Extremal extensions of positive maps and bound entangled quantum states

Thesis

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Declaration

The work presented in this thesis has been carried out by me under the guidance of Prof Arvind at the Indian Institute of Science Education and Research Mohali.

This work has not been submitted in part or in full for a degree, diploma or a fellowship to any other University or Institute. Whenever contributions of others are involved, every effort has been made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Ritabrata Sengupta

Place :
Date :

In my capacity as the supervisor of the candidate's PhD thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

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List of Publications

Out of this Thesis

1. **R Sengupta** and Arvind. Extremal extensions of entanglement witnesses and their connections with UPB. *Phys. Rev. A*, 87, 012318. (arXiv:1211.3294 [quant-ph])

2. **R Sengupta** and Arvind. Extremal extensions of entanglement witnesses : Unearthing new bound entangled states. *Phys. Rev. A*, 84, 032328 (arXiv:1106.4279 [quant-ph])

3. **R Sengupta**, Debmalya Das and Arvind. Entanglement detection and filtration. (In preparation).

4. **R Sengupta**, Arvind and Ajit Iqbal Singh. Projection operators on entangled subspaces. (In preparation).

Other Publications

1. L. Jeganathan, R.Rama and **Ritabrata Sengupta**. Note on Unlabelled Finite graphs. In *Languages and classification, Cryptography, Automata and Learning Theory*, pp. 188–191, Narosa, 2011.

2. L. Jeganathan, R. Rama, **Ritabrata Sengupta**. Application of Generalised sequential crossover of languages to generalised splicing. arXiv:0902.3818 [cs.DM], (Submitted in journal).

3. L. Jeganathan, R. Rama, **Ritabrata Sengupta**. Generalised sequential crossover of words and languages. arXiv:0902.3503 [cs.DM], (Submitted in journal).

4. L. Jeganathan, R. Rama, **Ritabrata Sengupta**. A proposal to a generalised splicing with a self assembly approach. arXiv:0802.0260 [cs.DM]. Accepted as a poster in "International Meeting on DNA Computing" (DNA 2008); Prague, Czech Republic.

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Chapter 1

Introduction

Quantum mechanics is one of the most important developments in physical sciences which took place in the first half of the twentieth century. The early pioneers like Born, Schrödinger, Heisenberg and Bohr used tools from Hamiltonian classical mechanics, developed in the nineteenth century, to formulate a theory describing microscopic phenomena. We follow the formulation of the theory as given in standard books such as Sakurai (Sakurai (1993)). and Peres (Peres (1993)). For a historical development of quantum mechanics, see Mehra & Rechenberg (1982a,b,c). For the development of quantum information and computation, refer to Nielsen and Chuang (Nielsen & Chuang (2000)).

Quantum entanglement plays a central role in quantum theory from a conceptual as well as a practical point of view. On the conceptual front, entanglement is intimately connected with the notions of non-locality and violation of Bell's inequalities Bell (1964, 1995), which is at the heart of the way the quantum mechanical description of the world differs from the classical one. On the practical front, quantum entanglement is essential in providing computational advantage to quantum computers over their classical counterparts Aaronson (2013); Nielsen & Chuang (2000).

The central question in this study is to determine whether a given arbitrary (pure or mixed) bipartite state ρ is entangled or separable. The problem has a simple solution for the case of pure states. A pure bipartite state is separable if and only if the reduced density operator obtained by tracing over one of the systems is pure. In fact the entropy of the reduced density operator can be used to quantify the amount of entanglement. However, for the case of mixed states such a characterization is not possible and only partial solutions are available. While there are methods to uncover entangled states, all of them are one-way conditions whose violation indicates entanglement. However satisfying a finite number of these conditions cannot guarantee separability. Thus the solution to this problem has remained elusive. Indeed from a computational perspective, the *s*eparability problem belongs to the class NP-hard Gurvits (2003); Ioannou (2007). A vast body of literature exists in this field. See Horodecki et al Horodecki *et al.* (2009) or Gühne and Tóth Gühne & Tóth (2009) for an exhaustive review.

Positivity is an important concept in quantum mechanics. The set of all states in a given dimension forms a closed cone. Maps which preserve positivity are called positive maps. Quantum evolution is described by a subclass of positive maps, called completely positive maps. Maps which are positive but not completely positive, are also important and will appear in this work, as they are the ones which can 'detect' entangled states. This study is important from a mathematical perspective as well, where positivity plays a key role. Indeed, in a list of fundamental tools used in mathematics, Fields medalist Alain Connes put positivity on the top of the list Connes (2004). It plays a key role in probability theory, in quantum mechanics and in operator algebra. Positivity of operators, used in quantum mechanics, is related to the positivity of Hilbert space operators. Thus in the study of quantum entanglement, we encounter non-commutative positivity, which Blecher Blecher (2007) coined as *Quantum Positivity*. In this thesis, we study positive maps which are not completely positive, with a view to explore quantum entanglement. We construct extensions of such maps, available in literature, discover new classes of entangled states, unearth a connection with unextendable product basis and quantum filtering.

1.1 Background & Motivation

In this section we give the basic ideas of quantum mechanics. We also discuss the developments of quantum computation and information in the past twenty years. The treatment is not exhaustive and the topics discussed here are to provide a context for our work.

1.1.1 Postulates of quantum mechanics

Quantum mechanics can be studied from an axiomatic point of view. A quantum system is described by a separable complex Hilbert space.

Axiom I The state space of a quantum system is the set of all positive semidefinite operators $\rho \in \mathcal{B}(\mathcal{H})$, with unit trace. $\mathcal{H}$ is a separable complex Hilbert space.

Such operators are called *states* or *density operators*.
A state of rank 1 is called a *pure state*. Otherwise it is called a *mixed state*.

Axiom II Physical observables can be represented by Hermitian operators $A \in \mathcal{B}(\mathcal{H})$. The expectation value of the observable A for the system represented by the state ρ is $\mathrm{Tr}[\rho A]$.

Axiom III The most general quantum operation between two systems $\mathcal{H}$ and $\mathcal{K}$ is given by a linear map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ such that:

1. ϕ is positive. For any density operator $\rho \in \mathcal{B}(\mathcal{H})$, $0 \leq \phi(\rho) \in \mathcal{B}(\mathcal{K})$
2. $0 \leq \mathrm{Tr}(\rho) \leq 1$.
3. For any natural number k, the natural extension of the map ϕ, written as

$$\begin{aligned} \mathbb{1}_k \otimes \phi : \mathcal{B}(\mathbb{C}^k) \otimes \mathcal{B}(\mathcal{H}) &\to \mathcal{B}(\mathbb{C}^k) \otimes \mathcal{B}(\mathcal{K}) \\ A \otimes B &\mapsto A \otimes \phi(B), \end{aligned}$$

is positive. Such a map is called a *completely positive map*.

Such completely positive maps can be described by the structure theorem given independently by Sudarshan (Sudarshan *et al.* (1961)), Kraus (Kraus (1971)) and Choi (Choi (1975a)). Any such quantum operation ϕ can be represented by a set of operators $\{V_k \in \mathcal{B}(\mathcal{H}, \mathcal{K})\}$, where $\sum V_k^\dagger V_k \leq I$ and

$$\phi(\rho) = \sum_k V_k^\dagger \rho V_k. \tag{1.1}$$

Such operators V_k are called *Kraus operators*. More details about such maps are given later in Section 1.2.1.2 and in the Appendix B.2.

The time evolution of a quantum system is given by a completely positive map which is trace preserving and unital. Any such map is described by a Kraus operator, which is a one parameter unitary operator $U(t)$. The generator of this group corresponds to the Hamiltonian of the system.

Axiom IV Quantum measurement is described by a set $\{M_k\}$ of measurement operators, which satisfy the completeness equation

$$\sum_k M_k^\dagger M_k = I. \tag{1.2}$$

If the state of a quantum system is in the state ρ before measurement, then the probability that the result k occurs is

$$P(k) = \text{Tr}(M_k^\dagger \rho M_k), \tag{1.3}$$

and the state after the measurement is

$$\rho' = \frac{M_k^\dagger \rho M_k}{\sqrt{P(k)}}. \tag{1.4}$$

By completeness of probability

$$\sum_k P(k) = 1.$$

1.1.1.1 Quantum states

Although quantum mechanical state space can be of infinite dimensions, in our work in this thesis only finite dimensional quantum systems are considered. In the finite dimensional case quantum states are described by finite dimensional operators i.e. matrices on a suitable space. We write down some definitions coming out of the axioms.

Definition 1.1.1 (State). *A (finite dimensional) quantum system is represented by a complex Hilbert space $\mathcal{H} = \mathbb{C}^n$. A state $\rho \in \mathcal{B}(\mathbb{C}^n)$ is a Hermitian positive semidefinite operator with unit trace.*

A state of rank one is called a pure state *otherwise it is called a mixed state. Hence a pure state can be represented by the corresponding nonzero eigen vector which is also called a state vector.*

The simplest quantum system is a qubit which is represented by a two-dimensional complex Hilbert space. A pure state of a qubit can be written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1,$$

where $|0\rangle$ and $|1\rangle$ denote the standard basis.

For simplicity, the word ‘positive’ is used in the place of ‘positive semidefinite’ throughout this thesis, unless explicitly mentioned otherwise.

It is useful to know the structure the state space of a quantum system. For the simplest case, i.e. for a two level system, a state $\rho \in \mathcal{B}(\mathbb{C}^2)$ can be written as,

$$\rho = \frac{1}{2}(I_2 + x.\sigma_x + y.\sigma_y + z.\sigma_z); \tag{1.5}$$

where $x,\ y$ and z are real numbers and

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.6}$$

are the Pauli matrices. Hence any state can be uniquely represented by the triplet (x, y, z). Positivity of ρ implies that $x^2 + y^2 + z^2 \leq 1$. The set of all the states are represented as points of a unit ball in $\mathbb{R}^3$, which in literature is known as Bloch-Poincare sphere. The surface of this sphere corresponds to pure states.

The problem of determining structure of all states becomes computationally difficult for higher dimensions. For dimension 3, it was solved by Arvind *et al.* (1997). The problem remains open for the higher dimensions.

1.1.1.2 Composite systems and Entanglement

If there are $k > 1$ distinguishable quantum systems, then the state space of the composite quantum system is given by the tensor product of the state space of individual systems.

Definition 1.1.2. *Let there be k distinct quantum systems whose state spaces are given by operators on the Hilbert spaces $\mathcal{H}_1, \cdots, \mathcal{H}_k$. Then the state space of the composite system is given by the operators on the Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$. If $\rho_j \in \mathcal{H}_j$ is a quantum state in the jth system, then the combined state in the composite system is given by $\rho_1 \otimes \cdots \otimes \rho_k$.*

In particular, for a bipartite system, i.e. when $k = 2$, any state ρ is a positive semidefinite Hermitian operator of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, whose trace is unity.

Any arbitrary state ρ acting on a composite system $\mathcal{H}_1, \cdots, \mathcal{H}_k$ need not be of the form of $\rho_1 \otimes \cdots \otimes \rho_k$. This leads to the important concepts of separability and entanglement. We give the definitions for the bipartite systems. These definitions can easily be extended for the general k as well.

Definition 1.1.3. *A state $\rho \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is said to be* separable *if it can be written as a finite sum*

$$\rho = \sum_{j=1}^{k} p_j \rho_j^{(1)} \otimes \rho_j^{(2)}, \quad p_i > 0,\ \sum_{j=1}^{k} p_j = 1; \tag{1.7}$$

where $\rho_j^{(i)} \in \mathcal{B}(\mathcal{H}_i)$ are the states in the respective sub-systems.

A state is called entangled *if it cannot be written in the above form.*

In this thesis we restrict our study to bipartite entanglement.

Example 1.1.1. *The simplest example of an entangled quantum state is the pure state* $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$. $\{|0\rangle, |1\rangle\}$ *denotes the standard orthonormal basis of the Hilbert space* $\mathbb{C}^2$.

Entanglement is one of the fundamental aspects which distinguishes quantum systems from classical systems. Due to the developments of quantum information theory in past two decades, entanglement has been used increasingly as an asset (Nielsen & Chuang (2000)).

Entanglement was initially considered as a counterexample to show the impossibility of quantum theory by Einstein, Podolsky and Rosen (Einstein *et al.* (1935)). Let Alice and Bob share an entangled state as given in Example 1.1.1. Each one of them can control only one of the subsystems since they are separated. If Alice makes a measurement in the $\{|0\rangle, |1\rangle\}$ basis of her system, and gets the value $|0\rangle$, then Bob's state also collapses to $|0\rangle$. Thus Alice can perfectly predict the outcome of Bob's measurement even if they are space like separated and hence cannot communicate. This, according to Einstein, Podolsky and Rosen, cannot take place. Hence they concluded the incompleteness of quantum theory. This conjecture is named as EPR conjecture, after the initials of the authors.

The above work created a long debate regarding validity of quantum mechanics, in which both physicists and philosophers contributed. A local hidden variable model of quantum mechanics was proposed to solve the above problem. This came to a logical conclusion with the seminal paper of Bell (Bell (1964)). Bell showed the impossibility of local hidden variable theories. He proved that entanglement is indeed a prime feature of quantum theory which separates quantum mechanics from its classical counterpart. Historical development of the theory as well as contributions of others can be seen in the book by Bell (Bell (1987)).

1.1.2 Quantum computation

Quantum computation is one of the most recent techniques of computation. In a sense, the researchers were interested to use materials beyond silicon and get a boost in the computational power. It was also wishfully thought that, by using new techniques one can bypass the problems related to NP-completeness. Among many proposals, two of them are noteworthy. The first was by Adleman (Adleman (1994)), who had introduced DNA computer and solved experimentally an instance of Hamiltonian path problem: This is a well known NP-complete problem. At the same time, Peter

Shor used quantum computation to give an algorithm for integer factorization problem. Hardness of factorization is the key point of the security of RSA crypto system. Shor's algorithm showed that a quantum computer can possibly perform computations which a classical computer cannot achieve.

1.1.2.1 Brief history

As the name suggests, a quantum computer uses laws of quantum mechanics for performing computational tasks. Feynman worked on the difficulty of simulating quantum states by classical computers. The complexity increases exponentially with the number of systems. Given copies of an n-qubit state and applying a unitary operator on it, we can determine the output state by repeated measurement or by tomography. However determining them by simulation takes exponential resources. Feynman wanted to use this phenomenon directly for computation, in which the input(s) and output(s) are quantum states and the classical data is extracted by (repeated) measurement (Feynman (1981/82)). [1]

Bennett and Brassard (Bennett & Brassard (1985, 1984)) used the works of Wiesner and quantum systems to create a novel scheme for key distribution of Shanon's perfect (classical) crypto system. Security of the method comes from the fact that any eavesdropping in this system is active in nature and can be determined by the sender and receiver. A different crypto system was created by Ekert (Ekert (1991)) by sharing of maximally entangled states.

The lectures of Feynman was first taken seriously by David Deutsch and Richard Jozsa when they proposed their algorithm (Deutsch & Jozsa (1992)). It showed that quantum computers are faster than (deterministic) classical computers in computing certain symmetric functions.

Computer scientists seriously started considering quantum computation when the first quantum algorithm for factorizing integers was announced by Shor (Shor (1994, 1997)). Factorizing an integer is considered as a difficult problem, though its complexity status is unknown till date. A major part of the public key cryptography is based on assumptions that certain problems like factorization, discrete logarithm etc. are difficult to compute. It is shocking to realize that an experimental realization of quantum

[1]Historically it was first proposed by the famous Soviet mathematician Ya Manin in a series of lectures from Moscow radio (see bibliographic notes in the book of Nielsen and Chuang (Nielsen & Chuang (2000))). Due to iron curtain, it was not known to most of the western block nations and hence not properly advertised. Indeed Feynman was the first to independently speculate and advertise, what is today known as quantum computation.

computer can theoretically break most of the existing crypto systems. As a result, both physicists and computer scientists started working on this newly discovered area.

It was quickly pointed out by the sceptics, that any such computer is highly error prone.[1] Quantum computation requires a level of purity of the state which may not be achievable in any practical setup. Even making a single qubit and isolating it for sufficient time from environment to manipulate, is not an easy job for the experimentalists. Thus making a quantum computer was considered an impossible task and the whole achievement was labeled as an interesting intellectual exercise. However in the very next year the first quantum error correcting code was announced by Shor (Shor (1996)). This showed that an error induced by environment or anything else can be corrected by quantum algorithms/codes. In due course, quantum error correcting codes became an increasing subject, just like its classical counterpart (for latest developments, see the book by Parthasarathy (Parthasarathy (2013))).

Of course all those good things has limitations. We are yet to create a scalable quantum computer. Neither an impossibility result has been proved theoretically.[2]. The first quantum computer used NMR techniques, which using Shor's algorithm, has managed to find out that $15 = 3 \times 5$, with a high probability. Nevertheless, many of the points of sceptics were countered by both experimentalists and theorists. First of all, with increasing sophistication of technology it is now possible to isolate a system for sufficiently longer time. It was clear that a scalable quantum computer is possible to construct if and only if the postulates of quantum mechanics are correct. This creates a serious problem for computer scientists and physicists. For example, if quantum mechanics is nonlinear (and such a model was considered), a classical computer can exploit that nonlinearity (however small) and can solve NP-complete problems in polynomial time. This is difficult to accept for most of the computer scientists. A more serious problem may arise with the definition of Turing machine and computability itself. For a lively discussion one can look at the recent book of Aaronson (Aaronson (2013)). Nevertheless we are in the same boat of Charles Babbage, when he failed to make a classical computer as he did not have the correct technologies.

[1]Incidentally the same thing holds for other computation methods like DNA computer.

[2]For the historical reason of keeping records, the company named D-Wave needs to be mentioned. They manufacture and market gigantic quantum computers (based on quantum annealing). Whether it is truly scalable, and the methodology of the machines are a matter of ongoing dispute

1.1.2.2 Grover search algorithm

How powerful is a quantum computer? Regardless of whatever the popular science magazines claim, till date, it cannot solve any known NP-complete problems in polynomial time.This can be explained the bounds provided by Grover search algorithm (Grover (1996, 1997)). In classical computation, searching unsorted database cannot be done less than linear time. Grover showed that, by using a suitable quantum oracle, it can be done sub-linear times, and its complexity is $O(\sqrt{n})$. This bound is optimal. Impressing at the first glance, it shows that : removing the structure of an NP-complete problem, solving can be reduced to a searching problem on correct solution on the solution space. In this case the present quantum computer which use discrete Fourier transformation can only give quadratic speed up. This concept was further extended in the seminal paper by Bennett et al (Bennett *et al.* (1997)), which also formally started the beautiful research area of quantum complexity. Such complexity theoretic arguments and bounds are not only mathematical jargons, but having physical meanings - as shown by Ambainis (Ambainis (2002)). In this paper a quantum argument for the search bounds has been given. This also gives us hope that the traditional complexity problems (like P versus NP, and various other similar ones) can have a physical meaning and perhaps the common belief that P$\neq$NP can be proved by using 'laws of nature'.

Grover search algorithm was nevertheless used for many special cases. Most important among them are graphs (for instance rooted trees). Just to mention, various improvements of the above algorithm and tighter bounds have been constructed which are beyond the scope of this thesis. However, we need to mention that the searching problem has been used to understand and explain the dynamics of electrons inside certain light harvesting molecules in plants (Lee *et al.* (2007)).

1.1.2.3 Quantum teleportation

Regardless of the negative result of Grover, quantum technology had a boom in the past decade. One remarkable idea is the concept of teleportation, Proposed by Bennett et al (Bennett *et al.* (1993)) and experimentally verified as well in which newer records of distances are achieved by increasing advancement of technology. The theory in a nutshell is that Alice sends an unknown state $|\psi\rangle$ to Bob, makes a few measurements in the process and transfers the measurement outcomes to Bob classically. Bob in turn, based on those outcomes, makes a series of local transformations on the state he receives and successfully recovers $|\psi\rangle$. Notice that, since Alice destroys the state, the quantum no-cloning theorem has not been violated. Quantum teleportation is a fundamental tool to build a quantum network. The present record is of distance 143 kilometer by using optical fiber (Ma *et al.* (2012)). Moreover it has been successfully done between clouds of gas atoms which are macroscopic objects (Krauter *et al.* (2013)).

1.1.2.4 Possible powers of quantum computation

Complexity class NP consists of all the languages which have a polynomial time verifier. The quantum analogue is the class QMA which are decided by a quantum polynomial time verifier with quantum state as certificate. By definition $\mathsf{NP} \subseteq \mathsf{QMA}$. Though their precise relation is unknown and the above relation is based on standard complexity hierarchy. At least two possible cases have been identified, in which quantum information can give extra advantages.

Informally, the class #P can be defined as, "How many solutions exists of a given problem". If the given problem is NP, then #P deals with the number of different solutions. Thus, $\mathsf{NP} \subseteq \#\mathsf{P}$. Valiant (Valiant (1979)) showed that finding permanent of a arbitrary matrix whose entries are coming from a given distribution (say, Gaussian) is #P. Recently Aaronson and Arkhipov (Aaronson & Arkhipov (2013)) had shown that while classical computers cannot do such a computation, such computations are very natural for bosonic particles. In particular, the Bosonic sampling problem is a doable experiment which shows the power of quantum computers over classical computers, assuming the standard hypothesis of quantum mechanics.

1.2 The problem

It is interesting to note that all of the above protocols of quantum algorithm or cryptography exploit quantum entanglement. Entanglement is used in quantum information and computation as a resource. For most of the studies, pure entangled states are employed. However for all practical purposes, we may not be able to keep the state pure and may be forced to use a mixed state. Hence it is important to know the structure of the state space of entangled states.

A central problem in quantum information theory is the following: Given an unknown quantum state ρ in a bipartite (or multipartite) system, determine whether it is an entangled state or separable. As stated earlier, the structure of set of states is not known. Even for simple $2 \otimes 2$ states, of which we have better knowledge, a proper geometric structure of the set of separable (as well as entangled) states is not completely known. (For recent developments in the $2 \otimes 2$ case, see Kye (Kye (2013))).

For the case of (bipartite) pure states of any dimensions, the problem is easy. Let $\rho = |\psi\rangle\langle\psi| \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ be a pure state. ρ is separable if and only if the reduced density operator $\mathrm{Tr}_{1/2}\rho$ is a pure state. (The subscript of Tr denotes the partial trace with respect to the first or the system respectively). The problem appears for the mixed states, which most of the experimentalists are expected to encounter.

1.2.1 Previous Work

For the mixed states, it has been proven to be NP-hard by Gurvits (Gurvits (2003)). There is some indication that the problem can be of NP-complete as well (Ioannou (2007)). In this situation, there are several methods to check the separability of a given state. It is to be noted that, the above result hold for the states whose density matrix is known. For an actual experimental set up, the problem is even more complicated.

1.2.1.1 Witnessing entanglement

Lemma 1.2.1 (Horodecki *et al.* (1996))**.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite dimensional complex Hilbert spaces. For any inseparable state $\rho \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$, there exists a Hermitian operator A such that $\mathrm{Tr}(A\rho) < 0$ but $\mathrm{Tr}(A\sigma) \geq 0$ for all separable state σ.*

The set of quantum states forms a closed convex set. The set of separable states forms a closed convex subset of the set of states. Hence, by using Hahn-Banach separation theorem, we can 'separate' a given entangled state from the set of separable states by a linear functional. More formally, if $\mathcal{S}$ is the set of separable states and ρ is an entangled state, then there exists a linear functional f over the real space of Hermitian matrices, such that $f(\rho) < 0$ whereas $f(\sigma) \geq 0$ for all $\sigma \in \mathcal{S}$. Such a functional is represented by a Hermitian operator A. It follows that A should not be a positive operator. Properties of A are given in the following theorem.

Theorem 1.2.1 (Horodecki *et al.* (1996))**.** *A state $\rho \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$ is separable if and only if $\mathrm{Tr}(A \cdot \rho) \geq 0$ for any Hermitian operator A satisfying $\mathrm{Tr}(A \cdot P \otimes Q) \geq 0$, where P and Q are projection operators acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.*

In other words, it should not have a separable state as an eigenvector corresponding to a negative eigenvalue. Such an operator is known as an *entanglement witness*.

1.2.1.2 Positive maps

The above construction leads to the concept of positive maps which are not completely positive and their connection with entanglement. A simple way to check, whether a given positive map is completely positive or not is given by Choi (Choi (1975a)) and Jamiołkowski (Jamiołkowski (1972)).

Let $\{|0\rangle, \cdots, |n-1\rangle\}$ be the standard basis of the space $\mathbb{C}^n$, and

$$|\psi\rangle = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} |j, j\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$$

be a maximally entangled state in $\mathbb{C}^n \otimes \mathbb{C}^n$.

Theorem 1.2.2 (Jamiołkowski (1972), Choi (1975a))**.** *Let $\phi : \mathcal{H}(\mathbb{C}^n) \to \mathcal{H}(\mathbb{C}^n)$ be a positive map. ϕ is completely positive if and only if the operator*

$$\mathbb{1} \otimes \phi(|\psi\rangle\langle\psi|) = \sum_{i,j=0}^{n-1} |i\rangle\langle j| \otimes \phi(|i\rangle\langle j|)$$

is positive semidefinite.

The same result holds, even if we replace $|\psi\rangle$ by any other maximally entangled state. (See the original works of Jamiołkowski (Jamiołkowski (1972)) and Choi (Choi (1975a)), which were done in a finite dimensional C^* algebra settings). The most generalized version of this theorem is given in Appendix Theorem B.2.3.

The above theorem is known as Choi-Jamiołkowski isomorphism. Using the inverse of the isomorphism, on an entanglement witness arising from lemma 1.2.1, we get a map which is positive but not completely positive.

Theorem 1.2.3. *Let $W \in \mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ be an entanglement witness as given in the Lemma 1.2.1. Then the map*

$$\begin{aligned} \phi : \mathcal{B}(\mathbb{C}^n) &\to \mathcal{B}(\mathbb{C}^m) \\ X &\mapsto \mathrm{Tr}_{\mathbb{C}^n}(X^T \otimes I_m \cdot W), \end{aligned}$$

is a positive map which is not completely positive. T denotes the transpose operator.

Thus the theorem 1.2.1 reduces to the following condition.

Theorem 1.2.4. *A state $\rho \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is separable if and only if for any positive map $\phi : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$, the operator $(\mathbb{1} \otimes \phi)\rho$ is positive.*

Thus we find that, the structure of entangled states is related with the structure of positive maps which are not completely positive. Though the structure of completely positive maps are well known, we do not know the structure of positive maps which are not completely positive, except for the following case. Størmer (Størmer (1963)) and Woronowicz (Woronowicz (1976b)) has shown that the positive maps $\mathcal{B}(\mathbb{C}^m) \to \mathcal{B}(\mathbb{C}^n)$, where $(m, n) = (2, 2),\ (2, 3),\ (3, 2)$, can be written as sum of a completely positive (CP) and transpose of a completely positive (called co-Cp or CCP) maps. In other words, for the above dimensions, the set of positive maps is a real linear combination of the set of CP maps and the transpose. Thus for the states in dimension ≤ 6 transpose is the only map which determines whether the state is separable or entangled. A state is separable if and only if it is positive under the map $\mathbb{1}\otimes$transpose, called in literature as partial transpose.

The above result does not hold for higher dimensions. Indeed, in all other dimensions, there exists maps which can not be written in the above form. Such maps are called indecomposable. Similarly, for any bipartite or multi-partite systems of dimension greater than 6 there are entangled states which are positive under partial transpose (PPT). Thus PPT is necessary and sufficient for separability of the systems $2 \otimes 3$ and $2 \otimes 2$ but only necessary in all other dimensions. Indecomposable maps are the only ones which can detect these states. The first example of such maps was discovered by Choi (Choi (1975b)). Since our work is based on this method; we give a detailed description in the appendix B.2.3.

Existence of PPT entangled states was known in the functional analysis literature. The first example of such states was given by Choi (Choi (1980a)). In fact, Størmer (Størmer (1982)) gave an alternate proof of in-decomposability of Choi's map, by producing a PPT entangled operator which is detected by Choi's map. Independently, Woronowicz (Woronowicz (1976a)) also discovered a similar PPT entangled state in $2 \otimes 4$ system, which was later used in the seminal paper of Horodecki et al (Horodecki *et al.* (1996)). Existence of PPT entangled states has serious consequences in quantum information theory, in particular for the existence of bound entangled states and on quantum key distribution.

1.2.1.3 Computable cross norm criteria (CCNR)

For a density matrix $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, its Schmidt decomposition in the operator space is given as

$$\rho = \sum_k \lambda_k G_k^A \otimes G_k^B \tag{1.8}$$

where $\lambda_k \geq 0$ and $\{G_k^A\}$ and $\{G_k^B\}$ are orthonormal basis of $\mathcal{B}(\mathcal{H}_A)$ and $\mathcal{B}(\mathcal{H}_B)$ respectively.

Theorem 1.2.5. *Chen & Wu (2003); Rudolph (2004, 2005) If a state is separable, then sum of all λ_k of 1.8 is less than equal to 1.*

The Schmidt decomposition defines a norm. For separable states the norm is 1, and for entangled states, the triangle inequality gives the result.

The above criteria can detect all entangled states in all dimensions except for the two qubit case (see Rudolph (2003)). The difficulty is generally to compute the necessary decomposition.

1.2.1.4 Schmidt rank

CCNR criteria is one of the generalisations of Schmidt number of pure states. There are many other generalisations as well (Aniello & Lupo (2009)). Generally computing this rank for an arbitrary state is difficult. We give one important generalisation given by Terhal and Horodecki (Terhal & Horodecki (2000)).

Given a pure bipartite state $|\psi\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ we write the Schmidt decomposition as

$$|\psi\rangle = \sum_{j=i}^{k} \sqrt{\lambda_k} |a_k\rangle \otimes |b_k\rangle; \tag{1.9}$$

where $\{|a_k\rangle\}$ and $\{|b_k\rangle\}$ are part of some orthonormal basis of $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively and $\lambda_k > 0$, $\sum_k \lambda_k = 1$. The number k is called as Schmidt rank of $|\psi\rangle$. It can be shown that Schmidt rank is 1 if and only if the state is separable. Moreover, $k \leq \min\{m, n\}$.

Definition 1.2.1 (Schmidt rank for density matrix)**.** *A bipartite state ρ has Schmidt rank k if*

1. *for any ensemble decomposition of ρ as $\{p_j \geq 0, |\psi_j\rangle\}$ where $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$; at least one of the vectors $|\psi_j\rangle$ has at least Schmidt rank k, and*

2. *there exists a decomposition of ρ with all vectors $\{|\psi_j\rangle\}$ has Schmidt rank at most k.*

It is clear that any state (pure or mixed) is separable if and only if the Schmidt rank is 1. Again it is not easy to compute this term for any state. However there is an important connection between the states of Schmidt rank and k-positive maps.

Theorem 1.2.6. *Let $\rho \in \mathcal{B}(\mathbb{C}^{n\otimes n})$ be a density matrix. Then it has Schmidt rank at least $k+1$ if and only if there exists a k-positive map $h : \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^n)$ such that* $\mathbb{1} \otimes h(\rho) \ngeq 0$.

In other words, if an entangled state ρ has Schmidt rank $k+1$, no l-positive but not $(l+1)$-positive map, where $l < k$ can detect the entanglement of ρ. This number is invariant under local operations, and hence is used as a measure of entanglement.

1.2.1.5 Range criteria

This was introduced by Horodecki (Horodecki (1997)). Apart from positive maps, this is perhaps the second such method constructed for detecting PPT entangled states. If a state ρ is separable, then there is a set of product vectors $\{|u_j, v_j\rangle\}$ which spans the range of ρ such that the set $\{|a_j^*, b_j\rangle\}$ spans ρ^{T_A}. T_A denotes the partial transpose.

The weakness is, if the state is of full rank entangled, the above criteria does not work as all the above conditions are already fulfilled. Physically, entanglement of a state affected by noise can not be detected by the above criteria.

One important byproduct of range criteria is the object called *Unextendible Product Basis* (UPB) (Bennett *et al.* (1999)). This is an incomplete basis whose each term is a product vector, and it cannot be extended to a full basis by introducing more product vectors. In other words all (normalised) vectors in the complement of the convex hull of such UPB are entangled. Bennett et al (Bennett *et al.* (1999)) constructed PPT entangled sates which are in this complement which can be detected by range criteria. In this thesis, we discover entanglement of PPT entangled states coming from UPBs. Such states are robust under noise (for instance, see Bandyopadhyay *et al.* (2008)). A detailed descriptions on UPB and beyond is given in Chapter 6.

