Combinatorial properties of some monomial ideals induced by graphs and permutations

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Declaration

The work presented in this thesis has been carried out by me under the guidance of Professor Kapil Hari Paranjape at the Indian Institute of Science Education and Research, Mohali. This work has not been submitted in part or in full for a degree, a diploma, or a fellowship to any other university or institute. Whenever contributions of others are involved, every effort is made to indicate this clearly, with due acknowledgement of collaborative research and discussions. This thesis is a bonafide record of original work done by me and all sources listed within have been detailed in the bibliography.

Amit Roy.

Amit Roy

In my capacity as the supervisor of the candidate's thesis work, I certify that the above statements by the candidate are true to the best of my knowledge.

Professor Kapil Hari Paranjape

(Supervisor)

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Contents

Abstract

Monomial ideals provide a bridge between combinatorics and commutative algebra. In this thesis we consider three families of monomial ideals: 1-skeleton ideal of the G-parking function ideal $\mathcal{M}_G$, monomial ideals induced by permutation avoiding patterns, and the edge ideals of circulant graphs. The 1-skeleton ideal $\mathcal{M}_G^{(1)}$ is a subideal of $\mathcal{M}_G$. Postnikov and Shapiro showed that the number of standard monomials of $\mathcal{M}_G$ is also given by $\det \widetilde{L}_G$, where $\widetilde{L}_G$ is the truncated Laplace matrix of G. We prove that number of standard monomials of $\mathcal{M}_G^{(1)}$ is bounded below by $\det \widetilde{Q}_G$, where $\widetilde{Q}_G$ is the truncated signless Laplace matrix of G. We have also given examples of some families of graphs for which this lower bound is attained.

Next, we consider monomial ideals induced by some permutation avoiding patterns. We show that number of standard monomials of Alexander dual of the monomial ideal induced by 132 and 312 avoiding patterns are also enumerated by number of rooted labeled forests avoiding 213 and 312 patterns. Formulas for number of standard monomials for other permutation avoiding patterns are also obtained.

Finally, we study edge ideals of the following three families of circulant graphs $C_n(1, \ldots, \widehat{j}, \ldots, \lfloor \frac{n}{2} \rfloor)$, $C_{lm}(1, 2, \ldots, \widehat{2l}, \ldots, \widehat{3l}, \ldots, \lfloor \frac{lm}{2} \rfloor)$ and $C_{lm}(1, 2, \ldots, \widehat{l}, \ldots, \widehat{2l}, \ldots, \lfloor \frac{lm}{2} \rfloor)$ and obtain all $\mathbb{N}$-graded Betti numbers of these ideals. Other algebraic and combinatorial properties such as when these graphs are well-covered, shellable, Cohen-Macaulay, Buchsbaum etc.

are also discussed.

The results are based on research done in collaboration with C. Kumar, G. Lather and S. Anand.

Notations

$[n]$: $\{1, 2, \ldots, n\}$.

$|S|$: cardinality of a set S.

$\#A$: number of elements in a finite set A.

K_{n+1}: complete simple graph on $n+1$ vertices $\{0, 1, \ldots, n\}$.

$K_{n+1}^{a,b}$: complete multigraph on $n+1$ vertices $\{0, 1, \ldots, n\}$.

$\mathbb{K}$: a field.

$\mathbb{N}$: set of natural numbers containing zero.

$\mathbb{R}$: set of real numbers.

$\mathbb{C}$: set of complex numbers

$\mathcal{R}_i$: the i^{th} row of a matrix.

$\mathcal{C}_i$: the i^{th} column of a matrix.

$\mathcal{M}_G$: the G-parking function ideal of a graph G.

$\mathcal{M}_G^{(1)}$: the 1-skeleton ideal.

$\widetilde{Q}_G$: reduced (truncated) signless Laplace matrix of a graph G.

$\mathfrak{S}_n$: set of all permutations of $[n]$.

I_S: monomial ideal induced by a set $S \subseteq \mathfrak{S}_n$.

C_n: cycle graph of length n.

$C_n(S)$: circulant graph on the generating set $S \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$.

$\beta_{i,j}$: i^{th} $\mathbb{N}$-graded Betti numbers in degree j.

Chapter 1

Introduction

Monomial ideals provide a bridge between commutative algebra and combinatorics. Combinatorial problems are encoded into monomial ideals, which then enable us to use techniques and tools in commutative algebra to solve the original problem. Richard Stanley [49] was the first mathematician who successfully applied such methods. To each simplicial complex he associated a square-free monomial ideal, called the Stanley-Reisner ideal (Section 2.3.1). Combinatorial properties of the simplicial complex are intimately related to the algebraic properties of this ideal. In this thesis we aim to understand certain algebraic and combinatorial properties of three families of monomial ideals: 1-skeleton ideals of the graphical parking function ideals (Section 3.2), monomial ideals induced by permutation avoiding patterns (Section 4.1), and the edge ideals of circulant graphs (Section 5.1).

The main object of study in Chapter 3 is the graphical parking function ideals and their skeleton ideals. Parking functions were introduced by Konheim and Weiss [26] in relation to hashing problems. This concept turned out to have connections and applications to many areas of mathematics. For example, it was shown in [27] that the parking functions of size n are in one to one correspondence with trees in $n+1$ labeled vertices. There are many generalizations of parking functions; one such is the G-parking functions or the graphical parking functions defined for a directed graph G. Interestingly, the graphical parking functions are related to the abelian sandpile model introduced by Dhar [12]. It was shown by Gabrielov [16] that the number of G-parking functions equals the number of oriented spanning trees of the directed graph G. Postnikov and Shapiro [45] introduced (graphical) parking functions in an algebraic context. Given a directed graph G on $n+1$ vertices, they associated a monomial ideal $\mathcal{M}_G$, called the G-parking function ideal. The standard monomials of $R/\mathcal{M}_G$ are given by the G-parking functions, where R is the polynomial ring in n variables over a field. The *Matrix-Tree Theorem* [50, Theorem 5.6.8] says that the number of oriented spanning trees N_G of a digraph G is equal to $\det \widetilde{L}_G$, where $\widetilde{L}_G$ is the truncated Laplace matrix of G. Thus the number of G-parking functions of a digraph G is given by $\det \widetilde{L}_G$.

The ideals $\mathcal{M}_G$ have connections to 'chip firing' [5] and a discrete Riemann-Roch theory for graphs [4, 36]. Motivated by certain constructions in 'hereditary chip firing' models, Dochtermann [14] introduced the notion of k-skeleton ideals $\mathcal{M}_G^{(k)}$ of $\mathcal{M}_G$. These are by definition, subideals of $\mathcal{M}_G$. In this thesis we focus on the 1-skeleton ideal $\mathcal{M}_G^{(1)}$ of an undirected multigraph G. Dochtermann showed that if $G = K_{n+1}$, the complete simple graph, then the number of standard monomials of $R/\mathcal{M}_G^{(1)}$ equals $\det \widetilde{Q}_G$, where $\widetilde{Q}_G$ is the truncated signless Laplace matrix of G. He also asked whether it is true that for a simple graph G, $\#\{\text{standard monomials of } R/\mathcal{M}_G^{(1)}\} \geq \det \widetilde{Q}_G$. We have shown that this is indeed true for all multigraphs. More generally, we have associated a monomial ideal $\mathcal{J}_H$ to a certain class of symmetric matrices H defined over nonnegative integers (see Section 3.2) and showed the following in [33, Theorem 3.3].

