Doodles and Twin Groups

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A dissertation submitted for the partial fulfilment of MS degree in Mathematical Science

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April,2019

Certificate of Examination

This is to certify that the dissertation titled **Doodles and Twin Groups** submitted by **Pooja** (MP16017) for the partial fulfilment of MS degree programme of IISER Mohali has been examined by the thesis committee duly appointed by IISER Mohali. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Pranab Sardar Dr. Shane D'mello Dr. Mahender Singh

(Supervisor)

Dated: April 26, 2019

Declaration

The work presented in this dissertation has been carried out by me under the guidance of **Dr. Mahender Singh** at Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. This is a bonafied record of expository work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements made by the candidate are true to the best of my knowledge.

Dr. Mahender Singh
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Acknowledgement

I would like to first thank my thesis supervisor Dr. Mahender Singh from the department of Mathematical Sciences at Indian Institute of Science Education and Research Mohali, for his consistent support and guidance throughout my one year course of thesis. I sincerely thank him for patiently resolving my doubts and giving valuable advices.

I would also thank my committee members Dr. Shane D'mello and Dr.Pranab Sardar who were involved in the validation of my thesis.

I am grateful to IISER Mohali for providing me the financial support along with all possible infrastructural facilities to carry out my thesis work. I also acknowledge the entire community of IISER Mohali : the dean, the director, faculty and students for providing a good atmosphere in the campus.

I sincerely thank my family and friends for the moral support and encouragement. I would also like to thank my seniors Neha Nanda and Manpreet Singh who were very helpful and supportive throughout my thesis.

Contents

Abstract

A doodle is a collection of piecewise-linear closed curves without triple intersections on a closed oriented surface. Two doodles are equivalent if there exists a homotopy from collection of curves representing one to the collection of curves representing other without creating triple points. Theory of doodles resembles theory of classical links. There is a group called the fundamental group of doodle associated with a doodle on a closed oriented surface. The fundamental group of a doodle resembles the fundamental group of a link complement. There is an associated group called twin group which plays the role that the braid group plays for classical links.

This MS thesis is an exposition of the paper of Mikhail Khovanov on Doodle Groups. We compute fundamental groups of some doodles and find some abelian subgroups of doodle groups. We construct examples of doodles on the 2-sphere whose fundamental groups have non-trivial center. Also, for some special types of doodles, we prove that their fundamental groups are automatic.

Chapter 1

TWIN GROUPS

1.1 Configuration of n-arcs

Consider two parallel lines $y = 0$ and $y = 1$ on the Euclidean plane $\mathbb{R}^2 = \{(x, y)|x, y \in \mathbb{R}\}$. Pick n points on $y = 0$, say $(1, 0),\ (2, 0),\ ....,\ (n, 0)$ and corresponding n points on $y = 1$ with the same x-coordinate. We define a *topological interval* to be a space homeomorphic to $I = [0, 1]$.

Definition. A *configuration* on n arcs is a set $C \subset \mathbb{R} \times I$ formed by n disjoint topological intervals (called arcs or strings of C) such that

$$C \cap (\mathbb{R} \times \{0\}) = \{(1, 0),\ (2, 0), \ldots,\ (n, 0)\}$$

and

$$C \cap (\mathbb{R} \times \{1\}) = \{(1, 1),\ (2, 1), \ldots,\ (n, 1)\}.$$

Consider configuration of n arcs connecting points $(1, 1),\ (2, 1), \ldots,\ (n, 1)$ with points $(1, 0),\ (2, 0), \ldots,\ (n, 0)$ in some order, such that

(i) The projection $\mathbb{R} \times I \to I$ maps each arc homeomorphically onto I,

(ii) No three arcs have a common point.

It is straightforward to check that each string of C satisfying (i) and (ii) meets each plane $\mathbb{R} \times \{t\}$ at atmost two points.

Definition. Two configurations C_1 and C_2 satisfying (i) and (ii) are said to be *equivalent* if one can be deformed into the other by homotopy of arcs in $\mathbb{R} \times [0, 1]$ such that throughout the homotopy, conditions (i) and (ii) are satisfied and end-points of the arcs are fixed.

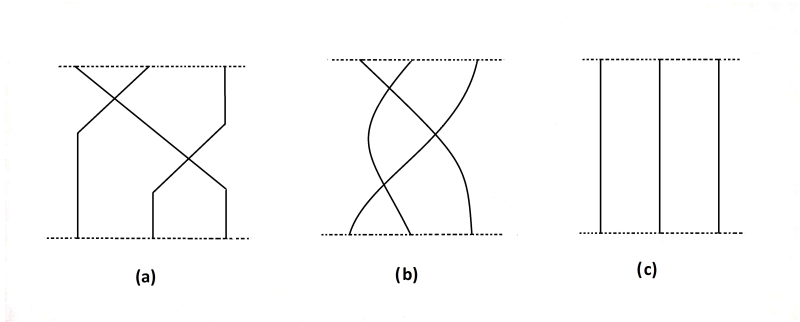

Figure 1.1: Examples of configurations satisfying (i) and (ii).

More explicitly, configurations C_1 and C_2 satisfying (i) and (ii) are equivalent if there exists a continuous map $F : C_1 \times I \to \mathbb{R} \times I$ such that for each $s \in I$,

$$F_s : C_1 \to \mathbb{R} \times I$$

$$x \to F_s(x) := F(x, s)$$

is an embedding whose image is a configuration on n-strings satisfying (i) and (ii) and

$$F_0 = \mathrm{Id}_{C_1} : C_1 \to C_1,$$

$$F_1(C_1) = C_2.$$

It is easy to see that, the relation defined previously is an equivalence relation on the set of all configurations satisfying (i) and (ii).

Definition. An equivalence class of configurations satisfying (i) and (ii) is defined as a *twin*.

The set of all twins on the same number of arcs forms a group under the operation defined as follows:
Let C_1 and C_2 be two twins on the same number of arcs. The product $C_1.C_2$ of twins C_1 and C_2 is defined as twin C which we get by putting C_1 on top of C_2 and then shrinking the interval $[0, 2]$ to $[0, 1]$.

More precisely, we define $C_1.C_2$ to be the set of points $(x, t) \in \mathbb{R} \times I$ such that

$$(x, 2t) \in C_2, \quad 0 \leq t \leq \frac{1}{2}$$

and

$$(x, 2t - 1) \in C_1, \quad \frac{1}{2} \leq t \leq 1.$$

With this operation, the set of all twins on the same number of arcs is turned into a group with identity element to be the twin given by a configuration in which arcs do not intersect (Figure 1.1(c)).

This operation is associative and it follows from the definition of equivalence. The only thing remains to be checked is the existence of inverse. To find that, we need to observe a few things.

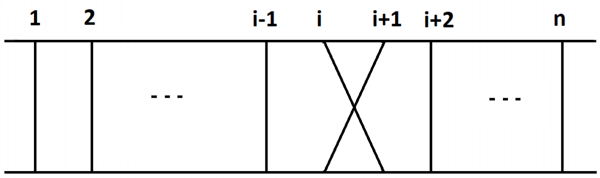

Figure 1.2: The twin p_i.

Let p_i be the twin given in Figure 1.2, that is, twin with only one double point. Observe that p_i^2 is equal to the unit twin, as the corresponding configuration can be homotoped to a configuration without intersection satisfying (i) and (ii) (Figure 1.3). Therefore every p_i is its own inverse.

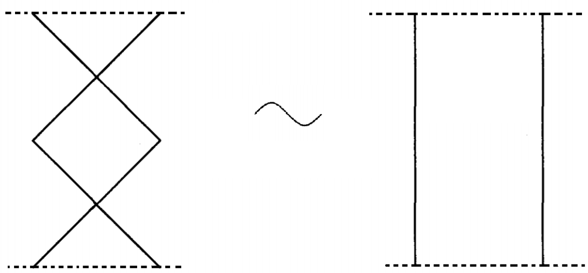

Figure 1.3

Also every twin can be written as finite product of these p_i's. It follows from the fact that every twin can be represented by a configuration such that it has finite layers with each layer containing exactly one crossing. Conditions (i) and (ii) and the compactness of strand implies that it has finite layers. Since each p_i is invertible, we get that every twin is invertible.

Therefore, the set of all twins with n arcs forms a group. It is called *the twin group on n arcs*. We denote it by T_n.

