Resolution of Curves and Surfaces

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Certificate of Examination

This is to certify that the dissertation titled "RESOLUTION OF CURVES AND SURFACES" submitted by Ms. Shikha Bhutani (Reg. No. MP17011) for the partial fulfillment of MS degree program of the Institute, has been examined by the thesis committee duly appointed by the Institute. The committee finds the work done by the candidate satisfactory and recommends that the report be accepted.

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Declaration

The work presented in this dissertation has been carried out by me under the guidance of Dr. Vaibhav Vaish at the Indian Institute of Science Education and Research Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgment of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

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In my capacity as the supervisor of the candidate's project work, I certify that the above statements by the candidate are true to the best of my knowledge.

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Chapter 0

Basic notions

This chapter is a quick recap of some notions of basic algebraic geometry. It is not a complete guide for the same. One may look at other references, such as [3] and [4]. In the later chapters, we will concentrate only on resolving singularities of projective varieties, generalizing it to other algebraic structures will obscure the general picture. Still, one can look for analogue techniques there.

In section one, we give basic definitions of the object of study. Section two is about the evolution of the notion of singular points on a variety. Finally, we will talk about divisors and then intersection theory. Throughout the text, intersection theory will play an essential role in measuring the improvement of local invariant after suitable transformation. For example, if $C \subset S$ is a curve and K_S is the canonical divisor on the surface S, then it is the intersection number $C \cdot (C + K_S)$ that decreases in subsequent blow-ups. We show that it is bounded below, and the sequence of transformations terminates after that.

0.1 Geometric prerequisites

0.1.1 Category of varieties

Objects

For the following, k will denote an algebraically closed field of any characteristic.

Definition 0.1.1 (Affine n-space)**.** $\mathbb{A}_k^n := k \times k \times ... \times k$

Definition 0.1.2 (Algebraic Sets)**.** Given $S \subset k[x_1, x_2, ..., x_n]$, we define

$$V(S) := \{ \text{ common zeroes of } f \in S\}$$

to be the *algebraic set* on $\mathbb{A}^n$.

An algebraic set defined by the zeroes of a single polynomial is called a *hypersurface*.

Remark. It is easy to verify that common zeroes of a collection of polynomials are same as the common zeroes of the ideal generated by the same(Hilbert basis theorem). Therefore, we can redefine the algebraic set to be zeroes of a finite number of polynomials.

Definition 0.1.3 (Zariski topology on affine space)**.** We give a topology on $\mathbb{A}^n$ by defining algebraic sets to be the closed sets. The topology is called Zariski topology.

Definition 0.1.4 (Affine variety)**.** We call a closed set X, reducible if there exist proper closed subsets X_1 and X_2 of X such that $X = X_1 \cup X_2$. Otherwise, X is said to be irreducible. An irreducible algebraic set is defined to be an *affine variety.*

Definition 0.1.5 (Projective n space)**.** *The projective n-space* is by definition the set

$$\mathbb{P}_k^n := \left(\mathbb{A}_k^{n+1} - (0, 0, ..., 0)\right) / \sim$$

where $' \sim'$ is defined as follows:
$(a_0, a_1, ..., a_n) \sim (b_0, b_1, ..., b_n)$ if there exists $\lambda \in k^*$ such that $a_i = \lambda b_i$ for all $i's$. We denote the equivalence class by $(a_0 : a_1 : ... : a_n)$.

Remark. Similar to the notion of algebraic sets in affine space, we can define algebraic sets in projective space to be the zeroes of homogenous polynomials (modulo the equivalence relation). As before, these sets induce a topology on the projective space called the Zariski topology on $\mathbb{P}^n$.

Definition 0.1.6 (Projective variety)**.** An irreducible algebraic subset of projective space is said to be a *projective variety.*

$\mathbb{P}_k^n$ possess an affine open cover given by $\mathbb{P}_k^n = \bigcup_{i=0}^n U_i$ where $U_i = V(x_i)^c$ for $i = 0, ..., n$. It is not hard to see that if $V \subset \mathbb{P}_k^n$ is a projective variety then $V \cap U_i$ is an affine variety.

Definition 0.1.7 (Quasi projective variety)**.** We define *quasi projective varieties* to be the open subsets of closed projective variety.

Remark. The idea of quasi projective variety is to generalize the notion of affine and projective variety. There are beautiful examples of quasi projective varieties that are neither projective nor affine.

Morphisms

Definition 0.1.8. (i) Let $V \subset \mathbb{A}_k^n$ be an affine variety. Then the function $\varphi : V \to \mathbb{A}_k^1$ said to be is a regular function on V if there exists an $f \in k[x_1, ..., x_n]$ such that $f|_V = \varphi$ on V.

(ii) Let $V \subset \mathbb{A}_k^n$ and $W \subset \mathbb{A}_k^m$ be affine varieties. Then the map $\varphi : V \to W$ is said to be a regular map on V if $p_i \circ \varphi$ is a regular function for each projection $p_i : \mathbb{A}_k^m \to \mathbb{A}_k^1$.

(iii) Let $Y \subset \mathbb{P}_k^m$ be a projective variety. Then the map $\varphi : V \to Y$ is said to be a regular map on Y if $i \circ \varphi = (f_0 : f_1 : ... : f_m)$ where f_i are homogenous polynomial in n variables and $i : Y \to \mathbb{P}_k^m$ is the inclusion map.

(iv) Let $X \subset \mathbb{P}_k^n$ be another projective variety. Then the map $\varphi : X \to Y$ is said to be a regular map on X if $\varphi^i = \varphi|_{X \cap U_i} : X \cap U_i \to Y$ is regular a regular map on $X \cap U_i$ for each $i = 0, ..., n$.

Theorem 0.1.9. *Suppose $X \subset \mathbb{P}^n$ is a quasi projective variety. Then for every point $x \in X$, there exists a neighborhood U_x that is isomorphic to an affine variety.*

Proof. Let $\bar{X}$ denote the projective closure of X in $\mathbb{P}^n$. Then X is an open subset of $\bar{X}$ and $X \cap U_i$ is an open subset of the affine variety $\bar{X} \cap U_i = X_i$. Let $X \cap U_i = X_i / Y_i$ for some closed set Y_i and $x \in X \cap U_1$ be a point. Then there exists an $f \in I(Y_1)$ with $f(x) \neq 0$. We define $D(f) := Y_1 - V(f)$ and claim that the neighborhood $D(f)$ of the point x is isomorphic to an affine variety. Indeed, it is the zero set of the following equations,

$$g_1(x_1, ..., x_m) = ... = g_l(x_1, ..., x_m) = 0 = y.f(x_1, ..., x_m) - 1$$

inside $k[x_1, x_2, ..., x_m, y]$ where $g_i's$ are the defining equations of X_1. $\square$

Definition 0.1.10. Let X and Y be quasi projective varieties. Then the map $\varphi : X \to Y$ is said to be *regular* at the point $x \in X$ if there exists an affine neighborhood U_x of x such that φ is regular on U_x. The varieties X and Y are said to be isomorphic if there are regular maps $\phi : X \to Y$ and $\varphi : Y \to X$ such that $\phi \circ \varphi = Id|_Y$ and $\varphi \circ \phi = Id|_X$.

Remark. Let X and Y be affine varieties. Then the regular map $\phi : X \to Y$ induces a ring map

$$\phi^* : k[Y] \to k[X]$$

$$f \mapsto f \circ \phi.$$

It is easy to verify that affine varieties are isomorphic if and only if the induced map between the coordinate rings is an isomorphism.

Rational maps

In high school calculus, we have always been interested in local pictures. For example, the notions of limits, continuity, and differentiability at a point are ineffective of the neighborhood chosen. On the other hand, regular functions on projective space $\mathbb{P}^n_k$ is just k, "the constants." Hence, we want to define the maps that are regular on an open subset of the variety if not on the whole of variety.

Let $V \subset \mathbb{A}^n$ be an algebraic set. We define,

$$k[V] := \{ \text{ regular functions on } V\} = \{[f]; f \in k[x_1, x_2, ..., x_n]\}$$

where $[f]$ denotes an equivalence class defined by the relation

$$[f] \sim [g] \text{ if and only if } f - g \text{ vanishes on } V.$$

We see that $k[V] \simeq k[x_1, x_2, ..., x_n]/I$ where I is the ideal of V. Let V be a variety, then $k[V]$ is an integral domain. We define

$$k(V) := \text{Quotient field of } k[V]$$

and call the elements in $k(V)$ to be rational functions.

Definition 0.1.11 (Rational maps between affine varieties)**.** Let $V \subset \mathbb{A}_k^n$ and $W \subset \mathbb{A}_k^m$ be two affine varieties. Then the map $\varphi : V \to W$ is said to be rational if $p_i \circ \varphi$ is a rational function for each projection function $p_i : W \to \mathbb{A}^1$.

Definition 0.1.12. We call a map $\phi : X \to Y$ to be *birational*, if ϕ is rational and there exists a rational map $\xi : Y \to X$ such that $\phi \circ \xi|_U = Id$ and $\xi \circ \phi|_V = Id$ for some open sets $U \subset Y$ and $V \subset X$.

Remark. The map $\phi : X \to Y$ is birational if and only if the induced map between the function fields is an isomorphism.

0.1.2 Category of schemes

To each point $(a_1, ..., a_n)$ in the affine space $\mathbb{A}^n$, we can associate a unique maximal ideal $(x_1 - a_1, x_2 - a_2, ..., x_n - a_n) \subset k[x_1, x_2, ..., x_n]$. Similarly, for $a = (a_1, ..., a_n) \in V(I)$ there exists a unique maximal ideal $(x_1 - a_1, x_2 - a_2, ..., x_n - a_n) + I$ in the ring $k[x_1, x_2, ..., x_n]/I$. There is no harm in defining an affine variety to be the MaxSpec of its coordinate ring. However, MaxSpec of a ring is not functorial, since the preimage of a maximal ideal may not be maximal, and hence, we work with prime ideals of a ring.

Objects

Definition 0.1.13 (Presheaf of Rings)**.** Let X be a topological space. We define the category $Top\,(X)$ as follows:
Objects: Open subsets of X,
Morphisms: $Hom\,(U, V) = \begin{cases} \text{inclusion if } U \subset V \text{ and} \\ \emptyset \text{ otherwise .} \end{cases}$
Then *presheaf of rings* is defined to be a contravariant functor $\mathfrak{F}$ from $Top\,(X)$ to the category *Rng* of commutative rings.

Definition 0.1.14 (Sheaf of rings)**.** We define a *sheaf* $\mathfrak{F}$ on a topological space X to be the presheaf that satisfies the following two additional properties:

(i) Let U be an open subset of X and $\{V_i\}$ be an open cover of U. If $f \in \mathfrak{F}(U)$ maps to zero under inclusion maps for all i, then f is zero.

(ii) Let $f_i \in \mathfrak{F}(V_i)$ be such that image of f_i under the inclusion maps to $\mathfrak{F}(V_i \cap V_j)$ is same for all i, j, then there exists an $f \in \mathfrak{F}(U)$ such that f restricted to V_i is f_i.

Example 1 (Affine variety with Zariski topology)**.** Let X be an affine variety. We have seen that X is a topological space with Zariski topology. Then for each open set U in X, we associate the subring of rational functions that consists of functions regular on U. It is easy to verify that this gives a sheaf structure on X.

Definition 0.1.15 (Spectrum of a ring)**.** Let R be a ring. Consider the topology on the set of its prime ideals $\operatorname{Spec} R$ defined by setting

$$V(I) = \{P \in \operatorname{Spec} R; I \subset P\}$$

to be the closed sets. Then, to this topological space, we define a sheaf as follows:

Let $U \subset \operatorname{Spec} R$ be an open subset. We define $\mathcal{O}_U$ to be the set of functions $\{s : U \to \amalg_{P \in U} R_P\}$ such that $s(P) \in R_P$ and s is locally a quotient of elements in R where R_P is the localization of the ring R at P (prime ideal). *Spectrum of a Ring* is defined to be the pair $(X = \operatorname{Spec} R, \mathcal{O}_X)$ where $\operatorname{Spec} R$ is the topological space and $\mathcal{O}_X$ is the sheaf that we have just defined.

Definition 0.1.16 (Scheme)**.** Let X be a topological space and $\mathcal{O}_X$ be a sheaf on X. A *scheme* is the pair $(X, \mathcal{O}_X)$ such that every point has a neighborhood U that is isomorphic to $\operatorname{Spec} R$ for some ring R.

Morphisms

Morphisms between two sheaves on a topological space are natural transformations.

Definition 0.1.17. A morphism of schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is the pair $(f, f\sharp)$ of a continuous map $f : X \to Y$ and a map of sheaf of rings $f\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ where $f_*\mathcal{O}_X$ is the sheaf defined on Y as, to each open set U in Y assign the ring $\mathcal{O}_X(f^{-1}(U))$ to it.

0.1.3 Local ring

The first local invariant of a point x of a variety is the local ring $\mathcal{O}_x$. It contains all the functions that are regular in some neighborhood of x. Let X be a variety. Then

$\mathcal{O}_x$ is the subring function field $k(X)$ that consists of the functions $f/g \in k(X)$ that are regular at x i.e. $f, g \in k[X]$ and $g(x) \neq 0$

Definition 0.1.18 (Local ring at a point)**.** Suppose X is a variety, $x \in X$ and $m_x \subset k[X]$ is the ideal defined as follows:

$$m_x := \{f \in k[X] \mid f(x) = 0\}.$$

Local ring at the point $x \in X$ is defined to be the localization of $k[X]$ at m_x.

0.1.4 Dimensions

Let V be a topological space. We define the dimension of V as

$$\dim V := max \{i \mid C_0 \subsetneq C_1 \subsetneq ... \subsetneq C_i; C_i \text{ are irreducible closed subsets of V }\}.$$

But this definition may not always be feasible to work with. The definition that is generally made is as follows:

Definition 0.1.19. Let V be an affine variety. We define the dimension of V to be the transcendental degree of the function field $k(V)$. For a quasi projective variety X, dimension of X is defined to be the dimension of an affine open subset $U \subset X$.

The following theorem gives an equivalence of the two definitions.

Theorem 0.1.20. *The Krull dimension of an integral domain is the same as the transcendental degree of its function field.*

Proof. For proof, refer [4] □

Remark. Let $X = X_1 \cup X_2 \cup ... \cup X_n$ be the decomposition of X into irreducible components. Then we define the dimension of X to be the maximum of the dimensions of $X_i's$.

Theorem 0.1.21. *Let X, Y be algebraic sets. If $Y \subset X$, then $\dim Y \leq \dim X$. Further, if X is irreducible and $\dim Y = \dim X$ then $X = Y$.*

Proof. We assume X and Y to be irreducible. Let $t_1, ..., t_l$ be rational functions on X. Suppose they are algebraically dependent. Then there exist a polynomial g such that $g(t_1, t_2, ..., t_l) = 0$ on X. But $g = 0$ on X implies $g = 0$ on Y. Therefore,

$t_1, t_2, ..., t_l$ are algebraically dependent on Y and $\dim Y \leq \dim X$. Now suppose, $Y \subset X$ and $\dim X = \dim Y = n$, we show that $k[X] = k[Y]$. Let $f \in k[X]$ be a non-zero element and $t_1, ..., t_n$ be the transcendental basis for $k(X)$. We can assume, without any loss of generality, that $t_i \in k[X]$ for all i. Then there exists a polynomial $p(x)$ such that $p(f) = f^k + f^{k-1}a_{k-1}(t_1, ..., t_n) + a_0(t_1, ..., t_n) = 0$ where k is the smallest such natural number. Now, if $f = 0$ on Y the $a_0(t_1 \ldots t_n) = 0$ on Y. But that is not possible since $\dim Y = n$. Hence, $k[X] = k[Y]$. □

Theorem 0.1.22. *A hypersurface in $\mathbb{A}^n$ or $\mathbb{P}^n$ is an $n-1$ dimensional subvariety. The converse is true if X is an affine variety.*

Proof. We can assume X to be an affine variety such that $X \subset \mathbb{A}^n$. Let $X = V(f)$ be the hypersurface such that x_n appears in f. We show that $x_1, ..., x_{n-1}$ are algebraically independent in $k(X)$. On contrary, let us suppose that, $x_1, ..., x_{n-1}$ are algebraically dependent. Then there exists an irreducible polynomial g such that $g(x_1, ..., x_{n-1}) = 0$ on X. This implies, f divides g which is a contradiction as g does not involve x_n. Conversely, let $X \subset \mathbb{A}^n$ be an affine variety of dimension $n-1$. We choose a non-zero polynomial $f \in k[x_1, ..., x_n]$ such that $f \equiv 0$ on X. If f is reducible, then one of its factors vanishes on X, so we can assume f to be irreducible. Now $X \subset V(f) \subset \mathbb{A}^n$ and $\dim X \leq \dim V(f) < n$. Since $\dim X = n-1$, this implies $X = V(f)$. □

0.2 Smooth versus non-singular

In high school Calculus, the notion of singularity appeared as the point of discontinuity. Now we call a point p to be non-singular if the dimension of tangent space at p is greater than the dimension of variety.

Definition 0.2.1. Let X be a scheme. A point $x \in X$ is said to be *non-singular* if the local ring around x is regular.

Remark. We will come back to this definition after the following discussion.

0.2.1 Tangent space

We want to define tangent space at x to be the collection of lines that are tangent to X at x. We say that a line L through $x = (0, 0, ..., 0)$ is tangent to X if the "intersection multiplicity" of L with X at x is more than one. Equivalently, the multiplicity of $t = 0$ in the expression $\gcd(f_1(tx), f_2(tx), ..., f_l(tx))$ is ≥ 2 where f_i are the defining equations of X.

We will write the conditions for a line L to be tangent at x. Let L_i be the linear part of f_i. Then $f_i(xt) = tL_i(x) + G_i(tx)$. We see that $G_i(tx)$ is divisible by t^2. Therefore, $f_i(xt)$ is divisible by t^2 if and only if $L_i(x) = 0$. Thus, the conditions for tangency are the equations $L_1(x) = ... = L_m(x) = 0$.

Definition 0.2.2 (Tangent space)**.** Let $X \subset \mathbb{A}^N$ be a variety defined by the equations $f_1, f_2, ..., f_m$. Then we define the tangent space of X at $x = (0, 0, ..., 0)$ to be the subvariety of $\mathbb{A}^N$ defined by $L_i(x_1, x_2, ..., x_N) = 0$ where L_i are the linear part of $f_i's$. It is denoted by θ_x. Since the linear part of a polynomial can also be written as

$$d_x f = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(x)(x_i - x_i).$$

Tangent space of X at x is therefore defined by

$$d_x f_1 = d_x f_2 = ... = d_x f_m = 0$$

where f_i are the defining equations of X.

Suppose that $g \in k[X]$ has a polynomial representation G. We want to define $d_x g$. If we set $d_x g = d_x G$, we see that it depends on the choice of the polynomial G. Further, if G and G' are two representative for g then $G - G' \in I(X)$. This implies that $d_x G - d_x G' = d_x F$ for some $F \in I(X)$. Let the ideal $I(X) = (F_1, ..., F_m)$. Then, $F = A_1 F_1 + ... + A_m F_m$, and therefore, $d_x F = A_1(x) d_x F_1 + ... + A_m(x) d_x F_m$. We know that linear forms $\{d_x F \mid F \in I(X)\}$ vanish on the tangent space θ_x. Hence, $d_x g = d_x G$ is a well-defined function on the tangent space θ_x.

This induces a homomorphism $d_x : k[X] \to \theta_x^*$ as follows

$$g \mapsto d_x G.$$

Since the map takes constants to zero, we can reduce the study of this map to the ideal m_x

Theorem 0.2.3. *The map $d_x : m_x/m_x^2 \to \theta_x^*$ defines an isomorphism.*

Proof. The proof is easy and is left to the readers. $\square$

Corollary 0.2.3.1. *Let* $\phi : X \to Y$ *be an isomorphism such that* $\phi(x) = y$*. Then* $\theta_x \simeq \theta_y$*. In particular* $\dim \theta_x = \dim \theta_y$*.*

Definition 0.2.4 (Singular point)**.** Let X be a variety and $l = \min \{\dim \theta_x \mid x \in X\}$. We say that a point $x \in X$ is non-singular if $\dim \theta_x = l$. A variety X is said to be non-singular if it is non-singular at every $x \in X$. If $\dim \theta_x > l$ then x is said to be a singular point of X.

Theorem 0.2.5. *Let* X *be a variety and* $x \in X$ *be a non-singular point. Then* $\dim \theta_x = dimX$*.*

Proof. We show that $\dim \theta_x \geq \dim X$ for every point $x \in X$ and that the set $\{x \in X \dim \theta_x = \dim X\}$ is non-empty. We can assume X to be affine. Over an algebraically closed field, an affine variety X is birational to a hypersurface $Y = V(f) \subset \mathbb{A}^n$. Let $\phi : X \to Y$ be the birational map and let $U \subset X$ and $V \subset Y$ be the open sets such that ϕ is an isomorphism on U. The tangent space of the hypersurface $V(f) = Y$ at x is given by

$$d_x f = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(x)(x_i - x_i).$$

Unless $\frac{\partial f}{\partial x_i}'s$ are zero functions, there exists a point y such that the tangent space at y is an $n-1$ dimensional variety. But $\frac{\partial f}{\partial x_i} = 0$ implies $d_x f = 0$ which further implies f is either a constant if char $k = 0$ or $f = f_1^p$ if char $k = p$ for some f_1. Since f is irreducible and non-constant, we conclude that the set $W = \{x \in V \mid \dim \theta_x = \dim X\}$ is non-empty and open. Indeed, W is the set of all the points where at least one of the partial derivative doesn't vanish. Hence, $\phi^{-1}(W \cap V) \subset U$ is also open and non-empty. Since, the dimension of tangent space remains same under isomorphism, $\dim \theta_x = \dim X$ for $x \in \phi^{-1}(W \cap V)$. □

Definition 0.2.6 (Local parameters of a point x)**.** The functions $u_1, ..., u_n \in \mathcal{O}_x$ are said to be the local parameters of x if each $u_i \in m_x$ and the images of $u_1, ..., u_n$ form a basis of the vector space m_x/m_x^2.

As a consequence of Nakayama's lemma, we see that local parameters of x generate the maximal ideal m_x of $\mathcal{O}_x$

Another equivalent definition of a non-singular point thus can be, a point $x \in X$ is said to be non-singular if the local ring $\mathcal{O}_x$ is regular local ring and hence, definition 5.2.3.

We call a variety X smooth over the point x if the dimension of its tangent space is same as that of the variety, and call a point x non-singular if the local ring $\mathcal{O}_x$ around x is regular local ring. We have seen that X is smooth over K at the point x then x is a non-singular point of X. But converse may not be true always. However, if the residue field at the point x is separable over k then non-singularity implies smoothness.

0.3 Divisors

We can describe a rational function $f \in k\left(\mathbb{A}^1\right)$ through its zeroes and poles. In the same way, we want to assign a linear combination of codimension one subvariety to a rational function. Let ζ be the collection of all the codimension one subvarieties of V. We define a divisor on V to be the sum $D := \sum_{i=1}^{n} k_i C_i$ where $k_i' s$ are non-zero for finitely many $i's$. Let

$$\text{Div X} = \left\{ \sum_{i=1}^{n} k_i C_i \text{ where } k_i \text{ is non zero for only finitely many i's } | C_i \in \zeta \right\},$$

then Div X forms an abelian group. We define $SuppD := \bigcup_{i=1}^{n} C_i$ and call a divisor $D = \sum_{i=1}^{n} k_i C_i$ to be effective if $k_i \geq 0$ for all i. Let $f \in k\left(V\right)$ be a rational function. We wish to assign an element of Div X to f. Assume that the set of singular points of X has codimension ≥ 2. Let $C \in \zeta$, and U be an affine open neighborhood such that $C \cap U \neq \emptyset$. It contains non-singular point of X. It follows from theorem 5.2 that C in U is defined by some u. Let $f \in \mathcal{O}_U$, then there exists an integer $k \geq 0$ such that $f \in \left(u^k\right)$ and $f \notin \left(u^{k+1}\right)$. We denote it by $v_C(f)$. We write $divf := \sum v_C(f)C$. Since each $g \in \mathcal{O}_x$ has a unique Taylor series expansion (x is non-singular). This will guarantee the existence of k above. We extend it to the rational function $f = \frac{g}{h} \in k(V)$ as follows:

$$divf = divg - divh.$$

We call the divisors of rational function to be the principal divisors. This forms a subgroup of Div X. The natural question is, Is every divisor is principal? The answer is no.

