 4-coordinates of the central point of the detection regions.
- T_S, time when the overlap of the wave packets of particles participating in neutrino production is maximal.
- X_S, position of the central point of the overlap region at time T_S.
- T_D, time when the overlap of the wave packets of particles participating in neutrino detection is maximal.

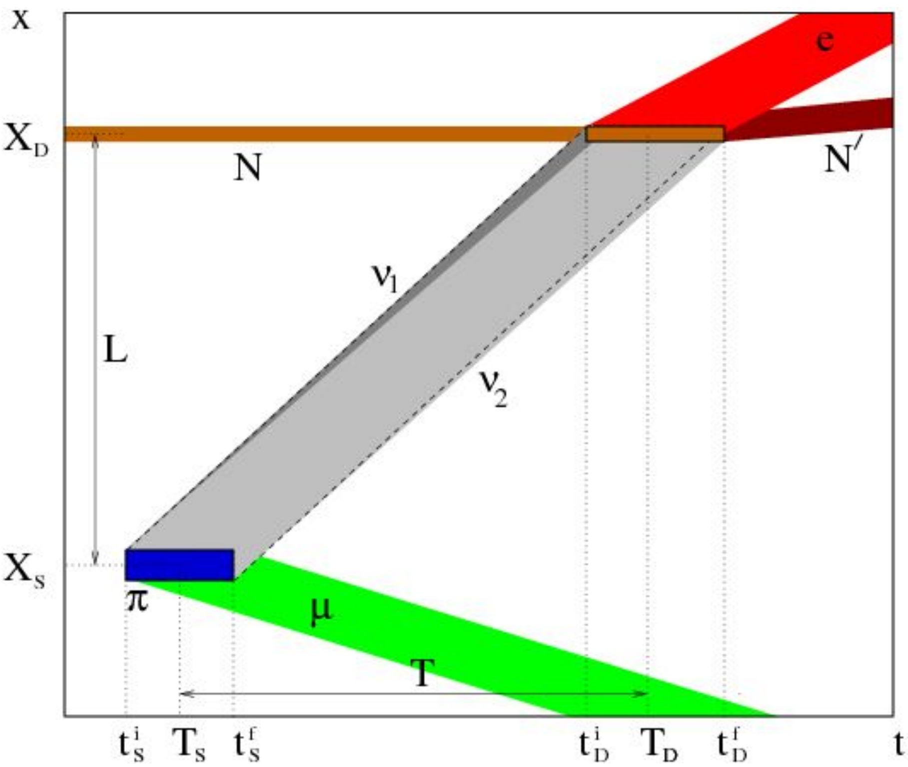

Figure 2.4: Space-time diagram illustrating Neutrino Oscillation.
Schematic representation of neutrino production in pion decay, propagation and observation of the oscillated neutrino due to its charged-current interaction with a target nucleon or nucleus. For more details see reference[3].

- X_D, position of the central point of the overlap region at time T_D.
- L = X_D - X_S, mean distance between the neutrino production and detection.
- T = T_D - T_S, mean time elapsed between the neutrino production and detection.
- colored bands correspond to space-time localization of the participating particles described by wave packets and the rectangular regions show the overlap domains of the wave packets at the neutrino production and detection regions.

Now, as per S-matrix formalism, the transition amplitude is

$$A_i = \int_{,} d^4x_1 \int_{,} d^4x_2 \prod_j \psi_j(x_1, X_S, T_S) \prod_l \psi_l(x_2, X_D, T_D) M_i(x_1, x_2) \tag{2.21}$$

where,

- $j(l)$ denotes the no. of the external particles interacting at the production point x_1.

- $\psi_j(x_1, X_S, T_S)$ and $\psi_l(x_2, X_D, T_D)$ are the wave packets of the external particles in the source and detector regions.

- The quantity $M_i(x_1, x_2)$ is the transition matrix element.