1.2.1.6 Majorisation criteria

Let $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$. $\mathbf{x}^{\downarrow}$ (respectively $\mathbf{x}^{\uparrow}$) is the vector where the coordinates are written in decreasing (respectively increasing) order. We say $\mathbf{x}$ is majorised by $\mathbf{y}$, and write as $\mathbf{x}^{\downarrow} \prec \mathbf{y}^{\downarrow}$ (respectively $\mathbf{x}^{\uparrow} \prec \mathbf{y}^{\uparrow}$) if

$$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j, \quad 1 \leq k \leq n. \tag{1.10}$$

This is an important tool in matrix analysis. For more details and applications regarding majorisation see the book on Matrix Analysis by Bhatia (Bhatia (2007)).

Let $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$. Let $\mathbf{p}$ be the set of eigenvalues of ρ and $\mathbf{q}$ be the eigenvalues of reduced density matrix $\rho_A = \mathrm{Tr}_B(\rho)$. The majorisation criteria of Nielsen and Kempe (Nielsen & Kempe (2001)) says that if ρ is separable the $\mathbf{p}^{\downarrow} \prec \mathbf{q}^{\downarrow}$. Similarly the same is true for the reduced density matrix $\rho_B = \mathrm{Tr}_A(\rho)$.

1.2.1.7 Covariance matrix

Given a set of observables $\{A_\alpha\}$ the covariance matrix γ is defined to be

$$\gamma = [[\langle M_p M_q \rangle - \langle M_p \rangle \langle M_q \rangle]].$$

Let $\{G_j^A\}$ and $\{G_k^B\}$ are sets of mutually orthogonal sets of observables in $\mathcal{H}_A$ and $\mathcal{H}_b$ respectively. Consider the sets of observables $\{G_j^A \otimes I_B, I_A \otimes G_k^B\}$. The covariance matrix criteria (Gühne *et al.* (2007)) says that, if the state ρ is separable then there exist matrices $\kappa_{A/B} = \sum p_k \gamma(|\psi^{A/B}\rangle)$ such that

$$\gamma_\rho = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \geq \begin{pmatrix} \kappa_A & 0 \\ 0 & \kappa_B \end{pmatrix}.$$

This condition is very powerful, as it can detect PPT entangled states as well.

1.2.1.8 Bell inequality

Since Bell inequalities show the difference between classical and quantum systems, it is only natural to use them for detecting entanglement. Quantum entanglement is necessary for the violation of Bell inequalities (Bell (1964)). However the relationship between them is not fully understood. Peres conjecture states that no bound entangled state can violate Bell inequality. This is not true for multipartite systems (see Vértesi & Brunner (2012) and the references therein). As we know, the bipartite case is still unsolved. Such a violation, if exists, is very small, as shown by Moroder et al (Moroder *et al.* (2013)).

Gisin (Gisin (1996)) showed that for some such states violation can be created by local filters. An entangled state which does not violate Bell inequality can be made to violate it under this transformation. (For a recent work in this line, see Hirsch et al (Hirsch *et al.* (2013))).

In the above section we gave the most popular approaches of detecting entangled states. It has been shown that in certain dimensions some of the approaches are related to each other. It is generally believed that the same property holds for any dimension. However a general theory is yet to be discovered. A vast body of literature exists in this field. See the survey articles by Horodecki et al (Horodecki *et al.* (2009)) or by Gühne and Tóth (Gühne & Tóth (2009)) for an exhaustive review.

1.2.2 PPT entangled states

In this thesis, we concentrate the problem of detecting entanglement for PPT entangled states. Basic definitions are already given in the section 1.2.1.2. In this section, we give a short summary of such states.

Transpose map on a matrix algebra is defined to be

$$T : [[x_{ij}]] \mapsto [[x_{ji}]]. \tag{1.11}$$

This is a positive map which is not completely positive. In fact, this map is 1-positive but not 2-positive. Hence it can be used for detecting entangled states. The extended map $\mathbb{1} \otimes T$ is not positive is applied on any state $\rho \in \mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$. With respect to some basis ρ can be expressed in the block matrix form as,

$$\rho = [[\rho_{ij}]]_{p,q=1,\cdots,m};$$

where each ρ_{ij} is a $n \times n$ matrix. The action of the map is given as

$$\mathbb{1} \otimes T(\rho) = [[T(\rho_{ij})]]_{p,q}.$$

In other words, transpose operation is applied on each block. Such an operation is called *partial transpose*.

It turns out that, positivity under partial transpose is a necessary and sufficient condition for all separable states in the composite systems of dimensions less than or equal to 6. However, this is only a necessary condition, not sufficient, for separability in all other dimensions. Entangled states which are positive under partial transpose (in short, PPT entangled states) appear for composite systems of all dimensions greater than 6.

PPT entangled states are also known as *bound entangled states*, and sometimes the terms are used interchangeably in the literature. Most of the quantum information protocols requires pure entangled states. However, it is not possible to eliminate noise completely. Thus the question of so called *entanglement distillation* comes up. Let Alice and Bob share a finite number of copies of a state ρ. Distillation protocols give a way to generate a singlet state by using LOCC operations. For the detail protocols, see the works of Bennett et al (Bennett *et al.* (1996)) and Deutsch at al (Deutsch *et al.* (1996)). A state ρ is said to be distillable if the above protocols can give maximally entangled states as outcome. Otherwise it is called as *bound entangled*.

It has been shown that no PPT entangled state is distillable. For the details of the proof and further developments see the works of Horodecki et al (Horodecki &

Horodecki (1999); Horodecki *et al.* (1998)) and Hiroshima (Hiroshima (2003)). The existence of a NPT state which is not distillable, is a long standing open problem.

Detection of PPT entangled states is a computational challenge. Most of the methods described earlier are difficult to apply for detecting such states. The approach of positive maps requires an appropriate indecomposable positive map to detect entanglement. Similar difficulty arises for choosing the correct observables for applying the covariance matrix criteria. Range criteria requires detection of an appropriate product basis in the image space of the state. If the state is of low rank, detecting the existence or non-existence of such basis is comparatively easy. With the increasing dimensions of the system or rank of the state, the dimension of the range space increases. As a result, it becomes difficult to apply this method. It is not known whether any PPT entangled state can be detected by majorization criteria. The method of Bell inequality is again difficult to apply. First of all, there are families of Bell type inequalities, all of them detect nonlocality. There are maximally entangled states which violates one version of Bell inequality but do not violate other versions. Thus to check Bell inequality violation of any state, one needs to check *all* possible Bell type inequalities. Further, Gisin (Gisin (1996)) has given examples of pure entangled states (and hence NPPT) which do not violate some version of the inequality. Whether there exists any bipartite PPT entangled state which violates some version of Bell inequality, is an open problem and is known as *Peres conjecture*. The multi-partite version of this conjecture is false.

A different line of research is to create PPT entangled states. Since PPT entanglement is difficult to detect, such states are prepared in a way so that they violate at least one of the conditions written above. The first method to create PPT entangled states was given by Bennett et al (Bennett *et al.* (1999)) by using unextendable product bases. (We have discussed this approach in details at Chapters 4 and 6).

The set of PPT states also form a closed convex set. It has been proved that if a given PPT states violates range criterion then the smallest face containing it has no separable states in the interior. (See Choi and Kye (Choi & Kye (2012)) and also the Subsection 1.2.1.5) is an extremal point of this set. In this approach, a state is constructed which is positive under partial transpose and violates the range criteria. Often such states have some extra properties. For instance the states constructed by Clarisse (Clarisse (2006)) and by Kye and Osaka (Kye & Osaka (2012)) have certain fixed matrix rank. The work of Chen and Doković (Chen & Doković (2013)) gives the latest developments of this topic.

1.3 Arrangement of the thesis

In chapter 2, we give a numerical method to construct a PPT entangled state, which can be detected by a given positive map. The output states are symmetric under partial transpose.

In chapter 3 we discuss the positive biquadratic forms and their connection with the positive maps. A method for generating new maps from the given forms are shown. Moreover, we have constructed a new method for generating extremal maps from the forms. We also construct a numerical method to construct PPT entangled state, which are detected by the new maps.

In chapter 4 we generate a theory of local isomorphism and using it, generate new classes of extremal positive maps. We also show that these modified maps can detect entanglement of the PPT entangled states coming from UPB.

The automorphisms which are not unitary are difficult to implement. These are known as the filtering operations. In the chapter 5, we have developed a POVM approach of implementing such operation.

PPT entangled states coming from the UPB construction of Bennett et al (Bennett *et al.* (1999)) is used extensively throughout the thesis. These bound entangled states are normalised projection operators on some subspace. Such subspaces are studied extensively by Parthasarathy (Parthasarathy (2004)) and Bhat (Bhat (2006)). We have studied the properties of the projection operators on such subspaces in the chapter 6.

In the process of the above work, we have encountered several negative results. These results, though not published, are included in appendix A. Here we try to develop a system to identify potential examples of extremal points of the set of positive maps. It gives greater attention to the case of the space $\mathbb{C}^3 \otimes \mathbb{C}^3$. We note that given two positive maps ϕ_1 and ϕ, is the difference $\phi_1 - \phi$ is a completely positive map, then the map ϕ is more powerful than ϕ_1 in terms of detecting entangled states. Since we are considering the detection of PPT entangled states, the difference can be any constant times a decomposable map to hold the general theory. Though this construction is weaker than the one given by Lewenstein et al (Lewenstein *et al.* (2000)), we can show that this technique is easily applicable. Using it we can show that most of the positive maps available in the literature are actually interior points of the cone of positive maps. We also identify the potential candidates of extremal points as well.

An appendix B is added to give some basic informations regarding the C^* algebra concept of positivity. It gives a list of available positive maps till date, which are not completely positive and indecomposable. The question of extremal and exposed maps are also discussed in brief. The examples of positive maps which are proven to be extremal are also mentioned. The theory of positive maps is not well understood. A subclass of the theory, namely the theory of completely maps is rather well studied, as it has a structure theorem which is Kraus decomposition. We give the necessary background of the C^* algebra and the positive maps in the appendix B. This contains a survey of the subject along with a list of known classes of such maps which will be used extensively in the text.

Chapter 2

Cholesky decomposition and numerical construction of PPT entangled states

PPT entangled states are exotic quantum states which are found in bi-partite systems of dimensions $3 \otimes 3$ and higher. The entanglement in these states is not distillable and is also referred to as bound entanglement. Even for the simplest bi-partite system namely, the system with Hilbert space dimension $3 \otimes 3$, the set of PPT entangled states is not fully characterized. There is no systematic way of finding such states. The examples of PPT entangled states found in literature are either isolated cases or are generated from UPB. Typically, the entanglement of a PPT entangled state is detected by a positive map which is not completely positive. However, given such a map, it is not possible to find all states whose entanglement is implicated by this map.

In this chapter, a method for generating states which are symmetric under partial transpose and are detected by a given positive map, is given. By construction, these states are PPT and their entanglement is detected by the given map. To achieve this, we perform a numerical search over symmetric states and employ the Cholesky decomposition to selectively scan the states in the Hilbert space. We also make use of Choi Jamiołkowski isomorphism to create an entanglement witness for the state. Although we employ this method for specific maps, the method of searching for PPT entangled states detected by a given positive but not completely positive map is in fact more general and can be tried for other maps as well.

2.1 Cholesky decomposition

Any operator A is positive if and only if it can be written as $A = B^\dagger B$. Given A, such a B need not be unique. One can restrict the choice of B to an upper triangular matrix. Such a decomposition is known as Cholesky decomposition.

Theorem 2.1.1 (Cholesky decomposition (See Bhatia (Bhatia (2007))). *A matrix $A\mathcal{B}(\mathbb{C}^n)$ is positive if and only if $A = T^\dagger T$ for some upper triangular matrix T. Further, T can be chosen to have nonnegative diagonal entries. If A is strictly positive (i.e. all eigenvalues are greater than zero), then T is unique. This is called the Cholesky decomposition of A. A is strictly positive if and only if T is nonsingular.*

The application of Choi-Jamiołkowski isomorphism to construct entanglement witnesses comes from the Theorems B.2.4 and B.2.4. Consider a composite system $H_1 \otimes H_2$ where both the subsystems are of dimension d. Using the standard basis $|i\rangle$ in both the H_1 and H_2 we define a maximally entangled state

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle; \tag{2.1}$$

Given a positive map Φ that is not completely positive we define an operator

$$W_\Phi = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|). \tag{2.2}$$

Such an operator is not positive. By construction, such a W acts as an entanglement witness given in the theorem 1.2.1. Given a density matrix ρ defined on $H_1 \otimes H_2$ the operator W provides a sufficient condition for entanglement

$$\mathrm{Tr}(W\rho) < 0 \Longrightarrow \quad \rho \quad \text{Entangled} \tag{2.3}$$

This is a one-way condition. Given a witness W such that $\mathrm{Tr}(W\rho) \geq 0$ does not imply that the state ρ is separable. The witness W is dependent on the choice of maximally entangled state. A different choice of the maximally entangled state will give rise to a different witness, which can potentially detect a completely different set of entangled states.

The above condition for entanglement given in (2.3) helps us in quickly identifying states whose entanglement is revealed by the map Φ and we employ this condition in our search for PPT entangled states revealed by Osaka family of maps.

2.1.1 The scheme

Let the space be $\mathbb{C}^d \otimes \mathbb{C}^d$. A positive, not completely positive, indecomposable map $\Phi : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^d)$ is given.

Step 1. Consider the $d^2 \times d^2$ upper triangular operator $T = ((a_{i,j}))$, where $a_{i,j} = 0$ for $i < j$. This requires $\frac{1}{2}d^2(d^2+1)$ indeterminate $a_{i,j}$s.

Step 2. Let $A = T^\dagger T = [[\alpha_{i,j}]]_{i,j=1,\cdots,d^2}$.

Step 3. Let $A^{PT} = [[\beta_{i,j}]]_{i,j=1,\cdots,d^2}$ be the partial transpose of the operator A.

Step 4. Set $A = A^{PT}$ and write down the equations

$$\alpha_{i,j} = \beta_{i,j}, \quad i,j = 1, \cdots, d^2. \tag{2.4}$$

In the above numeration, some of the equations will be repeated. A can be considered as a $d \times d$ matrix $[[A_{p,q}]]$ where each $A_{p,q}$ is a $d \times d$ matrix. Thus A^{PT} can be written as a transpose operation on each such $d \times d$ block, i.e $A^{PT} = [[A^T_{p,q}]]$. Since transpose does not alter the diagonal elements of each block, we get $(d^2 - d) = d(d-1)$ equations from each block. Maximum number of possible distinct equations is $d^3(d-1)$.

Step 5. Choose a suitable maximally entangled state as in equation 2.1, and construct the witness W_Φ corresponding to the map Φ as shown in the equation 2.2.

Step 6. Consider the inequality $\mathrm{Tr}(W_\Phi \cdot A) < 0$ and add it in the list of equations coming from Step 4.

Step 7. Solve the system of equations and inequalities.

Step 8. If a solution exists then, based on the values of $a_{i,j}$'s, reconstruct T. Else: declare, "No such symmetric state exists".

Step 9. Declare the state $\rho = \frac{1}{\mathrm{Tr}(A)} A$.

It can happen that, we want a particular state, which is not detected by a certain map ϕ. In that case :

Step 10. Construct W_ϕ as in the previous way.

Step 11. Consider the inequality $\mathrm{Tr}(W_\phi \cdot A) \geq 0$ and add it in the list of equations and inequalities given after Step 6.

Step 12. Goto Step 7.

The above algorithm can easily be modified if there are more than one maps to consider. Further, the case when the dimensions are different, i.e. the case of the system $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ can also be accommodated easily by changing the maximally entangled state.

2.2 Implementation and examples

Although the scheme presented above seems simple and straightforward, the complexity of the algorithm increases with increasing dimensions. Moreover an increase in the number of variables decreases the chances of finding numerical solutions. To deal with the first problem, we can restrict the search space of each a_{ij}. A state ρ is always normalised, and so $\|\rho\|_1 \leq 1$. Thus we can restrict ourselves to the case when each entry of the matrix ρ can be written as $\rho_{i,j} = r_{i,j}\exp(\imath\theta_{i,j})$, where $r_{i,j} \leq 1$ and $-\pi \leq \theta \leq \pi$. We can further restrict the number of variables by inserting zeros for specific values of $a_{i,j}$. As we will see in the following example, the algorithmic scheme presented above combined with intuitive understanding of the map, leads to unearthing bound entangled states.

An interesting class of positive maps was given by Kye (1992), who extended the result of Choi and Lam, and generated a class of maps whose extremality was proved by Osaka (1992). Kye – Osaka's map $\Phi_O(x, y, z)$ is defined as

$$\Phi_O(x,y,z) : \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} + xa_{33} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} + ya_{11} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} + za_{22} \end{pmatrix}, \quad (2.5)$$

where $x, y, z > 0$ and $xyz = 1$. Osaka (1992) showed that this class of maps is extremal. If $x = y = z = 1$, we get back the map given by Choi. The examples of PPT entangled states detected by the Choi map are known in the literature. The interesting question is whether the Osaka generalization of the Choi map can detect states which are not detected by the Choi map. We address this question by employing our numerical scheme where we find states whose entanglement is detected by the generalized Osaka map. Further, we show that this search processs leads to states whose entanglement is not detected by the Choi map. As an example, we want to find a PPT entangled state which is detected by the above map, where the values $x,\ y,\ z$ are not all equal to 1. In other words, the PPT entangled state which could not be detected by the Choi map, is now detected by the generalised Kye–Osaka map.

Following the algorithm outlined in the previous section, we first construct an upper triangular matrix T. We only consider the case where $a_{i,j} \in \mathbb{R}$. This amounts to generating a nontrivial solution of the equation

$$(T^tT)^{PT} - T^tT = 0. \tag{2.6}$$

For example in the $3 \otimes 3$ situation we take T to be of the form

$$\left(\begin{array}{ccc|ccc|ccc}
a_{1,1} & 0 & 0 & 0 & a_{1,5} & 0 & 0 & 0 & a_{1,9} \\
0 & a_{2,2} & 0 & a_{2,4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{3,3} & 0 & 0 & 0 & a_{3,7} & 0 & 0 \\
\hline
0 & 0 & 0 & a_{4,4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{5,5} & 0 & 0 & 0 & a_{5,9} \\
0 & 0 & 0 & 0 & 0 & a_{6,6} & 0 & a_{6,8} & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & a_{7,7} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{8,8} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{9,9}
\end{array}\right) \tag{2.7}$$

Using this as T we can reconstruct A as

$$\left(\begin{array}{ccc|ccc|ccc}
a_{1,1}^2 + a_{1,5}^2 + a_{1,9}^2 & 0 & 0 & 0 & a_{1,5}a_{5,5} + a_{1,9}a_{5,9} & 0 & 0 & 0 & a_{1,9}a_{9,9} \\
0 & a_{2,2}^2 + a_{2,4}^2 & 0 & a_{2,4}a_{4,4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{3,3}^2 + a_{3,7}^2 & 0 & 0 & 0 & a_{3,7}a_{7,7} & 0 & 0 \\
\hline
0 & a_{2,4}a_{4,4} & 0 & a_{4,4}^2 & 0 & 0 & 0 & 0 & 0 \\
a_{1,5}a_{5,5} + a_{1,9}a_{5,9} & 0 & 0 & 0 & a_{5,5}^2 + a_{5,9}^2 & 0 & 0 & 0 & a_{5,9}a_{9,9} \\
0 & 0 & 0 & 0 & 0 & a_{6,6}^2 + a_{6,8}^2 & 0 & a_{6,8}a_{8,8} & 0 \\
\hline
0 & 0 & a_{3,7}a_{7,7} & 0 & 0 & 0 & a_{7,7}^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_{6,8}a_{8,8} & 0 & a_{8,8}^2 & 0 \\
a_{1,9}a_{9,9} & 0 & 0 & 0 & a_{5,9}a_{9,9} & 0 & 0 & 0 & a_{9,9}^2
\end{array}\right).$$

By construction this is a positive matrix. For it to become a quantum state we need to normalise it. To make it symmetric under partial transpose (and hence PPT) we use equation 2.6 and get the following equations.

$$\begin{cases} a_{1,5}a_{5,5} + a_{1,9}a_{5,9} = a_{2,4}a_{4,4} \\ a_{1,9}a_{9,9} = a_{3,7}a_{7,7} \\ a_{6,8}a_{8,8} = a_{5,9}a_{9,9} \end{cases} \tag{2.8}$$

It is not always possible to find a non-trivial solution to the above equation. The possibility of finding a solution depends upon the choice of T. Therefore, if we do not find a solution, we begin with a new T. However, if we begin with a sparse matrix T like in 2.7, it is very likely that a solution of the above system of equations can be found. Now that we have a state which is invariant under partial transpose, it is enough to check whether the state is entangled or not. Given an indecomposable map Φ we now use the Choi-Jamiołkowski isomorphism. Thus a set of solutions satisfying the set of equation 2.6 and the inequalities 2.3 gives us PPT entangled states detected by the given map Φ. To complete our example of a state in a $3 \otimes 3$ system which is detected only by Osaka's map not Choi's map; at every stage of the search process, we impose the following conditions:

$$\begin{cases} \mathrm{Tr}\,(W\Phi_O\rho) < 0 \\ \mathrm{Tr}\,(W\Phi_{C^I}\rho) \geq 0 \end{cases} \tag{2.9}$$

This means that we restrict ourselves to those PPT states whose entanglement is revealed by Osaka's map but is not revealed by Choi's map.

We now address the question of constructing PPT entangled states for which Osaka's map acts as an entanglement witness. Although the three-parameter family of maps due to Osaka as described in equation 2.5 has been defined, and is known to be positive but not completely positive, there has not been an explicit construction of PPT entangled states whose entanglement is revealed by this class of maps.

The above methodology guides us in our numerical search to look for classes of states whose entanglement is revealed by the Osaka family of maps by finding the solution space of equations 2.8 and the inequalities 6.3. Using this method of numerical search, we construct an example of a PPT entangled state for a $3\otimes 3$ system. The upper triangular matrix

$$T = \begin{pmatrix} \sqrt{10}y & 0 & 0 & 0 & \frac{2+5y}{\sqrt{10}} & 0 & 0 & 0 & \frac{3(1+y)}{\sqrt{10}} \\ 0 & y & 0 & 2+5y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3y & 0 & 0 & 0 & 1+y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4+5y}{\sqrt{10}} & 0 & 0 & 0 & \frac{1+y}{\sqrt{10}} \\ 0 & 0 & 0 & 0 & 0 & 1+2y & 0 & 1+y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.10}$$

leads to a one parameter family of density operators, parameterized by a positive parameter y.

$$\rho(y) = \frac{1}{N} \times \begin{bmatrix} 10y^2 & 0 & 0 & 0 & y(2+5y) & 0 & 0 & 0 & 3y(1+y) \\ 0 & y^2 & 0 & y(2+5y) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9y^2 & 0 & 0 & 0 & 3y(1+y) & 0 & 0 \\ 0 & y(2+5y) & 0 & (2+5y)^2 & 0 & 0 & 0 & 0 & 0 \\ y(2+5y) & 0 & 0 & 0 & 2+6y+5y^2 & 0 & 0 & 0 & (1+y)(1+2y) \\ 0 & 0 & 0 & 0 & 0 & (1+2y)^2 & 0 & (1+y)(1+2y) & 0 \\ 0 & 0 & 3y(1+y) & 0 & 0 & 0 & (1+y)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1+y)(1+2y) & 0 & (1+y)^2 & 0 \\ 3y(1+y) & 0 & 0 & 0 & (1+y)(1+2y) & 0 & 0 & 0 & (1+y)^2 \end{bmatrix} \tag{2.11}$$

where $N = 10 + 36y + 57y^2$ is the normalization factor such that $\mathrm{Tr}(\rho(y)) = 1$. We apply a one parameter subfamily of Osaka's map defined in (2.5) $\Phi_O\left(1, x, \frac{1}{x}\right)$ to the family of states $\rho(y)$ and compute the eigen values of the resultant operator. We do a similar computation of the eigenvalues of the operator which is obtained by the action Choi's maps Φ_C^I and $\Phi_C^I I$ for comparison. In Figure 2.1 the least eigenvalue is

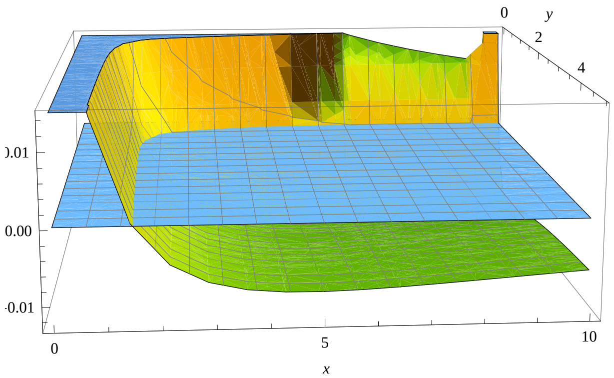

Figure 2.1: Application of Osaka's map $\Phi_O\left(1, x, \frac{1}{x}\right)$ to the density operator $\rho(y)$. x and y are the two variables, and the vertical axis denotes the eigenvalues of $\rho(y)$ under the map. The curved surface represents the variation of the minimum eigenvalue. The plane in the center is the plane $xy = 0$, which highlights the portion the surface with negative eigenvalue.

plotted as a function x and y. Here the curved surface denotes the minimum eigenvalue corresponding to the $\Phi_O\left(1, x, \frac{1}{x}\right)$. The middle plane denotes the xy plane which is placed to indicate the place when the surface becomes negative.

In Figure 2.2 we have taken a fixed value of x. The eigenvalues are plotted along the vertical axis and y varies along the horizontal axis. We apply the map $\Phi_O\left(1, 6, \frac{1}{6}\right)$ to this state $\rho(y)$ and plot the minimum eigenvalue which is denoted by the continuous curve in Figure 2.2. The dashed curve denotes the minimum eigenvalue achieved by the Choi's map. The plot highlights that approximately after point $y = 0.326402$ the minimum eigen value under $\Phi_O\left(1, 6, \frac{1}{6}\right)$ becomes negative. Thus there is a range of values, where Osaka's map can identify more PPT entangled states where the Choi's map fails to do so.

In the next Chapter 3 we have given a method to construct a new positive map from a given one. In particular we consider the Choi map and modify it by using parameters

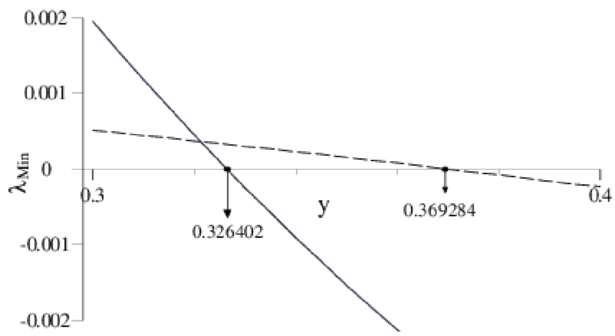

Figure 2.2: Application of Osaka's map with $x = 6$. y is the horizontal axis and eigenvalues are in the vertical axis. The continuous line denotes the variation of eigenvalues under $\Phi_O\left(1, 6, \frac{1}{6}\right)$. The dashed line corresponds to Choi's map. It clearly shows that approximately after $y = 0.326402$ onward, PPT entanglement is revealed by $\Phi_O(\left(1, 6, \frac{1}{6}\right)$. However Choi's map can reveal entanglement approximately from $y = 0.369284$. This shows the superiority of Osaka's map over Choi's map for this instance of $\rho(y)$.

$(2, 1, 1)$. Using the method described above we construct the upper triangular matrix

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{9x}{16} & 0 & 0 & 0 & \frac{3x}{2} \\ 0 & 0 & 0 & -\frac{9x}{16} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3x}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{91x}{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{9x}{4} & 0 & 0 & 0 & 2x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x \end{pmatrix}$$

We get the (not normalised) positive operator as

$$\left(\begin{array}{ccc|ccc|ccc} \frac{657x^2}{256} & 0 & 0 & 0 & \frac{273x^2}{64} & 0 & 0 & 0 & \frac{3x^2}{2} \\ 0 & \frac{81x^2}{256} & 0 & \frac{273x^2}{64} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{9x^2}{4} & 0 & 0 & 0 & \frac{3x^2}{2} & 0 & 0 \\ \hline 0 & \frac{273x^2}{64} & 0 & \frac{8281x^2}{144} & 0 & 0 & 0 & 0 & 0 \\ \frac{273x^2}{64} & 0 & 0 & 0 & \frac{145x^2}{16} & 0 & 0 & 0 & 2x^2 \\ 0 & 0 & 0 & 0 & 0 & 4x^2 & 0 & 2x^2 & 0 \\ \hline 0 & 0 & \frac{3x^2}{2} & 0 & 0 & 0 & x^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2x^2 & 0 & x^2 & 0 \\ \frac{3x^2}{2} & 0 & 0 & 0 & 2x^2 & 0 & 0 & 0 & x^2 \end{array}\right)$$

For the modified Choi map with parameters $(2, 1, 1)$ we see from the Figure 2.3 that the state is detected for some values of x where as the original Choi map fails to detect the entanglement.

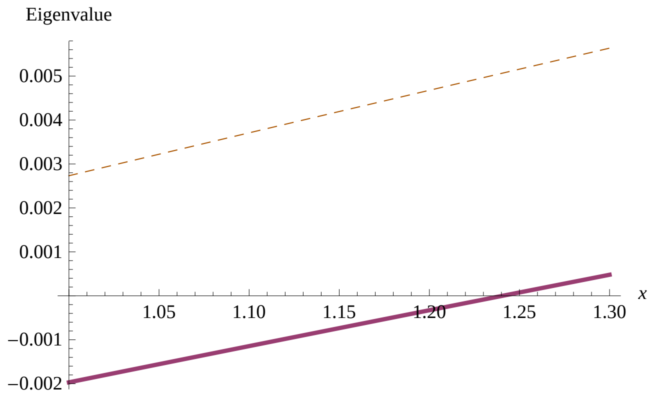

Figure 2.3: Comparison between Choi's map and the map corresponding to the parameters $(2, 1, 1)$. The red line denotes the change of eigenvalues corresponding to the map of $a = 2$. The blue line shows the change by using the Choi's map

It can be shown that the with the parameterisation $(a, 1, 1)$ of modified Choi map we can detect more entangled states as shown in the next figure 2.4.

2.3 Conclusions

In this chapter we have developed an algorithm which can potentially construct PPT entangled states detected by certain positive but not completely positive, indecomposable maps. The PPT nature of the state is ensured throughout the search process and by a suitable choice of the initial T matrix one can adjust the degree of difficulty of the search. The algorithm can also be setup in a way that we do not get into undue difficulties when we go to higher dimensions. One limitation of the method is that it cannot be used to construct PPT entangled states which are not symmetric under partial transpose.

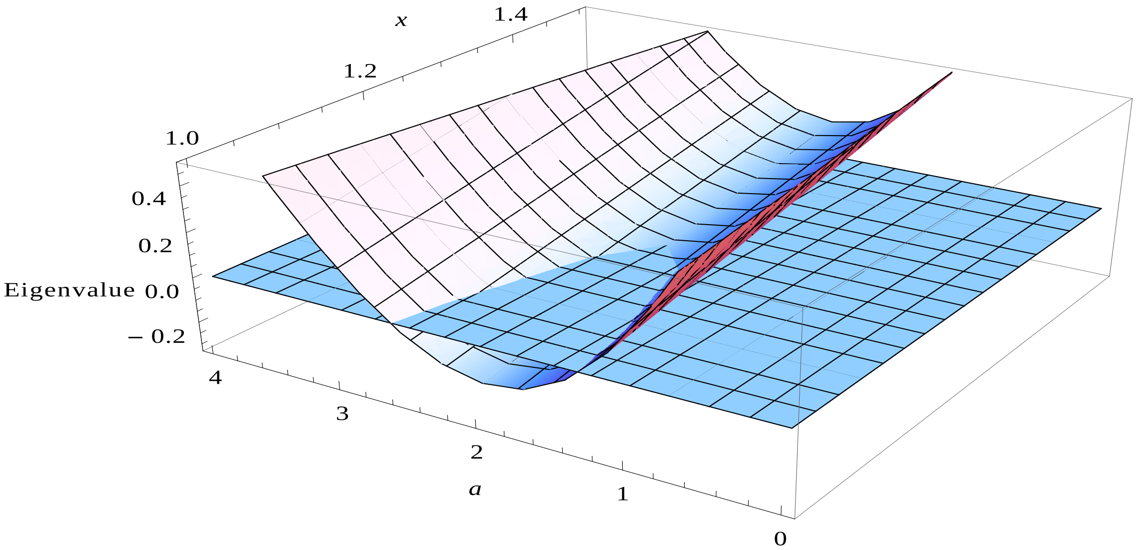

Figure 2.4: 3D plot of entangled states detected by the modified Choi map corresponding to the parameters $(a, 1, 1)$. The middle plane shows the plane for which minimum eigenvalue is equal to 0. As seen from the plot, corresponding to $a = 1$, which is the original Choi map, no entanglement is detected for any value of x.