We define an equivalence on Div X as follows: we call $D_1 \sim D_2$ if and only if $D_1 - D_2 = divf$ for some $f \in k$. The quotient group $DivX/P(X)$ is called as *divisor class group of* X.

Suppose X is a non-singular variety. Let C be a codimension one subvariety and U be an affine neighborhood around $x \in C \cap U$. Then there exists a local equation u of C in U. It follows that $D = \sum_{i=1}^{n} k_i C_i$ has a local equation $f = \prod_i^n g_i^{k_i}$ in U, where g_i are the local equations of C_i. Since closed varieties are quasi compact, there exists a finite open cover $\bigcup_i U_i$ for X. In each of the open set U_i, let D be defined by f_i. Then these $f_i's$ satisfy the following two conditions:
1 f_i/f_j are regular in $U_i \cap U_j$.
2 f_i/f_j are nowhere zero in $U_i \cap U_j$.
We call the system $\{(U_i, f_i)\}$ to be compatible system of functions on X. Any divisor D on a non-singular variety gives a compatible system of functions and vica versa. Suppose X and Y are non-singular varieties, $\phi : X \to Y$ be a morphism and D be a divisor on Y such that $\phi(X) \nsubseteq SuppD$. Then we can define a pull-back of D to be a divisor $\phi^*(D)$ on X as follows. Given a divisor D we know that it is locally principal. Let $\{(U_i, f_i)\}$ be the compatible system of functions associated with D. Then $\{(\phi^{-1}(U_i), \phi^*(f_i))\}$ gives a compatible system of functions on X. Let the associated divisor be D'. We define

$$D' = \phi^*(D)$$

and call it the pull-back of D. Further note that, ϕ^* defines a homomorphism

$$\phi^* : DivY \to DivX.$$

Principal divisors are mapped to principal divisors under ϕ^*. This induces a map

$$\phi^* : DivY/P(Y) \to DivX/P(X).$$

The divisors defined by a compatible system of functions are scheme theoretic analogue of Cartier divisors.

We define the degree of a divisor $D = \sum_i k_i C_i$ as $\deg D := \sum_i k_i$.

0.4 Intersection theory

The multiplicity of a point in a variety is one of the first invariant that we look at while resolving them. Intersection numbers are highly involved in the text, so we discuss it here. We have encountered such terms while defining the tangent space of a variety at a point. Intersection numbers are motivated by the general intersections, but there are several differences that we will see soon.

Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n. We choose a form F_1 that doesn't vanish on X and denote $X \cap V(F_1) = X_{F_1}$. Note that, it is a codimension 1 subvariety of X. We choose another form F_2 that does not vanish on X_{F_1} and denote $X_{F_1} \cap V(F_2) = X_{F_1F_2}$ and continue doing this. At each step, the dimension is decreased by one. We see that $X_{F_1F_2...F_{n+1}}$ is empty. Our study of intersection numbers is limited up to codimension one subvariety, and we are interested in the case where intersections are finite. Hence, we take n divisors that have no common components where n is the dimension of X. We define the intersection numbers for effective divisors and extend them linearly.

Suppose X is an n-dimensional projective variety, $D_1, D_2, ..., D_n$ be effective divisors with no common components and $x \in \bigcap_i SuppD_i$. Let U_x be an affine neighborhood of x and $f_1, f_2, ..., f_n$ be the local equations of $D_i's$ in U_x. We can assume that $f_i's$ are regular in U_x and have no common zeroes other than x there. It follows that

$$(f_1, f_2, ..., f_n) \subset m_x^k \tag{1}$$

for some k. Consider the quotient $\mathcal{O}_x/(f_1, f_2, ..., f_n)$ as a vector space over k. It follows from (1) that it is finite dimensional.

Definition 0.4.1. Let $D_1, D_2, ..., D_n$ be divisors on an n-dimensional projective variety with no common component. Then intersection multiplicity of $D_1, D_2, ..., D_n$ at x is defined as follows:

$$(D_1D_2...D_n)_x = \dim_k(\mathcal{O}_x/(f_1, ..., f_n))$$

Now, suppose that $D_1, D_2, ..., D_n$ are not effective. We can write $D_i = D_i' - D_i''$, then

$$(D_1D_2...D_n)_x := (\prod_{i=1}^{n}(D_i' - D_i''))_x = \sum_{i_1,i_2,...,i_n} \sum_{k=0}^{n} (-1)^{n-k}(D_{i_1}'...D_{i_k}'D_{i_{k+1}}''...D_{i_n}'')_x.$$

Definition 0.4.2. Let X be a projective variety on dimension n and $D_1, D_2, ..., D_n$ be divisors on X with no common component. Then, we define the intersection number of $D_1, D_2, ..., D_n$ as follows:

$$D_1 D_2 ... D_n = \sum_{x \in \cap_i D_i} (D_1 D_2 ... D_n)_x.$$

Example 2. Let $C_1, C_2, ..., C_n$ be codimension 1 subvariety of X and $f_1, f_2, ..., f_n$ be the local equations of C_i at x. Then $(C_1 C_2 ... C_n)_x = 1$ implies that $(f_1, f_2, ..., f_n) = m_x$. Therefore, the condition $(C_1 C_2 ... C_n)_x = 1$ implies that the functions $f_1, f_2, ..., f_n$ are a system of local parameters of x.

Definition 0.4.3. Suppose X is an n-dimensional variety and $D_1, D_2, ..., D_k$ are divisors on X such that $\cap_i SuppD_i$ is an $n - k$ dimensional algebraic set. Let C be a component of $\cap_i SuppD_i$, $x \in C$ be a point and $f_1, f_2, ..., f_k$ be the local equations of $D_1, D_2, .., D_k$ in U_x. We define the intersection multiplicity of $D_1, ..., D_k$ at C to be

$$(D_1 D_2 ... D_k)_C = l_{\mathcal{O}_C} (\mathcal{O}_C / I)$$

where $I = (f_1, ..., f_k)\mathcal{O}_C$ and $l_{\mathcal{O}_C}(N)$ is the length of the $\mathcal{O}_C$-module N.

Theorem 0.4.4. *Suppose $X \subset \mathbb{P}^n$ be a projective variety and $D_1, D_2, ..., D_{n-1}, D_n$ and $D_1, D_2, ...D_{n-1}, D_{n+1}$ are divisors with no common components. Then,*

$$(D_1 D_2 ... D_{n-1}(D_n + D_{n+1}))_x = (D_1 D_2 ... D_{n-1} D_n)_x + (D_1 D_2 ... D_{n-1} D_{n+1})_x.$$

Proof. It is enough to prove the theorem for effective divisors. Let $f_1, f_2, ..., f_n, f_{n+1}$ be the local equations of $D_1, D_2, ..., D_n, D_{n+1}$ respectively in a neighborhood of x. We wish to show that,

$$\dim_k \mathcal{O}_x/(f_1, ..., f_{n-1}, f_n f_{n+1}) = \dim_k \mathcal{O}_x/(f_1, ..., f_{n-1}, f_n) + \dim_k \mathcal{O}_x/(f_1, ..., f_{n-1}, f_{n+1}).$$

Note that,

$$0 \to (f_{n+1})/(f_1, ..., f_n f_{n+1}) \to \mathcal{O}_x/(f_1, ..., f_n f_{n+1}) \to \mathcal{O}_x/(f_1, ..., f_{n-1}, f_{n+1}) \to 0$$

is an exact sequence of k-vector spaces where the first map is the inclusion of the ideal $(f_{n+1})/(f_1, ..., f_n f_{n+1})$ in the ring $\mathcal{O}_x/(f_1, ..., f_n f_{n+1})$. It follows that

$$\mathcal{O}_x/(f_1, ..., f_{n-1}, f_{n+1}) \simeq \mathcal{O}_x/(f_1, ..., f_n f_{n+1})/(f_{n+1})/(f_1, ..., f_n f_{n+1})$$

. Therefore,

$$\dim_k \mathcal{O}_x/(f_1, ..., f_n, f_{n+1}) = \dim_k(f_{n+1}/(f_1, ..., f_n f_{n+1})) + \dim_k \mathcal{O}_x/(f_1, ..., f_{n-1}, f_{n+1})$$

It remains to show $\dim_k \mathcal{O}_x/(f_1, ..., f_{n-1}, f_n) = \dim_k(f_{n+1})/(f_1, ..., f_n f_{n+1})$. It follows directly after we show f_{n+1} is a non-zero divisor in $\mathcal{O}_x/(f_1, f_2, ..., f_{n-1})$. We show that if $f_1, f_2, ..., f_{n-1}, f_{n+1}$ are local equations of divisors $D_1, ..., D_{n-1}, D_{n+1}$ that have no common components, then f_{n+1} is a non-zero divisor in $\mathcal{O}_x/(f_1, f_2, ..., f_{n-1})$. The functions $f_1, f_2, ..., f_n$ are said to be a regular sequence, if f_i is a non zero divisor in $\mathcal{O}_x$ for all i. We proceed by induction on the dimension of the variety X. It is trivially true for $n = 1$. So we suppose that the result is true for $\dim X = n - 1$. Let H be a hyperplane given by f such that it does not contain $\theta_{x,X}$ or any component of $\cap_{i=1}^{n-1} SuppD_i$. Then $D_1, ..., D_{n-1}$ restricted to H have a finite intersection and possess no common component. By the induction hypothesis, f_i is a non-zero divisor in $\mathcal{O}_x/(f_1, ..., f_{i-1})$ for $i = 1, ..., n-1$. Since f is non-zero divisor in $\mathcal{O}_x$ we see that $f, f_1, f_2, ..., f_{n-1}$ is a regular sequence. Regular sequences are invariant of order. It follows that $f_1, f_2, ..., f_{n-1}, f$ form a regular sequence. In a neighborhood of x, $D_1, D_2, ..., D_{n-1}, D_{n+1}$ have no common component other than x. So,

$$m_x^k \subset (f_1, f_2, ..., f_{n-1}, f_{n+1})$$

for some k. This implies $f^k \in (f_1, f_2, ..., f_{n-1}, f_{n+1})$ i.e. $f^k \equiv a f_{n+1} \mod (f_1, f_2, ..., f_{n-1})$. Therefore, f_{n+1} cannot be a zero divisor in $\mathcal{O}_x/(f_1, f_2, ..., f_{n-1})$. □

Theorem 0.4.5. *Suppose X is non-singular projective variety and $D_1, ..., D_n, D_n'$ are divisors on X such that $D_n \sim D_n'$ and none of them have common components. Then,*

$$D_1 ... D_{n-1} D_n = D_1 ... D_{n-1} D_n'$$

Proof. We show it for the case where $D_1, D_2, ..., D_n, D_n'$ are effective. Since $D_n \sim D_n'$, there exists an f such that $D_n - D_n' = divf$. We will show that $D_1 D_2 ... D_{n-1} divf = 0$. Let $x \in \left\{\cap_{i=1}^{n-1} SuppD_i\right\} \cap Supp(divf)$ be a point. And U_x is an affine neighborhood of x. Further, let $D_1, D_2, ..., D_{n-1}$ are defined by the equations $f_1, f_2, ..., f_{n-1}$ in U_x. We write $\bar{\mathcal{O}} = \mathcal{O}_x/(f_1, f_2, ..., f_{n-1})$, then

$$\dim_k \mathcal{O}_x/(f_1, ..., f_{n-1}, f) = l_{\bar{\mathcal{O}}}(\bar{\mathcal{O}}/(f)).$$

Note that $\bar{\mathcal{O}}$ is one-dimensional local ring that has one maximal ideal m_x and l prime ideals $p_{C_1}, p_{C_2}, ..., p_{C_l}$ where $C_i's$ are the component of $\cap_{i=1}^{n-1} SuppD_i$ that pass through x. It follows that,

$$l_{\bar{\mathcal{O}}}\mathcal{O}/(f) = \sum_{p_i} l_{\bar{\mathcal{O}}_{p_i}} \bar{\mathcal{O}}_{p_i} \times l_{\bar{\mathcal{O}}}(\bar{\mathcal{O}}/(p_i + f\bar{\mathcal{O}}))$$

$$= \sum_i (D_1...D_{n-1})_{C_i} \times l_{\bar{\mathcal{O}}}(\mathcal{O}_{C_i,x}/(f)).$$

Let E_i denote the restriction of $divf$ to C_i then,

$$(D_1...D_{n-1}divf)_x = \sum_{i=1}^{l} (D_1...D_{n-1})_{C_i} \times v_x(E_i)$$

Let $\eta : C_i' \to C_i$ be the normalization of C_i(see chapter 3), then

$$(E_i)_x = \sum_{y \in \eta^{-1}(x)} (\eta^*(E_i))_y.$$

We see that,

$$(D_1...D_{n-1}divf) = \sum_{i=1}^{l} (D_1...D_{n-1})_{C_i} \times \sum_{x \in C_i} (\sum_{y \in \eta^{-1}(x)} v_y(\eta^*(E_i)))$$

We know that E_i are principal divisors on the curves C_i and so are $\eta^*(E_i)$. The normalization C_i' are non-singular projective curves (see chapter 3). It is easy to verify that degree of a principal divisor on a non-singular projective curve is zero. Hence, the result follows. □

Theorem 0.4.6. *Suppose X, Y be non-singular projective surface and $f : X \to Y$ be a birational morphism. Let D_1, D_2 are divisors on Y then*

$$f^*(D_1)f^*(D_2) = D_1D_2.$$

Further, if $\bar{D}$ is an exceptional divisor on X, then for any divisor D on Y

$$f^*(D)\bar{D} = 0.$$

Proof. Since $\phi = f^{-1}$ is a rational map on a surface, we have seen that ϕ is not regular at finitely many points. Let Z be the finite set where ϕ is not regular. Then $\phi : Y - Z \to X$ is an isomorphism. If D_1, D_2 does not contain Z then we are done otherwise, we can move away D_1 and D_2 from Z due to the following lemma:

Lemma 0.4.7. *Let X be a non-singular projective variety D be a divisor and $x \in X$ be a point then there exists a divisor D' such that $D' \sim D$ and $x \notin SuppD'$.*

So we assume D_1 and D_2 have no intersection with Z and hence. $D_1 D_2 = f^*(D_1) f^*(D_2)$

$\square$

Chapter 1

Introduction

The concept of singularity have appeared in the subject in one form or the other. In High school, one calls a point singular if tangent space is not defined at that point. Generalizing these definitions, we have seen that a point p in a variety X is said to be singular if local ring around p is not regular. The idea of resolving singularities is to replace this singular locus by a subvariety in such a way that one can study the properties of singular point p through them. The problem of resolution of singularities have been changing its essence since the time it was initiated. First being, to produce a non-singular variety X' such that there exists a birational proper morphism $\pi : X' \to X$ from X' to X.
Why proper?
Before that, what is a proper morphism?. A map $f : X \to Y$ between topological spaces X and Y is said to be proper if preimage of every compact set is compact. Scheme theoretic definition of a proper morphism is as follows:

Definition 1.0.1 (Proper morphism)**.** A morphism $f : X \to Y$ of schemes is said to be *universally closed* if for every scheme Z and the morphism $g : Z \to Y$, the projection from the product

$$X \times_Y Z \to Z$$

is a closed map for the underlying topological spaces. A morphism of schemes is said to be proper if it is separated, universally closed and is of finite type.

By taking proper map we want to get rid of trivial resolutions such as "variety minus the singular locus". Indeed, let $X = Y \backslash \{$ singular locus$\}$ and $Z = Y$ with

$g = Id|_Y$. Then $X \times_Y Y = Y$, but the projection is not closed, and therefore we get rid of trivial resolution in such case. The definition may not always be easy to work with, so we take into account the valuative criteria of properness:

Let $f : X \to Y$ be a morphism of Noetherian scheme of finite type. Then f is a proper map if and only if for all discrete valuation ring R with field of fractions K and for any $K - valued$ point $x \in X$ that maps to a point $f(x)$ defined over R, there exists a unique lift of x of $\bar{x} \in X(R)$.

There are other notions for resolution too. They are as follows:

Definition 1.0.2 (Embedded resolution)**.** Let $Y \subset X$ be a variety embedded in a non-singular variety X. An embedded resolution of Y is a birational proper morphism $\pi : X' \to X$ from a non-singular variety X' to X such that π is an isomorphism over $X \backslash Sing(Y)$ and the birational transform Y' of Y is non-singular.

Definition 1.0.3 (Strong resolution)**.** Given a variety X we want to find a smooth variety X' such that there exists a birational proper map $\pi : X' \to X$ such that π^{-1} is an isomorphism on $X \backslash \{Sing(S)\}$ and $\pi^{-1}(Sing(S))$ is a simple normal crossing.

Definition 1.0.4 (Functorial resolution)**.** Suppose X is a variety, we wish to find a resolution that is functorial with respect to smooth morphisms. That is for any morphism $f : X \to Y$ there exists an $f' : X' \to Y'$ such that the following diagram commute.

Definition 1.0.5 (Weak resolution)**.** For a variety X, we wish to find a smooth variety X' birational to X.

We are mostly interested in the first definition. After Hironaka has given a proof for resolution in arbitrary dimensions, the focus has been shifted from finding new methods of resolution to a more in-depth understanding of the existing ones with a motive of applying these techniques in positive characteristic. In this document, we talk about the various method and what they do to the invariants. Chapter one is about blow-ups, where we provide a basic definition and idea of blow-ups, followed by its applications. Finally, we show that singularity of curves and surfaces are resolved after finitely many blow-ups centered at the singular locus. Chapter two talks about normal varieties and what normalization do to a variety. We also talk about one of the interesting kinds of singularity i.e., Abelian quotient singularities. In chapter three, we talk about projection and discuss one of the significant results by S. Abhyankar which says that every projective variety of dimension n is birational to a normal variety whose top locus can have multiplicity at most $n!$. And finally, in chapter four, we give a proof of surface resolution in positive characteristic given by Cutkosky.

Chapter 2

Blow up

The very first example of resolution of singularity is that of a cone

$$z^2 = y^2 + x^2$$

Cone can also be seen as an image of the cylinder $x^2 + y^2 = 1$ under the map $\phi : \mathbb{A}^3 \to \mathbb{A}^3$ defined by

$$(x, y, z) \mapsto (xz, yz, z).$$

The map contracts $xy - plane$ to a point. Notice that, it is an isomorphism for all the points other than $\phi^{-1}(0, 0, 0)$. To further illustrate the idea, let us look at another example of the curve $y^2 = x^2 + x^3$. This curve is singular at $(0, 0)$, the singularity of the point is due to the two branches that pass through the point having different tangents. In order to separate the two branches of the curve, we lift two lines in $\mathbb{A}^2$ by associating to each line, the height z given by the slope of the line. More precise treatment is done in section one, where discuss the notion of blow up. In section two, we talk about some of its beautiful applications. Resolution of curve and surfaces in characteristic 0 are discussed in section three and four. Finally, we discuss some examples where these methods behave weird, if not carefully done. We will mostly talk about embedded resolution in a neighborhood, but the global case can be taken care of by patching up the local charts.

2.1 Blow up

2.1.1 Tangent cone

Tangent space is an invariant of an algebraic variety, its dimension tells us how far a singular point is, from being non-singular. However, for the curves like $y^2 = x^3$, the tangent space at $(0,0)$ is the whole $\mathbb{A}^2$ which is far away from the actual picture of tangent lines at the origin. This motivates the concept of tangent cones. We want to define the tangent cones similar to the limiting position of secant lines.

For a variety $X \subset \mathbb{A}^n$, we define $X' \subset \mathbb{A}^n \times \mathbb{A}^1$ to be the set

$$X' := \{(a,t) \mid ta \in X\}.$$

We observe that it is a closed subset of $\mathbb{A}^{n+1}$. Let

$$\phi : X' \to \mathbb{A}^1 \text{ and } \varphi : X' \to \mathbb{A}^n$$

be the natural projection maps.

Notice that X' is reducible, indeed

$$X' = \{(a,0) \mid a \in \mathbb{A}^n\} \quad \bigcup \quad \overline{\phi^{-1}\left(\mathbb{A}^1 - \{0\}\right)}. \tag{2.1}$$

Let ϕ_1 and φ_1 be the restriction of ϕ and φ respectively to $\overline{\phi^{-1}\left(\mathbb{A}^1 - \{0\}\right)}$. The set $\varphi_1\left(\overline{\phi^{-1}\left(\mathbb{A}^1 - \{0\}\right)}\right)$ is the closure of set of all the points on the secants through $x = (0,0,...,0)$. We call $\varphi_1\left(\phi_1^{-1}(0)\right)$ as the *tangent cone* to X at origin and denote it by $T_{(0,0,...,0)}$. Another commonly referred definition of tangent cone to the variety $X = V(I) \subset \mathbb{A}^n$ at $(0,0,...,0)$ is the variety $V(I^*)$ where $I^* = \{f^* \mid f \in I\}$ and $f^* = f_k$ is the leading term of $f = f_k + f_{k+1} + ... + f_r$.

We show that the two definitions are equivalent:

The equations of X' are $\{f | f(at) = 0 \text{ for } (a,t) \in X' \text{ and } f \in I\}$. Suppose that $f = f_k + f_{k+1} + ... + f_r$ where f_j is homogenous polynomial of degree j with $f_k \neq 0$. Then $f(at) = t^k\left(f_k(a) + tf_{k+1}(a) + ... + t^{l-k}f_l(a)\right)$. Since $f(0) = 0$ implies $k \geq 1$. Therefore, f decomposes into two components. The component $\{t = 0\}$ corresponds to $\{(a,0) \mid a \in \mathbb{A}^n\}$ and the other component is $V(f_k(a) + tf_{k+1}(a) + ... + t^{l-k}f_l(a))$. Comparing it with 2.1, we see that $\varphi_1\left(\phi_1^{-1}(0)\right)$ is given by $f_k = 0$ for $f \in I$. The form f_k is the leading form of f. Thus, tangent cone T_x at x, is defined by setting 0, the leading forms of all polynomials $f \in I$. We give the scheme theoretic definition here.

Definition 2.1.1 (Tangent cone of X at x)**.** Let k be an algebraically closed field, X be a scheme of finite type over k and $x \in X$ be a closed point. We define the *tangent cone* at x to be

$$T_x = \operatorname{Spec} \frac{k[x_1, x_2, ..., x_n]}{I^*}$$

where $\operatorname{Spec} \frac{k[x_1, x_2, ..., x_n]}{I}$ is an affine neighborhood around x in X and $I^* = (\{$ leading form of f for $f \in I\})$.

At first, it might look like that the definition is dependent upon the neighborhood we choose for x but we will relate it to the local ring $\mathcal{O}_x$ and the independence of choice of neighborhood will follow.

We define

$$gr\,(\mathcal{O}_x) := \sum_{n=0}^{\infty} m_x^n / m_x^{n+1}$$

where m_x is the maximal ideal of local ring $\mathcal{O}_x$. It is sufficient to show the equivalence for the point $(x_1, x_2, ..., x_n)/I$. Then,

$$m_x^k / m_x^{k+1} \simeq \sum (x_1, x_2, ..., x_n)^k / (x_1, x_2, ..., x_n)^{k+1} + I \cap (x_1, x_2, ..., x_n)^k$$

$$m_x^k / m_x^{k+1} \simeq \sum (x_1, x_2, ..., x_n)^k / (x_1, x_2, ..., x_n)^{k+1} + I_k^*.$$

But,

$$(x_1, ..., x_n)^k / (x_1, ..., x_n)^{k+1} + I_k^* \text{ is the } k^{th} \text{ graded piece of } k[x_1, x_2, ..., x_n]/I^*.$$

That is,

$$gr\,(\mathcal{O}_x) \simeq k[x_1, x_2, ..., x_n]/I^*$$

Definition 2.1.2 (Final definition)**.** Let X be a scheme and $x \in X$ be a closed point. We define $\operatorname{Spec} gr(\mathcal{O}_x)$ to be the tangent cone at $x \in X$.

Remark. Since tangent cone is defined through a homogenous ideal I^*, it is natural to projectivize it. The ideal I^* defines a subscheme T in $\mathbb{P}^{n-1}$, that we call as *projectivized tangent cone*. There is a natural way to put $X - x$ and T together in a new scheme $B(\xi)$ in such a way that locally on $B(\xi)$, T is a subscheme defined by vanishing of single function.