1.2 Twin and pure twin groups

In this section, we study the group of twins on n arcs T_n and the kernel of natural surjection from T_n to S_n, the group of permutations on n symbols.
Let G_n be an arbitrary group generated by ρ_i, $i = 1, 2, \ldots, n-1$, with relations

$$\rho_i^2 = 1, \quad i = 1, 2, \ldots, n-1, \tag{1.1}$$

$$\rho_i \rho_j = \rho_j \rho_i, \quad |i-j| > 1, i, j = 1, 2, \ldots, n-1. \tag{1.2}$$

Lemma 1.2.1. *If $s_1,\ s_2, \ldots,\ s_{n-1}$ are elements of a group G satisfying the above relations, then there exists a unique group homomorphism $f : G_n \to G$ such that $s_i = f(\rho_i)$ for all $i = 1, 2, \ldots, n-1$.*

Proof. Let F_n be a free group generated by the set $S = \{\rho_1,\ \rho_2,\ \ldots,\ \rho_{n-1}\}$. Let $\bar{f}$ be a set theoretic map from S to G, that maps ρ_i to s_i for each i.
Then by definition of free group, for the function $\bar{f}$, there exists a unique group homomorphism $f : F_n \to G$ such that $f(\rho_i) = s_i$ for all $i = 1, 2, \ldots, n-1$.

The group homomorphism from free group F_n to G, induces a homomorphism from G_n to G (where G_n is a group obtain from F_n by adding some relations).

In our case, the homomorphism $f : F_n \to G$ induces a homomorphism from G_n to G if $f(r) = f(r')$ for all relations $r = r'$ in G_n. It is straightforward to check for the relations (1.1).
For relation (1.2) we have

$$f(\rho_i.\rho_j) = f(\rho_i).f(\rho_j) = s_i.s_j = s_j.s_i = f(\rho_j).f(\rho_i) = f(\rho_j.\rho_i).$$

This concludes our lemma. □

The following result gives us the presentation of T_n.

Proposition 1.2.2. *T_n is generated by p_i, $i = 1, 2, \ldots, n-1$, with defining relations*

$$p_i^2 = 1, \quad i = 1, 2, \ldots, n-1,$$

$$p_i p_j = p_j p_i, \quad |i-j| > 1, i, j = 1, 2, \ldots, n-1.$$

Proof. Since the generators of twin group T_n satisfing the defining relations of G_n, the previous lemma implies that we have homomorphism $f : G_n \to T_n$. We are only left to show that it is indeed an isomorphism.
Surjectivity of f implies from the fact that $p_1, p_2, p_3, \ldots, p_{n-1}$ generates T_n and belongs to image of f.
Now we construct a set theoretic map $g : T_n \to G_n$ such that $g \circ f = \mathrm{Id}_{G_n}$. That will imply that f is injective.
Let

$$g : T_n \to G_n$$

be defined by sending $p_i \to s_i$. For this g, we have $g \circ f = \mathrm{Id}_{G_n}$.

□

Note that T_2 is the group generated by $\{p_1\}$ such that $p_1^2 = 1$. Therefore, $T_2 \cong \mathbb{Z}/2\mathbb{Z}$. Similarly, T_3 is the group generated by $\{p_1, p_2\}$ such that $p_1^2 = p_2^2 = 1$. Therefore, $T_3 \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, which is the infinite dihedral group.

Definition. *The pure twin group on n arcs*, is a subgroup of the twin group T_n consisting of twins with arcs connecting pairs of points $(i, 0)$ and $(i, 1)$, $1 \leq i \leq n$. It is denoted by PT_n.

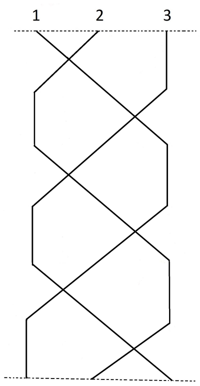

Figure 1.4: Example of a non-trivial element of PT_3.

Consider the natural homomorphism from the twin group T_n to S_n, the group of permutations of the set $\{1, 2, 3, \ldots, n\}$, that sends twin p_i to the transposition $(i, i+1)$. The *pure twin group* PT_n is actually the kernel of this homomorphism.

Chapter 2

SURFACES

In this chapter, we recall basic notions of surfaces which will need in later chapters.

2.1 Surfaces

Definition. An *n-dimensional manifold* is a second countable Hausdorff space X such that each $x \in X$ has an open neighbourhood U_x which is homeomorphic to $\mathbb{R}^n$.

Definition. An *n-dimensional manifold with boundary* is a second countable, Hausdorff space in which every point has a neighbourhood homeomorphic to an open subset of the closed n-dimensional upper half space $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \forall\ 1 \leq i \leq n\}$.

Definition. A *surface* is a 2-dimensional manifold.

Example.
1. *$\mathbb{R}^2$ is a non-compact surface without boundary.*
2. *S^2 is a compact surface without boundary.*
3. *$S^1 \times [0, 1]$ is a compact surface with boundary. It has two boundary components $S^1 \times \{0\}$ and $S^1 \times \{1\}$.*

Definition. Any surface is said to be *closed* if it is compact and does not have a boundary.

Definition. A subset $A = \{a_0, a_1, \ldots, a_k,\ k \geq 1\}$ of $\mathbb{R}^n$ is said to be *geometrically independent* if the set $S = \{a_1 - a_0,\ a_2 - a_0, \ldots, a_k - a_0\}$ of vectors of $\mathbb{R}^n$ is linearly independent.
We assume a set having only one point to be geometrically independent.

So by previous definition we see that

- $\{a_0, a_1\}$ is geometrically independent iff $a_0 \neq a_1$.
- $\{a_0, a_1, a_2\}$ is geometrically independent iff these three points are not collinear.
- $\{a_0, a_1, a_2, a_3\}$ is geometrically independent iff these points do not lie on a plane.

2.2 Simplicial Complexes

Definition. Let $A = \{a_0, a_1, \ldots, a_k,\ k \geq 1\}$ be a geometrically independent set of points in $\mathbb{R}^n$, $n \geq k$. Then a *k-dimensional geometric simplex* or *k-simplex* spanned by the set A is the set of all those points $x \in \mathbb{R}^n$ such that

$$x = \sum_{i=1}^{k} \alpha_i a_i, \text{ where } \sum_{i=1}^{k} \alpha_i = 1, \alpha_i \geq 0,$$

for each $i = 0, 1, 2, \ldots, k$.

We write $\sigma^k = < a_0, a_1, \ldots, a_k >$ to indicate that σ^k is the k-simplex with vertices $a_0, a_1, \ldots, a_k$.

Note that

(i) 0-*simplex* in $\mathbb{R}^n$ is simply a singleton set or a point.

(ii) If a_0, a_1 be any two distinct points of $\mathbb{R}^n$, then *1-simplex* determined by $\{a_0, a_1\}$ is a straight line segment joining a_0 and a_1.

(iii) If a_0, a_1, a_2 be any three distinct points of $\mathbb{R}^n$ not all lying on a line, then the 2-*simplex* determined by $\{a_0, a_1, a_2\}$ is a triangle spanned by these points.

(iv) If a_0, a_1, a_2, a_3 be any four distinct points of $\mathbb{R}^n$ not all lying on a plane, then 3-*simplex* determined by $\{a_0, a_1, a_2, a_3\}$ is a tetrahedron spanned by these points.

Note that if $A = \{a_0, a_1, \ldots, a_k\}$ is a geometrically independent set of points in $\mathbb{R}^n$, $n \geq k$,then the simplex $\sigma^k = < a_0, a_1, \ldots, a_k >$ is the convex hull of the set A.

Definition. Let σ^r, σ^s be two simplexes in $\mathbb{R}^n$ such that $r \leq s \leq n$. We say that σ^r is a r-dimensional face of σ^s or a r-*simplex* of σ^s if each vertex of σ^r is also a vertex of σ^s. If σ^r is a face of σ^s and $r < s$, then σ^r is a *proper face* of σ^s.

Example. *Consider* $\sigma^3 = < a_0, a_1, a_2, a_3 >$.

