On Knots and the Alexander Polynomial

Jnanajyoti Bhaumik

A dissertation submitted for the partial fulfilment of MS degree in the Integrated PhD program in Mathematics

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH MOHALI

May 2020

Certificate of Examination

This is to certify that the dissertation titled 'On Knots and the Alexander Polynomial' submitted by Mr. Jnanajyoti Bhaumik (Reg. number – MP17008) for the partial fulfillment of MS degree in the Integrated PhD program of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

Dr. Shane D'Mello (supervisor)	Dr. Mahender Singh	Dr. Pranab Sardar

Date: May 2020

Declaration of Authorship

The work presented in this dissertation has been carried out by me under the guidance of Dr. Shane D'Mello at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a work of exposition and all sources listed within have been detailed in the bibliography.

Jnanajyoti Bhaumik
(Candidate)

Date: May 2020

In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

Dr. Shane D'Mello
(Supervisor)

Date: May 2020

"It is a nuisance that knowledge can only be acquired by hard work."

-W. Somerset Maugham

Acknowledgements

First and foremost, I would like to acknowledge my deepest respect and love for my parents who have made numerous sacrifices for me to be in the position that I am and to be able to be typing this.

I would like to acknowledge the unfailing source of guidance and direction that my thesis guide, Dr Shane D'Mello has been for the past 18 months. I, for one, am not personally acquainted with anyone more readily dedicated to the subject. He has led me by example and there's little more I could have asked from a thesis guide.

I would like to thank Dr Mahender Singh and Dr Pranab Sardar for agreeing to be the committee members of my MS thesis. I have had the privilege of attending multiple courses taught by them and have had numerous fruitful mathematical discussions with them during the course of my time at IISER. I would like to thank the Department of Mathematics of IISER Mohali, for having provided me with this exceptional platform to know and grow.

As this thesis is the result of three years of my education at IISER Mohali, I feel the need to acknowledge the insights and knowledge that I have gained from my daily discussions on math with my roommate George Shaji.

Last but definitely not the least, I would like to thank Sukanya Dutta for lending me her laptop (and constant support) without which this thesis would not be possible.

Contents

Abstract

The main aim of this thesis is to study the Alexander's Polynomial and it's construction. This polynomial is a knot invariant, that means, if we pick isotopic knots, they will have the same value. We will look at two methods of construction of the infinite cyclic cover of a knot group and in the process come up with an invariant - The Alexander's Polynomial as well as deduce a lower bound for the unknotting number of a knot. The subsequent chapters deal with applications of the Alexander Polynomial and alternate procedures through which we can construct the Alexander Polynomial.

Chapter 0

Introduction

The main theme of this thesis is the study of the Alexander Polynomial, which is an invariant for knots. In Chapter 1, we begin with the problem of trying to determine if a knot has unknotting number. This leads us to the construction of the cyclic cover of the knot exterior, for which, we are going to describe two methods. The Alexander Polynomial arises during this process and as it turns out is a very useful tool for distinguishing knots. We introduce slice knots at the end of Chapter 1 and try characterising the Alexander Polynomials of slice knots. This will show that Alexander polynomials are also useful in ascertaining which knots cannot be slice.

Chapter 2 deals with more invariants of knots, namely colourings. One of this colourings turns out to be the Alexander Colouring. While, in Chapter 1, the method of arriving at the Alexander's Polynomial was topological in nature, the method of calculating the Alexander's polynomial in chapter 2 is combinatorial and can be calculated using any diagram of the knot.

We introduce Braid groups in Chapter 3 and define homological representations on the braid groups. One such representation, the Burau representation leads to the Alexander Polynomial. We study the Burau representation in detail.

Chapter 4 provides yet another method of construction of the Alexander's polynomial, this time using only a presentation of the knot group. This method of computation of the Alexander's Polynomial uses the techniques of Fox Differential Calculus or Free Differential Calculus devised by RH Fox. Using Free Differential Calculus, the computations are eased largely for certain classes of knots.

Chapter 1

Cyclic coverings of the knot complement

1.1 Unknotting a knot

This section is primarily taken from [Rob15]. Given any knot K in S^2 (or $\mathbb{R}^2$), we can obtain a knot diagram in S^2 (or $\mathbb{R}^2$), which is simply the projection of the knot along some plane (with some constraints of course). The information regarding over and under arcs can be suitably provided with the aid of broken arcs, i.e., when an arc goes under another arc, we represent the under arc as two broken arcs on either side of the over arc. The simplest non-trivial knots that we might think of would probably be the trefoil or perhaps the figure-8 knot.

An observation to be made here is that the knot diagram would be rendered totally useless if the over and under crossing information were to be absent. In fact, given any knot diagram of a non trivial knot, it is possible for us to construct a knot diagram of the unknot that would emulate it except at the crossings. A way to see this has been described in [Rob15]. We describe the method in brief.

Without loss in generality, we can consider that the knot has been projected along the $x-y$ plane, since the knot is a 1 dimensional manifold, we can parametrize it by some variable t. Say $(x(t), y(t), 0)$ denotes the equation of the knot diagram, then we pick some point from this set (not a crossing point), say $(x(t_0), y(t_0), 0)$, we consider the curve $(x(t), y(t), t)$ that starts at $(x(t_0), y(t_0), 0)$ and ends at $(x(t_0), y(t_0), t_0)$, we join these to points by a segment perpendicular to the $x-y$

plane. Projecting along the $y - z$ plane, we have the equation $(y(t), t)$ for the curve. This doesn't have any common points as the last co-ordinate is always different. Its easy to see that this closed curve is in fact the unknot.

Remark 1.1.1. A diagram of a knot, can be unknotted using $c/2$ or less crossing changes.

The reasoning behind this is as follows, if the curve $(x(t), y(t), t)$ flips more than $c/2$ crossings of the knot, then look at the curve $(x(t), y(t), -t)$ instead. In this case, the number of crossing changes would be less than $c/2$, thus proving our claim. Intuitively the first curve traverses strictly upwards while the second curve traverses downwards.

A natural question now would be how many crossings do we need to switch to give us the unknot. The previous argument tells us that given a knot diagram with c crossings, the maximum number of crossing changes needed to perform this would be $c/2$. Which of these knots can we turn into the unknot by performing just one crossing change? As it turns out, there are no easy answers to this seemingly simple question. Initial topological attempts to answer this question might include the perusal of the knot complement which would consequently lead us to the fundamental group and first homology of the knot complement, both of which are redundant to our cause, albeit for contrasting reasons. The Wirtinger's Presentation for knot complements gives us a presentation for the fundamental group of the knot complement (or the knot group). This presentation also makes it clear that the abelianization of the knot group would be infinite cyclic. While the Wirtinger's presentation does provide a presentation for the knot group, it is in general hard to tell if two presentations represent the same group. Thus, the former would almost always be too complicated and the latter just oversimplifies things.

We briefly state the Wirtinger's presentation here. Let D be a diagram for a knot K, let the arcs in D be labelled as α_i. Fig 2.3 shows a crossing in the knot. The Wirtinger's Presentation says that the knot group is generated by loops a_i, where each a_i is a loop from a point P taken perpendicular to the plane, can be viewed as the eye of the reader. The loop a_i goes under the arc α_i at the crossing, as shown in Fig 2.3. The relation between these loops is given by $a_j a_i a_i^{-1} = a_{i+1}$. Thus $\pi_1(S^3 - K) = (a_1, a_2 \ldots, a_n; r_1, r_2, \ldots, r_n)$, where r_i is the relation at each crossing. It is easy to see that each relation becomes $a_i = a_{i+1}$ if the knot group is abelianzed.

Thus, the homology and fundamental groups are not very useful apparently in helping find the 'unknotting number' of a knot. At this point, let us formally define the unknotting number.

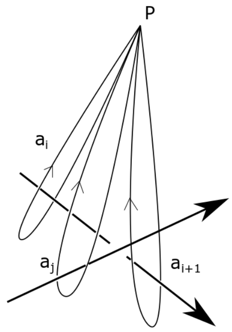


FIGURE 1.1: A crossing in D

Definition 1.1.2 (Unknotting number)**.** Given a knot K, the unknotting number of K is the minimum number of crossing changes needed to turn K into the unknot; this minimum is taken over all knot diagrams.

What we could instead do, is take a look at a covering space of the $S^3 - K$. The classification of covering spaces tells us that for each subgroup B of $\pi_1(S^3 - K)$, there exists a covering space E (with a covering map p), such that $p_*(\pi_1(E)) = B$. If we were to take the commutator subgroup of $\pi_1(S^3 - K)$, then the group of deck transformations would be the abelianization of $\pi_1(S^3 - K)$, i.e., $\mathbb{Z}$. Also, since the commutator group is unique, this infinite cyclic cover, too, is unique. We shall look at two methods of construction of the cyclic cover. The first approach is a standard method of creating a cyclic cover for any space whereas in the second approach we will proceed to create multiple candidates for the cyclic cover by removing the solid tori from the knot complement (and glue them back, albeit, in a different manner). The uniqueness of the infinite cyclic cover the ensures, both of these are the same object (upto homeomorphism).

1.2 The First Approach

This section is primarily taken from [Rob15] and [LL97].

Definition 1.2.1 (Seifert Surface)**.** A Seifert Surface F of a link L is a surface with boundary in S^3 such that $\partial F = L$

We will require a Seifert surface to construct a cover for the knot exterior. It is clear that a Seifert surface for a knot is not unique as we can keep adding handles to the surface (which do not alter the boundary of the surface) and get a different Seifert surface.

Example 1.2.2. The simplest example of a Seifert surface is a disk bounding the unknot.

Example 1.2.3. The Seifert surface of the trefoil is shown in Figure 1.2.3.

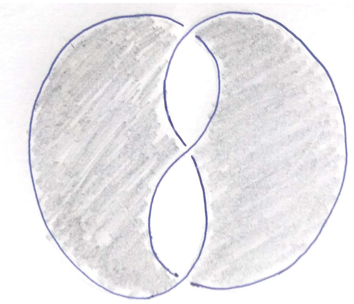


FIGURE 1.2: The Seifert surface of a trefoil

The following construction of the cyclic cover of a knot has been outlined in [LL97] and was originally presented in [Ale28]. Let L be a link in S^3. Consider a thickening of the link, say N. N is then a union of disjoint solid tori. If F is a Seifert surface of L, then N intersects F. Consider a regular neighbourhood $F \times [-1, 1]$ of F, we can identify $F \times \{0\}$ with F . We now look at the space $X = S^3 - N$, i.e., the complement of thickened link. $F \cap X$ can also be identified as a copy of F (since this is effectively just shrinking F slightly). Similarly we can identify $F \times [-1, 1] \cap X$ as a copy of $F \times [-1, 1]$ (from here on referred to as F^- and F^+ respectively).

Cut X along F, call this new space Y. Cutting along F is equivalent to removing a regular neighbourhood $F \times (-1, 1)$ around F. Hence, $Y/\sim = X$ where $\sim$ denotes the identification of $F \times \{-1\}$ with $F \times \{+1\}$. Take countable copies of Y indexed by $\mathbb{Z}$ and call them Y_i. Each Y_i then has two copies of F, namely, F_i^- and F_i^+. There is a natural homeomorphism between F, F^+, F^- for all i, call this ϕ. Similarly, let h_i be the homeomorphism between Y and Y_i. We construct the space $X_\infty$by taking a disjoint union of the Y_i and gluing F_i^- to F_{i+1}^+ by means of the homeomorphism $h_{i+1}\phi h_i^{-1}$. We will briefly state a few results of covering space theory before proceeding. We assume $p : \widetilde{X} \to X$ is a covering map in the following theorems. All of these can be found in [HPoM02] and [Mun18].

Theorem 1.2.4. (**Existence of lifts**) If $p : \widetilde{X} \to X$ if a covering map and $p_* : \pi_1\left(\widetilde{X}\right) \to \pi_1(X)$ is the induced map at the level of fundamental groups and if Y is a path-connected, semi-locally connected space such that there exists a map $f : Y \to X$, such that $f_*(Y) \subseteq p_*(\pi_1\left(\widetilde{X}\right))$, then there exists a lift $\widetilde{f} : Y \to \widetilde{X}$.

Theorem 1.2.5. (**Unique Lifitng property of covering spaces**) If we have a covering map $p : \widetilde{X} \to X$ and there exist two lifts $\widetilde{f_1}, \widetilde{f_2}$ of a map $f : Y \to X$ such that they agree at a point p, then they agree everywhere, that is, $\widetilde{f_1} = \widetilde{f}_2$.

Definition 1.2.6. Given a covering map $p : \widetilde{X} \to X$, a deck transformation or a covering transformation is a homeomorphism f of $\widetilde{X}$ that preserves the covering map, i.e., $p \circ f = p$.

The set of deck transformations is a group under composition of function. We denote this group as $G(D)$. Since, deck transformations can be seen as lifts from $\widetilde{X} \to \widetilde{X}$, we immediately deduce from Theorem 1.2.5 that two deck transformations that agree at a point must agree everywhere.

Definition 1.2.7. Regular covering $\widetilde{X}$ is said to be a regular cover for X if for any two points a and b in a given fibre of a point $x \in X$ under the covering map, there exists a (unique) deck transformation f such that $f(a) = b$.

Theorem 1.2.8. The group of deck transformations of $\widetilde{X}$ is isomorphic to ${}^{N(H)}/_{H}$, where $H = p_*(\pi_1\left(\widetilde{X}\right))$ and $N(H)$ is the normaliser of H in $\pi_1(X)$

There's a natural self-homeomorphism t of X_∞ given by $t|_{Y_i} = h_{i+1}h_i^{-1}$, that preserves the commutative diagram 1.2. What this homeomorphism does is it shifts a point in Y_i to the corresponding point in Y_{i+1}, i.e., shifts the point by 'one unit right'.

$$\begin{tikzcd}
X_\infty \arrow[r, "t"] \arrow[d, "p"'] & X_\infty \arrow[dl, "p"] \\
X &
\end{tikzcd}$$

Definition 1.2.6 tells us that t is a deck transformation. Under the projection map, the fibre of a point $x \in X$ is the corresponding element in each copy of Y_i. If there are two points $x_i \in Y_i, x_j \in Y_j$, then without loss of generality we can assume $i \leq j$, then $Y_j = Y_{i+k}($ and $x_j = x_{i+k})$ for some integer k. Then, $t^k|_{Y_i}(x_i) = h_{i+k}h_i^{-1}(x_i) = x_{i+k}$. This cover is ,thus ,a regular cover. Clearly, the group $< t >$ is infinite-cyclic, which means that the group of deck transformations of X_∞ is infinite-cyclic. We can further deduce the structure of the group $H = p_*(\pi_1(X_\infty))$ using Theorem 1.2.8. By Theorem 1.2.8, $\mathbb{Z} \cong {N(H)}/{H}$. We have seen through the Wirtinger's presentation that the Abelianization of the knot group is infinite-cyclic. Since, the commutator subgroup of a group is unique, this forces $H = [\pi_1()X, \pi_1(X)]$ and thus $N(H) = \pi_1(X)$. We thus, have our infinite cyclic cover X_∞. At this point we will take a detour and examine this more closely. We will see how this gives rise to an useful knot invariant, the Alexander polynomial, which was the first knot polynomial to be discovered. This homeomorphism induces a group homomorphism t_*, of $H_1(X_\infty, \mathbb{Z})$. $H_1(Y, \mathbb{Z})$ is finitely generated, this will be proved shortly, but the $H_1(X_\infty, \mathbb{Z})$ is not finitely generated as a group as it contains countably many of these Y_i. We can, however, use this induced action of t, to give a module structure which would make it a finitely generated module. If $[e_i]$ denoted the generators of Y_0, then any element of $H_1(X_\infty, \mathbb{Z})$ can be written as finite sum $\sum_{i,j} a_{ij}t_*^i[e_j]$, here $\sum_{i,j} a_{ij}t_*^i$ is an element of the group $\mathbb{Z}[t, t^{-1}]$. Thus there is a well defined $\mathbb{Z}[t, t^{-1}]$-module structure on $H_1(X_\infty, \mathbb{Z})$. Before proceeding further, it would be wise to describe the first homology of compact connected surfaces.

1.2.1 Classification of compact surfaces

We briefly classify compact surfaces with boundary. One can refer to [Rob15] and [Mun00] for more details about the classification of surfaces.

Definition 1.2.9. For a compact surface F and any triangulation of F, the quantity χ_F = (number of vertices - number of edges + number of faces) is constant and is called the Euler characteristic of F.

Theorem 1.2.10. Given a compact oriented surface F with boundary, the surface is determined up to homeomorphism by it's Euler characteristic and number of boundary components and the Euler characteristic is given by $\chi_F = 2 - 2g - n$, where g denotes the genus or the number of handles attached to the surface and n denotes the number of boundary components.

We take a look at a few explicit examples. Figure 1.3 shows 2-discs being glued

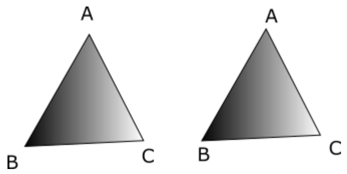


FIGURE 1.3: Two 2-discs being identified along the edges to give rise to a sphere

together along the edges to form a sphere. Using this triangulization, post gluing, there are 2 faces, 3 vertices and 3 edges. Thus the Euler characteristic of the sphere is 2. Since the Euler characteristic is calculated using the formula $2 - 2g - n$, 2 is the highest Euler characteristic of any compact oriented surface. Thus the sphere, up to homeomorphism, is the only compact oriented surface that has Euler characteristic 2. If we remove a disc from the sphere, then the resultant surface has one boundary component- the boundary of the disc removed. This surface is nothing but the 2-disc, which is homeomorphic to one of the triangles shown in Figure 1.3. As we can see there are 3 vertices, 3 edges and one face. Therefore, it's Euler characteristic is 1. Observe that the Euler characteristic drops by 1 when we remove a disc (or introduce a boundary component). If we remove 2-discs from

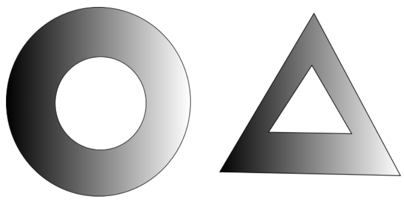


FIGURE 1.4: The annulus

the sphere, we get the annulus, which is shown in Figure 1.4, any triangulation of the 2nd diagram would show that the Euler characteristic of the annulus is 0. One such triangulization is shown in Figure 1.5. Here, as we can see there are 6 vertices, 6 faces and 12 edges, showing that the Euler characteristic of the annulus is 0.

Theorem 1.2.11. If A, and B are two compact oriented surfaces, then $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$.