2.1.2 Blow up of $\mathbb{A}^n$ at origin

Definition 2.1.3 (Blow up of $\mathbb{A}^n$ at $\xi = (0, 0, ...0)$)**.** Let $p : \mathbb{A}^n - \{0\} \to \mathbb{P}^{n-1}$ be the projection map that sends $(a_1, a_2, ..., a_n)$ to its homogenous coordinate. And $B(\xi) \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ be the closure of the graph of projection map p. Then the projection map $\pi : B(\xi) \to \mathbb{A}^n$ is defined to be the *blow up* of $\mathbb{A}^n$ at origin.

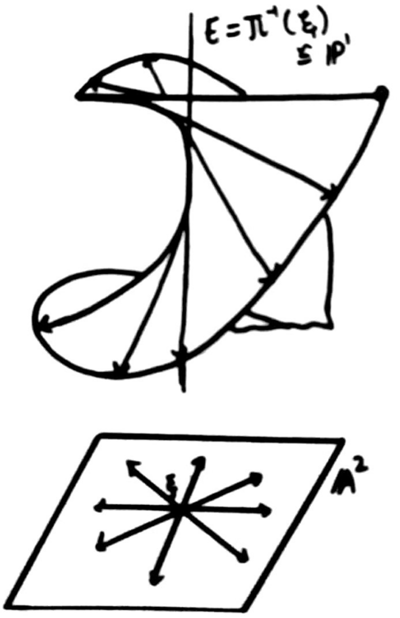


Figure 2.1: Blow up of $\mathbb{A}^2$ at origin

Since the graph is already closed in $(\mathbb{A}^n - \{0\}) \times P^{n-1}$, all the points of $B(\xi)$ that are not in the graph lie above the origin in $\mathbb{A}^n$. It is worth noting that the projection π is birational map, inverse of which is defined for all the point except origin.

The geometric interpretation is as follows:

Affine space $\mathbb{A}^n$ can be visualized as the union of lines through $(0, 0, ...0)$. Each line is further a union of origin and $p^{-1}(t)$ for some $t \in k$. Blowing up $(0, 0, ...0)$ associate to each line corresponding to the slope t, a new origin $\{0\} \times t$. In particular the closure, $B(\xi)$ contains all the points of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ that lie over the origin in $\mathbb{A}^n$. Therefore, we have effectively replaced the one origin in $\mathbb{A}^n$ by a whole variety of origins in $B(\xi)$, one for each line. The locus of origins $0 \times \mathbb{P}^{n-1}$ is called the *exceptional divisor* of $B(\xi)$

Affine open cover for $B(\xi)$

The blow up variety $B(\xi)$ is not always easy to work with, so we work with the pieces. $B(\xi)$ is a subvariety of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ defined by the equations

$$x_i y_j - x_j y_i = 0 \text{ for } i, j = 1, ..., n$$

where $x_1, x_2, ..., x_n$ are affine coordinates of $\mathbb{A}^n$ and $y_1, y_2, ..., y_n$ are the homogenous coordinates of $\mathbb{P}^{n-1}$. The open cover for $\mathbb{P}^{n-1}$ given by $\bigcup_{i=1}^n U_i$ where $U_i = \{(x_1, x_2, ..., x_n, y_1 : y_2 : ... : y_n) \mid y_i \neq 0\}$ induces an open cover for $\mathbb{A}^n \times \mathbb{P}^{n-1}$ as follows:

$$\mathbb{A}^n \times U_i = \left\{ \left(x_1, x_2, ..., x_n, \frac{y_1}{y_i}, \frac{y_2}{y_i}, ..., 1 ..., \frac{y_n}{y_i} \right) |y_i \neq 0 \right\} \simeq \mathbb{A}^n \times \mathbb{A}^{n-1}$$

The definition of p demands the ratios $\{x_1 : x_2 :, ..., : x_n\} = \{y_1 : y_2 : ... : y_n\}$. Therefore,

$$B(\xi) \cap (\mathbb{A}^n \times U_i) = V\left(\left(..., x_i . \frac{y_j}{y_i} - x_j, ... \right) \right) \text{ and}$$

$$B(\xi) \cap (\mathbb{A}^n \times U_i) = \text{MaxSpec } k \left[x_1, x_2, ..., x_n, \frac{y_1}{y_i}, \frac{y_2}{y_i}, ..., \frac{y_n}{y_i} \right] / \left(..., x_i . \frac{y_j}{y_i} - x_j, ... \right)$$

See that $\frac{y_j}{y_i}$ and $\frac{x_j}{x_i}$ are equal in the function field of $B(\xi)$. So we identify the affine ring $\Gamma\left(B(\xi) \cap (\mathbb{A}^n \times U_i), \mathcal{O}_{B(\xi)}\right)$ with its isomorphic image in $k\left(x_1, x_2, ..., x_n, \frac{y_1}{y_i}, ..., \frac{y_n}{y_i}\right)$ to obtain

$$B(\xi) \cap (\mathbb{A}^n \times U_i) = \text{MaxSpec}\, k \left[x_i, \frac{x_1}{x_i}, \frac{x_2}{x_i}, ..., \frac{x_n}{x_i} \right]$$

Replacing MaxSpec by Spec gives an open cover for X by affine schemes. Further, these affine pieces of $B(\xi)$ satisfies the gluing condition. Indeed,

$$[B(\xi)^i]_{\frac{x_j}{x_i}} = \text{Spec}\, k \left[x_i, \frac{x_1}{x_i}, \frac{x_2}{x_i}, ..., \frac{x_n}{x_i}, \frac{x_i}{x_j} \right]$$

$$= \text{Spec } k \left[x_j, \frac{x_1}{x_j}, \frac{x_2}{x_j}, ..., \frac{x_n}{x_j}, \frac{x_j}{x_i} \right] = [B(\xi)^j]_{\frac{x_i}{x_j}}$$

Therefore, we have a scheme structure on $B(\xi)$. The exceptional divisor is given by MaxSpec$k\left[\frac{x_1}{x_i}, \frac{x_2}{x_i}, ..., \frac{x_n}{x_i}\right]$. Indeed, the blow up morphism π induces the inclusion map

$$\mathrm{k}\left[x_1, x_2, ..., x_n\right] \subset \mathrm{k}\left[x_i, \frac{x_1}{x_i}, ..., \frac{x_n}{x_i} \right]$$

defined by $x_j \mapsto \left(\frac{x_j}{x_i}\right) . x_i$. Origin in MaxSpec $k\left[x_1, x_2, ..., x_n\right]$ is given by the ideal $(x_1, x_2, ..., x_n)$. Since all the x_j's are the multiple of x_i we see that,

$$E \cap B(\xi)^i = V\left((x_i)\right) \simeq \text{MaxSpec}\, k \left[\frac{x_1}{x_i}, \frac{x_2}{x_i}, ..., \frac{x_n}{x_i} \right]$$

Remark. Having defined the blow up for $\mathbb{A}^n$, next what follows is the blow up of projective space along a point. The definition is on the same lines but, we define it here for the completeness.

Definition 2.1.4 (Blow up of $\mathbb{P}^n$ along the point $\xi = (1:0:...:0)$)**.** Consider the closed subvariety $B(\xi) \subset \mathbb{P}^n \times \mathbb{P}^{n-1}$ defined by the equations

$$x_i y_j = y_i x_j, i, j = 1, ..., n$$

where $(x_0 : ... : x_n)$ and $(y_1 : ... : y_n)$ are homogenous coordinates for $\mathbb{P}^n$ and $\mathbb{P}^{n-1}$ respectively. Then the map $\pi : B(\xi) \to \mathbb{P}^n$ given by the restriction of the first projection $\pi : \mathbb{P}^n \times \mathbb{P}^{n-1} \to \mathbb{P}^n$ is defined to be the blow up of $\mathbb{P}^n$ centered at $\xi = (1:0:...:0)$

2.1.3 Birational transform of a blow up

What happens to the subvariety $X \subset \mathbb{A}^n$ under point blow up?
Consider the curve $y^2 = x^3 + x^2 \subset \mathbb{A}^2$. The image of this curve under the inverse of birational map π is the consist of two components $\{x\} \bigcup \{y^2 - x - 1\}$. Indeed, let π^i be the restriction of π to the affine piece $B(\xi)^i = \text{Spec k}[x_i \frac{x_1}{x_i}, ..., \frac{x_n}{x_i}]$. Then the map

$$\pi_i^* : k[x_1, x_2, ..., x_n] \to k[x_i, \frac{x_1}{x_i}, ..., \frac{x_n}{x_i}]$$

is defined as follows

$$x_i \to x_i$$

$$x_j \to \frac{x_j}{x_i} x_i.$$

We denote $\frac{x_j}{x_i} = x'_j$ and $x_i = x'_i$. Then for the curve $f(x, y) = y^2 - x^2 - x^3$

$$\pi_x^* (f) = (y'x')^2 - x'^2 - x'^3$$

implies

$$V (\pi_x^* (f)) = V (x') \cup V \left(y'^2 - 1 - x'\right)$$

The set $x' = 0$ constitutes the exceptional curve $0 \times \mathbb{P}^1$ while the other component is said to be the *birational transform* of the curve $C = y^2 - x^2 - x^3$ in the first chart. Notice that the birational transform of C is smooth here.

In fact, for any curve $f(x,y) \subset \mathbb{A}^2$, image of f under π^* is $f(x', x'y') = x'^{mult_{(0,0)}f} f'(x', y')$ and we define $f'(x', y')$ to be the birational transform of f in the first chart. In general, if $X \subset \mathbb{A}^n$ is an irreducible affine variety with $X \neq \mathbb{A}^n$. Then the inverse image $\pi^{-1}(X)$ of X under the blow up of $\mathbb{P}^n$ centered at ξ is reducible and consists of the two components

$$\pi^{-1}(X) = \left(\xi \times \mathbb{P}^{n-1}\right) \cup Y.$$

We define Y to be the birational transform of X. The restriction of π to the component Y defines a regular map $\pi : Y \to X$ which is an isomorphism on $Y \backslash \pi^{-1}(\xi)$.

2.1.4 Blow up along a subvariety

Let $Y \subset X$ be a non-singular subvariety of a non-singular variety X. Then there exist a neighborhood U and functions $u_1, u_2, ..., u_m \in \mathcal{O}_X(U)$, where $m = codim_X Y$ such that the ideal $a_Y \subset \mathcal{O}_X(U)$ is given by $a_Y = (u_1, u_2, ..., u_m)$ and $d_x u_1, d_x u_2, ..., d_x u_m$ are linearly independent. Suppose X is an affine variety and $Y \subset X$ is defined by the equations $u_1, u_2, ..., u_m$. Consider the closed subvariety $X' \subset X \times \mathbb{P}^{m-1}$ defined by the equations

$$t_j u_i(x) - u_j(x) t_i.$$

Let $\pi : X \times \mathbb{P}^{m-1} \to X$ be the natural projection.

Definition 2.1.5 (Blow up along a subvariety)**.** The restriction map $\pi : X' \to X$ is defined to be *blow up of X along Y*.

Remark. It may appear as if the blow up variety is dependent upon the selection of local parameters $u_1, u_2, ..., u_m$. But it is not the case.

Lemma 2.1.6. *Suppose $\pi_1 : X_1 \to X$ and $\pi_2 : X_2 \to X$ be the blow up of X along Y by different system of parameters say $u_1, u_2, ..., u_m$ and $v_1, v_2, ..., v_m$. Then X_1 and X_2 are isomorphic such that the following diagram commutes.*

$$\begin{tikzcd} X_1 \arrow[r, "\phi"] \arrow[dr, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ & X \end{tikzcd}$$

We have defined the blow up of X along a subvariety Y in a neighborhood of a point $p \in Y$. Let $X = U_1 \cup U_2 \cup ... \cup U_n$ be an open affine cover of X. For each U_α we can define X_α as above. The idea of global blow up is to glue together these pieces. But we are mostly interested in the local pictures.

Example 3. Consider the surface $S \subset \mathbb{A}^3$ given by the following equation:

$$f(x, y, z) = x^2 - y^3 z^2.$$

Clearly, maximum possible multiplicity of a point in S is 2. The partial derivatives are as follows:

$$\frac{\partial f}{\partial x} = 2x, \ \frac{\partial f}{\partial y} = 3y^2 z^2, \ \frac{\partial f}{\partial z} = 2zy^3.$$

The singular locus is given by the algebraic subset,

$$x = 0, yz = 0.$$

We blow up along the curve $C := \{x = 0, y = 0\}$. The two open pieces of birational transform are as follows:

1. $x_1 = x$, $y_1 = \frac{y}{x}$, $z_1 = z$

Total transform: $x_1^2 - y_1^3 x_1^3 z_1^2$ Birational Transform: $1 - y_1^3 x_1 z_1^2$

2. $x_1 = \frac{x}{y}$, $y_1 = y$, $z_1 = z$

Total transform: $x_1^2 y_1^2 - y_1^3 z_1^2$ Birational Transform: $x_1^2 - y_1 z_1^2$

We see that, the second piece of the birational transform possess singularity of same multiplicity at the curve $C_1 := \{x_1 = 0, z_1 = 0\}$, so we blow up along C_1.

1. $x_2 = x_1$, $y_2 = y_1$, $z_2 = \frac{z_1}{x_1}$

Total transform: $x_2^2 + y_2^2 z_2^2 x_2^2$ Birational transform: $1 - y_2^2 z_2^2$

2. $x_2 = \frac{x_1}{z_1}$, $y_2 = y_1$, $z_2 = z_1$

Total transform: $x_2^2 z_2^2 - y_2 z_2^2$ Birational transform: $x_2^2 - y_2$

We see that, after two blow ups, the singularity of surface S has been resolved.

2.1.5 Rees-algebra and Proj

Definition 2.1.7 (Blow up of a Noetherian scheme along an ideal sheaf I)**.** Let X be a Noetherian scheme and let I be the sheaf of ideals on X. Consider the sheaf of graded algebras $\zeta = \oplus_{d \leq 0} I^d$. Then $Proj\zeta$ is defined to be the *blow up of X with respect to the ideal I.*

Example 4. Let $X = \mathbb{A}^n$, $p \in X$ be the origin and $I = (x_1, x_2, ..., x_n)$,

$$B(I) = ProjA \text{ where } A = \oplus_{d \leq 0}(x_1, x_2, ..., x_n)^d.$$

See that, the blow up variety here is same as the one previously defined. Indeed, the map

$$\phi : k[x_1, x_2, ..., x_n][y_1, y_2, ..., y_n] \to A$$

$$y_i \mapsto (0, x_i, 0, ...)$$

induces

$$\bar{\phi} : k[x_1, x_2, ..., x_n][y_1, y_2, ..., y_n] \backslash ker\phi \to A$$

where $Ker\phi = (\{x_i y_j = x_j y_i | i, j = 1, ..., n\})$. We see that the latter is an isomorphism.

Proposition (Universal property of blow ups). Let X be a Noetherian scheme, I be an ideal sheaf on X and $\pi : B(I) \to X$ be the blow up of X along the ideal sheaf I. If $f : Z \to X$ is a regular map such that $f^{-1}(I).\mathcal{O}_Z$ is an invertible sheaf of ideal on Z then, there exists a unique regular map $g : Z \to B(I)$ factoring f

Proof. For proof, refer [4] □

One can also define blow ups through universal property.

Definition 2.1.8. The blow up of a variety X along a closed subvariety Y is a regular map $\pi : X' \to X$ such that $E = \pi^{-1}(Y)$ is a divisor in X' and that for any morphism $f : Z \to X$ with $f^{-1}(Y)$ divisor in Y there exists a unique morphism $\phi : Z \to X'$ such that $f = \pi \circ \phi$.

2.2 Applications

Rational maps from a quasi projective variety X to $\mathbb{P}^n$ are not defined on the whole of X. Through blow ups we can extend these rational maps in some way to the whole of the variety.

Theorem 2.2.1 (Resolution of indeterminacy)**.** *Let X be a non-singular projective surface and $\varphi : X \to \mathbb{P}^n$ be a rational map. Then there exists a chain of blow ups $X_m \to ... \to X_1 \to X$ such that the composite rational map $\psi = \varphi \circ \pi_1 ... \circ \pi_m : X_m \to \mathbb{P}^n$ is regular.*

$$\begin{tikzcd}
X_m \arrow[d, "\phi_m"] \arrow[ddddr, "\psi"] & \\
\vdots \arrow[d, "\phi_2"] & \\
X_1 \arrow[d, "\phi_1"] & \\
X \arrow[r, "\varphi"] & Y
\end{tikzcd}$$

Proof. The proof will follow through several steps:

(i) We show that φ fails to be regular at finitely many points

Indeed, we will show it in a neighborhood of x. Since projective varieties are quasi compact the claim will follow. Let φ be given by $(f_0, ..., f_n)$ where $f_i \in k(X)$, then the point of indeterminacy is either where f_i fails to be regular or where all of them vanish. The first one can be dealt by multiplying all the $f_i's$ with a common factor to make sure that all f_i's are in $\mathcal{O}_x$ and have nothing in common. Next, we show that the points where all f_i's vanish is at most zero dimensional algebraic set. Suppose on contrary, there exists a curve in the locus of indeterminacy of φ, then in the neighborhood of x, it is generated by single element, say $C := V(f)$. Then, $f_i = fg_i$ for some $g_i \in \mathcal{O}_x$. But this is contradiction to the assumption that they have no factor in common.

(ii) We give explicit description of such points

Let $\bar{D} = \gcd\left(div\left(f_0\right), div\left(f_1\right), ..., div\left(f_n\right)\right)$ and $D_i = div\left(f_i\right) - \bar{D}$. The set of irregular point for φ is $\bigcap_i SuppD_i$

(iii) We associate an invariant to the rational map

For the rational map φ, we define $D(\varphi) := D_i^2$. We show that the invariant is bounded below by 0 and decreases eventually under blow up at suitable centers Notice that $d(\varphi') = 0$ implies that intersection of $SuppD_i'$ and $SuppD_j'$ is empty that proves the theorem.

(iv) We show that $d(\varphi) \geq 0$ and that it decreases under blow up:

Define $D_\lambda := div(\sum_{i=0}^{n} \lambda_i f_i) - \bar{D}$ where $\lambda = (\lambda_0, ..., \lambda_n) \in k_{n+1} - \{(0, 0, ..., 0)\}$ and $D_{(0,0,...,0)} = 0$. The set $\{\lambda | v_C(D_\lambda) > 0 \text{ for } C \subset SuppD_0\} \cup \{(0, 0, ..., 0)\}$ forms a proper subspace of k^{n+1}. Indeed, $D_i's$ possess no common component, therefore choose for every irreducible component $C \subset D_0$, D_i such that $v_C(D_i) = 0$. Let g_i be the local equations for the D_i in a neighborhood of some point $c \in C$ then $v_C(D_\lambda) > 0$ if and only if $\sum_{i=0}^n \lambda_i(g_i|C) = 0$.

So there exists a λ_0 such that $v_C(D_{\lambda_0}) = 0$ for every component $C \subset D_0$. Since D_0 and D_{λ_0} are effective divisors that have no common components, therefore, $D_i^2 = D_0.D_{\lambda_0} \geq 0$.

(v) $d(\varphi') < d(\varphi)$

We associate to each point ξ, multiplicity of a divisor $D = \sum_{i=1}^n l_i C_i$ as $k = \sum_{i=1}^n l_i k_i$, where k_i are the multiplicities of the C_i at ξ. Let v_i be the multiplicity of D_i at ξ and $v = \min\{v_i\}$. The map φ' is given by the functions $f_i' = \sigma^*(f_i)$, Then

$$div(f_i') = \sigma^* div(f_i) = \sigma^*(D_i + \bar{D}) = \sigma'(D_i) + v_i L + \sigma^*(\bar{D})$$

$$= \sigma'(D_i) + (v_i - v)L + vL + \sigma^*(\bar{D}).$$

Let $D_i' = div(f_i') - \gcd\{div(f_1'), div(f_2'), ..., div(f_n')\}$ then, $D_i' = \sigma'(D_i) + (v_i - v)L$. We choose i such that $v_i = v$. By definition, $d(\varphi') = (D_i')^2 = (\sigma'(D_i))^2$

$$= (\sigma^*(D_i) - vL)^2 = (\sigma^*(D_i))^2 - v^2 = (D_i)^2 - v^2,$$

and hence $d(\varphi') = d(\varphi) - v^2$. □

Theorem 2.2.2 (Castelnuovo's contractibility criterion). *Suppose X is a smooth projective surface. Let $\mathbb{P}^1 \simeq E \subset X$ a curve such that $(E.E) = -1$. Then there exists a smooth projective surface Y and a birational regular map $\phi : X \to Y$ such that $\phi(E) = y$ and $\phi : X \backslash E \to Y \backslash y$ is an isomorphism.*

Theorem 2.2.3 (Factorization as a chain of blow ups). *Let X and Y be projective surfaces and $\varphi : X \to Y$ be a birational map. Then there exists a surface Z and maps $\phi : Z \to X$ and $\psi : Z \to Y$ such that ϕ and ψ are composition of blow ups. In other words, there exists a commutative diagram.*

$$
\begin{tikzcd}
Z \arrow[d, "\phi_{m+1}"] \arrow[dr, "\psi_{m+1}"] & & & \\
X_m \arrow[d, "\phi_m"] & Y_m \arrow[dr, "\psi_m"] & & \\
\vdots \arrow[d, "\phi_1"] & & \ddots \arrow[dr, "\psi_1"] & \\
X \arrow[rrr, "\varphi"] & & & Y
\end{tikzcd}
$$

where $X_i's$ and $Y_i's$ are surfaces and $\phi_i's$ and $\psi_i's$ are blow up maps.

Proof. We shall prove it in several steps:

1. By previous theorem, it is sufficient to show that a birational morphism is composition of blow downs:

Let Z be the surface constructed in theorem 2.2.1, $\phi : Z \to X$ be the composition of blow ups and $\psi = \varphi.\phi$. We show that ψ is composition of finitely many blow ups, i.e. there exist surfaces Y_i and blow up maps $\psi_i : Y_i \to Y_{i-1}$ with $Y_0 = Y$ such that $\psi = \psi_n.\psi_{n-1}...\psi_1 : Z \to Y$.

2. For each point $y \in Y$ where ψ^{-1} fails to be regular, there exists a curve C such that $\psi(C) = y$:

Let $U \subset Z$ and $V \subset Y$ be open sets where ψ is an isomorphism and W be the closure of graph of $\psi : U \to V$ in $Z \times Y$. The first and second projection $p : W \to Z$ and $q : W \to Y$ are regular birational maps. Let $y \in Y$ be a point where ψ^{-1} fails to be regular. Then, q^{-1} also fails to be regular at y. There exists $w \in W$ such that $q(w) = y$. We look at the affine neighborhood S of w. In this neighborhood, $g = q^{-1}$ is given by $(g_1, g_2, ..., g_m)$. Then one of g_i say g_1 fails to be regular at y. If $g_1 = \frac{u}{v}$ where $u, v \in \mathcal{O}_{Y,y}$, then $v(y) = 0$. This implies, $q^*(v)(w) = v(y) = 0$, so that $w \in V(q^*(v))$. Set $D = V(q^*(v))$. Since q is regular, $q^*(u) = 0$ on D. So, $q(D) \subset V(u) \cap V(v)$. $\mathcal{O}_{Y,y}$ is a UFD, Y is non-singular, so we can assume u and v have no common factor which implies $q(D)$ is a point. Let $C = p(D)$, then C is a codimension one closed set.

3. Blow up at the indeterminacy locus of ψ^{-1}:

Consider the blow up $\sigma_1 : Y_1 \to Y$ at y and define $\psi_1' = \sigma_1^{-1}.\psi$. We see that, ψ' maps the subvariety $\psi^{-1}(y)$ to $\sigma^{-1}(y) = L \simeq \mathbb{P}^1$.

4. We show that ψ_1' is regular:

Suppose ψ_1' is not regular at $z \in Z$, then there exists a curve in Y_1 such that $\psi_1'^{-1}(Y_1) = z$. Notice that $Y_1 = L$, in which case there exists a finite subset of L where $\psi_1'^{-1}$ fails to be regular. It follows that, $\psi(z) = y$, and that the tangent spaces at z and x are isomorphic, isomorphism being given by,

$$d_z\psi : \theta_{Z,z} \to \theta_{Y,y}.$$

But that is the contradiction to the fact that there exists a curve C such that $\psi(C) = y$. Hence, ψ_1' is a regular map.