Then the probability amplitude of the complete process of production & propagation of ν_α and detection of ν_β is given as

$$A_{\alpha\beta} = \sum_i U^*_{\alpha i} U_{\beta i} A_i \tag{2.22}$$

Here, we are describing all the incoming and outgoing particles at the source and the detector location respectively by employing them as wave packets where the complete process is integrated out over finite time intervals $t_i^S \leq t_1 \leq t_f^S$ and $t_i^D \leq t_2 \leq t_f^D$. Because if we describe each of them by plane waves then we can not localize the neutrino source and detector due to which neutrino oscillations would be non-observable.

Then, using quantum field theory, we can calculate eq.(2.22) in the following two methods:-

- $a.$) In this method, We will treat all the particles involved in both the processes described in eq.(2.19) and eq.(2.20) as wave packets (localized parts of the wave functions in configuration-space). The transition matrix element, $M_i(x_1, x_2)$ is proportional to the propagator of ν_i and is invariant under space-time translations i.e. $M_i(x_1, x_2)$=$M_i(x_2 - x_1)$ represented by the Fourier integral as:-

$$M_i(x_2 - x_1) = \int_, \frac{d^4q}{2\pi} M_i(q) e^{-iq(x_1 - x_2)} \tag{2.23}$$

. Similarly, the Fourier transforms of the incoming and outgoing wave functions are:-

$$\psi_j(x_1, X_S, T_S) = \int_, \frac{dp}{(2\pi)^3} f_j(p, \bar{p}_0) e^{-i\epsilon_j E_j(p)(t - T_S) + i\epsilon_j (x - X_S)} \tag{2.24}$$

and

$$\psi_l(x_2, X_D, T_D) = \int_, \frac{dp}{(2\pi)^3} f_l(p, \bar{q}_0) e^{-i\epsilon_l E_l(p)(t - T_D) + i\epsilon_l (x - X_D)} \tag{2.25}$$

where,

$\epsilon_j = \epsilon_l = +1$ for initial state of the particles & $\epsilon_j = \epsilon_l =$ -1 for final state particles.

$f_j(p, \bar{p}_0)$ & $f_l(p, \bar{q}_0)$ are the momentum distribution functions of the jth and lth particles taking part in neutrino production and detection processes respectively.

σ_{xj}, spatial length of the wave packet $\psi_j(x_1, X_S, T_S)$.

σ_{xl}, spatial length of the wave packet $\psi_l(x_2, X_D, T_D)$.

$f_j(p, \bar{p}_0)$ is peaked at $p = \bar{p}_0$ and has width $\sigma_{pj} \approx \sigma_{xj}^{-1}$.

$f_l(p, \bar{q}_0)$ is peaked at $p = \bar{q}_0$ and has width $\sigma_{pl} \approx \sigma_{xl}^{-1}$.

Now, using the fact that

$$F_j(p, \bar{p}_0; T_S, X_S) = f_j(p, \bar{p}_0) e^{i\epsilon_j [E_j(p) T_S - p X_S]} \tag{2.26}$$

&

$$F_l(p, \bar{q}_0; T_D, X_D) = f_l(p, \bar{q}_0) e^{i\epsilon_j [E_l(p) T_D - p X_D]} \tag{2.27}$$

we can write eq.(2.21) as:-

$$A_i = \int_{,} d^4x_1 \int_{,} d^4x_2 \prod_j \int_{,} \frac{dp_j}{(2\pi)^3} F_j(p, \bar{p}_0; T_S, X_S) \prod_l \int_{,} \frac{dp_l}{(2\pi)^3} F_l(p, \bar{q}_0; T_D, X_D) M_i(x_2 - x_1) e^{-i\epsilon_j p_j x_1} \tag{2.28}$$

this equation will lead to the exact energy-momentum conservation only when $F_j(p, \bar{p}_0; T_S, X_S)$ & $F_l(p, \bar{q}_0; T_D, X_D)$ are δ-functions having peak values at $\bar{p}_0$ and $\bar{q}_0$. But in actual practice instead of having these momentum distribution functions as δ-functions we have some peak width, σ_p peaked at some value $\bar{p}$ ($\sigma_p \ll \mid \bar{p} \mid$). This signifies that we cannot determine the exact energy and momentum of all particles involved in neutrino production and detection process which leads to the uncertainty in the energy and momentum of the neutrino as well. This uncertainty in energy and momentum of the neutrino implies the idea of neutrino state as a superposition of the mass eigenstates proving the the validity of neutrino oscillations.