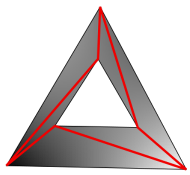


FIGURE 1.5: One triangulization of the annulus

Proof. Consider a triangulization of $A \cap B$ and extend this to a triangulization on $A \cup B$. We denote the number of vertices, edges and faces of the spaces as $v(*), e(*), f(*)$ respectively. Then $v(A \cup B) = v(A) + v(B) - v(A \cap B)$, since we will end up counting the number of vertices of $A \cap B$ twice. Similarly, $e(A \cup B) = e(A) + e(B) - e(A \cap B)$ and $f(A \cup B) = f(A) + f(B) - f(A \cap B)$ Combining these three equations we get our desired proof. $\square$

Theorem 1.2.11 is a very useful tool in computing the Euler characteristic of surfaces as it allows us to break down surfaces into simpler components. To demonstrate this let us look at the case when we add a handle to a sphere. Figure

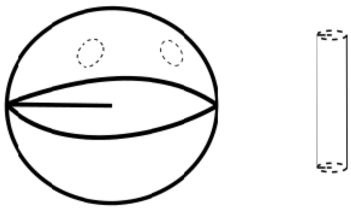


FIGURE 1.6: Sphere with one handle

1.6 displays this space. The sphere with one handle is attached by removing two discs from the sphere and attaching a cylinder as shown in the figure. We use Theorem 1.2.11 to calculate it's Euler characteristic. Take A to be the sphere with the two open discs removed and B to be the cylinder. We have seen previously that each disc removed from the sphere drops it's Euler characteristic by 1 and that the Euler characteristic of an annulus (which is homeomorphic to a cylinder) is 0. $A \cap B$ is just the disjoint union of two circles, whose Euler characteristic is zero(as they can be triangulized as a 1-dimensional triangle with three vertices, three edges and zero faces). Thus $\chi(A) + \chi(B) - \chi(A \cap B) = 0$. What we observe here is that adding a genus to the sphere, drops the Euler characteristic by 2. The kind of surfaces we will be interested in this chapter are Seifert surfaces, the boundary of which is a knot or a link. This means that there must exist at least one boundary component. Since a surface is determined up to homeomorphism by the number of boundary components and the Euler characteristic, any two surfaces that we construct which have the same Euler characteristic and number of boundary components are bound to be homeomorphic. Figures 1.7 and 1.8 are examples of surfaces with one boundary component (a disc) which are attached with handles. Each handles drops the Euler characteristic by 2. Figure 1.9

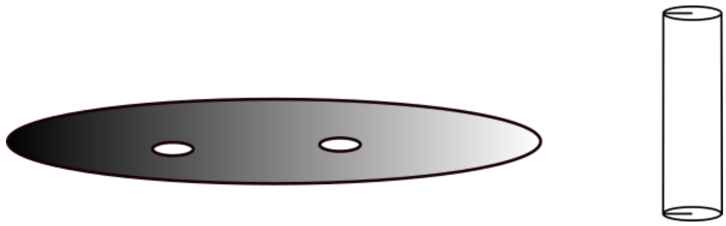


FIGURE 1.7: A disc with a handle attached

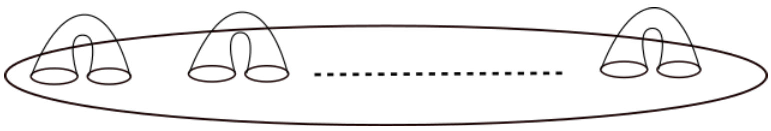


FIGURE 1.8: A disc with n handles attached

shows another description of disc with a handle attached. Tracing the outline of the figure on the left would show that there is only one boundary component. On the right we have a splitting of this surface into three subspaces. We will use Theorem 1.2.11 on these subspaces. Each of these subspaces are homeomorphic to a disc and the pairwise intersection of these spaces are two pair of disjoint segments (and an empty set) as shown in the diagram. Segments have Euler characteristic one as they can be represented by an edge between two vertices. Thus the Euler characteristic is $3 - 4 = -1$, which coincides with a disc with a handle attached. The final step is constructing an arbitrary surface (with at least one boundary

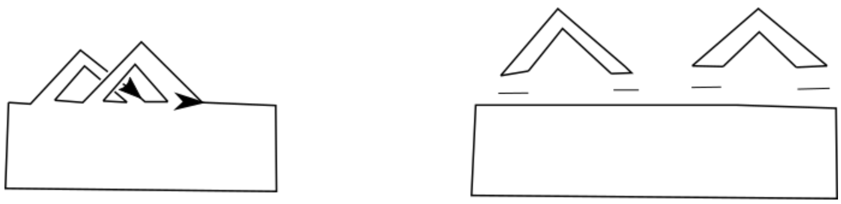


FIGURE 1.9: An alternative construction of a disc with a handle attached

component). To add more boundary components to the surface in 1.9, we can just remove discs and to increase it's genus, as we have seen, it suffices to add a 'band' on the surface. This is displayed in Figure 1.10.

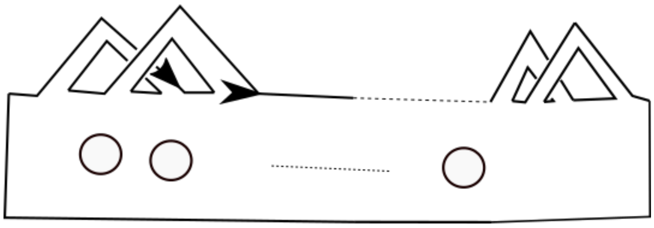


FIGURE 1.10: A surface with g genus and n boundary components

1.2.2 First homology of a surface

This section is taken from [LL97].

We would to like to define a bilinear form on the first homology group of X (with integral coefficients) and relate it with linking number(which we shall define shortly). As it happens there's a theorem that precisely does this job. We describe the theorem and delineate it's proof.

First we describe the setup, we consider an oriented surface F embedded in S^3. Let F be a 2-manifold with boundary with n boundary components and genus g. Without loss in generality, our surface looks as displayed in Fig 1.10.

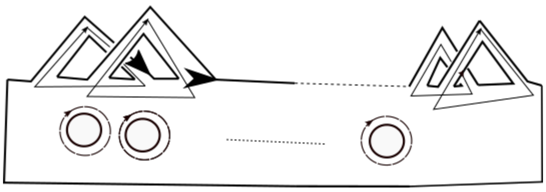


FIGURE 1.11: A surface with genus g and n boundary components

Theorem 1.2.12. Let F be as described above. Then, the first homology groups of F and $S^3 - F$ are isomorphic. Furthermore there is a unique bilinear from β

$$\beta : H_1 (F, \mathbb{Z}) \times H_1 \left(S^3 - F, \mathbb{Z}\right) \longrightarrow \mathbb{Z}$$

such that $\beta (a, b) = lk (a, b)$, where $lk (a, b)$ denotes the linking number of a with b. Here a and b are any two closed oriented simple curves.

We provide (one of) the definition of linking number in brief here.

Definition 1.2.13 (Linking Number)**.** Let a and b be two disjoint closed simple curves in S^3. As, $H_1 (S^3 - b, \mathbb{Z}) \cong \mathbb{Z}$. In $H_1 (S^3 - b, \mathbb{Z})$, the class $[a] = n\gamma$ where γ is a generator of $H_1 (S^3 - b, \mathbb{Z})$, for some integer n. The linking number $lk (a, b)$ is defined to be n.

Proof. Consider a regular neighbourhood V of F in S^3. Since V deformation retracts to F, their homologies are the same. Now consider the closure of the complement of V in S^3, call it V'. Applying the Mayer-Vietoris sequence to $V$and V', we get the following long exact sequence:

$$\cdots \longrightarrow H_2\left(S^3, \mathbb{Z}\right) \longrightarrow H_1\left(\partial V, \mathbb{Z}\right) \longrightarrow H_1\left(V, \mathbb{Z}\right) \bigoplus H_1\left(V', \mathbb{Z}\right) \longrightarrow H_1\left(S^3, \mathbb{Z}\right) \longrightarrow \ldots$$

Since, $H_m\left(S^n, \mathbb{Z}\right)$ is 0 for $m \neq n$, the first and last homology groups in the sequence are 0 and consequently $H_1\left(\partial V, \mathbb{Z}\right) \cong H_1\left(V, \mathbb{Z}\right) \bigoplus H_1\left(V', \mathbb{Z}\right)$. We now closely inspect $H_1\left(\partial V, \mathbb{Z}\right)$, V is homeomorphic to a 3-sphere with $(2g+n-1)$ handles. Therefore, ∂V is the boundary of such a sphere. Its first homology is isomorphic to $\bigoplus\limits_{2g+n-1} \mathbb{Z} \bigoplus\limits_{2g+n-1} \mathbb{Z}$. Let $\{f_i : 1 \leq i \leq 2g+n-1\}$ and $\{e_j : 1 \leq j \leq 2g+n-1\}$ be generators of $H_1\left(\partial V, \mathbb{Z}\right)$. These generators can be chosen in such a way that each f_i goes around the meridian of the i^{th} handle of the sphere (the handles of the sphere labelled in an appropriate manner), thus being the boundary of a disc in V. The e_is are consequently chosen so that each e_i goes around the i^{th} handle and each e_i intersects each f_i exactly once. Now, as these f_is form the boundary of a disc in V, the inclusion map from ∂V to V, takes it to the trivial element, as f_i would now be contractible in V. Thus there would have to be an isomorphism between $H_1\left(V, \mathbb{Z}\right)$ and the subgroup generated by $\{f_i\}_{i=1}^n$. Thus $H_1\left(V', \mathbb{Z}\right)$ would be isomorphic to the subgroup generated by the $\{e_i\}_{i=1}^n$, i.e., $\bigoplus\limits_{2g+n-1} \mathbb{Z}$. Thus giving us our intended result that $H_1\left(V, \mathbb{Z}\right) \cong H_1\left(V', \mathbb{Z}\right)$.

We now define the bilinear form β. First we define β on the generators, $\beta\left(e_i, f_j\right) = \delta_{ij}$, where δ_{ij} is the Kronecker-Delta function. This can be extended linearly. Observe that in S^3, e_i only links with f_i. For any two curves a and b in F and $S^3 - F$ respectively, $[a]$ and $[b]$ can be expressed as $\sum\limits_{i=1}^{2g+n-1} c_i\left[e_i\right]$ and $\sum\limits_{i=1}^{2g+n-1} d_i\left[f_i\right]$. Hence, $\beta\left(a, b\right) = \sum c_i d_i$. As, $a \in F$ and $b \in S^3 - F$, $a \in S^3 - \{b\}$. Thus, $lk(a, b) = \beta\left(a, b\right)$.

□

To verify that $\{f_i\}$ indeed generates $H_1\left(F, \mathbb{Z}\right)$, we can take an annulus, in which one of the band is contained, as shown in Fig 1.12 and call it U. Let V be the closure of $F - U$ in F, then $F \cap U$ is homeomorphic to a closed interval whose homologies are zero. The Mayer-Vietoris sequence then gives us an isomorphism between $H_1(U,\mathbb{Z}) \bigoplus H_1(V,\mathbb{Z})$ and $H_1(F,\mathbb{Z})$.

Since U is an annulus which deformation retracts to a circle, it's first homology is a copy of $\mathbb{Z}$. Repeatedly applying this argument gives us an isomorphism between $\bigoplus_{2g+n-1}\mathbb{Z} \bigoplus H_1(W,\mathbb{Z})$ and $H_1(F,\mathbb{Z})$, where W is a surface as shown in Fig 1.11 with no boundaries, i.e., only the bands on the top would be present.

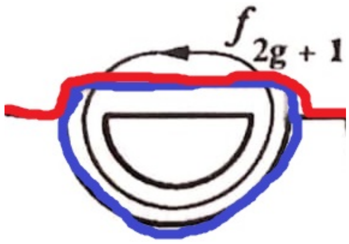


FIGURE 1.12: A description of U: the blue line being the boundary of U and the read line being the boundary of the rest of the surface

Next we shift our focus to the bands on top, as described in Fig 1.13. Take U' to be the annulus enclosed by the blue boundary and U'' be the annulus enclosed by the red boundary, take $V' = W \cap U^c \cup (U' \cap U'')$. Then $U' \cup V' = W$. Applying the Mayer-Vietoris sequence to this decomposition we get: $\cdots \longrightarrow H_2(U' \cup V',\mathbb{Z}) \longrightarrow H_1(U' \cap V',\mathbb{Z}) \longrightarrow H_1(U',\mathbb{Z}) \bigoplus H_1(V',\mathbb{Z}) \longrightarrow H_1(U' \cup V',\mathbb{Z}) \longrightarrow \ldots$ Since $U' \cap V'$ is homeomorphic to 2-disc, $H_1(U' \cap V',\mathbb{Z})$ is trivial, thus we get an isomorphism between $H_1(U',\mathbb{Z}) \bigoplus H_1(V',\mathbb{Z})$ and $H_1(W,\mathbb{Z})$. Here, $H_1(U',\mathbb{Z})$, owing to the fact that it is homeomorphic to an annulus, is a copy of $\mathbb{Z}$. Shifting our focus to W', we notice that it has two boundaries, one of them inside the annulus enclosed by the red boundary, by previous line of argument we find that this annulus contributes one generator to $H_1(F,\mathbb{Z})$. Repeatedly applying this argument we will end up with $H_1(F,\mathbb{Z}) = \bigoplus_{2g+n-1}\mathbb{Z} \bigoplus H_1(T,\mathbb{Z})$ where T is homeomorphic to a 2-disc, hence, $H_1(F,\mathbb{Z}) = \bigoplus_{2g+n-1}\mathbb{Z}$. This verifies our initial claim.

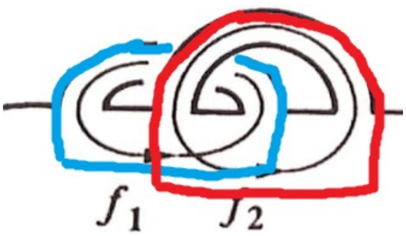


FIGURE 1.13: A description of U' and V'

We are now ready to resume our discussion on defining a bilinear form F. Recall the setup as described in Section Let D be a 2-disk with boundary with n marked points in the interior of the disk. We denote this set of n points as Q. Let i_- and i_+ denote the inclusion maps from $F \times \{0\}$ to $F \times \{-1\}$ and $F \times \{+1\}$ respectively. Then, define

$$\alpha : H_1(F, \mathbb{Z}) \times H_1(F, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$\alpha([a], [b]) = \beta(i_{*-}[a], [b]) = lk\left(f_i^-, f_j\right)$$

.

On picking a basis $\{f_i\}$ for the Seifert Surface, we are able to define the Seifert Matrix A. The entries A_{ij} of A are given by $\alpha([f_i], [f_j])$. Using the description of the Seifert Matrix we can find an expression for the f_i^-s and f_i^+s in terms of the corresponding $[e_j]$s :

$$\begin{aligned} \left[f_i^-\right] &= \sum_j a_{ij}[e_j] \\ \implies lk(f_i^-, f_j) &= a_{ij} = A_{ij} \\ \implies \left[f_i^-\right] &= \sum_j A_{ij}[e_j] \\ \text{Similarly, } \left[f_j^+\right] &= \sum_i A_{ij}[e_i] \end{aligned}$$

1.2.3 Presentation Matrices

This is the final pit stop on our route to getting to the Alexander's polynomial. Let R be a ring and M be a module over R. If E and F are free modules of finite rank over R, then a finite presentation for M is given by the following exact sequence:

$$R^m \xrightarrow{\alpha} R^n \xrightarrow{\beta} M \to 0$$

We know that every module is the quotient of a free module,hence there is a surjective map.

$$R^n \xrightarrow{\beta} M \to 0$$

In case we are fortunate enough to a have a finitely generated kernel, we have the following map:

$$R^m \xrightarrow{\alpha} \mathrm{Ker}\beta$$

Thus giving us our initial exact sequence:

$$R^m \xrightarrow{\alpha} R^n \xrightarrow{\beta} M \to 0$$

The basis elements of E can thus be thought of as the generators for M, while those of F can be thought of as relations between those generators.

A presentation matrix of a module, as it happens, is far from unique. In fact, any elementary operation on the matrix A of the linear map α, would give us another presentation. We make the following claim, any two finite presentation matrices A and A_1 for the module M differ by the following moves:

1. Switching rows or columns
2. Replacing A by $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$
3. Replacing A by $\begin{pmatrix} A & 0 \end{pmatrix}$
4. Adding a scalar multiple of a row or column to another row or column

To prove this consider two presentations for M:

$$\begin{tikzcd}
R^m \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & R^n \arrow[r, "\phi"] \arrow[d, "\beta"] & M \arrow[r] \arrow[d, "Id"] & 0 \\
R^{m_1} \arrow[r, "\alpha_1"] & R^{n_1} \arrow[r, "\phi_1"] & M \arrow[r] & 0
\end{tikzcd}$$

Take a basis $\{e_1, e_2, \ldots, e_n\}$ for R^n, then $\{\phi(e_i)\}$ generates M. As ϕ_1 is surjective, we can pick elements $e_i^{'} \in R^{n_1}$ such that $\phi_1\left(e_i^{'}\right) = \phi(e_i)$ and extend this map linearly, call this map β. Note that $\phi = \phi_1\beta \implies \text{Ker}\phi = \text{Ker}(\phi_1\beta) \implies \text{Ker}\beta \subseteq \text{Ker}\phi_1$ Similarly, given a basis $\{f_i\}$ of R^m, $\beta\alpha \subseteq \text{Ker}\phi_1$ as $\text{Ker}\phi = \text{Im }\alpha$. Also we have $\text{Im }\alpha_1 = \text{Ker }\phi_1$. Therefore arguing as above, we get a map γ such that $\alpha_1\gamma = \beta\alpha$. If the matrix of these modules maps $\alpha, \alpha_1, \gamma\beta$ were A, A_1, C, B respectively, then $BA = A_1C$. By a completely symmetric argument we would be able to find maps β_1 and γ_1 from R^{n_1} to R^n and from R^{m_1} to R^m (and their matrix B_1 and C_1) respectively. In this case $B_1A_1 = AC_1$

Under our 4 equivalence moves, on application of the second move, we have:

$$A \sim \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

Performing the fourth move (adding scalar rows), we can get:

$$A \sim \begin{pmatrix} A & B_1 \\ 0 & I \end{pmatrix}$$

The matrix on the right is now a $(n + n_1) \times (m + n_1)$. We then add $m + 1$ columns of zeros and perform the fourth operation (addition of scalar multiple of the columns of $\begin{pmatrix} B_1 \\ I \end{pmatrix}$ to the columns of zeros) to get:

$$A \sim \begin{pmatrix} A & B_1 & B_1 A_1 \\ 0 & I & A_1 \end{pmatrix}$$

Since $AC_1 = B_1 A_1$, we can use the fourth operation (column operation) to get:

$$A \sim \begin{pmatrix} A & B_1 & 0 \\ 0 & I & A_1 \end{pmatrix}$$

Again on addition of zero columns and addition of scalar multiple of the columns of $\begin{pmatrix} B_1 \\ I \end{pmatrix}$ to the columns of zeros, we get:

$$A \sim \begin{pmatrix} A & B_1 & 0 & B_1 B \\ 0 & I & A_1 & B \end{pmatrix}$$

Note $\phi\beta_1\beta = \phi\beta = \phi$, Since Ker $\phi =$ Im α, $\beta_1\beta - Id_E \subseteq$ Im α. As E is free, there is map $\partial : E \to F$ such that $\beta_1\beta - Id_E = \alpha\partial$, thus if D is the matrix representing ∂, we have $AD = B_1 B - I$. Now on applying property 4, we have:

$$A \sim \begin{pmatrix} A & B_1 & 0 & B_1 B \\ 0 & I & A_1 & B \end{pmatrix} \sim \begin{pmatrix} A & B_1 & 0 & I \\ 0 & I & A_1 & B \end{pmatrix}$$

A simple permutation of the rows and columns gives us: $A \sim A_1$

Definition 1.2.14. If M is a module over a ring R , and A is an $m \times n$ presentation matrix for M, then the ideals generated by the $(m-r+1) \times (n-r+1)$ minors are called the **r-th Elementary Ideals**

We can observe that the four equivalent operations do not affect the minors and hence, elementary ideals are invariant under different presentations of the same module.

1.2.4 The Alexander's Polynomial

We now have all the tools to obtain the Alexander's Polynomial. Our line of attack will be applying the Mayer-Vietoris sequence to a decomposition of $X_\infty$and proving that we indeed get a presentation matrix for $H_1(X_\infty, \mathbb{Z})$.