It has four 0*-faces, six* 1*-faces (or edges), four* 2*-faces and one* 3*-face (*σ^3 *itself).*

Definition. A *simplicial complex* or a *geometric complex* K is a finite collection of simplexes of $\mathbb{R}^n$, where n is sufficiently large and satisfies the following conditions:

1. If $\sigma \in K$, then all the faces of σ are also in K.

2. If σ and τ are in K, then either $\sigma \cap \tau = \phi$ or $\sigma \cap \tau$ is a common face of both σ and τ.

Definition. The *dimension* of simplicial complex K(denoted by dim$K$) is defined to be

$$\begin{cases} -1 & \text{if } K = \mathbb{Q}, \\ n \geq 0 & \text{if } n \text{ is the largest integer s.t. } K \text{ has an } n\text{-}simplex \end{cases}$$

2.3 PL-manifolds

Definition. Let K be a simplicial complex. Let $|K| = \bigcup_{\sigma \in K} \sigma$ be the union of all simplexes of K. Then $|K| \subseteq \mathbb{R}^n$ for some n, is a topological space with the topology induced from $\mathbb{R}^n$. This space $|K|$ is called the *geometric carrier* of K. A subspace of $\mathbb{R}^n$, which is a geometric carrier of some simplicial complex, is called a *rectilinear polyhedron.*

Definition. A topological space X is said to be a *polyhedron* if there exists a simplicial complex K such that $|K|$ is homeomorphic to X. In this case, the space X is said to be *triangulable* and K is called a *triangulation* of X.

Cube, cuboid and tetrahedron are a few examples of polyhedron.

Definition. A map $g : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *affine* if $g(x) = \lambda f(x) + y$, where $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$.

Example. *(i) The map*

$$L : \mathbb{R}^3 \to \mathbb{R}^3$$

defined by $(x_1, x_2, x_3) \mapsto (x_1 + 3x_2 - 2x_3 + 9, 2x_1 + 3x_2 - 5, x_2 + x_3)$ *is an affine map.*

(ii) The map

$$P : \mathbb{R}^2 \to \mathbb{R}^3$$

defined by $(x_1, x_2) \mapsto (x_1 - x_2 + 1, x_2 + 2, x_1 + x_2 + 3)$ *is an affine map.*

Definition. Let $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^m$ be polyhedra.

(1) We will say that a map $f : K \to \mathbb{R}^m$ is *linear* if it is the restriction of an *affine* map from $\mathbb{R}^n$ to $\mathbb{R}^m$. We say that f is *piecewise-linear (PL)* if there exists a triangulation $\{\sigma_i \subset K\}$ such that restriction of f to each σ_i is linear.

(2) We say that a map $f : K \to L$ is *piecewise-linear (PL)* if the underlying map $f : K \to \mathbb{R}^m$ is piecewise-linear.

Let $f : K \to L$ be a piecewise-linear homeomorphism between polyhedra. Then the inverse map $f^{-1} : L \to K$ is again piecewise-linear. To see this, choose any triangulation of K such that the restriction of f to each simplex of the triangulation is linear. Taking the image under f , we obtain a triangulation of L such that the restriction of f^{-1} to each simplex is linear.

Definition. Let M be a polyhedron. We say that M is a *piecewise-linear manifold* of dimension n or *PL-manifold* if for every point $x \in M$, there exists an open neighbourhood $U \subset M$ containing x and a piecewise linear homeomorphism from U to $\mathbb{R}^n$.

Example. *$\mathbb{S}^2$ and torus (space homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$) are piecewise-linear manifolds.*

If M is a PL-manifold of dimension n, then the underlying topological space of M is an n-manifold. We can think of a PL-manifold as a topological manifold equipped with some additional structure.

Definition. An *orientation* of closed surface X with some triangulation is an ordering of it's vertices (upto cyclic permutation) such that any two face glued along an edge receive same local orientation. X is called *orientable* if it has an orientation.

Example. *$\mathbb{S}^2$ and torus are examples of oriented surfaces. Möbius band and Klein bottle are examples of non-oriented surface.*

Chapter 3

DOODLES ON SURFACES

Hereafter, we assume that all manifolds and maps between them are piecewise-linear. We begin this chapter with the definition of a doodle.

3.1 Doodles

Definition. A *doodle* Δ is a collection of piecewise-linear closed curves $C_1, \ldots, C_n$ without triple points on a closed oriented surface.

Here by *triple point* we mean a point at which three curves intersect, or triple self-intersection point of a curve, or a self-intersection point of a curve which lies on another curve.

Definition. Two doodles Δ and Δ' on a surface M are called *equivalent* if there exists a homotopy in M from the collection of curves representing Δ to the collection of curves representing Δ' such that there are no triple intersection points throughout the homotopy.

Another way to see whether two doodles are equivalent is through local moves. These local moves are given by elementary transformations in Figure 3.1. Two doodles are *equivalent* if and only if one can be obtained from the other by a finite sequence of these moves.

Definition. If each component of a doodle has an orientation then it is called an *orientable doodle*.

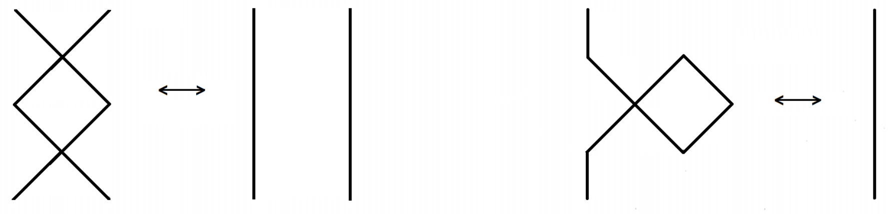

Figure 3.1: Two elementary transformations of doodles.

3.2 Doodles on 2-sphere

We know that 2-sphere is a closed orientable surface, so we can talk about doodles on a 2-sphere. Before going further, we define the closure operation on a twin. It is illustrated in Figure 3.2.

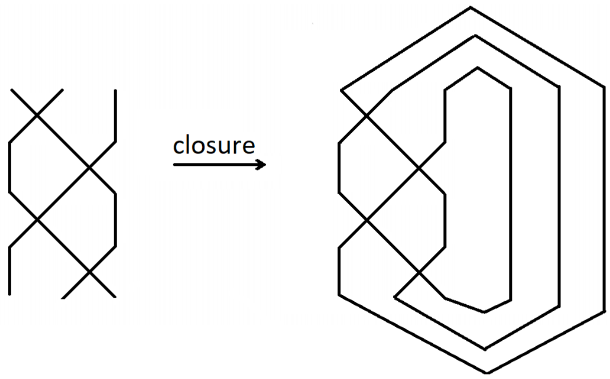

Figure 3.2: Closure of a twin.

Clearly, the closure of a twin on a 2-sphere is a doodle. But the following theorem proves that the converse holds for oriented doodles.

Theorem 3.2.1. *Every oriented doodle on a 2-sphere is the closure of a twin.*

Proof. View S^2 as $\mathbb{R}^2 \cup \{\infty\}$. Let Δ be an oriented doodle on S^2 and $a \in \mathbb{R}^2$. We will deform Δ so that it will lie in $\mathbb{R}^2 \setminus \{a\}$ and each segment is oriented clockwise around a. If we show that such a deformation exists, then cutting $\mathbb{R}^2 \setminus \{a\}$ along a ray emanating from a would be a twin whose closure is Δ. We choose a diagram Δ_1 of Δ such that

1. $\Delta_1 \in \mathbb{R}^2 \setminus \{a\}$.
2. No double point or angle point of Δ_1 is collinear with a or another angle point or double point.

A point of Δ_1 is an *angle point* if it is a vertex of an arc of Δ_1 when we view it as a polygon. Figure 3.3 shows an angle point of Δ_1.

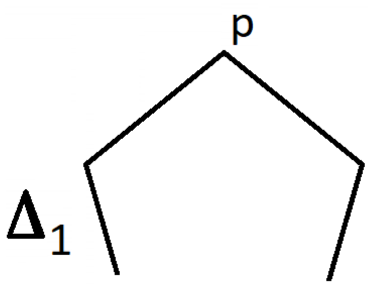

Figure 3.3: An angle point of Δ_1.

Let I be any straight line segment of Δ_1. If I is oriented clockwise with respect to a, no need to do anything. But if it is oriented counter-clockwise, we will change the segment into a configuration of clockwise segments in the following manner.

We consider the triangle formed by the segment I and the point a. We denote this triangle by $T(I, a)$.

(a) If there are no double points of Δ_1 inside $T(I, a)$, we change I into two clockwise segments as shown in Figure 3.4.

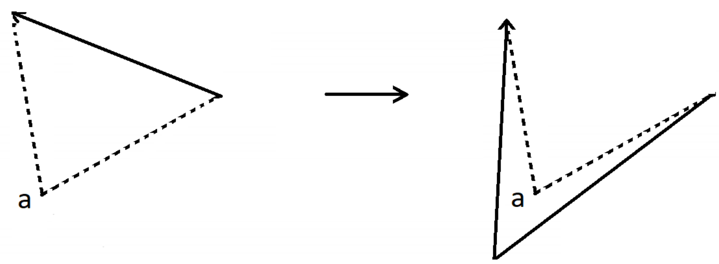

Figure 3.4: Changing a counter-clockwise segment into two clockwise segments when there is no double point in the triangle.