Theorem 1.0.1. *If H is positive semidefinite, then*

$$\#\{\textit{standard monomials of}\, R/\mathcal{J}_H\} \geq \det H.$$

In case $H = \widetilde{Q}_G$, the ideal $\mathcal{J}_H = \mathcal{M}_G^{(1)}$ so that the above theorem answers the question of Dochtermann. Moreover, we have characterized the subgraphs of the complete multigraph $K_{n+1}^{a,1}$, in particular all simple graphs G such that $\#\{\text{standard monomials of}\, R/\mathcal{M}_G^{(1)}\} = \det \widetilde{Q}_G$ (see Theorem 3.3.16). Examples of some families of multigraphs for which this equality holds are also given in Chapter 3.

In Chapter 4 we consider monomial ideals induced by permutation avoiding patterns. Let $\mathfrak{S}_n$ be the set of all permutations of $[n] = \{1, 2, \ldots, n\}$. For various $S \subseteq \mathfrak{S}_n$, the monomial ideals $I_S = \langle \mathbf{x}^\sigma : \sigma \in S\rangle$ and its Alexander dual $I_S^{[\mathbf{n}]}$ have many interesting combinatorial properties. For example, the ideal $I_{\mathfrak{S}_n}$ is called a permutohedron ideal and the Alexander dual $I_{\mathfrak{S}_n}^{[\mathbf{n}]}$ is the tree ideal $\mathcal{M}_{K_{n+1}}$. The i^{th} Betti number $\beta_i(I_{\mathfrak{S}_n}^{[\mathbf{n}]}) = (i!)S(n+1, i+1)$, where $S(n, r)$ is the Stirling number of the second kind, i.e., the number of set-partitions of $[n]$ into r blocks. Further, $\#\{\text{standard monomials of}\, R/I_{\mathfrak{S}_n}^{[\mathbf{n}]}\} = \#\{\text{parking functions of size } n\} = (n+1)^{n-1}$. When $S = \mathfrak{S}_n(\tau_1, \ldots, \tau_r)$ is a set of permutations avoiding patterns, the ideals I_S and their Alexander duals $I_S^{[\mathbf{n}]}$ also have many pleasing structures. For example, when S is the set of permutations avoiding 132 and 231-patterns, it is shown in [30] that the number of standard monomials of $R/I_S^{[\mathbf{n}]}$ is given by the integer sequence ($A000262$) in OEIS [48]. The monomial ideals I_S induced by permutation avoiding patterns $\mathfrak{S}_n(123, 132)$ and $\mathfrak{S}_n(123, 132, 213)$ are investigated in [31]. We have considered the monomial ideal I_W and its Alexander dual $I_W^{[\mathbf{n}]}$, where $W = \mathfrak{S}_n(132, 312)$. The ideal I_W also appeared in [29], where it is called a hypercubic ideal. We have identified the standard monomials of $R/I_W^{[\mathbf{n}]}$ with the integer sequence ($A007840$) in OEIS [48].

Theorem 1.0.2. [35, Theorem 2.7] *For $n \geq 1$, we have* $\dim_{\mathbb{K}}\left(\frac{R}{I_W^{[\mathbf{n}]}}\right) = \sum_{r=1}^{n}(r!)s(n,r)$*, where $s(n,r)$ is the signless Stirling number of the first kind.*

Thus the standard monomials of $R/I_W^{[\mathbf{n}]}$ are also enumerated by rooted-labeled forests on $[n]$ avoiding 213 and 312-patterns [3]. For $S_1 = \mathfrak{S}_n(123, 132, 312)$, $S_2 = \mathfrak{S}_n(123, 213, 231)$ and $S_3 = \mathfrak{S}_n(132, 213, 231)$, we have identified the standard monomials of $R/I_{S_a}^{[\mathbf{n}]}$ $(1 \leq a \leq 3)$ with the integer sequence ($A001710$) [35, Theorem 3.5]. Similarly, for $T_1 = \mathfrak{S}_n(123, 132, 231)$ and $T_2 = \mathfrak{S}_n(213, 312, 321)$, the standard monomials of $R/I_{T_b}^{[\mathbf{n}]}$, $b = 1, 2$, are identified with the integer sequence ($A000254$) [35, Theorem 3.7].

In Chapter 5 we study edge ideals of circulant graphs. Edge ideals of a graph are introduced by Villarreal [56]. They are mainly investigated to study the relationship between algebraic properties of the ideals and combinatorial properties of the graphs [20, 25]. Recently there has been an increased interest to study edge ideals of circulant graphs [1, 15, 41, 54, 55]. Circulant graphs are Cayley graphs over the simplest family of groups, the cyclic groups. In the literature they have appeared in a number of applications such as networks [7], connectivity [8], error-correcting codes [46] and even music [9] because of their regular structure. Brown and Hoshino studied the independence polynomial of a circulant graph in [9]. They have shown that it is in general a co-NP-complete problem to characterize all well-covered circulant graphs [10]. Moreover, they have classified all well-covered circulant graphs of the form $C_n(1, 2, \ldots, d)$, $C_n(d+1, d+2, \ldots, \lfloor \frac{n}{2} \rfloor)$ and 3-regular circulant graphs. Using these results Van Tuyl et al determined which circulant graphs of the above form are vertex decomposable, shellable, Cohen-Macaulay or Buchsbaum [15, 55]. In this thesis we study three families of circulant graphs, $C_n(1, \ldots, \widehat{j}, \ldots, \lfloor \frac{n}{2} \rfloor)$, $C_{lm}(1, 2, \ldots, \widehat{2l}, \ldots, \widehat{3l}, \ldots, \lfloor \frac{lm}{2} \rfloor)$ and $C_{lm}(1, 2, \ldots, \widehat{l}, \ldots, \widehat{2l}, \ldots, \lfloor \frac{lm}{2} \rfloor)$. We have given formulas for all the $\mathbb{N}$-graded Betti numbers of their edge ideals [2, Theorems 3.8, 4.3, 4.4, 4.5, 5.2]. The circulant graphs are expressed as a join of some well-known families of graphs. In [2] we have shown the following.

Proposition 1.0.3. [2, Proposition 2.8] *Let $d \geq 2$ be an integer. Suppose $G_1, \ldots, G_d$ are d number of finite simple graphs with disjoint vertex sets. Let $G = G_1 * \cdots * G_d$. Then*

(i) The induced matching number $\nu(G) = \begin{cases} \max_i\{\nu(G_i)\} & \text{if } \nu(G_i) \neq 0 \text{ for some } i, \\ 1 & \text{otherwise.} \end{cases}$

(ii) G is well-covered if and only if all G_j's are well-covered and for each $i \neq j$

$$\alpha(G_i) = \alpha(G_j).$$

(iii) G is vertex decomposable/shellable/Cohen-Macaulay (or S_2) if and only if G_j's are complete graphs for all j.

(iv) G is sequentially Cohen-Macaulay if and only if G_t is sequentially Cohen-Macaulay for some $1 \leq t \leq d$ and G_j's are complete for all $j \neq t$.