- $b.$) Let us all the particles as plane waves and then transition amplitude is:-

$$A_i^{pw}(p_j, p_l) = \int_, d^4x_1 \int_, d^4x_2 M_i(x_2 - x_1) e^{-i(\sum_j \epsilon_j p_j)x_1 - i(\sum_l \epsilon_l p_l)x_2} \tag{2.29}$$

Now, using eq.(2.23) in the above equation we get:-

$$A_i^{pw}(p_j, p_l) \propto \delta^{(4)}\left(\sum_j \epsilon_j p_j + \sum_l \epsilon_l p_l\right) \tag{2.30}$$

But the above equation does not support the phenomenon of neutrino oscillation as all the particles are having form of plane waves which are not localized in space and time. To solve this problem of localization of particles we convolute eq.(2.29) with the actual momentum distribution functions of all the particles as follows:-

$$A_i = \prod_j \int_, \frac{dp_j}{(2\pi)^3} F_j(p, \bar{p}_0; T_S, X_S) \prod_l \int_, \frac{dp_l}{(2\pi)^3} F_l(p, \bar{q}_0; T_D, X_D) A_i^{pw}(p_j, p_l) \tag{2.31}$$

This convolution localizes all the particles by integrating their respective delocalized plane waves over small intervals of width $\approx\sigma_p$ of momenta leading to the constructive interference of plane waves only in certain space-time intervals having width $\sigma_x \approx \frac{1}{\sigma_p}$. This makes the oscillation of neutrino-flavours possible along with the emphasization that conservation of energy and momentum are very fundamental and exact laws of nature.

2.2.4 Standard Oscillation Probability

Let a flavour eigenstate ν_a be produced during a time interval Δt_S centered at $t = 0$ in a source centered at $x = 0$. The wave packet describing the evolved neutrino state at a point with the coordinates (t, x) is then

$$| \nu_a(x,t)\rangle = \sum_i U_{ai}^* \psi_i(x,t) \mid \nu_i\rangle \tag{2.32}$$

Here, $\psi_i(x,t)$ is the wave packet describing a free propagating neutrino of mass m_i produced in the source:-

$$\psi_i(x,t) = \int_{,} \frac{dp}{(2\pi)^{\frac{3}{2}}} f_i^S(p-p_0) e^{ipx - iE_i(p)t} \tag{2.33}$$

where,

- $f_i^S(p-p_0)$, is the momentum distribution function with mean momentum p_i.
-
$$E_i(p) = \sqrt{p^2 + m_i^2} \tag{2.34}$$

Expanding $E_i(p)$ around the mean momentum p_0, we get:-

$$E_i(p) = E_i(p_0) + \frac{\partial E_i(p)}{\partial p^j} |_{p_0} (p-p_0)^j + \frac{1}{2}\frac{\partial^2 E_i(p)}{\partial p^j \partial p^k} |_{p_0} (p-p_0)^j (p-p_0)^k + ... \tag{2.35}$$

Now, using eq.(2.35) into eq.(2.33) and after solving, we get:-

$$\psi_i(x,t) \approx e^{ip_0 x - iE_i(p_0)t} g_i^S(x - v_{gi}t) \tag{2.36}$$

where,

$$g_i^S(x - v_{gi}t) = \int_{,} \frac{dp}{(2\pi)^{\frac{3}{2}}} f_i^S(p) e^{ip(x - v_{gi}t)} \tag{2.37}$$

is the known as shape-factor and

$$v_{gi} = \frac{\partial E_i(p)}{\partial p} |_{p_0} = \frac{p}{E_i} |_{p_0} \tag{2.38}$$

is called the group velocity of the wave packet.

Note that the wave packets corresponding to different neutrino mass eigenstates ν_i are in general described by different momentum distribution functions $f_i^S(p-p_i)$ and therefore by different shape factors $g_i^S(x - v_{gi}t)$. Also the shape factor i.e. eq.(2.37) depends on time and coordinate only through the combination $(x - v_{gi}t)$ this means that the wave packet propagates with the velocity v_{gi} without changing its shape.