5. After finitely many blow ups, ψ_m' is an isomorphism:

Since, ψ_1' maps Z onto the whole of Y', it follows that it maps $\psi^{-1}(y)$ onto the whole of L. There exists at least one component of $\psi^{-1}(y)$ that maps onto L and therefore, the number of components of $(\psi')^{-1}(y')$ is less than the number of components of $\psi^{-1}(y)$ for any $y' \in L$. Since the components are finitely many, we see that after finitely many blow ups, there are no exceptional divisors in X. $\square$

Remark. The statement that every rational map is a composition of blow ups is not true, counter example is given by Hironaka. Theorem 2.2.3 in arbitrary dimension is an open problem. While, resolution of indeterminacy in arbitrary dimension has been proved, The proof is due to Hironaka.

Example 5 (Quadratic/Cremona transformation)**.** Let $p_1 = (0 : 0 : 1), p_2 = (0 : 1 : 0), p_3 = (0 : 0 : 1) \in \mathbb{P}^2$ be three points. We call the three lines $L_1 := V(x_0), L_2 := V(x_1), L_3 := V(x_2)$ as exceptional lines. Notice that L_i and L_j intersect at p_k and the line L_i passes through p_j and p_k for $i \neq j \neq k$.

Definition 2.2.4. Consider $\mathbb{P}^2$ with homogenous coordinate $(x_0 : x_1 : x_2)$. The birational involution of $\mathbb{P}^2$

$$\tau : \mathbb{P}^2 \to \mathbb{P}^2$$

given by

$$(x_0 : x_1 : x_2) \mapsto (x_0^{-1}, x_1^{-1}, x_2^{-1}) = (x_1x_2 : x_2x_0 : x_0x_1)$$

Observations:

The map is not defined at the points p_1, p_2, p_3, while the lines L_1, L_2, L_3 are contracted to p_1, p_2, p_3 respectively. The map τ can be seen as composition of blow up at the three points and then contraction of the three lines. Consider the open set $U = \mathbb{P}^2 \setminus \{L_1 \cup L_2 \cup L_3\}$. The τ is an isomorphism on U inverse of which is τ itself. Indeed,

$$(x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_0x_2 : x_0x_1) \mapsto (x_0x_2x_0x_1 : x_1x_2x_0x_1 : x_1x_2x_0x_2).$$

This is the classic example where the birational map τ can be written as composition of blow ups and blow downs.

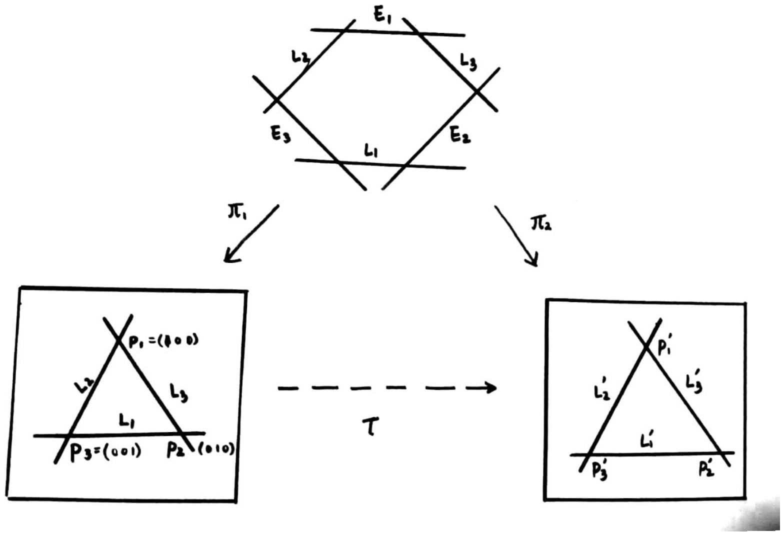


Figure 2.2: Cremona transformation

Cremona transformation and Resolution

Theorem 2.2.5. *Suppose $C \subset \mathbb{P}^2$ be a plane curve over an algebraically closed field of characteristic zero. Let $p \in C$ be a point of multiplicity $m_0 \geq 2$. Then there exists a finite sequence of Cremona transformation $\tau = \tau_1 \circ ... \circ \tau_n : \mathbb{P}^2 \to \mathbb{P}^2$ centered at p_i such that the birational transform p_n is an ordinary multiple point where p_i is the preimage of p under π_i.*

Proof. Let $p \in C$ be a singular point with multiplicity m_0. We take two general lines through p that are not contained in the tangent cone of p and have only transversal intersection with C. Choose another line that does not pass through p and intersect C transversely. Such choices are possible because tangent cones at different points are proper closed subvarieties of dual space $\mathbb{P}^{2^*}$. We mark the points of intersection as p, q, r. We can assume the points to be the standard points $p = (0 : 0 : 1), q = (0 : 1 : 0), r = (1 : 0 : 0)$. Let the multiplicity of the three point in C be m_0, m_1, m_2 respectively, and $f(x, y, z)$ be the equation of C and f' be the birational transform of f, then,

$$f'(x, y, z) = x^{-m_2} y^{-m_1} z^{-m_0} f(yz, xz, xy)$$

The generosity condition makes sure that the singularities of C outside p are unchanged. But three new ordinary singularities introduced, have multiplicities $\deg C$, $\deg C - mult_p C$ and $\deg C - mult_p C$ each. So, if we assume embedded resolution for curves, we are done.

But we introduce another invariant i.e. *apparent genus*. Let $C \subset \mathbb{P}^2$ be a curve, we associate a number

$$g_{app} = \frac{d-1!}{2(d-3!)} - \sum_i \frac{m_i!}{2(m_i-2)!}$$

where m_i are the multiplicities of singular points and d is the degree of the curve. With the observation as above, see that,

$$\begin{aligned} g_{app}(C_1) &< \frac{(2d-m_0-1)!}{2(2d-m_0-3)!} - \sum_{i>0} \frac{m_i!}{2(m_i-2)!} - \frac{d!}{2(d-2)!} - 2\frac{(d-m_0)!}{2(d-m_0)!} \\ &= \frac{d-1!}{2(d-3!)} - \sum_i \frac{m_i!}{2(m_i-2)!} = g_{app}(C) \end{aligned}$$

Now, we show that $g_{app}(C) \geq 0$ and the result will follow. Let W be the vector space of all the curves that have degree $d-1$ and multiplicity of $p_i \in C$ is $m_i - 1$. Notice that $W \neq \emptyset$, since, $\frac{\partial f}{\partial x} \in W$ where f is the defining equation for C. Let $C' \in W$. Then the restriction of C' on C denotes a divisor on C such that

$$v_{p_i}(C) \geq m_i(m_i - 1).$$

Consider the divisor, $D = \sum m_i(m_i-1)p_i$. Then,

$$\deg(C - D) = d(d-1) - \sum m_i(m_i-1) = 2g_{app}(C) + 2(d-1).$$

On the other hand,

$$\dim_k L(C-D) = \frac{(d+1)!}{2(d-1)!} - 1 - \sum_i \frac{m_i!}{2(m_i-1)!} = g_{app}(C) + 2(d-1).$$

Now since,

$$\dim(C-D) \leq \deg(C-D) + 1,$$

we have,

$$g_{app} \geq 0.$$

$\square$

2.3 Resolving through blow ups

2.3.1 General discussion

This has been overly stated, but we intend to prove that blow ups resolve singularity. Before we move on to the proof, we show that the blow up map is proper.

Lemma 2.3.1. $B(p) \to \operatorname{Spec}(R)$ *is a proper map.*

Proof. Suppose V is a valuation ring containing R. Then $\frac{x}{y}$ or $\frac{y}{x} \in V$. Say $\frac{y}{x} \in V$ Then, $R[\frac{y}{x} \subset V]$ and we have a morphism

$$\operatorname{Spec} V \to \operatorname{Spec}\left[\frac{y}{x}\right] \subset B(p)$$

which lifts the morphism $\operatorname{Spec} V \to \operatorname{Spec} R$ $\square$

We will give a proof of embedded resolution i.e. we resolve singularities for hypersurfaces. A bridging between general resolution and embedded resolution is as follows:

Theorem 2.3.2 (Principalisation of ideal sheaf)**.** *Let V be a smooth surface over perfect field k and $I \subset \mathcal{O}_V$ be an ideal sheaf. Then there exists a birational proper morphism $\pi : V' \to V$ such that V' is smooth and $I\mathcal{O}_{V'}$ is locally principal.*

Theorem 2.3.3. *Resolution of singularities for projective hypersurface of dimension n and principalisation of ideal of non-singular varieties of dimension n implies the resolution of singularities of projective variety of dimension n.*

Proof. Let V be a projective variety over a perfect field k. Then V is birational to a hypersurface of some projective space $\mathbb{P}^m$ say $W = V(f)$. Let $\phi : W \to V$ be the birational map. Let Γ be the closure of the graph ϕ in $V \times W$. Let $f_1 : \Gamma \to V$ and $f_2 : \Gamma \to W$ be the natural projections.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p"] & \Gamma \arrow[d, "f_2"] \arrow[dr, "f_1"] & \\
W' \arrow[r, "\pi"] & W \arrow[r, "\phi"] & V
\end{tikzcd}$$

Let W' be the resolution of singularity for the hypersurface W. Γ is blow up of W at the ideal J where J is the indeterminacy locus of ϕ. Let $p : X \to W'$ be the principalisation of ideal $\pi^{-1}(V(J))$. Then there exists a morphism $f : X \to \Gamma$ by universal property of blow ups and therefore, $f_1 \circ f$ is the resolution for V. $\square$

Definition 2.3.4. Let $Y \subset X$ be a hypersurface given by the equation $f(x_1, x_2, ..., x_n) = 0$. Let p be the singular point of $V(f)$, we assume that $p = (0, 0, ..., 0)$. We write f as:

$$f(x_1, x_2, ..., x_n) = f_m(x_1, x_2, ..., x_n) + f_{m+1}(x_1, x_2, ..., x_n) + ...f_l(x_1, x_2, ..., x_n)$$

where f_i is the homogenous polynomial of degree i. Then m is said to be the multiplicity of Y at $(0, 0, ..., 0)$.

Consider a hypersurface $Y \subset X$ with equation $(f = 0)$. Let $(0, 0, ..., 0) \in Y$, and $(x_1, x_2, ..., x_n)$ be the local parameters of $(0, 0, ..., 0)$. Then there is a largest power $(x_1, x_2, ..., x_n)^m$ of $(x_1, x_2, ..., x_n)$ that contains f, this m is the multiplicity of Y at $0 \in X$. Similarly, for higher dimensional singularity Z, we say that Z sits in the hypersurface $V(f)$ with multiplicity m if m is largest such that $f \in (u_1, u_2, ..., u_n)^m$ but $\notin (u_1, u_2, ..., u_k)^{m+1}$ where $\{u_1, u_2, ..., u_k\}$ are the local equations of Z in a neighborhood of some point.
What happens to the curve $C \subset \mathbb{A}^2$ as divisor under point blow up?
We have seen in 2.1.3 that on the chart $B_0\mathbb{A}^2 = \operatorname{Spec} k[\frac{y}{x}, x]$, the pull-back of $f \in k[x, y]$ is $f' \in k[y_1, x_1]$ given by $f'(x_1, y_1) = f(x_1, x_1y_1) = x_1^m f_1(x_1, y_1)$. Thus, the preimage of C contains the exceptional curve $E = V(x_1)$ with multiplicity m (defined by $(x^m = 0)$ on the $v \neq 0$ chart), and the birational transform of C, denoted by C_1, is defined by $(f_1 = 0)$ in the chart $v \neq 0$:

$$\pi^*(C) = (mult_0 C) \cdot E + C_1.$$

Lemma 2.3.5. *Let the notation be as above. The intersection points $C_1 \cap E$ are the roots of $(f_m(1, y) = 0) \subset \mathbb{P}^1 \simeq E$. More precisely, when counted with multiplicities, $C_1 \cap E = |(f_m(1, y) = 0)|$ where $|.|$ denote the cardinality of the set. Thus,*

(1) the intersection number $(C_1 \cdot E)$ equals $mult_0 C$, and

(2) If $p \in C_1 \cap E$, then $mult_p C_1 \leq mult_0 C$

Proof. Let $f(x, y) = f_m(x, y) + r(x, y)$, where $r(x, y) \in (x, y)^{m+1}$ $f_1(x_1, y_1) = f_m(y_1, 1) + x_1 r_1(x_1, y_1)$. Hence, the intersection points of C_1 and E is the solutions of $f_m(1, y_1) = x_1 = 0$, are the points $f_m(y_1, 1) = 0$ on E. Furthermore, multiplicity of C_1 at $(y_1 = a, x = 0)$ is less than the multiplicity of $y_1 = a$ as a root of $f_m(y_1, 1) = 0$. □

Theorem 2.3.6 (Weierstrass preparation theorem)**.** *Let $F(x, y) \in K[[x, y]] = K[[y]][[x]]$ be a power series. Assume that y^m appears in F with non-zero coefficient and m is the smallest such exponent. Then one can write $F(x, y)$ uniquely as*

$$F(x, y) = (y^m + g_{m-1}(x)y^{m-1} + \dots + g_0(x)).(unit)$$

where $g_i \in K[[x]]$.

Lemma 2.3.7 (Abstract Hensel's lemma)**.** *Let R be a ring and $F(x) := \sum_{i=0}^{\infty} r_i x^i$ a power series in x with coefficient in R. Assume that,*

$$r_0 = g_0 h_0$$

and g_0 and h_0 are coprime. Then there exist $H(x)$ and $G(x)$ with $H(0) = h_0$ and $G(0) = g_0$ such that,

$$F(x) = G(x)H(x).$$

Moreover, if we fix representatives r^* for every residue class $R/(g_0)$, then there is a unique solution where $G(x) = g_0 + \sum_{i \geq 1} g_i^* x^i$.

Proof. We find $G(x)$ and $H(x)$ inductively. Suppose $G_m(x)$ and $H_m(x)$ are already defined such that,

$$G_m(x)H_m(x) \equiv F(x) \mod (x^{m+1}).$$

We wish to define G_{m+1} and H_{m+1}. For this we solve,

$(G_m(x) + r_{m+1}x^{m+1})(H_m(x) + s_{m+1}x^{m+1}) \equiv F(x) \mod (x^{m+2})$

$G_m(x)H_m(x) + x^{m+1}(r_{m+1}H_m(x) + s_{m+1}G_m(x)) + x^{2m+2}(r_{m+1}s_{m+1}) \equiv$
$F(x) \mod (x^{m+2})$

But since

$$G_m(x)H_m(x) \equiv F(x) + f_{m+1} \mod x^{m+2}$$

for some f_{m+1}. We need to solve,

$$r_{m+1}h_0 + s_{m+1}g_0 = f_m$$

Since, h_0 and g_0 are relatively coprime, we have

$$a_0h_0 + b_0g_0 = 1$$

for some $a_0, b_0 \in R$. Hence, $r_{m+1} = a_0f_m$ and $s_{m+1} = b_0f_m$ are the required candidates for r_{m+1} and s_{m+1}. Now let

$$ug_0 + vh_0 = f_m$$

is any other solution then,

$$v = g_0s + g_{m+1}.$$

Put this in 2.3.1 and see that,

$$ug_0 + g_0h_0s + g_{m+1}h_0 = f_m.$$

This g_{m+1} is unique because we have set a unique representative for every element r in the residue class $R/(g_0)$. □

Proof of theorem 2.3.6. According to the hypothesis,

$$F(0, y) = y^m + f(y) + x(r(x, y)))$$

where $\deg(f(y)) \geq (m + 1)$. This implies

$$F(0, y) = y^m.u(y)$$

for some unit $u(y)$. Since $u(y)$ is a unit, all the hypothesis of Abstract Hensel's lemma are fulfilled, and we have $g(x, y) = y^m + x(r(x, y))$ for some $r(x, y)$ and $h(x, y) =$ some unit. Since, $\deg_y r(x, y) \leq m$, the result follows. □

Remark. We have seen that the multiplicity at a point as an invariant may not always decrease under blow up. Therefore, to read the transformations better, we have to introduce new invariants. We shall invoke Weierstrass preparation theorem, and do the process in an analytic neighborhood of a point. A global resolution can then be obtained by patching up the local charts.

2.3.2 Resolution of curves

Let k be an algebraically closed field of characteristic 0.

Theorem 2.3.8 (Embedded resolution for curves :Local)**.** *Let $C \subset S$ be a curve embedded in a non-singular surface S defined by $f = 0$ over k. Then the sequence of blow up of S_i*

$$\pi_n : S_n \to ... \to S_1 \to S$$

at singular points of the curve C_i where C_i is the birational transform of the curve $C \subset S$ in S_i, terminate.

Proof. We denote $Sing_m S := \{p \in C | v_p(C) \geq m\}$.

Let $m = max \{r | Sing_r(C) \neq \phi\}$, then we call $Sing_m S$ as *top Locus*. Clearly $1 < m$. We will show that after finitely many steps the maximum multiplicity drops. So, let us suppose that there exists an infinite sequence of blow up

$$... \to S_n \to S_{n-1} \to ... \to S_1 \to S$$

along the points $p_i \in C_i \subset S_i$ and p_i sits in C_i with multiplicity m such that $\pi_i(p_i) = p$ where $\pi_i : S_i \to S$ is the composition of blow ups. This induces an infinite sequence of completion of local rings around p_i

$$R_0 \to R_1 \to ... \to R_n \to ...$$

Let (x, y) denote the local parameters for the point p and (x_i, y_i) be the local parameters for the points p_i. By linear change of parameters, assume $y^m = f_m$. We call f to be of the normal form in such case. By Weierstrass preparation theorem and Tschirnhausen transformation, f has the form

$$f = y^m + a_2(x)y^{m-2} + ... + a_m(x).$$

We see that, normal forms are preserved under blow ups. So we let the birational transform f_i of f be given by:

$$f = y_i^m + a_{2,i}(x_i)y_i^{m-2,i} + ... + a_{m,i}(x_i)$$

where x_i, y_i are the local parameters of p_i given by either of the following:

1. $x_i = \frac{x_{i-1}}{y_{i-1}}, y_i = y_{i-1}$

2. $x_i = x_{i-1}, y_i = \frac{y_{i-1}}{x_{i-1}}$
We claim that the invariant $n_j := \min\left\{mult(\frac{a_{ij}(x_i)}{i})\right\}$ decreases under blow up. Indeed, we see that $n \geq 1$ for $j = 0$. The part of birational transform in the two affine open sets are as follows:
1. $f_1(x_1, y_1) = y_1^m + \frac{a_2(x_1)}{x_1^2} y_1^{m-2} + ... + \frac{a_m(x_1)}{x_1^m}$
2. $f_1(x_1, y_1) = 1 + \frac{a_2(x_1 y_1)}{y_1^2} + ... + \frac{a_m(x_1 y_1)}{y_1^m}$
Note that, f_1 is a unit in the second chart. This implies $v_{p_1}(C_1) = 0$

In the first chart, we see that, $n_1 \leq n_0 - 1$. We iterate the process till $n_i < 1$. The birational transform of the final curve have no points of multiplicity m. Therefore, the result follows by induction on the maximum multiplicity m. $\square$

Theorem 2.3.9 (Embedded resolution for curves : Global)**.** *Let $C_0 \subset S_0$ be a projective curve inside a smooth surface over a algebraically closed field k. Then the sequence of blow ups of S_i*

$$... \rightarrow S_n \rightarrow S_{n-1} \rightarrow ... \rightarrow S_1 \rightarrow S$$

along singular points of C_i terminates.

Proof. Let $C_i \subset S_i$ be the birational transform at i_{th} step, K_S denote the canonical divisor associated to the surface S. We prove the theorem in two steps:
(i) the intersection number $C \cdot (K_S + C)$ decreases at each step: Let p be the point of singularity then $m = mult_p C > 1$. Let C' be the birational transform of C, then

$$(C' \cdot C') = (\pi^* C + mE) \cdot (\pi^* C + mE)$$

$$= \pi^* C \cdot \pi^* C + 2m(\pi^* C \cdot E) + m^2 (E \cdot E)$$

$$= C \cdot C - m^2$$

And,

$$C' \cdot K_{S'} = ((\pi^* C - mE) \cdot (\pi^* K_S + E))$$

$$= (C \cdot K_S) - m(E \cdot E)$$

Combining the two we get,

$$(C' \cdot C') + C' \cdot K_{S'} = C \cdot C - m^2 + (C \cdot K_S) + m$$

$$C' \cdot (C' + K_S') = C \cdot (C + K_S) - m(m-1)$$

(ii) the intersection number $C \cdot (C + K_S)$ is bounded below:

Let $f : C \subset \mathbb{P}^n$, $\pi : \mathbb{P}^n \to \mathbb{P}^1$ be a composition of projection through points outside C. Then, $\pi|_C : C \to \mathbb{P}^1$ is a separable morphism of degree n. We have the following lemma

Lemma 2.3.10. *Let $f : X \to Y$ be a separable morphism. Then there exists an exact sequence of sheaves on X.*

$$0 \to f^*\Omega_Y \to \Omega_X \to \Omega_{XY} \to 0$$

Proof. Refer [4]. □

It implies the injectivity

$$f^*\Omega_{\mathbb{P}^1} \hookrightarrow \Omega_C.$$

The blow up map induces a separable morphism of degree n

$$f.\pi_n : C_n \to \mathbb{P}^1.$$

Now since, $f^*\mathcal{O}_{\mathbb{P}^1}(-2) \simeq f^*\Omega_{\mathbb{P}^1}$,

$$\deg \Omega_{C_i}/(torsion) \geq -2n,$$

Further we have, the injection

$$\Omega_C/(torsion) \hookrightarrow \mathcal{O}_S(C + K_S)|_C$$

Indeed, We note that, $\mathcal{O}_S(K_S + C)|_C$ is locally generated by $f^{-1}dx \wedge dy$. Its residue along C is as follows:

$$\frac{1}{f}dx \wedge dy = \frac{df}{f} \wedge \sigma.$$

It follows that $\mathcal{O}_S(K_S+C)|_C$ is locally generated by $\sigma|_C$. We can identify $\mathcal{O}_S(K_S+C)|_C$ with Ω_C along the smooth points. We show that it does not possess poles.

Since, $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$, we have,

$$\frac{1}{f}dx \wedge dy = \frac{df}{f} \wedge \frac{dy}{\frac{\partial f}{\partial x}} = -\frac{\partial f}{f} \wedge \frac{dx}{\frac{\partial f}{\partial y}}.$$

It follows that,

$$\sigma|_C = \frac{dy}{\frac{\partial f}{\partial x}} = -\frac{dx}{\frac{\partial f}{\partial y}}.$$

Finally, $dx|_C = -(\frac{\partial f}{\partial y})\sigma_C$ and $dy|_C = (\frac{\partial f}{\partial x})\sigma_C$ both have zeroes. □

Simple Normal Crossing Divisor

Definition 2.3.11 (Simple Normal Crossing Divisor)**.** Let X be a smooth variety and $D \subset X$ be a divisor. We say that D is a *simple normal crossing divisor* if every irreducible component of D is smooth and all the intersections are transversal. Two components C_1, C_2 are said to intersect transversally if the intersection number $C_1 \cdot C_2 = 1$. The other definition of simple normal crossing divisor is as follows:
we call a divisor $D = \sum_{i=1}^{n} k_i C_i \subset X$ to be simple normal crossing if each component C_i is smooth and for every point $x \in D$, there exist regular parameters $\{x_1, x_2, ..., x_n\} \in m_{x,X}$ such that D is defined by $x_1 x_2 ... x_n = 0$ in a neighborhood of x.

Example 6. The curve $y^2 = x^3 + x^2$ is not a simple normal crossing divisor. But the divisor $xy = 0$ is simple normal crossing divisor.

Theorem 2.3.12. *Let $C \subset S$ be a curve embedded in the smooth surface S over an algebraically closed field. Then after finitely many blow ups the preimage $\pi_m^{-1}(C)$ is a simple normal crossing divisor in S_m.*

Proof. The theorem is achieved by double application of embedded resolution. First, let $C_n \subset S_n$ be non-singular for some suitable n and then to the divisor $C_n + E_n \subset S_n$ where E_n is the Exceptional Curve for the map $\pi_n : S_n \to S$ □

2.3.3 Resolution of surfaces

Theorem 2.3.13 (Ultimate theorem)**.** *Let S be a projective surface over an algebraically closed field k of characteristic zero then there exists a resolution of singularity for S.*

Theorem 2.3.14. *Let $S \subset V$ be a hypersurface (projective surface) in smooth three-dimensional variety V over an algebraically closed field of characteristic zero. Then there exists a sequence of blow ups along non-singular curves/points on the S_i i.e. birational transforms of S in V_i in the sequence*

$$V' = V_n \to V_{n-1} \to ...V_1 \to V$$

such that S', birational transform of S in V' is non-singular.