Chapter 3

Bi-quadratic forms, entanglement witnesses and bound entangled states

The study of positive maps is closely related with the study of positive forms. A form is a homogeneous polynomial in several variables. The study and structure of positive forms is an important concept in both geometry and algebra. The relation between positive semidefinite bi-quadratic forms with positive maps was given by Choi (Choi (1975b)). Choi gave an example of a positive semidefinite bi-quadratic form which cannot be written as a sum of squares of quadratic forms. Using this form, Choi constructed the first example of a positive indecomposable map. Motivated by this result, a large volume of work on the study and construction of positive indecomposable maps on matrix algebras has been carried out since then. The subsequent positive indecomposable maps discovered by Kye (Kye (1992)), Cho et al (Cho *et al.* (1992)), Chruściński and Kossakowski (Chruściński & Kossakowski (2008)) extended and enriched the initial work of Choi to discover new examples of such maps.

In this chapter, we follow the above tradition and discover new classes of positive semidefinite bi-quadratic forms which can not be written as a sum of squares. We further construct the positive maps corresponding to these forms. Given a positive form, we give a method to construct new examples of such forms. We also examine the extremality of such objects. Using the forms, we construct positive indecomposable maps. We give examples of PPT entangled states detected by such maps. The chapter concludes with a robustness analysis of the examples of PPT entangled states.

3.1 Positive forms and Minkowski's conjecture

The theory of forms arises in algebra (and hence in geometry) as a study of homogeneous polynomials. From a geometric point of view, it is interesting to study such polynomials which are positive semi-definite.[1] Given such a form $P(x_1, \cdots, x_n) \geq 0,\ \forall (x_1, \cdots, x_n) \in \mathbb{R}^n$ of even degree d, the question whether P can always be written as a sum of squares of polynomials has been around for a long time. Minkowski conjectured that in general the answer should be 'no'. It was proved by Hilbert that, except for three exceptional cases ($n = 1$, d arbitrary; n arbitrary, $d = 2$; and one non-trivial case $n = 2$, $d = 4$), there always exist positive semi-definite polynomials which cannot be written as a sum of squares of polynomials. This proof was by an indirect method and did not provide actual examples of such polynomials. The generalized version of this problem on rational polynomials, is known as Hilbert's 17th problem, for which the answer is 'yes'. For a survey and development of the original problem, see Rudin (Rudin (2000)).

Hilbert expected that it would be reasonably easy to construct a counter example following his proof, but he did not give an explicit construction. The first counterexample was a ternary sextic

$$z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2, \tag{3.1}$$

constructed by Motzkin (Motzkin (1967)). Slightly later, Robinson independently constructed similar examples on ternary sextic and quaternary quartics (Robinson (1973)).

Choi (Choi (1975b)) came up with a different example of such objects and connected it with the problem of finding positive maps which are not completely positive and indecomposable. He considered a positive semi-definite bi-quadratic form $F_\mu(X : Y)$ (each term having degree four), with six variables, divided into two sets denoted by $X = \{x_1, x_2, x_3\}$, and $Y = \{y_1, y_2, y_3\}$ given by:

$$\begin{aligned} F_\mu(X : Y) = (x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2) - 2(x_1x_2y_1y_2 + x_2x_3y_2y_3 + x_3x_1y_3y_1) \\ + \mu(x_1^2y_2^2 + x_2^2y_3^2 + x_3^2y_1^2), \quad \text{where} \mu \geq 1. \end{aligned} \tag{3.2}$$

Theorem 3.1.1 (Choi (1975b))**.** *For $\mu > 1$, the bi-quadratic form 3.2 is positive semidefinite and can not be written as a sum of squares of quadratic forms.*

Proof. Since the indeterminates are all real numbers, one of the following cases, $|x_1| \leq |x_2|,\ |x_2| \leq |x_3|,\ |x_3| \leq |x_1|$, is possible. Moreover, any cyclic permutation of the

[1] As earlier, we will use the term positive, in place of positive semidefinite.

subscripts of two sets of variables X and Y do not change F. Without loss of generality we may assume $|x_1| \leq |x_2|$. Hence

$$F_\mu(X:Y) = (x_1y_1 - x_2y_2 + x_3y_3)^2 + 2x_1^2y_2^2 + \mu(x_2^2y_3^2 + x_3^2y_1^2 - 2x_3x_1y_3y_1).$$

The last term is positive provided $|x_1| \leq |x_2|$. This shows that F is a positive form.

It remains to show that F can not be written as a sum of squares of quadratic forms. To show this, let us assume it is possible. Let $F = \sum f_i^2$, where f_i are quadratic forms, then f_i^2's can not contain the terms $x_1^2y_3^2$, $x_2^2y_1^2$ and $x_3^2y_2^2$, as they are missing in F. Let $f_i = g_i + h_i$, where each g_i contains terms of x_1y_1, x_2y_2 and x_3y_3; each h_i contains terms x_1y_2, x_2y_3 and x_3y_1 only. Equating $F = \sum(g_i + h_i)^2$, we get

$$\sum g_i^2 = (x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2) - 2(x_1x_2y_1y_2 + x_2x_3y_2y_3 + x_3x_1y_3y_1) \quad (3.3)$$

$$\sum 2g_ih_i = 0 \quad (3.4)$$

$$\sum h_i^2 = \mu(x_1^2y_2^2 + x_2^2y_3^2 + x_3^2y_1^2). \quad (3.5)$$

Equation 3.3 is not possible. If $x_i = y_j = 1$ for all $i,\ j$, then the right hand side is negative whereas the left hand side is the sum of real squares. Hence F can not be written as a sum of squares of quadratic forms. □

Choi's method has been modified and extended, and different examples of such positive semi-definite bi-quadratic forms were found. Among these, the results by Kye (Kye (1992)), Osaka (Osaka (1992)), Cho et al (Cho *et al.* (1992)), and Ha (Ha (1998, 2002, 2003)), Ha and Kye (Ha & Kye (2013)) are important. Generalizations of these methods for generating such forms in arbitrary dimensions were developed by Chruściński and Kossakowski (Chruściński & Kossakowski (2007)).

3.1.1 Connection with positive maps

We now describe the connection between positive maps and positive forms. This was also discovered by Choi (Choi (1975b)).

The connection between the Hermiticity preserving maps and bi-quadratic forms can be established as follows. Consider a Hermiticity preserving linear map

$$S : \mathcal{B}(\mathbb{C}^m) \to \mathcal{B}(\mathbb{C}^n), \quad (3.6)$$

We can construct the corresponding bi-quadratic form $F(X:Y)$ as

$$F(X:Y) = \langle Y|S(X \cdot X^T)|Y\rangle \quad (3.7)$$

where T is the transpose operation, $X = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.

On the other hand, let $F(X : Y)$ be a bi-quadratic form. Notice that, it is a quadratic form with respect to Y (as well as X). So we can write it in the form $\langle Y|A_X|Y\rangle$, where A_X is a symmetric matrix associated with X. Thus we get a map which takes any one-dimensional projection $X.X^T$ to A_X. Using linearity and Hermiticity, we can extend it to a map which preserves Hermiticity. It was shown by Choi, that given any positive semi definite form the corresponding map is a positive map and vice-versa (Choi (1975b, 1980b)). Since the choices of x_i's and y_j's are arbitrary, every one-dimensional real projection $P_X = X.X^T$ is mapped to a positive operator. Hence the corresponding map is a positive map.

Thus there is a bijective relation between the set of positive semi definite forms and positive maps between matrix algebras. The property of complete positivity can also be translated easily. If a map is completely positive, the corresponding bi-quadratic form can be written as a sum of squares of quadratic forms and vice versa. Put differently, if a map is positive but not completely positive, the corresponding bi-quadratic form will be positive semi definite but can not be written as a sum of squares of quadratic forms. Thus each such form gives rise to a unique map between the space of real symmetric operators, which can be trivially extended to the set of all Hermitian operators, and then to all operators. This also connects with the work of Arveson (Arveson (1969, 1974)) and Størmer (Størmer (1963, 1982)) who were exploring the set of positive maps between C^*-algebras. Since then, other examples and classes of such maps have been discovered.

By the above correspondence, the Choi quadratic form given in equation (3.2) leads to the following map for 3×3 matrices.

$$\Phi_C^I : \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} + \mu a_{33} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} + \mu a_{11} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} + \mu a_{22} \end{pmatrix}, \tag{3.8}$$

with $\mu \geq 1$. Interchanging X and Y variables we get another map -

$$\Phi_C^{II} : \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} + \mu a_{22} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} + \mu a_{33} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} + \mu a_{11} \end{pmatrix}. \tag{3.9}$$

As explained in section 1.2.1.2, our interest in these positive but not completely positive maps is because of their ability to detect entanglement of quantum states. For the maps that are to be used as entanglement witnesses, two notions, namely decomposability and extremality are very important. We define these notions below.

Definition 3.1.1. *A positive but not completely positive map is called decomposable, if it can be written as a sum of a completely positive and a completely co-positive map (combination of transpose and a completely positive map).*

This property was discussed first independently by Woronowicz (Woronowicz (1976b)) and Choi (Choi (1975a)). Since decomposable maps are obtained by combining a completely positive map with a transposed completely positive map, it is clear that they are weaker than partial transpose in terms of their ability to detect entanglement and therefore are not of interest. The interesting point however is that, given a map which is positive but not completely positive, there is no standard way to check if it is decomposable or not!

Proposition 3.1.1. *Given a decomposable map $\phi = \psi_1 + T \circ \psi_2$ and a completely positive map $\phi' = \psi_1 + \psi_2$, where $\psi_1,\ \psi_2$ are completely positive map, the corresponding forms F_ϕ and $F_{\phi'}$ are equal.*

Proof. It is easy to notice that the maps transpose T and identity $\mathbb{1}$ gives the same form. Hence the proof. □

The above proposition gives hope that given an unknown positive map, we just need to create the corresponding form and check whether it can be written as a sum of squares. Unfortunately, for a general form, it is not easy to check this property.

Choi and Lam define an extremal map using the corresponding bi-quadratic form Choi and Lam (Choi & Lam (1977/78)) as follows.

Definition 3.1.2 (Extremal form)**.** *A positive semi-definite bi-quadratic form F is said to be extremal, if for any decomposition of $F = F_1 + F_2$ where F_i's are positive semi-definite bi-quadratic forms, $F_i = \lambda_i F$, where $\lambda_1,\ \lambda_2$ are non-negative real numbers with $\lambda_1 + \lambda_2 = 1$.*

Since the set of positive maps is a convex set it can be described by its extremal elements. Therefore, it is most natural to study the extremal positive maps.

Proposition 3.1.2. *From the point of view of detecting entanglement, any extremal map is more powerful than the maps which are internal points of the set of positive maps.*

Proof. Let h be a positive map which is not extremal. Then there exists finite number of extremal positive maps h_i and $\lambda_i > 0$ such that $h = \sum_i \lambda_i h_i$ where $\sum_i \lambda_i = 1$. Now for an entangled state ρ detected by h, i.e $\mathbb{1} \otimes h(\rho) \ngeq 0$. Then $\sum_i \lambda_i \mathbb{1} \otimes h_i(\rho) \ngeq 0$. By Weyl's inequality (see Bhatia (Bhatia (1997))) there exists at least one i such that $\mathbb{1} \otimes h_i(\rho) \ngeq 0$. Hence the result. $\square$

It was shown by Choi and Lam that in the case $\mu = 1$, the form F_1 defined in equation (3.2) is extremal.

Since the set of positive semi-definite forms is a convex set, it is enough to identify the set of such extremal forms. At this stage it is useful to change the notation to original Choi Lam notation for the bi-quadratic forms and we therefore denote $F(X : Y)$ as $F\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$.

3.1.2 Modification of a given map

For a given bi-quadratic form, it is possible to define extensions. Let $F = F\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$ be an extremal positive semi-definite bi-quadratic form. For a given set of non-zero positive real numbers a, b, c we define

$$G\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} = F\begin{pmatrix} ax_1 & bx_2 & cx_3 \\ y_1 & y_2 & y_3 \end{pmatrix}. \tag{3.10}$$

Proposition 3.1.3. *The form G is positive semi-definite and extremal.*

Proof. We first prove the positivity. Let us assume that the proposition is not true and there exists real numbers $p_1, p_2, p_3, q_1, q_2, q_3$ such that $G\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix} < 0$. But then by definition $F\begin{pmatrix} p'_1 & p'_2 & p'_3 \\ q_1 & q_2 & q_3 \end{pmatrix} < 0$; for a set of real numbers $p'_1, p'_2, p'_3, q_1, q_2, q_3$ where $p'_1 = ap_1,\ p'_2 = bp_2$ and $p'_3 = cp_3$. This contradicts the assumption that F is a positive semi-definite form for all real values of x_i's and y_j's.

For extremality, let us assume $G = G_1 + G_2$, where G_1 and G_2 are positive semi-definite bi-quadratic forms.

Notice that

$$\begin{aligned} F\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} &= G\begin{pmatrix} \frac{x_1}{a} & \frac{x_2}{b} & \frac{x_3}{c} \\ y_1 & y_2 & y_3 \end{pmatrix} \\ &= G_1\begin{pmatrix} \frac{x_1}{a} & \frac{x_2}{b} & \frac{x_3}{c} \\ y_1 & y_2 & y_3 \end{pmatrix} + G_2\begin{pmatrix} \frac{x_1}{a} & \frac{x_2}{b} & \frac{x_3}{c} \\ y_1 & y_2 & y_3 \end{pmatrix}, \end{aligned}$$

as we have assumed. Define two positive semi-definite forms

$$F_i\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} = G_i\begin{pmatrix} \frac{x_1}{a} & \frac{x_2}{b} & \frac{x_3}{c} \\ y_1 & y_2 & y_3 \end{pmatrix},$$

for $i = 1, 2$. We can now write,

$$F = F_1 + F_2.$$

However, F is extremal. Therefore, $F_i = \lambda_i F,\ i = 1,\ 2$, and λ_1 and λ_2 are positive real numbers with $\lambda_1 + \lambda_2 = 1$. Thus $G_i\begin{pmatrix} \frac{x_1}{a} & \frac{x_2}{b} & \frac{x_3}{c} \\ y_1 & y_2 & y_3 \end{pmatrix} = \lambda_i F\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$. Hence

$$\begin{aligned} G_i\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} &= G_i\begin{pmatrix} a\frac{x_1}{a} & b\frac{x_2}{b} & c\frac{x_3}{c} \\ y_1 & y_2 & y_3 \end{pmatrix} \\ &= \lambda_i F\begin{pmatrix} ax_1 & bx_2 & cx_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \\ &= \lambda_i G. \end{aligned}$$

Since $\lambda_1,\ \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$; the form G is an extremal form. $\square$

The family of maps corresponding to the bi-quadratic forms G defined above are positive maps which are extremal. Therefore we have a three parameter family of extremal maps originating from the original map corresponding to the bi-quadratic form F. It is straightforward to extend this construction to higher dimensions.

Proposition 3.1.4. *Given any extremal positive semi definite bi-quadratic form,*

$$F = F\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix},$$

for any non zero positive real $a_1, a_2, \cdots, a_n$; $G = F\begin{pmatrix} a_1x_1 & a_2x_2 & \cdots & a_nx_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$ is positive semi-definite and extremal.

3.2 Examples of new bound entangled states

3.2.1 Modification of a positive map and the states detected by it

We now turn to the map corresponding to the extremal bi-quadratic form G defined in equation 3.2. Let us choose three non-zero real numbers a, b, c, and consider the form

$$\begin{aligned} G\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} &= F_1\begin{pmatrix} ax_1 & bx_2 & cx_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \\ &= a^2x_1^2y_1^2 + b^2x_2^2y_2^2 + c^2x_3^2y_3^2 - 2(abx_1x_2y_1y_2 + bcx_2x_3y_2y_3 + cax_3x_1y_3y_1) \\ &\quad + a^2x_1^2y_2^2 + b^2x_2^2y_3^2 + c^2x_3^2y_1^2 \end{aligned} \tag{3.11}$$

The corresponding positive map is then given as,

$$\Phi(a,b,c) : \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \mapsto \begin{pmatrix} a^2x_{11} + c^2x_{33} & -abx_{12} & -acx_{13} \\ -abx_{21} & b^2x_{22} + a^2x_{11} & -bcx_{23} \\ -acx_{31} & -bcx_{32} & c^2x_{33} + b^2x_{22} \end{pmatrix}; \tag{3.12}$$

where $a, b, c \neq 0$. By Choi and Lam (Choi & Lam (1977/78)), the form F_1 of equation 3.2 is extremal. Using the proposition 3.1.3, we see that the form G of equation 3.11 is extremal. Hence, the map $\Phi(a, b, c)$ is an extremal positive map which is not completely positive for all nonzero a, b and c and it can act as an entanglement witness.

We now construct a set of PPT entangled states for which the above map acts as an entanglement witness. Consider a density operator for a $3 \otimes 3$ system defined by two parameters t and x.

$$\rho(x,t) = \frac{1}{4 + \frac{3}{t} + 4t}\left(\begin{array}{ccc|ccc|ccc} 1+t & 0 & 0 & 0 & x & 0 & 0 & 0 & x \\ 0 & t & 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{t} & 0 & 0 & 0 & x & 0 & 0 \\ \hline 0 & x & 0 & \frac{1}{t} & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 1+t & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 & t & 0 & x & 0 \\ \hline 0 & 0 & x & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & \frac{1}{t} & 0 \\ x & 0 & 0 & 0 & x & 0 & 0 & 0 & 1 \end{array}\right). \tag{3.13}$$

This ρ is a unit trace density operator for $t > 0$ and $0 \leq x \leq 1$.

The action of the map $\Phi(a, b, c)$ on the density operator $\rho(x, t)$ leads to the transformed density operator $\rho^{x,}$.

$$\rho'(x,t) = (\Phi(a,b,c) \otimes \mathbb{1}_3)\rho(x,t). \tag{3.14}$$

We compute eigenvalues of $\rho(x,t)'$ and use the negativity of the least eigen value as an indicator of entanglement of $\rho(x,t)$. It is useful to note that the map $\Phi(a,b,c)$ with $a=b=c=1$, reduces to Choi's map $\Phi_C^I(1)$ while for other values of a,b and c it is still an extremal map with a potential to reveal entanglement of quantum states.

A computation of eigenvalues of $\rho'(x,t)$ reveals that for this example, the maps $\Phi(a,b,c)$ have more potential than the Choi's map in unearthing the entanglement of PPT quantum states. The results are displayed graphically in Figures 3.1 and 3.2. We take the parameter values to be $a = 1+\frac{3}{5}$, $b=c=1$ and calculate the minimum eigenvalues of $\rho(x,t)$ for the range $x\in[0,1]$ and $t\in(0,1]$. The results are displayed in Figure 3.1. The curved surface denotes the minimum eigenvalue of $\rho(x,t)$ after the action $\Phi(1+\frac{3}{5},1,1)$. To show the power of this map clearly, we display a section of the above graph where we fix the parameter $x=\frac{1}{20}$ and plot the minimum eigen value as a function of t. We compare our result with Choi's maps. The result is shown in Figure 3.2. The continuous line here denotes the minimum eigenvalue corresponding to the action $\Phi(1+\frac{3}{5},1,1)$ while the other two dashed lines are the minimum eigenvalues corresponding to Φ_C^I and Φ_C^{II}. It turns out that after approximately $x=0.604428$, the minimum eigenvalue becomes negative under $\Phi(1+\frac{3}{5},1,1)$ while the minimum eigenvalues under $\Phi_C^I(1)$ and $\Phi_C^{II}(1)$ still remain positive. It is only after x crosses the value 0.66 that the Choi maps begin to detect entanglement for this class of states. Therefore, for $\rho(x,\frac{1}{20})$, there is a clear window of x values where the entanglement is revealed by $\Phi(1+\frac{3}{5},1,1)$ and is not revealed by any of the Choi maps.

The map $\Phi(1+\frac{3}{5},1,1)$ was chosen as a representative example. In fact the class of maps $\Phi(a,b,c)$ can reveal entanglement of a large class of PPT entangled states and therefore provide a genuine extremal extension of the Choi's maps.

3.2.2 Robustness analysis

We have constructed two different families of PPT entangled states. In chapter 2, we constructed the family of states detected by Osaka's map and in this chapter we have constructed the family detected by extremal extensions of the Choi map. We now consider the robustness of these PPT entangled states given in equations (2.11) and (3.13). Let ρ be an arbitrary entangled state. We consider the convex combination of ρ with a maximally mixed state. We consider the following convex combinations;

$$\rho'(\varepsilon) = \frac{\varepsilon}{9}\mathbb{I}_9 + (1-\varepsilon)\rho;$$

where ε is a real positive parameter less than one and $\mathbb{I}_9$ denotes the identity matrix of dimension 9. To explore how robust is the entanglement of state ρ, we compute

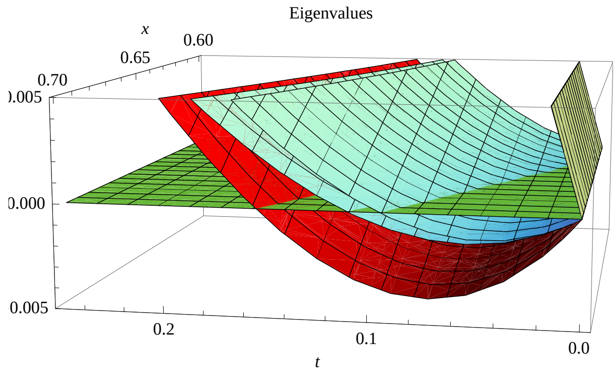

Figure 3.1: The plot of least eigen values of $\rho'(x,t)$ as a function of x and t. The curved surface corresponds to the case where $\rho'(x,t)$ was generated by the action of $\Phi(1+\frac{3}{5},1,1)$ upon $\rho(x,t)$. The middle plane is the plane $xy=0$, given a referral plane for highlighting the negativity of the eigenvalues represented by the curved surface.

the range of ε for which ρ' is entangled. Since the identity matrix represents noise, this calculation indicates as to how much noise can be added to the state ρ without destroying its entanglement. Typically, the map which detects entanglement of ρ is used on ρ' as well.

We begin with $\rho(y)$ of the example (2.11). Using the process previously described, we obtain the new state

$$\rho'(\varepsilon,y)=\frac{\varepsilon}{9}\mathbb{I}_9+(1-\varepsilon)\rho(y).$$

For $y=\frac{5}{2}$, $\rho(y)$ is an entangled state, whose entanglement is revealed by Φ_O. We use the map Φ_O on the family of states $\rho'\left(\varepsilon,\frac{5}{2}\right)$ and can see that there is a continuous range of ε for which $\rho'\left(\varepsilon,\frac{5}{2}\right)$ remains entangled. The change in eigenvalues is shown in Figure 3.3. It shows that approximately up to $\varepsilon=0.045$, the state remains entangled.

We now use the same procedure for $\rho(x,t)$ of the example in (3.13). The family of states is given by;

$$\rho'(\varepsilon,x,t)=\frac{\varepsilon}{9}\mathbb{I}_9+(1-\varepsilon)\rho(x,t).$$

We use $\rho\left(\frac{7}{10},\frac{3}{40}\right)$, which is an entangled state and its entanglement is revealed by $\Phi\left(1+\frac{6}{10},1,1\right)$. Now the family $\rho'\left(\varepsilon,\frac{7}{10},\frac{3}{40}\right)$ is dependent on ε. We plot the change of minimum eigenvalue of this family. Figure 3.4 shows that up to approximately $\varepsilon=0.012$ the state remains entangled.

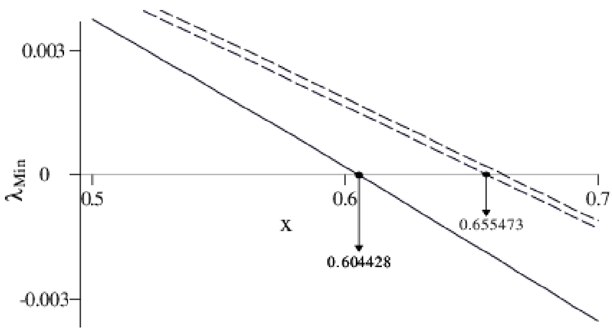

Figure 3.2: The section corresponding to $t = \frac{1}{20}$ of Figure 3.1 is displayed here. The blue line corresponds to the map $\Phi(1 + \frac{3}{5}, 1, 1)$ while the other two curves correspond to the two Choi maps. The window of values of x (approximately $x >= 0.6025$ and $x < 0.66$) where the map $\Phi(1 + \frac{3}{5}, 1, 1)$ is able to reveal the entanglement of $\rho(x, t)$ and where the Choi maps do not reveal the entanglement is clearly visible.

3.3 Conclusions

We have generated a family of extremal extensions of Choi's original map and shown that these extremal extensions are capable of revealing the entanglement of new classes of entangled states.

After publication of the above results, Ha (Ha (2013)) pointed out that, contrary to previously published results (example Kim & Kye (1994); Osaka (1992); Robertson (1985)), the extension from the positive semidefinite bi-quadratic form to the positive maps on real or complex matrices need not be unique. There may exist different positive maps which have the same bi-quadratic form, and gave examples of such maps. Furthermore, the extremality of the forms need not make the corresponding maps to be extremal. If ϕ is an extremal map, then the map $\frac{1}{2}(\phi + T \circ \phi)$ also haves the same form. We note that, the extremality of the maps given by our construction is further validated by the content of the next chapter 4. Further details about the above important paper by Ha (Ha (2013)) is discussed in appendix B.

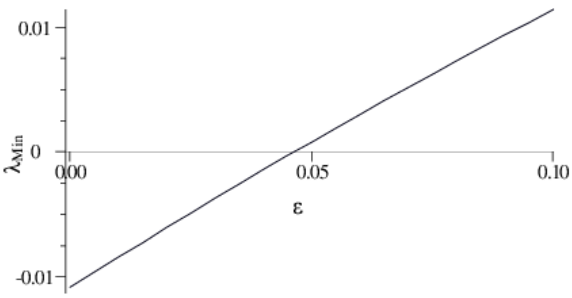

Figure 3.3: Application of Osaka's map Φ_O on the convex combination $\rho'\left(\varepsilon, \frac{5}{2}\right) = \frac{\varepsilon}{9}\mathbb{I}_9 + (1-\varepsilon)\rho\left(\frac{5}{2}\right)$. ε is plotted along the horizontal axis, and eigenvalues along the vertical axis. The blue line shows the change in minimum eigenvalue. The state remains entangled approximately up to $\varepsilon = 0.045$.

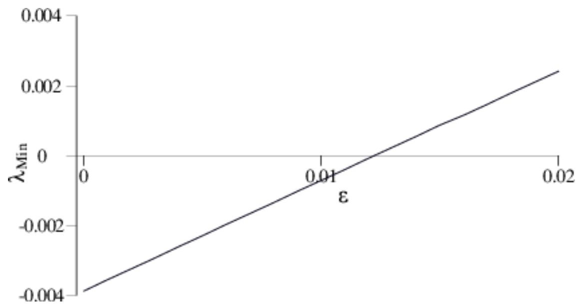

Figure 3.4: Application of $\Phi\left(1+\frac{6}{10}, 1, 1\right)$ on $\rho'\left(\varepsilon, \frac{7}{10}, \frac{3}{40}\right)$. ε is plotted along the horizontal axis, and eigenvalues along the vertical axis. The blue line shows the change in minimum eigenvalue. The state remains entangled upto approximately $\varepsilon = 0.012$.

Chapter 4

Automorphism of positive maps, extremal extensions and unextendable product basis

The set of positive maps forms a closed convex cone. For such sets the interior points can be expressed as convex sums of extremal points. Therefore to study the set of positive maps it is sufficient to study the structure of its extremal points namely the extremal positive maps. In the context of composite quantum systems and action of positive maps on their states it is not easy to enumerate all the extremal points. Given an extremal point, finding other extremal points related to it is of interest. This may also lead to new entanglement witnesses and help unearth new entangled states and enhance our understanding of quantum entanglement.

In Chapter 3 we have given examples of extremal extensions of known extremal positive maps. We have shown that such examples can be generalised for arbitrary finite dimensions. In this chapter, we further advance the methods. We use the method of automorphism in the state space to construct new examples of positive maps. We show that there are two different types of automorphisms, which we call *inner automorphism* and *outer automorphism*. We also show the ability of detecting new entangled states by these modified maps.

One of the most important classes of PPT entangled states comes from the unextendable product basis (Bennett *et al.* (1999)). It can be shown that the PPT states constructed by Bennett et al (Bennett *et al.* (1999)) are not detectable by the standard Choi map or the variations of Choi maps available in literature. The only map which can detect such states were constructed by Terhal (Terhal (2001)) which uses the same

UPB and a nontrivial optimization.

Using the inner automorphism, we can show that the examples of the such PPT entangled states are detectable by the standard Choi's map. Entanglement of such states given by DiVincenzo et al (DiVincenzo *et al.* (2003)) has been verified by identifying suitable positive maps for certain cases.

One of the most useful generalizations of the Choi map was done by Cho-Kye-Lee (Cho *et al.* (1992)). Recently, Ha and Kye (Ha & Kye (2011)) had shown that a subclass of this class gives the optimal entangled witness in the sense of Lewenstein et al (Lewenstein *et al.* (2000)). Further, this class can be written as a one-parameter family, where the parameter $t \in [0, \infty]$. Further, the exposed nature of the maps of this class have been verified recently by Ha and Kye (Ha & Kye (2011)). We show that the class $0 \leq t < 1$ and $1 < t \leq \infty$ are related to each other by a combination of inner and outer unitary operations.

4.1 Extremal extensions of Positive Maps

In this section, starting with a P map (which is not CP) and a CP map, we construct a composite map. This composite map turns out to be extremal if the original map is extremal and under certain conditions has more power to detect entanglement as compared to the original map. Consider $\varphi : \mathcal{B}(\mathbb{C}^n) \longrightarrow \mathcal{B}(\mathbb{C}^n)$ to be a positive indecomposable map. For any $A \in Gl_n(\mathbb{C})$ we can define a map

$$\begin{aligned} A : \mathcal{B}(\mathbb{C}^n) &\longrightarrow \mathcal{B}(\mathbb{C}^n) \\ X &\longmapsto AXA^\dagger \quad \text{For } X \in \mathcal{B}(\mathbb{C}^n) \end{aligned} \tag{4.1}$$

By definition, A is a CP map. Note that we are using the same symbol A for the $Gl_n(\mathbb{C})$ element and the corresponding map. To make it a valid quantum operation, we impose the condition $AA^\dagger \leq I$ where I denotes the identity element of $\mathcal{B}(\mathbb{C}^n)$.

We can then define the two automorphisms as the compositions

$$\begin{aligned} \varphi \circ A =& \varphi_A \\ A \circ \varphi =& \varphi^A \end{aligned} \tag{4.2}$$

The former is called inner automorphism while the latter is called outer automorphism. The outer automorphism is not useful for us as it does not strengthen the entanglement detection capability of φ. However as we will see below and in the next sections, the inner automorphism is useful.

It is worth noting that the set of positive maps is a convex set and can be described by its 'extremal points', in our case 'extremal maps'. Recall that a positive map h is said to be extremal, when for any decomposition $h = h_1 + h_2$, where h_1 and h_2 are positive maps, $h_i = \lambda_i h$, where $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 = 1$.

Theorem 4.1.1. *For any positive map $\varphi : \mathcal{B}(\mathbb{C}^n) \longmapsto \mathcal{B}(\mathbb{C}^n)$, and for any full rank operator A, (such that $AA^\dagger \leq I$) φ_A is a positive map. Moreover, if φ is not completely positive and extremal, so is the map φ_A.*

Proof: The map $A : X \longmapsto AXA^\dagger$, when A is a non-singular operator defines an automorphism on $\mathcal{B}(\mathbb{C}^n)$. If X is Hermitian, so is $AXA^\dagger$ and if X is positive, so is the image as the map A is completely positive. Thus the map A is a bijection map from the set of positive semi-definite operators onto itself.