(b) Suppose $T(I, a)$ contains k double point $d_1, d_2, \ldots, d_k$. We cut I into $2k+1$ segments $I_1, \ldots, I_{2k+1}$ such that

(i) The triangles $T(I_{2i+1}, a)$ formed by odd numbered segments do not contain any double point.

(ii) There is only one double point d_i lying in $T(I_{2i}, a)$.

(See Figure 3.5, case $k = 2$).

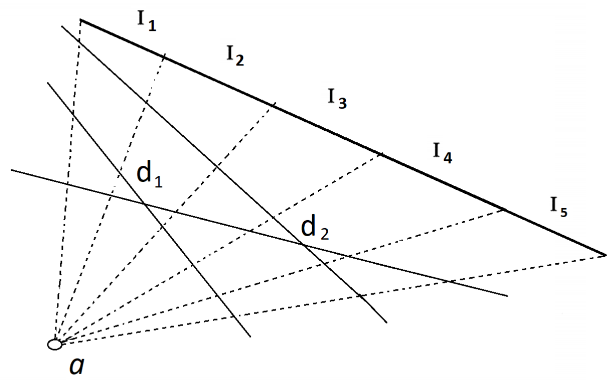

Figure 3.5: Subdividion of I when there are double points in the triangle.

Such a subdivision of I is possible because of condition (2) on the diagram Δ_1, otherwise we might have a situation where two double point and a are collinear, then we won't be able to put these two double points in two different triangles.

Deform I_{2i+1} into two segments going clockwise around a as in Figure 3.4 such that no double points appear in any of the $k-i$ triangles bounded by a and I_{2j+1}, $i < j \leq k$ while varying i from 0 to k.

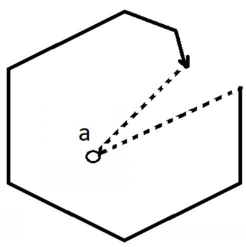

Figure 3.6: Deformation of I_{2i}.

In case of I_{2i} whose triangle $T(I_{2i}, a)$ contains a double point, move one of its point through ∞. It will change its configuration to the one shown in Figure 3.6 where each segment is oriented clockwise relative to a.

Thus, we have deformed I into union of segments such that each segment is oriented clockwise relative to a. After this deformation a new double point may appear and the new diagram might not satisfy conditions (1) and (2). But that problem can be easily solved by making a slight change in the diagram. The number of counter-clockwise segments in this new diagram is one less than in the diagram Δ_1. After repeating this process several times we get a diagram in which every segment is oriented clockwise around a.This concludes our theorem. $\square$

3.3 Minimal diagram of a doodle

Definition. A doodle Δ is *rigid* if it does not have a diagram such that one of the components is a simple curve which does not intersect other components and bounds an open disk in S^2.

Figure 3.7 shows a local diagram for a *non-rigid* doodle.

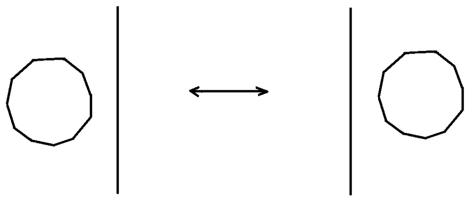

Figure 3.7: A transformation for non-rigid doodle.

Theorem 3.3.1. *A doodle has a unique (up to the transformation in Figure 3.7) diagram with a minimal number of intersection points (called vertices). This diagram can be constructed from any other doodle diagram by applying only the local moves as in Figure 3.1 that reduces the number of intersection points.*

Proof. Denote the local moves in Figure 3.1 by ± 1,± 2 depending on the number of double points that it is creating or annihilating. Thus, by our convention, $+1, +2$

moves are those which creates one and two double points, respectively while $-1, -2$ moves annihilates one and two double points, respectively.

Let Δ' and Δ'' be two diagrams of the same doodle. Then there is a sequence of diagrams $\Delta' = \Delta_1, \Delta_2, \ldots, \Delta_k = \Delta''$ such that any two consecutive diagrams are connected by one of ± 1, ± 2 moves.

The following lemma will help us to establish the theorem.

Lemma 3.3.2. *Let Δ be a rigid doodle. Let Δ' and Δ'' be any two diagrams representing Δ. Then there exists a sequence of diagrams $\Delta' = \Delta_1, \Delta_2, \ldots, \Delta_k = \Delta''$ connected by ± 1 ,± 2 moves and with no $+$ move preceding a $-$ move. That is, for some j with $1 \leq j \leq k$,*

$$|\Delta_1| > |\Delta_2| \cdots > |\Delta_j| < |\Delta_{j+1}| < \cdots < |\Delta_k|, \tag{3.1}$$

where $|\Delta_s|$ denotes the number of double points of the diagram Δ_s for some s.

Proof. Let $\Delta' = \Delta_1, \Delta_2, \ldots, \Delta_k = \Delta''$ be any sequence of diagram connecting Δ' to Δ''. Suppose that the i^{th} move m_i from Δ_i to Δ_{i+1} is a $+$ move and the $i+1^{th}$ move m_{i+1} from Δ_{i+1} to Δ_{i+2} is a – move. Then the move m_i will either create one double point or two double points. Now if m_{i+1} does not destroy at least one double point created by m_i, we can change their order and can apply m_{i+1} first and then m_i. But if m_{i+1} has destroyed point created by m_i, then we have following cases:

1. If m_i is a $+1$ move and m_{i+1} is a -1 move then there is only one possibility since the doodle Δ is rigid (see Figure 3.8). So m_{i+1} cancels m_i.

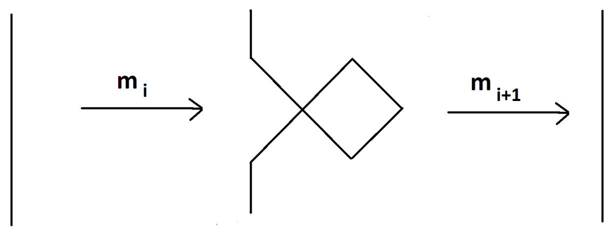

Figure 3.8: $m_{i+1} \circ m_i$ when m_i is a $+1$ move and m_{i+1} is a -1 move.

2. If m_i is a $+1$ move and m_{i+1} is a -2 move, then the composition $m_{i+1} \circ m_i$ is a -1 move.

3. If m_i is a $+2$ move and m_{i+1} is a -1 move, then the composition $m_{i+1} \circ m_i$ is a $+1$ move.

4. If m_i is a $+2$ move and m_{i+1} is a -2 move, then these two moves cancel each other.

Thus, in every case these two moves either cancel each other or they can be replaced by another single move. Applying induction on n concludes the lemma. □

Lemma 3.3.2 tells that for a rigid doodle there exists a diagram with a minimal number of double points and any other diagram can be obtained from that diagram by applying only + moves. Thus minimal diagram will be unique because if there are two minimal diagrams, then on applying lemma we will get an intermediate diagram Δ_j for both the diagrams which will contradict the minimality of these two diagrams. Therefore, minimal diagram is unique.

Lemma 3.3.2 is not applicable on *non-rigid* doodles. Consider doodle diagrams Δ', Δ'' of a doodle Δ as shown in Figure 3.9.

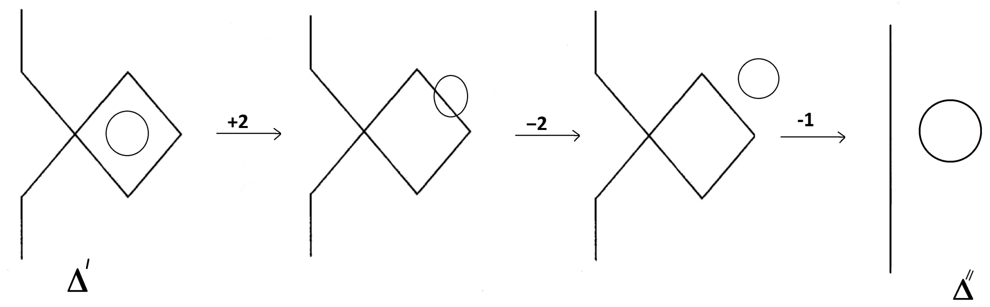

Figure 3.9: Example.

In order to bring the circle out, we have to apply a $+2$ move before a -2 move. So we a can not get a sequence where no positive move proceeds negative moves unless we permit transformation given in Figure 3.7.