(v) G is Buchsbaum if and only if each G_i is Buchsbaum for $1 \leq i \leq d$.

Using this we have determined under what conditions the above three families of circulant graphs are vertex decomposable, shellable, Cohen-Macaulay, sequentially Cohen-Macaulay, Buchsbaum or if they satisfy Serre's condition S_2 [2, Theorems 3.9, 4.7, 5.4]. Moreover, formulas for Castelnuovo-Mumford regularity, projective dimension and induced matching numbers are also given in [2].

The thesis is organized as follows. In Chapter 2 we recall some definitions and results in order to use them in subsequent chapters. In Chapter 3 we give a lower bound for the number of standard monomials of the 1-skeleton ideal of a multigraph and characterize all subgraphs of $K_{n+1}^{a,1}$ which attains the lower bound. In Chapter 4 we identify the standard monomials of some Artinian monomial ideals induced by permutation avoiding patterns with some well-known integer sequences. In Chapter 5 we first compute the graded Betti numbers of the edge ideals of three families of circulant graphs and then determine certain algebraic and combinatorial properties of these ideals.

Chapter 2

Preliminaries

In this chapter we recall some definitions and basic results in commutative algebra. The general references for this chapter are the books by Matsumura [37], Bruns and Herzog [11], Miller and Sturmfels [39], and Peeva [43]. We also discuss some properties of positive semidefinite matrices following the book by Horn and Johnson [23].

2.1 Graded modules and Betti numbers

Let $\mathbb{K}$ be a field and $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n variables. Let $\mathfrak{m}$ be the maximal ideal $\langle x_1, x_2, \ldots, x_n \rangle$ in R. We define two gradings on R in the following way. For $i \in \mathbb{N}$, let R_i be the $\mathbb{K}$-vector space of homogeneous polynomials of degree i. Then R can be written as a direct sum of $\mathbb{K}$-vector spaces, $R \cong \oplus_{i \in \mathbb{N}} R_i$. We refer to this grading as the $\mathbb{N}$-*grading* of R. Another grading for R is called the $\mathbb{N}^n$-*grading* and is defined in the following way. A *monomial* in R is of the form $x^{\mathbf{a}} := \prod_{i=1}^{n} x_i^{a_i}$ for some $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$. For $\mathbf{a} \in \mathbb{N}^n$, consider the one-dimensional $\mathbb{K}$-vector space $R_{\mathbf{a}} = \{cx^{\mathbf{a}} \mid c \in \mathbb{K}\}$. Then $R \cong \oplus_{\mathbf{a} \in \mathbb{N}^n} R_{\mathbf{a}}$ as a $\mathbb{K}$-vector space and this gives the $\mathbb{N}^n$-grading for R.

Let S denote the Abelian semigroup $\mathbb{N}$ or $\mathbb{N}^n$. A graded R-module M is an R-module together with a direct sum decomposition $M = \oplus_{i \in S} M_i$ satisfying $R_i M_j \subseteq M_{i+j}$. An element $x \in M_i$ is called a *homogeneous* element of M with $\deg x = i$. The $\mathbb{N}^n$-grading is finer than the $\mathbb{N}$-grading in the sense that $\mathbb{N}^n$-graded modules are naturally $\mathbb{N}$-graded. Note that, R is a graded module over itself. For $j \in \mathbb{N}$ (respectively, $\mathbf{a} \in \mathbb{N}^n$), let $R(-j)$ (respectively, $R(-\mathbf{a})$) denote the free $\mathbb{N}$-graded (respectively, $\mathbb{N}^n$-graded) R-module of rank one with a homogeneous generator of degree j (respectively, degree $\mathbf{a}$).

Let M be an R-module. A *free resolution* of M is a sequence

$$\mathcal{F}: \quad \cdots \to F_p \xrightarrow{d_p} F_{p-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \to 0$$

of free R-modules F_i such that $d_i \circ d_{i+1} = 0$ with coker $d_1 = M$ and $\mathcal{F}$ is exact except at the 0^{th} position. Therefore, the homology groups $H_0(\mathcal{F}) \cong M$ and $H_i(\mathcal{F}) = 0$ for $i \geq 1$. A resolution $\mathcal{F}$ is called a graded free resolution if R is a graded ring, M is a graded R-module, the F_i are graded free modules in the same grading as of M and all the d_i are homogeneous of degree 0, i.e., for all $i \geq 1$, and for all $y \in F_i$, $\deg d_i(y) = \deg y$.

Let M be a (graded) R-module. A (graded) free resolution $\mathcal{F}$ of M over R is called a *minimal (graded) free resolution* if, for all $i \geq 1$, $d_i(F_i) \subseteq \mathfrak{m}F_{i-1}$. Every graded module M has a minimal graded free resolution. Moreover, Hilbert's syzygy theorem [43, Theorem 15.2] says that every finitely generated graded R-module M has a finite graded minimal free resolution of length at most n.

Let M and N be finitely generated graded R-modules. Let $\mathcal{F}$ be a graded free resolution of M. Consider the complex

$$\mathrm{Hom}_R(\mathcal{F}, N) : 0 \to \mathrm{Hom}_R(F_0, N) \longrightarrow \mathrm{Hom}(F_1, N) \to \dots .$$

The R-module $H_i(\mathrm{Hom}_R(\mathcal{F}, N))$ is denoted by $\mathrm{Ext}^i_R(M, N)$ for $i \geq 0$. Note that, $\mathrm{Ext}^0_R(M, N) = \mathrm{Hom}_R(M, N)$. Since $\mathcal{F}$ is a graded free resolution the complex $\mathrm{Hom}_R(\mathcal{F}, N)$ is a graded complex. So, its homology is graded. Thus $\mathrm{Ext}^i_R(M, N)$ is a graded R-module with the same type of grading as $\mathbb{N}$ or $\mathbb{N}^n$. Further, they do not depend on the choice of resolution of M (see [43, Theorem 38.2]).

Given two graded free resolutions $\mathcal{F}$ of M and $\mathcal{F}'$ of N consider the complexes

$$\mathcal{F} \otimes_R N : \ldots \to F_i \otimes_R N \to F_{i-1} \otimes_R N \to \ldots \to F_0 \otimes_R N \to 0$$

and

$$M \otimes_R \mathcal{F}' : \ldots \to M \otimes_R F'_i \to M \otimes_R F'_{i-1} \to \ldots \to M \otimes_R F'_0 \to 0.$$

Theorem 2.1.1. [58, Theorem 2.7.2] *Let M and N be graded R-modules with graded free resolutions $\mathcal{F}$ and $\mathcal{F}'$ respectively. Then*

$$H_i(\mathcal{F} \otimes_R N) \cong H_i(M \otimes_R \mathcal{F}').$$

Define

$$\mathrm{Tor}^R_i(M, N) := H_i(\mathcal{F} \otimes_R N) \cong H_i(M \otimes_R \mathcal{F}'),$$

for $i \geq 0$. Note that, $\mathrm{Tor}^R_0(M, N) \cong M \otimes_R N$. Since $\mathcal{F}$ is a graded free resolution the complex $\mathcal{F} \otimes_R N$ is also a graded complex. So, its homology is graded. Thus every $\mathrm{Tor}^R_i(M, N)$ is a graded R-module with the same type of grading as $\mathbb{N}$ or $\mathbb{N}^n$. Further, they do not depend on the choice of resolutions of M and N (see [43, Theorem 38.1]).