Let φ be a P but not CP map. Assume that φ_A is a CP map. Then by Kraus decomposition, there exists a finite set of operators $\{V_i\}$ which represents the map and we can write for any $X \in \mathcal{B}(\mathbb{C}^n)$

$$\varphi_A(X) = \sum_i V_i X V_i^\dagger. \tag{4.3}$$

Now $\varphi(X) = \varphi_A(A^{-1}XA^{\dagger^{-1}})$ since A is a non-singular operator. We thus have

$$\varphi(X) = \varphi_A(A^{-1}XA^{\dagger^{-1}}) = \sum_i V_i A^{-1}XA^{\dagger^{-1}} V_i^\dagger. \tag{4.4}$$

implying that φ is a CP map. This is a contradiction. Hence φ_A is a P but not CP map.

For the second part, let φ be extremal and let us assume that φ_A is not extremal. Then there exist positive maps φ^1 and φ^2 so that $\varphi_A = \varphi^1 + \varphi^2$. Using a similar argument as above we can write

$$\begin{aligned} \varphi(X) &= \varphi_A(A^{-1}XA^{\dagger^{-1}}) \\ &= \varphi^1(A^{-1}XA^{\dagger^{-1}}) + \varphi^2(A^{-1}XA^{\dagger^{-1}}) \\ &= \varphi^1_{A^{-1}}(X) + \varphi^2_{A^{-1}}(X). \end{aligned} \tag{4.5}$$

But the map φ is an extremal map. By definition of extremality, if $\varphi = \varphi_1 + \varphi_2$, where φ_i are positive maps, then $\varphi_i = \lambda_i \varphi$, where $\lambda_1 + \lambda_2 = 1$. Hence

$$\begin{aligned} \varphi^i_{A^{-1}} = \lambda_i \varphi &\Rightarrow \varphi^i_{A^{-1}} \circ A = \lambda_i \varphi \circ A \\ &\Rightarrow \varphi^i = \lambda_i \varphi_A. \end{aligned} \tag{4.6}$$

Hence φ_A is an extremal map.

A special case of interest is when A is unitary which we denote by U. A number of special results are available for this case. By the Russo-Dye theorem (see Bhatia (Bhatia (2007))) we can show that for any unitary operator U,

$$\|\varphi^U\| = \|\varphi_U\| = \|\varphi\|. \tag{4.7}$$

It is obvious that if φ is unital, so are φ_U and φ^U.

Further, positivity under partial transpose is invariant under inner unitary automorphism. In other words, for the transpose map T and any unitary operator U, $(I \otimes T)\rho \geq 0$ implies $(I \otimes T_U)\rho \geq 0$ for any state ρ.

This can be proved as follows: Let $\rho \in \mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^n)$ be a PPT state. Let us write $\rho = ((\rho_{ij}))$ in the block form, where for each i and j, $\rho_{ij} \in \mathcal{B}(\mathbb{C}^n)$. Then $(I \otimes T)\rho = ((T(\rho_{ij})) = ((\rho_{ij}^T))$. Hence

$$\begin{aligned}(1 \otimes T_U)\rho &= ((T(U\rho_{ij}U^\dagger)) \\ &= ((\overline{U}T(\rho_{ij})\overline{U}^\dagger)) \\ &= (I \otimes \overline{U})(1 \otimes T)\rho(I \otimes \overline{U})^\dagger.\end{aligned} \tag{4.8}$$

Where $U = ((u_{ij}))$ and its complex conjugate $\overline{U} = ((\overline{u_{ij}}))$ are unitary operators. Since eigenvalues remain invariant under unitary transformations (local unitary in our case), the result follows.

Theorem 4.1.2.

1. *For any positive map $\varphi : \mathcal{B}(\mathbb{C}^n) \longrightarrow \mathcal{B}(\mathbb{C}^n)$, and any unitary operator U, the outer automorphism φ^U is a positive map.*

2. *Any entangled state ρ detected by φ^U is detected by φ and vice versa.*

Proof: Let $x \in \mathcal{B}(\mathbb{C}^n)$ be any positive semi-definite Hermitian operator. Since φ is positive, $\varphi(x) \geq 0$. Since the unitary operators do not change eigenvalues, we have $U\varphi(x)U^\dagger \geq 0$, i.e. $\varphi^U = U(x) \circ \varphi \geq$. Hence φ^U is a positive map.

For the second part, notice that the eigenvalues are invariant under unitary operators. Hence,

$$\begin{aligned}(1 \otimes \varphi)\rho \ngeq 0 &\iff (I \otimes U)(1 \otimes \varphi)\rho(I \otimes U)^\dagger \ngeq 0 \\ &\iff (1 \otimes U\varphi U^\dagger)\rho \ngeq 0 \\ &\iff (1 \otimes \varphi^U)\rho \ngeq 0.\end{aligned} \tag{4.9}$$

This means that for the entanglement detection application, unitary outer automorphisms are not useful and therefore we should focus only on the inner automorphism.

In the next section we discuss the power of such extensions. We will consider PPT entangled states discovered through UPB construction due to Bennett et al (Bennett *et al.* (1999)) and apply one-parameter sub-families of unitary inner automorphisms to them.

4.2 Extensions of Choi map and UPB construction

4.2.1 The Choi Map

The first non-trivial example of a P map which is not CP and can provide a witness for the entanglement of some PPT entangled states was discovered by Choi (Choi (1975b)). This map comes in two variants and they are defined on a 3-dim Hilbert space as follows:

$$\varphi_{C_1} : ((x_{ij})) \longmapsto \frac{1}{2}\begin{pmatrix} x_{11}+x_{22} & -x_{12} & -x_{13} \\ -x_{21} & x_{22}+x_{33} & -x_{23} \\ -x_{31} & -x_{32} & x_{33}+x_{11} \end{pmatrix} \tag{4.10}$$

and

$$\varphi_{C_2} : ((x_{ij})) \longmapsto \frac{1}{2}\begin{pmatrix} x_{11}+x_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{22}+x_{11} & -x_{23} \\ -x_{31} & -x_{32} & x_{33}+x_{22} \end{pmatrix} \tag{4.11}$$

Both these maps as defined in (4.10) and (4.11) are useful in unearthing entanglement of PPT entangled states and are extremal points in the space of maps (Choi & Lam (1977/78)). There are only a few examples of extremal maps and apart from Choi maps, there have been extensions of Choi maps by Kye (Kye (1992)) which were shown to be extremal by Osaka (Osaka (1992)). We are interested in unitary inner automorphisms of the Choi maps which are defined as the composition $\varphi_{C_{1,2}} \circ U$ where $U \in SU(3)$ is a unitary operator. For every $U \in SU(3)$ we have an extremal map generated from the Choi map. For example, for every one-parameter subgroup of $SU(3)$ we will have a family of maps which can help us unearth entanglement of PPT entangled states.

4.2.2 The TILES construction

The unextendable product basis, the 'TILES' construction was proposed by Bennett et al (Bennett *et al.* (1999)). Given a composite system with Hilbert space $\mathbb{C}^3 \otimes \mathbb{C}^3$, we consider the normalized orthogonal states

$$\begin{aligned}
|\psi_0\rangle &= \frac{1}{\sqrt{2}}|0\rangle\left(|0\rangle - |1\rangle\right), \\
|\psi_2\rangle &= \frac{1}{\sqrt{2}}|2\rangle\left(|1\rangle - |2\rangle\right), \\
|\psi_1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)|2\rangle, \\
|\psi_3\rangle &= \frac{1}{\sqrt{2}}\left(|1\rangle - |2\rangle\right)|0\rangle, \\
|\psi_4\rangle &= \frac{1}{3}\left(|0\rangle + |1\rangle + |2\rangle\right)\left(|0\rangle + |1\rangle + |2\rangle\right)
\end{aligned} \tag{4.12}$$

Bennet et. al. showed that there is no product state in the orthogonal complement of these states. Therefore, the state

$$\rho = \frac{1}{4}\left(I_9 - \sum_{i=0}^{4}|\psi_i\rangle\langle\psi_i|\right). \tag{4.13}$$

is entangled. Further, by construction this state is PPT and therefore we have a PPT entangled state. We can apply the maps $I \otimes \varphi_{C_{1,2}}$ to the state and it turns out that the state remains positive and does not reveal its entanglement. Consider a one-parameter

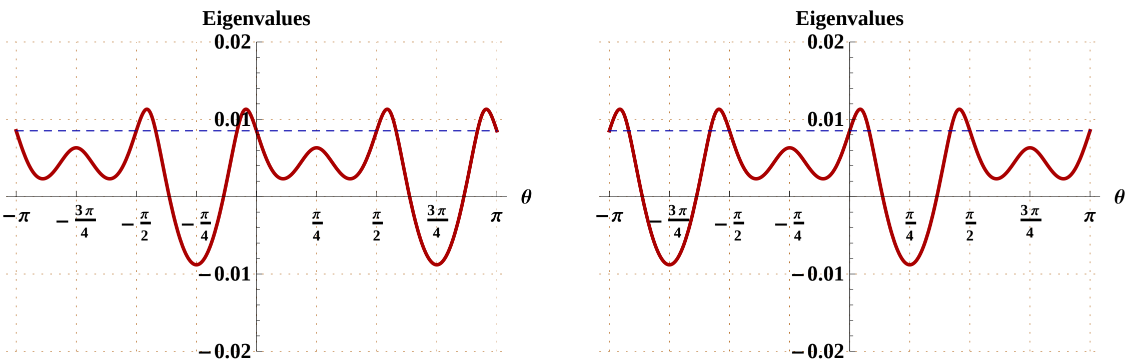

Figure 4.1: Plot of minimum eigenvalue as a function of θ of the operators $\rho_1'(\theta)$ and $\rho_2'(\theta)$ obtained after the action of $I_3 \otimes \varphi_{C_{1,2}}(\theta)$ on the state ρ defined in equation (4.13). The upper graph corresponds to $\rho_1'(\theta)$ while the lower one corresponds to $\rho_2'(\theta)$. The straight line represents the minimum eigenvalue corresponding to the operators obtained after the action of the corresponding Choi map through the operators $I_3 \otimes \varphi_{C_{1,2}}$. The negativity of the minimum eigen value, which occurs in both the graphs in a similar way but for shifted values of θ, indicates that the map has revealed the entanglement of the state.

family of extremal extensions of the Choi maps $\varphi_{C_{1,2}}(\theta) = \varphi_{C_{1,2}} \circ U(\theta)$ with

$$U(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{4.14}$$

These two families of maps defined via the unitary inner automorphism can now be tried on the PPT entangled states defined in Equation (4.13) to see if they can reveal its entanglement. We apply the maps $I \otimes \varphi_{C_{1,2}}(\theta)$ to the state defined in Equation (4.13).

$$I_3 \otimes \varphi_{C_{1,2}}(\theta) \,:\, \rho \to \rho'_{1,2}(\theta) \tag{4.15}$$

We compute the eigen values of $\rho'_1(\theta)$ and $\rho'_2(\theta)$. It turns out that the smallest eigen value becomes negative for a range of θ values indicating that the resultant operator is not a state, thereby revealing the entanglement of the original state ρ. The plot of minimum eigen values of $\rho'_1(\theta)$ and $\rho'_2(\theta)$ are shown in Figure 4.1. The upper graph corresponds to the case $\rho'_1(\theta)$ and the lower one corresponds to the case $\rho'_2(\theta)$.

Both families of maps are able to reveal the entanglement of the state ρ defined in Equation (4.13). However the θ ranges for which the map reveals the entanglement are different in each case. The lower graph can be superimposed on the upper graph by a shift of $\frac{\pi}{2}$ in θ. In each graph the straight lines show the positive minimum eigen value obtained after application of the corresponding non-modified Choi map.

4.2.3 The PYRAMID construction

Another interesting UPB construction for the $3 \otimes 3$ Hilbert space is the PYRAMID construction (Bennett *et al.* (1999)). We first define five vectors in a three dimensional Hilbert space as:

$$v_i = N\left(\cos\frac{2\pi j}{5}, \sin\frac{2\pi i}{5}, h\right) \quad j = 0, \cdots, 4; \tag{4.16}$$

where $h = \frac{1}{2}\sqrt{1+\sqrt{5}}$ and $N = \frac{2}{\sqrt{5+\sqrt{5}}}$. Using these vectors we define the UPB set as

$$|\psi_j\rangle = |v_j\rangle \otimes |v_{2j \mod 5}\rangle, \quad j = 0, \cdots, 4. \tag{4.17}$$

The corresponding PPT entangled state is obtained by substituting the UPB states given in Equation (4.17) above into Equation (4.13). We carry out an identical analysis to the TILES case and find that the entanglement of this state is again detected by the modified Choi maps. The plots are shown in Figure 4.2 where the minimum eigen

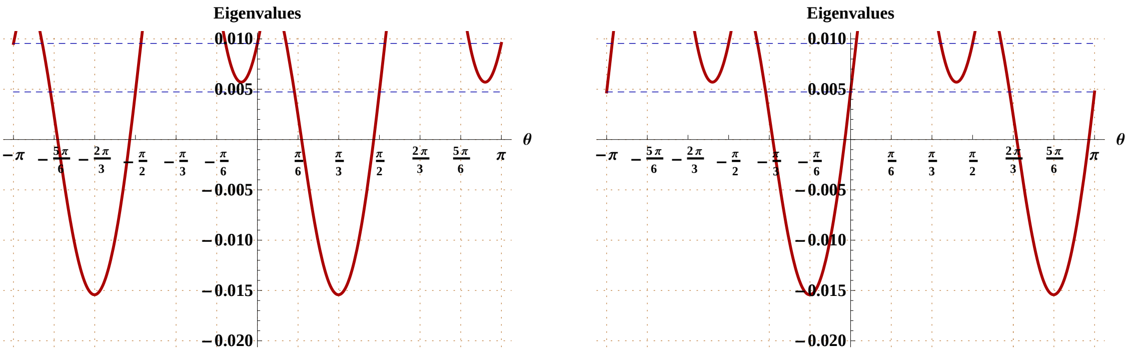

Figure 4.2: Plot of minimum eigen value of operators $\rho'_{1,2}(\theta)$ as a function of θ. The operators $\rho'_{1,2}(\theta)$ are obtained from the PPT entangled states in the orthogonal complement of the PYRAMID UPB construction by the action of families of extremal extensions of two Choi maps on the second system. The negativity of the minimum eigenvalues shows that the map is able to detect entanglement of the states. The straight line in each graph shows the minimum eigenvalue in the case of the original Choi map which remains positive and therefore does not reveal the entanglement.

value is displayed as a function of θ for the operator obtained after action of modified Choi Both the families of maps reveal the entanglement of the state and the graphs (Figure 4.2) also display an invariance under a shift of $\frac{\pi}{2}$ in θ, as was seen for the TILES case. However in this case, the range of values over which the minimum eigen value is negative is different. This means that the extremal maps which reveal the entanglement of the state in this case are different from the ones in the TILES case.

Recent results independently given by Skowronek (Skowronek (2011)) and by Chen and Doković (Chen & Doković (2011)), gives a curious connection between PPT entangled states and UPBs.

Theorem 4.2.1 (Chen & Doković (2011); Skowronek (2011))**.** *Positive partial transpose rank* 4 *states in* $3 \otimes 3$ *systems are either separable or are of the form*

$$\rho = (A \otimes B)^{\dagger} \left(I - \sum_{i=1}^{5} |\alpha_i\rangle\langle\alpha_i| \otimes |\beta_i\rangle\langle\beta_i| \right) (A \otimes B), \tag{4.18}$$

where $A,\ B \in Sl_3(\mathbb{C})$ *and* $\{|\alpha_i \otimes \beta_i\rangle\}_{i=1}^{5}$ *forms an orthonormal unextendible product basis.*

Since any such basis is local unitarily equivalent with any other known UPB's, for example the UPB of 4.12, and the operators A and B are invertible, we can combine our earlier results with the above theorem and conclude that

Theorem 4.2.2. *Any rank 4 PPT entangled state in $3 \otimes 3$ system can be detected by a positive map which is inner automorphism of Choi's map.*

4.3 Further examples of extermal extensions

To demonstrate the usefulness of the extensions based on automorphisms we describe below three insightful results. The first result is that the two maps due to Choi described in Equations (4.10) and (4.11) naturally get connected via a combination of inner and outer unitary automorphisms. The map φ_{C_1} thus gets related to φ_{C_2}.

$$\varphi_{C_1} = U\left(\frac{3\pi}{2}\right) \circ \varphi_{C_2} \circ U\left(\frac{\pi}{2}\right). \tag{4.19}$$

Secondly, the construction that we had described in the previous chapter 3 where we have generated extremal maps as candidate entanglement witnesses from the existing ones turns out to be a nonunitary inner automorphism. This extremal extension can be recast as an inner automorphism of the original map given below

$$\varphi_{(a_1,\cdots,a_n)} = \varphi \circ A \tag{4.20}$$

where A is an operator given by the diagonal matrix

$$A = \mathrm{Diag}(a_1, a_2 \cdots, a_n) \tag{4.21}$$

This is clearly a non-unitary inner automorphism and connects our earlier result with the present formulation.

In the third example we turn to a generalization of the Choi map defined by Cho, Kye and Lee (Cho *et al.* (1992)) as

$$\varphi_m((x_{ij})) \mapsto \frac{1}{2}\begin{pmatrix} ax_{11}+bx_{22}+cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & ax_{22}+bx_{33}+cx_{11} & -x_{23} \\ -x_{31} & -x_{32} & ax_{33}+bx_{11}+cx_{33} \end{pmatrix} \tag{4.22}$$

where a, b, c satisfy certain conditions given in detail in their paper.

It has been shown by Ha and Kye (Ha & Kye (2011)) that a sub-class of the above family of maps, given by

$$0 < a < 1, \quad a+b+c = 2, \quad bc = (1-a)^2; \tag{4.23}$$

are extremal maps. It has been further shown by Ha and Kye (Ha & Kye (2011); Ha & Kye (2011)) that these extremal maps can be written as a one-parameter family of maps φ_t with $0 \leq t < \infty$. The parameters $a(t), b(t)$ and $c(t)$ are given by

$$a(t) = \frac{(1-t)^2}{1-t+t^2},\ b(t) = \frac{t^2}{1-t+t^2},\ c(t) = \frac{1}{1-t+t^2}. \tag{4.24}$$

We have $\varphi_{t=0} = \varphi_{C_1}$, $\varphi_{t\to\infty} = \varphi_{C_2}$ while $\varphi_{t=1}$ is a decomposable map. Using the unitary automorphism defined through the one-parameter family of unitary transformations given in Equation (4.14), we are able to relate the maps in the interval $[0, 1]$ to maps in the interval $[1, \infty)$ as follows:

$$\varphi_t = U\left(\frac{3\pi}{2}\right) \circ \varphi_{\frac{1}{t}} \circ U\left(\frac{\pi}{2}\right). \tag{4.25}$$

This mean that we need to consider only the maps in the interval $[0, 1]$ if we are interested in using them as entanglement witnesses and the others can be generated via the automorphism given above. The above examples show that the automorphisms provide us with a way to connect various seemingly unrelated maps.

A natural question is whether any two extremal maps are related to each other by such an isomorphism. It can be shown that there does not exist any such local isomorphism. Consider the maps given in Equations 4.10 and 4.22. It is clear that there is no local isomorphism which can convert one to the other.

4.4 Conclusions

In this chapter we have described extremal extensions of P maps which are not CP via their composition with quantum operations. Two kinds of automorphisms are described and it is shown that only one of them, namely, the inner automorphism has the ability to enhance the entanglement detection power of the original map. This construction opens up new possibilities of extremal extensions of P maps which are CP. Focusing on the famous Choi map and its extensions via a one-parameter family of unitary transformations, we have discovered a useful and interesting connection with UPB. We discover that for a certain parameter range the map begins to unearth the entanglement of states in the orthogonal complement of UPB.

The exposedness of maps has been discussed and used in the entanglement context in a recent interesting development Ha and Kye (Ha & Kye (2011)). It turns out that the automorphisms described in our work preserve the exposed property and thus if we start with an exposed map we can construct families of exposed maps. In this context extensions of extermal exposed maps have also been considered by Sarbicki and Chruściński (Sarbicki & Chruściński (2013)).

In the context of UPB there is a way to interpolate between TILES and PYRAMID (DiVincenzo *et al.* (2003)). This possibility provides us with a rich variety of PPT entangled states. There are several possible extensions of this work. For one, the possibility of detecting these states with extensions of already known P but not CP maps or implicating the non-CP character of certain maps using these states. Further, there could be interesting consequences of these results in higher dimensions.

Chapter 5

Quantum Filtering and detection of PPT entangled states

The concept of inner (and also outer) automorphism give an important tool to discover entangled quantum states. These automorphisms are transformations on positive maps where a completely positive map is combined with a positive map which is not completely positive to give new entanglement witnesses. In a dual approach, the completely positive map that we use, can be thought of as an action on the quantum states. The operations are only on one subsystem of the bipartite system and therefore come under the general ambit of LOCC. This operation can be unitary as well as non-unitary. In the unitary case, it is in fact a Hamiltonian quantum evolution, while in the non-unitary case it is in general a POVM on the subsystem. In this case after the POVM, we retain a sub-ensemble of the original ensemble corresponding to a particular value of the measurement. Therefore, it amounts to implementing a quantum filter. In this chapter we re-intepret the automorphisms described in Chapter 4 from this point of view. In the examples of entanglement witnesses given in Chapters 4 and 3, we have used unitary as well as non-unitary automorphisms. In general these automorphisms can be described by an invertible operator S and the map is of the form $\rho \mapsto \frac{1}{\mathrm{Tr}(S\rho S^\dagger)}(S\rho S^\dagger)$. Such operations are known as *quantum filtering*.

All completely positive maps acting on quantum systems can have a POVM representation. We start with a brief description of POVM. We find out the differences of local and global operations and their action on the entangled states. We show that the operations used in Chapter 4 are the ones which preserve entanglement. We further discuss the general form of such filtering. Gisin (Gisin (1996)) has shown that, such maps are possible in an experimental set up. This construction was up on the states of two spin $\frac{1}{2}$ particles. We start with the example of Gisin and represent it in terms of projection operators. We further consider the general case of arbitrary finite dimension. Since most of the examples of entangled states given in this thesis are for $3 \otimes 3$ systems, we give special importance to this case. We further show that given a bipartite system, the general filtration scheme can be achieved by using local projection operations. Thus we give a theoretical recipe for practical implementation of filters.

5.1 Super-positive maps

Let $\phi : (\mathbb{C}^n) \to (\mathbb{C}^n)$ be a completely positive map. Then it can be represented by a Kraus operators $\{L_1, \cdots, L_s\}$.

$$\phi(X) = L_1 X L_1^\dagger + \cdots + L_s X L_s^\dagger.$$

It is important to know the minimal number s required to represent a completely positive map ϕ.

Definition 5.1.1 (Ando (2004))**.** *Let ϕ be a completely positive map and $\{L_1, \cdots, L_s\}$ is the minimal Kraus representation. Let*

$$k = \max\{rank(L_j), j = 1, 2, \cdots, s\}.$$

Then the map ϕ is called to be k-super positive .

The structure of such maps is still not fully understood. One of the most important classes of maps is the case when $k = 1$. Such a map is called *entanglement breaking channel*. For such channels, given any entangled state $\rho \in \mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^n)$, the output state $(\mathbb{1}_n \otimes \phi)\rho$ is a separable state. Maps corresponding to other values of k are also studied by Szarek et al (Szarek *et al.* (2008)) and by Skowronek and Størmer (Skowronek & Størmer (2012)).

In our case, we consider the maps which has only one Kraus operator L and that operator is of rank n. In other words, we consider the n-super positive maps.

Proposition 5.1.1. *Let L_1 and L_2 be two full rank operators. Then the map $\rho \mapsto (L_1 \otimes L_2)\rho(L_1 \otimes L_2)^\dagger$ does not change the Schmidt number of the state.*

Proof. Recall that, a bipartite state ρ has Schmidt rank k if

1. for any ensemble decomposition of ρ as $\{p_j \geq 0, |\psi_j\rangle\}$ where $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$; at least one of the vectors $|\psi_j\rangle$ has at least Schmidt rank k, and

2. There exists a decomposition of ρ where all vectors $\{|\psi_j\rangle\}$ in the decomposition have a Schmidt rank at most k.

Hence it is sufficient to check that given any state written in its Schmidt decomposition

$$|\psi\rangle = \sum_j \lambda_j |e_j\rangle \otimes |f_j\rangle$$

the Schmidt number remains invariant under the operation $|\psi\rangle \mapsto L_1 \otimes L_2 |\psi\rangle$ (for time being, we ignore the normalisation factor). Notice that Schmidt rank(SR) of $|\psi\rangle$ is the matrix rank of $\sum_j \lambda_j |f_j\rangle \langle e_j|$. Thus

$$SR(L_1 \otimes L_2(|\psi\rangle)) = rank \sum_j \lambda_j L_2 |f_j\rangle \langle e_j| L_1^\dagger.$$

Let $L_1 = U_1 D_1 V_1$ and $L_2 = U_2 D_2 V_2$ be the respective singular value decompositions, where U_1, V_1, U_2, V_2 are unitary operators. Then

$$\begin{aligned} SR(L_1 \otimes L_2(|\psi\rangle)) &= rank \sum_j \lambda_j U_2 D_2 V_2 |f_j\rangle \langle e_j| V_1^\dagger D_1^\dagger U_1^\dagger \\ &= rank \sum_j \lambda_j D_2 |f_j'\rangle \langle e_j'| D_1^\dagger \end{aligned}$$

where $|e_j'\rangle = V_1 |e_j\rangle$ and $|f_j'\rangle = V_2 |f_j\rangle$ are mutually orthogonal basis of first and second system respectively. Since L_1 and L_2 are of full rank, the diagonal matrices D_1 and D_2 are also of full rank, and the above assertion holds, i.e. $SR |\psi\rangle$ is invariant under these operations. □

The above proposition further shows that, entanglement is not created or destroyed by the above operations.

Proposition 5.1.2. *Any PPT entangled state remains PPT entangled after the invertible local operations described above. Similarly any NPPT state remains NPPT.*

Proof. Follows from the previous proposition. □

The above proposition connects with the concept of inner and outer automorphisms described in Chapter 4. Since the nature of entanglement is not changed, a given state ρ can be converted to another state by using local filters, whose entanglement can be checked by a given map. This gives us a handle to generate new PPT entangled states from the old ones via quantum filtering.

5.2 Quantum measurement

In Chapter 1, we have given the basic axioms of quantum mechanics. Axiom IV deals with quantum measurement and its outcomes, which we recapitulate here.

Axiom IV Quantum measurement is described by a set $\{M_k\}$ of measurement operators which satisfy the completeness equation

$$\sum_k M_k^\dagger M_k = I. \tag{5.1}$$

If the state of a quantum system is in the state ρ before measurement, then the probability that the result k occurs is

$$P(k) = \mathrm{Tr}(M_k^\dagger \rho M_k), \tag{5.2}$$

and the state after the measurement is

$$\rho' = \frac{M_k^\dagger \rho M_k}{\sqrt{P(k)}}. \tag{5.3}$$

By completeness of probability

$$\sum_k P(k) = 1.$$

If we denote $E_k = M_k^\dagger M_k$, then $\sum_k E_k = I$. The operators $\{E_k\}$ are known as POVM elements associated with the measurement. The finite version of Neumark dilation theorem (Paulsen (2002)) gives that

Theorem 5.2.1. *Let $\mathcal{E} = \{E_j\}$ be a set of POVM elements on $\mathcal{B}(\mathcal{H})$. Then we can embed $\mathcal{H}$ to a larger Hilbert space $\mathcal{H}'$ such that $\mathcal{E}$ is extended to a set of projection operators. In other words, the POVM elements can be represented as a projective measurements in a higher dimensional space.*

This extension is not unique. In the subsequent sections we discuss the way to extend and write the local filtrations in terms of projection operators. In this way, such quantum operations can be be implemented for a practical purpose. We further show that any such projection operations can also be performed locally. The way this implementation happens is that for a given outcome of the measurement, we select the corresponding state. By repeated such selection, we create a sub-ensemble represented by a new density operator. This process is called quantum filtering.

5.3 Filtration of two qubit systems

An interesting example of quantum filtration was introduced by Gisin Gisin (1996) for an entangled mixed state of two spin -$\frac{1}{2}$ particles. He showed that by using a polarised beam splitter one can convert the input state to an output state which remains entangled but its entanglement can not be detected by a Bell inequality violation. Borrowing quantum optics language, such local operations are called local filters.