Let $Sing_m S = \{p \in S | v_p(S) \geq m\}$. The aim is to blow up the surface along curves and points in $Sing_m S$. We will prove that after finitely many blow ups $Sing_m S'$ is empty, and therefore the result will follow by induction. Before we move on to the proof of theorem 5.2.3, let us set the notations. Let $S \subset V$ be the surface embedded in a smooth 3-fold defined by $f(x, y, z)$. Suppose that z^m appear in f with positive coefficient, then by Weierstrass preparation theorem and Tschirnhausen transformation, we have,

$$f(x, y, z) = z^m + a_2(x, y)z^{m-2} + ... + a_m(x, y) \tag{2.2}$$

We say that f is in the *normal form.* Notice that, $p \in S$ is a point with multiplicity m if and only if it lies on the hyperplane $\{z = 0\}$ and sits in the coefficient curve $a_i(x, y)$ with multiplicity at least i. We denote $V_1, V_2, ...$ as blow up of V and $S_1, S_2, ...$ denote the birational transform of S defined by f_i in V_i. Also note that the normal form is preserved under blow up.

Blow ups along points and curves on S

1. Point blow up

Let p be a point of multiplicity m on S and (x, y, z) be the local parameters of p. The three possible local parameters for the points in $\pi^{-1}(p) \cap \bar{S}$ be (x_1, y_1, z_1) where x_1, y_1, z_1 are as follows:

1. $x_1 = x$, $y_1 = \frac{y}{x}$ $z_1 = \frac{z}{x}$
2. $x_1 = \frac{x}{y}$, $y_1 = y$ $z_1 = \frac{z}{y}$
3. $x_1 = \frac{x}{z}$, $y_1 = \frac{y}{z}$ $z_1 = z$

The birational transform corresponding to the three neighborhoods are as follows:

1. $z_1^m + \frac{a_2(x_1, y_1x_1)}{x_1^2} z_1^{m-2} + ... + \frac{a_m(x_1, y_1x_1)}{x_1^m}$
2. $z_1^m + \frac{a_2(x_1y_1, y_1)}{y_1^2} z_1^{m-2} + ... + \frac{a_m(x_1y_1, y_1)}{y_1^m}$
3. $1 + \frac{a_2(x_1z_1, y_1z_1)}{z_1^2} + ... + \frac{a_m(x_1z_1, y_1z_1)}{z_1^m}$

Since the third equation is a unit, the first two covers the whole birational transform, and we will often just work with them without stating.

2. Curve blow up

The curve lie on the hyperplane $H := \{z = 0\}$. By change of coordinates, assume the curve is given by $x = 0$. Let $q \in \pi^{-1}(p) \cap \bar{S}$, local parameters for q be $(x_1, y_1.z_1)$

where x_1, y_1, z_1 are as follows:

1. $x_1 = x$, $y_1 = y$, $z_1 = \frac{z}{x}$
2. $x_1 = \frac{x}{z}$,$y_1 = y$,$z_1 = z$

The birational transform corresponding to the two neighborhoods are as follows:

1. $z_1^m + \frac{a_2(x_1,y_1)}{x_1^2} z_1^{m-2} + ... + \frac{a_m(x_1,y_1)}{x_1^m}$
2. $1 + \frac{a_2(x_1 z_1,y_1)}{z_1^2} + ... + \frac{a_m(x_1 z_1,y_1)}{z_1^m}$

Again, since the second equation is a unit, therefore any point $q \in \pi^{-1}(p) \cap \bar{S}$ has local parameters $x_1 = x$, $y_1 = y$, $z_1 = \frac{z}{x}$ under the blow up of S along the curve $V(x, z)$

Lemma 2.3.15. *Let $C \subset Sing_m S$ be a non-singular curve and $\pi : V_1 \to V$ be the blow up at C and $q \in \pi^{-1}(p) \cap S_1$ for some $p \in C$. Then $v_q(S_1) \leq m$. Also there is at most one point $q \in \pi^{-1}(p) \cap \bar{S}$ with $v_q(S_1) = m$. In particular if $E = \pi^{-1}(C)$ then $E \cap Sing_m(S_1)$ is either a curve that maps isomorphically to C under π or is a set with only finite number of points.*

Proof. We show it in an analytic neighborhood of p, i.e., we look at the completion of the local ring $\mathcal{O}_p$. Let $I_{C,p} = (x, z)$ then, $f \in (I_{C,p})^m$ and x^i divides $a_i(x, y)$. Let $q \in \pi^{-1}(p) \cap \bar{S}$, the local parameters of q be (x_1, y_1, z_1) where $x_1 = x, y_1 = y, z_1 = \frac{z}{x}$. Then the birational transform is given by $z_1^m + \frac{b_2(x_1,y_1)}{x_1^2} z_1^{m-2} + ... + \frac{b_m(x_1,y_1)}{x_1^m}$. Clearly, $v_q(f_1) \leq m$ □

Lemma 2.3.16. *Suppose $p \in Sing_m S$ be a point and $\pi : B(p) \to S$ be the blow up at p. Let $\bar{S}$ be the birational transform of S in $B(p) and E = \pi^{-1}(p)$. Then $v_q(\bar{S}) \leq m$ for all $q \in \pi^{-1}(p)$. In particular, $E \cap Sing_m(\bar{S})$ is either a curve or is finite number of points.*

Proof. Proof is the same as the case of curve. □

Definition 2.3.17 (Good points)**.** We call a point $p \in S$ to be a pre good point if in a neighborhood of p, $Sing_m S$ is either empty, a non-singular curve through p or simple normal crossing there. Further we call a pre good point p, good if for any sequence

$$V_n \to V_{n-1} \to ... \to V_1 \to V$$

of blow ups of non-singular curves in $Sing_m(S_i)$ where S_i here is the birational transform of $S \cap \mathrm{Spec}(\mathcal{O}_{V,p})$

If a point is not good, we call it a bad point.

Theorem 2.3.18. *Suppose all the points of S are good. Then there exists a sequence of blow ups*

$$V' = V_n \to V_{n-1} \to ... \to V_1 \to V$$

of non-singular curves in $Sing_m(S_i)$ such that $Sing_m(S') = \emptyset$ where S' is the birational transform of S in V'.

Proof. On contrary let us suppose, there exists an infinite sequence

$$... \to V_n \to V_{n-1} \to ... \to V_1 \to V \tag{2.3}$$

of non-singular curves in $Sing_m S_i$, then there exist curves $C \subset Sing_m S$ and $C_i \subset Sing_m S_i$ such that C_i maps to C under π_i where $\pi_i : V_i \to V$ is composition of blow up maps. Consider the two-dimensional regular local ring $R = \mathcal{O}_{V,C}$. Let I_S be the height one prime ideal and I_C be the maximal ideal. Then by dimension formula,

$$\dim R + trdeg(R/P) = 3$$

therefore, $trdeg(R/P) = 1$ over k. Let t be the basis, then $k[t] \cap P = \emptyset$ implies $k(t) \subset R$. Then the ring $R = A_Q$ for some finitely generated $k(t)-$algebra A where Q is the maximal ideal in A. Therefore, R is a local ring of a non-singular point q on $k(t)$-surface $\operatorname{Spec} A$ and q sits in the curve I_S with multiplicity m. Then the infinite sequence 2.3 induces a sequence of blow up of non-singular $k(t)$-surface at points q_i that sits in the curve I_{S_i} with multiplicity m as follows:

$$... \to V_n \times_V \operatorname{Spec}(R) \to V_{n-1} \times_V \operatorname{Spec}(V) \to ... \to \operatorname{Spec}(R)$$

This contradicts embedded resolution for curves. □

Lemma 2.3.19. *Number of bad points are finitely many.*

Proof. Let $B_i = \{\text{isolated points of } Sing_m S_i\} \bigcup \{\text{singular points of } Sing_m S_i\}$ where S_i is the birational transform of S under the blow ups of V_{i-1} along the open subset(subscheme) $Sing_m(S_{i-1}) - B_{i-1}$. Note that, $Sing_m S_i - B_i$ is non-singular one dimensional subscheme of S_i. We show that the sequence terminates. On contrary let us suppose

$$... \to V_n - B_n \to ... \to V_1 - B_1 \to V \tag{2.4}$$

is an infinite sequence of blow up along the curves in $Sing_mS_i - B_i$. We denote the map $\pi_n : V_n - B_n \to V - T_n$ where

$$T_n = B_0 \cup \pi_1(B_1) \cup ... \cup \pi_n(B_n)$$

Let $C \subset Sing_mS$ be a curve. We have seen that $\operatorname{Spec} \mathcal{O}_{S,C}$ is a singularity of a curve of multiplicity m. The sequence 2.4 induces a sequence by base change an infinite sequence as follows:

$$S_n \times_S \operatorname{Spec}(\mathcal{O}_{S,C}) \to S_{n-1} \times_S \operatorname{Spec}(\mathcal{O}_{S,C}) \to ... \to S_1 \times_S \operatorname{Spec}(\mathcal{O}_{S,C}) \to \operatorname{Spec}(\mathcal{O}_{S,C})$$

The above is an infinite sequence of open subset of blow up of points over $\operatorname{Spec} \mathcal{O}_{S,C}$. We know that, the sequence terminates after finite number steps . Hence no curve in $Sing_mS_n$ dominate C. For such n, $Sing_mS_n \bigcap V_n - B_n$ is empty. $\square$

Next target is to resolve bad points.

Theorem 2.3.20. *The sequence of blow up of V_i at bad points terminate after finitely many steps.*

Proof. On contrary, let us suppose, there exists an infinite sequence of blow ups at bad points p_i in $S_i \subset V_i$:

$$... \to V_n \to V_{n-1} \to ... \to V_1 \to V.$$

This induces an infinite sequence of completion of local rings around p_i

$$R_0 = \mathcal{O}_{V,p} \to R_1 \to ... \to R_n \to ...$$

Let (x, y, z) be the local parameters of the point p in V and the local parameters of the points p_i be (x_i, y_i, z_i) where x_i, y_i, z_i are as follows:
1. $x_{i-1} = x_i$, $y_{i-1} = x_iy_i$, $z_{i-1} = x_iz_i$
2. $x_{i-1} = x_iy_i$, $y_{i-1} = y_i$, $z_{i-1} = y_iz_i$
3. $x_{i-1} = x_iz_i$, $y_{i-1} = y_iz_i$, $z_{i-1} = z_i$
By Weierstrass preparation theorem, we write

$$f_i = z_i^m + a_{2,i}(x_i, y_i)z^{m-2} + ... + a_{m,i}(x_i, y_i)$$

where $a_{ji}(x_i, y_i) = \left\{ \frac{a_{j,i-1}(x_i, x_iy_i)}{x_i^j} \text{ or } \frac{a_{j,i-1}(x_iy_i, y_i)}{y_i^j} \right\}$ for all j. Without any loss of generality, we assume the union of coefficient curves is simple normal crossing divisor in

R_i. Let $f_i = z_i^m + b_{2,i}(x_i,y_i)x_i^{a_{2,i}}y_i^{b_{2,i}}z_i^{m-2} + ... + b_{m,i}(x_i,y_i)x_i^{a_{m,i}}y_i^{b_{m,i}}$ where $b_{j,i}(x_i,y_i)$ are units in R_i and $a_{ji}+b_{ji} \geq j$ for all $b_{j,i} \neq 0$.

Claim: $b_{ji} \neq 0, a_{k,i} \neq 0$ for some j and k.

Proof of claim: On contrary suppose there exists an i such that $b_{ji} = 0$ for all j. Then $x_i^j/a_{j,i}(x_i,y_i)$ and $\hat{I}(Sing_m(S_i)) = (x_i,z_i)$. Let $\lambda : W \to V_i$ be the blow up at the curve (x_i,z_i). Then, $q \in \lambda^{-1}(p_i)$ have regular parameter (u,v,y) where $x_i = u, z_i = uv$, or $x_i = uv, z_i = v$ and we denote $f' = v^m + d_{2,i}(u,uv)u^{a_{2,i}-2}v^{m-2} + ... + d_{m,i}(u,uv)u^{a_{m,i}-m}$ to be the equation of the birational transform S' of S_i in W for the first neighborhood, while f' is a unit in the other neighborhood, in which case $v_q(S') < m$ For the first chart, we see that the number $n = min\left\{\frac{a_{j,i}}{j} | 2 \leq j \leq r\right\}$ is decreased by one. After finitely many blow ups of non-singular curves in $Sing_m$, the multiplicity must reduce to be strictly less than m for all the points p_i. This implies p_i is not a bad point which is a contradiction. It follows that, $b_{j,i} \neq 0$ for some j. Similarly, $a_{i,k} \neq 0$ for some k.

Let $a \in \mathbb{R}$ and $\{a\}$ denote the fractional part of a. Observe that a point $p_i \in S_i \subset V_i$ is good if there exists j such that $b_{j,i} \neq 0$ with

$$\frac{a_{j,i}}{j} \leq \frac{a_{k,i}}{k}, \ \frac{b_{j,i}}{j} \leq \frac{b_{k,i}}{k} \tag{2.5}$$

for all $2 \leq k \leq m$ and $d_{k,i} \neq 0$ and $\left\{\frac{a_{j,i}}{j}\right\} + \left\{\frac{b_{j,i}}{j}\right\} < 1$. So, the motive is to reach to an S_i defined by f_i such that the above is achieved for the coefficient curves.

Consider the number

$$\alpha_{i,j,k} = (\frac{a_{j,i}}{j} - \frac{a_{k,i}}{k})(\frac{b_{j,i}}{j} - \frac{b_{k,i}}{k})$$

. Note that

$$\alpha_{i+1,j,k} = \alpha_{i,j,k} + (\frac{b_{j,i}}{j} - \frac{b_{k,i}}{k})^2$$

If $\alpha_{i,j,k} < 0$ we must have $(\frac{b_{j,i}}{j} - \frac{b_{k,i}}{k}) \neq 0$ Thus,

$$(\frac{b_{j,i}}{j} - \frac{b_{k,i}}{k})^2 \geq \frac{1}{j^2k^2} \geq \frac{1}{r^4}.$$

We note that after finitely many steps there exist an l such that for all $i \geq l$, $\alpha_{i,j,k} \geq 0$ for all i and j. So we have achieved condition 2.5. The other one can be easily obtained after finitely many blow ups. □

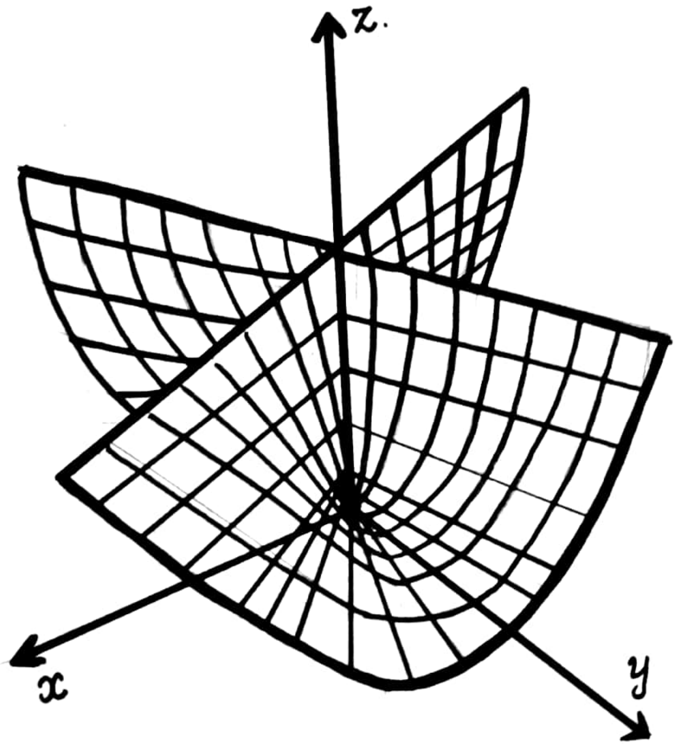
2.4 Examples

2.4.1 Whitney umbrella

The example is due to Herwig Hauser. Consider the surface $S := V(f = x^2 + y^2 z)$. The partial derivatives of f are as follows:

$$f_x = 2x, f_y = 2yz, f_z = y^2.$$

This implies that the singular locus of S is the $Z = z - axis$. Since, Z sits in S with multiplicity 2, we see that $Sing_2 S = Z$. Following 5.2.3, possible two centers of blow up are origin and the curve Z.



Blow up at the origin produces the following three pieces:

1. $1 + y_1^2 x_1 z_1$ where $x_1 = x$, $y_1 = \frac{y}{x}$, $z_1 = \frac{z}{x}$

2. $x_1^2 + z_1 y_1$ where $x_1 = \frac{x}{y}$, $y_1 = y$, $z_1 = \frac{z}{y}$

3. $x_1^2 + y_1^2 z_1$ where $x_1 = \frac{x}{z}$, $y_1 = \frac{y}{z}$, $z_1 = z$

And blowing up at Z produces the following two pieces:

1. $1 + y_1^2 z_1$ where $x_1 = x$, $y_1 = \frac{y}{x}$, $z_1 = z$

2. $x_1^2 + z_1$ where $x_1 = \frac{x}{y}$, $y_1 = y$, $z_1 = z$

Notice that, blowing up at origin reproduces the original surface in the third piece but blowing up Z resolves the singularity. Though, complexity of singularity is more at origin than the curve Z, yet taking origin as center could not solve the problem. Nevertheless, a good idea while choosing centers for blow ups is to blow up the centers of maximal possible dimension. Notice that, the proof of 5.2.3 have included curve blow ups for the resolution, while point blow up was operated to achieve good points. It is necessary to make sure that, each multiple point is included in the blow up centers at least once.

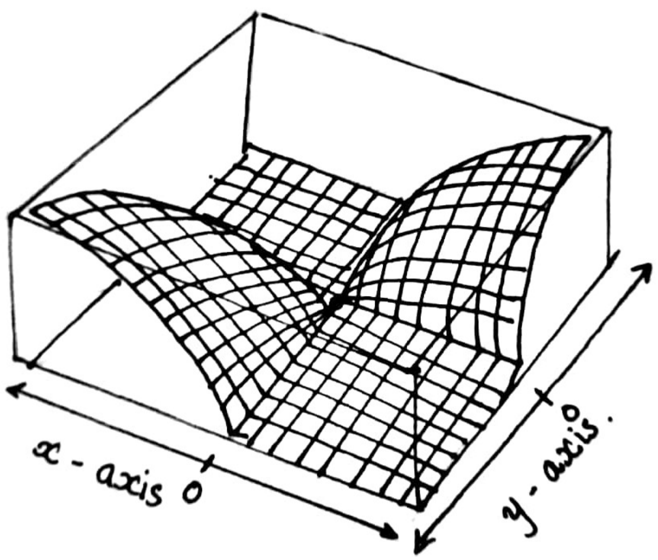
Sometimes, choosing wrong centers may worsen the singularity. For example,

$$f = x^2 + y^3 z,$$

blow up at origin, in the third open set is given by $f' = x_1^2 + y_1^3 z_1^2$.

2.4.2 Blow up along a non reduced subscheme

We invoke here Rees-proj definition of blow up to show that blow up of a non-singular variety is not always non-singular. Indeed, blow ups centered at reduced subscheme may not show such phenomena and therefore the following example:



Consider the blow up of $\mathbb{A}^2$ at the ideal (x, y^2). The blow up scheme is given by:

$$B_{(x,y^2)}\mathbb{A}^2 = V(uy^2 - vx) \subset \mathbb{A}^2_{x,y} \times \mathbb{P}^1_{u,v}.$$

Then the two affine charts are as follows:

1. For $u \neq 0$, the surface is defined by:

$$y^2 - wx = 0$$

where $w = \frac{v}{u}$.

2. For $v \neq 0$, the surface is defined by:

$$zy^2 - x$$

where $z = \frac{v}{u}$.

Clearly, the surface has a singularity at the point $(0, 0)$ point, more precisely at the point $((0, 0), (1 : 0))$.

We wish to understand how transversal intersections behave under this blow up. Consider the following curves:

$L_1 := \{x + y = 0\}$

$L_2 := \{x - y = 0\}$

$L_3 := \{y - ax = 0\}$

$L_4 := \{x = 0\}$

$C_1 := \{x = y^2\}$

$C_2 := \{x = y^3\}$

We claim that, L_1 and L_2 are not separated even though they have transversal intersections at origin. Also, L_4 and C_1 are separated but L_4 and C_2 are not. We will come back to these assertions.

The above blow up has a description as follows:

Let $\pi : B(x_1) \to \mathbb{A}^2$ be the blow up of $\mathbb{A}^2$ at $x_1 = (0, 0)$ and $E = \pi^{-1}(x_1)$. Suppose x_2 is the point of intersection of E and the birational transform of $y-$axis. We blow up at x_2, and call the exceptional locus to be F. Then $B_{(x,y^2)}$ is obtained by a blow-down of the birational transform of E. A more precise treatment is done in [**?**]. We now give a geometric explanation of the assertions made before.

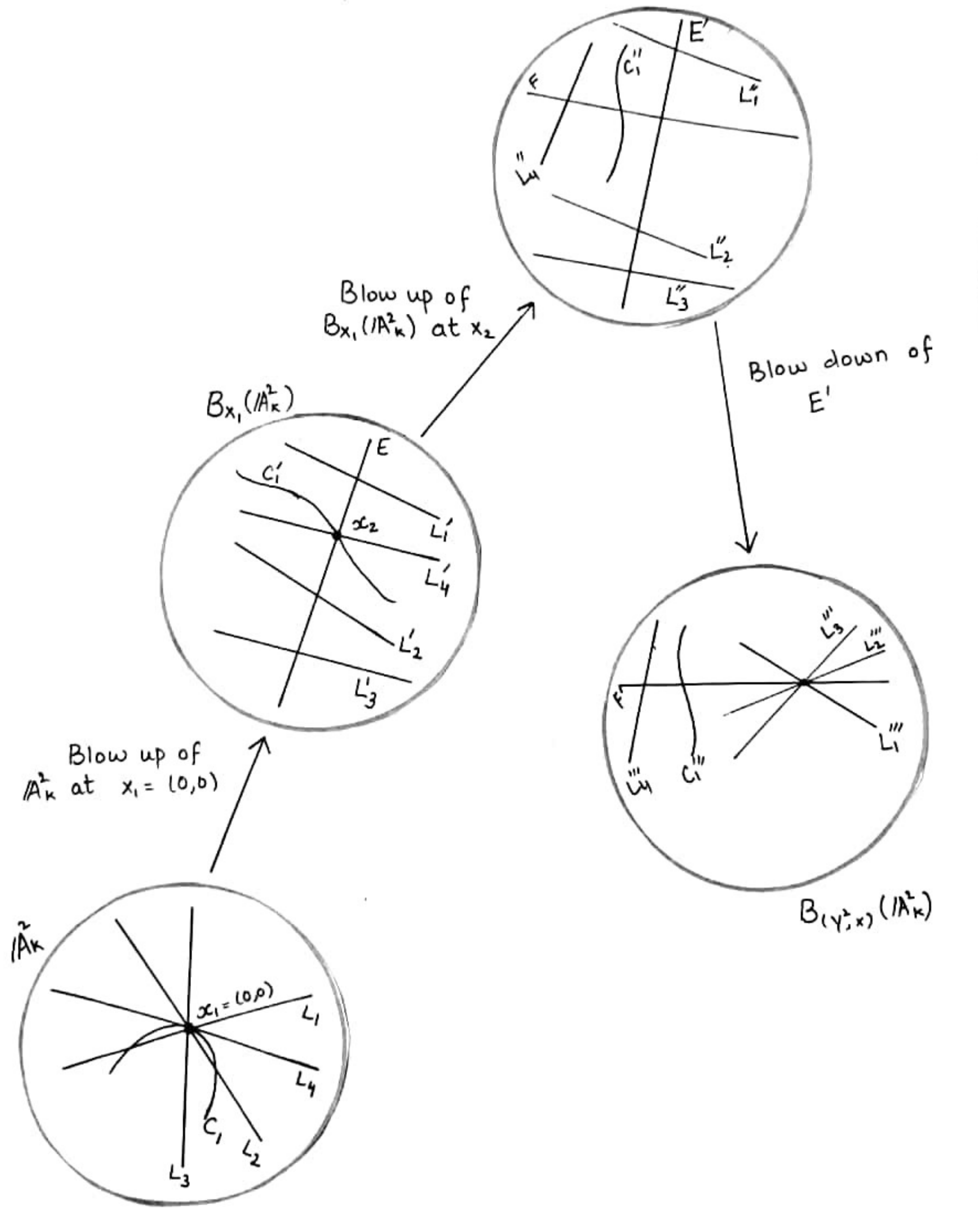


Figure 2.3: Description of $B_{(y^2,x)}(\mathbb{A}^{\nvDash})$

Chapter 3

Normalization and Jung's method

Hironaka has given a proof of resolution in the arbitrary dimension. For positive characteristics, it is still an open problem. In arbitrary dimensions, the method involves the notion of blow-up of ideal sheaves. We have seen in chapter two that it works well for curves and surfaces. However, for small dimensions, there exist some of the quickest ways to resolve singularity. Normalization is one of them for dimension one. We will show that a normal variety can have the singular locus of dimension at most $n-2$, which makes the notion of normality and non-singularity equivalent for curves. Moreover, for the surfaces, it reduces the singular locus to a finite set.