We define the *graded Betti numbers* and *multigraded Betti numbers* as follows.

Definition 2.1.2 (Betti numbers)**.** *Let M be a finitely generated $\mathbb{N}$-graded R-module. For $0 \le i \le n$ and $j \in \mathbb{N}$, the graded Betti numbers $\beta_{i,j}(M)$ (or simply $\beta_{i,j}$), of M, are defined as $\beta_{i,j}(M) := \dim_{\mathbb{K}} \operatorname{Tor}_i^R(M,\mathbb{K})_j$. Similarly, if M is an $\mathbb{N}^n$-graded R-module then for $0 \le i \le n$ and $\mathbf{a} \in \mathbb{N}^n$, the multigraded Betti numbers $\beta_{i,\mathbf{a}}(M) := \dim_{\mathbb{K}} \operatorname{Tor}_i^R(M,\mathbb{K})_{\mathbf{a}}$. The Betti numbers $\beta_i(M)$ (or simply β_i), of M, are defined as $\beta_i(M) := \dim_{\mathbb{K}} \operatorname{Tor}_i^R(M,\mathbb{K})$.*

Proposition 2.1.3. [39, Lemma 1.32] *Let $\mathcal{F}: \; 0 \to F_r \xrightarrow{d_r} F_{r-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \to 0$ be a minimal $\mathbb{N}$-graded free resolution of an R-module M. Then $F_i = \oplus_{j\in\mathbb{N}} R(-j)^{\beta_{i,j}(M)}$ for each $0 \le i \le r$. Similarly, if M is $\mathbb{N}^n$-graded then for each i, $F_i = \oplus_{\mathbf{a}\in\mathbb{N}^n} R(-\mathbf{a})^{\beta_{i,\mathbf{a}}(M)}$.*

Since every $\mathbb{N}^n$-graded module is naturally an $\mathbb{N}$-graded module we get the following.

Corollary 2.1.4. *Let M be a finitely generated $\mathbb{N}^n$- graded R-module. Considering M as an $\mathbb{N}$-graded module we have $\beta_{i,j}(M) = \sum_{|\mathbf{a}|=j} \beta_{i,\mathbf{a}}(M)$, where for $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, $|\mathbf{a}| = \sum_{i=1}^n a_i$.*

Definition 2.1.5. *Let M be a finitely generated $\mathbb{N}$-graded R-module. The length of a minimal graded free resolution of M is called as the projective dimension of M and is denoted by $\operatorname{pd}(M)$, i.e., $\operatorname{pd}(M) = \max\{i \mid \beta_{i,j}(M) \neq 0$ for some $j\}$. The Castelnuovo-Mumford regularity (or simply the regularity) of M, denoted by $\operatorname{reg}(M)$, is defined as $\operatorname{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\}$.*

Note that $\operatorname{reg}(M)$ is well-defined since a finitely generated graded R-module has a finite free resolution.

2.2 Cohen-Macaulay rings and some related notions

In this section we recall some properties of Noetherian rings following the book by H. Matsumura [37].

Let A be a commutative ring with identity and M an A-module. An element $x \in A$ is said to be M-regular if $xm = 0$ for some $m \in M$ implies $m = 0$.

Definition 2.2.1 (Regular sequence)**.** *A sequence $x_1, \ldots, x_n$ of elements of A is said to be an M-sequence (or an M-regular sequence) if the following two conditions hold:*

1. *x_1 is M-regular, x_2 is (M/x_1M)-regular,..., x_n is $(M/\sum_{i=1}^{n-1} x_iM)$-regular;*

2. *$M/\sum_{i=1}^{n} x_iM \neq 0$.*

Let A be a Noetherian ring, I an ideal of A and M a finite A-module such that $M \neq IM$; then by [37, Theorem 16.7] the length of a maximal M-sequence in I is a well-defined integer which we denote by $\operatorname{depth}(I, M)$.

For an A-module M, the ideal $\{x \in A : xM = 0\}$ is denoted by $\operatorname{ann}(M)$. Moreover, we define $\dim(M)$ to be the Krull dimension of the ring $A/\operatorname{ann}(M)$.

Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring with the unique maximal ideal $\mathfrak{m}$ and the residue field k. We call $\operatorname{depth}(\mathfrak{m}, M)$ simply the depth of M.

Definition 2.2.2. *Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring, and M a finitely generated A-module. We say that M is a Cohen-Macaulay module if $M \neq 0$ and $\operatorname{depth} M = \dim M$, or if $M = 0$. If A is a Cohen-Macaulay module over itself then we say that A is a Cohen-Macaulay local ring.*

A Noetherian ring A is said to be a Cohen-Macaulay ring if $A_{\mathfrak{p}}$ is a Cohen-Macaulay local ring for every prime ideal $\mathfrak{p}$ in A.

Next we define the well-known Serre's conditions (S_i) for $i \geq 0$ and see how it is related to the Cohen-Macaulay condition.

Definition 2.2.3 (Serre's condition (S_i))**.** *Let A be a Noetherian ring. We say that A satisfies Serre's condition (S_i) for $i \geq 0$ if for each prime ideal $\mathfrak{p}$ in A, we have* $\operatorname{depth}(A_{\mathfrak{p}}) \geq \min(\operatorname{ht}(\mathfrak{p}), i)$, *where* $\operatorname{ht}(\mathfrak{p})$ *is the height of the prime ideal $\mathfrak{p}$.*

From the definition we see that a Noetherian ring A is Cohen-Macaulay if and only if it satisfies Serre's condition (S_i) for each $i \geq 0$. For rings of Krull dimension ≤ 2, Cohen-Macaulay condition is same as (S_2).

Next we define the notion of Buchsbaum ring. First recall that for a Noetherian local ring $(A, \mathfrak{m}, k)$ of dimension r, there exists an $\mathfrak{m}$-primary ideal generated by r elements, but none generated by fewer [37, Theorem 13.4]. If $a_1, \ldots, a_r \in \mathfrak{m}$ generate an $\mathfrak{m}$-primary ideal, then $\{a_1, \ldots, a_r\}$ is said to be a *system of parameters* of A.

Definition 2.2.4 (Buchsbaum ring)**.** *A Noetherian local ring $(A, \mathfrak{m}, k)$ is said to be a Buchsbaum local ring if every system of parameters of A is a weak sequence, i.e., if $\{a_1, \ldots, a_r\}$ is a system of parameter of A, then $\mathfrak{m} \cdot ((a_1, \ldots, a_{i-1}) : a_i) \subset (a_1, \ldots, a_{i-1})$ for all i.*

A Noetherian ring A is said to be a Buchsbaum ring if $A_{\mathfrak{p}}$ is a Buchsbaum local ring for every prime ideal $\mathfrak{p}$ in A.

Note that a Cohen-Macaulay ring is always a Buchsbaum ring.