Interpreting the Gisin filter in our formalism reveals that in his case $\rho \mapsto (I_2 \otimes A)\rho(I \otimes A^\dagger)$, where the operator A is given by

$$A = \begin{pmatrix} \sqrt{\frac{\beta}{\alpha}} & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.4}$$

where α and β are two real numbers such that $\alpha > \beta > 0$ and $\alpha^2 + \beta^2 = 1$. Clearly A is a non-unitary operator. We have ignored the normalisation here, which can be put back to ensure that the filtered state has unit trace. Since the operator A is acting locally, we need to look for the map

$$\mathcal{B}(\mathbb{C}^2) \ni \sigma \mapsto A\sigma A^\dagger.$$

We write the spectral decomposition of A as

$$A = A_1 + A_2, \tag{5.5}$$

where

$$A_1 = \begin{pmatrix} \sqrt{\frac{\beta}{\alpha}} & 0 \\ 0 & 0 \end{pmatrix} \tag{5.6}$$

and

$$A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.7}$$

Now all POVM elements must sum up to identity. Hence we introduce two more POVM elements A_3 and A_4 given by

$$A_3 = A_4 = \begin{pmatrix} \frac{1}{2}(1 - \sqrt{\frac{\beta}{\alpha}}) & 0 \\ 0 & 0 \end{pmatrix} \tag{5.8}$$

so that $\sum_i A_i = I$. According to Neumark's theorem any POVM acting on a Hilbert space $\mathcal{H}$ can be realized by a projective measurement on a larger Hibert space $\mathcal{H} \otimes \mathcal{H}_{aux}$. Let us suppose that we have a system in state ρ and an ancilla in the state

ρ_{aux}. Let us perform a projective measurement given by the projective operator P on the Hilbert space comprising the system and the ancilla. The corresponding POVM element A is given by a matrix whose elements are calculated using the formula (see Peres (Peres (1993))),

$$\langle \mathbf{m}| A |\mathbf{n}\rangle = \sum_{r,s} \langle \mathbf{m}r| P |\mathbf{n}s\rangle \langle s| \rho_{aux} |r\rangle \tag{5.9}$$

In the above equation, the bold indices **m** and **n** denote the system indices and the indices r and s denote those of the ancilla. If the ancilla is taken to be a quantum system in the pure state $|0\rangle\langle 0|$, then the above formula suggests that a projection operator should contain the corresponding POVM element as its top leftmost block, i.e.

$$P = \left(\begin{array}{c|c} A & \\ \hline & \end{array}\right)$$

Thus the entire problem of constructing a projection operator out of a POVM element reduces to one of matrix completion. For each A_j, we need to construct a projection operator $P_j = \left(\begin{array}{c|c} A_j & \\ \hline & \end{array}\right)$. Since each A_j is a rank one operator, it can be easily be extended to a projection operator. Such an extension need not be unique. We construct the following operators P_1, P_2, P_3 and P_4 for A_1, A_2, A_3 and A_4, respectively:

$$P_1 = \begin{pmatrix} \sqrt{\frac{\beta}{\alpha}} & 0 & 0 & -\sqrt{\sqrt{\frac{\beta}{\alpha}} - \frac{\beta}{\alpha}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\sqrt{\sqrt{\frac{\beta}{\alpha}} - \frac{\beta}{\alpha}} & 0 & 0 & 1 - \sqrt{\frac{\beta}{\alpha}} \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} \frac{1}{2}(1 - \sqrt{\frac{\beta}{\alpha}}) & 0 & \frac{1}{2}\sqrt{1 - \sqrt{\frac{\beta}{\alpha}}} & \frac{1}{2}\sqrt{\sqrt{\frac{\beta}{\alpha}} - \frac{\beta}{\alpha}} \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}\sqrt{1 - \sqrt{\frac{\beta}{\alpha}}} & 0 & \frac{1}{2} & \frac{1}{2}(\frac{\beta}{\alpha})^{\frac{1}{4}} \\ \frac{1}{2}\sqrt{\sqrt{\frac{\beta}{\alpha}} - \frac{\beta}{\alpha}} & 0 & \frac{1}{2}(\frac{\beta}{\alpha})^{\frac{1}{4}} & \frac{1}{2}\sqrt{\frac{\beta}{\alpha}} \end{pmatrix}$$

$$P_4 = \begin{pmatrix} \frac{1}{2}(1-\sqrt{\frac{\beta}{\alpha}}) & 0 & -\frac{1}{2}\sqrt{1-\sqrt{\frac{\beta}{\alpha}}} & \frac{1}{2}\sqrt{\sqrt{\frac{\beta}{\alpha}}-\frac{\beta}{\alpha}} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}\sqrt{1-\sqrt{\frac{\beta}{\alpha}}} & 0 & \frac{1}{2} & -\frac{1}{2}(\frac{\beta}{\alpha})^{\frac{1}{4}} \\ \frac{1}{2}\sqrt{\sqrt{\frac{\beta}{\alpha}}-\frac{\beta}{\alpha}} & 0 & -\frac{1}{2}(\frac{\beta}{\alpha})^{\frac{1}{4}} & \frac{1}{2}\sqrt{\frac{\beta}{\alpha}} \end{pmatrix}$$

Notice that the projection operators P_j are mutually orthogonal and $\sum_j P_j = I$. Further

$$P_1 + P_2 = \left(\begin{array}{c|c} A & \\ \hline & \end{array}\right)$$

Taking a general density operator $\sigma = \frac{1}{2}\begin{pmatrix} 1+z & x-\iota y \\ x+\iota y & 1-z \end{pmatrix}$ as the state of the system and $\rho_{aux} = |0\rangle\langle 0|$ as that of the ancilla, let us operate the POVM elements on it. The desired outcome is

$$(A_1 + A_2)\sigma(A_1+A_2)^\dagger = \begin{pmatrix} \frac{(1+z)\beta}{2\alpha} & \frac{1}{2}(x-\iota y)\sqrt{\frac{\beta}{\alpha}} \\ \frac{1}{2}(x+\iota y)\sqrt{\frac{\beta}{\alpha}} & \frac{1-z}{2} \end{pmatrix} \tag{5.10}$$

On the other hand, on the combined Hilbert space of the system and ancilla the result of the transformation dictated by the projection operators is

$$(P_1 + P_2)(|0\rangle\langle 0| \otimes \sigma)(P_1+P_2) =$$
$$\left(\begin{array}{cc|cc} \frac{(1+z)\beta}{2\alpha} & \frac{1}{2}(x-\iota y)\sqrt{\frac{\beta}{\alpha}} & 0 & -\frac{1}{2}(1+z)\sqrt{\frac{\beta}{\alpha}}\sqrt{-\frac{\beta}{\alpha}+\sqrt{\frac{\beta}{\alpha}}} \\ \frac{1}{2}(x+\iota y)\sqrt{\frac{\beta}{\alpha}} & \frac{1-z}{2} & 0 & -\frac{1}{2}(x+\iota y)\sqrt{-\frac{\beta}{\alpha}+\sqrt{\frac{\beta}{\alpha}}} \\ \hline 0 & 0 & 0 & 0 \\ -\frac{1}{2}(1+z)\sqrt{\frac{\beta}{\alpha}}\sqrt{-\frac{\beta}{\alpha}+\sqrt{\frac{\beta}{\alpha}}} & -\frac{1}{2}(x-\iota y)\sqrt{-\frac{\beta}{\alpha}+\sqrt{\frac{\beta}{\alpha}}} & 0 & \frac{1}{2}(1+z)(-\frac{\beta}{\alpha}+\sqrt{\frac{\beta}{\alpha}} \end{array}\right)$$

We can see that the top left most block of the above matrix is the same as that obtained in the previous equation. This part cannot be extracted using partial trace. However, the trick in the physical realization of the POVM A lies in the following. Given the outcome of the projective measurement $P_1 + P_2$, if a projective measurement in the σ_z basis is carried out in the ancilla space only, we will get outcomes $|0\rangle\langle 0| \otimes \rho_{\text{top left block}}$ and $|1\rangle\langle 1| \otimes \rho_{rest}$. Now we will select the first kind of outcomes only and discard the rest. In this way we actually get what is obtained by applying the POVM element.

5.4 General scheme

Our basic aim is to write down quantum filtration from a physical point of view. Any filtration operation can be written as $\mathcal{B}(\mathbb{C}^n) \ni \sigma \mapsto S\sigma S^\dagger$ where $S \in Sl_n(\mathbb{C})$. Any such invertible operator S can be written as $S = UDV$, where U and V are unitary operators and D is a diagonal matrix where each diagonal entry is real and strictly positive. Since unitary operators correspond to Hamiltonian evolutions and can be realized in an experiment in a straightforward way, we focus on the diagonal matrices. In this section, we give a scheme to write a physical realisation of the operator in terms of projection operators. Notice that, we have not normalised the outcome.

We consider the mapping $\sigma \mapsto L\sigma L^\dagger$ where $L = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$, and $0 < d_j \leq 1$ for each j. Our scheme is as follows: We want to show that corresponding to such an L there exists a projection operator $P \in \mathcal{B}(\mathbb{C}^{n^2})$ of the form

$$P = \left(\begin{array}{c|c} L & \cdots \\ \hline \vdots & \ddots \end{array}\right)_{n^2 \times n^2} \tag{5.11}$$

Using this projection operator P on the state $|0\rangle\langle 0| \otimes \sigma$ (where $|0\rangle \in \mathbb{C}^n$ is the standard basis vector i.e. we use ancilla of the same size as that of the system), we get the outcome

$$P\left(|0\rangle\langle 0| \otimes \sigma\right)P = \frac{1}{L\sigma L^\dagger}\left(\begin{array}{c|c} L\sigma L^\dagger & \cdots \\ \hline \vdots & \ddots \end{array}\right) \tag{5.12}$$

Projection operators are physical. Moreover the normalisation is also taken care of after the projection. Now taking the partial trace with respect to the ancillia system (i.e. the first system in this situation), we get the required result (including the normalisation).

We consider a set of operators $\{L_j\}$ such that

$$L_j = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & d_j & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}; \quad L_{n+j} = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & 1-d_j & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}; \quad j = 1, \cdots, n. \tag{5.13}$$

Notice that $\sum_j L_j = I$. Our objective is to construct a set of projection operators $\{P_j\}$ corresponding to each L_j such that $P = \sum_j P_j$. For each j, L_j is a rank 1 operator which can be written as $(\sqrt{d_j}|j-1\rangle)(\sqrt{d_j}\langle j-1|)$ (and similarly for L_{n+j}). We write down a $n \times n^2$ matrix where the first $2n$ column vectors in the appropriate order are the column vectors $\sqrt{d_j}|j-1\rangle$, $\sqrt{1-d_j}|j-1\rangle$ and the rest of the columns are the zero columns. Thus the matrix looks like

$$\left(\begin{array}{cccc|cccc|c} \sqrt{d_1} & & & & \sqrt{1-d_1} & & & & \\ & \sqrt{d_2} & & & & \sqrt{1-d_2} & & & \\ & & \ddots & & & & \ddots & & \\ & & & \sqrt{d_n} & & & & \sqrt{1-d_n} & \end{array}\right); \tag{5.14}$$

where the first two blocks are of length n and the last block consists of zeros and is of length n^2-2n. Let $\mathbf{u_j}$ be the jth row vector of the above matrix. We use Gram-Schmidt ortho normalisation on the set $\{\mathbf{u_j}\}_{j=1}^{n}$ to construct a set of orthonormal basis vectors of $\mathbb{C}^{n^2}$. (This process need not be unique. We have used standard basis along with the above set to construct a orthonormal set of vectors). Let $\{\mathbf{u_k}\}_{k=1}^{n^2}$ be the set of orthonormal vectors where $k=j$ for $1 \leq k \leq n$. Writing the basis vectors we get the matrix

$$\begin{pmatrix} \mathbf{u_1} \\ \vdots \\ \mathbf{u_n} \\ \vdots \\ \mathbf{u_{n^2}} \end{pmatrix} =$$

$$\left(\begin{array}{cccc|cccc|c} \sqrt{d_1} & & & & \sqrt{1-d_1} & & & & \\ & \sqrt{d_2} & & & & \sqrt{1-d_2} & & & \\ & & \ddots & & & & \ddots & & \\ & & & \sqrt{d_n} & & & & \sqrt{1-d_n} & \\ \hline -\sqrt{1-d_1} & & & & \sqrt{d_1} & & & & \\ & -\sqrt{1-d_2} & & & & \sqrt{d_2} & & & \\ & & \ddots & & & & \ddots & & \\ & & & -\sqrt{1-d_n} & & & & \sqrt{d_n} & \\ \hline & & & & & & & & I_{n^2-2n} \end{array}\right);$$

where I denotes the identity matrix. Let $\mathbf{v_j}$ be the jth column of this matrix and $P_j = \mathbf{v_j}\mathbf{v_j}^\dagger$ be the corresponding projection operator. We notice that the matrix $P = P_1 + \cdots + P_n$ is the required projection operator.

5.5 The case of $3 \otimes 3$ systems

This thesis for the most part looks at entangled states of $3 \otimes 3$ systems. Therefore we delineate the filtration process for such systems in detail. Let us begin by discussing the filtrations on a single three level system. Later we will consider composite systems where each part is a three level system. For simplicity, we start with an example of such a filtration. The general scheme will be discussed in the same line.

Example 5.5.1. *Let us consider the map*

$$\rho \mapsto D\rho D^{\dagger},$$

where $\rho \in \mathcal{B}(\mathbb{C}^3)$ *and*

$$D = \begin{pmatrix} \frac{1}{2} & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

The operators are given by

$$A_1 = \begin{pmatrix} \frac{1}{2} & & \\ & 0 & \\ & & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \quad A_4 = \begin{pmatrix} \frac{1}{2} & & \\ & 0 & \\ & & 0 \end{pmatrix};$$

such that $A_1 + A_2 + A_3 + A_4 = I_4$*. We write down the* 3×9 *dimensional matrix as given earlier.*

$$\left(\begin{array}{ccc|ccc|ccc} \sqrt{\frac{1}{2}} & & & \sqrt{\frac{1}{2}} & & & & & \\ & 1 & & & 0 & & & & \\ & & 1 & & & 0 & & & \end{array}\right);$$

Completion of the matrix will be of the form.

$$\left(\begin{array}{ccc|ccc|c} \sqrt{\frac{1}{2}} & & & \sqrt{\frac{1}{2}} & & & \\ & 1 & & & 0 & & \\ & & 1 & & & 0 & \\ \hline -\sqrt{\frac{1}{2}} & & & \sqrt{\frac{1}{2}} & & & \\ & 0 & & & 1 & & \\ & & 0 & & & 1 & \\ \hline & & & & & & I_3 \end{array}\right).$$

The required projection operators are of the form

$$P_1 = \frac{1}{2}\left(\begin{array}{ccc|ccc|ccc} 1 & & & -1 & & & & & \\ & 0 & & & & & & & \\ & & 0 & & & & & & \\ \hline -1 & & & 1 & & & & & \\ & & & & 0 & & & & \\ & & & & & 0 & & & \\ \hline & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{array}\right),$$

$$P_2 = \left(\begin{array}{ccc|ccc|ccc} 0 & & & & & & & & \\ & 1 & & & & & & & \\ & & 0 & & & & & & \\ \hline & & & 0 & & & & & \\ & & & & 0 & & & & \\ & & & & & 0 & & & \\ \hline & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{array}\right),$$

and

$$P_3 = \left(\begin{array}{ccc|ccc|ccc} 0 & & & & & & & & \\ & 0 & & & & & & & \\ & & 1 & & & & & & \\ \hline & & & 0 & & & & & \\ & & & & 0 & & & & \\ & & & & & 0 & & & \\ \hline & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{array}\right),$$

where the empty spaces represent zeros. The required projection operator is then

$$P_1 + P_2 + P_3 = \left(\begin{array}{ccc|ccc|ccc} \frac{1}{2} & & & -\frac{1}{2} & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ \hline -\frac{1}{2} & & & \frac{1}{2} & & & & & \\ & & & & 0 & & & & \\ & & & & & 0 & & & \\ \hline & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{array}\right),$$

We now consider the most general case for three level systems. Let us consider the following transformation $\rho \mapsto D\rho D^\dagger$ where $D = \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & d_3 \end{pmatrix}$ and $d_i \in \mathbb{R}+$. To construct the corresponding projective measurements, we first write the equations as

$$A_1 = \begin{pmatrix} d_1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & & \\ & d_2 & \\ & & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & d_3 \end{pmatrix}$$
$$A_4 = \frac{1}{2}\begin{pmatrix} 1-d_1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \quad A_5 = \frac{1}{2}\begin{pmatrix} 1-d_1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \quad A_6 = \frac{1}{2}\begin{pmatrix} 0 & & \\ & 1-d_2 & \\ & & 0 \end{pmatrix}$$
$$A_7 = \frac{1}{2}\begin{pmatrix} 0 & & \\ & 1-d_2 & \\ & & 0 \end{pmatrix} \quad A_8 = \frac{1}{2}\begin{pmatrix} 0 & & \\ & 0 & \\ & & 1-d_3 \end{pmatrix} \quad A_9 = \frac{1}{2}\begin{pmatrix} 0 & & \\ & 0 & \\ & & 1-d_3 \end{pmatrix}$$

so that

$$\sum_{j=1}^{9} A_j = I_3.$$

The corresponding projection operators are given by

$$P_1 = \begin{pmatrix} d_1 & 0 & 0 & 0 & 0 & 0 & -\sqrt{d_1(1-d_1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{d_1(1-d_1)} & 0 & 0 & 0 & 0 & 0 & 1-d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 & 0 & 0 & -\sqrt{d_2(1-d_2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{d_2(1-d_2)} & 0 & 0 & 0 & 0 & 0 & 1-d_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3 & 0 & 0 & 0 & 0 & 0 & -\sqrt{d_3(1-d_3)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{d_3(1-d_3)} & 0 & 0 & 0 & 0 & 0 & 1-d_3 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} \frac{1}{2}(1-d_1) & 0 & 0 & \frac{\sqrt{1-d_1}}{2} & 0 & 0 & \frac{1}{2}\sqrt{-(d_1-1)\,d_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{1-d_1}}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{\sqrt{d_1}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\sqrt{-(d_1-1)\,d_1} & 0 & 0 & \frac{\sqrt{d_1}}{2} & 0 & 0 & \frac{d_1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_5 = \begin{pmatrix} \frac{1}{2}(1-d_1) & 0 & 0 & -\frac{1}{2}\sqrt{1-d_1} & 0 & 0 & \frac{1}{2}\sqrt{-(d_1-1)\,d_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}\sqrt{1-d_1} & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{\sqrt{d_1}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\sqrt{-(d_1-1)\,d_1} & 0 & 0 & -\frac{\sqrt{d_1}}{2} & 0 & 0 & \frac{d_1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(1-d_2) & 0 & 0 & \frac{\sqrt{1-d_2}}{2} & 0 & 0 & \frac{1}{2}\sqrt{-(d_2-1)\,d_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{1-d_2}}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{\sqrt{d_2}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{-(d_2-1)\,d_2} & 0 & 0 & \frac{\sqrt{d_2}}{2} & 0 & 0 & \frac{d_2}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(1-d_2) & 0 & 0 & -\frac{1}{2}\sqrt{1-d_2} & 0 & 0 & \frac{1}{2}\sqrt{-(d_2-1)\,d_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\sqrt{1-d_2} & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{\sqrt{d_2}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\sqrt{-(d_2-1)\,d_2} & 0 & 0 & -\frac{\sqrt{d_2}}{2} & 0 & 0 & \frac{d_2}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(1-d_3) & 0 & 0 & \frac{\sqrt{1-d_3}}{2} & 0 & 0 & \frac{1}{2}\sqrt{-(d_3-1)\,d_3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{1-d_3}}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{\sqrt{d_3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{-(d_3-1)\,d_3} & 0 & 0 & \frac{\sqrt{d_3}}{2} & 0 & 0 & \frac{d_3}{2} \end{pmatrix}$$

$$P_9 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(1-d_3) & 0 & 0 & -\frac{1}{2}\sqrt{1-d_3} & 0 & 0 & \frac{1}{2}\sqrt{-(d_3-1)\,d_3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\sqrt{1-d_3} & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{\sqrt{d_3}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sqrt{-(d_3-1)\,d_3} & 0 & 0 & -\frac{\sqrt{d_3}}{2} & 0 & 0 & \frac{d_3}{2} \end{pmatrix}$$

Given the POVM diagonal matrix D, we get the corresponding projective measurement P given by

$$\begin{aligned} P &= P_1 + P_2 + P_3 \\ &= \begin{pmatrix} d_1 & 0 & 0 & 0 & 0 & 0 & -\sqrt{d_1(1-d_1)} & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 & 0 & 0 & -\sqrt{d_2(1-d_2)} & 0 \\ 0 & 0 & d_3 & 0 & 0 & 0 & 0 & 0 & -\sqrt{d_3(1-d_3)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{d_1(1-d_1)} & 0 & 0 & 0 & 0 & 0 & 1-d_1 & 0 & 0 \\ 0 & -\sqrt{d_2(1-d_2)} & 0 & 0 & 0 & 0 & 0 & 1-d_2 & 0 \\ 0 & 0 & -\sqrt{d_3(1-d_3)} & 0 & 0 & 0 & 0 & 0 & 1-d_3 \end{pmatrix} \end{aligned}$$

For any operator $\sigma \in \mathcal{B}(\mathbb{C}^3)$ we use the ancillary system $|0\rangle\langle 0|$ and use projection on this system as $P(|0\rangle\langle 0| \otimes \sigma)P$. Then the required operation is

$$D\sigma D^\dagger = Tr_1\left[(P(|0\rangle\langle 0| \otimes \sigma)P) \cdot (|0\rangle\langle 0| \otimes I_3)\right], \tag{5.15}$$

where Tr_1 is the partial trace with respect to the first system.

If we had used the extended system as $\sigma \otimes |0\rangle\langle 0|$ then we would have had to exchange the order by using a SWAP operator which is given as

$$U_{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The modification of equation 5.15 is

$$D\sigma D^{\dagger} = Tr_1 \left[(PU_{SWAP}(\sigma \otimes |0\rangle\langle 0|)U_{SWAP}^{\dagger}P) \cdot (|0\rangle\langle 0| \otimes I_3) \right], \tag{5.16}$$

5.5.1 For a composite $\rho \in 3 \otimes 3$ system

Let us consider the most general filtration map $\rho \mapsto (C \otimes D)\rho(C \otimes D)^{\dagger}$, where C and D are both real diagonal matrices.[1] Let Alice and Bob be the two parties who are sharing a state $\rho \in \mathcal{B}(\mathbb{C}^3 \otimes \mathbb{C}^3) = \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Both of them use the auxiliary system $|0\rangle\langle 0|$ and apply the corresponding projection operators P and Q on their respective system. More precisely, the extended system is $|0\rangle_A\langle 0| \otimes \rho \otimes |0\rangle_B\langle 0|$. $|0\rangle_{A,\ B} \in \mathcal{H}^{aux}_{A,\ B} = \mathbb{C}^3$ are the respective auxiliary systems (and spaces) of Alice and Bob.

Let P and Q be the projective measurements for the transformations$\sigma \mapsto C\sigma C^{\dagger}$ and $\sigma \mapsto D\sigma D^{\dagger}$ respectively. On the system $\mathcal{H}^{aux}_A \otimes \mathcal{H}_A$, Alice use the projection operator P. Similarly on $\mathcal{H}_B \otimes \mathcal{H}^{aux}_B$ Bob uses the projection operator Q. Then the transformation is

$$|0\rangle_A\langle 0| \otimes \rho \otimes |0\rangle_B\langle 0| \mapsto (P \otimes Q)(I_9 \otimes U_{SWAP})(|0\rangle_A\langle 0| \otimes \rho \otimes |0\rangle_B\langle 0|)(I_9 \otimes U_{SWAP})^{\dagger}(P \otimes Q). \tag{5.17}$$

Notice that after U_{SWAP} operation, the state is in the space $\mathcal{H}^{aux}_A \otimes \mathcal{H}_A \otimes \mathcal{H}^{aux}_B \otimes \mathcal{H}_B$. The required state is recovered by taking the product with $|0\rangle_A\langle 0| \otimes I_3 \otimes |0\rangle_B\langle 0| \otimes I_3$

[1] A generalised invertible operator S can be written in the polar decomposition $S = HU$ where H is a positive operator and U is a suitable unitary. Further H should have a spectral decomposition where the spectrum is positive. Hence $S = V_1 D V_2$ for some positive diagonal matrix D and unitary operators V_1 and V_2. Thus we need to consider the diagonal transformations only.

and then tracing out all the auxiliary systems.

$$
\begin{aligned}
&(C \otimes D)\rho(C \otimes D)^{\dagger} = \\
&\mathrm{Tr}_{\mathcal{H}_A^{aux},\mathcal{H}_B^{aux}}\left[((P \otimes Q)(I_9 \otimes U_{SWAP})(|0\rangle_A\langle 0| \otimes \rho \otimes |0\rangle_B\langle 0|)\right. \\
&\left.(I_9 \otimes U_{SWAP})^{\dagger}(P \otimes Q)\big) \times (|0\rangle_A\langle 0| \otimes I_3 \otimes |0\rangle_B\langle 0| \otimes I_3)\right].
\end{aligned}
$$

The scheme has been shown in the following Figure 5.1.

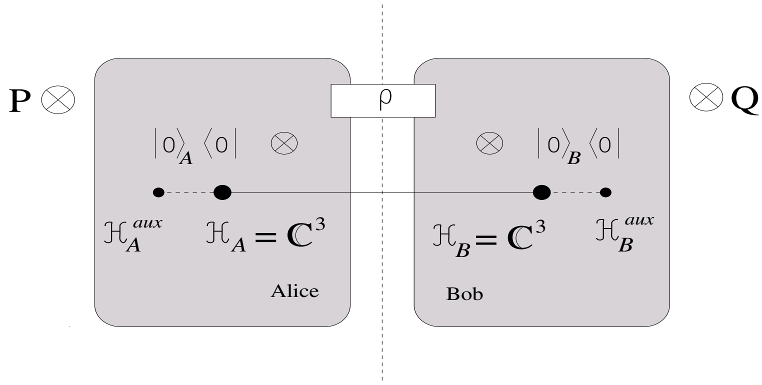

Figure 5.1: Schematic diagram for performing the measurement in the $3 \otimes 3$ shared entangled state ρ

5.6 Conclusions

In this chapter, we have discussed the connection of automorphisms of positive maps with quantum operations. In this dual picture, the state transforms from one PPT entangled state to another PPT entangled state. It may turn that the entanglement of the new state is detectable by a given map, while that of the old one is not detectable. The simple case of such a transformation is a local unitary. However, the non-trivial cases involve application of general quantum filters. These quantum filters are shown to correspond to quantum measurements which need not be projective. We have shown explicitly how these quantum filters can be physically implemented by introducing ancilla spaces. This gives a concrete physical interpretation of the automorphisms described in the earlier chapters.

Chapter 6

Role of projection operators on entangled subspaces

In quantum information processing ideally pure states are used as inputs and we get pure states as output (Nielsen & Chuang (2000)). For non-trivial situations, these states invariably involve quantum entanglement. The structure and geometry of pure entangled states is well studied in literature (see Bengtsson and Życzkowski (Bengtsson & Życzkowski (2006))). Though these states are rank one operators, their structure can be complicated. For the case of mixed states, the situation is qualitatively more difficult.

In Chapter 4 we discussed the concept of UPBs. These are incomplete product bases of a composite system, which cannot be extended by adding any further product vector. Bennett et al (Bennett *et al.* (1999)) gave a construction of bound entangled states which are normalised projection operators in the orthogonal complement of the subspace spanned by the UPB. Shortly thereafter, Wallach (Wallach (2002)) and Parthasarathy (Parthasarathy (2004)) independently looked at the structure of the subspace where each normalised vector is an entangled (pure) state. This study was further extended by Cubitt, Montanaro and Winter (Cubitt *et al.* (2008)), where they have given the maximal dimensions of subspaces where the Schmidt rank of each entangled state is greater than or equal to some given number. In this Chapter, motivated by the work of Bennett et al (Bennett *et al.* (1999)), we study the properties of such subspaces and the projection operators onto them. We start with a description of UPB and give the constructions of Parthasarathy (Parthasarathy (2004)) and Bhat (Bhat (2006)). The subspaces constructed from these structures are written in terms of the orthonormal basis vectors.

The projection operators on such spaces when normalized, correspond to density operators of entangled quantum states. We study the nature of entanglement of such states. We also construct the subspaces of the above spaces and the projection operators onto them. The entanglement structure of the states which are normalised projection operators are also discussed. Using the techniques of Davidson, Marcoux and Radjavi (Davidson *et al.* (2008)), we give an independent proof of the main result of (Cubitt *et al.* (2008)).

6.1 Unextendable product bases (UPB)

There is no explicit structure of entangled states as such, though partial results regarding the structure of PPT and NPPT states are available. Moreover there is no systematic way to generate such states. One well studied way to produce PPT entangled states was produced by Bennett et al (Bennett *et al.* (1999)) by using UPB.

An incomplete product bases set $\mathcal{B}$ in the Hilbert space $\mathcal{H} = \otimes_{j=1}^{k}\mathcal{H}_j$ is called unextendable if the space $\langle\mathcal{B}\rangle^{\perp}$ does not contain any product vector. To be explicit, we give the key theorem of (Bennett *et al.* (1999)).

Theorem 6.1.1. *Bennett* et al. *(1999) If in the Hilbert space $\mathcal{H}$ of dimension D, as defined earlier, there is a mutually orthonormal set of unextendible product basis $\{|\psi_j\rangle : j = 1, \cdots, d\}$, then the state*

$$\rho = \frac{1}{D-d}\left(I_D - \sum_{j=1}^{d} |\psi_j\rangle\langle\psi_j|\right), \tag{6.1}$$

where I_D is the identity operator, is an entangled state which is positive under partial transpose.

The proof depends heavily on the orthogonality of the basis vectors. If $|\alpha\rangle \otimes |\beta\rangle$ is a product vector in the UPB, then under partial transpose the projection operator

$$|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta| \xrightarrow{\text{partial transpose}} |\alpha\rangle\langle\alpha| \otimes |\bar{\beta}\rangle\langle\bar{\beta}|$$

is also a product projection operator. (Throughout this chapter, $|\bar{z}\rangle \in \mathbb{C}^n$ will denote the vector whose terms are complex conjugates of the corresponding terms of $|z\rangle \in \mathbb{C}^n$). Hence the sum of the right hand side of equation 6.1 remains positive under partial transpose. The general multi-partite case is also similar.

The above theory was further extended by DiVincenzo et al (DiVincenzo *et al.* (2003)), where some of the above examples are further generalised. Also a characterisation of the existence of UPBs with the Ramsay number of the orthogonality graph was established. Inspite of all these results, there is no systematic method of constructing such bases.

6.1.1 Entangled subspaces

Let $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$, where $\mathcal{H}_j = \mathbb{C}^{d_j}$ for some finite d_j. Wallach (Wallach (2002)) considered the question of the maximal possible dimension of the subspace $\mathcal{S}$ of $\mathcal{H}$ where each nonzero vector is an entangled state. We call such subspaces as entangled subspaces, as they do not contain any nonzero product vector. He showed that

Theorem 6.1.2. *Wallach (2002)* $\dim \mathcal{S} \leq d_1 \cdots d_k - (d_1 + \cdots d_k) + k - 1$. *Furthermore, this upper bound is attained.*

6.1.2 Parthasarathy's construction

Parthasarathy's proof of the above theorem is mostly existential, which uses Noether normalising lemma. Parthasarathy (Parthasarathy (2004)) gave an explicit construction of such entangled subspaces where the maximal dimension is attained, and thus gave an independent proof of the theorem.

We follow the above notation and let $N = d_1 + \cdots + d_k - k + 1$. Choose N distinct complex numbers $\lambda_1, \cdots, \lambda_N$. Let

$$|u_{ij}\rangle = \begin{pmatrix} 1 \\ \lambda_i \\ \lambda_i^2 \\ \vdots \\ \lambda_i^{d_j - 1} \end{pmatrix}, \quad 1 \leq i \leq N, \quad 1 \leq j \leq k.$$

For $1 \leq i \leq N$, consider $|u_{i1}\rangle \otimes \cdots \otimes |u_{ik}\rangle$. Denote $\mathcal{F} = \text{span}\{|u_i\rangle, 1 \leq i \leq N\}$. Consider the subspace $\mathcal{S} = \mathcal{F}^{\perp}$.

It has been shown by Parthasarathy (Parthasarathy (2004)) that the above space does not contain any product vector. Indeed if a nonzero product vector $|v\rangle = |v_1\rangle \otimes \cdots \otimes |v_k\rangle \in \mathcal{S}$ where each $|v_j\rangle \in \mathcal{H}_j$, then $\prod_{j=1}^{k} \langle v_j | u_{ij}\rangle = 0$ for $1 \leq i \leq N$. By using van der Monde determinant, the vector $|v\rangle$ has to be equal to zero.

If $E_j = \{i : \langle v_j | u_{ij}\rangle = 0\} \subseteq \{1, \cdots, N\}$, then the above construction gives that $\bigcup_{j=1}^{k} E_j = \{1, \cdots, N\}$. Hence $N \leq \sum_{j=1}^{k} |E_j|$. By using van der Monade determinant, it follows that

$$\dim \mathcal{S} = d_1 \cdots d_k - (d_1 + \cdots + d_k) - k + 1.$$

Simple computations show that the basis vectors of $\mathcal{F}$ need not be orthogonal, but the subspaces of $\mathcal{F}$ can contain orthonormal basis of product vectors. This is true for multi-partite systems as well.

6.1.3 Bhat's construction

For notational convenience, he starts with an infinite dimensional space with a basis $\{e_0, e_1, \cdots\}$. Identify $\mathcal{H}_r = \langle\{e_0, \cdots, e_{d_r-1}\}\rangle,\ 1 \leq r \leq k$, where k is the total number of partitions of $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$.

Let $N = \sum_{r=1}^{k}(d_r - 1)$. For $0 \leq n \leq N$, let $\mathcal{I}_n = \{\mathbf{i} = (i_r)_{r=1}^{k},\ 0 \leq i_r \leq d_r - 1,\ \sum_{r=1}^{k} i_r = n\}$.
Let $\mathcal{I} = \bigcup_{n=0}^{N} \mathcal{I}_n$. For $\mathbf{i} \in \mathcal{I}$, let $e_{\mathbf{i}} = \otimes_{r=1}^{k} e_{i_r}$.
For $0 \leq n \leq N$, let $\mathcal{H}_n = \langle\{e_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}_n\}\rangle$, and $\{e_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}_n\}$ is an orthonormal basis for $\mathcal{H}_n$. Further $\mathcal{H} = \bigoplus_{n=0}^{N} \mathcal{H}_n$ and $\{e_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$ is an orthonormal basis for $\mathcal{H}$. Let $u_n = \sum_{\mathbf{i}\in\mathcal{I}_n} e_{\mathbf{i}}$, $0 \leq n \leq N$. Let $\mathcal{T}^{(n)} = \mathbb{C}u_n$, then $\mathcal{H}^{(n)} = \mathcal{S}^{(n)} \bigoplus T^{(n)}$, where

$$\mathcal{S}^{(n)} = \text{span}\{e_{\mathbf{i}} - e_{\mathbf{j}} : \mathbf{i}, \mathbf{j} \in \mathcal{I}_n\}.$$

Clearly $\mathcal{S}^{(n)}$ is also equal to all the sums $\sum_{\mathbf{i}\in\mathcal{I}_n} \alpha_{\mathbf{i}} e_{\mathbf{i}}$ such that $\sum \alpha_{\mathbf{i}} = 0$. Let $\mathcal{S} = \bigoplus \mathcal{S}^{(n)}$ and $\mathcal{T} = \bigoplus \mathcal{T}^{(n)}$. The $\mathcal{H} = S \oplus \mathcal{T}$ and $\mathcal{S}^{\perp} = \mathcal{T}$ which is the $\mathcal{F}$ in the Parthasarathy's construction as shown in the previous section.