So for *non-rigid* doodles lemma holds upto the transformation in Figure 3.7. We can bring all the circles out from both the diagrams then apply lemma to the remaining rigid parts. This concludes the theorem.

□

Chapter 4

DOODLES AND 2-COMPLEXES

4.1 2-Complex of a doodle

A 2-*dimensional complex* or a 2-*complex* is a topological space homeomorphic to a two-dimensional finite CW-complex.

Let M be a closed oriented surface and Δ be a doodle on it. To any diagram Δ^1 of doodle Δ we associate a 2-*dimensional complex* $R(\Delta^1)$. We are not calling it a 2-dimensional CW-complex even though it is homeomorphic to a CW-complex because cell decompositions in it are not the canonical cell decompositions (where $0, 1$ and 2 dimensional cell is point, line segment and disk, respectively).

Consider any diagram Δ^1 of Δ. Suppose there are d double points of Δ^1 denoted by $pt_1, \ldots, pt_d$, q edges denoted by $edg_1, \ldots, edg_q$ and s regions (connected components of $M \backslash (edg_1 \cup \cdots \cup edg_q)$) denoted by $reg_1, \ldots, reg_s$. The 2-complex $R(\Delta^1)$ consists of a surface PL-homeomorphic to M with 1-dimensional cells and 2-dimensional compact surfaces with boundary attached to it.

Construction of $R(\Delta^1)$:

Take d 1-cells (one 1-cell for every double point of Δ^1) and denote them by p_1, $p_2, \ldots, p_d$ and take s surfaces $r_1, \ldots, r_s$, where surface r_j is homeomorphic to the region reg_j, for $1 \leq j \leq s$.

We glue the 1-cells to M in following way:

Glue both the ends of the 1-cell p_i to the double point $pt_i \in M$, for every $i = 1, 2, \ldots, d$ (see Figure 4.1). We denote this complex by $PR(\Delta^1)$. Denote image of p_i in $PR(\Delta^1)$ by the same symbol p_i. Now fix an orientation of p_i for every i.

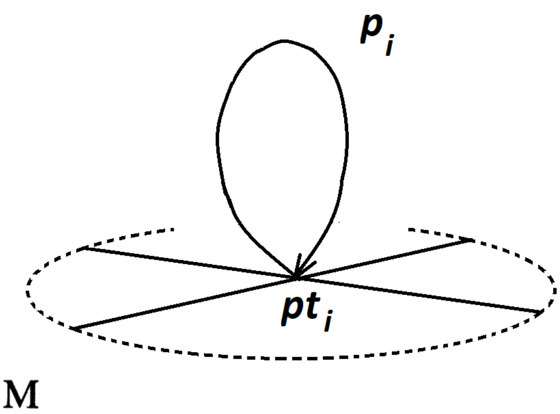

Figure 4.1: Gluing p_i to the double point pt_i.

Now we glue the surfaces r_j, $j = 1, 2, \ldots, s$ to $PR(\Delta^1)$ in following way:
Case(1): When r_j is a disk.
If we move along the boundary of the 2-cell $reg_j \in M$ in the clockwise direction, we will meet some double point and edges of Δ_1. Denote them in a unique order (up to permutation) by $pt_1, edg_1, pt_2, edg_2, \ldots, pt_k, edg_k$. See Figure 4.2.

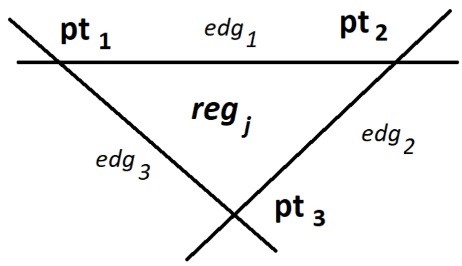

Figure 4.2: case $k = 3$.

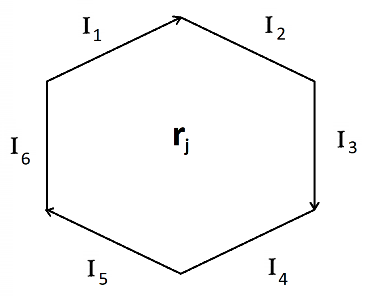

Figure 4.3: Separating boundary of r_j

Separate the boundary of the 2-cell r_j into $2k$ segments, denote them $I_1, I_2, \ldots, I_{2k}$ while moving clockwise along the boundary. Orient the segments $I_1, I_3, \ldots, I_{2k-1}$ clockwise (see Figure 4.3). Now identify oriented segments I_1 and p_1, I_3 and $p_2, \ldots, I_{2k-1}$ and p_k. Then identify I_2 and edg_1, I_4 and $edg_2, \ldots, I_{2k}$ and edg_k. These operations are illustrated in Figure 4.4. Dashed arrows shows how r_j is glued to $PR(\Delta^1)$.

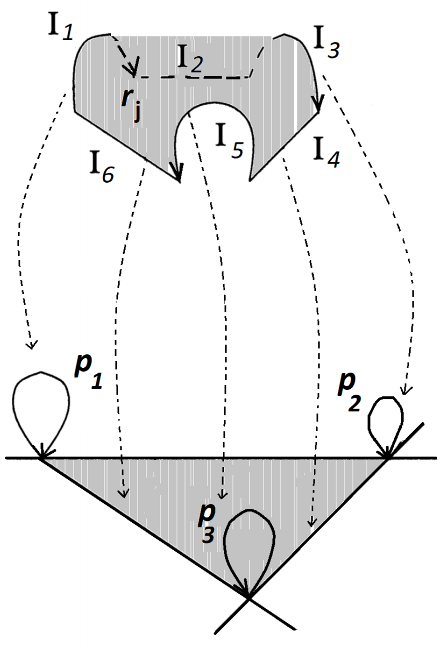

Figure 4.4: Gluing r_j to $PR(\Delta^1)$.

Case (2): When r_j is not a disk.

If r_j is not a disk, then it has more than one boundary component (see Figure 4.5). Glue r_j to $PR(\Delta^1)$ in a similar way along each boundary components.

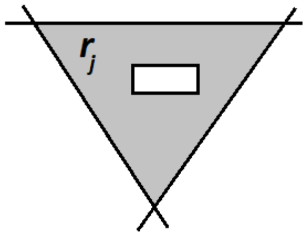

Figure 4.5: Example of r_j having more than one boundary components.

If diagram Δ^1 has a component C with no double points, then none of the 1-cells gets attach to C. In this case, glue part of the boundary of the corresponding r_j homeomorphically to C (see Figure 4.6).

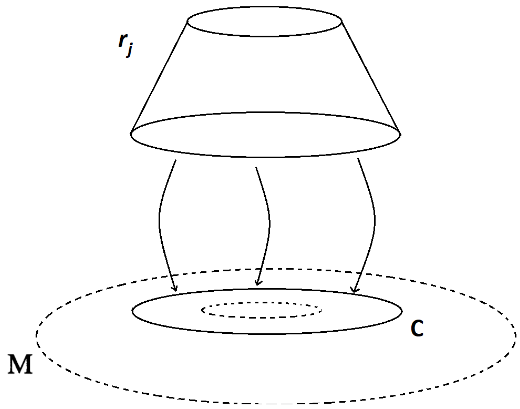

Figure 4.6: Gluing when Δ^1 has a component C with no double points.

After gluing $r_i, \ldots, r_s$ to $PR(\Delta^1)$ in the way described above, we obtain a complex. Denote it by $R(\Delta^1)$. Observe that $R(\Delta^1)$ contains the surface M as a subcomplex.

We define an equivalence relation on $R(\Delta^1)$ given by:

For $x_1,\ x_2 \in R(\Delta^1)$, $x_1 \sim x_2$ iff $x_1 = x_2$ or $x_1,\ x_2 \in M$. We define $\bar{R}(\Delta^1) \cong R(\Delta^1)/\sim$. Notice that $\bar{R}(\Delta^1)$ is actually $R(\Delta^1)$ with M contracted to a point. The topology on $\bar{R}(\Delta^1)$ is the quotient topology induced from $R(\Delta^1)$ by the equivalence relation.

We call $R(\Delta^1)$ *the geometric realization* of the diagram Δ^1 and $\bar{R}(\Delta^1)$ *the reduced geometric realization* of Δ^1.

4.2 Invariants of doodles

Definition. Let K_1, K_2 be finite CW-complexes. Then there is an *elementary expansion* from K_1 to K_2 if K_2 is obtained by gluing an n-disk D^n to K_1 through its boundary i.e., $K_2 = K_1 \cup_f D^n$ where $f : S^{n-1} \to K_1$ is a map from boundary of D^n to K_1. K_2 is said to be an *expansion* of K_1 and K_1 is said to be *contraction* of K_2.