Definition 2.2.5 (Gorenstein ring)**.** *Let $(A, \mathfrak{m}, k)$ be a Noetherian local ring of dimension r. A is said to be a Gorenstein local ring if A is a Cohen-Macaulay local ring and* $\operatorname{Ext}^r_A(k, A) \cong k$.

A Noetherian ring A is said to be a Gorenstein ring if $A_{\mathfrak{p}}$ is a Gorenstein local ring for every prime ideal $\mathfrak{p}$ in A.

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$ and I a homogeneous ideal in R. We have the following criterion for R/I to be Gorenstein.

Theorem 2.2.6. [43, Theorem 25.7] *Let I be a homogeneous ideal in R. Let R/I has Krull dimension q. The quotient R/I is Gorenstein if and only if $\operatorname{pd}(R/I) = n - q$ and $\beta_{n-q}(R/I) = 1$.*

2.3 Monomial ideals

In this section we recall some basic results related to monomial ideals that we need in the subsequent sections.

2.3.1 Simplicial complex and Stanley-Reisner ring

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables. An (abstract) simplicial complex Δ on the vertex set $V = \{x_1, \ldots, x_n\}$ is a collection of subsets of V called faces or simplices closed under taking subsets, i.e., if $F \in \Delta$ is a face and $G \subset F$, then $G \in \Delta$. The maximal elements of Δ, with respect to inclusion, are called the *facets* of Δ. If $\{F_1, F_2, \ldots, F_t\}$ is a complete list of the facets of Δ, we will sometimes write $\Delta = \langle F_1, \ldots, F_t \rangle$. The *dimension* of a face $F \in \Delta$, denoted by $\dim F$, is given by $\dim F = |F| - 1$, where we make the convention that $\dim \emptyset = -1$. The *dimension* of Δ, denoted by $\dim \Delta$, is defined to be $\dim \Delta = max_{F \in \Delta}\{\dim F\}$. The dimension of Δ is $-\infty$ if $\Delta = \{\}$, the void complex with no face.

Example 2.3.1. *Let $V = \{x_1, x_2, x_3, x_4\}$. Then an example of a simplicial complex on V is $\Delta = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_2, x_3\}, \{x_3, x_4\}\}$.*

Let Δ be a simplicial complex on $\{x_1, \ldots, x_n\}$. For each integer i, let $F_i(\Delta)$ be the set of i-dimensional faces of Δ, and let $\mathbb{K}^{F_i(\Delta)}$ be the vector space over $\mathbb{K}$ whose basis elements e_σ correspond to i dimensional faces $\sigma \in F_i(\Delta)$.

The reduced chain complex of Δ over $\mathbb{K}$ is the complex $\widetilde{C}.(\Delta; \mathbb{K})$:

$$0 \to \mathbb{K}^{F_{n-1}(\Delta)} \xrightarrow{\partial_{n-1}} \cdots \to \mathbb{K}^{F_i(\Delta)} \xrightarrow{\partial_i} \mathbb{K}^{F_{i-1}(\Delta)} \to \cdots \xrightarrow{\partial_0} \mathbb{K}^{F_{-1}(\Delta)} \to 0.$$

The *boundary maps* ∂_i are defined by setting $\operatorname{sign}(x_j, \sigma) = (-1)^{r-1}$ if x_j is the r^{th} element of the set $\sigma \subseteq \{x_1, \ldots, x_n\}$, written in the increasing order $x_1 < x_2 < \cdots < x_n$, and $\partial_i(e_\sigma) = \sum_{x_j \in \sigma} \operatorname{sign}(x_j, \sigma) e_{\sigma \smallsetminus x_j}$.

If $i < -1$ or $i > n-1$, then $\mathbb{K}^{F_i(\Delta)} = 0$ and $\partial_i = 0$ by definition. It can be checked that $\partial_i \circ \partial_{i+1} = 0$. For each integer i, the $\mathbb{K}$-vector space

$$\widetilde{H}_i(\Delta; \mathbb{K}) := \ker(\partial_i) / \operatorname{im}(\partial_{i+1})$$

is defined to be the i^{th} *reduced homology* of Δ over $\mathbb{K}$.

Example 2.3.2. *Let Δ be the simplical complex from Example 2.3.1. The simplical complex Δ has all its reduced simplical homology zero, i.e., $\widetilde{H}_i(\Delta, \mathbb{K}) = 0$ for each integer i.*

Given a simplicial complex Δ on the vertex set $V = \{x_1, \ldots, x_n\}$, we can associate it with a square-free monomial ideal I_Δ in the polynomial ring $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$ in the following way. For every subset F of V, we define a monomial $x_F := \prod_{x_i \in F} x_i$ in R. Then the ideal $I_\Delta := \langle x_F : F \notin \Delta \rangle$ is called the *Stanley-Reisner ideal* of Δ and the quotient ring $\mathbb{K}[\Delta] = R/I_\Delta$ is called the *Stanley-Reisner ring*. Conversely, if I is a square-free monomial ideal in R, then it can be associated to a simplicial complex $\Delta(I) = \{F \subseteq V : x_F \notin I\}$. This correspondence between simplicial complexes on $V = \{x_1, \ldots, x_n\}$ and square-free monomial ideals in R is called the *Stanley-Reisner correspondence*.

Example 2.3.3. *Let Δ be the simplicial complex from Example 2.3.1. Then the Stanley-Reisner ideal of Δ is* $I_\Delta = \langle x_1x_2, x_1x_3, x_1x_4, x_2x_4, x_2x_3x_4\rangle$.

In order to describe some algebraic and combinatorial properties of the simplicial complex Δ and its Stanley-Reisner ring $\mathbb{K}[\Delta]$, we first recall some definitions related to simplicial complexes [57]. A simplicial complex is called *pure* if all its facets have the same dimension. A *simplex* is a simplicial complex having exactly one facet. Let Δ and Δ' be two simplicial complexes with vertex sets V and V' respectively. The union $\Delta \cup \Delta'$ is a simplicial complex with vertex set $V \cup V'$ and F is a face of $\Delta \cup \Delta'$ if and only if F is a face of Δ or Δ'. For a simplicial complex Δ, if $F \in \Delta$ is a face then the *link of* F is the simplicial complex $\mathrm{lk}_\Delta(F) = \{H \in \Delta \mid H \cap F = \emptyset \text{ and } H \cup F \in \Delta\}$, and *deletion of* F is the simplicial complex $\mathrm{del}_\Delta(F) = \{H \in \Delta \mid H \cap F = \emptyset\}$. When $F = \{x_i\}$, then we simply write $\mathrm{lk}_\Delta(x_i)$ or $\mathrm{del}_\Delta(x_i)$.

Example 2.3.4. *Let* $\Delta = \langle\{a, b\}, \{a, c, d\}, \{c, d, e\}\rangle$ *be a simplicial complex on the vertex set* $\{a, b, c, d, e\}$. *Take* $x = b$. *Then* $\mathrm{lk}_\Delta(b) = \langle\{a\}\rangle$ *is a simplex. Moreover,* $\mathrm{del}_\Delta(b)$ *is the simplicial complex* $\langle\{a, c, d\}, \{c, d, e\}\rangle$.