Theorem 6.1.3. *Bhat (2006) $\mathcal{S}$ is a completely entangled subspace of maximal dimension.*

Theorem 6.1.4. *Bhat (2006) The set of product vectors in $\mathcal{S}^{\perp}$ id $\{cz^{\lambda} : c \in \mathbb{C}, \lambda \in \mathbb{C} \cup \{\infty\}\}$ where*

$$\begin{aligned} z^{\lambda} &= \bigotimes_{r=1}^{k}(e_0 + \lambda e_1 + \cdots + \lambda^{d_r-1} e_{d_r-1}), \quad \lambda \in \mathbb{C} \\ z^{\infty} &= \bigotimes_{r=0}^{k} e_{d_{r-1}}. \end{aligned}$$

6.1.4 Subspace with bounded Schmidt rank

Motivated by the above works, Cubitt, Montanaro and Winter (Cubitt *et al.* (2008)) considered the subspaces where each vector is entangled and of Schmidt rank $\geq r$. Since Schmidt rank of a state vector is uniquely defined for the bipartite cases only, they confined their work to $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. It was shown that

Theorem 6.1.5. *Cubitt* et al. *(2008) The maximal dimension of the subspace $\mathcal{S}$ of $\mathcal{H}$ where each nonzero vector is of Schmidt rank is $\geq r$ is given by $(d_1-r+1)(d_2-r+1)$.*

6.2 Projection for bipartite systems

Theorem 6.2.1. *Let $\mathcal{H}$ be the bipartite system defined as above.*

1. *The (normalised) projection operator onto the maximally entangled subspace $\mathcal{S}$ is not positive under partial transpose.*

2. *$\mathcal{F}$ does not contain any unextendable orthonormal product basis.*

To prove this, we need the explicit construction of the basis vectors of Parthasarathy's system Parthasarathy (2004). When $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^n$, Parthasarathy gave an explicit construction of the orthonormal set of basis vectors of $\mathcal{S}$. For completeness, we give the construction. As mentioned above choose $(2n-1)$ distinct complex numbers $\lambda_1, \cdots, \lambda_{2n-1}$. Let

$$u_{\lambda_j} = \sum_{x=0}^{n-1} \lambda_j^x |x\rangle.$$

Consider $\mathcal{F} = \text{span}\{u_{\lambda_j} \otimes u_{\lambda_j} : 1 \leq j \leq 2n-1\}$ and set $\mathcal{S} = \mathcal{F}^\perp$. By the construction $\mathcal{S}$ is a completely entangled subspace of dimension (n^2-2n+1). The basis vectors are given by:

- Antisymmetric basis vectors:

$$\frac{1}{\sqrt{2}}(|xy\rangle - |yx\rangle), \quad 0 \leq x < y \leq n-1. \tag{6.2}$$

- For $2 \leq j \leq n-1$ and j is even, vectors are of the forms

$$\frac{1}{\sqrt{j(j+1)}} \left(\sum_{m=0}^{\frac{j}{2}-1} (|m, j-m\rangle + |j-m, m\rangle) - j \left| \frac{j}{2}, \frac{j}{2} \right\rangle \right), \quad \text{and} \tag{6.3}$$

$$\frac{1}{\sqrt{j}} \sum_{m=0}^{\frac{j}{2}-1} \exp\left(\frac{4\pi\imath mp}{j}\right) (|m, j-m\rangle + |j-m, m\rangle), \quad 1 \leq p \leq \frac{j}{2} - 1. \tag{6.4}$$

- For $2 \leq j \leq n-1$ and j is odd, vectors are of the form:

$$\frac{1}{\sqrt{j+1}} \sum_{m=0}^{\frac{j-1}{2}} \exp\left(\frac{4\pi\imath mp}{j+1}\right) (|m, j-m\rangle + |j-m, m\rangle), \quad 1 \leq p \leq \frac{j-1}{2}. \tag{6.5}$$

- For $n \leq j \leq 2n-4$ and j is even, vectors are of the form:

$$\frac{1}{\sqrt{(2n-2-j)(2n-1-j)}} \left(\sum_{m=0}^{\frac{2n-2-j}{2}-1} (|j-n+m+1, n-m-1\rangle \right.$$
$$\left. +|n-m-1, j-n+m+1\rangle) - (2n-2-j) \left| \frac{j}{2}, \frac{j}{2} \right\rangle \right), \quad \text{and} \tag{6.6}$$

$$\frac{1}{\sqrt{2n-2-j}} \sum_{m=0}^{\frac{2n-2-j}{2}-1} \exp\left(\frac{4\pi\imath mp}{2n-2-j}\right) (|j-n+m+1, n-m-1\rangle$$
$$+|n-m-1, j-n+m+1\rangle), \quad 1 \leq p \leq \frac{2n-2-j}{2} - 1. \tag{6.7}$$

- For $n \leq j \leq 2n-4$ and j is odd, vectors are of the form:

$$\frac{1}{\sqrt{2n-1-j}} \sum_{m=0}^{\frac{2n-1-j}{2}-1} \exp\left(\frac{4\pi\imath mp}{2n-1-j}\right) (|j-n+m+1, n-m-1\rangle$$
$$+|n-m-1, j-n+m+1\rangle), \quad 1 \leq p \leq \frac{2n-1-j}{2} - 1. \tag{6.8}$$

Proof. Let $\{P_j : 1 \leq j \leq (d-1)^2\}$ be the projection operators for the corresponding the above basis vectors written in some order. Let $P_{\mathcal{S}} = \sum_{j=1}^{(d-1)^2} P_j$ be the projection onto the maximally entangled space $\mathcal{S}$. Let $P_1 = \frac{1}{2}(|0\rangle\langle 0| \otimes |1\rangle\langle 1| - |0\rangle\langle 1| \otimes |1\rangle\langle 0| - |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|$ be one of the projection operators corresponding to $x = 0$ and $y = 1$ in the equation (6.2). We use the following lemmas whose proofs are obvious.

Proposition 6.2.1. *A square matrix A is positive semidefinite if and only if all its principal sub-minors are positive semidefinite.*

Proposition 6.2.2. $\begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & * \end{pmatrix} \ngeq 0$, *where $*$ denotes any arbitrary number.*

We use this to show that in the partial transpose of $P_{\mathcal{S}}$, there is a 2×2 principal sub-minor of the form as in the proposition (6.2.2).

Suppose $P_{\mathcal{S}} = \sum_{q,r,s,t=0}^{n-1} p_{q,r,s,t} |q\rangle\langle r| \otimes |s\rangle\langle t|$. Since $j \geq 2$ none of the projection operators coming from equations (6.2 – 6.8) contributes any nonzero coefficient of

$|0\rangle\langle 0| \otimes |0\rangle\langle 0|$. In other words $p_{0,0,0,0} = 0$. Similar reasons show that the coefficients $p_{0,1,1,0} = p_{1,0,0,1} = -\frac{1}{2}$.

Under partial transpose,

$$P_{\mathcal{S}}^{PT} = \sum_{q,r,s,t=0}^{n-1} p_{q,r,s,t}|q\rangle\langle r| \otimes |t\rangle\langle s| = \sum_{q,r,s,t=0}^{n-1} p_{q,r,t,s}|q\rangle\langle r| \otimes |s\rangle\langle t|.$$

Thus we get a principal sub-minor of the form

$$0|0\rangle\langle 0| \otimes |0\rangle\langle 0| - \frac{1}{2}|0\rangle\langle 1| \otimes |0\rangle\langle 1| - \frac{1}{2}|1\rangle\langle 0| \otimes |1\rangle\langle 0| + p_{1,1,1,1}|1\rangle\langle 1| \otimes |1\rangle\langle 1|.$$

As shown in the proposition (6.2.2), this matrix is not positive definite. Hence by proposition (6.2.1), the result follows.

The second part of the theorem follows immediately by using the first part. If there exists such a UPB, then the right hand side of the equation (6.1) would give a positive multiple of the projection operator $P_{\mathcal{S}}$ which will be positive under partial transpose, contradicting the theorem (6.2.1). Hence the result follows. □

Let $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^n$. Let the basis vectors due to the construction of Parthasarathy (2004) are enumerated as $\{|v_1\rangle, \cdots, |v_{(d-1)^2}\rangle\}$ where the first $\frac{1}{2}n(n-1)$ are the antisymmetric vectors. Let $P_{v_j} = |v_j\rangle\langle v_j|$ are the corresponding projection operators. The above computation shows that

Corollary 6.2.1. *Any positive operator of the form* $\frac{1}{\sum_{j=1}^{(d-1)^2} p_j} \sum_{j=1}^{(d-1)^2} p_j P_{v_j}$*, where* $p_j > 0$ *for all* j*, is not positive under partial transpose.*

Proof. The theory follows from the above proof. Under partial transpose, we are going to get a sub-matrix of the form $\begin{pmatrix} 0 & -p_1\frac{1}{2} \\ -p_1\frac{1}{2} & * \end{pmatrix}$, where $*$ denotes any number. Such submatrix can not be positive semidefinite. Hence, the result follows. □

6.3 Orthonormal basis of product vectors

Remark 6.3.1. *The space* $\mathcal{F}$ *is independent of the choice of* λ*'s.*

Though the basis vectors of $\mathcal{F}$ are not orthonormal, we can still extract a orthonormal set out of it. These vectors span a subspace of $\mathcal{F}$.

Theorem 6.3.1. *This subspace is of dimension* d.

We prove the theorem by a series of lemmas. We begin with a set up as in the previous subsection 6.1.2 and try to check whether a UPB exists for the space orthogonal complement of the space of entangled vectors. Take

$$\begin{aligned} v_\lambda &= \left(\sum_{j_1=0}^{d_1-1} \lambda^j e_{j_1}\right) \otimes \cdots \otimes \left(\sum_{j_k=0}^{d_k-1} \lambda^j e_{j_k}\right) \\ &= \sum_{j_1,\cdots,j_k} \lambda^{j_1+\cdots+j_k} e_{j_1} \otimes \cdots \otimes e_{j_k} \\ &= \sum_{n=0}^{\nu} \lambda^n \left(\sum_{\substack{j_1,\cdots,j_k \\ j_1+\cdots+j_k=n}} e_{j_1} \otimes \cdots \otimes e_{j_k} \right) \\ &= \sum_{n=0}^{\nu} \lambda^n \left(\sum_{\mathbf{i}\in\mathcal{I}_n} e_{\mathbf{i}} \right) \\ &= \sum_{n=0}^{\nu} \lambda^n u_n \quad \text{where } u_n = \sum_{\mathbf{i}\in\mathcal{I}_n} e_{\mathbf{i}}. \end{aligned}$$

Lemma 6.3.1. $\langle v_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$ *if and only if* $\bar{\lambda}\mu$ *is a root of unity.*

Proof. Given $v_\lambda = \sum_{n=0}^{\nu} \lambda^n u_n$, where $u_n = \sum_{\mathbf{i}\in\mathcal{I}_n} e_{\mathbf{i}}$, we have

$$\begin{aligned} \langle v_\lambda, v_\mu \rangle &= \sum_n (\bar{\lambda}\mu)^n \|u_n\| \\ &= (\bar{\lambda}\mu)^n |\mathcal{I}_n|. \end{aligned}$$

This shows that $(\bar{\lambda}\mu)^n$ is a root of the equation

$$0 = \sum_{n=0}^{\nu} X^n |\mathcal{I}_n| = (1 + X + \cdots + X^{d_1-1}) \cdots (1 + X + \cdots + X^{d_k-1}).$$

Roots of this equation are respectively $d_1, \cdots, d_k$th roots of unity (except 1 itself). $\square$

Let us consider the bipartite case $\mathbb{C}^d \otimes \mathbb{C}^d$. If we choose the set $\Lambda = \{\lambda, \lambda_1, \cdots, \lambda_d\}$ then $\bar{\lambda}\lambda_j$'s are roots of unity. It is easy to check that the orthogonality relation requires the $\lambda = \exp(\imath\theta)$.

Theorem 6.3.2. *Projection onto this subspace is positive under partial transpose and is a separable state.*

Proof. The PPT property follows directly from theorem 6.1.1. □

Remark 6.3.2. *Moreover the analogous theorem for multipartite setup with a general* $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k$ *is also true.*

Let us consider the case when $d = 3$ and the λs are from the set $\{1, \omega, \omega^2\}$, where ω is a complex cube root of unity. Then the orthogonal vectors are $|u_1\rangle = \begin{pmatrix}1\\1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\\1\end{pmatrix}$, $|u_2\rangle = \begin{pmatrix}1\\\omega\\\omega^2\end{pmatrix} \otimes \begin{pmatrix}1\\\omega\\\omega^2\end{pmatrix}$ and $|u_3\rangle = \begin{pmatrix}1\\\omega^2\\\omega\end{pmatrix} \otimes \begin{pmatrix}1\\\omega^2\\\omega\end{pmatrix}$. The projection operator onto the orthogonal complement of the space spanned by $|u_j\rangle$s is given by

$$P = I_9 - (P_{u_1} + P_{u_2} + P_{u_3});$$

where $P_{u_j} = |u_j\rangle\langle u_j|$. The range of this projection is spanned by the following product vectors.

$$\begin{pmatrix}1\\1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\\omega\\\omega^2\end{pmatrix}, \quad \begin{pmatrix}1\\1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\\omega^2\\\omega\end{pmatrix}$$
$$\begin{pmatrix}1\\\omega\\\omega^2\end{pmatrix} \otimes \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\\omega^2\\\omega\end{pmatrix} \otimes \begin{pmatrix}1\\1\\1\end{pmatrix}$$
$$\begin{pmatrix}1\\\omega\\\omega^2\end{pmatrix} \otimes \begin{pmatrix}1\\\omega^2\\\omega\end{pmatrix}, \quad \begin{pmatrix}1\\\omega^2\\\omega\end{pmatrix} \otimes \begin{pmatrix}1\\\omega\\\omega^2\end{pmatrix}$$

Thus the projection P (and hence the state formed by the normalised projection operator) is a separable state. The general result follows in the same way. In this case the space is spanned by the set of product vectors $\{|u_{\lambda_j}\rangle \otimes |u_{\lambda_k}\rangle : \ j \neq k, \ 1 \leq j, k \leq d-1\}$.

For the general bipartite case, as above, let

$$\Lambda_\theta = \{\exp(\imath\theta), \exp(-\imath\theta)\omega, \cdots, \exp(-\imath\theta)\omega^{d-1}\},$$

where ω is a d-th root of unity (not equal to 1). There are infinitely many different choices of the set based on the choice of θ. However, projection operators corresponding to the above set is local unitarily equivalent with projection operator corresponding to the set $\{1, \omega, \cdots, \omega^{d-1}\}$, with the local unitary operator is given as

$$\begin{pmatrix} 1 & & & \\ & e^{-\imath\theta} & & \\ & & \ddots & \\ & & & e^{-\imath(d-1)\theta} \end{pmatrix} \otimes \begin{pmatrix} 1 & & & \\ & e^{-\imath\theta} & & \\ & & \ddots & \\ & & & e^{-\imath(d-1)\theta} \end{pmatrix}.$$

For the case $\mathcal{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, where $d_1 \neq d_2$, we have the $\bar{\lambda}\lambda_j$'s are the complex roots of the equation $(1 + X + \cdots + X^{d_1-1})(1 + X + \cdots + X^{d_2-1}) = 0$. Let ω_{d_1} is a d_1th root of unity, and ω_2 is a d_2th root of unity. It can be seen easily that the vectors $\begin{pmatrix} 1 \\ \omega_{d_1} \\ \vdots \\ \omega_{d_1}^{d_1-1} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \omega_{d_1} \\ \vdots \\ \omega_{d_1}^{d_2-1} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \omega_{d_2} \\ \vdots \\ \omega_{d_2}^{d_1-1} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \omega_{d_2} \\ \vdots \\ \omega_{d_2}^{d_2-1} \end{pmatrix}$ can not be orthogonal to each other unless they are powers of each other. This can only happen if $\gcd(d_1, d_2) \neq 1$. Even then, the separability condition holds for the above construction. Further, the above constructions yield the product vectors which spans the projection operator. Hence we can conclude that

Theorem 6.3.3. *Any normalised projection operator constructed by the method of theorem 6.3.2 (as well as of the remark 6.3.2) is separable.*

6.4 Subspaces of fixed Schmidt rank

Schmidt number is an important tool in quantum information and is a measure of entanglement, which we have discussed in 1.2.1.4. As mentioned, there is no generalisation in multi-partite setting. In this section, we try to determine the maximal dimensions of subspaces of $\mathcal{H}_1 \otimes \mathcal{H}_2$ where each state vector is of Schmidt rank greater than or equal to a fixed number r.

Let $|\psi\rangle = \sum_{i,j} c_{ij} |i\rangle \otimes |j\rangle$ be the Schmidt decomposition of the vector $|\psi\rangle$, where $|i\rangle$ and $|j\rangle$ are orthonormal basis of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. This space can be identified with the $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$. Thus the vector $|\psi\rangle$ is mapped to the operator $M(\psi) = [[c_{ij}]]_{i=1,\cdots,d_1,\ j=1,\cdots,d_2}$.

To move further, we need the following identification. Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ be of finite dimensional space. Then

$$\begin{array}{ccc} \mathcal{H}_1 \otimes \mathcal{H}_2 & \xrightarrow{\Gamma} & \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1) \\ |u\rangle \otimes |v\rangle & \leftrightarrow & |v\rangle \langle u| \end{array}$$

Since we are dealing with a finite dimensional situation, the right hand side gives a matrix of order $d_2 \times d_1$.

Lemma 6.4.1. *The set of all states in $\mathcal{H}_1 \otimes \mathcal{H}_2$ with Schmidt rank r is isomorphic to the set of all $d_1 \times d_2$ density matrices with rank r.*

Proof. If $|\psi\rangle = \sum_{j=1}^{k} p_j |u_j\rangle \otimes |v_j\rangle$, where $p_i \geq p_2 \geq \cdots \geq p_k > 0$. $\{|u_j\rangle\}$ and $\} |v_j\rangle\}$ forms orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.

Consider unitary operators U and V of order d_1 and d_2, and the map $Z \mapsto UZV4$ in $\mathbb{M}_{d_1,d_2}$. Notice that

$$\langle UZ_1V|UZ_2V\rangle = \mathrm{Tr}V^{\dagger}Z_1^{\dagger}Z_2V = \mathrm{Tr}Z_1^{\dagger}Z_2 = \langle Z_1|Z_2\rangle.$$

Clearly $rank\ Z = rank\ UZV$. Thus

$$\begin{aligned} rank\,\Gamma\,|\psi\rangle &= rank\,\sum_{j=1}^{k} p_j\,|u_j\rangle\,\langle v_j| \\ &= rank\,\sum_{j} p_j U\,|u_j\rangle\,\langle v_j|\,V^{\dagger} \\ &= rank\,\sum_{j} p_j\,|e_j\rangle\,\langle f_j| \\ &= rank \left(\begin{array}{cccc|c} p_1 & & & & \\ & p_2 & & & \\ & & \ddots & & \\ & & & p_k & \\ \hline & & & & \end{array}\right) \\ &= k \end{aligned}$$

□

We are going to calculate the exact dimension of the maximal subspace where every vector is of Schmidt rank greater than or equal to k.

Definition 6.4.1 (Davidson *et al.* (2008))**.** *A subspace $\mathcal{V}$ of $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ is called k-transitive if for every choice of k-linearly independent vectors $|x_1\rangle, \cdots, |x_k\rangle$ in $\mathcal{H}_1$, there exists k linearly independent vectors $|y_1\rangle, \cdots, |y_k\rangle$ in $\mathcal{H}_2$, there is a $A \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that $A|x_j\rangle = |y_j\rangle$.*

If $\mathcal{V}$ is a $(k-1)$ transitive subspace. Then the space $\mathcal{V}^{\perp}$ maps to a subspace of $\mathcal{H}_1 \otimes \mathcal{H}_2$ where each vector is of Schmidt rank k.

Theorem 6.4.1 (Davidson *et al.* (2008))**.** *The minimal dimension of a* $(k-1)$*-transitive subspace of the above space is* $(k-1)(d_1 + d_2 - k + 1)$.

This shows that

Theorem 6.4.2. *Maximal dimensional of the subspace of* $\mathcal{H}_1 \otimes \mathcal{H}_2$*, where each vector is of Schmidt rank greater than or equal to k, is* $d_1 d_2 - (k-1)(d_1 + d_2 - k + 1)$.

Proof. Follows from the theorem above. □

6.5 Conclusions

In this chapter, we have shown the characterisation of the states coming from the normalised projection operators of subspaces of Hilbert spaces. Further we have given a proof of the maximal dimension of the subspace where vectors are of rank greater than or equal to some fixed number k.

Appendix A

Families of positive maps which are not completely positive and their power to detect entanglement

In the course of our work, we have analyzed a large number of positive maps which are not completely positive, with a view to study their ability to detect quantum entanglement. In this process, we have gained insights and have found a number of interesting results. Some of these results are not included in the main course of the thesis and therefore we discuss them in this appendix. As it turns out, some of these results are negative.

We begin with a survey of the geometric properties of positive maps. The set of Hermitian operators in the space $\mathbb{C}^n$ forms a real vector space. This space can be identified with $\mathbb{R}^{n^2}$. The set of states forms a closed space Ω in the subspace of $\mathbb{R}^{n^2-1}$. Any positive map can be considered as a map $\mathbb{R}^{n^2} \rightarrow \mathbb{R}^{n^2}$ which embeds Ω to itself. Even though the complete structure of Ω is not known in general, partial insights are available. These in turn have been exploited to generate positive maps. Such maps can be completely positive, and decomposable as well.

Any linear map which embeds Ω to itself gives rise to a linear map at the matrix level. We notice that any affine map with the above property also gives rise to a linear map at the matrix level. This indicates the complexity of the geometry of maps. Using the above geometric construction and the methodology developed in the Chapter 4, we identify some of the maps as completely positive and some as decomposable. Hence these maps cannot be used for detecting PPT entangled states.

In the set of positive maps we define a partial ordering "$>$". This ordering can be used to compare the powers of two different maps in terms of detecting entanglement. Given a map $\phi : \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^n)$ let $\mathcal{D}_\phi = \{\rho \in \mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^n)$ is a state : $(\mathbb{1} \otimes \phi)\rho \not\geq 0\}$ be the set of entangled states detected by ϕ. We show that, if $\phi_1 > \phi$, then the $\mathcal{D}_{\phi_1} \subset \mathcal{D}_\phi$. We apply it on some of the known classes of positive maps and identify the maps ϕ which gives the largest $\mathcal{D}_\phi$ in size. This process is weaker than the method of finding extremal points of the set of maps. However, it is easier to use. Thus it can be used for identifying the potential classes of extremal points of positive maps.

A.1 Geometry of maps for three-dimensional systems

We present a discussion of the geometry of maps in three dimensions with a view to get insights into the Choi map which we have used extensively in our work. This construction was initially proposed by Kossakowski (Kossakowski (2003)) and later used by other authors (example Simon *et al.* (2006), Simon *et al.* (2009)). This geometrical construction also works for generalizations of the Choi map due to Ha and Kye.

We can represent all 2×2 density matrices as:

$$\rho = \frac{1}{2}(I + x.\sigma),$$

where $x = (x_1, x_2, x_3)$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. Hermiticity gives that the vector $x \in \mathbb{R}$ and positivity gives that $\|x\| \leq 1$.
The picture is more difficult for higher dimension. We extend this idea in higher dimension by representing $n \times n$ density matrices as a real linear combination of $n^2 - 1$ traceless matrices along with identity. We need to understand structure of the convex space $\Omega \subset \mathbb{R}^{n^2-1}$. The traceless matrices are of the following types:

- σ_3 type of matrices.

$$\begin{gathered} J_1 = \mathrm{diag}(1, -1, 0, \cdots, 0) \\ \sqrt{3}J_2 = \mathrm{diag}(1, 1, -2, 0, \cdots, 0) \\ \vdots \\ \sqrt{\tfrac{(n-1)(n-2)}{2}}J_{n-2} = \mathrm{diag}(1, 1, 1, \cdots, 1, -(n-2), 0) \\ \sqrt{\tfrac{n(n-1)}{2}}J_{n-1} = \mathrm{diag}(1, 1, 1, \cdots, 1, -(n-1)) \end{gathered}$$

- σ_1 and σ_2 types: For $1 \leq i_0 < j_0 \leq n$

$$\begin{aligned} (M(i_0, j_0))_{i,j} &= \delta_{ii_0}\delta_{jj_0} + \delta_{ij_0}\delta_{ji_0} \\ (N(i_0, j_0))_{i,j} &= i(\delta_{ii_0}\delta_{jj_0} - \delta_{ij_0}\delta_{ji_0}) \end{aligned}$$

- Rename and relabel all n^2-1 matrices as $J_1, \cdots, J_{n^2-1}$.

Thus any Hermitian operator A can be represented as

$$\begin{aligned} A &= x_0\left(I+\sum_{k=1}^{n^2-1} x_k J_k\right) \quad \text{(A.1)}\\ \text{where } x_0 &= \frac{1}{n}Tr(\rho)\\ x_k &= \frac{1}{2x_0}Tr(J_k\rho), \quad k=1,2,\cdots,n^2-1. \end{aligned}$$

In the three dimensional case the corresponding basis of generalised Pauli matrices gives the trace zero Gell-Mann matrices along with identity matrix I_3. Gell-Mann Matrices are given by

$$\lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix} \quad \lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}$$
$$\lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix} \quad \lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$$
$$\lambda_5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix} \quad \lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$$
$$\lambda_7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix} \quad \lambda_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$$

The representation looks like

$$A = x_0\left(I_3+\sum_{i=1}^{8} x_i\lambda_i\right);$$

where $x_i \in \mathbb{R}$ and are given by

$$\begin{aligned} x_0 &= \frac{1}{3}\mathrm{Tr}(A)\\ x_i &= \frac{1}{2x_0}\mathrm{Tr}(A\lambda_i) \qquad i=1,\cdots,8. \end{aligned}$$

If we restrict ourselves to the study of density operators only, then $x_0 = 1$ and each state is mapped to a unique point $(x_1, \cdots, x_8)$. Let

$$\Omega = \{\mathbf{x} = (x_1,\cdots,x_8) \leftrightarrow \ \rho \text{ is a trace class operator}\}. \quad \text{(A.2)}$$

Let $((a_{ij})) \in \mathcal{M}_3(\mathbb{C})$. The major coefficients are given as

$$\begin{cases} x_0 = \frac{a_{11}+a_{22}+a_{33}}{3} \\ x_3 = \frac{a_{11}-a_{22}}{2x_0} \\ x_8 = \frac{a_{11}+a_{22}-2a_{33}}{2x_0\sqrt{3}} \end{cases} \tag{A.3}$$

The other coefficients are the real or imaginary parts of the off diagonal elements divided by $2x_0$.

Conversely given $x = (x_1, \cdots, x_8)$, and $x_0 = \frac{1}{3}$; the corresponding density operator is of the form

$$\frac{1}{3}\begin{pmatrix} 1 + x_3 + \frac{x_8}{\sqrt{3}} & x_1 - ix_2 & x_4 - ix_5 \\ x_1 + ix_2 & 1 - x_3 + \frac{x_8}{\sqrt{3}} & x_6 - ix_7 \\ x_4 + ix_5 & x_6 + ix_7 & 1 - 2\frac{x_8}{\sqrt{3}} \end{pmatrix}. \tag{A.4}$$

Proposition A.1.1. *Any affine transformation ϕ on $\mathbb{R}^8$ gives rise to a linear map T_ϕ*

Proof. Any affine transformation can be written as $\mathbf{x} \mapsto B.\mathbf{x} + \mathbf{c}$, where B is a real operator acting on $\mathbb{R}^8$ and $\mathbf{c} = (c_1, \cdots, c_8)$ is a fixed translation.

Let $\phi : \mathbf{x} \mapsto B\mathbf{x}$, By linearity of the above transformations, T_ϕ is a linear map on $\mathcal{B}(\mathbb{C}^3)$.

Let $\phi : \mathbf{x} \mapsto \mathbf{x} + \mathbf{c}$; where $\mathbf{c} = (c_1, \cdots, c_8)$ is a fixed translation. Now for any arbitrary Hermitian matrix $A \in \mathcal{M}_3$ we have

$$T_\phi : A \mapsto A + C\mathrm{Tr}(A), \qquad \text{where } C = \begin{pmatrix} c_3 + \frac{c_8}{\sqrt{3}} & c_1 - ic_2 & c_4 - ic_5 \\ c_1 + ic_2 & -c_3 + \frac{c_8}{\sqrt{3}} & c_6 - ic_7 \\ c_4 + ic_5 & c_6 + ic_7 & -2\frac{c_8}{\sqrt{3}} \end{pmatrix}. \tag{A.5}$$

C is a constant matrix determined by c and trace is linear map. Combining, we get the result. □

The above method can easily be extended to any arbitrary finite dimension n. Further, it shows that any such affine map gives rise to a Hermiticity preserving map. Notice that, the above proposition does not deal with the positivity of the map.

Now any map $\Omega \hookrightarrow \Omega$ maps a positive operator to a positive operator. Since any positive semi-definite Hermitian operator can be diagonalised by a unitary transformation, we need to consider only the diagonal cases. Apart from the identity matrix, there are two diagonal basis matrices available, namely λ_3 and λ_8. Now any positive semi-definite trace class operator can be written as a convex combination of three matrices $\mathrm{Diagonal}[1, 0, 0]$, $\mathrm{Diagonal}[0, 1, 0]$, and $\mathrm{Diagonal}[0, 0, 1]$. These three matrices correspond to three points $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(0, -\sqrt{3}\right)$ in the $\lambda_3 - \lambda_8$ plane in $\mathbb{R}^8$, which are the three vertices of an equilateral triangle centred at origin denoted by $\triangle$. The origin represents the maximally mixed state. Now we need to find all the maps $\phi : \triangle \hookrightarrow \triangle$ so that the lifting in the operator space T_ϕ is positive and linear. The lifting of the maps is given in the following diagram.

$$\begin{array}{ccc}
\mathcal{M}_3(\mathbb{C}) & \xrightarrow{T_{\tilde{h}}} & \mathcal{M}_3(\mathbb{C}) \\
\big\downarrow & & \big\downarrow \\
\Omega & \underset{\subset}{\xrightarrow{\tilde{h}}} & \Omega \\
\cup\big\uparrow & & \cup\big\uparrow \\
\triangle & \underset{\subset}{\xrightarrow{\phi}} & \triangle
\end{array}$$

We are going to use T_h, instead of $T_{\tilde{h}}$ when the embedding is obvious, i.e. it is a reflection in (λ_3, λ_8) plane.

The largest disk which can be embedded in this equilateral triangle $\triangle$ is of radius $\frac{\sqrt{3}}{2}$. Similarly the smallest disk, in which this triangle can be embedded is of radius $\sqrt{3}$. For $n = 3$ the radii of in-sphere and out-sphere of the convex subset $\Omega \subset \mathbb{R}^{n^2-1}$ is in $1 : 2$.

In general, Ω is a simplex in $\mathbb{R}^{n^2-1}$. The same argument as above gives,

Proposition A.1.2. *The radii of in-sphere and out-sphere of the convex subset* $\Omega \subset \mathbb{R}^{n^2-1}$ *is in* $1 : (n-1)$.

It is clear from the correspondence between $n \times n$ Hermitian unit trace matrices and points in $\mathbb{R}^{n^2-1}$, that any linear map preserving trace and Hermiticity are in in 1-1 correspondence with the homogeneous elements of $\mathcal{L}(\mathbb{R}^{n^2-1}, \mathbb{R}^{n^2-1})$- which maps Ω to itself. $SU(n)$ operations give a rotation of in $\mathbb{R}^{n^2-1}$. These rotations form a proper subset of orthogonal group $\mathcal{O}(n^2-1)$. Now given any $(n^2-1) \times (n^2-1)$ orthogonal matrix; testing whether it maps Ω to itself is in general nontrivial. However in certain cases it is easy.

Any element $R \in \mathcal{O}(n^2-1)$ maps the out-sphere of Ω onto itself. Now reducing it by a scale factor $n-1$ it maps the out sphere to in sphere. Now any point inside the in-sphere is a valid state and no point outside the out-sphere is a valid state. So we have:

Theorem A.1.1. *Kossakowski (2003); Simon* et al. *(2006); Simon* et al. *(2009) Every map represented by a matrix of the form* $\frac{1}{n-1}R$ *where* $R \in \mathcal{O}(n^2-1)$ *is positive.*

The importance of the above theorem is that, many of the important classes of maps are of the above form. In particular for Choi's map (Choi (1975b)), rotation matrix is given by a rotation in $(\lambda_3,\ \lambda_8)$ plane combined with a reflection with respect to the $(\lambda_3,\ \lambda_8)$ plane. Similarly a general rotation in $(\lambda_3,\ \lambda_8)$ gives rise to the class of maps given by Ha and Kye (Ha & Kye (2012)).