Definition. Two CW-complexes are said to be *simple homotopy equivalent* if they are related by a sequence of expansions and contractions.

Theorem 4.2.1. *If Δ^1 and Δ^2 are two diagrams of a doodle Δ, then the 2-complex $R(\Delta^1)$ is simple homotopy equivalent to $R(\Delta^2)$ and the 2-complex $\bar{R}(\Delta^1)$ is simple homotopy equivalent to $\bar{R}(\Delta^2)$.*

Proof. It is sufficient to check simple homotopy invariance of $R(\Delta^1)$ and $R(\Delta^2)$ under the two elementary transformations of doodles given in Figure 3.1. Let Δ^1 and Δ^2 be two diagrams of doodle Δ where Δ^2 be obtained from Δ^1 by adding a curl. Let p be 1-cell of $R(\Delta^2)$ corresponding to the new double point. Denote the region of M bounded by the curl by *reg* and by r the corresponding disk of $R(\Delta^2)$ glued to p and to the boundary of *reg* (See Figure 4.7).

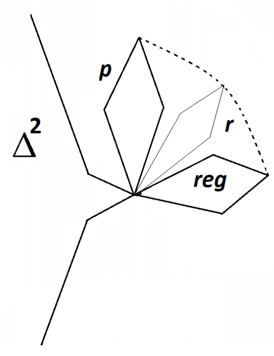

Figure 4.7: Part of the diagram Δ^2 where dashed line are showing parts of the disk r.

Note that $r \cup reg$ is a subcomplex of $R(\Delta^2)$ which is homeomorphic to a disk. If we contract $r \cup reg$ to a point, then we get a complex which is homeomorphic to $R(\Delta^1)$. So $R(\Delta^1)$ is a contraction of $R(\Delta^2)$. Therefore, $R(\Delta^1)$ and $R(\Delta^2)$ are simple homotopy equivalent under move (b). Similarly, move (a) can be verified. $\square$

We now define fundamental group of doodles.

Definition. The fundamental group of the 2-complex $R(\Delta^1)$ is called the *fundamental group of the doodle* Δ represented by the diagram Δ^1 and is denoted by $\pi_1(\Delta^1)$. The fundamental group of the 2-complex $\bar{R}(\Delta^1)$ is called the *reduced fundamental group of the doodle* Δ and is denoted by $\bar{\pi}_1(\Delta^1)$.

By theorem 4.2.1, fundamental group and reduced fundamental group are invariants of doodle. Observe that if Δ is a doodle on a 2-sphere, then the groups $\pi_1(\Delta)$ and $\bar{\pi}_1(\Delta)$ are isomorphic.

Chapter 5

FUNDAMENTAL GROUP OF A DOODLE

We have seen construction of 2-complex $R(\Delta)$ in last chapter. This construction translates to an algorithm that describes $\pi_1(\Delta)$ in terms of generators and relations. We are restricting ourselves to the case of a doodle on the 2-sphere.

The algorithm goes as follows:
Fix an orientation of S^2. Let Δ be a doodle on S^2.

Definition. A *disk diagram* Δ^1 of doodle Δ is a diagram such that union of curves $C_1, C_2, \ldots, C_n$ which represent Δ^1 cuts the 2-sphere into a union of disks.

Let Δ^1 be any disk diagram of doodle Δ. Suppose it has k vertices. We denote them by a_1,$a_2, \ldots, a_k$. By abuse of notation, we denote generators of doodle group $\pi_1(\Delta)$ by same notation $a_1, a_2, \ldots, a_k$. Denote regions separated by Δ^1 by $reg_1, reg_2, \ldots, reg_p$. To each region we associate a relation among $a_1, a_2, \ldots, a_k$ such that the vertices of reg_i taken in the counter-clockwise order be a_{i1},$a_{i2}, \ldots, a_{is}$ (up to a cyclic permutation). Then the relation associated to reg_i is

$$a_{i1}a_{i2}\cdots a_{is} = 1.$$

The fundamental group $\pi_1(\Delta)$ of Δ is a group with generators a_1,$a_2, \ldots, a_k$ with defining relations

$$a_{i1}a_{i2}\cdots a_{is} = 1,$$

for all regions reg_i, $i = 1, 2 \ldots, p$, of Δ_1.

Theorem 4.2.1 implies that $\pi_1(\Delta) \cong \pi_1(R(\Delta))$ is independent of the choice of disk diagram Δ^1 of Δ.

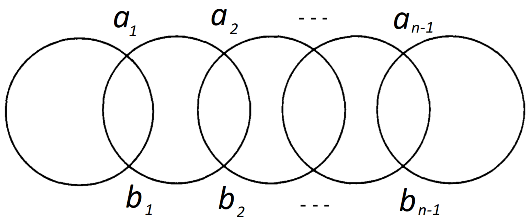

Figure 5.1: A diagram of trivial n-component doodle on the 2-sphere .

Example. *Let Δ be a trivial (that is, without self-intersections and bounding a disc) n-component doodle on the 2-sphere. Consider the diagram of Δ given in Figure 5.1. There are $2n-2$ intersection points of this diagram. Therefore we have $2n-2$ generators of fundamental group $\pi_1(\Delta)$, denote them by $a_1, b_1, \ldots, a_{n-1}, b_{n-1}$. Also there are $2n$ regions which gives $2n$ relations among $a_1, b_1, \ldots, a_{n-1}, b_{n-1}$ that reduces to $n-1$ relations given by $a_1b_1 = 1$, $a_2b_2 = 1$, $a_{n-l}b_{n-1} = 1$. As b_i is the inverse of a_i for all $i = 1, 2, \ldots, n-1$ and there are no relations among a_i's so $\pi_1(\Delta)$ is a free group of rank $n-1$.*

More generally,

Proposition 5.0.1. *Suppose that a doodle Δ_1 is obtained from a doodle Δ by adding a trivial component. Then $\pi_1(\Delta_1)$ is the free product of $\pi_1(\Delta)$ and $\mathbb{Z}$.*

Proof. Take diagram of Δ_1 in which trivial component intersects only one arc of Δ say at b and b'. Region bounded between this arc and trivial component gives us relation $b.b' = 1$. So b' is inverse of b and there is no relation between generators of $\pi_1(\Delta)$ an b. So adding a trivial component only adds a new generator which has no relation with other generators. This implies $\pi_1(\Delta^1)$ is the free product of $\pi_1(\Delta)$ and $\mathbb{Z}$. □

Remark. For a trivial one component doodle Δ on a closed oriented surface M, we have:

(a) The reduced fundamental group of Δ is isomorphic to the fundamental group of the surface M.

(b) The fundamental group of Δ is isomorphic to $\pi_1(M) * \pi_1(M)$.

This observation follows directly from the definitions.

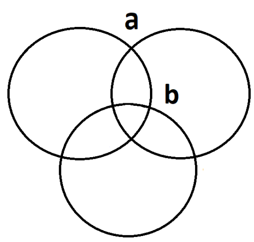

Figure 5.2: A doodle with three components.

Example. *Consider the doodle* Δ *on* 2*-sphere as shown in Figure 5.2. If we take generators* a, b *for* $\pi_1(\Delta)$*, we get the following defining relations*

$$\langle a, b \mid a^2b = ba^2, ab^2 = b^2a, abab = baba\rangle.$$

Proof. Denote the vertices by a, b, c, d, e, f and regions by $reg_1, reg_2, \ldots, reg_8$.

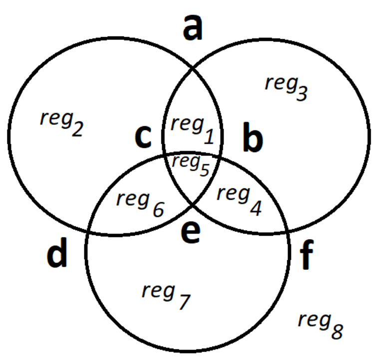

Figure 5.3: A doodle with three components.