Let us now turn to the detected flavour-eigenstate neutrino ν_b. We describe its state by a wave packet peaked at the coordinate L of the detecting particle:-

$$| \nu_a(x-L)\rangle = \sum_i U^*_{bi} \psi^D_i(x-L) \mid \nu_i\rangle \tag{2.39}$$

This state has no time dependence because the detection process is essentially time independent on the time scale of the inverse energy resolution of the detector. The wave function $\psi^D_i(x-L)$ can be written as:-

$$\psi^D_i(x-L) = \int_, \frac{dp}{(2\pi)^{\frac{3}{2}}} f^S_i(p-q) e^{ipx-iE_i(p)t} \tag{2.40}$$

where, $f^S_i(p-q)$ is the momentum distribution function of the wave packet characterizing the detection state, with q being the mean momentum.
We can also write eq.(2.40) as:-

$$\psi^D_i(x-L) = e^{iq(x-L)} g^D_i(x-L) \tag{2.41}$$

with

$$g^D_i(x-L) = \int_, \frac{dp}{(2\pi)^{\frac{3}{2}}} f^D_i(p) e^{ip(x-L)} \tag{2.42}$$

The transition amplitude A describing neutrino oscillations is obtained by projecting the evolved state given by eq.(2.32) onto eq.(2.30):-

$$A_{ab}(L,t) = \int_, d^3x \langle \nu_b(x-L) \mid \nu_a(x,t)\rangle = \sum_i U^*_{ai} U_{bi} \int_, d^3x \psi^{D*}_{\ i}(x-L)\psi_i(x,t) \tag{2.43}$$

Now, putting eqs.(2.36) & eq.(2.40) in the above equation, we get:-

$$A_{ab}(L,t) = \sum_i U^*_{ai} U_{bi} G_i(L - v_{gi}t) e^{-i(E_i(p_i)t - ip_i L)} \tag{2.44}$$

where

$$G_i(L - v_{gi}t) = \int_{,} d^3x g^{D*}_i(x - L) g^S_i(x - v_{gi}t) e^{i(p_0 - q)(x - L)} \tag{2.45}$$

Then, the probability of finding a ν_b at the detector site provided that a ν_a was emitted by the source at the distance L from the detector is then:-

$$P_{ab}(L) = \int_{-\infty}^{\infty} dt \mid A_{ab}(L,t) \mid^2 = \sum_{i,k} U^*_{ai} U_{bi} U_{ak} U^*_{bk} I_{ik}(L) \tag{2.46}$$

where,

$$I_{ik}(L) = \int_{-\infty}^{\infty} dt G^*_i(L - v_{gi}t) G_i(L - v_{gi}t) e^{-i\Delta\phi_{ik}(L,t)} \tag{2.47}$$

The quantity $\Delta\phi_{ik}(L,t)$ is the phase differences between the ith and kth mass eigenstates and

$$\Delta\phi_{ik}(L,t) = (E_i - E_k)t - (p_i - p_k)L = \Delta E_{ik} t - \Delta p_{ik} L \tag{2.48}$$

Now, let us assume that

$$\delta = p_0 - q \tag{2.49}$$

using it in eq.(2.45) and then putting that result into eq.(2.47); we get:-

$$I_{ik}(L) = \frac{2\pi}{v_k} e^{i(\Delta p_{ik} - \frac{\Delta E_{ik}}{v_k})L} \int_{-\infty}^{\infty} dp f^S_i(p) f^{D*}_i(p) f^S_k(\bar{p}) f^{D*}_i(\bar{p}) e^{ip(1-r)L} \tag{2.50}$$

where

- $r = \frac{v_i}{v_k} \approx 1$

- $\bar{p} = rp$

Using $E = \sqrt{p^2 + m^2}$≈E=p^2 + $\frac{m^2}{2E}$ up to the first order, we get

$$e^{i(\Delta p_{ik} - \frac{\Delta E_{ik}}{v_k})L} = e^{-i\frac{\Delta m_{ik}^2}{2E}L} \tag{2.51}$$