A.2 Comparison of maps

We first define a partial order on the set of positive maps. This in turn gives a way to compare power of detecting entanglement between maps.

Definition A.2.1. *Given two positive maps ϕ_1 and ϕ_2, we denote $\phi_1 > \phi_2$ if $(\phi_1 - \phi_2)$ is a completely positive map.*

Definition A.2.2. *Let*

$$\mathcal{D}_\phi = \{\rho \in \mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^n) \text{ is a state} : (\mathbb{1} \otimes \phi)\rho \ngeq 0\}$$

be the set of entangled states detected by ϕ. ϕ is said to be more powerful than another positive map ϕ_1, if $\mathcal{D}_{\phi_1} \subset \mathcal{D}_\phi$. This shows that for any positive semidefinite operator ρ $(1 \otimes \phi_1)\rho \ngeq 0$ implies that $(1 \otimes \phi)\rho \ngeq 0$, i.e. any entangled state detected by the map ϕ_1 will always be detected by ϕ.

Using this definition, we can see that if the difference $(\phi_1 - \phi)$ is a completely positive map, we have for any density operator ρ acting an appropriate product space

$$\begin{aligned} & \mathbb{1} \otimes (\phi_1 - \phi)\rho \geq 0 \\ \Rightarrow \quad & \langle x|\mathbb{1} \otimes (\phi_1 - \phi)\rho|x\rangle \geq 0 \qquad \forall x \\ \Rightarrow \quad & \langle x|(\mathbb{1} \otimes \phi_1)\rho|x\rangle \geq \langle x|(\mathbb{1} \otimes \phi)\rho|x\rangle \quad \forall x \end{aligned}$$

Hence any entangled state identified by ϕ_1 can always be identified by ϕ. The converse need not be true in general. In case the converse is true, we call ϕ and ϕ_1 are of equal power to detect entanglement.

It can be possible that this difference is not a positive map. However, if ϕ is a positive map, then for any $\varepsilon > 0$, $\varepsilon\phi$ is also a positive map. Hence we can use the same argument with the difference $(\phi_1 - \varepsilon\phi)$, by some suitable choice of ε such that the difference is a CP map. Downside is, we may have to deal with non-diagonal elements. This of course breaks the partial order structure. However, this gives an advantage if we restrict our attention to detecting PPT entangled states only. Thus we modify the definition A.2.2 as

Definition A.2.3. *An indecomposable positive map ϕ is stronger than a map ϕ_1 for detecting PPT entangled states if there exists a constant ε such that $(\phi_1 - \varepsilon\phi)$ is a decomposable map.*

Notice that, for any PPT state ρ, we have $1 \otimes (\phi_1 - \varepsilon\phi)\rho \geq 0$. Thus, we can use the same logic once again. However, in general, it is difficult to determine whether a given map is decomposable or not.

The above technique is weaker than detecting extremality. However, since it is easier, it can be used to detect stronger maps from a class of maps from the point of view of entanglement detection. We can show various classes of maps and detect the stronger ones from them. We give a few examples of the classes of maps available in literature and try to detect the stronger members of the class.

Example A.2.1. *We consider the class of maps discovered by Choi (1975b) (written in example B.2.3). The map is given as follows*

$$S_\mu : \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{pmatrix} + \mu \begin{pmatrix} a_{33} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{22} \end{pmatrix}. \quad \text{(A.6)}$$

The class of maps is positive for all $\mu \geq 1$. Størmer discovered a class of operators dependant on μ which are positive under partial transpose but not positive under the above map. The map for $\mu = 1$ was shown to be extremal by Choi & Lam (1977/78). Using the above method, we have

$$(S_\mu - S_1) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (\mu - 1) \begin{pmatrix} a_{33} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{22} \end{pmatrix}.$$

For $\mu > 1$ right hand side gives a completely positive map ($\mu - 1 > 0$ and for any positive operator the diagonal elements are always positive, and hence their rearrangements as well). Hence the map S_1 is more powerful to detect entangled state than all other S_μ and any entangled state detected by S_μ is detectable by S_1.

Example A.2.2. *Kye (1992) made one of the generalisation of Choi's map and extremality of one of the subclass was shown by Osaka (1992) (shown in the B.2.5). A further generalisation of the above was made by Ha (2002, 2003). For completeness,* *we write the map as follows.*

$$S_{x,y,z} : \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} xa_{33} & 0 & 0 \\ 0 & ya_{11} & 0 \\ 0 & 0 & za_{22} \end{pmatrix}. \tag{A.7}$$

Ha showed that the above map is positive provided $xyz \geq 1$ where $x,\ y,\ z > 0$. The example of Choi (1975b) S_1 is achieved when $x = y = z = 1$ and Kye - Osaka's extremal map is derived when $xyz = 1$. We want to show that the map $S_{x,y,z}$ for $xyz > 1$ is weaker than the map $S_{x',y',z'}$ when $x'y'z' = 1$ for a suitable choice of $x',\ y',\ z'$; in the above sense of detecting entanglement.

It is sufficient to prove that there exists a point (x', y', z') with $x' \leq x,\ y' \leq y,\ z' \leq z$ and $x'y'z' = 1$. Using the method as above, we can easily see that the bi quadratic form corresponding to $S_{x,y,z} - S_{x',y',z'}$ is a sum of squares of quadratic forms. Hence the map $S_{x,y,z} - S_{x',y',z'}$ is completely positive.

Without loss of generality we can assume that $x \geq y \geq z$. Set $z' = \min\{1, z\}$. Two possibilities may arise.
Case I: $z' = 1$. That means $x \geq y \geq z \geq 1$. We can simply put $x' = y' = z' = 1$, to get the required condition.
Case II: Let $z' = z$, i.e. $z < 1$. Choose $y' = y$ and put $x' = \frac{1}{yz} \leq x$, to satisfy all the conditions.[1]

This example was further generalised by Ha for arbitrary dimensions, as shown in the example B.2.6. We use the same notation and consider the case for dimension $d = 3$. Here we have two situations:

1. *When $p_0 = 1$ we have $p_1p_2p_3 \geq 1$. This is the same case as discussed earlier, with x, y, z is replace by p_1, p_2, p_3. The extremal maps are given by the $p_1p_2p_3 = 1$.*

2. *When $1 < p_0 < 2$, then $2 - p_0 < 1$. Put $(2 - p_0)^3 = \frac{1}{s}$. Clearly $s \geq 1$. We have $p_1p_2p_3 \geq \frac{1}{s}$; i.e. $sp_1p_2p_3 \geq 1$. Now put $x = sp_1$, $y = p_2$ and $z = p_3$. Hence the case is as in the theorem 3. So we can have $x',\ y',\ z'$ with the required properties. Now replace $p'_1 = \frac{x'}{s}$, $p'_2 = y'$ and $p'_3 = z'$. Since $s > 1$ the extremal points lie on the surface $p_1p_2p_3 = \frac{1}{s}$.*

[1] The proof can also follow the line shown by Lewenstein *et al.* (2000).

By repeatedly combining the variables as above we get the general result for dimension d.

Theorem A.2.1. *For the generalised Ha map, as given in example B.2.6, we have the following two classes of maps which are stronger than the other maps of the same category.*

1. *If $p_0 = d - 2$, then $p_1 \cdots p_d = 1$.*

2. *If $p_0 > d - 2$, then $p_1 \cdots p_d = (d - 1 - p_0)^d$.*

It can happen that two given maps are not comparable. Since our object is to detect PPT entangled states, we can drop the condition of completely positivity from the definition A.2.2. Suppose the difference is a positive decomposable map. Then for any PPT state ρ, we have $\mathbb{1} \otimes (\phi_1 - \varepsilon\phi)\rho \geq 0$. Thus, we can use the same logic once again. However, in general, it is difficult to determine whether a given map is decomposable or not.

Example A.2.3. *Another important generalisation of Choi's map is given by Cho* et al. *(1992) and written in B.2.4. Let us denote the map as $\phi_{a,b,c}$. Recently it has been shown that a subclass of the maps, namely the case when $0 < a < 1,\ a + b + c = 1,\ bc = (1 - a)^2$, has been shown to be optimal Ha & Kye (2011), and exposed in Ha & Kye (2011). Our method gives that the above class is 'the' strongest among this class of maps, and all other maps are weaker than the these maps. We achieve it by steps.*

Case I *Let $a = 1$. Then from the conditions we get that $b + c \geq 1,\ 0 \leq bc < \frac{1}{4}$. Assume that $b \geq 1$. Then for any matrix $((x_{ij})$, we get*

$$(\phi_{1,b,c} - \phi_{1,1,0})\left((x_{ij})\right) = \begin{pmatrix} (b-1)x_{22} + cx_{33} & & \\ & (b-1)x_{33} + cx_{11} & \\ & & (b-1)x_{11} + cx_{22} \end{pmatrix}.$$

The blank spaces denote zero. By our choice of $b \geq 1,\ c > 0$ Since for any positive semi-definite operator the diagonal elements are always greater than or equal to zero, we can see that the map $(\phi_{1,b,c} - \phi_{1,1,0})$ is a completely positive map. Hence by the earlier argument, $\phi_{1,1,0}$ is more powerful than $\phi_{1,b,c}$ when $b \geq 1$. $\phi_{1,1,0}$ is a standard example of Choi's map which is extremal. Similarly when $c \geq 1$ we use the map $\phi_{1,0,1}$ (which is again extremal) and come to the similar conclusion.

Case II *By the previous case we need to look at the points $a,\ b,\ c$ so that $a=1,\ b+c=1$. We know that, the two extremal maps are $\phi_{1,1,0}$ and $\phi_{1,0,1}$. Moreover the map $\phi_{1,\frac{1}{2},\frac{1}{2}}$ is a decomposable map.* [1] *Consider the case of $a=1,\ b>\frac{1}{2},\ c<\frac{1}{2}$.*

$$\begin{aligned}
&\left(\phi_{1,b,c}-\varepsilon\phi_{1,1,0}\right)((x_{ij}))\\
&= \begin{pmatrix}(1-\varepsilon)x_{11}+(b-\varepsilon)\,x_{22}+cx_{33} & -(1-\varepsilon)x_{12} & -(1-\varepsilon)x_{13}\\ -(1-\varepsilon)x_{21} & (1-\varepsilon)x_{22}+(b-\varepsilon)x_{33}+cx_{11} & -(1-\varepsilon)x_{23}\\ -(1-\varepsilon)x_{31} & -(1-\varepsilon)x_{32} & (1-\varepsilon)x_{33}+(b-\varepsilon)x_{11}+cx_{22}\end{pmatrix}\\
&= (1-\varepsilon)\begin{pmatrix}x_{11}+\frac{b-\varepsilon}{1-\varepsilon}x_{22}+\frac{c}{1-\varepsilon}x_{33} & -x_{12} & -x_{13}\\ -x_{21} & x_{22}+\frac{b-\varepsilon}{1-\varepsilon}x_{33}+\frac{c}{1-\varepsilon}x_{11} & -x_{23}\\ -x_{31} & -x_{32} & x_{33}+\frac{b-\varepsilon}{1-\varepsilon}x_{11}+\frac{c}{1-\varepsilon}x_{22}\end{pmatrix}
\end{aligned}$$

We need to find whether there exists any $\varepsilon>0$ satisfying the above conditions. It can be checked that the map $\phi_{1,\frac{b-\varepsilon}{1-\varepsilon},\frac{c}{1-\varepsilon}}$ is a P map of Kye type as it satisfies the first three conditions of B.2.4. However, to taste the decomposability we note that the conditions

$$\frac{(b-\varepsilon)c}{(1-\varepsilon)^2}\geq\frac{1}{4},\qquad \textit{provided}\qquad \begin{cases} b+c=1\\ \frac{1}{2}<b\leq 1\\ 0\leq c<\frac{1}{2}\end{cases};$$

is satisfied by the solution $\varepsilon=(b-c)$. Thus the corresponding subtraction gives a decomposable map and the conditions are satisfied.

Case III *Suppose $a>1$, and $a+b+c=2$. Without loss of generality, assume $b>c$, and use $\phi_{1,1,0}$.*

$$\left(\phi_{a,b,c}-\varepsilon\phi_{1,1,0}\right)((x_{ij}))=(1-\varepsilon)\phi_{\frac{a-\varepsilon}{1-\varepsilon},\frac{b-\varepsilon}{1-\varepsilon},\frac{c}{1-\varepsilon}}((x_{ij})).$$

[1] Decomposition of $\phi_{1,\frac{1}{2},\frac{1}{2}}$ is given by

$$\phi_{1,\frac{1}{2},\frac{1}{2}}((x_{ij}))=\begin{pmatrix}x_{11} & -\frac{x_{12}}{2} & -\frac{x_{13}}{2}\\ -\frac{x_{21}}{2} & x_{22} & -\frac{x_{23}}{2}\\ -\frac{x_{31}}{2} & -\frac{x_{32}}{2} & x_{33}\end{pmatrix}+\frac{1}{2}\begin{pmatrix}x_{22}+x_{33} & -x_{12} & -x_{13}\\ -x_{21} & x_{33}+x_{11} & -x_{23}\\ -x_{31} & -x_{32} & x_{11}+x_{22}\end{pmatrix}.$$

The first map $((x_{ij}))\mapsto\begin{pmatrix}x_{11} & -\frac{x_{12}}{2} & -\frac{x_{13}}{2}\\ -\frac{x_{21}}{2} & x_{22} & -\frac{x_{23}}{2}\\ -\frac{x_{31}}{2} & -\frac{x_{32}}{2} & x_{33}\end{pmatrix}$ is a CP map, where as the second map $((x_{ij}))\mapsto$ $\begin{pmatrix}x_{22}+x_{33} & -x_{12} & -x_{13}\\ -x_{21} & x_{33}+x_{11} & -x_{23}\\ -x_{31} & -x_{32} & x_{11}+x_{22}\end{pmatrix}$ is a well known CCP map known in literature as reduction map.

Notice that it is of the type of Kye map and satisfies the first two conditions of positivity. Again Choosing the point $\varepsilon = (b - c)$*, we see that the equality* $bc = \frac{(2-a)^2}{4}$ *holds which makes the resultant map decomposable. Hence the result.*

Case IV *Hence the most important part seems the case when* $0 < a < 1$ *and* $a+b+c = 2$*. We need to check the conditions for this case only, which seems not possible.*

For a single ε *the calculations are like the earlier ones. However, the third condition gives that* $b'c' \geq (1-a')^2$*, which implies* $(b-\varepsilon)c \geq (1-a)^2$*. However if we assume* $bc = (1-a)^2$ *from the beginning, as in Ha & Kye (2011), then this method does not work for any* ε*.*

A.3 Comparison between affine maps

In this section we use the methods of Section A.1 to produce examples of positive maps which corresponds to affine transformations. We use the techniques of Section A.2, we can identify stronger maps in terms of detecting entanglement. As a side result, we show that 'all' maps of example B.2.4 correspond to a certain rotation in $\mathbb{R}^8$.

We use this method to check the decomposability of some of the entanglement witnesses described by Chruściński & Kossakowski (2009). In this paper they discussed about the most generalised form of witnesses generated by rotation and affine transformation (also see Kimura & Kossakowski (2004); Kossakowski (2003)). Let $\tilde{\rho}$ be a strictly positive quantum state.

$$\tilde{\rho} = \tilde{\lambda}_1 P_1 + \cdots + \tilde{\lambda}_n P_n, \ \text{ where } \ \tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_n > 0;$$

and P_i's are the rank 1 projections corresponding to the eigenvectors. The above equation can be written as

$$\tilde{\rho} = \lambda_1 P_1 + \cdots + \lambda_{n-1} P_n + \lambda_n \frac{\mathbb{I}}{n};$$

where

$$\begin{aligned} \lambda_i &= \tilde{\lambda}_i - \tilde{\lambda}_n \qquad \text{for} \quad i = 1, \cdots, n-1 \\ \lambda_n &= n\tilde{\lambda}_n \end{aligned}$$

The basic idea is the following. There is a ball $B(\tilde{\rho}, r)$ centred at $\tilde{\rho}$ which can be inscribed in the space of density operators of $\mathcal{B}(\mathbb{C}^n)$. The maximal radius of the ball is $r_{\max} = \frac{\lambda_n}{\sqrt{n(n-1)}}$, using the notations as above. The first map defined is

$$\phi_\mu : A \longmapsto \mu A + (1-\mu)\mathrm{Tr}(A)\tilde{\rho}.$$

$\mathrm{Tr}(a)$ denote the trace of a matrix a, and μ is a real parameter.

Theorem A.3.1 (Chruściński & Kossakowski (2009))**.** *If μ satisfies $|\mu| \leq \mu_{\max}$; where*

$$\mu_{\max} = \frac{r_{\max}}{\sqrt{1+\lambda_1^2+\cdots+\lambda_{n-1}^2-\frac{\lambda_n^2}{n}}}$$

and r_{max} is defined as above; then ϕ_μ is a positive map.

It is obvious that if $\mu \in [0,1]$ then ϕ_μ is a completely positive map as it is a convex combination of two positive maps. It is less obvious that for any n, and any choice of completely positive state $\tilde{\rho}$, $r_{\max} \leq \frac{1}{n-1}$ To prove this, observe that $\lambda_i \geq 0$ for $i = 1,\cdots,n-1$. Also, the smallest eigenvalue of $\tilde{\rho}$, denoted by $\tilde{\lambda}_n$ can not be more than $\frac{1}{n}$, as all the eigenvalues are non zero and sum of them is 1. Thus we have $\lambda_n = n\tilde{\lambda}_n \leq 1$. Now putting it back in the above expression, we get the required relation. Moreover, this upper bound is attained when $\tilde{\rho} = \frac{\mathbb{I}_n}{n}$. Physically, it is the maximal radius of sphere which can be inscribed in the manifold of quantum states in dimension n, centred at the maximally mixed state.

Consider the case of $n = 3$, $\tilde{\rho} = \frac{\mathbb{I}_3}{3}$. We can see that in this case $|\mu_{\max}| = \frac{1}{2}$. As discusses above, for $\mu \in [0, \frac{1}{2}]$, the map ϕ_μ is a completely positive map. We want to consider the case when $\mu \in [-\frac{1}{2}, 0]$. Set $\mu = -s$ for $0 \leq s \leq \frac{1}{2}$. Thus we rewrite the above map as

$$\phi_s : A \longmapsto (1+s)\mathrm{Tr}(A)\frac{\mathbb{I}_3}{3} - sA = \frac{1+s}{3}\left(\mathrm{Tr}(A)\mathbb{I}_3 - \frac{3s}{1+s}A\right).$$

Notice that, when $s = -\mu_{\max} = \frac{1}{2}$, the above map becomes $\phi_\mu(a) = \frac{1}{2}(\mathrm{Tr}(a)\mathbb{I}_3 - a)$, which is known in literature as reduction map (or criteria).

We define a scaled map $\Phi_s = \frac{3}{1+s}\phi_s$. Then for any arbitrary non negative definite Hermitian operator A, we have

$$(\Phi_s - \Phi_{\frac{1}{2}})A = \left(1 - \frac{3s}{1+s}\right)A.$$

Since $s \in [0, \frac{1}{2}]$, we have the coefficient of A in the right hand side as a positive quantity. Hence the map $\Phi_s - \Phi_{\frac{1}{2}}$ is a completely positive map. Using the notation and procedures as used earlier, we get that for any arbitrary $|x\rangle$, and any fixed density operator ρ in suitable dimension

$$\langle x|I \otimes \Phi_s \rho|x\rangle \geq \left\langle x \left| I \otimes \Phi_{\frac{1}{2}} \rho \right| x \right\rangle .$$

Replacing the value of Φ_s and using the notation $\langle x|I \otimes \Phi_s \rho|x\rangle = \alpha$ and $\langle x|I \otimes \Phi_{\frac{1}{2}} \rho|x\rangle = \beta$, we get $\frac{3}{1+s}\alpha \geq 2\beta$, i.e. $\beta \leq \frac{3}{2(1+s)}\alpha$. Since $0 \leq s \leq \frac{1}{2}$, we have $\frac{3}{2(1+s)} > 0$. Hence we can conclude that if $\alpha < 0$, the value of $\beta < 0$, but not the converse. In other words, for any arbitrary state ρ, by applying $I \otimes \phi_s$ for $0 \leq s \leq \frac{1}{s}$, if we get any negative eigen value λ with the eigenvector $\lambda\rangle$; the quantity $\langle \lambda|I \otimes \phi_{\frac{1}{2}} \rho|\lambda\rangle$ will be forced to be less than 0. However the converse does not hold, i.e. if $\langle x|I \otimes \phi_{\frac{1}{2}} \rho|x\rangle < 0$ for any $|x\rangle$, that does not imply that $\langle x|I \otimes \phi_s \rho|x\rangle < 0$.

It is shown in the literature that the reduction map denoted here as $\phi_{\frac{1}{2}}$ here, is completely co-positive and decomposable. Thus it is not stronger than the partial transpose (see Bengtsson & Życzkowski (2006)). The above calculations show that the map ϕ_μ where $\mu \in (-\frac{1}{2}, 0]$ is weaker than $\phi_{\frac{1}{2}}$ which is the reduction map. Combining these two results we get

Lemma A.3.1. *For $n = 3$ and $\tilde{\rho} = \frac{\mathbb{I}_3}{3}$, none of the positive but not completely positive maps ϕ_μ defined as above are stronger than partial transpose.*

We can generalise it in higher dimensional too. In that case we have to define Φ_s as

$$\Phi_s = \frac{1+s}{n}\phi_s.$$

Remaining part follows exactly as above for $0 \leq s \leq \frac{1}{n-1}$. Hence we can summarise as:-

Lemma A.3.2. *For any arbitrary n and $\tilde{\rho} = \frac{\mathbb{I}_n}{n}$, none of the positive but not completely positive maps ϕ_μ where $|\mu| \leq \frac{1}{n-1}$, defined as above are stronger than partial transpose.*

Instead of maximally mixed state, we can now consider for diagonalised states $\tilde{\rho} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$ where $\lambda_i > 0$ for all i. In this case $\mu_{\max}$ is not known for any arbitrary choice of $\tilde{\rho}$. As above, we substitute $-\mu_{\max} = s$. Since we are not interested in the positive side of the combination, we have $0 \leq s \leq s_{\max} = \mu_{\max} \leq \frac{1}{n-1}$. Define

$$\Phi_s = \frac{1}{1+s}\phi_s : a \longmapsto \left(\mathrm{Tr}(a)\tilde{\rho} - \frac{s}{1+s}a \right).$$

Doing the usual calculations for any arbitrary a we find that

$$(\Phi_s - \Phi_{s_{\max}})a = \left(\frac{s_{\max} - s}{(1+s_{\max})(1+s)}\right) a.$$

Since $0 \leq s \leq s_{\max} \leq \frac{1}{n-1}$, we have the map $(\Phi_s - \Phi_{s_{\max}})$ is completely positive. Thus we can repeat the above calculations and arrive at a similar conclusion as above. Notice that, instead of diagonal states we could take any arbitrary strictly positive state, and the calculations remains unaffected. Hence we can conclude

Theorem A.3.2. *Let $\tilde{\rho}$ be a strictly positive state with only nonzero entries are the diagonals. Consider the map*

$$\phi_\mu : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$$

defined as

$$a \longmapsto \mu a + (1-\mu)\mathrm{Tr}(a)\tilde{\rho},$$

where $|\mu| \leq \mu_{\max}$ defined as earlier in theorem A.3.1. These maps are positive maps, and they are not completely positive when $-\mu_{\max} \leq \mu < 0$. Moreover the maps ϕ_μ for $-\mu_{\max} < \mu < 0$ are weaker than $\phi_{-\mu_{\max}}$, in the sense that any entangled state detected by ϕ_μ can be detected by applying $\phi_{-\mu_{\max}}$, but not the converse.

To check the effectiveness of the above map, we check the case when for $n = 3$, and $\tilde{\rho} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$, with $\lambda_i > 0$ for $i = 1, 2, 3$ and $\lambda_1 + \lambda_2 + \lambda_3 = 1$. The general case for diagonal matrices follows the same proof pattern. By the earlier theorem, it is sufficient to check for the map

$$\phi_{-\mu_{\max}} : a \mapsto -\mu_{\max} a + (1+\mu_{\max})\mathrm{Tr}(a)\tilde{\rho} = (1+\mu_{\max})\left(\mathrm{Tr}(a)\tilde{\rho} - \frac{\mu_{\max}}{1+\mu_{\max}} a\right).$$

Denote $\frac{\mu_{\max}}{1+\mu_{\max}}$ by ε. The following inequality can be checked easily.

Proposition A.3.1. *$\lambda_i \geq \varepsilon$, for all $i = 1, 2, 3$.*

Proof. Without loss of generality we may assume $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$. Then, it is sufficient to prove that $\lambda_3 \geq \varepsilon$.

$$\mu_{\max} = \frac{\frac{3\lambda_3}{\sqrt{6}}}{\sqrt{\lambda_1^2 + \lambda_2^2 + (1-\lambda_3)^2}}.$$

Now by simple calculations and using the relations between λ_i's,the proposed inequality boils down to the identity

$$(\lambda_1 - \lambda_2)^2 \geq 0;$$

which validates the correctness of the above inequality. □

In general, if we have $\lambda_1 \geq \cdots \geq \lambda_n > 0$, with $\sum \lambda_i = 1$, the above inequality boils down to verifying the inequality

$$(n-1)\sum_{\substack{i=1\\ i\neq j}}^{n-1} \lambda_i^2 \geq 2\sum_{\substack{i=1\\ i\neq j}}^{n-1} \lambda_i \lambda_j.$$

This directly follows from the standard arithmetic mean and geometric mean inequality upon λ_i^2 and λ_j^2, for $1 \leq i, j \leq n-1$, and $i \neq j$.Thus we can extend the above proposition as:

Proposition A.3.2. *$\lambda_i \geq \varepsilon$, for all $i = 1, \cdots, n$.*

Given the map for dimension 3, in the above form, the corresponding Choi matrix will be of the form

$$C = \left(\begin{array}{ccc|ccc|ccc}
\lambda_1-\varepsilon & 0 & 0 & 0 & -\varepsilon & 0 & 0 & 0 & -\varepsilon \\
0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\
-\varepsilon & 0 & 0 & 0 & \lambda_2-\varepsilon & 0 & 0 & 0 & -\varepsilon \\
0 & 0 & 0 & 0 & 0 & \lambda_3 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \lambda_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 \\
-\varepsilon & 0 & 0 & 0 & -\varepsilon & 0 & 0 & 0 & \lambda_3-\varepsilon
\end{array}\right),$$

without considering the normalisation factor, which is just a scalar multiple. This matrix can be positive or negative, depending upon the choice of λ_i's. In particular, if we choose $\tilde{\rho}$ as the maximally mixed state, we get back the reduction map, which is P, but cot CP, and in fact co-CP. Thus its power is less than or equal to the power of partial transpose. In other words, it can not detect PPT entangled states. To check, whether it is CcP map, we take C^{T_2}, which is the partial transpose on the second system of the Choi operator. Notice that, under row-column transformation, eigenvalues of any matrix remains invariant, as the resultant matrix is a similar matrix. Using row-column transformation on the C^{T_2}, we get the following matrix

$$\left(\begin{array}{ccc|cccccc}
\lambda_1-\varepsilon & & & & & & & & \\
 & \lambda_2-\varepsilon & & & & & & & \\
 & & \lambda_3-\varepsilon & & & & & & \\
\hline
 & & & \lambda_1 & -\varepsilon & & & & \\
 & & & -\varepsilon & \lambda_2 & & & & \\
 & & & & & \lambda_2 & -\varepsilon & & \\
 & & & & & -\varepsilon & \lambda_3 & & \\
 & & & & & & & \lambda_3 & -\varepsilon \\
 & & & & & & & -\varepsilon & \lambda_1
\end{array}\right).$$

Since $\lambda_i \geq \varepsilon$ by our earlier proposition, we have $\lambda_i - \varepsilon \geq 0$. Hence the positivity of the matrix is determined by the the matrix of the lower right block. Since all entries are real, the matrix is Hermitian. A Hermitian matrix is positive if and only if all its principal minors are positive. In this case, it is positive if and only if each block matrix $\begin{pmatrix} \lambda_i & -\varepsilon \\ -\varepsilon & \lambda_j \end{pmatrix}$ is positive for each pair $(i, j),\ i \neq j$. Calculating the eigenvalues, we can see that they are positive if and only if $\lambda_i\lambda_j \geq \varepsilon^2$ for all $i,\ j,\ i \neq j$. But by the earlier proposition(s), this is true for all choice of i and j. Thus C^{T_2} is always positive as long we choose $\tilde{\rho}$ as diagonal matrix. The general case for any dimension also follows the same line of argument. Hence we conclude that:

Theorem A.3.3. *For any density matrix* $\tilde{\rho} = \begin{pmatrix} \lambda_1 & & \\ & \cdots & \\ & & \lambda_n \end{pmatrix}$*, with* $\lambda_i > 0$*, the corresponding maps* ϕ_μ*, as defined earlier, are either CP or co-CP.*

We can now handle the general case also, instead of a diagonal matrices.

Theorem A.3.4. *For any strictly positive density matrix* $\tilde{\rho}$ *the corresponding maps* ϕ_μ*, as defined earlier are CP or co-CP.*

Proof. First notice that, for any fixed unitary operator U, the map $A \mapsto UAU^\dagger$ is a bijection on $M_n(\mathbb{C})$. Since $\tilde{\rho}$ is Hermitian, there exists unitary operator U such that $U\tilde{\rho}U^\dagger$ is a diagonal matrix.

Now the maps

$$\phi_\mu : A \to \mu A + (1-\mu)\mathrm{Tr}(A)\tilde{\rho},$$

and

$$\phi_{\mu,U} : A \to \mu U A U^\dagger + (1-\mu)\mathrm{Tr}(A) U \tilde{\rho} U^\dagger,$$

are local unitary equivalent, $\phi_{\mu,U}(A) = U\phi(A)U^\dagger$ for all $A \in M_n(\mathbb{C})$. Hence the set of entangled states detected by them are the same. But the second map $\phi_{\mu,U}$ deals with a diagonal matrix $U\tilde{\rho}U^\dagger$. Hence by the previous theorem $\phi_{\mu,U}$ is either CP or co-CP. Hence so is ϕ_μ. $\square$

A.4 Conclusions

In this appendix, we have studied the geometric structures of positive maps. We have reviewed the matrix representations of maps and certain cases which guarantee the map to be positive.

We have also developed a partial ordering in the set of positive maps. A modified version of this ordering gives rise to a simple checking criteria for the comparative entanglement detection power of maps. We have used it on some known classes of positive maps. We have identified subclasses of maps which can detect more entangled states than other maps of the bigger class. In certain cases, we have identified exactly the classes which are known to be extremal. Though this method can not identify extremal maps directly, it can produce the subclasses of maps which are better candidates for being extremal.

We have studied some of the non unital positive maps. We have considered the power of such maps for detecting entanglement. Using the inner automorphism concept of Chapter 4 we show that shifting the origin does not give us any advantage, as the maps produced are all completely positive (hence cannot detect entanglement) or decomposable (which cannot detect PPT entangled states).

A. Families of positive maps which are not completely positive and their power to detect entanglement

Appendix B

Positive maps

The concept of operators (and their finite dimensional avatar - matrices) was well known in the mathematics community by the time of discovery of quantum mechanics. The major use of matrices and operators was in the theory of solving linear equations and differential equations. Matrix groups also frequently appeared in the abstract theory of groups and its actions of spaces. Heisenberg pioneered the development of quantum mechanics as a subclass of (infinite) matrix theory and named it as Matrix Mechanics. This mathematical scheme was immediately noticed and carried forward by Weyl (1977), von Neumann (1996) and Mackey (1963). [1] These works also revealed the intriguing connection of quantum mechanics with functional analysis and helped mutual development. This is evident in the more advanced (and recent) works in quantum mechanics like Gustafson & Sigal (2003), Takhtajan (2008), Strocchi (2005), Teschl (2009); more advanced uses - Thirring (1979) in the atomic systems, Prugovečki (1971) on Hilbert space operators, and in quantum statistical mechanics by Bratteli & Robinson (1987, 1997). Last but not the least, we must mention the monumental treatises of Reed & Simon (1975, 1978, 1979, 1980).

Another important reference is a recent book by Størmer Størmer (2013) dealing the same subject with more detail. We mention the key points of this area, and spend more time on the areas least covered in the above mentioned books. In particular,

1. we give a list of available positive maps which are not completely positive,

2. some detailed discussions regarding the exposed maps.