3.1 Normal variety

Definition 3.1.1 (Normal variety)**.** Let X be an affine variety. Then it is said to be normal if the coordinate ring $k[X]$ is integrally closed. A quasi projective variety X is said to be normal if every point has a normal affine neighborhood.

Theorem 3.1.2. *A non-singular variety is normal.*

Proof. Let X be a non-singular variety. We claim that the local ring $\mathcal{O}_x$ of a point $x \in X$ is a UFD. Indeed, let $\hat{\mathcal{O}}_x$ denote the completion of the local ring $\mathcal{O}_x$ of x. Then the completion map $\Phi : \mathcal{O}_x \to \hat{\mathcal{O}}_x$ defined by

$$f \mapsto \oplus(f + m_x^i)$$

is an inclusion. This is because the point x is non-singular. It follows that $\hat{m}_x \cap \mathcal{O}_x = m_x$. Note that, the formal power series ring $k[[T]]$ is a UFD (Weierstrass preparation theorem). Therefore, lemma 3.1.3 implies that $\mathcal{O}_x$ is a UFD. □

Lemma 3.1.3. *Suppose that a Noetherian local ring A is contained in a local ring $\hat{A}$ which is a UFD. Suppose that the maximal ideals $m \subset A$ and $\hat{m} \subset \hat{A}$ satisfy the following conditions:*

(a) $m\hat{A} = \hat{m}$;

(b) $(\hat{m}^n A) \cap A = m^n$ for $n > 0$;

(c) for any $\alpha \in A$ and any integer $n > 0$ there exists $a_n \in A$ such that $\alpha - a_n \in m^n\hat{A}$.

Then A is also a UFD.

Proof. Refer [8] □

Theorem 3.1.4. *Let X be a normal variety of dimension n and S be the singular locus of X. Then,* $\dim S \leq n - 2$.

Proof. On contrary, let us suppose that $S = S' \cup S_2 \cup ... \cup S_m$ is the decomposition of S into irreducible components and $\dim S' = n - 1$. Then there exists an affine open subset U of X such that $\bar{S} = U \cap S'$ is non-empty. Let f be the equation of $\bar{S}$ in U, x is a non-singular point of $\bar{S}$ and $u_1, u_2, ..., u_{n-1}$ be the local parameters of x in $\bar{S}$. We claim that the ideal $m_{U,x} = (u_1, u_2, ..., u_{n-1}, f)$. Indeed, the inclusion map $\phi : \bar{S} \hookrightarrow U$

induces the ring map $\phi^* : k[U] \to k[\bar{S}]$. We see that $ker\phi^* = (f)$. The map ϕ^* induces the map between local rings as follows,

$$\phi^* : \mathcal{O}_{U,x} \to \mathcal{O}_{\bar{S},x}.$$

We see that $ker\phi^* = (f)\mathcal{O}_{U,x}$, since localization is an exact functor. Then, $m_{\mathcal{O}_{U,x},x}$ is the preimage of maximal ideal $m_{\mathcal{O}_{\bar{S},x},x}$ under ϕ^*.

This implies that $\dim\left(m_{\mathcal{O}_{X,x}}/m_{\mathcal{O}_{X,x}}{}^2\right) = \dim\left(m_{\mathcal{O}_{U,x},x}/m_{\mathcal{O}_{U,x},x}{}^2\right) \leq n$. Therefore, x is a non-singular point of X which is a contradiction. □

Corollary 3.1.4.1. *For algebraic curves, the notion of non-singularity and normality are equivalent.*

Remark. We call a scheme X normal if all of its local rings are integrally closed.

The natural question is, does there exist a way to normalize a curve? In which case, does it produce a resolution method? The answer is yes.

3.2 Normalization

Definition 3.2.1. Let X be a variety. Then we define a normal variety X' together with a finite, birational morphism $\eta : X' \to X$ to be *normalization of* X.

Suppose $X \subset \mathbb{A}^n$ be an affine variety. Let A be the integral closure of $k[X]$ inside $k(X)$. Then A is reduced and finitely generated. So, there exists X' such that the coordinate ring of X' is A. It is easy to verify that X' is the normalization of affine variety X.

Lemma 3.2.2. *Let X be a quasi projective curve. Then there exist a normalization X' that is quasi projective.*

Proof. Suppose $X = \bigcup_i U_i$ be an affine open cover for X. Let $\eta_i : U_i' \to U_i$ be the normalization of U_i and $V_i \subset \mathbb{P}^{n_i}$ be their closure. Note that all the varieties defined above are birational. Let the birational map of U_i' and V_j is given by $\phi_{ij} : U_i' \to V_j$. Note that $\phi_{ij}'s$ are regular, since the curves U_i' are non-singular. Further, let

$$W := \prod_j V_j \text{ and } \phi_i := \prod_j \phi_{ij} : U_j' \to W.$$

We claim that, $X' = \bigcup_i \phi_i(U_i') \subset W$ is the normalization of X. Indeed, $U_0 := \bigcap_i U_i$ is an open subset of X. We see that, $U_0' := \eta_i^{-1}(U_0) \subset U_i'$ for all i. Also $\phi_i's$ coincide on U_0' for all i. We write ϕ for the common restriction. Then,

$$\phi(U_0') \subset \phi_i(U_i') \subset \phi_i(\bar{U_i'})$$

Obviously, $\phi(U_0')$ is an irreducible quasi projective curve with only finitely many points less than its closure. And hence, $\phi_i(\bar{U_i'})\backslash X'$ is also finite. It follows that, X' is quasi projective and irreducible. Next, we show that X' is normal and there exists a finite birational morphism from X' to X.

Let $x \in X'$ then $x \in \phi_i(U_i')$ for some i. We see that $\phi_i : U_i' \to \phi_i(U_i') \subset W$ is an isomorphism. It follows that the neighborhood $\phi_i(U_i')$ of x is normal. Hence, X' is normal. Next, we write $g_i = \eta_i \circ \phi_i^{-1} : \phi_i(U_i') \to U_i \subset X$. Note that, g_i are finite maps and g_i and g_j coincide on the open set of $\phi(U_0')$.. Thus, g_i and g_j coincide at all points where they are defined. Hence, they define a regular map $\eta : X' \to X$ which is birational and finite. □

Remark. Let X be a reduced, irreducible scheme of finite type over k. Moreover, let $\bigcup_i^n U_i$ be an affine open cover for X. We know that, normalization of an affine variety exists. So let $\eta_i : U_i' \to U_i$ be the normalization of U_i. It is easy to see that the normalization is unique if it exists. It follows that, $\eta_i^{-1}(U_i \cap U_j)$ and $\eta_j^{-1}(U_i \cap U_j)$ are isomorphic. So we can glue together the normal affine pieces U_i'. The glued up scheme is reduced, irreducible and finite type over k. It is not difficult to see that it is the normalization of X. Hence, normalization exists for quasi projective varieties.

Remark. Theorem 3.2.2 is true for arbitrary dimension. But we shall not prove it here. It can be found in [4].

Theorem 3.2.3. *Let X be a projective curve. Then the normalization X is projective.*

Proof. Let $\eta : X' \to X$ be the normalization of X as in theorem 3.2.2 and $X' \subset \mathbb{P}^n$. We call Y to be the closure of X' in $\mathbb{P}^n$. Let $y \in Y\backslash X'$ and U be an affine neighborhood around y. We define $\eta_U : U' \to U$ to be the normalization of U. We have the commutative diagram:

$$\begin{array}{ccccc} U' & \xrightarrow{f} & X' & \xrightarrow{\eta} & X \\ \downarrow{\scriptstyle \eta_U} & & \downarrow{\scriptstyle i_1} & & \\ U & \xhookrightarrow{i_2} & Y & & \end{array}$$

There exists a birational map $g = \eta \circ i_1^{-1} \circ i_2 \circ \eta_U : U' \to X$. Since U' is non-singular, g is regular. By uniqueness of normalization g lifts to f. It follows that $i_1 \circ f(U') = i_2 \circ \eta_U(U')$ which is a contradiction to the assumption that $y \notin X'$. $\square$

3.3 Jung's method

Let k be an algebraically closed field of characteristic zero and $Z \subset \mathbb{P}_k^N$ be a projective surface. We project $\mathbb{P}_k^N$ along a point $z \notin Z$ to a hyperplane H. Let $p' : \mathbb{P}_k^N \to H$ be the map. Restriction of p' to Z is a finite morphism (see 4). We continue to project through points outside Z for $n-2$ times and call it $p : Z \to \mathbb{P}_k^2$. Consider the commutative diagram:

$$\begin{array}{ccc} X = N_{k(Z)} B_{P_k^2}(D) & \xrightarrow{\eta} & B_{P_k^2}(D) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{\quad p \quad} & P_k^2 \end{array}$$

Let $D \subset \mathbb{P}_k^2$ be the branch locus for p and π be the blow up of $\mathbb{P}_k^2$ such that the total transform $D' = \pi^{-1}(D)$ of D is a simple normal crossing divisor. Let X be the normalization of $B_{\mathbb{P}_k^2}(D)$ in the function field $k(Z)$.

Definition 3.3.1. Let X be an affine variety and Z be a normal variety. Then X is said to be the quotient of Z by a finite group G, if there exists a Galois extension $K/k(X)$ with group G such that the normalization of X in K is Z.

Definition 3.3.2 (Quotient singularities)**.** Let X be a variety over an algebraically closed field. We say that X has abelian quotient singularities if there exists an open affine cover $X = \bigcup X_i$ such that X_i is a quotient of smooth affine variety Z_i by a finite abelian group G_i for each i.

We will show that X possess abelian quotient singularities. Consider the following commutative diagram:

$$\begin{tikzcd}
 & Y \arrow[dl] \arrow[d] \arrow[dr] & & & \\
V = N_K(U) \arrow[dr] & U_X \arrow[r, hook, "i"] \arrow[d, "\eta"] & X \arrow[dr, "\eta"] \arrow[dd, "i"] & & D_X \arrow[ll, hook', "i"'] \arrow[d, "\eta"] \\
 & U \arrow[rr, "\pi"] & & B_{P^2_k}(D) \arrow[d, "\pi"] & \pi^{-1}(D) \arrow[l, hook', "i"] \arrow[d, "\pi"] \\
 & & Z \arrow[r, "p"] & P^2_k & D \arrow[l, hook', "i"]
\end{tikzcd}$$

In order to show that X have abelian quotient singularity, we define an open affine cover $\bigcup_i V_i$ of X such that each V_i is a quotient of a smooth normal variety by an abelian Galois group. We consider the open cover $\bigcup_i U_i$ of the blow up variety $B_{\mathbb{P}^2_k}(D)$ that we discussed in chapter 2. Let $U = U_1$ and u_1, u_2 be the local parameters of $0 \in D'$ such that D' is defined by $(u_1 u_2)$ in the neighborhood U of 0. Consider the field

$$K = k(U, u_1^{\frac{1}{m}}, u_2^{\frac{1}{m}})$$

where $m = \deg\ \eta!$. We look at the normalization of U in K and call it V. Finally, let Y be the normalization of $U_X = \eta^{-1}(U)$ in $k(V)+k(X)$. We show that Y is smooth. If Y is a normal variety over $\mathbb{C}$ then the complex space associated with Y is the normal in analytic sense. The statement is a non-trivial theorem, and we shall assume it here. We shift to the analytic picture to show that Y is smooth. We will not change here the notation to indicate the analogy but all the items here present the idea of what happens in the analytic neighborhood. Let U be a small disc in $\mathbb{C}^2$ and u_1, u_2 be the coordinate functions. Then $\eta : U_X \backslash (\eta^{-1}(U \cap D')) \to U \backslash (U \cap D')$ is a finite map. Let $D_0 := U \backslash (U \cap D')$ and $D_0^X := \eta^{-1}(U \backslash (D'))$. Then D_0^X is a finite cover of D_0. We know the only possible m-cover of D_0 is D_0 and the covering map is given by

$$(x, y) \mapsto (x^m, y^m).$$

This implies $D_0^X \simeq D_0$. The fundamental group of D_0 is $\mathbb{Z} \times \mathbb{Z}$, let $\mathbb{Z} \times \mathbb{Z} \to G$ be the finite quotient. If m is a multiple of $|G|$ then, $(m\mathbb{Z})^2 \subset Ker(\mathbb{Z}^2 \to G)$. By Riemann extension theorem, we extend it to whole of the disc $\Delta^2_{(x,y)}$. Then $\Delta^2_{(x,y)}$ is the analytic picture of Y. We conclude that Y is smooth. And therefore, we see that X possess abelian quotient singularities.

$$\begin{array}{ccc} \Delta^2_{(x,y)} & \longrightarrow & U_X \\ \downarrow{\simeq} & & \downarrow{\eta} \\ \Delta^2_{(x,y)} & \longrightarrow & U \end{array}$$

Next we resolve abelian quotient singularities.

Lemma 3.3.3. *Suppose Y is a normal affine variety and X be quotient of Y by the abelian group G. Let $x \in X$ be a point such that $y_1, y_2, ..., y_m$ are preimages of y under normalization. Then*

$$\hat{\mathcal{O}}_{x,X} = \hat{\mathcal{O}}^{G_i}_{y_i,Y}$$

where $G_i \subset G$ is stabilizer of y_i.

Proof. Left to readers. $\square$

We intend to resolve $\mathcal{O}_{x,X}$, for which it is sufficient to resolve $\hat{\mathcal{O}}_{x,X}$. Lemma 3.3.3 implies that it is equivalent to resolve $\hat{\mathcal{O}}^{G_i}_{y_i,Y}$. We show that it reduces to resolving singularities of the form $\mathbb{A}^2/G$. Indeed, $\hat{\mathcal{O}}_{y_i,Y}$ is a complete local ring. More precisely, it is a $\hat{\mathcal{O}}_{y_i,Y}/m$-algebra. The action of G_i on m/m^2 is completely reducible. The sequence of $k-$vector spaces

$$0 \to m^2 \to m \to m/m^2 \to 0$$

is exact and the maps are $G-invariant$. This induces a map between the polynomial ring

$$k[m/m^2] \to \hat{O}_{x,X}$$

We claim that, $\hat{O}_{x,X}$ is the completion of $k[m/m^2]$. This will imply that $\hat{O}^{G_i}_{x,X}$ is the completion of $k[m/m^2]^{G_i}$, since a power series is $G - invariant$ if and only if its homogenous components are $G - invariant$.

Proof of claim: The map $k[m/m^2] \to \hat{O}_{x,X}$ induces the map

$$S^i \to m^i/m^{i+1}.$$

Note that these maps are isomorphism since $\hat{O}_{x,X}$ are regular. This gives an isomorphism of completion of $k[m/m^2]$ and $\hat{O}_{x,X}$. But the latter is already complete and hence the claim is true.

Now we focus on resolving singularities of the form $\mathbb{A}^2/G$ where G is an abelian group acting on $\mathbb{A}^2$ linearly. There are certain assumptions that we make. But before that, notice $G \subset GL(2,k)$ is a finite abelian group and therefore it is simultaneously diagonalizable. We assume that the elements g of G are diagonal matrices in $GL(2,k)$. Also, we can assume that $g = diag(\mu, \eta)$ where neither of them is 1. Indeed,

$$\mathbb{A}_{x,y}/(diag(\mu, 1)) \simeq \mathbb{A}^2_{x^m,y}$$

where μ is the m^{th} root of unity and therefore,

$$\mathbb{A}^2_{x,y}/G \simeq \mathbb{A}^2_{x^m,y}/(G/(diag(\mu, 1))).$$

So, we assume that G is free from elements of the form $diag(\mu, 1)$ or $diag(1, \eta)$. Next, we observe that G is cyclic due to previous assumption. Let $G = (g)$ and $g = diag(\mu^a \mu^b)$ where μ is an m^{th} root of unity. We denote the element g as $\frac{1}{n}(a, b)$. There is no harm in assuming $a = 1$, in which case $\gcd(n, b) = 1$. It finally reduces to resolving

$$\mathbb{A}^2_{x,y}/\frac{1}{n}(1, a)$$

for some $a \neq 0$. We blow up $\mathbb{A}^2_{x,y}$ at the ideal (x^a, y). Then,

$$B_{(x^a,y)}(\mathbb{A}^2_{x,y}) = (ux^a = vy) \subset \mathbb{A}^2_{x,y} \times \mathbb{P}^1_{u,v}$$

The action on $\mathbb{A}^2_{x,y}$ in $\mathbb{A}^2_{x,y} \times \mathbb{P}^1_{u,v}$ is induced by the action down at the base $\mathbb{A}^2_{x,y} \backslash (0, 0)$ while we take trivial action on $\mathbb{P}^1_{u,v}$. We look at the affine pieces of the blow up variety.
1. Let $v \neq 0$ and $s = u/v$,

$$(sx^a = y)/\frac{1}{n}(1, a, 0) \simeq \mathbb{A}^2_{y,s}.$$

2. Let $u \neq 0$ and $t = v/u$, then blow up variety is given by

$$(x^a = yt)/\frac{1}{n}(1, a, 0).$$

We see that $(x^a = yt)$ is already singular. However,

$$(x^a = yt) \simeq \mathbb{A}^2_{w,z}/\frac{1}{a}(1, -1)$$

where $x = wz$, $y = z^a$, $t = u^q$. The action of $\frac{1}{n}(1, a, 0)$ lifts to $\mathbb{A}^2_{u,v}$ as action of $\frac{1}{n}(0, 1)$. Therefore,

$$(x^a = yt)/\frac{1}{n}(1, a, 0) \simeq \mathbb{A}^2_{w,z}/(\frac{1}{a}(1, -1) \times \frac{1}{n}(0, 1))$$

But,

$$\mathbb{A}^2_{w,z}/(\frac{1}{a}(1, -1) \times \frac{1}{n}(0, 1)) \simeq \mathbb{A}^2_{w,z}/(\frac{1}{n}(0, 1))/(\frac{1}{a}(1, -1)) \simeq \mathbb{A}^2_{u,v}/\frac{1}{a}(1, -n).$$

Thus we have obtained after blowing up $\mathbb{A}^2_{u,v}/\frac{1}{n}(1, a)$, a variety X_1 is isomorphic to the surface

$$\mathbb{A}^2/\frac{1}{q}(1, r)$$

where $0 < r < a$ and $n = b_1 a - r$.

That is the order of group has been reduced. Hence, after finitely many blow ups we achieve a non-singular surface.

Chapter 4

Projection

Definition 4.0.1 (Projection)**.** Let $X \subset \mathbb{P}^n$ be a projective variety and $H \simeq \mathbb{P}^{n-1}$ be a hyperplane in $\mathbb{P}^n$.. Then we define *projection of X to H through p* to be the map $\pi_{p,H} : X \to H \simeq \mathbb{P}^{n-1}$

$$q \mapsto \{\text{line joining } p, q\} \cap H.$$

Remark. We can choose the coordinates on $\mathbb{P}^n$ such that $p = (0 : 0 : ... : 0 : 1)$ and $H := V(x_n)$. Then $\pi_{p,H}(x_0 : ... : x_n) = (x_0 : ... : x_{n-1})$. Indeed, let

$\begin{pmatrix} x_{0,0} \\ x_{1,0} \\ \vdots \\ x_{n,0} \end{pmatrix} \begin{pmatrix} x_{0,1} \\ x_{1,1} \\ \vdots \\ x_{n,1} \end{pmatrix} \ldots \begin{pmatrix} x_{0,n-1} \\ x_{1,n-1} \\ \vdots \\ x_{n,n-1} \end{pmatrix}$ be a basis for H' and $p' = \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{pmatrix}$ be the point of projection then,

$$D = \begin{pmatrix} x_{0,0} & x_{0,1} & \cdots & p_1 \\ x_{1,0} & x_{1,1} & \cdots & p_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,0} & x_{n,1} & \cdots & p_n \end{pmatrix}^{-1}$$

is the required projective transformation.

Example 7. Suppose $m_1 \leq m_2 \leq ... \leq m_r$, be a sequence of positive integers. Let $C \subset \mathbb{A}^r$ be the curve

$$t \to (t^{m_1}, t^{m_2}, t^{m_3}, ..., t^{m_r}).$$

We project C through $(1, 0, 0, ..., 0)$. Then the image curve is given by $C' \subset \mathbb{A}^{r-1}$

$$t \to (t^{m_2 - m_1}, t^{m_3 - m_1}, t^{m_4 - m_1}, ..., t^{m_r - m_1}).$$

It is intuitively clear that the projection improves the singularity, but it is hard to say exactly in what way.

Example 8. Consider the curve $y^2 = x^3 + x^2$

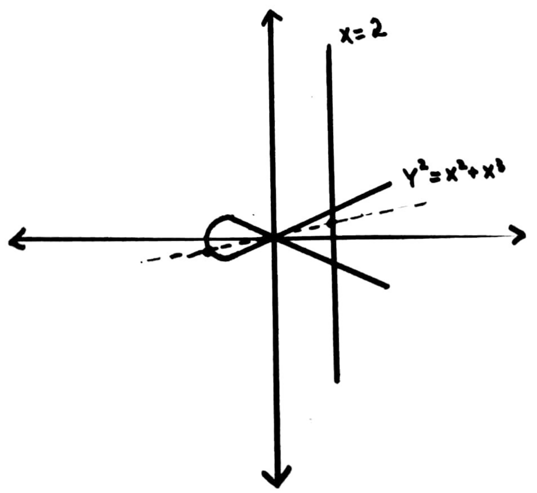


Figure 4.1: Projection of $y^2 = x^2 + x^3$ through $(0, 0)$

We project it through origin and take the closure. The image curve is non-singular.

Our aim will be to project through singular point to separate the different tangent directions on the point. Projection map can be finite, quasi-finite or neither. For example, projection of the cone $x^2 + y^2 = z^2$ to the hyperplane $z = 0$ through origin is not a finite map. Note that there is a decrease in the dimension in this case. It is finite in Example 8. Note that, projection through general point is a finite map, but the singular point may not always be a general point. We conclude that projection may create singularities.

Example 9. A non-singular cubic surface of $\mathbb{P}^3$ contains exactly 27 lines. An outline of the proof can be found in [9]. There exist at most three lines of S that can pass through a point. Let p be a point with at least two lines passing through p. Then, the projection through the point p is a finite map only on an open subset of S. Refer [3], [9] for more details.

4.1 Resolution through projection

For the following, k will denote an algebraically closed field of characteristic 0.

Theorem 4.1.1. *Let $X \subset \mathbb{P}^n$ be a projective variety of dimension N over k. Then there exists a normal variety X' birational to X, such that $mult_x(X') \leq N!$ for every $x \in X'$*

We will come back to this result.

Theorem 4.1.2. *Let $X_0 \subset \mathbb{P}^N$ be projective variety spanning $\mathbb{P}^N$ defined over an algebraically closed field. Let $\pi_i : X_i \to X_0$ be the composition of projections at points p_i where $mult_{p_i} X_i . \deg(X_i/X_0) > \dim X_0!$. If $\deg X_0 < (\dim X_0! + 1)(N + 1 - \dim X_0)$, then the sequence eventually stops with a variety X_i and a map $\pi_i : X_0 \to X_i$ such that*

(1) either $\deg(X_0/X_i).mult_p X_i \leq \dim X_0!$ for every $p \in X_i$,

(2) or X_i is a cone and $\deg(X_0/X_i) \leq \dim X_0!$.

Remark. Let $A \subset \mathbb{P}^n$ be a non-empty algebraic variety and $\pi : \mathbb{A}^{n+1} \backslash (0, 0, ..., 0) \to \mathbb{P}^n$ be the natural projection map. Then the projective closure $\pi^{-1}(A) \cup \bar{(}0, 0, ..., 0) \subset \mathbb{P}^{n+1}$ is the *projective cone over A*.