and then eq.(2.50) becomes as:-

$$I_{ik}(L) = \frac{2\pi}{v_k} e^{-i\frac{\Delta m_{ik}^2}{2E}L} \int_{-\infty}^{\infty} dp f_i^S(p) f^{D*}_i(p) f_k^S(\bar{p}) f^{D*}_i(\bar{p}) e^{ip(1-r)L} \tag{2.52}$$

using

$$I_{ik}(L) = \frac{2\pi}{v_k} e^{-i\frac{\Delta m_{ik}^2}{2E}L} \int_{-\infty}^{\infty} dp f_i^S(p) f^{D*}_i(p) f_k^S(\bar{p}) f^{D*}_i(\bar{p}) e^{ip\frac{\Delta m_{ik}^2}{2E^2}L} \tag{2.53}$$

then, the standard expression for probability of neutrino oscillation is:-

$$P_{ab}(L) = \sum_{i,k} U_{ai}^* U_{bi} U_{ak} U_{bk}^* \frac{2\pi}{v_k} e^{-i\frac{\Delta m_{ik}^2}{2E}L} \int_{-\infty}^{\infty} dp f_i^S(p) f^{D*}_i(p) f_k^S(\bar{p}) f^{D*}_i(\bar{p}) e^{ip\frac{\Delta m_{ik}^2}{2E^2}L} \tag{2.54}$$

2.2.5 Gaussian Wave Packet treatment in 2-Flavour Limit

Now, we will consider that the two flavoured neutrinos are described by gaussian wave packets[5.]. A neutrino with flavour α is created by a source S and is decomposed into massive neutrino states described by the wave packet $\psi_i^s(x,t)$ is:-

$$\mid \nu_\alpha(x,t) = \sum_{i=1}^{3} U_{\alpha i}^* \psi_i^s(x,t) \mid \nu_i\rangle \tag{2.55}$$

where,

$$\psi_i^s(x,t) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} f_i^s(p - p_i^s) e^{i(p.x - E_i t)} \tag{2.56}$$

&

$$f_i^s(p - p_i^s) = \frac{1}{\sqrt{2\pi\sigma_s^2}} e^{\frac{-(p-p_i^s)^2}{2\sigma_s^2}} \tag{2.57}$$

mean momentum $= p_i^s$

width of wave packet$= \sigma_s$

$v_{gi} = \dfrac{p_i^s}{E_i(p_i^s)}$(group velocity of wave packet)

The neutrinos with flavour β detected at $x = L$ are described by the state:-

$$| \nu_\beta(x-L)\rangle = \sum_{i=1}^{3} U_{\beta i}^{*}\psi_i^d(x-L) \mid \nu_i\rangle \tag{2.58}$$

where,

$$\psi_i^d(x-L) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} f_i^s(p-p_i^d)e^{i(p.x-E_it)} \tag{2.59}$$

&

$$f_i^d(p-p_i^d) = \frac{1}{\sqrt{2\pi\sigma_d^2}}e^{\frac{-(p-p_i^d)^2}{2\sigma_d^2}} \tag{2.60}$$

mean momentum $= p_i^d$

width of wave packet$= \sigma_d$

$r = \frac{v_{gi}}{v_{gj}}$

then Oscillation Amplitude is:-

$$A_{\alpha\to\beta}(L,t) = \int d^3p\langle\nu_\beta(x-L) \mid \nu_\alpha(x,t)\rangle \tag{2.61}$$

and Oscillation Probability is,

$$P_{\alpha\to\beta}(L) = \int_{-\infty}^{\infty} dt \mid A_{\alpha\to\beta}(L,t) \mid^2 \tag{2.62}$$

$$= \sum_{i,j=1}^{3} U_{\beta i}U_{\alpha i}^{*}U_{\alpha j}U_{\beta j}^{*}I_{ij} \tag{2.63}$$

where

$I_{ij} = \frac{1}{\sqrt{2\pi v_j(1+r^2)(\sigma_s+\sigma_d)\sigma_s^3\sigma_d^3}}e^{\frac{-i\Delta m_{ij}^2 L}{2E}}e^{\frac{-\Delta m_{ij}^4 L^2\sigma_s\sigma_d}{8E^4(1+r^2)(\sigma_s+\sigma_d)}}$