Consider the decomposition X_∞=$Y' \cup Y''$, where $Y' = \cup Y_{2i}$ and $Y'' = \cup Y_{2i+1}$. This gives rise to a short exact sequence of chain groups:

$$0 \to C_n(Y' \cap Y'') \xrightarrow{\alpha} C_n(Y') \bigoplus C_n(Y'') \xrightarrow{\beta} C_n(X_\infty) \to 0$$

To get a Long-Exact sequence of modules from this short exact sequence of groups, we need to verify that this exact sequence of groups can be made into a exact sequence of modules. First, we note that the action of t interchanges Y' and Y'', thus Y' and Y'' aren't $\mathbb{Z}[t, t^{-1}]$modules. However, each of the terms in the above exact sequence is closed under the action of t, thus we need to only check the maps α and β are $\mathbb{Z}[t, t^{-1}]$module maps. This is achieved if we define $\alpha(x) = (-x, x)$ and $\beta(a, b) = a + b$, then $\beta\alpha = 0$. This enables us to get the following long exact sequence of modules:

$$\cdots \longrightarrow H_1(Y' \cap Y'', \mathbb{Z}) \xrightarrow{\alpha_*} H_1(Y', \mathbb{Z}) \bigoplus H_1(Y'', \mathbb{Z}) \xrightarrow{\beta_*} H_1(X_\infty, \mathbb{Z}) \xrightarrow{\partial}$$

$$\xrightarrow{\partial} H_0(Y' \cap Y'', \mathbb{Z}) \xrightarrow{\alpha_*} H_0(Y', \mathbb{Z}) \bigoplus H_0(Y'', \mathbb{Z}) \to \dots$$

Consider $Y' \cap Y''$, it is the union of countable disjoint copies of F, we can, thus, identify $H_0(Y' \cap Y'', \mathbb{Z})$ with $\mathbb{Z}[t, t^{-1}] \bigotimes_{\mathbb{Z}} H_0(F, \mathbb{Z})$. Since F is connected, $H_1(F, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$. So, $1 \bigotimes 1$ is a generator of $\bigotimes_{\mathbb{Z}} H_0(F, \mathbb{Z})$. Similarly $H_0(Y', \mathbb{Z}) \bigoplus H_0(Y'', \mathbb{Z})$ can be written as $\mathbb{Z}[t, t^{-1}] \bigotimes_{\mathbb{Z}} H_0(Y, \mathbb{Z})$ as $H_0(Y', \mathbb{Z})$ and $H_0(Y'', \mathbb{Z})$ are both countable direct sum of $\mathbb{Z}$. Under this convention $\alpha * (1 \bigotimes 1)$ maps to $(-1 \bigotimes 1) + (t \bigotimes 1)$. This is because the generator of $H_0(F, \mathbb{Z})$ is included in Y_0 and Y_1 under the map α and the sign on the former is negated. This tells us that α_* is injective, as any element $\sum (a_n t^n \bigotimes 1)$ would be mapped by α to $\sum a_n ((-t^n \bigotimes 1) + (t^{n+1} \bigotimes 1))$. Since α_* is injective and the sequence is exact, β_* is surjective, implying that we have the following exact sequence:

$$\cdots \longrightarrow H_1(Y' \cap Y'', \mathbb{Z}) \xrightarrow{\alpha_*} H_1(Y', \mathbb{Z}) \bigoplus H_1(Y'', \mathbb{Z}) \xrightarrow{\beta_*} H_1(X_\infty, \mathbb{Z}) \xrightarrow{\partial} 0$$

This is a $\mathbb{Z}[t, t^{-1}]$module presentation for $H_1(X_\infty, \mathbb{Z})$. All we now need to do, is compute the matrix of the presentation map α_* between $H_1(Y' \cap Y'', \mathbb{Z})$ and $H_1(Y', \mathbb{Z}) \bigoplus H_1(Y'', \mathbb{Z})$. As done previously, we can rewrite the module $H_1(Y' \cap Y'', \mathbb{Z})$ as $\mathbb{Z}[t, t^{-1}] \bigotimes_{\mathbb{Z}} H_1(F, \mathbb{Z})$. If $x \in H_1(Y_0 \cap Y_1, \mathbb{Z})$, then x can be denoted as $1 \bigotimes x$, this can be translated appropriately by the action of t. Let $\{f_i\}$ be basis for F. Arguing in the same way $H_1(Y', \mathbb{Z}) \bigoplus H_1(Y'', \mathbb{Z})$ can be written as $\mathbb{Z}[t, t^{-1}] \bigotimes_{\mathbb{Z}} H_1(Y, \mathbb{Z})$. Recall that Y is the space X with a regular neighbourhood (a thickening) of F removed, hence a basis for $H_1(Y, \mathbb{Z})$ is given by the corresponding $\{[e_i]\}$s. Inclusion f_i into Y results in f_i^-

$$\alpha_* \left(1 \bigotimes [f_i]\right) = \sum_j \left(-A_{ij} \left(1 \bigotimes [e_j]\right) + A_{ji} \left(t \bigotimes [e_j]\right)\right)$$

The presentation matrix is thus given by $(A - tA^{tr})$ (upon multiplication by a unit). Here A is the Seifert matrix. We get a bunch of link invariants from the elementary ideals of this matrix, the most significant one being the first elementary ideal.

Definition 1.2.15. Alexander Polynomial A generator of the smallest principal ideal containing the first elementary ideal of $(A - tA^{tr})$ is called the **Alexander Polynomial** of a knot.

As we can see, the Alexander's polynomial is unique only upto the multiplication by a unit. We thus have gotten hold of a very computable invariant of knot.

1.3 The Second Approach

This section is taken from [LL97] and [Rol75]. Since the commutator subgroup is unique, the cyclic cover for a knot exterior is unique. Therefore any method we use to arrive at a cyclic cover, should provide us with the same results (up to homeomorphism). The next method we use to construct a cyclic cover is using the method of surgery. It is easier to see this using an explicit example. We will consider the knot 4_1.

Consider the knot 4_1 in the space S^3 as shown in Figure 1.14. Around a crossing of this knot, we consider a solid torus T and remove the solid torus from this space, i.e., we consider the space $S^3 - T$.

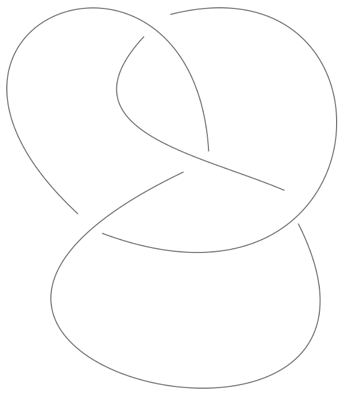


FIGURE 1.14: The Figure eight knot

A crossing of the knot 4_1 lies inside the cavity of the solid torus. It is then possible to find a neighbourhood of the crossing which is homeomorphic to the standard solid cylinder, which is bounded on the sides by the the solid tori. Now performing, the Dehn twist on this solid cylinder, flips the strands. If the standard solid cylinder is denoted as $D^2 \times I$, then, explicitly the Dehn twist is defined as follows:

$$\begin{aligned} f : D^2 \times I &\to D^2 \times I \\ (s, t) &\mapsto \left(se^{i2\pi t}, t\right) \end{aligned}$$

We have identified D^2 with the complex unit disc. Now this homeomorphism is isotopic to the identity since we can now decay it to the identity outside an open neighbourhood containing the cylinder in $S^3 - T$. This process would extend the homeomorphism f to a homeomorphism h of the entire space $S^3 - T$. It is easy to see simply by drawing a diagram of the figure eight knot that if we were to switch a crossing, then the resulting diagram would be that of the unknot, meaning the unknotting number of the figure eight is one. This means, in $S^3 - T$ it should be possible to now isotope the knot into the unknot. The catch here is that the homeomorphism h cannot be extended to all of S^3 because that would imply that the unknot is isotopic to the figure-8 knot (!).

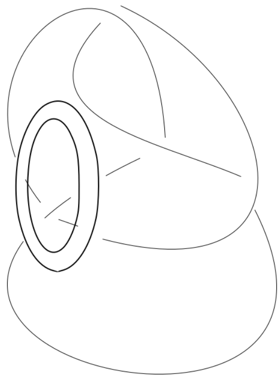


FIGURE 1.15: The Figure eight knot with the torus removed around a crossing

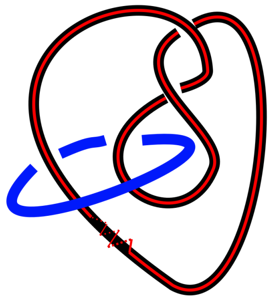


FIGURE 1.16: Upon isotopy

Fig 1.15 and Fig 1.16 describe this process. We obtain the space in Fig 1.16 after an isotopy. Let us denote these composition of homeomorphisms as g, then $g(K)$ is unknotted in this space. This is because of the fact that the unknotting number of the figure eight knot is 1. The homology of this space is generated by the meridian of the torus. To construct the cyclic covering of the figure eight knot exterior, we cut along the Seifert surface of the knot in this space, which is just a disc and glue back the solid tori sending meridian to the curve showed in Fig . We keep track of the meridian as we perform the isotopy. Then we attach countable such copies along the Seifert surface.

We denote the cyclic cover of $S^3 - g(K)$ by X_∞. This space is homeomorphic to $R^1 \times R^2$, since $S^3 - g(K) \cong S^1 \times R^2$. The covering map is simply $p : R \times R^2 \to$

$S^1 \times R^2, p(x,y) = (e^{2\pi x}, y)$. The group of deck transformations, thus, are all $t^k, k \in \mathbb{Z}$ where $t^k : R \times R^2 \times R^2, t^k(x,y) = (x+k, y)$.

To compute the homology of X_∞, we first compute the homology of the $X_\infty - \cup_k t^k T$, where T is the solid torus. This group is generated by the by the $t^k \alpha$. The effect that gluing in the solid tori has on the homology is killing the class of meridional loop which are denoted by $t\alpha - 3\alpha + t^{-1}\alpha$. Expressing $H_1(X_\infty, \mathbb{Z})$ as a module over $\mathbb{Z}[t, t^{-1}]$, we see that α is a generator while $t\alpha - 3\alpha + t^{-1}\alpha$ is a relation and a presentation matrix for $H_1(X_\infty, \mathbb{Z})$ is given by the 1×1 matrix $[t - 3 + t^{-1}]$, which is also the Alexander Polynomial of the figure-eight knot.

Even though this method was performed using the figure-eight, the method used can be generalized to all knots. The idea is to remove solid tori from around the crossings of knot till we get back the unknot, if we use m solid tori then we will get a presentation for $H_1(X_\infty, \mathbb{Z})$ with m generators and m relators. This immediately tells us that if we have a knot which can be unknotted with m crossing changes, then the r^{th} elementary ideals of the module for every $r > m$ is the entire ring $\mathbb{Z}[t, t^{-1}]$. Thus proving the following theorem.

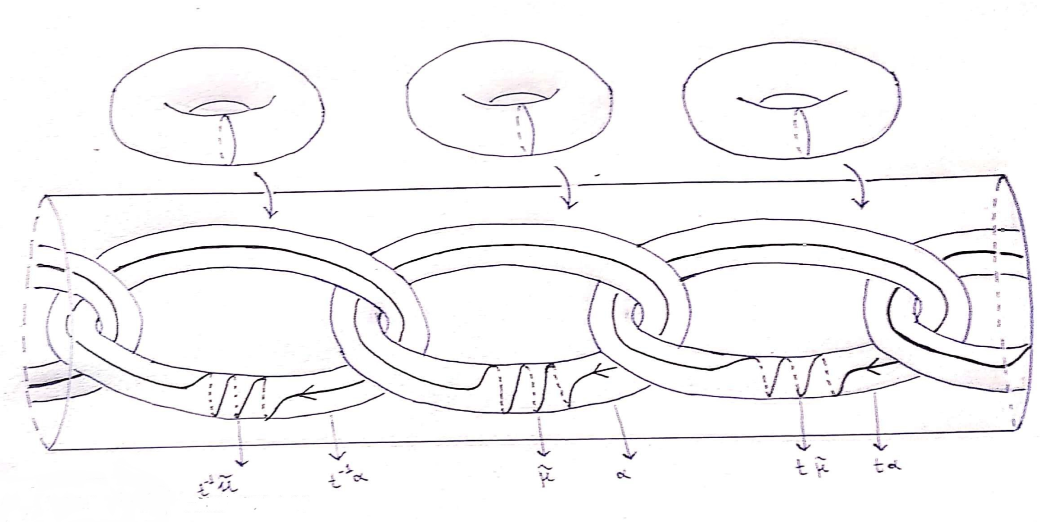


FIGURE 1.17: The Cyclic cover of the figure eight knot

Theorem 1.3.1. If the r^{th} elementary ideal of the Alexander module of a knot isn't $\mathbb{Z}[t, t^{-1}]$, then the unknotting number of the knot, $u(K) \geq r$.

The figure above shows a diagram of the Pretzel knot $(3, 3, -3)$. The diagram also shows a Seifert surface for the knot (the shaded area). f_1 and f_2 are two

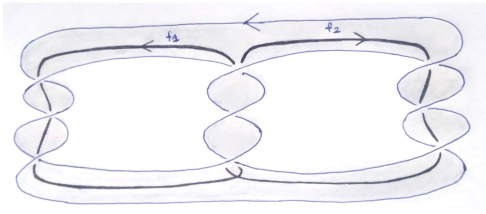


FIGURE 1.18: A diagram of the knot 9_{46} or the pretzel knot $P(3,3,-3)$.

generators of the first homology. The Seifert matrix is given by:

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$$

Then the presentation matrix for the Alexander Module is:

$$tA - A^{\text{tr}} = \begin{pmatrix} 3t-3 & 2t-1 \\ t-2 & 0 \end{pmatrix}$$

The first elementary ideal is generated by $(2t-1)(t-2) = -2t^2 - 5t + 2$, which is also the Alexander Polynomial of the knot. The second elementary ideal is generated by the polynomials $(2t-1)(t-2)$. If the ideal generated by these polynomials were to be the entire ring $\mathbb{Z}[t, t^{-1}]$, then the evaluation map at any integer would map the entire ring of integers. Evaluating at -1, we see that both these polynomials map to -3, hence the image of the ideal under this evaluation map would be $3\mathbb{Z}$ which isn't the entire ring. Hence this ideal is not the entire ring. By Theorem 1.3.1, unknotting number of this knot must be greater than 2. It is not difficult to see that we can undo the knotting by two crossing changes.

We have thus caught hold of a method that can help us in distinguishing which knots have unknotting number 1.

1.4 Slice Knots

This chapter is taken from [LL97], [Tei10] and [Kau87].

1.4.1 Introduction

This section is taken from [Tei10],[LL97] and [Kau87]. We will look at slice knots, which 'bound a disc' in the 4^{th} dimension and thus can be said to be the next best thing to the unknot in some sense. We will look at slice knots and their properties and try to ascertain what they look like. The main result of this section will be proving that all slice knots have Alexander Polynomials which split into factors which are functions of the variable and it's inverse respectively.

Definition 1.4.1. Given a manifold M of dimension m and a submanifold N of M of dimension n, N is said to be *locally flat* if at every point x of N, there exists a neighbourhood U of M such that the pair $(U, U \cap N)$ is homeomorphic to the pair (R^m, R^n).

A knot K in S^3 is an embedding of S_1. The unknot, amongst other reasons, is distinguished in the light that it bounds a 2-disc. This disc is *locally flat*. Every knot is unknotted in S^4 due to the extra dimension available for movement. A natural question now would be which knots bound a locally flat disc in B^4 (or $\mathbb{R}^4$). As it so happens, there's a 2-disc bounding every knot in B^4. We can simply view B^4 as the cone over S^3 and subsequently we would have a cone over any knot K lying in S^3. The problem would occur at the point of coning. Let us denote this point as P. Then any neighbourhood of B^4 around this point P would have a copy of K on its boundary. We seek to avoid these mishaps. Hence, we have the following definition.

Definition 1.4.2. (**Slice Knots**) A knot K lying in S^3 is said to be slice if there is a flat disc D in B^4 such that $\partial D = D \cap S^3 = K$.

This also means that D has a neighbourhood N that is homeomorphic to $D \times I^2$, where I is the unit interval, meeting S^3 in $\partial D \times I^2$. $\partial D \times I^2$ is just a thickening of the knot in S^3.

Intuitively, for a knot to be slice, it means we can arrange a sequence of concentric S^3 spheres such that the knot K moves through these spheres and forms either of the following:

1. A maxima or a minima (non-singularity).

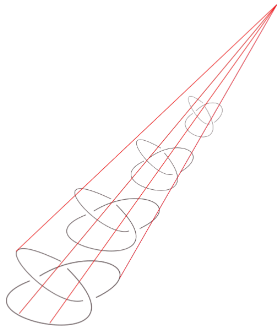


FIGURE 1.19: Cone over the trefoil

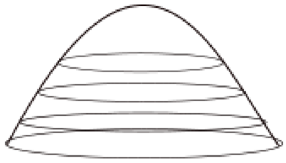


FIGURE 1.20: Maxima

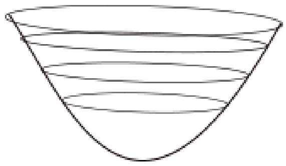


FIGURE 1.21: Minima

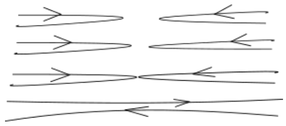


FIGURE 1.22: Singular Level

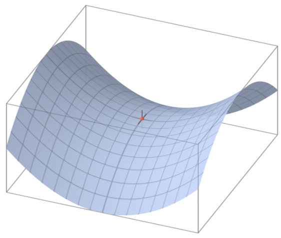


FIGURE 1.23: Saddle point, Source: https://commons.wikimedia.org/wiki/File:Saddle_point.png

2. A saddle point (singularity).

The simplest example of a slice knot is the Stevedore 6_1 knot. To see this, we construct what is called a 'movie'. Each frame shows a copy of the knot in a concentric sphere. This is shown in Figure 1.24. In the remainder of the chapter we will try figuring out what the Alexander Polynomials of slice knots look like.

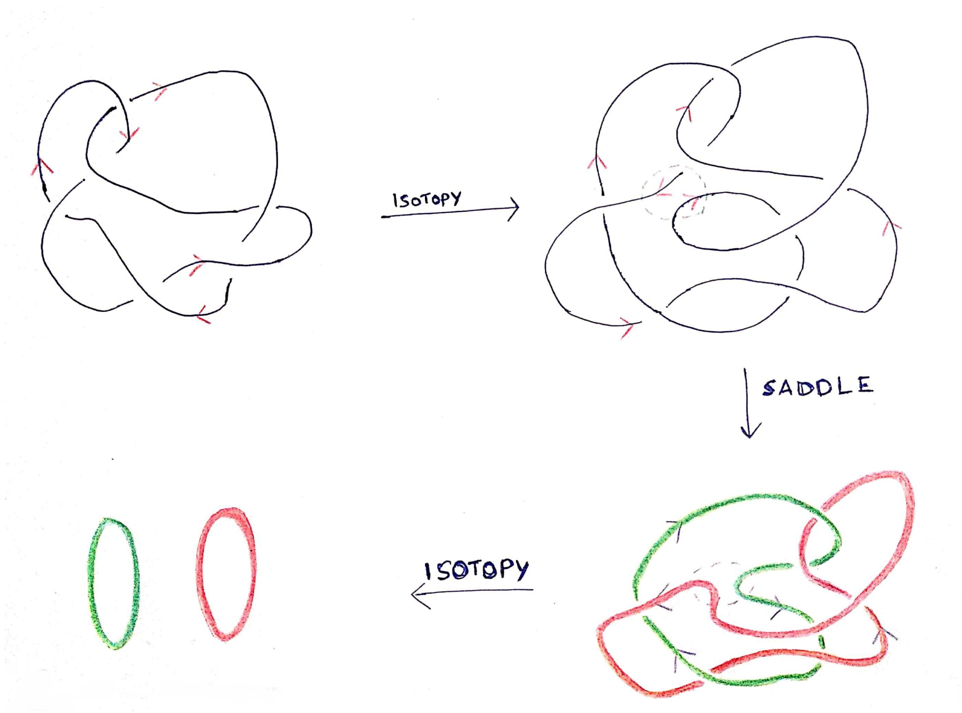


FIGURE 1.24: A Movie of the 6_1 knot being slice, Source: [Tei10]

Lemma 1.4.3. If D is a slicing disc of the slice knot, then the inclusion map $S^3 - K \hookrightarrow B^4 - D$ induces an isomorphism of the first homologies.