Proof. We proceed by induction on i for the formula:

$$\deg(X_0/X_i). \deg X_i < (\dim X! + 1)(N - i + 1 - \dim X) \tag{4.1}$$

Clearly true for $i = 0$. Suppose the formula is true for i. We note that, for a projective variety $X \subset \mathbb{P}^N$

$$\deg X \geq N + 1 - \dim X. \tag{4.2}$$

Indeed, we project it along a point p.

1. If X is a cone at p, then it follows by induction on dimension of X. Indeed,

$$\begin{aligned}\deg X_0 =& \deg X' + mult_p X_0 \\ \geq& N - 1 + 1 - (\dim X_0 - 1) + mult_p X_0 \\ =& N + 1 - (\dim X_0) + mult_p X_0\end{aligned}$$

2. Otherwise, induct on N i.e. if $\pi : X \to X_1$ is the projection,

$$\begin{aligned}&\deg X_0 \geq \deg X_1 . \deg(X_1/X_0) + mult_p X \\ &\deg X_0 \geq (N - n) \deg(X_1/X_0) + 1 \\ &\deg X_0 \geq N - \dim X_0 + 1\end{aligned}$$

Coming back to the proof, we see that, if X_i is a cone, the fact that $\deg(X_0/X_i) \leq \dim X!$ follows from inequality 4.1 and 4.2. If X_i is not a cone and there exists a point $p_i \in X_i$ such that $\deg(X_0/X_i).mult_p X_i \geq n!$, we project it through p_i. Let $\phi_{i+1} : X_i \to X_{i+1}$ be the projection map

$$\begin{aligned}\deg(X_0/X_{i+1}). \deg X_{i+1} &= \deg(X_0/X_i). \deg(X_i/X_{i+1}). \deg X_{i+1} \\ &\leq \deg(X_0/X_i). \deg X_i - \deg(X_0/X_i) mult_{p_i} X_i \\ &\leq \deg(X_0/X_i). \deg X_i - (\dim X_0! + 1) \\ &\leq (\dim X_0! + 1)(N - i + 1 - \dim X_0) - (\dim X_0! + 1) \\ &= (\dim X_0! + 1)(N - (i + 1) + 1 - \dim X_0)\end{aligned}$$

Now, maximum value for i is $N - \dim X_0$ where

$$\deg(X_0/X_i). \deg X_i < (\dim X! + 1)(1)$$

Since, $mult_{p_i} X_i < \deg X_i$,the result follows. $\square$

Corollary 4.1.2.1. *Let X be a projective variety over an algebraically closed field. Then X can be embedded into some $\mathbb{P}^N$ such that* $\deg X < (\dim X!+1)(N+1-\dim X)$ *and X spans $\mathbb{P}^N$. Thus theorem* 5.2.3 *is true for projective surfaces.*

Remark. This proves theorem 4.1.1 for a projective varieties of dimension n if resolution exists in dimension $n - 1$. Indeed,

(i) If $\pi_i : X \to X_i$ is a birational map in the proof of theorem 5.2.3, then we take the normalization X_i' of X_i. Note that, X_i' is birational to X and $mult_p X_i' \leq \dim X!$.

(ii) If $\pi_i : X \to X_i$ is not birational in the proof of theorem 5.2.3, we take normalization of X_i in $k(X)$ say X_i'. It is birational to X such that $mult_p X_i' \leq \dim X!$

(iii) If X_i is a cone over an algebraic variety A and $\pi_i : X \to X_i$ is birational. Let A' be resolution for A. Then X is birational to the smooth variety $\mathbb{P}^1 \times A'$.

(iv) If X_i is a cone over the algebraic variety A and $\pi_i : X \to X_i$ is not birational. Then the normalization X_i' of $\mathbb{P}^1 \times A$ in $k(X)$ is birational to X with $mult_p X_i' \leq \dim X!$.

Theorem 4.1.1 gives a resolution for curves, since a birational map from a non-singular curve is a morphism. We want to resolve double points now.

Theorem 4.1.3. *Let X be a normal variety of dimension n over an algebraically closed field of characteristic zero. Then completion of a local ring around a point $x \in X$ has the form*

$$\hat{O}_{x,X} \simeq \textit{ normalization of } k[[x_1, ..., x_n, y]]/f(x_1, ..., x_n, y),$$

where $f = y^m + a_1(x_1, x_2, ...x_n)y^{m-1} + ... + a_m(x_1, x_2, ..., x_n)$ and m is multiplicity of the point x in X.

Proof. Suppose $X \subset \mathbb{P}^l$ and $p' : X \to H \simeq \mathbb{P}^{l-1}$ be a projection through a point outside X to the hyperplane H. We continue to do this for $l - n - 1$ times to obtain a finite morphism $p : X \to \mathbb{P}^{n+1}$. Consider the affine open subset $Y = \mathbb{A}^{n+1} \cap p(X)$. Since, $\dim(p(X)) = \dim X$, Y defines a hypersurface in $\mathbb{A}^{n+1}$. Without any loss of generality, we assume $p(x) = (0, 0, ..., 0)$. Then by Weierstrass preparation theorem Y looks like

$$f = y^m + a_1(x_1, x_2, ...x_n)y^{m-1} + ... + a_m(x_1, x_2, ..., x_n)$$

in the completion of local ring around $p(x)$. Since the map $p : p^{-1}(Y) \to \mathbb{A}^{n+1}$ is finite, this induces

$$k[[x_1, x_2, ..., x_n, y]]/(f) \to \hat{\mathcal{O}}_{x,X}.$$

Finally, because completion of a normal local ring is normal, we see that

$$\hat{O}_{x,X} \simeq \text{ normalization of } k[[x_1, ..., x_n, y]]/f(x_1, ..., x_n, y).$$

□

Therefore, the completion of local ring at the double point x on a normal surface S has the form

$$\hat{\mathcal{O}}_{x,X} \simeq k[[x,y]]/(y^2 = r(x)) \simeq k[[x]][\sqrt{r(x)}]$$

Therefore, we resolve the ring $R[\sqrt{r}]$, where (R,m) is regular local $k-$algebra of dimension two. Consider the commutative diagram:

$$\begin{tikzcd} SpecR(\sqrt{\pi^* r}) \arrow[r] \arrow[d] & SpecR(\sqrt{r}) \arrow[d] \\ S' \arrow[r, "\pi"] & S = SpecR \end{tikzcd}$$

Let $S = \operatorname{Spec} R$ and $S[\sqrt{r}] = \operatorname{Spec} R[\sqrt{r}]$. We blow up S to make sure that total transform of r is a simple normal crossing divisor. Let $\pi : S' \to S$ be the blow up map. Here, $S'[\sqrt{\pi^* r}]$ denote the pull-back in the square.

$$\begin{tikzcd} \bar{S} \arrow[r, "\eta"] \arrow[dr] & S'[\sqrt{\pi^* r}] \arrow[d] \\ & S' \end{tikzcd}$$

Let $\bar{S}$ be the normalization of $S'[\sqrt{\pi^* r}]$. We look at the singularities of $\bar{S}$. Let $s \in S$ then there exist $x, y \in m_{S'}$ such that, $f^* r = x^a y^b (unit)$. Over a neighborhood of $s \in S'$, $\bar{S}$ is normalization

$$\mathcal{O}_{s,S'[z']}/(z'^2 - x^a y^b u)$$

where $0 \leq a, b \leq 1$. The ring is regular is either a or b is zero. If both $a = b = 1$, blowing up once resolves the singularity.

Chapter 5

In Positive characteristic

We can resolve curves in positive characteristics through normalization. We will give a proof of surface resolution in positive characteristic in this chapter. The proof has been sketched by Hironaka in his paper 'Desingularisation of Excellent Schemes' and is explained by Cutkosky. Herwig Hauser gives another description. We also present an example where the method of characteristic zero fails in positive characteristic. However, before we move to the proof, we show the existence of minimal resolution for surfaces.

5.1 Minimal Resolution

Theorem 5.1.1. *Let $X \subset \mathbb{P}^n$ be a projective surface. Then there exists a resolution $\pi : X' \to X$ of X which is minimal. In the sense that, for any resolution $\phi : Y \to X$ of X, there exists a map $\xi : X' \to Y$ such that the following diagram commute.*

$$\begin{tikzcd}
& X' \arrow[dl, "\xi"] \arrow[d, "\phi"] \\
Y \arrow[r, "\pi"] & X
\end{tikzcd}$$

Remark. We call the canonical divisor K_X of X to be $f - nef$ if the intersection number $K_X \cdot E \geq 0$ for every $f-$exceptional curve $E \subset X$. In general, we call a line bundle L on X nef (numerically- effective) if it is non-negative on any curve on the proper scheme X. We want to get rid of all such curves in our minimal model.

Proof. Let $\xi : Y \to X$ be a resolution for X. Suppose there exists an exceptional divisor E such that $K_X \cdot E \geq 0$. By Hodge index theorem and adjunction formula, the intersection number $E \cdot E < 0$. Hence, by Castelnuovo's contractibility criterion, we can contract the curve E. Since $\xi(E)$ is a point of indeterminacy for the map $\xi^{-1} : X \to Y$, the number of exceptional curves are finitely many. Let X' be the resolution with no exceptional curves. The uniqueness of X' follows from the Factorization Theorem. $\square$

5.2 Resolution in positive characteristic

5.2.1 Hypersurface of maximal contact

Definition 5.2.1 (Hauser)**.** Suppose $Y \subset X$ be a variety and X is smooth. Let $y \in Y$ be a point of maximum multiplicity m. Then *a hypersurface of maximal contact* at y is a smooth closed hypersurface Z that satisfy the following two conditions:

1. $Sing_m Y \subset Z$.
2. Let $\pi : X' \to X$ be the blow up X along a non-singular component of $Sing_m Y$. And Y', Z' be the birational transform of Y, Z under π respectively. Then $Sing_m Y' \cap \pi^{-1}(y) \subset Z'$.

Remark. Suppose $V(f)$ is a hypersurface of X over an algebraically closed field of characteristic zero. And $p \in V(f)$ be a point of multiplicity m. Let y be one of the local parameters of p such that y^m appears in f with non-zero coefficient. Then the $m-1^{st}$ partial derivative of f with respect to y defines a hypersurface of maximal contact at p.

5.2.2 An Example

The example is due to Hauser. Consider the surface $S \subset \mathbb{A}^3$ defined by the local equation

$$f(x,y,z) = x^2 + y^7 + y^4z^2 + yz^4$$

over an algebraically closed field of characteristic 2. See that, $(0,0,0)$ sits in S with multiplicity 2. The top locus of S is given by $(z^2+y^3, x+zy^2)$. Indeed,

$$f(x,y,z) = (x+zy^2)^2 + y(z^2+y^3)^2.$$

In characteristic 0, the hypersurface of maximal contact for f is given by $x=0$, we will show that it is not the case for char 2.

$$f(x,y,z) = x^2 + y^7 + y^4z^2 + yz^4$$

We blow up at $(0,0,0)$ and look the piece:

1. $x_1 = \frac{x}{y}$, $y_1 = y$, $z_1 = \frac{z}{y}$

   $f_1(x_1,y_1,z_1) = x_1^2 + y_1^3(y_1^2 + z_1^2y_1 + z_1^4)$

Next we blow up at $(0,0,0)$ again and look at the following piece:

2. $x_2 = \frac{x_1}{z_1}$, $y_2 = \frac{y_1}{z_1}$, $z_2 = z_1$

   $f_2(x_2,y_2,z_2) = x_2^2 + y_2^3z_2^3(y_2^2 + z_2y_2 + z_2^2)$

The coefficient curve has powers of y_2 and z_2 common, so we set the point $(0,1,0)$ on the exceptional locus as the new origin, the changed transformation are as follows:

3. $x_3 = \frac{x_2}{z_2}$, $y_3 = \frac{y_2}{z_2} - 1$, $z_3 = z_2$

   $f_3(x_3,y_3,z_3) = x_3^2 + z_3^6(y_3+1)^3((y_3+1)^2 + (y_3+1) + 1)$

We blow up the birational transform three times along the curve (x,z). The number six in the exponent of monomial z_3^6 reflects the reason for choosing three.

4. $x_4 = \frac{x_3}{z_3}$, $y_4 = y_3$, $z_4 = z_3$

$f_4(x_4, y_4, z_4) = x_4^2 + z_4^4(y_4+1)^3((y_4+1)^2 + (y_4+1) + 1)$

5. $x_5 = \frac{x_4}{z_4}$, $y_5 = y_4$, $z_5 = z_4$

$f_5(x_5, y_5, z_5) = x_5^2 + z_5^6(y_5+1)((y_5+1)^2 + (y_5+1) + 1)$

6. $x_6 = \frac{x_5}{z_5}$, $y_6 = y_5$, $z_6 = z_5$

$f_6(x_6, y_6, z_6) = x_6^2 + (y_6+1)^3((y_6+1)^2 + (y_6+1) + 1)$

The birational transform of x_1 is x_2, of x_2 is x_3 and so on. We see that the point $(1,0,0)$ sits in $S_6 = V(f_6)$ with multiplicity 2 and $(1,0,0) \in \pi_6^{-1}(0,0,0)$ but $(1,0,0) \notin V(x_6)$. Therefore, $x = 0$ fails to be a hypersurface of maximal contact.

5.2.3 Proof by Cutkosky

Theorem 5.2.2. *Embedded resolution of singularity exist for surfaces in positive characteristic.*

Theorem 5.2.3. *Embedded resolution of singularity exists for hypersurfaces of dimension two in positive characteristics.*

We shall prove theorem 5.2.3 in two steps :

(1) We will prove it for the case where $Sing_m S$ is a finite set.

(2) And then reduce all other cases to case (1)

Before we move on to the proof of case 1. Let us set the notations.

For the following, $S \subset V$ will denote a projective surface embedded in a smooth $3-fold$ V over an algebraically closed field of positive characteristic. Let $p \in S$ be a point then $B(p)$ denotes blow up of V at p and $B(C)$ denote blow up of V at the curve C. We often write V_i in general for both the blow ups at the i^{th} step. The points $q_i's$ denote the points in the fiber of p under π_i. The resolution we construct is local in nature, and we shall do it in an analytic neighborhood of p. Let $S \subset \mathrm{Spec}(\hat{O}_{V,p})$ be given by the equation $f = \sum_{i,j,k} a_{ijk} x^i y^j z^k$. And $f_1, f_2, ...$ will denote the birational transform of S under π_i or defining equations of S_i where (x, y, z) are the local parameters for the point p.

Definition 5.2.4 (Approximate manifold)**.** Suppose X is a smooth variety of dimension n and $Y \subset X$ be a hypersurface. Let $y \in Y$ be a point of multiplicity m and $U \subset X$ be an affine neighborhood of y. We assume Y is defined by $f = 0$ in the neighborhood U and

$$f = \sum_{i_1+...+i_n \geq m} a_{i_1,i_2,...,i_n} x_1^{i_1} x_2^{i_2} ... x_n^{i_n}$$

where $(x_1, x_2, ..., x_n)$ are the local parameter of y in X. Let

$$L(x_1, x_2, ..., x_n) = \sum_{i_1+...+i_n = m} a_{i_1,i_2,...,i_n} x_1^{i_1} x_2^{i_2} ... x_n^{i_n}$$

be the leading form of f. We define M to be the smallest subspace of $k[x_1, x_2, ..., x_n]$ spanned by $x_1, x_2, ..., x_n$ in $k[x_1, x_2, ..., x_n]$ such that $L \in k[M]$. We define the variety $N = V(M)$ in $\operatorname{Spec} \hat{\mathcal{O}}_{X,y}$ to be the *Approximate manifold at y.*

Let $\tau(y)$ denote the dimension of M.

Theorem 5.2.5. *Suppose that $Sing_m(S)$ is a finite set. Then Theorem* 5.2.3 *is true.*

Let us see how the multiplicity of a point p and the dimension $\tau(p)$ behave under point/curve blow up.

Point blow up:

Let p be the blown up point and the local parameters of q_1 be (x_1, y_1, z_1). Then two of the three charts are depicted as follows:

1. $x_1 = x$, $y_1 = \frac{y}{x} - a$, $z_1 = \frac{z}{x} - b$

$f_1(x_1, y_1, z_1) = L(1, y_1 + a, z_1 + b) + x_1 g_1(x_1, y_1, z_1)$ for some g_1.

Clearly $v_q(f_1) \leq m$. And

$$L(1, y_1 + a, z_1 + b) = \sum_{i+j+k=m} a_{ijk}(y_1 + a)^j (z_1 + b)^k, \tag{5.1}$$

$v_q(f') = m$ implies $i = 0$ in 5.1. Clearly $\tau(p) \leq 2$

If $\tau(p) = 2$, then 5.1 implies, $a = b = 0$. Therefore, $\tau(q_1) \geq \tau(p)$.

If $\tau(p) = 1$ i.e.

$$L = cz^m,$$

$$mult_{q_1} S_1 = m \Rightarrow b = 0$$

and $\tau(q_1) \geq \tau(p)$.

2. $x_1 = \frac{x}{y} - a$, $y_1 = y$ $z_1 = \frac{z}{y} - b$

$f_1(x_1, y_1, z_1) = L(x_1, 1, z_1) + y_1 g_1(x_1, y_1, z_1)$ for some g.

Clearly $v_q(f_1) \leq m$. And

$$L(x_1 + a, y_1, z_1 + b) = \sum_{i+j+k=m} a_{ijk}(x_1 + a)^j (z_1 + b)^k, \tag{5.2}$$

$v_q(f') = m$ implies $j = 0$ in 5.2. Clearly $\tau(p) \leq 2$

If $\tau(p) = 2$, then 5.1 implies, $a = b = 0$. Therefore, $\tau(q_1) \geq \tau(p)$.

If $\tau(p) = 1$ i.e.

$$L = cz^m,$$

$$mult_{q_1} S_1 = m \Rightarrow b = 0$$

and $\tau(q_1) \geq \tau(p)$.

Similarly, in the third case, we see that, if $\pi : B(p) \to V$ is blow up at p, $E = \pi^{-1}(p)$ and $q_1 \in E \cap S_1$ Then $v_p(S) \geq v_{q_1}(S_1)$. If $v_p(S) = v_{q_1}(S_1)$, then $\tau(p) \leq \tau(q_1)$.

Also see that, if $\tau(p) = 3$, then $N_1 \cap E = \phi$, $\tau(p) = 2$, then $N_1 \cap E$ is a point and is a line if $\tau(p) = 1$ where N_1 is the birational transform of N.

Curve blow up :

Without any loss of generality, we assume $C \subset Sing_m S$ be the curve defined by $x = y = 0$. Let $\pi : V_1 = B(C) \to V$ is the blow up of C and $q \in \pi^{-1}(p) \cap S_1$. Then the description of one of the affine chart of the blow up variety is as follows:

1. $x_1 = x$, $y_1 = \frac{y}{x} - a$, $z_1 = z$

The equation of exceptional divisor is $x_1 = 0$. Then,

$$f_1(x_1, y_1, z_1) = L(1, y_1 - a, z_1) + x_1 g_1(x_1, y_1, z_1)$$

for some g_1 where

$$L(1, y_1 - a, z_1) = \sum_{i+j+k=m} a_{ijk}(y_1 - a)^j z_1^k. \tag{5.3}$$

Clearly, $v_{q_1}(S_1) \leq v_p(S)$. If $v_{q_1}(S_1) = m$ then $a = 0, i = 0$ in 5.3. Therefore,

$$L(x, y, z) = cy^m.$$

This implies $\tau(p) = 1$ and clearly, $\tau(p) \leq \tau(q_1)$. Similarly, for the other piece we see that, If $\pi : V_1 \to V$ is blow up at curve C in the top locus of S and $E = \pi^{-1}(C)$. Then, $v_{q_1} \leq m$ and $v_{q_1} = m \Rightarrow \tau(p) \leq \tau(q_1)$ for $q_1 \in E \cap S_1$.

Also note that, $N_1 \cap E = \phi$ if $\tau(p) = 2$ and is a curve which maps isomorphically under π if $\tau(p) = 1$.

Proof of theorem 5.2.5. We will show that after finitely many blow ups the invariant $\tau(q_i)$ increases. The three cases are as follows:

Case 1 : $\tau(p) = 3$

We know that the multiplicity drops at the very first blow up in this case.

Case 2 : $\tau(p) = 2$

We show that after finitely many blow ups along the component in the top locus of S_i, all $q_n \in \{E = \pi_n^{-1}(p)\} \cap S_n$ with $v_{q_n}(S_n) = m$ satisfy the equality $\tau(q_n) = 3$. On contrary, let us suppose that there exists an infinite such sequence i.e. there exists an infinitely many points q_i (one at each step) that maps to p under π_i such that $v_{q_i}(S_i) = m$ and $\tau(q_i) = 2$. Without any loss of generality, we assume that the sequence consists of point blow ups. This induces an infinite sequence of completion of local ring around $q_i's$ as follows:

$$\hat{O}_{V,p} \to \hat{O}_{V_1,q_1} \to ... \to \hat{O}_{V_n,q_n} \to ...$$

Let $f(x,y,z) = \sum_{i+j+k \geq m} a_{ijk} x^i y^j z^k$ be written as

$$f(x,y,z) = L(x,y,z) + g(x,y,z)$$

where $L(x,y,z) = \sum_{i+j+k=m} a_{ijk} x^i y^j z^k$.

We define

$$\gamma_{xyz}(f) = \min\left\{ \frac{k}{m-(i+j)} \mid a_{ijk} \neq 0 \text{ and } i+j < m \right\}.$$

Observe that $\gamma_{xyz}(f) < 1$ if and only if $v_p(f) < m$. Next, we define,

$$[f]_{xyz} = \sum_{(i+j)\gamma + k - m\gamma} a_{ijk} x^i y^j z^k$$

where $\gamma = \gamma_{xyz}$. If

$$T_\gamma = \left\{ (i,j) \mid \frac{k}{m-(i+j)} = \gamma, i+j < m \text{ for some k such that } a_{ijk} \neq 0 \right\},$$

then, $[f]_{xyz} = L(x,y,z) + \sum_{(i,j)\in T_\gamma} a_{ij\gamma(m-i-j)} x^i y^j z^{\gamma(m-i-j)}$. We call $[f]_{xyz}$ as solvable if there exist $a, b \in k$ such that,

$$[f]_{xyz} = L(x - az^l, y - bz^l)$$

for some $l \in \mathbb{N}$.

We observe that:

1. $\gamma_{x_1,y_1,z_1}(f_1) = \gamma_{xyz}(f) - 1$
2. $[f_1]_{x_1,y_1,z_1} = \frac{1}{z^m}[f]_{xyz}$ so, $[f_1]_{x_1,y_1,z_1}$ is not solvable if $[f]_{xyz}$ is not solvable.
3. If $\gamma_{x_1,y_1,z_1}(f_1) > 1$ then $L_1(x_1, y_1, z_1) = \frac{1}{z^m}L(x, y, z)$.

Lemma 5.2.6. *There exists a set of transformations*

$$x_1 = x - \sum_{i=1}^{n} \alpha_i z^i \text{ and } y_1 = y - \sum_{i=1}^{n} \beta_i z^i$$

such that $[f]_{x_1,y_1,z_1}$ is not solvable.

Proof. We make the change of variable

$$x_1 = x - az^l,\ y_1 = y - bz^l$$

On contrary, let us suppose there does not exist a change of variable such that f is not solvable. Then $f_1(x_1, y_1)$ is not solvable. We continue to make the transformations

$$x_i = x_{i-1} - a_i z^{l_i}\ y_i = y_{i-1} - b_i z^{l_i}$$

to obtain the power series

$$x' = x - \sum_{i=1}^{\infty} a_i z^{l_i}\ y' = y - \sum_{i=1}^{\infty} b_i z^{l_i}$$

Since, $l_{i+1} > l_i$, we see that $\gamma_{x',y',z}(f) = \infty$ which implies $f \in (x', y')^m$ which is a contradiction to the assumption that p_i is an isolated singularity. $\square$

Coming back to the proof of theorem 5.2.5, Through lemma 5.2.6 we assume that f is not solvable, then after finitely many steps we have $\gamma_{x_n,y_n,z_n} \leq 1$.
If $\gamma_{x_n,y_n,z_n} < 1$ then multiplicity reduces at this step. If $\gamma_{x_n,y_n,z_n} = 1$ we show that $\tau(q_n) = 3$ for every $q_n \in E \cap S_n$. Indeed, $\tau(q_n) = 2$ implies that there exist a, b, c, d, e, f with $ae \neq bd$ such that,

$$L(x_n, y_n, z_n) = \psi(ax_n + by_n + cz_n, dx_n + ey_n + fz_n).$$

Putting $z_n = 0$, we have

$$L_n(x_n, y_n, 0) = \psi(ax_n + by_n, dx_n + ey_n).$$

This implies $[f_n(x_n, y_n, z_n)]_{x_n,y_n,z_n}$ is solvable. Indeed,

$$L(x_n + gz_n, y_n + hz_n) = \psi(a(x_n + gz_n) + b(y_n + hz_n), d(x_n + gz_n) + e(y_n + hz_n))$$

$$L(x_n + gz_n, y_n + hz_n) = \psi(ax_n + by_n + (ag + bh)z_n, dx_n + ey_n(dg + eh)z_n).$$

In order to show that $[f_n]_{x_n,y_n,z_n}$ is solvable, we need to solve the linear system

$$ag + bh = c$$

$$dg + eh = f.$$

But the system possess a solution as $ae - bd \neq 0$. This implies $[f_n]_{x_n,y_n,z_n}$ is solvable which is a contradiction.