$\Delta m_{ij}^2 = (m_i^2 - m_j^2)$

Now, as we know that in 2-flavour limit possible mixing unitary matrix is:-

$$U = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

and if $\sigma_s = \sigma_d = \sigma$ & $v_{gi} = v_{gj} = v \to r = 1$, then
oscillation probability after normalization:-

$$P_{\alpha\to\beta}(L) = \frac{1}{2}\sin^2(2\theta)(1 - Re[I_{12}]) \tag{2.64}$$

Again if we take the values of the fixed parameters in the above equation as:-

- E, energy of each neutrino
- p, momentum of each ν & E≈p≈ $1MeV$
  m_1, mass of mass 1st eigenstate of ν =0.001eV
- m_2, mass of mass 1st eigenstate of ν=0.005eV
- θ, angle of mixing of flavours of $\nu = \frac{\pi}{4}$
- σ, width of wave-packets describing $\nu = 0.001p$
- L, distance between the source & detector region of neutrinos

then variation of oscillation probability, $P_{\alpha\to\beta}(L)$ with L where, $L = \dfrac{10^{lgl}}{0.197\times 10^{-12}}$ is shown below:-

Here, from graph we see that in the wave-packet treatment oscillation process becomes constant after some time depending on the width of the wave-packets and for flavour oscillations to be feasible, this width should always be smaller than the distance between the source and detection regions of neutrinos.

2.2.6 Kinetic Entanglement

Let us consider a decay process:-

$$\pi \to \nu_\mu + \mu \tag{2.65}$$

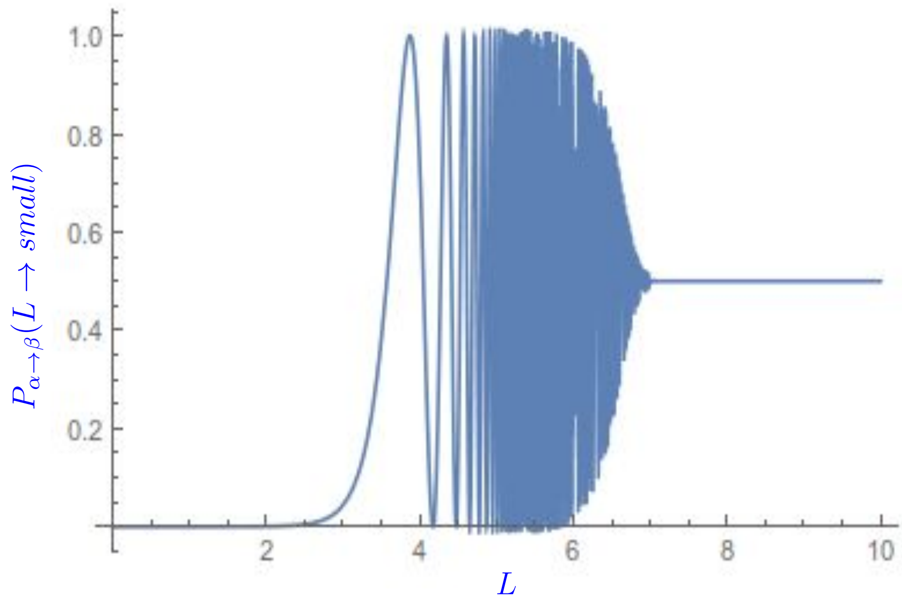

Figure 2.5: Variation of oscillation probability, $P_{\alpha\to\beta}(L)$ with L (Gaussian wave-packet approach)

Now, suppose that the 4-momentum of the pion p_π is well defined[3.]. Then, by exact energy-momentum conservation we will have the condition:-

$$p_{\nu i} + p_{\mu i} = p_\pi \tag{2.66}$$

where,

- i = 1 & 2.
- $p_{\nu i}$: 4-momentum of each emitted neutrino mass eigenstate ν_i.
- $p_{\mu i}$: 4-momentum of the muon.