Proof. We use the Mayer-Vietoris sequence applied to the subspaces N and the closure of it's complement in B^4. N is a regular neighbourhood of D, i.e., it is homeomorphic to $D \times I^2$. Additionally, N meets S^3 in $K \times I^2$. The intersection of these two spaces is homeomorphic to $D \times \partial I^2$. This gives rise to the following sequence.

$$H_2\left(B^4, \mathbb{Z}\right) \longrightarrow H_1\left(D \times \partial I^2, \mathbb{Z}\right) \longrightarrow H_1\left(N, \mathbb{Z}\right) \bigoplus H_1\left(\overline{B^4 - N}, \mathbb{Z}\right) \longrightarrow H_1\left(B^4, \mathbb{Z}\right)$$

Since, B^n is contractible, we have $H_n\left(B^m, \mathbb{Z}\right) = 0$. This implies that the second map is an isomorphism. $H_1\left(D \times \partial I^2, \mathbb{Z}\right) \cong H_1\left(D, \mathbb{Z}\right) \bigoplus H_1\left(\partial I^2, \mathbb{Z}\right)$. D is included in N, hence, $H_1\left(D, \mathbb{Z}\right) \cong H_1\left(N, \mathbb{Z}\right)$. This would imply that $H_1\left(\partial I^2, \mathbb{Z}\right) \cong H_1\left(\overline{B^4 - N}, \mathbb{Z}\right)$. Since ∂I^2 is simply a meridional loop around the knot, a generator of ∂I^2 would be homologous to a generator of $H_1\left(K, \mathbb{Z}\right)$. Since N is just a thickening of D, it deformation retracts to D, thus proving the statement of the theorem. $\square$

1.4.2 Alexander Polynomial of the slice knots

This section is primarily taken from [LL97]. Our attempt will be to find a general form of the Seifert matrix of a slice knot. Examples of slice knots are the $6_1, 8_8, 8_9, 8_{20}$. The Alexander Polynomial for these knots are:

- $6_1 = -2 + 5x - 2x^2 = -(t-2)(2t-1)$
- $8_8 = 2 - 6x + 9x^2 - 6x^3 + 2x^4 = (2 - 2x + x^2)(1 - 2x + 2x^2)$
- $8_9 = -1 + 3x - 5x^2 + 7x^3 - 5x^4 + 3x^5 - x^6 = -((-1 + x - 2x^2 + x^3)(-1 + 2x - x^2 + x^3))$
- $8_{20} = 1 - 2x + 3x^2 - 2x^3 + x^4 = (1 - x + x^2)^2$

Each of the Alexander polynomials of these knots can be factorised. This provides a hint about how the Alexander Polynomials of these slice knots might look like. We will rely on the following two theorems to arrive at a conclusion.

Theorem 1.4.4. Given a slice knot K, if F is a Seifert surface in S^3 and D is a slicing disc for K, then there exists a 3-manifold M with boundary in B^4, such that $\partial M = K \cup D$.

Theorem 1.4.5. If there are two maps $f_i : B^2 \to B^4$, for $i = 1, 2$, such that on ∂B^2 the maps are homeomorphisms and the images of the f_i are disjoint, the linking number of the boundary of each map is zero.

For any 3-manifold M with boundary we have the following theorem. We will need a few results of homology and cohomology to prove Theorem 1.4.9.

Theorem 1.4.6. (Poincaré Duality) If M is a closed orientable m-manifold, then then,

$$H^k(M, \mathbb{Z}) \cong H_{n-k}(M, \mathbb{Z})$$

for $1 \leq k \leq n$.

Theorem 1.4.7. (Lefschetz Duality) If M is a compact oriented n-manifold with boundary, then

$$H^k(M; \partial M, \mathbb{Z}) \cong H_{n-k}(M, \mathbb{Z})$$

for $1 \leq k \leq n$.

Theorem 1.4.8. (Universal Coefficient Theorem) Let X be a topological space and A be any Abelian group and G be a module over a principal ideal domain.

- (Universal Coefficient Theorem for Homology) There is a short exact sequence:

$$0 \to H_i(X, \mathbb{Z}) \otimes A \to H_i(X, A) \to \mathrm{Tor}_1(H_i(X, \mathbb{Z}), A) \to 0$$

- (Universal Coefficient Theorem for Cohomology) Then there is a short exact sequence:

$$0 \to \mathrm{Ext}^1_R(H_{i-1}(X, R), G) \to H^i(X, G) \to \mathrm{Hom}_R(H_i(X, R), G) \to 0$$

Theorem 1.4.9. Let ∂M be a surface with genus g. If the inclusion map is $i : \partial M \hookrightarrow M$, then the kernel of the induced homology map $i_* : H_1(\partial M, \mathbb{Q}) \hookrightarrow H_1(M, \mathbb{Q})$ is a subspace of dimension g.

Proof. Using Lefschetz-Duality theorem we have the following two isomorphisms:

$$H_2(M; \partial M, \mathbb{Q}) \cong H^1(M, \mathbb{Q})$$
$$H_1(M, \mathbb{Q}) \cong H^2(M; \partial M, \mathbb{Q})$$

and using the Poincaré Duality theorem we have the following:

$$H_1(\partial M, \mathbb{Q}) \cong H^1(\partial M, \mathbb{Q})$$

Now, we use the homology and cohomology exact sequences:

$$\begin{tikzcd}
H_2(M; \partial M, \mathbb{Q}) \arrow[r, "d"] \arrow[d] & H_1(\partial M, \mathbb{Q}) \arrow[r, "i_*"] \arrow[d] & H_1(M, \mathbb{Q}) \arrow[d] \\
H^1(M, \mathbb{Q}) \arrow[r, "i^*"] & H^1(\partial M, \mathbb{Q}) \arrow[r, "\partial"] & H^2(M; \partial M, \mathbb{Q})
\end{tikzcd}$$

All of these homologies are vector spaces as $\mathbb{Q}$ is a field. Therefore, the Universal coefficient theorem of Cohomology tells us that the spaces $H_1(\partial M, \mathbb{Q})$ and $H^1(\partial M, \mathbb{Q})$ & $H_1(M, \mathbb{Q})$ and $H^1(M, \mathbb{Q})$ are dual vector spaces. This is true as $\mathbb{Q}$

is torsion free and the exact sequence reduces to the following.

$$\begin{tikzcd}
0 \arrow[r] & H^1(M,\mathbb{Q}) \arrow[r] \arrow[d, "i_*"] & \mathrm{Hom}(H_1(M,\mathbb{Q})) \arrow[d] \\
0 \arrow[r] & H^1(\partial M,\mathbb{Q}) \arrow[r] & \mathrm{Hom}(H_1(\partial M,\mathbb{Q}))
\end{tikzcd}$$

Therefore i_* and i^* are dual maps. The best guess would be to use Rank-Nullity theorem in this case. We know that dual vector spaces of finite dimension have the same dimension and the rank and nullity of the dual maps are same. We see that $\mathrm{rank}\,(i^*) = \mathrm{rank}\,(i_*)$ since $H_1(M,\mathbb{Q})$ and $H^1(M,\mathbb{Q})$ are dual vector spaces. Due to the first vertical isomorphism we have $\mathrm{rank}\,(i^*) = \mathrm{rank}\,(d)$. By the isomorphism of $H_1(\partial M,\mathbb{Q})$ and $H^1(\partial M,\mathbb{Q})$ and the exactness of both the equations we have, $N(i_*) = N(\partial)$. We know the dimension of $H_1(\partial M,\mathbb{Q}) = 2g$, therefore from the upper exact equation and the Rank-Nullity theorem, we get, $\mathrm{rank}\,(i_*)+\mathrm{rank}\,(d) = \mathrm{rank}\,(i_*) + \mathrm{rank}\,(i^*) = 2\,\mathrm{rank}\,(i_*) = 2g \implies \mathrm{rank}\,(i_*) = g$ □

Corollary 1.4.10. There exists a basis $\{[f_1],[f_2],\ldots,[f_{2g}]\}$ over $\mathbb{Z}$ of $H_1(\partial M,\mathbb{Z})$ such that $[f_i]$ for $i = 1$ to g map to zero in $H_1(M,\mathbb{Q})$ under the inclusion map.

Proof. Using Theorem 1.4.9, we get a basis β for ker i_*. Now ker i_* is a g dimensional subspace of $H_1(\partial M,\mathbb{Q})$, which is a $2g$ dimensional vector space of over Q. This means $H_1(\partial M,\mathbb{Q}) \cong \bigoplus_{i=1}^{2g} Q$. We will use the Universal Coefficient theorem for homology. Since, $\mathbb{Q}$ is a field, the torsion is zero and the exact sequence reduces to the following.

$$0 \to H_1(X,\mathbb{Z}) \otimes \mathbb{Q} \to H_1(X,\mathbb{Q}) \to 0$$

The map $H_1(X,\mathbb{Z}) \to H_1(X,\mathbb{Q})$ is thus as injection. Each element of β can be viewed as an element of $\bigoplus_{i=1}^{2g}\mathbb{Z} \subset \bigoplus_{i=1}^{2g} Q$, by multiplying with suitable integers. Now consider the $\mathbb{Z}$-span of β (viewed as a subset of $\bigoplus_{i=1}^{2g}\mathbb{Z}$). Let us denote this span by S, since $\bigoplus_{i=1}^{2g}\mathbb{Z}$ is a module over a principal ideal domain, $\bigoplus_{i=1}^{2g}\mathbb{Z}/S = A/S \bigoplus B/S$, where A/S is the free summand and B/S is the torsion summand. If there exists an element $b \in B$, then for some integer n, $nb \in S$, i.e. $nb = \sum_i a_i[f_i] \iff b = \frac{1}{n}\sum_i a_i[f_i] \in \bigoplus_{i=1}^{2g}\mathbb{Q}$. This means a $\mathbb{Z}$-base for B is a Q-base for ker i_*. Extending this using a basis of A, we get the required basis. □

Theorem 1.4.11. It is possible to find a Seifert Matrix for a slice knot K which is of the form $\begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$.

Proof. Let F be a Seifert surface for the knot with g genus. If D is a slicing disc, then by Theorem 1.4.4 there is a manifold M which has a regular neighbourhood $M \times [-1, 1]$ in B^4. Corollary 1.4.10 provides us with a basis $\{[f_1], [f_2], \ldots, [f_{2g}]\}$ for $H_1(\partial M, \mathbb{Z})$. The entries of the Seifert Matrix S would be given by the linking numbers of f_i with f_{j^+}. For, $1 \leq i \leq g$, Corollary 1.4.10 tells us that there exists an integer n_i such that $n_i[f_i] = 0$ when included in $H_1(M, \mathbb{Q})$, hence in $H_1(M, \mathbb{Z})$ (by suitably multiplying with an integer). This f_i can be so chosen so that it bounds a surface in M, the surface that f_{j^+} bounds can similarly be moved into $M \times [-1, 1]$ thus these surfaces being disjoint. By Theorem 1.4.5 we get that lk $\left(f_i, f_j^+\right) = 0$ and hence the desired matrix form. $\square$

Corollary 1.4.12. The Alexander Polynomial of slice knots are of the form $f(t) f(t^{-1})$.

Proof. By Theorem 1.4.11, there's a certain Seifert surface for a slice knot with the associated Seifert matrix being of the form $\begin{pmatrix} 0 & A \\ B & C \end{pmatrix}$. Then $\det(tA - A^{\text{tr}}) =$

$$\det\left(\begin{pmatrix} 0 & tA - B^{\text{tr}} \\ tB - A^{\text{tr}} & tC - C^{\text{tr}} \end{pmatrix}\right) = \det\left(tB - A^{tr}\right)\left(t^{-1}B - A^{tr}\right). \qquad \square$$

Chapter 2

Colourability

2.1 Introduction

This section is taken from [Rob15]. In this chapter we will describe the motivation behind tricolourability and how the notion of colourability can be generalized by the notion of Fox Colourings. In this chapter we will study the results of Louis Kauffman and Pedro Lopes' paper titled 'Colourings Beyond Fox'([KL17]). One of these generalisations will be the Alexander's colouring, which, again, leads us to the Alexander's polynomial.

One of the simplest invariants for a knot is tricolourability. Given a knot K and a diagram of it, say D, being able to colour the arcs at every crossing with either distinct colours or the same colour is called tricolourability. Of course, every knot admits the monochrome colouring, hence at least two (hence three) colours must be used in the knot diagram for it to be tricolourable. For any knot K, the number of three-colourings is an invariant. To fully convinvce ourselves that this is indeed an invariant, all we need to do is check if tricolourability is invariant under the Reidemeister moves.

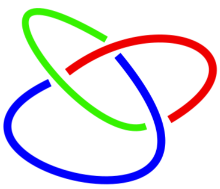


Figure 2.1: A tricoloured trefoil

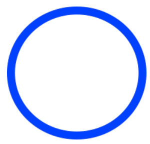


FIGURE 2.2: A monochrome unknot

At first glance, tricolourability might seem like a very random guess at obtaining an invariant for a knot. However, as it turns out, its the outcome of a very natural series of questioning. The knot group is the perhaps one of the first topological properties of a knot, one would be likely to think of and it is no surprise that the knot group is a knot invariant. Wirtinger's presentation of knot groups further tells us that it is also possible to obtain a finite presentation for a knot group, given a diagram for the knot. Isomorphism of groups then ensures that the number of group homomorphisms between the knot group, say K and a finite group, say G, is finite (since the number of elements the finite number of generators can be mapped to is finite). Since, the knot group is independent of the knot diagram, then $|\mathrm{Hom}\,(K, G)\,|$ is a knot invariant ([Rob15]). Let us briefly recall Wirtinger's representation: If D is a diagram for knot K, let the arcs in D be labelled as α_i. Fig 2.3 shows a crossing in the knot. The Wirtinger's Presentation says that the knot group is generated by loops a_i, where each a_i is a loop from a point P taken perpendicular to the plane, can be viewed as the eye of the reader. The loop a_i goes under the arc α_i at the crossing, as shown in Fig 2.3. The relation between these loops is given by $a_j a_i a_j^{-1} = a_{i+1}$. Thus $\pi_1\left(S^3 - K\right) = (a_1, a_2 \ldots, a_n; r_1, r_2, \ldots, r_n)$, where r_i is the relation at each crossing. If the knot group is abelianzed, then it is possible to commute the a_i and a_js, thus $a_i a_j a_j^{-1} = a_i = a_{i+1}$.

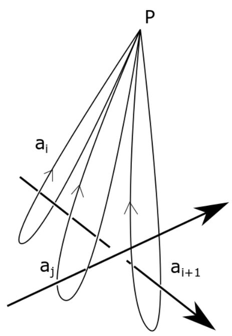


FIGURE 2.3: A crossing in D

If we view the arcs as the generators, at every crossing, the in going arc is conjugated by the overcrossing arc which then equals the outgoing arc. This implies precisely that the image of every generator of K lies in the same conjugacy class of G, thus giving us a refined invariant $|\mathrm{Hom}\,(K,G,C)\,|$, where the all the non-identity elements are mapped to the conjugacy class C. An obvious exclusion in any attempt to recoup a meaningful invariant from this process would be Abelian groups. The conjugacy classes of Abelian groups are the singleton sets containing every group element. If we had any non-trivial homomorphism $f : K \to G$, then $f\,(a_i) = f\,(a_{i+1}) = g \forall i$ and for some $g \in G$, which implies that there exists a unique homomorphism from K to G where a generator is sent to g. Thus, $|\mathrm{Hom}\,(K, G, \{g\})\,| = 1$, for any knot. The next best guess is to consider the smallest non-Abelian group S_3, the permutation group of 3 elements (or D_3, the group of symmetries of a triangle).

$$S_3 = \{(1)\,, (12)\,, (13)\,, (23)\,, (123)\,, (132)\}$$

With the conjugacy classes being as follows:

$$\{(1)\}, \qquad \{(123)\}, \qquad \{(132)\}, \qquad \{(12)\,, (13)\,, (23)\}$$

As we have seen above, if we were to have a homomorphism sending a generator of the knot group to an element whose conjugacy class is a singleton (just the element itself), then we do not get an invariant that can distinguish between knots. So, in the homomorphisms we will consider, the generators in the presentation of the knot group in consideration must not map to either $\{(123)\}$ or $\{(132)\}$. To get a non-trivial homomorphism that we hope leads us to an invariant, the only candidate is the conjugacy class $\{(12)\,, (13)\,, (23)\}$.

Consider a crossing where the arcs a_i, a_{i+1} and a_j (as shown in Figure 2.3), without loss in generality, consider the map where a_i is sent to the transposition (12). Using the relation $a_j a_i a_j^{-1} = a_{i+1}$, we deduce that one of a_j and a_{i+1} must be mapped to (23) and the other must be mapped to (13). This is because of the following :

$$(23)\,(12)\,(23) = (13) \qquad \text{and} \qquad (13)\,(12)\,(13) = (23)$$

Now we see that, a tricolouring of a knot K corresponds to a homomorphism of the knot group of K sending the generators to the transpositions. It is easy see that each transposition when thought of as a colouring on an arc coincides with the definition of tricolourability. If C is the conjugacy class containing the transpositions, $|\mathrm{Hom}\,(\pi_1\,(S^3 - K), G, C)\,|$ is the number of three colourings of the knot. This provides us with a rich host of knot invariants.

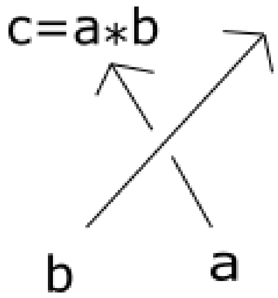
We can also interpret this result in terms of linear algebra. If each of the arcs were to be labelled by an element of $\mathbb{F}_3$, then at a crossing where the overcrossing arc is labelled x_i, ingoing arc is x_j , outgoing arc is labelled x_k, the condition $2x_i - x_j - x_k = 0$ mod 3 implies tricolourability. It is easy to see this, when x_i is 0, one of x_j or x_k must be 2 and the other must be 1; When x_i is 1, one of x_j or x_k must be 0 and the other must be 2. The number of tricolourings, then, becomes all possible solutions to the following system of linear equations:

$$2x_i - x_j - x_k = 0$$

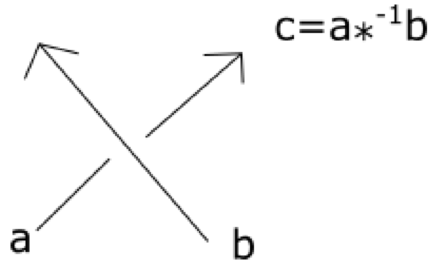
for every ordered triplet of arcs at a crossing (x_i, x_j, x_k). If there are n crossings then the number of such equations would be n. The set of solution of these system forms a subspace of $\mathbb{F}_3^n$. If the dimension of this subspace is k, then we have the following result.