Given a pure simplicial complex Δ, we say Δ is *vertex decomposable* if either Δ is a simplex, or there exists a vertex x such that $\mathrm{lk}_\Delta(x)$, $\mathrm{del}_\Delta(x)$ are vertex decomposable and every facet of $\mathrm{del}_\Delta(x)$ is a facet of Δ.

Example 2.3.5. *Let* $\Delta = \langle\{a, b, c\}, \{b, c, d\}\rangle$ *be a pure simplicial complex on the vertex set* $\{a, b, c, d\}$. *Since* $\mathrm{lk}_\Delta(a) = \langle\{b, c\}\rangle$ *and* $\mathrm{del}_\Delta(a) = \langle\{b, c, d\}\rangle$, *we see that Δ is vertex decomposable.*

A simplicial complex Δ is called *shellable* if Δ is pure and there exists an ordering of facets $F_1 < F_2 < \cdots < F_r$ such that for all $1 \leq j < i \leq r$, there is some $x \in F_i \setminus F_j$ and some $k \in \{1, \ldots, i-1\}$ for which $F_i \setminus F_k = \{x\}$.

Example 2.3.6. *Let* $\Delta = \langle\{a, b, c\}, \{a, c, d\}, \{c, d, e\}\rangle$ *be a pure simplicial complex on the vertex set* $\{a, b, c, d, e\}$. *Taking* $F_1 = \langle\{a, b, c\}\rangle$, $F_2 =$

$\langle\{a, c, d\}\rangle$ and $F_3 = \langle\{c, d, e\}\rangle$ we see that Δ is shellable. The simplicial complex in Example 2.3.5 is also shellable.

Let Δ be a pure simplicial complex on V. We say Δ is *Cohen-Macaulay* over a field $\mathbb{K}$ if the Stanley-Reisner ring $\mathbb{K}[\Delta]$ is a Cohen-Macaulay ring.

Theorem 2.3.7 (Reisner's criterion, [11, Corollary 5.3.9])**.** *A simplicial complex Δ is Cohen-Macaulay over a field $\mathbb{K}$ if and only if $\widetilde{H}_i(\mathrm{lk}_\Delta(F); \mathbb{K}) = 0$ for all $F \in \Delta$ and all $i < \dim \mathrm{lk}_\Delta(F)$ (here $\widetilde{H}_i(-; \mathbb{K})$ is the i^{th} reduced simplicial homology group).*

If $\mathrm{lk}_\Delta(x)$ is Cohen-Macaulay for all $x \in V$, then the pure simplical complex Δ is called a *Buchsbaum* simplicial complex. A simplicial complex Δ is said to satisfy *Serre's condition S_2* if $\mathbb{K}[\Delta]$ satisfies S_2.

Theorem 2.3.8. [42] *Let Δ be a simplicial complex.Then Δ satisfies Serre's condition S_2 if and only if Δ is pure and $\mathrm{lk}_\Delta(F)$ is connected for every face F of Δ having $\dim \mathrm{lk}_\Delta(F) \geq 1$.*

For a (not necessarily pure) simplicial complex Δ the *pure i^{th} skeleton* of Δ is the subcomplex $\Delta^{[i]}$ of Δ whose facets are the faces F of Δ with $\dim F = i$. If $\Delta^{[i]}$ is Cohen-Macaulay for all i then Δ is called a *sequentially Cohen-Macaulay* simplicial complex. Note that sequentially Cohen-Macaulay pure simplicial complexes are also Cohen-Macaulay. If Δ is Buchsbaum, then the ring $\mathbb{K}[\Delta]$ is called a *Buchsbaum ring.* Similarly, if Δ is sequentially Cohen-Macaulay, then the Stanley-Reisner ring $\mathbb{K}[\Delta]$ is called a *sequentially Cohen-Macaulay ring.*

2.3.2 Alexander duality

Let Δ be a simplicial complex on the vertex set $[n] = \{1, \ldots, n\}$. The *Alexander dual* Δ^* of Δ is the simplicial complex $\{F : F^c \notin \Delta\}$, where

$F^c = [n] \setminus F$. Let $I \subseteq R = \mathbb{K}[x_1, \ldots, x_n]$ be a square-free monomial ideal. Consider I as a Stanley-Reisner ideal I_Δ for some simplicial complex Δ. The Alexander dual of the square-free monomial ideal $I = I_\Delta$, denoted by I^*, is by definition the Stanley-Reisner ideal I_{Δ^*} of the Alexander dual Δ^*.

Example 2.3.9. *Let* $\Delta = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{2,3\}, \{3,4\}\}$ *be a simplicial complex. Then* $\Delta^* = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{2,3\}, \{3,4\}, \{1,3\}, \{2,4\}\}$ *and* $I_{\Delta^*} = \langle x_1x_2, x_1x_4, x_2x_3x_4\rangle$.

For a simplicial complex Δ on the vertex set $[n]$ and for $W \subseteq [n]$, the *restriction* of Δ to W is the subcomplex $\Delta[W] = \{F \in \Delta : F \subseteq W\}$. Given $W \subseteq [n]$, we associate it with a vector $\mathbf{a} \in \{0,1\}^n$ as follows. If $\mathbf{a} = (a_1, \ldots, a_n)$ then $a_i = 1$ if and only if $i \in W$. Further the restriction of Δ to $\mathbf{a}$ is $\Delta[\mathbf{a}] := \Delta[W]$. For a square-free monomial ideal $I = I_\Delta$ in the polynomial ring R the Betti numbers of R/I can be calculated using the following formula by M. Hochster.

Theorem 2.3.10 (Hochster's Formula, [22])**.** *Let* $\mathbb{K}[\Delta] = R/I_\Delta$ *be the Stanley-Reisner ring of the simplicial complex* Δ. *The non-zero Betti numbers of* $\mathbb{K}[\Delta]$ *are only in square-free degrees* $\mathbf{a}$ *and may be expressed as*

$$\beta_{i,\mathbf{a}}(\mathbb{K}[\Delta]) = \dim_{\mathbb{K}} \widetilde{H}_{|\mathbf{a}|-i-1}(\Delta[\mathbf{a}]; \mathbb{K}).$$

In particular, by Corollary 2.1.4 the $\mathbb{N}$-graded Betti numbers of $\mathbb{K}[\Delta]$ may be expressed as

$$\beta_{i,j}(\mathbb{K}[\Delta]) = \sum_{\substack{W \subset [n] \\ |W| = j}} \dim_{\mathbb{K}} \tilde{H}_{j-i-1}(\Delta[W]; \mathbb{K}). \tag{2.1}$$

Alexander duality for square-free monomial ideals has been extended to any monomial ideal by Miller [38] as follows. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and $\mathbf{x}^{\mathbf{a}}$ be the monomial $\prod_{i=1}^n x_i^{a_i}$ in $R = \mathbb{K}[x_1, \ldots, x_n]$. The monomial ideal $\langle x_i^{a_i} : a_i > 0\rangle$ is denoted by $\mathbf{m}^{\mathbf{a}}$. Now every monomial ideal I in R has a unique set of minimal monomial generators and the primary components of I are also a unique set of monomial ideals of the form $\mathbf{m}^{\mathbf{a}}$ (see [39, Lemma 5.18]). Let $\{\mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \mathcal{A}\}$ be the set of minimal generators of I and $\{\mathbf{m}^{\mathbf{a}} : \mathbf{a} \in \mathcal{A}'\}$ be the set of (monomial) primary components of I, where $\mathcal{A}$ and $\mathcal{A}'$ are finite subsets of $\mathbb{N}^n$. Then $I = \langle \mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \mathcal{A}\rangle = \cap\{\mathbf{m}^{\mathbf{a}} : \mathbf{a} \in \mathcal{A}'\}$.