Here, if there is any uncertainty in 4-momentum of either of ν_μ or μ then the state of the other particle will also have correlated uncertainty in the value of 4-momentum. The combined state of ν_μ and μ can be written as:-

$$| \mu\nu\rangle = \sum_{i=1}^{2} U^*_{\mu i} \mid \nu(p_{\mu i})\rangle \mid \nu_i(p_{\nu i})\rangle \tag{2.67}$$

We also call this state as an entangled state in the sense that we can not factorize the full quantum state of the products,ν_μ and μ as a product of their individual and independent states. In this case, if we know the exact 4-momentum of muon then due to their entanglement (or correlation) we can measure the exact 4-momentum of the neutrino too which will signify no flavour oscillation process for them.

But by all experimental evidences we know very well that the 4-momentum of parent pion can not be measured exactly. Whenever pion is localized in space and time then it's energy and momentum are always going to have uncertainties and we describe it by a wave packet characterized by a momentum distribution function of a width $\sigma_{\pi p}$. This means that there is no strict correlation between the 4-momentas of the neutrino and muon produced in the pion decay i.e. for a given value of $p_{\mu i}$, we can not precisely determine the value of $p_{\nu i}$ for ith neutrino mass eigenstate because it can take any value within a width of $\sigma_{\pi p}$. Or we can write it as:-

$$p_{\pi i} = p_{\mu i} + p_{\nu i} \tag{2.68}$$

for i = 1 & 2.

Now, consider the case when only the 4-momentum of muon is precisely known then we have:-

$$p_{\pi 1} = p_{\mu} + p_{\nu 1} \tag{2.69}$$

and

$$p_{\pi 2} = p_{\mu} + p_{\nu 2} \tag{2.70}$$

then as far as we have

$$\mid p_{\pi 1} - p_{\pi 2} \mid \leq \sigma_{\pi p} \tag{2.71}$$

eq.(2.68), eq.(2.69) & eq.(2.70) all of them are valid concluding that both neutrino mass eigenstates ν_1 and ν_2 can be produced with muon having the same momentum p_{μ}. This means that there is no kinetic entanglement or correlation between the 4-momenta of the neutrino and the muon and phenomenon of neutrino oscillation is possible as far as the pion decay region or the neutrino production region is small compared to the oscillation length of neutrino.

Chapter 3

curved spacetime

3.1 Neutrino Oscillations in Curved Space-time

3.1.1 Neutrino propagation in the Schwarzschild metric

In Schwarzschild Metric, line element in the coordinate frame $x_\mu = (t, r, \theta, \phi)$ is:-

$$ds^2 = B(r)dt^2 - \frac{dr^2}{B(r)} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{3.1}$$

where,

- $B(r) = \left(1 - \frac{2GM}{r}\right)$
- M, mass of the source of the gravitational field

Then, for isotropic gravitational field, we have the mass-shell condition satisfied by the canonical momenta is:-

$$m_k^2 = \frac{1}{B(r)}(p_t^k)^2 - B(r)(p_r^k)^2 - \frac{(p_\theta^k)^2}{r^2} \tag{3.2}$$

where,

- $p_t^k = E_k$ & $p_\phi^k = -J_k$ are constants of motion.

Now expression of quantum mechanical phase as mentioned in eq.(1) becomes:-

$$\Phi_k^L = \int_{r_A}^{r_B} \Big[E_k(\frac{dt}{dr}) - p_k(r) - J_k(\frac{d\phi}{dr})\Big] dr \tag{3.3}$$

Radial propagation

For relativistic neutrinos propagating in the radial direction in a weak field (i.e. $GM << r$) with energy E_k as a reference such that $m_k << E_k$, $d\phi = 0$, then:-

$$\Phi_k^L = \int_{r_A}^{r_B} \big(E_k - \sqrt{E_k^2 - B(r)m_k^2}\big) \frac{dr}{B(r)} \tag{3.4}$$

and phase shift of flavour oscillations is:-

$$\Delta\Phi_{kj}^L \approx \frac{\Delta m_{kj}^2 L_p(A,B)}{2E^{loc}_0(r_B)} \Big[1 - GM\Big(\frac{1}{L_p(A,B)} ln\frac{r_B}{r_A} - \frac{1}{r_B}\Big)\Big] \tag{3.5}$$

where,

$$L_p(A,B) \approx r_B - r_A + GMln(\frac{r_A}{r_B}) \tag{3.6}$$

is the proper distance.