Theorem 2.1.1. The number of three colourings of a knot is 3^k

There is no need to restrict ourselves to the case of $\mathbb{F}_3$, this line of reasoning can be extended to any number of colourings. However to avoid zero-divisors, we restrict ourselves to the case of p-colourings where p is a prime. The condition $2x_i - x_j - x_k = 0$ mod p is also termed as Fox Colouring. Observe that we have two kinds knot invariants now, first, the usual one, the number of permissible colourings, according to preceding conditions. Second, the minimum number of colourings used in such a colouring (where at least two colours are used in the diagram), this minimum is taken over all possible diagrams of the knot. Of course, the latter is a redundant distinction for tricolourability since minimum number of colourings for a tricolouring with at least two distinct colours used in the diagram, is 3. In the rest of this section we will be discussing results from Louis Kauffman's and Pedro Lopes' paper titled *Colourings beyond Fox* where the Fox Colouring condition is given a quandle structure which is later used to generalise these colourings. We will also see how the Alexander matrix can be obtained in terms of these colourings.



((A)) Positive crossing



((B)) Negative crossing

FIGURE 2.4

2.2 Quandles and Fox Colouring

This section is taken from [KL17].

Definition 2.2.1. (**Quandles**) A quandle $(Q, *)$ is a set of elements equipped with a binary operation $*$ such that the following properties are satisfied.

- If $x \in Q$, then $x * x = x$.
- If $x, y \in Q$, then there exists a unique element of Q, z, such that $z * x = y$.
- If $x, y, z \in Q$ $(x * z) * (y * z) = (x * y) * z$.

Quandles can be interpreted in terms of Reidemeister moves. To see this let us look at Fig 2.4 and 2.5.

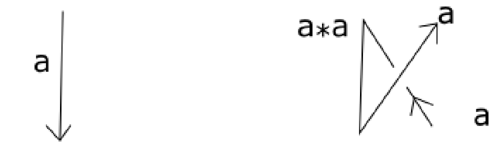
Example 2.2.2. *The Fundamental Quandle* of a Knot is defined using these operations. Every arc stands for a generator and the relations are precisely the ones shown in Fig 2.4.

Another example of a quandle is the Alexander quandle which gives rise to the Alexander Polynomial. The underlying set of elements is the set of Laurent Polynomials.

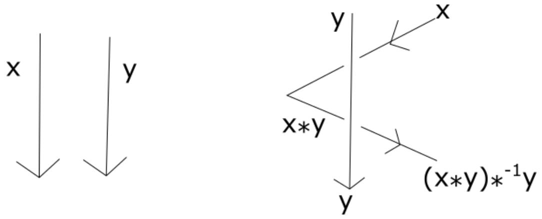
$$\Lambda = \mathbb{Z}\left[t, t^{-1}\right]$$

. The operation between two elements $a, b \in \Lambda$ is defined as follows.

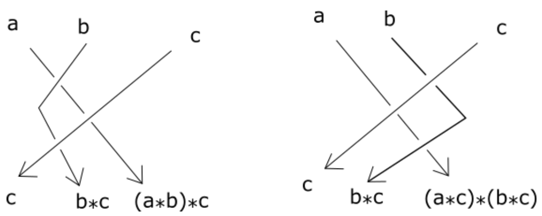
$$a * b = Ta + (1 - T)\, b$$



((A)) The first Reidemeister move



((B)) The second Reidemeister move



((C)) The third Reidemeister move

FIGURE 2.5

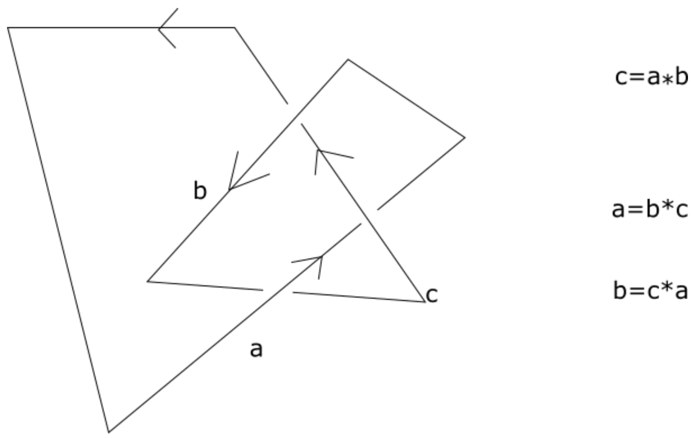


FIGURE 2.6: The Trefoil coloured using the Alexander Polynomial

Let us compute the trefoil's Alexander matrix using this method.

A colouring of the trefoil using the Alexander quandle is shown in Fig 2.6. We obtain the following sets of equations using these relations.

$$(1-T)\,a - b + Tc = 0$$
$$-a + Tb + (1-T)\,c = 0$$
$$Ta + (1-T)\,b - c = 0$$

The matrix of this system of equations is $A = \begin{bmatrix} (1-T) & -1 & T \\ -1 & T & (1-T) \\ T & (1-T) & -1 \end{bmatrix}$. This is the Alexander matrix of the trefoil. It's minor along the first entry of the matrix is the matrix $A_{11} = \begin{bmatrix} T & (1-T) \\ (1-T) & -1 \end{bmatrix}$.

It's determinant gives us the Alexander Polynomial of the trefoil (unique upto multiplication by T^n). $\Delta_0\,(T) = -T - (1-T)^2 = -T^2 + T - 1$.

What one might observe from the system of equations is that each equations has exactly on of each -1, T and $(1-T)$ as coefficient. Using column operations on the matrices, we can add up all the columns to get a column of only zeros, effectively telling us that the determinant of the matrix would be zero. Thus we have a lot of solutions to this system of equations however these solutions are just the *monochrome* ones, that is, they are multiples of the the constant colouring $(1, 1, 1, 1, \ldots, 1)$. It is clear that these elements would satisfy the preceding system of equations. To prove that these are the only solutions, let v be a n-column vector that satisfies $Mv = 0$, where M is the system of linear equations. Now consider any of the entries of the column Mv, we would have an equation of the following form.

$$-v_i + TV_J + (1-T)\,v_k = (-v_i + v_k) + T\,(v_j - v_k) = 0$$
$$\implies v_i = v_k \quad \& \quad v_i = v_k$$

Considering all the crossings of the knot, we'd see that all these v_i must be equal. To find a *polychromatic* solution, we will need to work with a quotient of Λ. This will be proved in the following theorem.

Theorem 2.2.3. A polychromatic solution of the Alexander matrix can be obtained by quotienting Λ by the ideal generated by the determinant of the first minor of the matrix.

Proof. Let v be a polychromatic solution of M, that is, there are at least 2 distinct entries in n. Then $Mv = 0$. Observe that the sum of a polychromatic and a monochromatic solution is still a solution of the system of solutions. To see this let the c be a column vectors with same entries. If at a crossing we have the equation $v_i * v_j = Tv_i + (1-T)\,v_j = v_k$, then

$$\begin{aligned}(v_i + c) * (v_j + c) &= T\,(v_i + c) + (1-T)\,(v_j + c)\\ &= Tv_i + (1-T)\,v_j + c\\ &= v_k + c\end{aligned}$$

Now, returning to the original equation $Mv = 0$, we add a monochromatic solution to v to make the i^{th} entry 0. We can delete the i^{th} column and any column. By abuse of notation, we still denote this new matrix and new column as M and v respectively. Then $Mv = 0 \implies \operatorname{adj}(A)\,A = \det A I_n v = 0$. Since v is polychromatic $\det A = 0$. □

2.3 Linear Alexander Quandles

This section is taken from [KL17]. Tricolourability can be interpreted in terms of quandles. First, we need the following definitions.

Definition 2.3.1. The *Dihedral quandle* D_n is a quandle with the underlying set $\mathbb{Z}_n$ and the binary operation $a * b = 2b - a \mod n$.

Definition 2.3.2. (Quandle homomorphism) Given two quandles $(Q, *)$ and $(Q', *')$, a quandle homomorphism $h : Q \to Q'$ is a map such that for any elements $x, y \in Q$, $h\,(x * y) = h\,(x) *' h\,(y)$.

Recall the definition of the fundamental quandle of a knot given in Example 2.2.2. The number of tricolourings of a knot K is the number of quandle homomorphisms between the fundamental quandle of the knot K and the dihedral quandle D_3. Clearly this is an invariant, for otherwise there wouldn't be an isomorphism between knot quandles.

In this section we introduce Linear Alexander quandles, of which, the Dihedral quandle is a special case.

Definition 2.3.3. *Linear Alexander Quandles* denoted as $\mathrm{LAQ}\,(n, m)$ consist of those quandles which have $\mathbb{Z}_n$ as their underlying set and the binary operation is defined as $a * b = ma + (1 - m)\, b \bmod m$ such that $(m, n) = 1$ and $m < n$.

It is clear that dihedral quandles is the special case of Linear Alexander Quandles when $m = -1$.

Definition 2.3.4. ((m,n)-colourings) If a knot diagram can be coloured using by the quandle $\mathrm{LAQ}\,(m, n)$, then this colouring is called a (m, n)-colouring.

We shall only deal with prime n and positive m. This ensures that for a distinct pair of (m, n), we have a distinct quandle. Our aim in this section is to find a lower bound for the minimum number of colourings needed for (m, n) colourings. The requirement $(m, n) = 1$ is chosen so that the operation is right invertible, i.e., so that the second axiom in Definition 2.2.1 is satisfied. We prove this claim.

Proof. We need to find an x such that $x * b = c$. Now, $x * b = mx + (1 - m)\, b \bmod m$. Since $(m, n) = 1$. By Euclid's Lemma there exists integers u, v such that $um + vn = 1$. Suitably multiplying by d, we have $udm \equiv d \mathrm{mod} n$. Choosing $x \equiv (c - (1 - m)\, b) \mod n$, we have $x * b = c$. □

The convention for these colourings as similar to what was shown in Fig 2.4. For dihedral quandles however, there is no distinction of how the diagram is oriented for $a * b = c = 2b - a \implies 2b - c = a = c * b$.

We will now show the existence of a preliminary lower bound for colourings under the present assumptions that we have made.

Remark 2.3.5. We consider only non split links, i.e., links, components of which, cannot be separated by disjoint 3-balls.

Theorem 2.3.6. Every (p, m) colouring of a non split link uses at least three colours.

Proof. We split the proof into cases. First we remark that this is not true in case of split links. We can provide two distinct monochrome colourings to each portion of the links in each distinct ball.

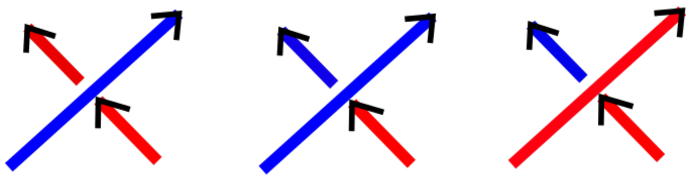


FIGURE 2.7: The possible cases that can occur while trying to colour a crossing with 2 colours

1. (Case I) Under-arc and the out coming arc have the same colour. Then we have the equation

$$\begin{aligned} a &= ma + (1-m)\,b \\ \implies a\,(1-m) &= (1-m)\,b \\ \implies a &= b \end{aligned}$$

   The third line follows from the second since there are no zero divisors in $\mathbb{Z}_p$.

2. (Case II) The over-crossing and the out coming arc have the same colour. Then we have the equation

$$\begin{aligned} a &= ma + (1-m)\,b \\ \implies a\,(1-m) &= (1-m)\,b \\ \implies a &= b \end{aligned}$$

3. (Case III) When the overcrossing arc has a distinct colour from the under arcs. Here, we have the following equation.

$$\begin{aligned} b &= ma + (1-m)\,a \\ \implies b &= ma + a - ma \\ \implies b &= a \end{aligned}$$

The proof is complete. □

The bound that is obtained in [KL17] is $2 + \lfloor ln_M p \rfloor \leq k$, where k is the minimum number of colourings needed in a (p, m) colouring and $M = \max\{|M|, |M-1|\}$. To go about proving this, we will need the aid of the palette graph of a colouring and its adjacency matrix.

2.4 Palette graphs and Adjacency Matrix

This section is taken from [KL17].

Definition 2.4.1. (Palette graph) Given a (p, m) colouring of a knot, the palette graph of the knot consists of vertices and edges. Vertices are labelled according to the colours of the under arcs at every crossing. Two vertices s_1 and s_3 are connected by and edge if there exists a colour s_3 such that $s_1 * s_2 = s_3$, that is, $ms_1 + (1 - m)\, s_2 = s_3$. The palette graph is a directed graph with the arrow being from s_1 to s_3 in the previous case.

Definition 2.4.2. (Adjacency Matrix) The Adjacency matrix for a (p, m) palette graph consists of the vertices as the columns and the edges as the row. The (i, j)th entry of the matrix is given as follows:

- If the edge e_i begins at v_j, then the (i, j)th entry will $-m$
- If the edge e_i ends at v_j, then the (i, j)th entry will 1
- If the edge e_i is coloured as v_j, then the (i, j)th entry will $m - 1$

Theorem 2.4.3. If we have a knot K that admits a (p, m) colouring, it's palette graph is denoted as G and it's adjacency matrix as A. We delete the jth column from A and denote this matrix as A_j. Then the following conditions are satisfied:

1. det A_j is divisible by p (can be zero).
2. det $A_j = \pm 1 \bmod |m - 1|$

Proof. (1.) If det $A_j = 0$, then we are done. If not, then we have det $A_j \neq 0$. Since A is the coefficient matrix of colourings, we have two linearly independent solutions in the form of the monochromatic solution (trivial colouring) and a polychromatic colouring (by assumption) modulo p. Thus, rank of A can be at most 2. This implies that det$A_j = 0 \bmod p$.

(2.) If we have reduce the matrix $A_j \bmod (m - 1)$, then each row can have at most 2 non-zero entries, namely, 1 and -1. Now the subgraph, generated by A_j' (this new matrix), has exactly one vertex less than the original graph G, now there is a unique bijection between the vertices and edges such that it maps an edge to a vertex adjacent to it, call this bijection σ, then $\det(A_j) = \det\left(A'_j\right) = a_{1,\sigma(1)} a_{2,\sigma(2)} \ldots a_{k-1,\sigma(k-1)} = \pm 1 \bmod m - 1$. ☐

We have a very interesting corollary due to this theorem.

Corollary 2.4.4. The only integer root of a non-identically zero Alexander polynomial of a knot is 2.

Proof. The Alexander polynomial for a knot K, $\Delta_K(m)$ is the first minor of the coefficient matrix. We are assuming that $\Delta_K(m) = 0$. Since the colouring matrix has only entries $-m, m-1$ and 1 in each row, the determinant of the matrix is zero. Now if we look at the Smith Normal form of this matrix, the diagonal will have two zero entries at least, owing to the fact that it doesn't have full rank and the first minor is zero. Thus there must exist at least one (p, m) colouring. Then we use Theorem 2.4.3, that is, $\det A_j = 0$ and $\det A_j = \pm 1 \mod |m-1|$. For this to be true, m must be equal to 2. □

To get the lower bound on colouring let us observe more closely the class of matrices of the colourings. These matrices have only at most 3 non-zero entries in each row. Then, have the following lemma.

Lemma 2.4.5. Let M be a square matrix of dimension n such that each row has at most one each of $-m, m-1$ and 1 as it's non-zero entries. Then $|\det M| \leq k^n$ where $k := \max\{|m|, |m-1|\}$.

Proof. The statement trivially holds if the dimension of the matrix is 1. We proceed by induction. If the statement holds for all $m \leq n$ and If m is positive, then we have the following cases.

1. *Case I*: If there is no $-m$ in one of the rows. In this case, expanding the along this row, we will get that,

$$\det(X) = (m-1)A + 1 \cdot B$$

where A and B are minors of dimension n-1. By induction hypothesis, then we have that A,$B \leq k^{n-1}$. Therefore, the following holds.

$$\det X \leq (m-1)k^{n-1} + k^{n-1} \leq k^n$$

2. *Case II*: If there is only a $-m$ as a non-zero entry in one of the rows. Then, expanding along this row, we get that,

$$\det(X) = (-m)\,A$$

where A is a minor of dimension $n-1$. Similarly, as in first step, using induction hypothesis we get the following.

$$\det X \leq mk^{n-1} = k^n$$

3. *Case III*: Finally, when every row has one $-m$ and atleast one of $m-1$ and -1. In this case by suitable column re-arranging we can have the $(1,1)$ entry in the matrix as $-m$. Now using the column operation $C_1 \to \sum_{i=1}^{n} C_i$, where C_i is the ith column. Now the entries in the first column can be $0, -m+1 or 1$. This leads us back to our previous cases. Hence,

$$\det X \leq k^n$$

In the case of the negative m, the exact same proof goes through with $m-1$ taking the role of $-m$. □

Corollary 2.4.6. If K is a knot which permits a (p,m) colourings then,

$$2 + \lfloor \ln_k p \rfloor \leq L$$

when L is the lower bound for colours required for the (p,m)-colouring.

Proof. By Lemma 2.4.5 and Theorem 2.4.3, $\det A_j \leq k^n$ and that $\det A_j$ is divisible by p. Since, a non-trivial colouring exists, $\det A_j \neq 0$. Therefore we have $p \leq \det A_j \leq k^n$. Therefore, $p \leq k^n$, taking natural logarithm on both sides we have the desired inequality. □

Chapter 3

Burau Representation

3.1 Introduction

The aim of this chapter is to establish a connection between braid groups and the Alexander Polynomial for links. We will introduce braid groups. Braid groups can also be thought of as the mapping class group of a disc. It is then possible to define homological representations. The Burau representation is one such homological representation. In the concluding section, it will be shown how the Alexander's Polynomial can be obtained from the matrices of the Burau representation. Most of the content we refer to in this chapter can be found in chapter 3 of [KT08].

3.2 Braid groups

We first give an algebraic definition.

Definition 3.2.1 (Artin Braid Group)**.** The Artin Braid group on n strings B_n is a free group with n generators, σ_i, $1 \leq i \leq n$, with the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2$$

$$\sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i$$

It can be easily seen that B_1 is the trivial group and B_2 is the infinite cyclic group with the generator σ_1. The Braid group on 3 strands B_3 is particularly interesting

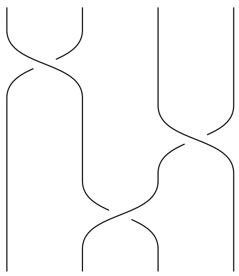


Figure 3.1: An example of a geometric braid

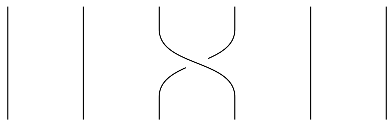


Figure 3.2: σ_i

as this is isomorphic to the knot group of the trefoil knot. To verify this, we recall a presentation of the trefoil's knot group (obtained by means of the Wirtinger's presentation):

$$\pi_1\left(S^3 - K,\ \right) \ = \ \{x, y \,|\, x^2 = y^3\}$$

Here K denotes the trefoil. B_3 has two generators σ_1 and σ_2. We set $x = \sigma_1\sigma_2\sigma_1$ and $y = \sigma_1\sigma_2$. Clearly $x \neq y$, we further claim that $x^2 = y^3$. This would prove that the two aforementioned groups are isomorphic.

$$\begin{aligned} x^2 &= (\sigma_1\sigma_2\sigma_1)(\sigma_1\sigma_2\sigma_1) \\ &= (\sigma_1\sigma_2\sigma_1)(\sigma_2\sigma_1\sigma_2) \\ &= (\sigma_1\sigma_2)(\sigma_1\sigma_2)(\sigma_1\sigma_2) \\ &= y^3 \end{aligned}$$

Definition 3.2.2 (Geometric Braids)**.** A geometric braid b on n strings is a subset of $\mathbb{R}^2 \times I$ that is homeomorphic to the disjoint union of n closed intervals. The projection of each string onto I under the natural projection $\mathbb{R}^2 \times I \to I$ is a homeomorphism. Also each braid string has it's endpoints in the set $\{(i,0,0), (j,0,1) \ |1 \leq i,j \leq n\}$ and no two strings intersect each other.