Let $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$ such that $\mathbf{b} \geq \mathbf{a}$, i.e., $b_i \geq a_i$ for $1 \leq i \leq n$. Then we define $\mathbf{b} \setminus \mathbf{a} = (b_1 \setminus a_1, \ldots, b_n \setminus a_n) \in \mathbb{N}^n$, where

$$b_i \setminus a_i = \begin{cases} b_i - a_i + 1 & \text{if } a_i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

If I is the monomial ideal in R generated by $\{\mathbf{x}^{\mathbf{a}} : \mathbf{a} \in \mathcal{A}\}$ then the Alexander dual $I^{[\mathbf{b}]}$ of I with respect to $\mathbf{b}$ is defined by Miller [38] as the monomial ideal

$$I^{[\mathbf{b}]} = \bigcap\{\mathbf{m}^{\mathbf{b}\setminus\mathbf{a}} : \mathbf{a} \in \mathcal{A}\} = \left\langle \mathbf{x}^{\mathbf{b}\setminus\mathbf{a}} : \mathbf{a} \in \mathcal{A}' \right\rangle.$$

Example 2.3.11. *Let $I = \langle x^2, xy^2\rangle = \langle x\rangle \cap \langle x^2, y^2\rangle$. Then the Alexander dual of I with respect to the vector $\mathbf{b} = (2, 4)$ is $I^{[\mathbf{b}]} = \langle x^2, xy^3\rangle = \langle x\rangle \cap \langle x^2, y^3\rangle$.*

Let I be a square-free monomial ideal in $R = \mathbb{K}[x_1, \ldots, x_n]$. Then $I = I_\Delta$ for some simplicial complex Δ on the vertex set $\{x_1, \ldots, x_n\}$. We show that the Alexander dual $I^* = I^{[\mathbf{1}]}$, where $\mathbf{1} = (1, 1, \ldots, 1)$, as follows. Let $F \in \Delta$ be a face. Identify F with a vector $\mathbf{a}_F \in \{0, 1\}^n$, where $\mathbf{a}_F = (a_1, \ldots, a_n)$ with $a_i = 1$ if and only if $x_i \in F$. We can write $I = I_\Delta = \cap_{F\in\Delta}\mathbf{m}^{\mathbf{a}_{F^c}}$ (see [39, Theorem 1.7]). Then $I^{[\mathbf{1}]} = \langle x_{F^c} \mid F \in \Delta\rangle$. Since $I^* = \langle x_F \mid F \notin \Delta^*\rangle$, where $\Delta^* = \{F \mid F^c \notin \Delta\}$, we see that $I^{[\mathbf{1}]} = I^*$. Thus Alexander duality of monomial ideals by Miller is indeed a generalization to the square-free case.

The Alexander duality is a duality in the following sense.

Theorem 2.3.12. [39, Theorem 5.24] *If all the minimal generators of a* *monomial ideal I divide $\mathbf{x^b}$, then all minimal generators of $I^{[\mathbf{b}]}$ divide $\mathbf{x^b}$ and $(I^{[\mathbf{b}]})^{[\mathbf{b}]} = I$.*

The minimal generators of the Alexander dual of a monomial ideal can be determined using the following result.

Proposition 2.3.13. [39, Proposition 5.23] *Suppose that all the minimal generators of a monomial ideal I divide $\mathbf{x^b}$. If $\mathbf{b} \geq \mathbf{a}$, then $\mathbf{x^a}$ lies outside of I if and only if $\mathbf{x^{b-a}}$ lies inside of $I^{[\mathbf{b}]}$.*

2.3.3 Cellular resolution

Let C be a subset of $\mathbb{R}^n$ such that for any two points $x, y \in C$ the line segment $\{\lambda y + (1-\lambda)x : \lambda \in \mathbb{R},\ 0 \leq \lambda \leq 1\}$ joining x and y is in C, then the set C is called a *convex set* in $\mathbb{R}^n$. Note that intersection of nonempty convex sets is again convex. For any nonempty subset X of $\mathbb{R}^n$, the convex hull $\mathrm{Conv}(X)$ of X is the intersection of all convex sets $C \subset \mathbb{R}^m$ $(m \leq n)$ containing X. A *polytope* $\mathcal{P}$ in $\mathbb{R}^n$ is the convex hull of a finite set of points.

A polytope can also be expressed as a finite intersection of closed half-spaces. Let $\mathbf{b} \in \mathbb{R}^m$ with $\mathbf{b} \neq \mathbf{0}$ and $c \in \mathbb{R}$. The set $H_{\mathbf{b},c} = \{\mathbf{x} \in \mathbb{R}^m : \langle \mathbf{b}, \mathbf{x} \rangle = c\}$, where $\langle \mathbf{b}, \mathbf{x} \rangle := \sum_{i=1}^m b_i x_i = 0$ is the *hyperplane* with normal vector $\mathbf{b}$. Then the closed half-space $H^+_{\mathbf{b},c}$ is defined as $H^+_{\mathbf{b},c} = \{\mathbf{x} \in \mathbb{R}^m : \langle \mathbf{b}, \mathbf{x} \rangle \geq c\}$. If $\mathcal{P}$ is a polytope then $\mathcal{P} = \bigcap_{i=1}^s H^+_{\mathbf{b_i},c_i}$ for some $\mathbf{b_i}, c_i$ and the converse is also true provided that the intersection is bounded. A subset $\mathcal{P}' \subseteq \mathcal{P}$ is a *face* of $\mathcal{P}$ if there are $\mathbf{b} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ and $c \in \mathbb{R}$ such that $\mathcal{P}' = H_{\mathbf{b},c} \cap P$ and $\mathcal{P} \subseteq H^+_{\mathbf{b},c}$. A zero-dimensional face is called a vertex of $\mathcal{P}$. Note that a face is also a convex polytope.

For many interesting properties of convex polytopes we refer to [60].

Definition 2.3.14. [39, Chapter 4] *A polyhedral cell complex X is a finite collection of convex polytopes in $\mathbb{R}^n$, called faces of X, satisfying two properties:*

- *If $\mathcal{P}$ is a polytope in X and F is a face of $\mathcal{P}$, then F is in X.*
- *If $\mathcal{P}$ and $\mathcal{Q}$ are in X, then $\mathcal{P} \cap \mathcal{Q}$ is a face of both $\mathcal{P}$ and $\mathcal{Q}$.*

Given a polyhedral cell complex X, Bayer and Sturmfels [6] defined a *monomial labeling* on X as follows:

(i) Each vertex v of X is labeled with a monomial $\mathbf{x}^{\mathbf{b}_v}$, where $\mathbf{b}_v \in \mathbb{N}^d$.

(ii) Each face F of X is labeled with the monomial $\mathbf{x}^{\mathbf{b}_F} = \operatorname{lcm}(\mathbf{x}^{\mathbf{b}_v} : v \in F$ is a vertex$)$.