So, We see that in above calculated phase shift, first term is analogous to the flat space–time oscillation phase and the second second term represents the correction due to the gravitational effects[6.]&[7.]. Also the proper oscillation length L_{osc}, given by eq.(3.6), is increased because of the gravitational field which implies decrease in the oscillation probability in the gravitational field.

Chapter 4

Summary & Conclusions

4.1 Concluding Remarks

1. We saw that the stationary source approximation is valid when the time-dependent features of the neutrino emission and absorption processes are either absent or irrelevant i.e. when we essentially deals with steady neutrinos.

2. In the wave-packet approach, the oscillation probability is independent of the production and detection processes provided the following conditions are satisfied:-

    - neutrino emission and absorption are coherent, and decoherence effects due to the wave packet separation are negligible.
    - the energy of neutrinos in the production and detection reactions is large compared to the neutrino mass (or compared to the mass differences).

3. Quantum-mechanical uncertainty relations are at the heart of the neutrino oscillations i.e. the energy and momentum uncertainties inherent in this processes must be large enough to prevent the determination of mass of neutrinos.

4. In radial propagation of neutrinos in the Schwarzschild metric, the proper oscillation length is increased due to the gravitational field leading to lesser flavour oscillations than flat space-time.

4.2 Future Outlook

4.2.1 Problems still Unresolved

- We saw that assumption of the existence of non-degenerate mass eigenstates of neutrinos gives us a probability-based model that accomodates the experimentally observed phenomenon of neutrino oscillations. The existence of this phenomenon itself denies the assumption of neutrinos being massless and forces us to look beyond the Standard Model.
- We also encounter one open question that whether more than three mass eigenstates exist or not and the presence of sterile neutrinos is true or not.
- One more important aspect to focus is that what are the explicit values of the mass eigenstates of neutrinos. Because by all assumptions made here, neutrino oscillation experiments can only give the relative squared-splittings of these values not the precise values.
- Also, we are not clear about the differences or similarities between neutrinos and anti-neutrinos yet i.e. whether neutrinos are Majorana particles (particle identical to its antiparticle) or Dirac particles (particles and antiparticles are distinct).

Thus, though the phenomenon of neutrino oscillation helped us to solve many problems regarding Standard Model in particle physics but still there are a lot of problems and challenges to be resolved yet which we can expect to become clear gradually with the new discoveries in this field.

Bibliography

[1] Super-Kamiokande Collaboration, *A Measurement of atmospheric neutrino oscillation parameters by SUPER-KAMIOKANDE*, Phys. Rev. D **71**, 112005 (2005)

[2] E. Kh. Akhmedov, *Neutrino physics* Centro de Fısica das Interacções Fundamentais (CFIF) Departamento de Fısica, Instituto Superior Técnico Av. Rovisco Pais, P-1049-001, Lisboa, Portugal.

[3] E. Kh. Akhmedov, A. Yu. Smirnov, *Neutrino Oscillations:Entanglement, energy-momentum conservation and QFT*, National Research Centre Kurchatov Institute Moscow (Russia).

[4] E. Kh. Akhmedov, A. Yu. Smirnov, *Paradoxes of neutrino oscillations*, Institute for Nuclear Research, Russian Academy of Sciences.

[5] Andrew Kobach, Aneesh V. Manohar, John Mcgreevy, *Neutrino Oscillations Measurements Computed in Quantum Field Theory*, Physics Department, University of California.

[6] N. Fornengo, C. Giunti, C. W. Kim, and J. Song, *Gravitational effects on the neutrino oscillation*, Department of Physics and Astronomy, The Johns Hopkins University, Baltimore, Maryland 21218 INFN, Sezione di Torino, via P. Giuria 1, 10125 Torino, Italy, Dipartimento di Fisica Teorica, Università di Torino, via P. Giuria 1, 10125 Torino, Italy.

[7] Luca Visinelli, *Neutrino flavour oscillations in a curved spacetime*, Department of Physics, Stockholm University.

[8] Wikipedia.

[9] http://www.hyper-k.org/en/neutrino.html.