A generator σ_i of the braid group B_n can be represented as shown in Fig 3.2.

Two geometric braids b and b' are said to be isotopic if there is a continuous family of maps $\{F_s\}_{s\in I}$ such that $F_0 = Id$ and $F_1(b) = b'$.

If D is a 2-disc and Q is a set of n points in the disc. A self-homeomorphism f of the pair (D, Q) is a homeomorphism of D which is identity on the boundary of the disc and maps the set Q to itself. The mapping class group of (D, Q), MCG(D, Q) is the group of isotopy classes of these self-homeomorphism.

Theorem 3.2.3. $B_n \cong \mathrm{MCG}(D, Q)$, where Q has n distinct points.

To understand what the generators of the mapping class group of the disc are, we will need a few definitions.

Definition 3.2.4. A *spanning arc* α of the disc D is a subset of D° which is homeomorphic to the closed unit interval and intersects Q only at it's end-points.

Definition 3.2.5. Given a spanning arc α, a *half-twist* τ_α is a self-homeomorphism of (D, Q) which switches the end points of alpha and is identity outside a ball containing α. To see this explicitly, we consider a neighbourhood V around the spanning arc α and identify it with the open unit disc in . Let α be identified with $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Then the half twist is defined as follows:

$$\tau_\alpha = \begin{cases} -z & |z| \leq \frac{1}{2} \\ e^{(-2\pi i|z|)}z & \frac{1}{2} \leq |z| \leq 1 \end{cases}$$

We can consider the disc D as a subset of $\mathbb{R}^2$ such that $Q = \{(1,0), (2,0), \ldots, (n,0)\}$ are contained in the disc. Let $\alpha_i = [i, i+1] \times \{0\}$. These are spanning arcs. Then the generators of the mapping class group of (D, Q) MCG(D, Q) is generated by the half-twists τ_{α_i}.

Our interest in braid groups stems primarily due to the following theorem.

Theorem 3.2.6. Alexander's Theorem Every link can be obtained as the closure of a braid.

'Closure' of a braid simply means identifying the end points of each strand of the braid. More formally speaking, we can view a braid β on n strands as a subset of a solid cylinder $D^2 \times I$ with each strand having it's end points at $(x, 0)$ and $(x, 1)$. Without loss of generality we can assume the braid lies in the interior of the solid cylinder (we can always ensure by taking a bigger cylinder or by isotopy). The closure is then obtained by identifying $D \times \{0\}$ with $D \times \{1\}$. One might now wonder which braid closures give rise to isotopic knots. The following theorem addresses this question.

Theorem 3.2.7. *Braids have isotopic closures if and only if they are related by Markov moves.*

The following are the *Markov Moves*:

1. $(M1)$ If β is a braid , conjugation of β by another braid γ, i.e., $\beta \mapsto \gamma\beta\gamma^{-1}$ is the $M1$ move.
2. $(M2)$ If β is a braid in n strands, then $\beta \mapsto \sigma_n i\,(\beta)$ is the $M2$ move.

It's easy to see that these moves create isotopic closure. The first move composes γ^{-1} on γ, producing the identity. The second move only contributes to the formation of a kink in the closed braid.

Definition 3.2.8. Markov function A collection $\{f_n : B_n \longrightarrow A\}$ where A is any set, such that:

1. $f_n\,(\alpha\beta) = f_n\,(\beta\alpha)$
2. $f_n\,(\beta) = f_{n+1}\,(\sigma_n\beta)$ and $f_n\,(\beta) = f_{n+1}\,(\sigma_n^{-1}\beta)$

for all $n \in \mathbb{N}$.

Every Markov function is a link invariant as braids that have isotopic closures will be mapped to the same element under the Markov functions. Braids are useful as now there's the additional group structure. This enables us to define representations of the braid group. In this chapter we will mainly focus on the Burau representation, which arises as a homological representation of the Braid group using the mapping class description of the Braid group. As it will eventually turn out, we will be able to define a Markov function using the Burau representation which will coincide with the Alexander Polynomial for links.

3.3 Homological Representations of mapping class groups of surfaces ([KT08])

We recall that a self homeomorphism h of a surface S with boundary and a set Q is homeomorphism of S which fixes the boundary of S pointwise and permutes the points of Q. Let $\mathrm{MCG}(S, Q)$ denote the mapping class group of self-homeomorphisms of the pair (S, Q). If two self-homeomorphisms h and g are in the same mapping class, then they are isotopic and in particular homotopic, hence they induce the same automorphisms on the homologies of S. For our purposes we consider Q to be an empty set. This leads us to the existence of a map

$$\mathrm{MCG}\left(S, Q\right) \to \mathrm{Aut}\left(H_1\left(S, \mathbb{Z}\right)\right)$$

Using this we can get a representation for braid groups (consider the mapping class group depiction of braid groups). However, upon closer inspection, one notices that this map would just be the isomorphism between the mapping class group depiction of braid groups and the braid automorphisms. A natural guess to remedy this would be attempting to define a map from the mapping class group to the homology of a covering space of our surface instead.

Let S have a non empty boundary, choose a base point $d \in \partial S$ (d would be invariant under the self homeomorphisms). If there exists a surjective group homomorphism $\phi : \pi_1\left(S, d\right) \to \ G$, for some group G, then covering space theory tells us that corresponding to the kernel of this map, there is a covering space of S, say $\tilde{S}$ and a covering map p such that $p_*\left(\pi_1\left(\tilde{S}\right)\right) = Ker\,\phi$. The group of deck transformations of this cover is given by $N\left(Ker\,\phi\right)/Ker\,\phi$. $N\left(Ker\,\phi\right) = \pi_1\left(\tilde{S}\right)$ as $Ker\,\phi$ is normal in $\pi_1\left(\tilde{S}\right)$. Thus, by first isomorphism theorem, the group of deck transformations is isomorphic to G. If $[x] \in H_1\left(\tilde{S}; G\tilde{d}, \mathbb{Z}\right)$, then the action of G on $[x]$ is defined as

$$g \cdot [x] = [gx]\,.$$

Here, $H_1\left(\tilde{S}; G\tilde{d}, \mathbb{Z}\right)$ is the relative homology of the covering space of S with respect to the set of pre-images of the base point under the covering map and $\tilde{d}$ is any fixed lift of the base point d. Under this relative homology, the lift of any loop representing a cycle in $H_1\left(S, \mathbb{Z}\right)$, is a cycle in $H_1\left(\tilde{S}; G\tilde{d}, \mathbb{Z}\right)$ and conversely, any loop representing a cycle in $H_1\left(\tilde{S}; G\tilde{d}, \mathbb{Z}\right)$ maps to a cycle in $H_1(S, \mathbb{Z})$ under the covering map.

Consequently, $\mathbb{Z}[G]$ also acts on $H_1\left(\tilde{S}; G\tilde{d}, \mathbb{Z}\right)$. Let $\mathrm{Aut}(H_1\left(\tilde{S}; G\tilde{d}, \mathbb{Z}\right))$ be the group of linear $\mathbb{Z}[G]$ transformations. This action imbibes $H_1\left(\tilde{S}; G\tilde{d}, \mathbb{Z}\right)$ with the structure of a left $\mathbb{Z}[G]$ module. S deformation retracts to a join of n circles, this is possible since it's boundary is non-empty. Then the rank of the $\mathbb{Z}[G]$ module $H_1\left(\tilde{S}; G\tilde{d}, \mathbb{Z}\right)$ is n.

We would ideally like to define a map between $\mathrm{MCG}(S, Q) \to \mathrm{Aut}(H_1\left(\tilde{S}; G\tilde{d}, \mathbb{Z}\right))$. This can be done by taking a representative of an isotopy class, let f be a homeomorphism that belongs to this class. The map f is defined on S and can be lifted uniquely to a map $\tilde{f}$ on $\tilde{S}$ (the base point d is lifted to some fixed pre image $\tilde{d}$). However, there is no reason for $\tilde{f}$ to induce a $\mathbb{Z}[G]$ linear map as $\tilde{f}$ might not commute with the action of G. To fix this we consider a subset of $\mathrm{MCG}(S, Q)$, consisting of isotopy classes of homemorphisms $[f]$ such that $\phi \circ f_{\#} = \phi$, where $f_{\#}$ is the induced map on $\pi_1(S, d)$.

$$\begin{tikzcd}
\pi_1(S,d) \arrow[rr, "f_{\#}"] \arrow[dr, "\phi"'] & & \pi_1(S,d) \arrow[dl, "\phi"] \\
& G &
\end{tikzcd}$$

We call this class of homeomorphisms, the homeomorphisms twisted by ϕ and this collection is denoted by $\mathrm{MCG}_\phi(S, Q)$. We then have a map from $\mathrm{MCG}_\phi(S, Q) \to \mathrm{Aut}(H_1\left(\tilde{S}; G\tilde{d}, \mathbb{Z}\right))$ that sends a class of homeomorphism $[f]$ to $\tilde{f}_*$, the induced homology map on $\mathrm{Aut}(H_1\left(\tilde{S}; G\tilde{d}, \mathbb{Z}\right))$.

We will now use the tools we have developed so far and use it to construct a Braid group representation. Let Σ be a 2-disk and Q be a set of n distinct points in it. Then the mapping class group $\mathrm{MCG}(\Sigma, Q)$ of the pair (Σ, Q) is isomorphic to the braid group on n strands as we have seen previously. We shall denote the punctured disc $\Sigma - Q$ as S. Our attempt to define a representation will involve defining a map from $\mathrm{MCG}(\Sigma, Q)$ to $\mathrm{MCG}_\phi(S, \varnothing)$ and then use the homological representation we have defined previously. Thus, we will have the following sequence of maps:

$$\mathrm{MCG}(\Sigma, Q) \longrightarrow \mathrm{MCG}_\phi(S, \varnothing) \longrightarrow \mathrm{Aut}(H_1\left(\tilde{S}; G\tilde{d}, \mathbb{Z}\right))$$

.

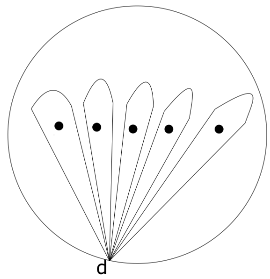


FIGURE 3.3: The generators of the fundamental group

The fundamental group of S, $\pi_1(S,d) \cong \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{\text{n times}}$. The generators of $\pi_1(S,d)$ are depicted in Fig 3.3. $\phi : \pi_1(S,d) \to \mathbb{Z}$ is defined on each generator as $\phi(x_i) \mapsto 1$, where 1 is a generator of the infinite cyclic group. $\phi[\gamma]$ for any loop γ, is in fact the sum of the winding numbers of the loop around each point in Q. If $[f] \in \mathrm{MCG}(\Sigma, Q)$, then the map g from $\mathrm{MCG}(\Sigma, Q) \to \mathrm{MCG}_\phi(S, \varnothing)$ is defined as the restriction of f to S, i.e., $\left[f_{|S}\right]$. The *twisting* condition remains to be checked, i.e., if there is a homeomorphism f whose isotopy class lies in $\mathrm{MCG}_\phi(S, \varnothing)$, then $\phi \circ f_\# = \phi$. It would suffice if we check whether or not the property holds on the generators of $\mathrm{MCG}(\Sigma, Q)$, namely τ_i, where $1 \leq i \leq n-1$. Fig 3.4 shows the action of a spanning arc on a transversal curve. $\tau_{i\#}([y])$, as we have seen previously, is defined as follows:

$$\begin{cases} [x_{i+1}], & [y] = [x_{i+1}] \\ [x_{i+1}]^{-1}[x_i][x_{i+1}], & [y] = [x_i] \\ Id, & \text{otherwise} \end{cases}$$

$\phi \circ \tau_{i\#}([x_i]) = \phi([x_{i+1}]) = 1 = \phi([x_i])$

$$\begin{aligned} \phi \circ \tau_{i\#}([x_{i+1}]) &= \phi\left([x_{i+1}]^{-1}[x_i][x_{i+1}]\right) \\ &= \phi\left([x_{i+1}]^{-1}\right)\phi([x_i])\phi([x_{i+1}]) \\ &= \phi\left([x_{i+1}]^{-1}\right)\phi([x_{i+1}])\phi([x_i]) \\ &= 1 = \phi([x_i]) \end{aligned}$$

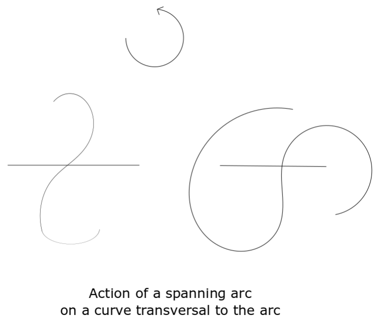


FIGURE 3.4: A spanning arc

The third equality in the second case holds as $\mathbb{Z}$ is Abelian. This shows that $\tau_{i\#}$ indeed satisfies the twisting condition. Thus the map $\mathrm{MCG}\,(\Sigma, Q) \xrightarrow{g} \mathrm{MCG}_{\phi}\,(S, \varnothing)$ is well-defined. Composing this with the homological representation we constructed previously, we have a representation for the braid groups.

3.4 A matrix representation

We will now attempt to obtain matrices for the representation we have constructed. S deformation retracts to a wedge of n circles, where n is the number of holes is the disc, which can be seen as a graph with one vertex d and n edges Y_i, $1 \leq i \leq n$. The first homology of $\tilde{S}$ only depends on the 1-skeleton of $\tilde{S}$, which is an infinite graph with vertices $t^k\tilde{d}$ and edges $t^k\tilde{x}_i$, where t is a generator of a the infinite cyclic group. An example of an infinite cyclic covering for S is shown in 3.5.

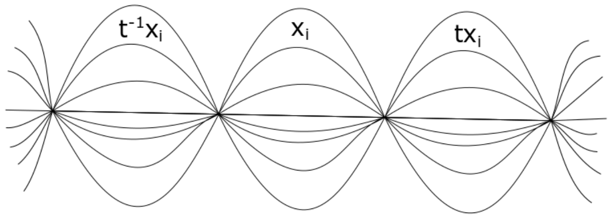


FIGURE 3.5: An infinite cyclic cover of S

$H_1\left(\tilde{S}; \sqcup_{k\in\mathbb{N}} t^k\tilde{d}, \mathbb{Z}\right)$ is thus a free $\mathbb{Z}\,[t, t^{-1}]$module with rank n with generators $[x_i]$. Thus $H_1\left(\tilde{S}; \sqcup_{k\in\mathbb{N}} t^k\tilde{d}, \mathbb{Z}\right) =$ To get matrices for our presentation, it suffices to check the generators. We pick the generators $\{\tau_i^{-1}\}_{i=1}^{n-1}$ of the braid group $\mathrm{MCG}\,(S, Q)$, σ_i is mapped to an element in $\mathrm{Aut}\,(H_1\left(\tilde{S}; G\tilde{d}, \mathbb{Z}\right))$. $\mathrm{Aut}\,(H_1\left(\tilde{S}; G\tilde{d}, \mathbb{Z}\right)) = \mathrm{GL}_n\,(\mathbb{Z}\,[t, t^{-1}])$, thus τ_i^{-1} is mapped to a $n \times n$ matrix with entries in $\mathbb{Z}\,[t, t^{-1}]$. To determine this matrix, let us examine the homeomorphism that $g\left(\tau_i^{-1}\right)$ induces. If Q is the set of points $\{(1,0),(2,0),\ldots,(n,0)\}$, then τ_i^{-1} interchanges the points $(i+1,0)$ and $(i,0)$.

$$\begin{aligned}\tau_{i*}^{-1}\left([x_{i+1}]\right) &= [x_i]\\ \tau_{i*}^{-1}\left([x_i]\right) &= [x_i]\,[x_{i+1}]\left[x_i^{-1}\right]\end{aligned}$$

The lift of this map, is thus defined as follows:

$$\begin{aligned}\tilde{\tau}_{i*}^{-1}\left([\widetilde{x_{i+1}}]\right) &= [\tilde{x}_i]\\ \tilde{\tau}_{i*}^{-1}\left([\tilde{x}_i]\right) &= [\tilde{x}_i]\,[t\widetilde{x_{i+1}}]\left[t\tilde{x}_i{}^{-1}\right]\\ &= (1-t)\left([\tilde{x}_i]\right)\bigoplus t\left([\widetilde{x_{i+1}}]\right)\end{aligned}$$

All the other generators are mapped to themselves. Thus the matrix corresponding to the generator $\tilde{\tau}_{i*}^{-1}$ is:

$$\begin{bmatrix} I_{i-1} & 0 & 0 & 0\\ 0 & 1-t & 1 & 0\\ 0 & t & 0 & 0\\ 0 & 0 & 0 & I_{n-i-1}\end{bmatrix}$$

The *Burau Representation* is obtained by taking the transpose of these matrices.

3.5 The Burau Representation

The Burau representation for Braid groups, as we have seen previously, is defined on each generator, $\sigma_i \mapsto U_i$, where

$$U_i = \begin{bmatrix} I_{i-1} & 0 & 0 & 0\\ 0 & 1-t & t & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & I_{n-i-1}\end{bmatrix}$$

As these matrices were mapped as images of generators of braid groups and these maps were homomorphisms as seen in previous sections, these matrices U_i satisfy the braid relations

$$\begin{aligned} U_iU_{i+1}U_i &= U_{i+1}U_iU_{i+1} \\ U_iU_j &= U_jU_i \ , |i-j| \geq 2 \end{aligned}$$

Since these are block matrices, it is easy to see that the second condition is satisfied. If $|i-j| \geq 2$ and without loss of generality $i < j$ (if $i = j$, this trivially holds)

$$U_iU_j = \begin{bmatrix} I_{i-1} & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & I_{n-i-2} \end{bmatrix} \begin{bmatrix} I_{j-1} & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & I_{n-j-2} \end{bmatrix} = \begin{bmatrix} I_{i-1} & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 \\ 0 & 0 & I_{n-|i-j|-2} & 0 & 0 \\ 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 & I_{n-j-2} \end{bmatrix} = U_jU_i$$

where $B = \begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix}$. In the other case , we need to verify that $U_iU_{i+1}U_i = U_{i+1}U_iU_{i+1}$, this boils down to proving that

$$\begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{bmatrix}$$

3.6 The Reduced Burau Representation

The Burau representation is reducible which means there exists a subspace such that when the Burau representation matrices when operated on this subspace map within this susbspace. If the basis elements are denoted by $\{[x_1]\ [x_2]\ \ldots, [x_n]\}$, consider the action of the matrices U_i^{tr} on the set $\{[x_1]\,[x_2]\,, [x_1]-[x_3]\,, \ldots, [x_1]-[x_n]\}$.

$$U_j\left([x_1]-[x_i]\right)=\begin{bmatrix} I_{i-1} & 0 & 0 & 0\\ 0 & 1-t & 1 & 0\\ 0 & t & 0 & 0\\ 0 & 0 & 0 & I_{n-i-1}\end{bmatrix}\begin{bmatrix}1\\0\\\ldots\\-1\left(i^{th}\text{ position}\right)\\\ldots\\0\end{bmatrix}=\begin{bmatrix}1\\0\\\ldots\\-1+t\left(i^{th}\text{ position}\right)\\-t\left(i+1^{th}\text{ position}\right)\\\ldots\\0\end{bmatrix}$$

$$\begin{aligned}&=\left([x_1]-[x_i]\right)+t\left([x_i]-[x_{i+1}]\right)\\&=\left([x_1]-[x_i]\right)+t\left([x_1]-[x_{i+1}]\right)-t\left([x_1]-[x_i]\right)\\&=(1-t)\left([x_1]-[x_i]\right)+t\left([x_1]-[x_{i+1}]\right)\end{aligned}$$

$$U_j\left([x_1]-[x_{i+1}]\right)=\begin{bmatrix} I_{i-1} & 0 & 0 & 0\\ 0 & 1-t & 1 & 0\\ 0 & t & 0 & 0\\ 0 & 0 & 0 & I_{n-i-1}\end{bmatrix}\begin{bmatrix}1\\0\\\ldots\\-1\left(i+1^{th}\text{ position}\right)\\\ldots\\0\end{bmatrix}=\begin{bmatrix}1\\0\\\ldots\\-1\left(i^{th}\text{ position}\right)\\\ldots\\0\end{bmatrix}$$

$$=\left([x_1]-[x_{i+1}]\right)$$

This shows that the subspace generated by the elements $\{[x_1]\,[x_2]\,,[x_1]-[x_3]\,,\ldots,[x_1]-[x_n]\}$ is invariant under the Burau representation. Thus the Burau representation is reducible. To see what this means geometrically , let us take the example the infinite cyclic cover shown in Fig 3.5.