Relations associated to the regions are:
For reg_1

$$acb = 1 \Rightarrow c = a^{-1}b^{-1}.$$

For reg_2

$$adc = 1 \Rightarrow d = a^{-1}ba^{-1}.$$

For reg_3

$$abf = 1 \Rightarrow f = b^{-1}a^{-1}.$$

For reg_4

$$efb = 1 \Rightarrow e = b^{-1}f^{-1} = b^{-1}ab.$$

Putting these values in rest of the four relations, we get
For reg_5

$$ebc = 1.$$

This implies $b^{-1}ab.b.a^{-1}b^{-1} = 1$, and hence $ab^2 = b^2a$.
For reg_6

$$ecd = 1.$$

This implies $b^{-1}ab.a^{-1}b^{-1}.a^{-1}ba = 1$, i.e., $abab = baba$.
For reg_7

$$edf = 1.$$

We have $b^{-1}ab.a^{-1}ba.b^{-1}a^{-1} = 1$, which implies $abab = baba$.
For reg_8

$$afd = 1,$$

which gives $a.b^{-1}a^{-1}.a^{-1}ba = 1$. Equivalently, $a^2b = ba^2$. $\square$

Proposition 5.0.2. *Let M be an oriented closed surface and let Δ be a doodle on M. Then the first homology groups $H_1(\pi_1(\Delta), \mathbb{Z})$ and $H_1(\bar{\pi}_1(\Delta), \mathbb{Z})$ of the fundamental group of Δ and of the reduced fundamental group of Δ depends only on the conjugacy classes of the components of Δ in the fundamental group of the surface M.*

Proof. It follows from the invariance of $H_1(\pi_1(\Delta), \mathbb{Z})$ and $H_1(\bar{\pi}_1(\Delta), \mathbb{Z})$ under the move (called triple point move) depicted in Figure 5.4. $\square$

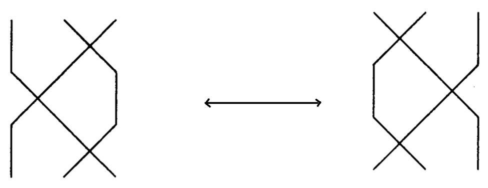

Figure 5.4: Triple point move.

Chapter 6

DOODLE GROUPS WITH ABELIAN SUBGROUPS

In this chapter we will see some examples of doodles and some of their free abelian groups. We will see doodles with infinite center.

Consider a doodle Δ. Let Δ^{min} be the diagram of Δ with the minimal possible number of double points. By Theorem 3.3.1 such a diagram is unique up to the move in Figure 3.7.

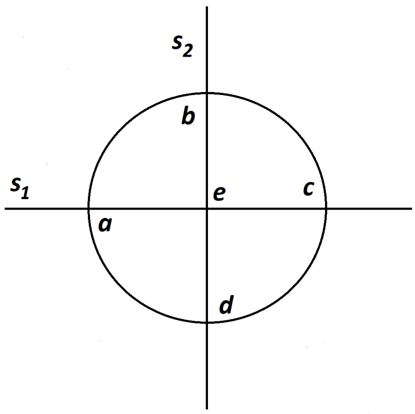

Figure 6.1: Subdiagram of Δ^{min}.

Proposition 6.0.1. *Let Δ be a doodle on the 2-sphere. Suppose that Δ^{min} contains a subdiagram depicted in Figure 6.1, such that the segments s_1, s_2 belong to different components of Δ. Then $\pi_1(\Delta)$ contains a free abelian subgroup of rank two.*

Proof. Let a, b, c, d, e be the elements of $\pi_1(\Delta)$ associated to double points in the Figure 6.1 part of Δ^{min}. Relations associated to the four regions given in Figure 6.1 are:

$$eba = 1, \ ecb = 1, \ edc = 1, \ ead = 1. \tag{6.1}$$

Expressing e and c in terms of other generators, we get $e = a^{-1}b^{-1}$ and $c = e^{-1}b^{-1} = bab^{-1}$.
Putting values of e and c in the remaining two relations, we get

$$edc = 1 \ \Rightarrow \ a^{-1}b^{-1}dbab^{-1} = 1 \ \Rightarrow \ d = baba^{-1}b^{-1},$$

$$ead = 1 \ \Rightarrow \ a^{-1}b^{-1}ad = 1 \ \Rightarrow \ d = a^{-1}ba.$$

Equating values of d, we get

$$baba^{-1}b^{-1} = a^{-1}ba,$$

$$\Rightarrow abab = baba. \tag{6.2}$$

Note that (6.2) is equivalent to any of the two relations

$$[abab, a] = 1, \tag{6.3}$$

$$[abab, b] = 1. \tag{6.4}$$

Let G be a subgroup of $\pi_1(\Delta)$ generated by $(ab)^2$ and a. Then the relation (6.3) implies that G is abelian, as

$$\begin{gathered}[abab, a] = 1, \\ \Rightarrow (abab)a(abab)^{-1}a^{-1} = 1, \\ \Rightarrow (ab)^2a(ab)^{-2}a^{-1} = 1, \\ (ab)^2a = a(ab)^2.\end{gathered}$$

Since generators of G commutes, therefore all elements commute.

Note that the segments s_1 and s_2 belongs to different components of doodle Δ. This implies that the image of G in $H_1(\pi_1(\Delta), \mathbb{Z})$ has rank 2. Therefore, G is a rank 2 abelian subgroup of $\pi_1(\Delta)$. $\square$

Our next goal is to construct doodles on 2-sphere whose fundamental groups have non-trivial center. With the help of relations (6.3) and (6.4) we can do that.

Let $\Delta(2n)$ be the doodle with $2n+2$ components as shown in Figure 6.2. Let $\Delta(2n-1)$ be the doodle with $2n+1$ components as in Figure 6.3.

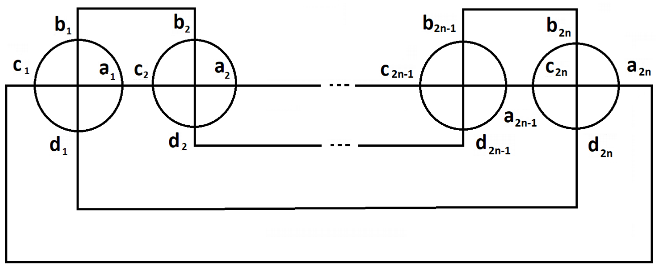

Figure 6.2: $\Delta(2n)$.

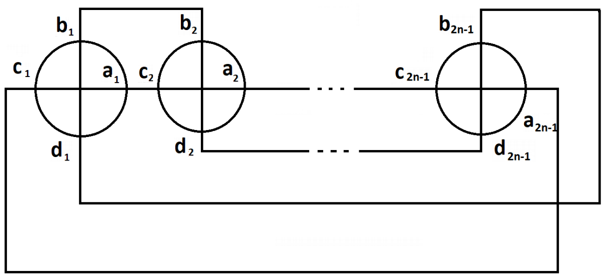

Figure 6.3: $\Delta(2n-1)$.

Proposition 6.0.2. *The fundamental groups of the doodles $\Delta(2n)$ and $\Delta(2n-1)$ have infinite center, for $n \geq 1$.*

Proof. Denote by $a_1, b_1, c_1, d_1, \ldots, a_{2n}, b_{2n}, c_{2n}, d_{2n}$ the elements of $\pi_1(\Delta(2n))$ associated with double points as shown in Figure 6.2.

Claim: The element $(b_1a_1)^2$ is in the center of the fundamental group of $\Delta(2n)$. Element $(b_ia_i)^2$ commutes with each of the four elements a_i, b_i, c_i, d_i for $i = 1, \ldots, 2n$ (by (6.3) and (6.4) and the fact that c_i and d_i are product of a_i, b_i and their inverses).

Observe that

$$(b_ic_i)^2 = (a_ib_i)^2. \tag{6.5}$$

(As $(b_ic_i)^2 = b_ib_ia_ib_i^{-1}b_ib_ia_ib_i^{-1} = b_ib_ia_ib_ia_ib_i^{-1} = b_ia_ib_ia_ib_ib_i^{-1} = b_ia_ib_ia_i = a_ib_ia_ib_i = (a_ib_i)^2$).

Also, we have $c_2b_2b_1a_1 = 1$.

This implies that,

$$(b_1a_1)^2 = (c_2b_2)^{-2}. \tag{6.6}$$

By using (6.5) for $i = 2$, we get

$$(c_2b_2)^2 = (b_2c_2)^2 = (a_2b_2)^2 = (b_2a_2)^2. \tag{6.7}$$

Now (6.5) and (6.6) gives,

$$(b_1a_1)^2 = (b_2a_2)^{-2}. \tag{6.8}$$

Similarly,

$$(b_ia_i)^2 = (b_{i+1}a_{i+1})^{-2}, i = 1, 2, \ldots, 2n-1. \tag{6.9}$$

By using (6.8) and recursive use of (6.9), we get

$$(b_1a_1)^2 = (b_ia_i)^{\pm 2}, i = 1, 2, \ldots, 2n-1. \tag{6.10}$$

Now (6.10) and the fact that $(b_ia_i)^2$ commutes with each of the four elements a_i, b_i, c_i, d_i for $i = 1, \ldots, 2n$. It implies that $(b_1a_1)^2$ commutes with a_i, b_i, c_i, d_i for $i = 1, \ldots, 2n$. Therefore, $(b_1a_1)^2$ is in the center of the fundamental group of $\Delta(2n)$. This means that the center is non-trivial. This concludes our claim.