[1] To cast the quantum mechanics foundations by a set of axioms was an Hilbertian goal for mathematicians, in particular von Neumann.

Functional analysis can be understood as a study of vector spaces under norms and the functionals on them. It started in the last decade of nineteenth century and carried forward in twentieth century by Hilbert, Riesz and the Polish group (Banach, Sz-Nagy, Foias, Stanisław Mazur), in USA by Barry Mazur, von Neumann, Halmos, Segal etc., and in USSR by a renowned subgroup of researchers under Kolmogorov, Gelfand, Naimark etc. It supplied tools and methods to describe newly developed quantum mechanics. In doing so, it also developed new areas, which are now known as C^* algebra, von Neumann algebra, spectral theory and perturbation theory of linear operators. Without going to this beautiful historical development we introduce and give an overview of the relevant areas necessary for the problem stated in the previous chapter.

B.1 Positivity and C^* algebra

The primary materials of the subject are covered in many excellent text books like Sunder (1997), Arveson (1976), Davidson (1996) and advanced references: like Takesaki (2002), Kadison & Ringrose (1997). We give a series of basic definitions to mantain the continuity before coming to the

B.1.1 C^* algebra

Definition B.1.1 (Normed algebra)**.** *Let $\mathcal{A}$ be a normed space (on $\mathbb{C}$). It is called as a normed algebra if there is a well defined multiplication structure, i.e. there is a well defined map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, denoted by $(x, y) \mapsto xy$, satisfying the following conditions for all $x, y, z \in \mathcal{A}$ and for all $\alpha \in \mathbb{C}$*

1. *(associativity) $(xy)z = x(yz)$,*
2. *(distributivity) $(\alpha x + y)z = \alpha xz + yz,\ z(\alpha x + y) = \alpha zx + zy$,*
3. *(sub-multiplicativity of norm) $\|xy\| \leq \|x\|.\|y\|$.*

Definition B.1.2 (Banach algebra)**.** *A Banach algebra is a normed algebra which is complete as a normed space. A normed (or Banach) algebra is said to be unital if it has a multiplicative identity - i.e., if there exists an element, which we shall denote simply by 1, such that $1x = x1 = x$, for all x.*

A mapping $\mathcal{A} \ni x \mapsto x^* \in \mathcal{A}$ is said to be *involution*, or adjoint operation of the algebra $\mathcal{A}$ if it satisfies the following conditions:

1. (conjugate linearity) $(\alpha x + y)^* = \overline{\alpha} x^* = y^*$,

2. (product reversal) $(xy)^* = y^* x^*$,

3. (order two) $x^{**} = x$.

A Banach algebra with involution is called a *Banach $*$-algebra*.

Remark B.1.1. *Adjoint of a square matrix, which is by definition transpose complex conjugate, is an example of involution. This is generally referred in mathematical texts as $*$ and in physics texts as $\dagger$. In the main text we follow the latter symbol.*

Definition B.1.3 (C^* algebra)**.** *A C^* algebra is a Banach $*$- algebra $\mathcal{A}$, with the property*

$$\|x^* x\| = \|x\|^2. \tag{B.1}$$

The above identity is also called the C^* identity.

One of the most important theorems in C^* algebra theory is the following

Theorem B.1.1. *For any C^* algebra $\mathcal{A}$, there exists a isometric representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

If $\mathcal{A}$ is separable, then the Hilbert space $\mathcal{H}$ can be chosen to be separable as well.

We omit the proof of the theorem, as it will not be used in the text. For further proof, see Sunder (1997) or Davidson (1996).

Remark B.1.2. *In some of the references (ex: Higson & Roe (2000)) the above theorem is considered as a definition of C^* algebra and called as* concrete C^* algebra. More precisely, a concrete C^* algebra is a Banach $*$-algebra which is isometrically $*$-isomorphic to a norm closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. The algebra given in definition B.1.3 is called *abstract C^*algebra.*

For quantum systems, the space $\mathcal{H}$ is some separable Hilbert space (of finite or infinite dimension), and the C^* algebra representation is isometric isomorphic with it. In other words, the C^* algebra *is* $\mathcal{B}(\mathcal{H})$. For more details, see Varadarajan (1985) or Parthasarathy (1992, 2005).

B.1.2 Positive elements

Positivity is one of the fundamental concepts in the present mathematical research. Based on various approaches stated above, there are different ways to define positivity, and they are all equivalent. In this subsection we show that, the set of positive elements of a C^* algebra $\mathcal{A}$ forms a cone.

Definition B.1.4. *Let $\mathcal{A}$ is a unital Banach algebra and $x \in \mathcal{A}$. The spectrum of x is the defined as the set*

$$\sigma(x) = \{\lambda| : (x - \lambda) \textit{is not invertible}\},$$

and the spectral radius of x is the defined as

$$r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

Definition B.1.5 (Positive element)**.** *An element x in a C^* algebra $\mathcal{A}$ is said to be positive (and written as $x \geq 0$) if there is a self adjoint element $y \in \mathcal{A}$ such that $x = y^2$.*

Lemma B.1.1. *Let $\mathcal{A}$ be a C^* algebra. $\mathcal{A} \ni x, y \geq 0$, implies $(x + y) \geq 0$.*

Proof. Without loss of generality we may assume $\mathcal{A}$ to be unital (by embedding it in a larger algebra with unity). Moreover we can also assume that $\|x\|,\ \|y\| \leq 1$ (by a suitable scaling). Then $r(x) \leq 1$ and $\sigma(x) \subseteq [0, 1]$. Hence $\sigma(1 - x) \subseteq [0, 1]$ and $(1 - x) \geq 0$ and $\|(1 - x)\| = r(1 - x) \leq 1$. Similarly $\|(1 - y)\| \leq 1$. Then $\|1 - \frac{x+y}{2}\| = \frac{1}{2}\|(1 - x) + (1 - y)\| \leq 1$. Since $\frac{x+y}{2}$ is self adjoint $\sigma(\frac{x+y}{2}) \subseteq [0, 2]$ i.e.$\sigma(x + y) \subseteq [0, 4]$, hence the proof. □

Proposition B.1.1. *For any element $z \in \mathcal{A}$, $z^*z \leq 0$ implies $z = 0$*

Proof. It can be easily seen from the definition that for any $x, y \in \mathcal{A}$, $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$. Using this we have $z^*z + zz^* \leq 0$. If $z = u + \imath v$ be the Cartesian decomposition. Then $0 \leq z^*z + zz^* = 2(u^2 + v^2) \geq 0$. This gives $z^*z + zz^* = 2(u^2 + v^2) = 0$. Hence the result. □

Lemma B.1.2. *In a C^* algebra $\mathcal{A}$ any element x is positive if and only if there exists $z \in \mathcal{A}$ such that $x = z^*z$.*

Proof. If x is positive set $z = x^{\frac{1}{2}}$.

For the converse, note that any self adjoint element x can be uniquely decomposed as $x = x_+ - x_-$, where x_+ and x_- are positive elements of $\mathcal{A}$ such that $x_+x_- = 0$. Hence we have $x_-xx_- = -x_-^3 \leq 0$. But $x_-xx_- = (zx_-)^*(zx_-)$. Hence by the previous proposition $x_-^3 = 0$ and $z^*z = x = x_+$. □

B.1.3 With operators

In the recent years it became clear to the operator algebraists that the structure of C^* algebra are too much restrictive. Operator algebraists relaxed conditions of C^* algebra and discovered objects called operator system and operator spaces.

Definition B.1.6 (Operator system)**.** *A $*$-closed subset of a unital C^* algebra $\mathcal{A}$ is said to be a operator system.*

Though the concept of operator system was already known (though not by the present name) since the seminal paper of Arveson (1969), it was subsided by courageous development of operator space in the late eighties pioneered by Ruan (1988).

Definition B.1.7 (Operator space)**.** *A closed subspace $E \subset \mathcal{B}(\mathcal{H})$ is called an operator space.*

Because of the simplicity of definition, research on operator spaces became more prominent. In recent years, the theory of operator systems made a comeback due to works of Johnston & Størmer (2012); Johnston *et al.* (2009, 2011, 2013); Paulsen *et al.* (2011). One of the reasons for its revival is the recent advancement of quantum information theory, entanglement and positive maps. Indeed the operator systems are conceptually closer to quantum systems, as to study the quantum systems, we need to study the states, which are positive semidefinite self adjoint operators.

The usefulness of the above notions will become clear in the next section when we study the positive maps between them.

B.2 Positive map

The structure of operator space and the study of positive maps between them is an important topic for various reasons.

Definition B.2.1 (Positive map)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ are two C^* algebras. A map $\phi : \mathcal{A} \to \mathcal{B}$ is said to be positive if for all $a \in \mathcal{A}$, $\phi(aa^*) \geq 0$.*

Definition B.2.2 (k-positive and Completely positive map)**.** *A positive map ϕ is said to be k-positive if the natural extension*

$$\begin{aligned} \phi_k &: M_k(\mathcal{A}) \to M_k(\mathcal{B}) \\ \phi_k((a_{i,j})) &\mapsto ((\phi(a_{i,j}))). \end{aligned}$$

A map is said to be completely positive if it is k-positive for all k.

The adverb 'completely' means all members of the sequence $\{\phi_n\}$ satisfies the same property. In a similar way, we can define *complete contraction* and *complete boundedness* when the original map ϕ is contractive or bounded, respectively.

The above definitions can be easily be converted to the more general operator space settings. For details see Paulsen (2002) or Pisier (2003).

The set of all completely bounded maps carry a natural norm associated with it. This norm is defined as

$$\|\phi\|_{cb} = \sup_k \{\|\phi_k : M_k(\mathcal{A}) \to M_k(\mathcal{B})\|\} < \infty,$$

and called as *cb norm*. Set of all completely bounded maps between $\mathcal{A}$ and $\mathcal{B}$ along with the cb norm form a Banach space.

B.2.1 Kraus representation

It turns out that the completely positive maps are the allowed maps between two quantum systems. Fortunately its structure is known due to Sudarshan *et al.* (1961), Kraus (1971), Choi (1975a) (for the finite dimensional cases). The more general case (i.e. when $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$) is given by Stinespring.

Theorem B.2.1 (Stinespring's dialation Stinespring (1955))**.** *Let $\mathcal{A}$ be a unital C^*-algebra and ϕ is a completely positive map.*

$$\begin{array}{ccc} \mathcal{B}(\mathcal{H}_1) & & \mathcal{B}(\mathcal{H}_2) \\ \uparrow & & \uparrow \\ \mathcal{A} & \xrightarrow{\phi} & \mathcal{B} \end{array}$$

Then there exists a Hilbert space $\mathcal{K}$, a unital $*$-homomorphism

$$\pi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{K}),$$

and operator $V : \mathcal{H} \to \mathcal{K}$ *with* $\|\phi(1)\|^2 = \|V\|^2 = \|V^*V\|$ *such that*

$$\phi(a) = V^*\pi(a)V, \quad \forall a \in \mathcal{A}.$$

Moreover ϕ *admits a completely positive extension* $\tilde{\phi}$

$$\begin{array}{ccc} \mathcal{B}(\mathcal{H}_1) & \xrightarrow{\tilde{\phi}} & \mathcal{B}(\mathcal{H}_2) \\ \uparrow & & \uparrow \\ \mathcal{A} & \xrightarrow{\phi} & \mathcal{B} \end{array}$$

such that $\|\tilde{\phi}\|_{cb} = \|\phi\|_{cb}$.

Theorem B.2.2 (Choi (1975a); Kraus (1971); Sudarshan *et al.* (1961))**.** *Let* $\phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ *be a unital completely positive map. The there exists a set of operators* $\{V_j : \mathcal{H}_2 \to \mathcal{H}_1\}$ *such that*

$$\phi(X) = \sum_j V_j^* X V_j, \tag{B.2}$$

where $\sum_j V_j^* V_j = I_{\mathcal{H}_2}$ *is strongly convergent sum.*

In our work, the Hilbert spaces are finite-dimensional. Hence the C^* structure is of a matrix algebra $\mathcal{B}(\mathbb{C}^n) = M_n$, where $n < \infty$ is the dimension. The following work assumes that finite dimensionality, unless stated otherwise.

Since $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional, we are dealing with complex matrix algebra. Hence the above Kraus form B.2 is consists of only finitely many operators. In fact, using Choi's theorem B.2.3 one can show that the maximal number of such operators required to represent any completely positive map is $\dim(\mathcal{H}_1)\dim(\mathcal{H}_2)$.

B.2.2 Positive maps which are not CP

The completely positive maps are the highest forms of positivity possible between operator systems.[1] Choi used an easy way to check the completely positivity of any given positive map. Let M_n be the complex matrix algebra of dimension $n < \infty$.

Theorem B.2.3. *Let* $\mathcal{A}$ *be a* C^* *algebra with unit. Let* $\mathcal{S}$ *be an operator system in* $\mathcal{A}$ *and* $\varphi : \mathcal{S} \to M_n$*. The following things are equivalent:*

1. φ *is completely positive.*

[1] Most of these theorems are true for operator space set up as well with possible modifications.

2. *φ is n-positive.*

3. **Choi - Jamiołkowski isomorphism** *Choi (1975a); Jamiołkowski (1972). There is a unique way to associate any map $\varphi : \mathcal{S} \to M_n$ with an element of $M_n(\mathcal{S})$.*

$$\begin{aligned} \mathcal{B}(\mathcal{S}, M_n) &\longrightarrow M_n(\mathcal{S}) \\ \varphi \mapsto (\mathbb{1}_n \otimes \varphi)\left(\sum_{i,j=1}^{n} E_{ij} \otimes E_{ij}\right) &= \sum_{i,j=1}^{n} E_{ij} \otimes \varphi(E_{ij}) \end{aligned} \tag{B.3}$$

where $E_{ij} = |i\rangle\langle j|$, written in the standard basis. φ is completely positive if and only if the corresponding Matrix in right hand side is positive semi-definite.

The matrix given in equation B.3 is known in literature as Choi matrix and is the most useful tool to decide the complete positivity of a given map. The inverse of the above isomorphism is also useful. Given a (Hermitian) operator $C \in M_n(\mathcal{S})$, the corresponding map ϕ_C is given by

$$\phi_C(X) = Tr_1\left(X^T \otimes I \cdot C\right), \tag{B.4}$$

where I is the identity element of M_n and T denotes the transpose operation. Tr_1 denotes the partial trace with respect to the first system, which is the operator system $\mathcal{S}$. Use of the above theorem will be shown in the subsection B.2.3.

If a map φ is positive but not completely positive, there is a least k such that the canonical extension of the map $\mathbb{1}_k \otimes \varphi$ (described above) is not positive. Notice that for any positive semidefinite $A \in M_k$ and $B \in \mathcal{S}$, $\mathbb{1}_n \otimes \varphi(A \otimes B) = A \otimes \varphi(B) \geq 0$. Real positive linear combination of positive operators is positive. Hence image of any positive operator which can be written as a positive linear combination of tensor product of positive operators (like $A \otimes B$) is also positive. Since the map $\mathbb{1}_k \otimes \varphi$ is not positive, there exists positive operators which can not be expressed as positive linear combinations of tensor product of positive operators. Hence positive maps which are not completely positive can be used to detect such positive operators. This clearly shows that positive but not completely positive maps can be used to detect entanglement of a system. In the next subsection B.2.3, we give the formal statements of the relevant results in this direction.

B.2.3 Entanglement and positive maps

The theorem B.2.4 connects the witness with the positive maps which are not completely positive Horodecki *et al.* (1996).

Theorem B.2.4. *A state $\rho \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is separable if and only if for any positive map $\phi : \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$, the operator $(\mathbb{1} \otimes \phi)\rho$ is positive.*

Suppose the map ϕ is positive. Then $\phi(\sigma) \geq 0$ for any state σ. The extended map $\mathbb{1} \otimes \phi$ sends the set of separable states to itself, as for any separable state $\sigma_1 \otimes \sigma_2$, the map $\mathbb{1} \otimes \phi(\sigma_1 \otimes \sigma_2) = \sigma_1 \otimes \phi(\sigma_2) \geq 0$. Hence it is true for any convex combination of separable states. However the map $\mathbb{1} \otimes \phi$ is not positive. So there exists non-separable and hence entangled, states ρ such that $\mathbb{1} \otimes \phi(\rho) \ngeq 0$. Thus, these maps are very important to detect entanglement.

B.2.4 List of examples

The structure of positive maps is closely related with the structure of quantum states. The structure for both of them are not well known. Though the structure of the completely positive maps are well understood, the general structure of positive maps has turned out to be very complicated. In the absence of a general structure, discovering particular examples of positive maps became an important branch of study. In fact, there are only few examples of such maps available in literature. We give a list of some well known maps. This list is not exhaustive.

B.2.4.1 Decomposable maps

Example B.2.1. *The simplest possible map is transpose $X \mapsto X^T$. This map is 1-positive not 2-positive.*

Definition B.2.3 (Decomposability)**.** *A positive map ϕ is said to be decomposable if it can be written as $\phi = \psi_1 + \psi_2 \circ T$, where ψ_1 and ψ_2 are completely positive maps and T is the transpose operation.*

A positive but not completely positive map, which can not be written in the above form is called a indecomposable map.

If a transpose of a map is completely positive, then the map is called completely co-positive (CCP). Thus these maps are of the form $T\circ$-some CP map. These are a subclass of decomposable maps.

Example B.2.2 (Reduction map)**.** *$M_n \ni X \mapsto \frac{1}{n-1}\left(Tr(X)I_n - X\right) \in M_n$, where I_n is the identity matrix, and $Tr(X)$ is the trace of X . This map is decomposable, and actually CCP. Hence it is unitarily equivalent to transpose. This map, surprisingly is* *stronger Hiroshima (2003) than majorisation criteria Nielsen & Kempe (2001). Moreover this map can be extended to an indecomposable map for all even dimensions more than* 2 *as shown in the example B.2.10.*

B.2.4.2 Indecomposable maps

The structure of all indecomposable maps is not known. In absence of a structure theory, different examples of such maps are discovered. To determine whether a map is CCP is rather easy, as it needs to be checked whether the map combined with transpose is completely positive or not. It should be noted that, though there are many known examples of positive maps, only a few of them are proved to be indecomposable.

There are two ways of proving in-decomposability. The first is by showing that there exists an entangled state which is positive under partial transpose, and whose entanglement is detected by the given map. In absence of the structure of the set of states, there is no systematic way to generate such examples. The second method is by using positive semidefinite bi-quadratic forms and is also not easy to determine.

A list of positive maps which are proved to be indecomposable, is given below.

Example B.2.3 (Choi Choi (1975b); Choi & Lam (1977/78))**.** *This is the first example of such maps. Choi showed that the map*

$$\mathcal{B}(\mathbb{C}^n) \ni ((x_{i,j})) \mapsto \frac{1}{2}\begin{pmatrix} x_{11}+\mu x_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{22}+\mu x_{11} & -x_{23} \\ -x_{31} & -x_{32} & x_{33}+\mu x_{22} \end{pmatrix}, \quad \textit{where } \mu \geq 1; \tag{B.5}$$

is positive, 2*-positive, but not* 3*-positive (hence not CP), and indecomposable. A PPT entangled state which can be determined by this map is given below.*

$$\rho(a) = \frac{1}{3\left(1+a+\frac{1}{a}\right)}\left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & a & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{a} & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 0 & \frac{1}{a} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{a} & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right). \tag{B.6}$$

It was shown that the state is positive under partial transpose (PPT), and entangled for $a > 0$. The above map detects its entanglement for all but $a = 1$. This example is due to Simon et al. *(2006); Simon* et al. *(2009). This can be also seen as a modified version of the example constructed by Størmer (1982).*

There are several modified versions of Choi's map. A few of them are proved to be indecomposable. Two most important versions are given below.

Example B.2.4 (Cho *et al.* (1992))**.**

$$((x_{i,j})) \mapsto \frac{1}{a+b+c}\begin{pmatrix} ax_{11}+bx_{22}+cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & ax_{22}+bx_{33}+cx_{11} & -x_{23} \\ -x_{31} & -x_{32} & ax_{33}+bx_{11}+cx_{22} \end{pmatrix}; \tag{B.7}$$

where a, b, c are positive real parameters. The map is positive and indecomposable if the following conditions are satisfied:

1. $0 \leq a < 2$.
2. $a + b + c \geq 2$.
3. $\begin{cases} (1-a)^2 \leq bc < \frac{(2-a)^2}{4}, & \text{if} \quad 0 \leq a \leq 1 \\ 0 \leq bc < \frac{(2-a)^2}{4}, & \text{if} \quad 1 \leq a < 2. \end{cases}$

For all $a \geq 2$ the above map is CP. The third condition gives the indecomposability criteria. For $(a, b, c) = (1, 0, 1)$, the map reduces to Choi's map.

Example B.2.5 (Kye (1992))**.**

$$((x_{i,j})) \mapsto \begin{pmatrix} x_{11}+ax_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{22}+bx_{11} & -x_{23} \\ -x_{31} & -x_{32} & x_{33}+cx_{22} \end{pmatrix}; \tag{B.8}$$

where a, b, c are positive real parameters and $abc \geq 1$. Notice that this map reduces to Choi's map (without normalisation) for $a = b = c = 1$. Moreover for this special choice of (a, b, c) (and their positive scalar multiples i.e. $(\lambda, \lambda, \lambda)$ where $\lambda \geq 1$) only, the map can be made as unital.

Example B.2.6. *This example can be generalised in the higher dimensions, i.e. a positive indecomposable map from $\mathcal{B}(\mathbb{C}^n)$ to itself. Define ϕ_p by choosing $n+1$ parameters $\mathbf{p} = (p_0, p_1, \cdots, p_n)$.*

$$\begin{aligned} \phi_{\mathbf{p}}(E_{11}) &= p_0 E_{11} + p_n E_{dd}, \\ \phi_{\mathbf{p}}(E_{22}) &= p_0 E_{22} + p_1 E_{11}, \\ \vdots \quad &\vdots \quad \vdots \\ \phi_{\mathbf{p}}(E_{nn}) &= p_0 E_{nn} + p_{n-1} E_{d-1,d-1}, \\ \phi_{\mathbf{p}}(E_{ij}) &= -E_{ij}, \qquad i \neq j. \end{aligned}$$

It has been shown in Ha (2003) that if

1. $p_1, \cdots, p_n > 0$,
2. $n - 1 > p_0 \geq n - 2$,
3. $p_1 \cdot \cdots \cdot n_d \geq (n - 1 - p_0)^n$,

then $\phi_{\mathbf{p}}$ is a positive indecomposable map.

Example B.2.7. *A family of maps constructed from unextendible product basis Bennett* et al. *(1999) by Terhal (2001).*

Example B.2.8. *A discrete family $\tau_{n,k},\ k = 1, \cdots, n-2$ (Ha (1998)) is one such extension of Choi's map. Let s be a unitary shift defined by,*

$$se_i = e_{i+1} \qquad i = 1, \cdots, n;$$

(where $e_{n+1} \equiv e_1$). The maps $\tau_{n,k}$ are defined as follows

$$\tau_{n,k}(X) = (n-k)\epsilon(X) + \sum_{i=1}^{k} \epsilon(s^i X s^{*i}) - X, \tag{B.9}$$

where $\epsilon(X)$ is the projector onto the diagonal part, i.e.

$$\epsilon(X) = \sum_{i=1}^{d} Tr[XE_{ii}]E_{ii}. \tag{B.10}$$

The map $\tau_{n,0}$ is completely positive and the map corresponding to $k = n-1$ is completely co-positive Ha (1998). Note that $\tau_{n,k}(I_n) = (n-1)I_n$ and $\tau_{n,k}(X) = (n-1)Tr(X)$. Hence the normalised map

$$\Phi_{n,k}(X) = \frac{1}{n-1}\tau_{n,k}(X),$$

are bi-stochastic. In particular $\Phi_{3,1}$ is the Choi's map.

Example B.2.9. *The following example is given by Robertson (1983a,b,c, 1985). The Robertson map, $\phi_R : \mathcal{B}(\mathbb{C}^4) \to \mathcal{B}(\mathbb{C}^4)$ can be written as follows:*

$$\phi_R\left(\begin{array}{c|c} X_{11} & X_{12} \\ \hline X_{21} & X_{22} \end{array}\right) = \frac{1}{2}\left(\begin{array}{c|c} I_2Tr(X_{22}) & -X_{12} - R(X_{21}) \\ \hline -X_{21} - R(X_{12}) & I_2Tr(X_{11}) \end{array}\right), \tag{B.11}$$

where, $R : \mathcal{B}(\mathbb{C}^2) \to \mathcal{B}(\mathbb{C}^2)$ is a reduction map defined as

$$R(X) = I_2Tr(X) - X.$$

Example B.2.10 (Breuer (2006), Hall Hall (2006))**.** *This map is only for the even dimensions $2n$ where $n > 1$. This is given by,*

$$\varphi_{BH} : X \mapsto I_{2n}Tr(X) - X - U^{\dagger}X^{T}U; \tag{B.12}$$

where U is an antisymmetric unitary operation ($U^T = -U$). Such operations are only possible for even dimensions. Hence such maps are possible in the even dimensional spaces.

These shows that there are only few known examples of indecomposable positive maps available in literature. A few examples of the above are generalised in arbitrary dimensions by Chruściński and Kossakowski (Chruściński & Kossakowski (2007)).

Example B.2.11. *The pattern of Choi's map and its generalisations gives rise to a natural question. Does there exist any positive map which does not disturb the diagonals? Mathematically, such objects will be of the form*

$$\phi_{A,B} : X \mapsto A \circ X + B \circ X^T + I \circ X, \quad X \in M_n(\mathbb{C}), \tag{B.13}$$

where A and B are operators with zero diagonals, I is the identity operator and $X \circ Y$ denotes the Hadamard product (product of term by term). Kye (1995) had shown that for the dimension three any such map will be decomposable. But in the higher dimensions, there exist indecomposable positive maps which fix the diagonals Kim & Kye (1994).

B.2.5 Extremal maps

The set of all positive maps forms a closed convex set. By the celebrated Krain - Milman theorem any closed convex set can be expressed as a convex hull of its extremal points. Hence it is important to know the extremal points of this set of all positive maps. We call this set as the set of extremal maps. This study of extremal maps started with the works of Choi and Lam (Choi & Lam (1977/78)). Recently, Eom and Kye (Eom & Kye (2000)), Ha and Kye (Ha & Kye (2011); Ha & Kye (2012, 2013)), and Chruściński (Chruściński (2011)) made efforts to discover new examples of extreme positive maps. In this section, we mention some of the definitions and the results given in the above mentioned papers.

Definition B.2.4 (Extremal map)**.** *A positive map φ is said to be extremal, when for any decomposition $\varphi = \varphi_1 + \varphi_2$, where φ_1 and φ_2 are positive maps, $\varphi_i = \lambda_i \varphi$, where $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 = 1$.*

There are only few maps available in literature which are extremal. Choi's map given in B.5 corresponding to $\mu = 1$ is shown to be extremal by Choi and Lam Choi & Lam (1977/78).

An important subclass of the set of extreme points are called the exposed points. We give a few definitions.

Definition B.2.5. *Let A be a convex subset of a real vector space. Let $\ell : A \to \mathbb{R}$ be a non constant linear functional such that $\sup_{x \in A} \ell(x) = \alpha < \infty$. Then the set $F = \{y | \ell(y) = \alpha\}$ is a tangent hyperplane and is called an exposed set. If F consists of a single point, then that point is called an exposed point.*

Extreme points are the boundary points of the closed convex set but the converse is not true. Similarly the exposed points are extreme points but not the converse.

Example B.2.12.

$$A = \{(x, y) : -1 \leq x \leq 1,\ -2 \leq y \leq 0\} \cup \{(x, y) : x^2 + y^2 = 1\}$$

The curve above $y = -2$ is a differentiable curve, hence there is unique supporting hyperplane through each of the boundary points. But the support hyperplane through the extreme point $(1, 0)$ is $x = 1$. So $(1, 0)$ is not an exposed point though it is extreme point.

The major importance of such points is that for a given closed convex body any extreme point can be considered as a limit of a sequence of exposed points. Thus we can write the Krein-Milman theorem as

Theorem B.2.5 (Krein-Milman)**.** *Let A be a compact convex subset of a locally convex Banach space X. Then $A =$ closed convex hull of $Exp(A)$. $Exp(A)$ denotes the set of all exposed points of A.*

Let $\mathcal{P} = \{\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) : \phi(AA^\dagger) \geq 0 \; \forall A \in \mathcal{B}(\mathcal{H})\}$ be the cone of positive maps. The dual cone $\mathcal{P}^\circ$ is defined as

$$\mathcal{P}^\circ = \text{conv}\{|x\rangle \langle x| \otimes |y\rangle \langle y| : \langle y|\phi(|x\rangle \langle x|)|y\rangle, \; \phi \in \mathcal{P}, \; |x\rangle \in \mathcal{H}, \; |y\rangle \in \mathcal{K}\}.$$

A *face* $\mathcal{F}$ of $\mathcal{P}$ is a subset such that if $\phi \in \mathcal{F}$ and $\phi = \lambda\phi_1 + (1-\lambda)\phi_2$, where $0 \leq \lambda \leq 1$ and $\phi_1, \; \phi_2 \in \mathcal{P}$, then $\phi_1, \; \phi_2 \in \mathcal{F}$. A *ray* is the set $\{\lambda\phi : \lambda > 0\}$ generated by the map ϕ and is written as $[\phi]$. A ray is said to be extreme, if it is a 1-dimensional face of the set $\mathcal{P}$.

Definition B.2.6. *A face $\mathcal{F}$ is exposed if there exists a supporting hyperplane H for a convex cone $\mathcal{P}$ such that $\mathcal{F} = H \cap \mathcal{P}$.*

It can further be shown that for positive maps,

Proposition B.2.1. *A face $\mathcal{F}$ is exposed if there exists $|x\rangle \otimes |y\rangle \in \mathcal{H} \otimes \mathcal{K}$ such that*

$$\mathcal{F} = \{\phi \in \mathcal{P} : \langle y|\phi(|x\rangle\langle x|)|y\rangle = 0\}.$$

The dual face of $\mathcal{F}$ can be defined in the similar way. If $\mathcal{F}$ is exposed, then the dual face $\mathcal{F}'$, a subset of $\mathcal{P}^\circ$, is defined as

$$\mathcal{F}' = \text{conv}\{|x\rangle \langle x| \otimes |y\rangle \langle y| : \langle y|\phi(|x\rangle\langle x|)|y\rangle = 0, \; \phi \in \mathcal{F}\}.$$

It has been proved by Eom and Kye (Eom & Kye (2000)) that

Theorem B.2.6 (Eom & Kye (2000))**.** *A face $\mathcal{F}$ is exposed if and only if $\mathcal{F}'' = \mathcal{F}$.*

Example B.2.13. *Transpose map is an example of exposed map.*

Using the above theorem, a few examples of positive exposed mas has been identified.

Example B.2.14. *Reduction map B.2.2 for single qubit (i.e.* $n = 2$*) is also an example of exposed map. Note that the fie* $n \geq 3$*, reduction map is not exposed.*

Example B.2.15. *One of the most important class of examples of generalisations of Choi's map was given by Cho, Kye and Lee (Cho* et al. *(1992)), given here in the Example B.2.4. This map is defined by three parameters* $[a, b, c]$*. It was known that the specific example of* $[1, 1, 0]$ *and* $[1, 0, 1]$ *gave two different versions of Choi's map (Choi (1975b)). The class* $[0, 1, 1]$ *is known as decomposable. However, this is an example of decomposable extermal map. These maps are known to be extremal (Choi & Lam (1977/78); Ha (2013)). A subfamily of this map is given by* $a + b + c = 2,\ a \leq 1$ *and* $bc = (1 - a)^2$ *is proved to be exposed by Ha and Kye (Ha & Kye (2013)). We have shown in the Appendix A, that for other values of* $[a, b, c]$ *we do not get any extremal or exposed maps.*

Example B.2.16. *It has been shown by Chruściński (Chruściński (2011)) that the Robertson map (Robertson (1983a,b,c, 1985), shown here in Example B.2.9), and Breuer-Hall map (Breuer (2006); Hall (2006), given here in Example B.2.10) are local unitarily equivalent to each other. Further this maps are exposed.*

B.3 Conclusions

In this appendix, we have given some basics of C^* algebra. We have reviewed the basics of positivity in the *non-commutative* context. We have given the major results concerning positive maps. Further, we have given some well known examples of positive maps which are not completely positive. We have concluded with some recent results regarding the extremality of positive maps.

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