Case 3: $\tau(p) = 1$

Similar to the previous case, we show that after finitely many curve/point blow ups $\tau(q_n) = 2$ for all $q_n \in E \cap S_n$. On contrary, let us suppose that there exists an infinite sequence of point/curve blow up in the top locus. We choose a sequence of points $q_n \in V_n$ on the blown up subvariety that maps to q under π_n such that $v_{q_n}(S_n) = m$ and $\tau(q_n) = 1$. Let $R_n := \hat{O}_{V_n,q_n}$ then this induces an infinite sequence of blow ups at either maximal ideals $m_{q_i}R_i$ or at prime ideals $p_{C_i}R_i$. We define for

$$f(x, y, z) = a_{ijk}x^i y^i z^k$$

a polygon

$$\Delta(f, x, y, z) = \left\{ \left(\frac{i}{m-k}, \frac{j}{m-k} \right) \in \mathbb{Q} \mid k < m \text{ and } a_{ijk} \neq 0 \right\}$$

Let Λ be the smallest set in $\mathbb{R}^2$ which is convex, $\Delta(f, x, y, z) \subset \Lambda$ and for every $c \geq 0, d \geq 0$, $(a + c, b + d) \in \Lambda$ for $(a, b) \in \Lambda$.

We define:

$S(a) :=$ line of slope -1 through $(a, 0)$.

$V(a) :=$ vertical line through $(a, 0)$

$\alpha_{xyz}(f)$ be the smallest real number such that $(\alpha_{xyz}(f), b) \in \Lambda(f, x, y, z)$ for some b.

$\beta_{xyz}(f)$ be the smallest real number such that $(a, \beta_{xyz}(f)) \in \Lambda(f, x, y, z)$ for some a.

$\gamma_{xyz}(f)$ be the first number γ such that $S(\gamma) \cap \Lambda \neq \phi$

$\delta_{xyz}(f)$ be such that $(\gamma_{xyz} - \delta_{xyz}, \delta_{xyz})$ is the first intersection of $S(\gamma)$ with $\Delta(f, x, y, z)$.

$\epsilon_{xyz}(f) =|$ Largest slope of a line through $(\alpha_{xyz}(f), \beta_{xyz}(f))$ such that there does not exist a points of $\Delta(f,x,y,z)$ below the line. |

Observe the following:

1. $(\alpha_{xyz}, b), (a, \beta_{xyz})$ and $(\gamma_{xyz} - \delta_{xyz}, \delta_{xyz})$ are the vertices of Λ.
2. Vertices of Λ are the points of $\Delta(f,x,y,z)$.
3. $\alpha_{xyz}(g) < 1$ if and only if $g \notin (x,z)^m$.
4. A vertex lies below the line $b = 1$ if and only if $g \notin (y,z)^m$.
5. Multiplicity of the top locus of $V(f)$ is less than m if and only if $S(c) \cap \Gamma \neq \phi$ for some $c < 1$ which is possible if and only if there exists a vertex (a,b) such that $a + b < 1$.

Further terminologies:

We call the parameters (x,y,z) to be good parameters if in the presentation of f as

$$f(x,y,z) = \sum_{i+j+k \geq m} a_{ijk} x^i y^j z^k,$$

$a_{00m} = 1$. In which case,

$$S_{(a,b)} = \left\{ k \mid \left(\frac{i}{m-k}, \frac{j}{m-k} \right) = (a,b) \text{ and } a_{ijk} \neq 0 \right\}.$$

We define $f_{xyz}^{ab} := z^m + \sum_{k \in S(a,b)} a_{a(m-k),b(m-k),k} x^{a(m-k)} y^{b(m-k)} z^k$

We say that (a,b) on $\Lambda(f,x,y,z)$ is not prepared if a and b are integers and

$$f_{xyz}^{ab} = \left(z - cx^a y^b\right)^m$$

for some constant c and call it prepared otherwise. We call $\Lambda(f,x,y,z)$, well-prepared if all its vertices are prepared.

Observe that,

1. If (x,y,z) are good parameters of f and $\Lambda(f,x,y,z)$ is well-prepared then $z = 0$ is approximate manifold of $f = 0$.

We want to see what happens to the vertices of Λ when we make transformation. Let the transformation be $z_1 = z - cx^a y^b$, then $x^i y^j z^k$ is replaced by the sum

$$\sum_{\lambda=0}^{k} c^{k-\lambda} \frac{k!}{\lambda!(k-\lambda)!} x^{i+(k-\lambda)a} y^{j+(k-\lambda)b} z_1^{\lambda}.$$

2. If $k < m$, then the monomial with non-zero coefficient in $x^i y^j (z_1 + cx^a y^b)^k$ correspond to distinct points on the line joining (a,b) and $(\frac{i}{m-k}, \frac{j}{m-k})$ if $(a,b) \neq (\frac{i}{m-k}, \frac{j}{m-k})$,

otherwise all the monomial with non- zero coefficient in the sum correspond to the point $(a, b) = (\frac{i}{m-k}, \frac{j}{m-k})$.

3.If $m \leq k$ and $(i, j, k) = (0, 0, m)$, the monomial in the sum correspond to (a, b), otherwise, all the monomials other than z_1^m correspond to points in $(a, b) + \mathbb{Q}^2_{\geq 0}$ (a, b).

4. Therefore, the transformation $z = z_1 + cx^a y^b$ makes the following changes to the graph:

(i) $\Lambda(f, x, y, z) \subset \Lambda(f, x, y, z_1) \backslash (a, b)$

(ii) And all the vertices of $\Lambda(f, x, y, z)$ is also a vertex of $\Lambda(f, x, y, z_1)$ and f^{ab}_{xyz} is transformed to $f^{a'b'}_{xyz_1}$

Lemma 5.2.7. *Suppose that (x, y, z) are good parameters for f at p. There exists $\phi(x, y) \in k[[x, y]]$ such that $\Lambda(f, x, y, z_1)$ is well-prepared and (x, y, z_1) are good parameters.*

Proof. Let u_1, v_1 be the vertex of $\Gamma(f, x, y, z)$ such that v_1 is the smallest such. We make the transformation

$$z_1 = z - c_1 x^{u_1} y^{v_1}.$$

Let (u_2, v_2) be the lower most vertex of $\Gamma(f, x, y, z_1)$. If (u_2, v_2) is prepared, we are done. Otherwise, if $v_2 = v_1$, we make the transformation,

$$z_2 = z_1 - c_2 x^{u_2} y^{v}.$$

We continue to do this process till we achieve one of the following:

1. (u_n, v_n) is prepared. In which case the process terminates.

2. $v_n > v_{n-1}$

3. We have an infinite sequence of transformation such that the lower most vertex of $\Gamma(f, x, y, z_i)$ is (u_i, v) and none of them is prepared. Since $u_{n+1} > u_n$, we make the transformation,

$$z' = z - \sum_{i=0}^{\infty} c_i x^{u_i} y^{v}$$

such that the lower most vertex of $\Gamma(f, x, y, z')$ satisfy $v' > v$. We apply the same process for the other coordinate to achieve a power series $\phi(x, y)$ such that either the lower most vertex $(\bar{u}, \bar{v})$ of $\Gamma(f, x, y, \bar{z})$ is prepared or $\Gamma(f, x, y, \bar{z}) = \emptyset$. But the later case implies that $f = g(unit)$ which is not true since f is reducible. $\square$

Final terminologies:
Let (x, y, z) be the good parameters. Then we call a well prepared graph $\Lambda(f, x, y, z)$ to be very well-prepared if either of the following holds:
(i) $(\gamma_{xyz} - \delta_{xyz}\delta_{xyz}) \neq (\alpha_{xyz}, \beta_{xyz})$, and the transformation

$$y_1 = y - cx \text{ and } z_1 = z - \phi(x, y)$$

for well preparation yields :
$\alpha_{x,y_1,z_1}(f) = \alpha_{x,y,z}(f)$, $\beta_{x,y_1,z_1}(f) = \beta_{x,y,z}(f)$, $\gamma_{x,y_1,z_1}(f) = \gamma_{x,y,z}(f)$ and

$$\delta_{x,y_1,z_1}(f) \leq \delta_{x,y,z}(f)$$

(ii) $(\gamma_{xyz} - \delta_{xyz}, \delta_{xyz}) = (\alpha_{xyz}, \beta_{xyz})$, and one of the following holds:
(a) $\epsilon_{x,y,z} \neq 0$
(b) $\epsilon_{x,y,z} = 0$ and $\frac{1}{\epsilon} \notin \mathbb{Z}$
(c) $\epsilon_{x,y,z} \neq 0, n = \frac{1}{\epsilon} \in \mathbb{Z}$, and for any $c \in k$ an the transformation:

$$y_1 = y - cx^n, z_1 = z - \phi(x, y),$$

$\epsilon_{xy_1z}(f) = \epsilon_{xyz}$. Let (c, d) be the lower most point of the line through $(\alpha_{xyz}, \beta_{xyz})$ with slope $-\epsilon$ and $\Lambda(f, x, y, z)$. Similarly let (c_1, d_1) be defined for f_1, then $d_1 \leq d$.

Lemma 5.2.8. *Suppose (x, y, z) are good parameters for f at p then there exists power series $\phi(x, y), \xi(x)$ such that after the transformation*

$$z_1 = z - \phi(x, y) \text{ and } y_1 = y - \xi(x)$$

$\Lambda(f, x, y_1, z_1)$ are very well-prepared.

Coming back to the proof of the theorem, we see that if (x, y, z) are local parameters of p and $q_1 \in S_1 \cap \pi_1^{-1}(p)$ is such that $v_{q_1} S_1 = m$, then the local parameters of (x_1, y_1, z_1) are given by one of the following:
1. $x_1 = x \; y_1 = \frac{y}{x} + \eta \; z_1 = \frac{z}{x}$
2. $x_1 = \frac{x}{y} \; y_1 = y \; z_1 = \frac{z}{x}$
3. $x_1 = x \; y_1 = y \; z_1 = \frac{z}{x}$
4. $x_1 = x \; y_1 = y \; z_1 = \frac{z}{y}$
This is not the complete list of the possible candidates for the local parameters at q_1. But, notice that for all the remaining ones, the multiplicity decreases and hence, we are not considering them here. We name the four transformations as $T1, T2, T3, T4$ respectively. We note that, for all the four transformations if $v_{q_1} S_1 = m$ and $\tau(q_1) = 1$

then (x_1, y_1, z_1) are good parameters for f.

We shall now discuss the effect that these transformations on the numbers $\alpha_{xyz}, \beta_{xyz}, \gamma_{xyz}, \delta_{xyz}, \epsilon_{xyz}$. The results have been collectively stated in table. We shall prove the boxes in afterwards discussion.

Transformation $\rightarrow$	T_1 I	T_2 II	T_3 III	T_4 IV
1.Local parameters	$x = x_1, y = x_1(y_1 + \eta)$, $z = z_1 x_1$	$x = x_1 y_1, y = y_1$, $z = z_1 y_1$	$x = x_1, y = y_1$, $z = z_1 x_1$	$x = x_1, y = y_1$, $z = z_1 y_1$
2.Vertices	$(a, b) \mapsto (a + b - 1, b)$	$(a, b) \mapsto (a, a + b - 1)$	$(a, b) \mapsto (a - 1, b)$	$(a, b) \mapsto (a, b - 1)$
3.If (a, b) is Prep then	$(a + b - 1, b)$ is Prep	$(a, a + b - 1)$ is Prep	$(a - 1, b)$ is Prep	$(a, b - 1)$ is Prep
4.very well-prepared	No Let (x_1, y', z') be the very well preparation	No Let (x_1, y', z') be the very well preparation	Yes	No Let (x_1, y', z') be the very well preparation
5.α_{xyz}	$(\beta_{xy'z'}, \delta_{xy'z'}, \frac{1}{\epsilon_{xy'z'}}, \alpha_{xy'z'})$ $< (\beta_{xyz}, \delta_{xyz}, \frac{1}{\epsilon_{xyz}}, \alpha_{xyz})$ if $\eta \neq 0$	-	decreased by 1	-
6.β_{xyz}	$(\beta_{xy'z'}, \delta_{xy'z'}, \frac{1}{\epsilon_{xy'z'}}, \alpha_{xy'z'})$ $< (\beta_{xyz}, \delta_{xyz}, \frac{1}{\epsilon_{xyz}}, \alpha_{xyz})$ if $\eta \neq 0$ $\exists$ a very well preparation such that $\beta_{x,y',z'} < \beta_{x,y'z}$ such that $\beta_{x,y',z'} < \beta_{x,y'z}$ if $\eta = 0$	decreased	remains same	decreased
7.γ_{xyz}	-	-	decreased by 1	-
8.δ_{xyz}	$(\beta_{xy'z'}, \delta_{xy'z'}, \frac{1}{\epsilon_{xy'z'}}, \alpha_{xy'z'})$ $< (\beta_{xyz}, \delta_{xyz}, \frac{1}{\epsilon_{xyz}}, \alpha_{xyz})$ if $\eta \neq 0$	-	remains same	-
9.ϵ_{xyz}	$(\beta_{xy'z'}, \delta_{xy'z'}, \frac{1}{\epsilon_{xy'z'}}, \alpha_{xy'z'})$ $< (\beta_{xyz}, \delta_{xyz}, \frac{1}{\epsilon_{xyz}}, \alpha_{xyz})$ if $\eta \neq 0$	-	remains same	-

We shall prove the boxes of the table. Having done that, the following theorem gives a contradiction to the assumption that there exists an infinite length as in the assertion.

Theorem 5.2.9. *Let $p \in S$ be a point in the top locus and (x_n, y_n, z_n) be good parameters of q_n and*

$$\sigma(n) = (\beta_{f_n,x_n,y_n,z_n}, \delta_{f_n,x_n,y_n,z_n}, \frac{1}{\epsilon_{f_n,x_n,y_n,z_n}}, \alpha_{f_n,x_n,y_n,z_n}),$$

then $\sigma(n+1) < \sigma(n)$ for all n in the lexicographic order in the lattice $\frac{1}{m!}\mathbb{N} \times \frac{1}{m!}\mathbb{N} \times (\mathbb{Q}_+ \cup \{\infty\}) \times \frac{1}{m!}\mathbb{N}$.

Proof. The proof follows from the table. □

Proof of the table:

1. Row 2 is elementary.
2. Row 3 is straight forward and we leave it to the readers.
3. We will start with column I and prove all the rows together:

Proof. Notice that the linear transformation

$$(x, y) \mapsto (x + y - 1, y)$$

maps the line with slope $t \neq -1$ to the lines with slopes $\frac{t}{t+1}$.
Case 1: $\epsilon_{xyz} \geq 1$
It follows that $(\alpha_{xyz}, \beta_{xyz}) \neq (\gamma_{xyz} - \delta_{xyz}, \delta xyz)$. Let L_1 be the line joining (α, β) and $(\gamma_{xyz} - \delta_{xyz})$. Then, L_1 is mapped to a line of positive slope or a vertical line, which implies,

$$\beta_{x_1y_1z_1} < \beta_{xyz}.$$

Because very well preparation lay no effect on β, we have, $\beta x_1y_1z_1 < \beta_{xyz}$ in this case.
Case 2: $\frac{1}{2} \leq \epsilon_{xyz} < 1$
It follows that $(\alpha_{xyz}, \beta_{xyz}) = (\gamma_{xyz} - \delta_{xyz}, \delta_{xyz})$. The slope of line joining $(\alpha + \beta - 1, \beta)$ and $(c + d - 1, d)$ is

$$-\epsilon_{x_1y_1z_1} = \frac{\epsilon_{xyz}}{-\epsilon_{xyz} + 1} < -1$$

where c and d are as in the definition of very well-prepared. This implies $(\alpha_{x_1y_1z_1}, \beta_{x_1y_1z_1}) = (\alpha_{xyz} + \beta_{xyz} - 1, \beta_{xyz})$ and

$$(\alpha_{x_1y_1z_1}, \beta_{x_1y_1z_1}) \neq (\gamma_{x_1y_1z_1} - \delta_{x_1y_1z_1}, \delta_{x_1y_1z_1})$$

and $(c_1, d) = (c+d-1, d)$ is the lower most point of the intersection of the line through $(\alpha_{x_1y_1z_1}, \beta_{x_1y_1z_1})$ with slope $-\epsilon_{x_1y_1z_1}$ and $\Lambda(f_1, x_1, y_1, z_1)$. We claim that $\Delta(f_1, x_1, y_1, z_1)$ is very well-prepared and leave the verification to the readers.

We have, $\beta_{x_1y_1z_1} = \beta_{xyz}, \delta_{x_1y_1z_1} = \delta_{xyz}$ and $\frac{1}{\epsilon_{x_1y_1z_1}} = \frac{1}{\epsilon_{xyz}} - 1$

Case 3 : $\epsilon_{xyz} = 0$

It follows that, $\beta_{x_1y_1z_1} = \beta_{xyz}$ and $\Lambda(f_1, x_1, y_1, z_1)$ is very well-prepared. Also $(\alpha_{x_1y_1z_1}, \beta_{x_1y_1z_1}) = (\gamma_{x_1y_1z_1} - \delta_{x_1y_1z_1}, \delta_{x_1y_1z_1}) = (\alpha_{x_1y_1z_1} - \beta_{x_1y_1z_1}, \beta_{x_1y_1z_1})$ and $\epsilon_{x_1y_1z_1} = 0$. Finally $\epsilon_{xyz} = 0$ implies that $\beta_{xyz} < 1$ and the result follows. $\square$

4. We prove here II6:

II6. Since, T2 sends (a, b) to $(a, a + b - 1)$, we see that such a linear transformation $T : m(x, y) \mapsto (x, x + y - 1)$ on k^2 sends the lines with slope t to the lines with slope $t + 1$. In particular,

$$(\alpha_{x_1,y_1,z_1}(f_1), \beta_{x_1,y_1,z_1}(f_1)) = (\alpha_{x_1,y_1,z_1}(f_1), \alpha_{x_1,y_1,z_1}(f_1) + \beta_{x_1,y_1,z_1}(f_1) - 1).$$

Notice that, $\alpha_{x,y,z}(f) < 1$, since the singularity was at (x, y, z) and therefore, $\beta_{x_1,y_1,z_1}(f_1) < \beta_{x,y,z}(f)$. We know that, $\Lambda(f_1, x_1, y_1, z_1)$ is well-prepared so, $(\alpha_{x_1,y_1,z_1}(f_1), \beta_{x_1,y_1,z_1}(f_1))$ is not effected by very well preparation. $\square$

5. Notice that T3 pushes the graph down by one unit therefore, proof of III5-III9 becomes obvious after we show III4. The proof of III4 is as follows:

III4. We know that $\Lambda(f_1, x_1, y_1, z_1)$ is well-prepared. Suppose that $\Lambda(f_1, x_1, y_1, z_1)$ is very well-prepared. If $(\gamma_{x_1y_1z_1} - \delta_{x_1y_1z_1}\delta_{x_1y_1z_1}) \neq (\alpha_{x_1y_1z_1}, \beta_{x_1y_1z_1})$, we wish to show that the transformation

$$y_1' = y_1 - cx_1$$

yields :

$$\alpha_{x_1,y_1',z_1}(f_1) = \alpha_{x_1,y_1,z_1}(f_1), \beta_{x_1,y_1',z_1}(f_1) = \beta_{x_1,y_1,z_1}(f_1), \gamma_{x_1,y_1',z_1}(f_1) = \gamma_{x_1,y_1,z_1}(f_1)$$

$$\text{and } \delta_{x_1,y'_1,z_1}(f_1) \leq \delta_{x_1,y_1,z_1}(f_1).$$

The part of f that lies on the line $S(\gamma)$ are as follows :

$$\sum_{i+j+\gamma k = m\gamma | k<m} a_{ijk} x^i y^j z^k. \tag{5.4}$$

After T3, the part of f that lies on the line $S(\gamma_1)$ is as follows:

$$\sum_{(i+k-m)+j+\gamma_1 k = m\gamma_1} a_{ijk} x_1^{i+j-m} y_1^j z_1^k \tag{5.5}$$

Notice that after T3 transformation, only exponents have changed, but a_{ijk} have remained the same i.e., if i, j, k that satisfy $\{i + j + \gamma k = m\gamma | k < m\}$ in 5.4 then it satisfies $(i + k - m) + j + \gamma_1 k = m\gamma_1$ in 5.5 since $\gamma_1 = \gamma - 1$. And therefore we have,

$$\delta_{xyz}(f) = \delta_{x_1 y_1 z_1}(f_1) = \min\left\{\frac{j}{m-k} \mid a_{ijk} \neq 0 \text{ in } 5.5\right\}.$$

A translation $y = y' + cx$ transform 5.4 into

$$\sum_{i+j+\gamma k = m\gamma | k<m} b_{ijk} x^i y'^j z^k$$

and the translation $y_1 = y'_1 - c_1 x$ transforms 5.5 into

$$\sum_{(i+k-m)+j+\gamma_1 k = m\gamma_1} b_{ijk} x^i y'^j z^k.$$

In the case, $(\gamma_{x_1 y_1 z_1} - \delta_{x_1 y_1 z_1} \delta_{x_1 y_1 z_1}) \neq (\alpha_{x_1 y_1 z_1}, \beta_{x_1 y_1 z_1})$, we see, that $\Lambda(f_1, x_1, y_1, z_1)$ is very well-prepared. Similarly, one can see it for the other case. □

6. We now give a proof of IV6:

IV6. Since T4 sends (a, b) to $(a, b - 1)$, it pushes the graph on the left and therefore,

$$\beta_{x_1,y_1,z_1}(f_1) < \beta_{x,y,z}(f).$$

After a very well preparation the vertex $(\alpha_{x_1,y_1,z_1}(f_1), \beta_{x_1,y_1,z_1}(f_1))$ remains unaffected an therefore,

$$\beta_{x_1,y'_1,z'_1}(f_1) = \beta_{x_1,y_1,z_1}(f_1) < \beta_{x,y,z}(f).$$

□

Proof of theorem 5.2.5 is a matter of constructing local parameters (x_n, y_n, z_n) for q_n such that $(x_n y_n z_n)$ are good parameters, $Sing_m S_n \subset V(x_n, z_n)$ or $V(y_n, z_n)$ and that $\Lambda(f_n, x_n, y_n, z_n)$ is well-prepared. But this follows from the table. $\square$

It remains to show that all the cases for surface singularity reduce down to case 1, i.e., for a surface S there exists another surface S' and a birational proper map $f : S' \to S$ such that the top locus of $Sing_m S'$ is just finitely many points.

Theorem 5.2.10. *Suppose the curves in the top locus of $S \subset V$ are non-singular, then the sequence of blow ups of V_i along curves in $Sing_m S_i$ terminate.*

Proof. On contrary, let us suppose there exists an infinite sequence of blow up along non-singular curves in the top locus

$$... \to V_n \to V_{n-1} \to ... \to V_1 \to V.$$

The sequence induces another infinite sequence of local rings as follows :

$$\mathcal{O}_{V,C} \to \mathcal{O}_{V_1,C_1} \to ... \to \mathcal{O}_{V_n,C_n} \to ...$$

There exist infinitely many curves $C_i's$ that maps to C under the blow up maps π_i. As in the case of characteristic 0, we can view $\mathcal{O}_{V_n,C_n}$ as a local ring a point C_n in a 2 dimensional regular surface, where c_n is singularity of a curve embedded in the surface $\operatorname{Spec} \mathcal{O}_{V_n,C_n}$. Since embedded resolution for curves is true in arbitrary characteristic, this brings a contradiction. $\square$

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