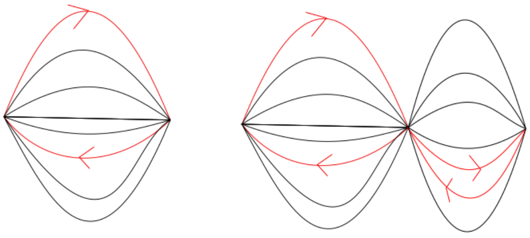


FIGURE 3.6: Action of the group MCG (S, Q) on $[x_1] - [x_i]$

We will now define what is called the reduced Burau representation. Let

$$C_n = \begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 0 & 1 & \cdots & \cdots & 1 \\ \vdots & 0 & \ddots & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Then,

$$\mathrm{C}_n^{-1} U_i C_n = \begin{bmatrix} V_i & 0 \\ 0 & 1 \end{bmatrix} = W, \text{ for } 1 \leq i \leq n-1$$

$$C_n{}^{-1} U_i C_n = \begin{bmatrix} V_i & 0 \\ i_* & 1 \end{bmatrix} = W,$$

when $i = n{-}1$ where i_* is the row with the last entry being 1 and the rest being zero

$$V_1 = \begin{bmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{bmatrix} V_{n-1} = \begin{bmatrix} I_{n-3} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & -t \end{bmatrix} V_i = \begin{bmatrix} I_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-i-2} \end{bmatrix}$$

To see that these identities are indeed true, we can verify that $U_i C_n = C_n W$

The mapping $\psi_n^r : B_n \to \mathrm{GL}_{n-1}\left(\mathbb{Z}\left[t, t^{-1}\right]\right)$ which maps σ_i to V_i is called the reduced Burau representation.

3.7 The Alexander-Conway Polynomial

The Alexander-Conway polynomial ∇, is Conway's normalization of the Alexander polynomial, it is uniquely determined by the following three properties:

- $\nabla(L)$, for some link L, is invariant under the three Reidemeister moves.
- $\nabla(\text{unknot}) = 1$
- $\nabla(L_+) - \nabla(L_-) = (s - s^{-1})\nabla(L_0)$

The third condition is called the *skein* relation. L_+ and L_- differ only at a crossing and L_0 is obtained by smoothening the crossing. L_+, L_-, L_0 together a called the Conway triple and they differ only in a small neighbourhood around the crossing.

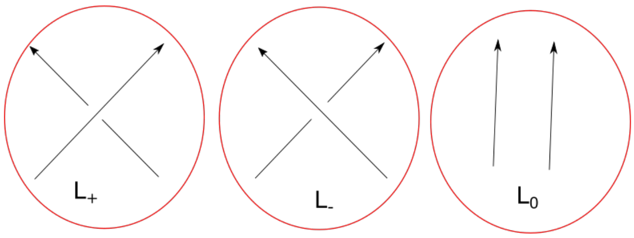


FIGURE 3.7: The Conway Triple

Theorem 3.7.1. The Alexander-Conway Polynomial is uniquely determined by the skein relation and the value on the unknot (and invariance under the Reidemeister moves).

Proof. The idea is to proceed by induction. However, note that we cannot simply induct on the crossing number, as both L_+ and L_- have the same crossing number. We induce on the complexity κ. κ of a knot diagram is defined as $\kappa = (c, d)$ where c is the crossing number and d is the unknotting number. The ordering on the complexity is the dictionary order, $(c_1, d_1) < (c_2, d_2)$ if either $c_1 < c_2$ or if $c_1 = c_2$ and $d_1 < d_2$. If we take a knot diagram with minimal crossing, then the unknotting number of L_+ and L_- differ by 1 since, we are just flipping a crossing, thus they are in a definite order. Also, L_0 has crossing number 1 less than both L_+ and L_-. Now, in Section 1.1 we have seen, it is possible to get a projection of the unknot that coincides with the diagram of any knot. The same line of reasoning would show that this is true even for the unlink of n components. Thus, by reducing the complexity of the knot diagrams, we will eventually end up with unlinks. Thus value of the unlink is completely determined by the value on the unknot. $\square$

Our aim here will to be define a Markov function that involves the Reduced Burau representation and satisfies the three aforementioned properties. We claim the following function does the job:

$$f_n(\beta) = (-2)^{(n+1)} \frac{s^{\langle\beta\rangle}\left(s - s^{-1}\right)}{s^n - s^{-n}} g\left(\det\left(\psi_n^r(\beta)\right) - I_{n-1}\right)$$

Here, $g : \mathbb{Z}\left[t, t^{-1}\right] \to \mathbb{Z}\left[s, s^{-1}\right]$ is defined as $g(t) = s^2$ and ψ_n^r is the reduced Burau representation.

Fact: $\{f_n\}$ form a Markov function. The first property is thus satisfied by as isotopic braid closures are generated only by the two Markov moves and the function f_n is invariant under the Markov moves. The second property is satisfied too, as if the braid closure is trivial, then it is a braid on a single strand , thus the second property trivially holds. To check the third property we will need to pick three braids that whose closure will form a Conway triple. Through Alexander's Theorem (Theorem 3.2.6) we know that given any knot, we can find a braid whose closure is isotopic to the knot. For any braids $\alpha, \beta \in B_n$, $\alpha\sigma_i\beta, \alpha\sigma^{-1}\beta$ and $\alpha\beta$ form a Conway triple. This can be seen from the definition itself, by keeping α and β separated by unknotted strands. We will then need to prove the following identity:

$$f_n(\alpha\sigma_i\beta) - f_n\left(\alpha\sigma^{-1}\beta\right) = \left(s - s^{-1}\right) f_n(\alpha\beta)$$

By property (ii) of Definition 3.2.8, we can multiply with α^{-1}, thus it suffices to assume $\alpha = 1$. Additionally, all the generators of the braids are conjugate to each other, meaning, we can work with σ_1. Thus the equation that we need to prove reduces to the following.

$$f_n(\sigma_1\beta) - f_n\left(\sigma_1^{-1}\beta\right) = \left(s - s^{-1}\right) f_n(\beta)$$

Using the Markov function that we have defined above, this equation simplifies to the following.

$$s^{-1} g(D_+) - s g(D_-) = \left(s^{-1} - s\right) g(D_0)$$

(3.7.2)

where $D_{\pm} = \det\left(\psi_n^r\left(\sigma_1^{\pm 1}\beta\right) - I_{n-1}\right)$ and $D_0 = \det\left(\psi_n^r(\beta) - I_{n-1}\right)$

$$\psi_n^r(\beta) = \begin{pmatrix} a & b & x \\ c & d & y \\ p & q & M \end{pmatrix}$$

Here M is a $n-3 \times n-3$ matrix, a, b, c, d are elements of the ring and p, q are columns while x, y are rows.

$$\psi_n^r(\sigma_1\beta) = \begin{pmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix} \begin{pmatrix} a & b & x \\ c & d & y \\ p & q & M \end{pmatrix} = \begin{pmatrix} -ta & -tb & -tx \\ a+c & b+d & x+y \\ p & q & M \end{pmatrix}$$

$$\psi_n^r\left(\sigma^{-1}\beta\right) = \begin{pmatrix} -t^{-1} & 0 & 0 \\ t^{-1} & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix} \begin{pmatrix} a & b & x \\ c & d & y \\ p & q & M \end{pmatrix} = \begin{pmatrix} -t^{-1}a & -t^{-1}b & -t^{-1}x \\ t^{-1}a+c & t^{-1}b+d & t^{-1}x+y \\ p & q & M \end{pmatrix}$$

Now consider the following:

$$\det\left(\psi_n^r(\beta) - I_{n-1}\right) = \det \begin{pmatrix} a-1 & b & x \\ c & d-1 & y \\ p & q & M - I_{n-3} \end{pmatrix}$$

$$\begin{aligned} \det\left(\psi_n^r(\sigma_1\beta) - I_{n-1}\right) &= -t\det \begin{pmatrix} a+t^{-1} & b & x \\ a+c & b+d-1 & x+y \\ p & q & M - I_{n-3} \end{pmatrix} \\ &= -t\det \begin{pmatrix} a+t^{-1} & b & x \\ c-t^{-1} & d-1 & x+y \\ p & q & M - I_{n-3} \end{pmatrix} \end{aligned}$$

$$\det\left(\psi_n^r\left(\sigma_1^{-1}\beta\right) - I_{n-1}\right) = -t^{-1}\det\begin{pmatrix} a+t & b & x \\ t^{-1}a+c & t^{-1}b+d & t^{-1}x+y \\ p & q & M \end{pmatrix}$$
$$= -t^{-1}\det\begin{pmatrix} a+t & b & x \\ c-1 & d-1 & t^{-1}x+y \\ p & q & M-I_{n-3} \end{pmatrix}$$

As we can see the matrices, the determinants of which are in consideration, differ only in the first column. If we denote these columns as C_0, C_+ and C_- respectively, it is easy to see that, they satisfy the following relation

$$-tC_+ + C_- = (1-t)C_0$$

Therefore, we have

$$D_+ - tD_- = (1-t)\,D_0$$

which matches with Eqn 3.7 and this completes the proof.

Chapter 4

Free Differential Calculus

This chapter is primarily taken from [BZ03], [CF08] and [LL97]. Fox Differential Calculus or Free Differential Calculus was originally introduced by R.H. Fox in [Fox53].

4.1 Introduction

This section is taken from [LL97]. One who attempts to compute the Alexander polynomial of a knot via the original definition, that is, with the use of Seifert matrices will soon realise that the computations become increasingly complicated with the increase in genus of the Seifert surface. Every additional genus introduces two new generators of the first homology and in turn increases the dimension of the Seifert matrix by two. In this section we will discuss the method of *Free Differential Calculus*, posited by RH Fox, to compute the Alexander Polynomial for a knot. As we shall see later, this method eases the computation for certain families of knots, for example, torus knots.

From covering space theory we know that if there is a covering map $p : E \to B$, then the induced map p_* on fundamental groups is an injection (this happens because if two classes of loops are homotopic to each other, then the path homotopy lifts to a path homotopy between the lifts of the two loops in E). Thus, if X denotes the knot exterior of a link L and X_∞ it's infinite cyclic cover, then $\pi_1(X_\infty)$ is isomorphic to the commutator subgroup $[\pi_1(X), \pi_1(X)]$ of $\pi_1(X)$. We have seen in previous sections, the group of deck transformations G of X_∞ acts on $H_1(X_\infty, \mathbb{Z})$. G is isomorphic to $\pi_1(X) \big/ [\pi_1(X), \pi_1(X)]$, while $H_1(X_\infty, \mathbb{Z})$ is isomorphic to G/G' where G' is the commutator subgroup of G. $\pi_1(X)$ acts on $\pi_1(X) \big/ [\pi_1(X), \pi_1(X)]$, via conjugation and this action passes on to G/G'. Thus all the information one needs for computing the Alexander polynomial of a link lies within the knot group itself.

4.2 Fox's Free Differential Calculus

This section is taken from [BZ03] and [CF08] while the original paper by RH Fox is [Fox53]. This section will be purely algebraic in nature and will introduce the notions of derivations and will be used in the next section to find a presentation of the *Alexander Module*. Let G denote a group and $\mathbb{Z}G$ denote the associated group ring, i.e., $\mathbb{Z}G = \{\sum_{\text{finite}} n_i g_i | g \in G\}$.

Definition 4.2.1. (Augmentation) The map $\epsilon : \mathbb{Z}G \to \mathbb{Z}$, $\epsilon(\sum n_i g_i) = \sum n_i$, is called the augmentation map.

Definition 4.2.2. (Derivations) A map $\zeta : \mathbb{Z}G \to \mathbb{Z}G$ is called a derivation if it satisfies the following properties:

1. (Linearity) $\zeta(\alpha + \beta) = \zeta(\alpha) + \zeta(\beta)$
2. (Product rule) $\zeta(\alpha \cdot \beta) = \zeta(\alpha) \cdot \epsilon(\beta) + \alpha \cdot \zeta(\beta)$

Lemma 4.1. Additionally, every derivation also satisfies the following properties:

1. $\zeta(1 \cdot e) = 0,$ *hence* $\zeta(n \cdot e) = 0$
2. $\zeta(g^{-1}) = -g^{-1} \cdot \zeta(g)$
3. $\zeta(g^n) = (1 + g + \cdots + g^{n-1}) \cdot \zeta(g)$

Proof. (Proof of 1) $\zeta(e) = \zeta(e) + \zeta(e) \implies \zeta(e) = 2\zeta(e) \implies \zeta(e) = 0$ Hence, $n \cdot \zeta(e) = 0$. (Proof of 2) From the proof of 1, we know that $\zeta(e) = \zeta(g \cdot g^{-1}) = 0$ Therefore, we have the following:

$$0 = \zeta(g) \cdot \epsilon\left(g^{-1}\right) + g \cdot \zeta\left(g^{-1}\right) = \zeta(g) + g \cdot \zeta\left(g^{-1}\right) \implies \zeta\left(g^{-1}\right) = -g^{-1} \cdot \zeta(g)$$

(Proof of 3) Here, we shall proceed with induction. Clearly, the statement holds when $n = 1$. Assuming it were true for $n = k - 1$,

$$\begin{aligned}
\zeta\left(g \cdot g^{k-1}\right) &= \zeta(g) \cdot \epsilon\left(g^{k-1}\right) + g \cdot \zeta\left(g^{k-1}\right) \\
&= \zeta(g) + g \cdot \zeta\left(g^{k-1}\right) \\
&= \zeta(g) + g \cdot \left(1 + g + \cdots + g^{k-2}\right) \cdot \zeta(g) \\
&= \left(1 + g + \cdots + g^{k-1}\right) \cdot \zeta(g)
\end{aligned}$$

A similar proof shows that $\zeta\left(g^{-n}\right) = -\left(g^{-1} + g^{-2} + \cdots + g^{-n}\right) \cdot \zeta(g)$ $\square$

On free groups, a derivation is uniquely determined by the images of the basis elements. We prove this in the following theorem.

Theorem 4.2.3. Let S be the free group generated by the elements S_i, where $1 \leq i \leq n$. Then a derivation $\zeta : \mathbb{Z}S \to \mathbb{Z}S$, is uniquely determined by the images of the generators $\zeta(S_i) = w_i$.

Proof. Let ζ be a derivation which maps S_i to w_i. Then ζ satisfies all the properties listed in Lemma 4.1. $\zeta(0) = 0$, where 0 is the empty word. The linearity and product rule ensure that this derivation is unique. For if there were another derivation ζ' satisfying all the aforementioned properties, then for words w_i, $\sum \zeta'(n_i w_i) = \sum n_i \zeta'(w_i)$. Observe that each word w_i is an element of the form $S_{i_1}^{n_1} S_{i_2}^{n_2} \ldots S_{i_k}^{n_k}$. $\zeta'\left(S_{i_1}^{n_1} S_{i_2}^{n_2} \ldots S_{i_k}^{n_k}\right) = \zeta'\left(S_{i_1}^{n_1}\right) + S_{i_1}^{n_1} \zeta'\left(S_{i_2}^{n_2}\right) + \cdots + S_{i_1}^{n_1} \ldots S_{i_{k-1}}^{n_{k-1}} \zeta'\left(S_{i_k}^{n_k}\right)$. Clearly, ζ and ζ' agree on the values of S_i^k for all i and k. Uniqueness is thus proved. Existence remains to shown. The proof of existence hints at how to define the function. We define $\zeta\left(S_{i_1}^{n_1} S_{i_2}^{n_2} \ldots S_{i_k}^{n_k}\right) = \zeta\left(S_{i_1}^{n_1}\right) + S_{i_1}^{n_1} \zeta\left(S_{i_2}^{n_2}\right) + \cdots + S_{i_1}^{n_1} \ldots S_{i_{k-1}}^{n_{k-1}} \zeta\left(S_{i_k}^{n_k}\right)$. All that remains to be checked is that this mapping is well-defined. It suffices if for words u, v we show that $\zeta(uv) = \zeta\left(u S_i^m S_i^{-m} v\right)$.

$$\begin{aligned}
\zeta(uv) = \zeta(u) + u\zeta(v)\, \zeta\left(u S_i^m S_i^{-m} v\right) \\
&= \zeta(u) + u\zeta\left(S_i^m\right) + u S_i^m \zeta\left(S_i^{-m}\right) + u S_i^m S_i^{-m} \zeta(v) \\
&= \zeta(u) + u\left(\zeta\left(S_i^m\right) + S_i^m \zeta\left(S_i^{-m}\right)\right) + u\zeta(v) \\
&= \zeta(u) + u\left(\zeta\left(S_i^m S_i^{-m}\right)\right) + u\zeta(v) \\
&= \zeta(u) + u\zeta(v).
\end{aligned}$$

This completes the proof. $\square$

Definition 4.2.4. (**Partial Derivations**) The maps $\frac{\partial}{\partial S_i} : \mathbb{Z}S \to \mathbb{Z}S$ defined as $\frac{\partial}{\partial S_i}(S_j) = \delta_{ij}$ and extended linearly are called partial derivations.

We observe that, $\frac{\partial\left(S_{i_1}S_{i_2}\cdots S_{i_k}\right)}{\partial S_{i_j}} = S_{i_1}S_{i_2}\cdots S_{i_{j-1}}$. We can use this observation to conclude the following result.

Lemma 4.2.5. For any derivation ζ, $\zeta = \sum_{i\in J} \frac{\partial}{\partial S_i}\zeta(S_i)$

Proof.

$$\begin{aligned}
\zeta\left(S_{i_1}^{n_1}S_{i_2}^{n_2}\dots S_{i_k}^{n_k}\right) &= \left(\sum_{i\in J}\frac{\partial}{\partial S_i}\cdot\zeta(S_i)\right)\left(S_{i_1}^{n_1}S_{i_2}^{n_2}\dots S_{i_k}^{n_k}\right)\\
&= \left(\sum_{i\in J}\frac{\partial\left(S_{i_1}^{n_1}S_{i_2}^{n_2}\dots S_{i_k}^{n_k}\right)}{\partial S_i}\zeta(S_i)\right)\\
&= \left(S_{i_1}^{n_1}S_{i_2}^{n_2}\dots S_{i_{j-1}}^{n_{j-1}}\frac{\partial S_{i_j}^{n_j}}{\partial S_j}\zeta\left(S_{i_j}\right)\right)\\
&= \left(S_{i_1}^{n_1}S_{i_2}^{n_2}\dots S_{i_{j-1}}^{n_{j-1}}\left(1+S_{i_j}+S_{i_j}^2+\dots+S_{i_j}^{n_j-1}\right)\zeta\left(S_{i_j}\right)\right)\\
&= \left(S_{i_1}^{n_1}S_{i_2}^{n_2}\dots S_{i_{j-1}}^{n_{j-1}}\zeta\left(S_{i_j}^{n_j}\right)\right)
\end{aligned}$$

□

These partial derivations will be used in finding a representation for the Alexander Module of a link.