Now, since the image of $(b_1a_1)^2$ in the first homology group of $\pi_1(\Delta(2n))$ is non-trivial. Therefore, space $(b_1a_1)^2$ has infinite order in $\pi_1(\Delta(2n))$. This proves the proposition for $\Delta(2n)$. The case of $\Delta(2n-1)$ has similar proof.

□

Remark. Another example of doodle whose fundamental group has infinite center is given in Figure 6.4. Our method is applicable to this infinite family of doodles as well.

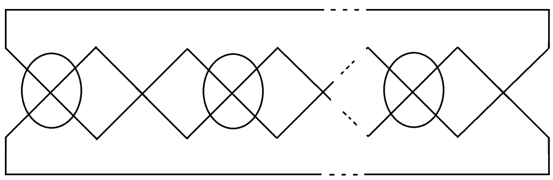

Figure 6.4: A doodle with infinite center.

Chapter 7

CURVATURE OF DOODLE GROUPS

In this chapter we will talk about doodles whose fundamental groups are automatic. An automatic group was first introduced in 1986 by Thurston, motivated by results of Jim Cannon on hyperbolic groups. Major work related to this important class of groups was done by David Epstein in recent years.

We will use some definitions and results from [E], [GS1] and [GS2] for this chapter.

Definition. An $\mathbf{A}_2$ *complex* is a 2-dimensional CW-complex equipped with a metric with all 2-cells isometric to equilateral triangles.

Definition. An $\mathbf{A}_2$ complex X has *non-positive curvature* if every cycle without backtracking in the link of the vertex has length greater than or equal to 6.

We will use the following theorem proved by Gersten and Short ([GS1] and [GS2]) to show that the fundamental group of some doodles are automatic.

Theorem 7.0.1. *The fundamental group of a finite $\boldsymbol{A}_2$ complex of non-positive curvature is automatic.*

We restrict ourselves to the case of doodles on 2-sphere for simplicity.

Definition. A doodle Δ on 2-sphere is said to be *reducible* if it can be represented as the disjoint union of two doodles. Otherwise it is *irreducible.*

Doodle shown in Figure 7.1 is a reducible doodle since it is a disjoint union of two doodles.

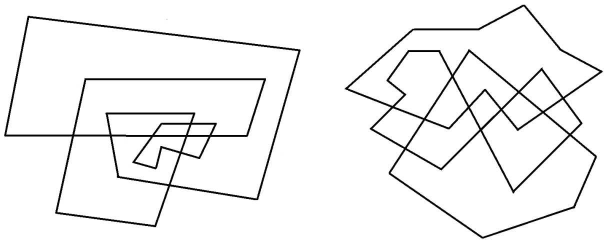

Figure 7.1: A *reducible* doodle on a 2-sphere.

Observe that every *irreducible* doodle is *rigid* as it can't have a free component. By Theorem 3.3.1, an irreducible doodle Δ has a unique minimal diagram. Let us denote it by Δ^{min}.

Definition. A doodle Δ on the 2-sphere is called *thick* if it is irreducible and each cycle of even length of Δ^{min}, without backtracking, has length greater than or equal to 6.

Theorem 7.0.2. *The fundamental group of any thick doodle Δ can be realized as the fundamental group of a finite $\boldsymbol{A}_2$ complex of non-negative curvature and is automatic.*

Proof. Let Δ be an irreducible doodle on 2-sphere. Let Δ^{min} be the minimal diagram of Δ. Recall that for doodles on 2-sphere, the fundamental group is isomorphic to the reduced fundamental group. Consider the reduced geometric realization $\bar{R}(\Delta^{min})$ of the minimal diagram of Δ. Observe that the minimal diagram of an irreducible doodle is a disk diagram. The reduced geometric realization $\bar{R}(\Delta^{min})$ is a 2-dimensional complex with only one 0-cell since 2-sphere can be shrunk to a point(by the definition of reduced geometric realization). Its 1-cells are in bijection with the double points of Δ^{min} and the 2-cells are in bijection with the regions of Δ^{min}. Observe that since Δ^{min} is the minimal diagram, each of the regions of Δ^{min} is bounded by at least 3 edges otherwise we can apply one of $-1, -2$ moves to it and get another diagram which will contradict the fact that Δ^{min} is minimal.

We want to make $\mathbf{A}_2$ complex out of $\bar{R}(\Delta^{min})$. For this, we triangulate each of the 2-cells of $\bar{R}(\Delta^{min})$ and then make all triangles equilateral. Also we triangulate it in such a way that it will not introduce new 0-cells. For example, if the region of Δ^{min} (2-cell of $\bar{R}(\Delta^{min})$)) is an n-gon ($n > 3$), then the triangulation is a partition of this n-gon into $n-2$ triangles. Triangulate all of the 2-cells of $\bar{R}(\Delta^{min})$ like that and then make all triangles equilateral.

Now we fix an arbitrary such triangulation of the 2-cells of $\bar{R}(\Delta^{min})$. We denote this $\mathbf{A}_2$ complex with this fix triangulation by $X(\Delta^{min})$. Observe that in this notation we suppress the dependence on triangulations of the 2-cell.

$X(\Delta^{min})$ has only one vertex. The *link* of the vertex is described as:
It is a 1-dimensional CW-complex such that for each double point v of Δ^{min} we have two 0-cells v^+ and v^- and if there is an arc connecting two double points v_1 and v_2 in Δ^{min}, then there are two 1-cells connecting v_1^+ with v_2^- and v_2^+ with v_1^-.

If we consider Δ^{min} as a 4-valent plane graph, then from above description cycles of even length without backtracking of the diagram Δ^{min} are in one-to-one correspondence with the cycles without backtracking in the link of the only vertex of $X(\Delta^{min})$. Therefore, $X(\Delta^{min})$ has non-negative curvature.

Therefore, if Δ is a thick doodle on a 2-sphere then for any triangulation of the 2-cells of $\bar{R}(\Delta^{min})$ as described above, the complex $X(\Delta^{min})$. As $\pi_1(\Delta) \cong \pi(X(\Delta^{min}))$, Theorem 7.0.1 implies that the fundamental group of a thick doodle is automatic. □

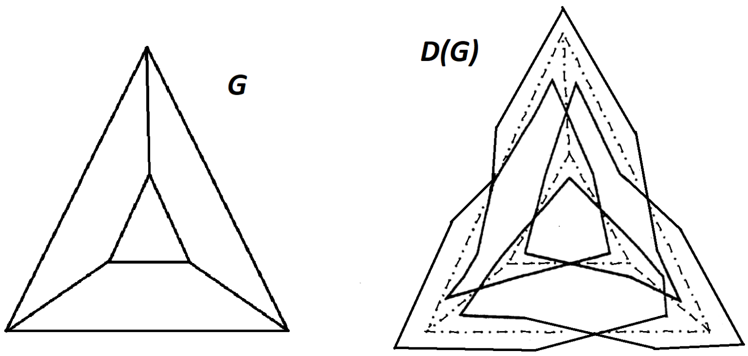

Figure 7.2: .

We can construct thick doodles on 2-sphere from trivalent graphs without loops, where loop is an edge that connects a vertex to itself. Let G be a trivalent graph without loops on 2-sphere. To any such graph we can attach a doodle in the following way:
On each edge of G, pick a point. If the two edges of G share a common point then connect their corresponding chosen points by an arc . If two edges of G have two points in common, connect the points corresponding to the edges by two arcs (See Figure 7.2) This gives a 4-valent graph on the sphere which also represents a doodle. We denote this doodle by $D(G)$.

Proposition 7.0.3. *If G is a trivalent graph on the 2-sphere such that it doesn't have cycles of length less than 5, then the associated doodle $D(G)$ is thick.*

Proof. By construction of $D(G)$, if G doesn't have cycles of length less than 5 then $D(G)$ doesn't have cycles of length less than 6. Therefore, the associated doodle $D(G)$ is thick. $\square$

So by constructing example of trivalent graphs on the 2-sphere without cycles of length less than 5, we can create thick doodle $D(G)$. Therefore, from proposition 7.0.3 we get an automatic group corresponding to every such example.

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