4.3 Regular Coverings and the Alexander Polynomial

This section is taken from [BZ03] and [LL97]. Given a topological space X with fundamental group G and a regular covering $\tilde{X}$ with a presentation $G = \{S_1, S_2, \dots, S_n | R_1, \dots, R_m\}$, where S_is are the generators and R_js are the relations. Let $p : \tilde{X} \to X$ be the covering map. We can construct a CW-complex with the same fundamental group and hence the same homological properties. For our purposes we consider a complex with one 0-cell V, n 1-cells attached to the 0-cell, one for each S_i and m 2-cells that are attached according to the relations R_j. These 1-cells and 2-cells are also denoted as S_i and R_j. The complex and it's regular covering, too, are denoted as X and $\tilde{X}$ respectively. Since, $\tilde{X}$ is a regular covering, therefore $p_*\left(\pi_1\left(\tilde{X}\right)\right) \lhd G$. Then the group of deck transformations D, is isomorphic to ${}^{G}/_{N}$, where $N = \pi_1\left(\tilde{X}\right)$. Let $\phi : G \to D$ be the canonical homomorphism and let $\tilde{X^0}$ denote the 0-cell complex of $\tilde{X}$. If $\tilde{V}$ is a fixed lift of the V, then $\tilde{X^0}$ is the set of points $D \cdot \tilde{V}$; by D here we refer to the group of deck transformations. Our aim here is to present $H_1\left(\tilde{X}; \tilde{X^0}, \mathbb{Z}\right)$ as a $\mathbb{Z}D$ module. Using this we will be able to find a presentation of the Alexander Module of a knot.

If w is a loop in X, then let $\tilde{w}$ be the lift of w beginning at $\tilde{V}$. $\tilde{w}$ is a special element in $H_1\left(\tilde{X}; \tilde{X^0}, \mathbb{Z}\right)$ in the sense that in the 1-skeleton of $\tilde{X}$, with $D \cdot \tilde{V}$ being vertices and $D \cdot \tilde{S_i}$ being edges, it would represent a connected graph. Conversely any finite connected sub-graph would represent an element of $H_1\left(\tilde{X}; \tilde{X^0}, \mathbb{Z}\right)$ that would project under the covering map to a loop w in X. A shorthand for the map

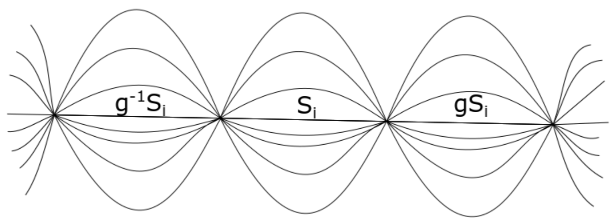


FIGURE 4.1: An example of a regular covering

$\psi \circ \phi : \mathbb{Z}S \to \mathbb{Z}D$ is $(x)^{\psi\phi}$.

Fig 4.1 is an example of a regular covering (in fact, a cyclic covering). Any chain $\tilde{w}$ in $Z\left(\tilde{X}, \tilde{X^0}\right)$, can be written as a linear combination of the lifts $\tilde{S_i}$ of the generators of $H_1(X, \mathbb{Z})$, S_i, i.e., $\tilde{w} = \sum a_i \tilde{S_i}$. If w_1 and w_2 are two loops starting at V, let us try and see what the lift of the concatenated loop $w_1 w_2$ looks like algebraically. The lift $\tilde{w_1}$ of w_1 beginning at V, ends at $\phi(w_1) \cdot V$. Therefore, the lift of w_2 would begin from $\phi(w_1) \cdot \widetilde{V}$ and end at $\phi(w_2) \cdot \left(\phi(w_1) \cdot \widetilde{V}\right)$. Thus the chain $\widetilde{w_1 w_2}$ can be written as $\widetilde{w_1} + \phi(\widetilde{w_1}) \cdot \widetilde{w_2}$. At this point, we have the following sequence of maps:

$$\mathbb{Z}S \xrightarrow{\phi} \mathbb{Z}G \xrightarrow{\psi} \mathbb{Z}D$$

.The maps being the natural projection maps. We recall the maps used in Definition 4.2.4, $\frac{\partial}{\partial S_i} : \mathbb{Z}S \to \mathbb{Z}S$. These maps can be used to reformulate how a chain is written. $\widetilde{w} = \sum a_i \widetilde{S}_i$, where $a_i \in \mathbb{Z}D$. To understand what these a_is are let us consider an explicit example.

Example 4.3.1. Consider the wedge of three circles, we can denote the generators by S_1, S_2, S_3. Let the intersection point of the three circles be P. We consider the loop $S_1S_2S_3S_1$, the lift of this loop in a cyclic cover of the space beginning at $\tilde{P}$, is given by

$$\begin{aligned}
&= \widetilde{S_1} + (\psi \circ \phi)(S_1) \cdot \widetilde{S_2} + (\psi \circ \phi)(S_1S_2) \cdot \widetilde{S_3} + (\psi \circ \phi)(S_1S_2S_3) \cdot \widetilde{S_1} \\
&= (1 + (\psi \circ \phi)(S_1S_2S_3)) \cdot \widetilde{S_1} + (\psi \circ \phi)(S_1) \cdot \widetilde{S_2} + (\psi \circ \phi)(S_1S_2) \cdot \widetilde{S_3}
\end{aligned}$$

$\left(\frac{\partial S_1S_2S_3S_1}{\partial S_1}\right)^{\psi\phi} = (1 + (\psi \circ \phi)(S_1S_2S_3))$, $\left(\frac{\partial S_1S_2S_3S_1}{\partial S_2}\right)^{\psi\phi} = (\psi \circ \phi)(S_1)$,
$\left(\frac{\partial S_1S_2S_3S_1}{\partial S_2}\right)^{\psi\phi} = (\psi \circ \phi)(S_1S_2)$

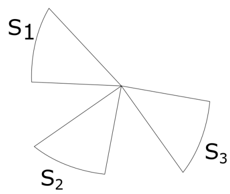


FIGURE 4.2: Example 4.3.1

We have expressed $H_1\left(\tilde{X}; \tilde{X}^0, \mathbb{Z}\right)$ as a $\mathbb{Z}D$ module.
$H_1\left(\tilde{X}; \tilde{X}^0, \mathbb{Z}\right) = \{\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_n | \tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_m\}$, if $w \in H_1\left(\tilde{X}; \tilde{X}^0, \mathbb{Z}\right)$ then $w = \sum \left(\frac{\partial w}{\partial S_i}\right)^{\psi\phi} \tilde{S}_i$. To find a presentation matrix for $H_1\left(\tilde{X}; \tilde{X}^0, \mathbb{Z}\right)$, we observe the following exact sequence:

$$\mathbb{Z}\tilde{R} \hookrightarrow \mathbb{Z}\tilde{S} \rightarrow H_1\left(\tilde{X}; \tilde{X}^0, \mathbb{Z}\right) \rightarrow 0$$

Here $\mathbb{Z}\tilde{R}$ denotes the group ring generated by the elements $\{\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_m\}$. The first map is the natural inclusion map. The second map is the map $\phi : \mathbb{Z}S \rightarrow \mathbb{Z}G$ composed with the lifting map, sending a loop in $\pi_1(X)$ to a cycle in $H_1\left(\tilde{X}, \tilde{X}^0, \mathbb{Z}\right)$. Now, $i\left(\tilde{R}_j\right) = \sum \left(\frac{\partial R_j}{\partial S_i}\right)^{\psi\phi} \tilde{S}_i$. This provides us with the entries of a presentation matrix. The ij^{th} entry of the matrix is $\left(\frac{\partial R_j}{\partial S_i}\right)^{\psi\phi}$.

4.3.1 Recovering the Alexander Module

We have thus far obtained a presentation for $H_1\left(\tilde{X}; \tilde{X}^0, \mathbb{Z}\right)$, to find the Alexander Polynomial, we need a presentation matrix for the $\mathbb{Z}D$ module $H_1\left(\tilde{X}, \mathbb{Z}\right)$.

Our problem would be simplified if the latter were a direct summand of the former. For if P were a presentation matrix of a R module M, then a presentation matrix for $M \bigoplus R$ would be P with an additional row of zeroes, since an extra generator would be added. The non-zero elementary ideals would remain invariant. The slight wrinkle here is that the r^{th} elementary ideals of M coincide with the $r+1^{th}$ elementary ideals of $M \bigoplus R$. As it turns out, in case of knot groups, $H_1\left(\tilde{X}; \tilde{X^0}, \mathbb{Z}\right)$ indeed splits with $H_1\left(\tilde{X}, \mathbb{Z}\right)$as a direct summand. Let t be a generator of the group of deck transformations of $H_1\left(\tilde{X}; \tilde{X^0}, \mathbb{Z}\right)$ To see this, let us look at the the homology sequence:

$$H_1\left(\tilde{X^0}, \mathbb{Z}\right) \xrightarrow{i_*} H_1\left(\tilde{X}, \mathbb{Z}\right) \to H_1\left(\tilde{X}, \tilde{X^0}, \mathbb{Z}\right) \xrightarrow{\partial_*} H_0\left(\tilde{X^0}, \mathbb{Z}\right) \xrightarrow{i_*} H_0\left(\tilde{X}, \mathbb{Z}\right) \to 0$$

There are no 1-cycles in $H_1\left(\tilde{X^0}, \mathbb{Z}\right)$, hence $H_1\left(\tilde{X^0}, \mathbb{Z}\right) = 0$. $H_0\left(\tilde{X^0}, \mathbb{Z}\right)$ is the space generated by all the translates of $\tilde{V}$. Hence $H_0\left(\tilde{X^0}, \mathbb{Z}\right) \simeq \mathbb{Z}\left[t, t^{-1}\right]$. Furthermore, since $\tilde{X}$ is connected thus $H_1\left(\tilde{X}, \mathbb{Z}\right) \simeq \mathbb{Z}$. The long exact sequence, thus, reduces to the following:

$$0 \xrightarrow{i_*} H_1\left(\tilde{X}, \mathbb{Z}\right) \to H_1\left(\tilde{X}, \tilde{X^0}, \mathbb{Z}\right) \xrightarrow{\partial_*} \mathbb{Z}\left[t, t^{-1}\right] \xrightarrow{i_*} \mathbb{Z} \to 0$$

We can deduce a short exact sequence from this.

$$0 \xrightarrow{i_*} H_1\left(\tilde{X}, \mathbb{Z}\right) \to H_1\left(\tilde{X}, \tilde{X^0}, \mathbb{Z}\right) \xrightarrow{\partial_*} \text{Ker } i_* \xrightarrow{i_*} 0$$

If $\tilde{w} \in H_1\left(\tilde{X}, \tilde{X^0}, \mathbb{Z}\right)$, then $\partial \tilde{w} = \partial\left(\sum \left(\frac{\partial w}{\partial S_i}\right)^{\psi\phi} \tilde{S}_i\right) = \left(\sum \left(\frac{\partial w}{\partial S_i}\right)^{\psi\phi} \partial \tilde{S}_i\right)$

$= \left(\sum \left(\frac{\partial w}{\partial S_i}\right)^{\psi\phi} \left(S_i^{\psi\phi} - 1\right) \cdot \tilde{V}\right)$

Ker i_* is thus generated by $\left(S_j^{\psi\phi} - 1\right)\tilde{V}$, $1 \leq j \leq n$. In case of the knot group, this becomes $(t-1)\tilde{V}$. Ker i_* is then the free $\mathbb{Z}\left[t, t^{-1}\right]$module generated by $(t-1)\tilde{V}$. To prove that $H_1\left(\tilde{X}, \tilde{X^0}, \mathbb{Z}\right) = H_1\left(\tilde{X}, \mathbb{Z}\right) \bigoplus \mathbb{Z}\left[t, t^{-1}\right](t-1) \cdot \tilde{V}$, we need to show the existence of a splitting map. This is easy, defining a map α which maps $(t-1) \cdot \tilde{V}$ to any of the $\tilde{S}_j$ does the job, since, $\partial \alpha (t-1) \cdot \tilde{V} = (t-1) \cdot \tilde{V}$.

The Alexander's Polynomial for any knot can now be obtained by deleting any column from the presentation matrix and computing the 1^{st} elementary ideal.

Remark 4.3.2. This proof goes through because the group of deck transformation is infinite-cyclic, this does not necessarily hold otherwise.

4.4 Computing the Alexander Polynomial for certain knots

The examples discussed in this section can be found in [LL97] and [BZ03]. As we had mentioned previously, this method of calculating the Alexander Polynomial eases computations.

The Trefoil Knot The simplest knot is the trefoil knot, A Wirtinger presentation of the trefoil knot is $\{x_1, x_2, x_3 | R_1 = x_1 x_2 x_3^{-1} x_2^{-1}, R_2 = x_2 x_3 x_1^{-1} x_3^{-1}, R_3 = x_3 x_1 x_2^{-1} x_1^{-1}\}$. We now compute the Jacobian $\left(\frac{\partial R_i}{\partial x_j}\right)$, the map $\psi\phi$ maps each S_i to the generator t of the group of deck transformations. Lets compute each of the entries:

$$\begin{array}{lll}
\dfrac{\partial R_1}{\partial x_1} = 1 & \dfrac{\partial R_1}{\partial x_2} = x_1 - x_1 x_2 x_3^{-1} x_2^{-1} & \dfrac{\partial R_1}{\partial x_3} = -x_1 x_2 x_3^{-1} \\
\dfrac{\partial R_2}{\partial x_1} = -x_2 x_3 x_1^{-1} & \dfrac{\partial R_2}{\partial x_2} = 1 & \dfrac{\partial R_2}{\partial x_3} = x_2 - x_2 x_3 x_1^{-1} x_3^{-1} \\
\dfrac{\partial R_3}{\partial x_1} = x_3 - x_1 x_2 x_3^{-1} x_2^{-1} & \dfrac{\partial R_3}{\partial x_2} = -x_3 x_1 x_2^{-1} & \dfrac{\partial R_3}{\partial x_3} = 1
\end{array}$$

The corresponding Jacobian matrix is as follows:

$$\begin{bmatrix} 1 & t-1 & -t \\ -t & 1 & t-1 \\ t-1 & -t & 1 \end{bmatrix}$$

The Alexander Polynomial $\Delta(t)$ can be computed by checking the 2×2 minors, which in this case is $t^2 - t + 1$, this is unique upto multiplication by a unit.

Torus Knots Torus knots comprise of the family of knots that occur as simple closed curves on a torus. The fundamental group of a torus is isomorphic to $\mathbb{Z} \bigoplus \mathbb{Z}$. If (a, b) denotes the class of a knot in the fundamental group of the torus, then a and b are necessarily coprime. The converse that there exists a knot which is a representative of (a, b) , where a and b are coprime is also true. Informally speaking, such a knot would wrap around the meridian a times and b times around the longitude. Van-Kampen's theorem tells us that a representation of a torus knot that represents the class (a, b) is $G = \{x, y | R = x^a y^{-b}\}$. Free Differential Calculus yields a very efficient method of computing torus owing to the fact that there are very few generators and relations. The Abelianization map $\pi : G \to G' =< t >$ is defined as $x \mapsto t^b$, $y \mapsto t^a$. The Jacobian is a 1×2 matrix.

$$\left(\frac{\partial R}{\partial x}\right)^{\psi\phi} = \left(1 + x + \ldots x^{a-1}\right)^{\psi\phi}$$

$$= \left(\frac{\partial R}{\partial x}\right)^{\psi\phi} = \left(1 + t^b + \ldots t^{b(a-1)}\right) = \frac{1 - t^{ab}}{1 - t^b}$$

$$\left(\frac{\partial R}{\partial y}\right)^{\psi\phi} = -x^a \left(y^{-1} + y^{-2} \ldots y^{-b}\right)^{\psi\phi}$$

$$= \left(\frac{\partial R}{\partial y}\right)^{\psi\phi} = -t^{ab} \left(t^{-1} + t^{-2} \ldots t^{a(-b)}\right) = \frac{-t^{ab} t^{-a} \left(1 - t^{ab}\right)}{1 - t^a}$$

$$J_{(a,b)} = \begin{bmatrix} \dfrac{1 - t^{ab}}{1 - t^b} & \dfrac{-t^{ab} t^{-a} \left(1 - t^{ab}\right)}{1 - t^a} \end{bmatrix}$$

The Alexander Polynomial is a generator of the smallest principal ideal, generated by the 1×1 minors. The smallest principal ideal would be generated by the $\gcd\left(\dfrac{1 - t^{ab}}{1 - t^b}, \dfrac{1 - t^{ab}}{1 - t^a}\right)$. We see that each of the terms is divisble by $\dfrac{1 - t^{ab}}{(1 - t^b)(1 - t^a)}$, with quotients of the division being $(1 - t^a)$ and $\left(1 - t^b\right)$ respectively. Each of these two polynomials are products of an irreducible polynomial with $1 - t$. Thus $\left(\dfrac{1 - t^{ab}}{1 - t^b}, \dfrac{1 - t^{ab}}{1 - t^a}\right) = \left(\dfrac{\left(1 - t^{ab}\right)(1 - t)}{(1 - t^a)\left(1 - t^b\right)}\right)$. Hence, the Alexander Polynomial of torus knots are given by $\Delta_{a,b}(t) = \left(\dfrac{\left(1 - t^{ab}\right)(1 - t)}{(1 - t^a)\left(1 - t^b\right)}\right)$.

Bibliography

[Ale28] James W Alexander. Topological invariants of knots and links. *Transactions of the American Mathematical Society*, 30(2):275–306, 1928.

[BZ03] G. Burde and H. Zieschang. *Knots*. De Gruyter studies in mathematics. Walter de Gruyter, 2003.

[CF08] R.H. Crowell and R.H. Fox. *Introduction to Knot Theory*. Dover books on mathematics. Dover Publications, 2008.

[Fox53] Ralph H Fox. Free differential calculus. i: Derivation in the free group ring. *Annals of Mathematics*, pages 547–560, 1953.

[HPoM02] A. Hatcher, Cambridge University Press, and Cornell University. Dept. of Mathematics. *Algebraic Topology*. Algebraic Topology. Cambridge University Press, 2002.

[Kau87] L.H. Kauffman. *On Knots*. Annals of mathematics studies. Princeton University Press, 1987.

[KL17] Louis H. Kauffman and Pedro Lopes. Colorings beyond fox: the other linear alexander quandles, 2017.

[KT08] Christian Kassel and Vladimir Turaev. *Braid groups*, volume 247. Springer Science & Business Media, 2008.

[LL97] W.B.R. Lickorish and W.B. Lickorish. *An Introduction to Knot Theory*. Graduate Texts in Mathematics. Springer New York, 1997.

[Mun00] J.R. Munkres. *Topology*. Featured Titles for Topology. Prentice Hall, Incorporated, 2000.

[Mun18] J.R. Munkres. *Elements Of Algebraic Topology*. CRC Press, 2018.

[Rob15] Justin Roberts. Knots knotes. 2015.

[Rol75] Dale Rolfsen. A surgical view of alexanderâs polynomial. In *Geometric topology*, pages 415–423. Springer, 1975.

[Tei10] Peter Teichner. Slice knots: Knot theory in the 4th dimension. 2010.