(iii) The empty face $\emptyset$ is labeled with 1.

Such a polyhedral cell complex is called a *labeled cell complex* and the exponent $\mathbf{b}_F$ is called the degree of face F.

Example 2.3.15. *Let X be the polyhedral cell complex consisting of faces of a filled pentagon $\langle v_1, v_2, v_3, v_4, v_5 \rangle$. We give the monomial labels on the vertices by defining the degrees as follows: $\mathbf{b}_{v_1} = (2,1), \mathbf{b}_{v_2} = (1,2), \mathbf{b}_{v_3} = (1,5), \mathbf{b}_{v_4} = (3,2)$ and $\mathbf{b}_{v_5} = (5,0)$. Then the degrees of the edges are $\mathbf{b}_{\langle v_1,v_2 \rangle} = (2,2), \mathbf{b}_{\langle v_2,v_3 \rangle} = (1,5), \mathbf{b}_{\langle v_3,v_4 \rangle} = (3,5), \mathbf{b}_{\langle v_4,v_5 \rangle} = (5,2), \mathbf{b}_{\langle v_5,v_1 \rangle} = (5,1)$ and the degree of the whole complex is given by $\mathbf{b}_{\langle v_1,v_2,v_3,v_4,v_5 \rangle} = (5,5)$.*

We are now in a position to define the *reduced chain complex* associated to a polyhedral cell complex X. Let us fix a total ordering on the vertices of X. An orientation of a face is a choice of ordering of its vertices. The reduced chain complexes for polyhedral chain complexes are defined the same way as the reduced chain complexes of simplicial cell complexes, except the signs depend on the orientation of the faces. The faces of X are oriented in such a way that for each oriented face F and its facet G, we define $\operatorname{sign}(G, F)$ is $+1$ if the orientation of G is induced by orientation of F, and -1 otherwise. Moreover, the *boundary chain* of a face F is given by $\partial(e_F) = \sum_{\text{facets } G \subset F} \text{sign}(G, F) \cdot e_G$, where the boundary maps ∂ have the property $\partial \circ \partial = 0$.

Let X be a oriented labeled polyhedral cell complex with vertex set V and I be the ideal $\left\langle \mathbf{x}^{\mathbf{b}_v} : v \in V \right\rangle$ generated by the monomial labeled on the vertices. Corresponding to X we define a free complex $\mathbf{F}_X$ and determine the conditions such that $\mathbf{F}_X$ becomes a free resolution of R/I.

Let $\mathbf{F}_{X,i}$ be the set of i-dimensional faces of X. Then we define the free R-modules

$$F_i = \bigoplus_{F \in \mathbf{F}_{X,i}} R(-\mathbf{b}_F) e_F.$$

Consider the chain complex

$$\mathbf{F}_X : \cdots \to F_i \xrightarrow{\partial_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0,$$

where the boundary maps $\partial_i(e_F) = \sum_{\text{facets } G \text{ of } F} \text{sign}(G, F) \mathbf{x}^{\mathbf{b}_F - \mathbf{b}_G} e_G$. The signatures $\text{sign}(G, F)$ is defined in such a way that $\partial_i \circ \partial_{i+1} = 0$. The maps ∂_i are in fact an $\mathbb{N}^n$-graded R-module homomorphisms. If $\mathbf{F}_X$ is an exact complex then it becomes a free resolution of R/I.

Definition 2.3.16. *Let I be a monomial ideal and X be a polyhedral cell complex labeled by the minimal generators of I. The free complex $\mathbf{F}_X$ is called a cellular resolution of R/I if $\mathbf{F}_X$ is exact (or acyclic).*

Given two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{N}^n$, we write $\mathbf{b} \preceq \mathbf{a}$ if $\mathbf{a} - \mathbf{b} \in \mathbb{N}^n$. Given $\mathbf{a} \in \mathbb{N}^n$ the two complexes, $X_{\preceq}$ consisting of faces of X with degree $\mathbf{b}_F \preceq \mathbf{a}$ and $X_{\prec}$ obtained from $X_{\preceq}$ by removing the faces of degree $\mathbf{a}$ of X are subcomplexes of X.

Theorem 2.3.17. [6, Proposition 1.2] *The cellular free complex $\mathbf{F}_X$ supported on X is a cellular resolution if and only if $X_{\preceq \mathbf{a}}$ is acyclic over $\mathbb{K}$ for all $\mathbf{a} \in \mathbb{N}^n$. When $\mathbf{F}_X$ is acyclic it is a free resolution of R/I, where*

$I = \left\langle \mathbf{x}^{\mathbf{b}_v} : v \in X \text{ is a vertex} \right\rangle$ *is generated by the monomial labels on vertices. Moreover the cellular resolution* $\mathbf{F}_X$ *is a minimal resolution if and only if any two comparable faces* $F' \subset F$ *of the complex* X *have distinct degrees* $\mathbf{b}_F \neq \mathbf{b}_{F'}$.

Example 2.3.18. [6, Example 1.9] *Let* $u_1, \ldots, u_n$ *be distinct integers and* I *be the ideal generated by the* $n!$ *monomials* $x_{\pi(1)}^{u_1} x_{\pi(2)}^{u_2} \cdots x_{\pi(n)}^{u_n}$ *in* $R = \mathbb{K}[x_1, \ldots, x_n]$*, where* π *runs over all permutations of* $\{1, 2, \ldots, n\}$*. Let* X *be the complex of all faces of the permutohedron* [60, Example 0.10]*, which is the convex hull of the* $n!$ *vectors* $(\pi(1), , \ldots, \pi(n))$ *in* $\mathbb{R}^n$*. The* i*-faces* F *of* X *are indexed by chains*

$$\emptyset = A_0 \subset A_1 \subset \ldots \subset A_{n-i-1} \subset A_{n-i} = \{u_1, u_2, \ldots, u_n\}.$$

The following monomial labels are assigned to the i*-face* F *indexed by this chain:*

$$\mathbf{x}^{\mathbf{b}_F} = \prod_{j=1}^{n-i} \prod_{r \in A_j \setminus A_{j-1}} x_r^{\max\{A_j \setminus A_{j-1}\}}.$$

It can be checked that $\mathbf{F}_X$ *is acyclic and for any two comparable faces* $F \subset F'$*, the monomial labels* $\mathbf{x}^{\mathbf{b}_F}$ *and* $\mathbf{x}^{\mathbf{b}_{F'}}$ *are different (see* [6] *for more details). Hence* $\mathbf{F}_X$ *is the minimal free resolution of* R/I*.*

2.4 Positive semidefinite matrices

Let $M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices over complex numbers. For $M = [a_{i,j}] \in M_n(\mathbb{C})$, the *adjoint* M^* of the matrix M is defined as $M^* = \bar{M}^t$, in which M^t is the *transpose* of M and $\bar{M}$ is the entrywise conjugate of M. A matrix $M \in M_n(\mathbb{C})$ is said to be *Hermitian* if $M^* = M$.

Definition 2.4.1. *A Hermitian matrix* $M \in M_n(\mathbb{C})$ *is called a positive* *semidefinite matrix if*

$$x^*Mx \geq 0 \quad \textit{for all } x \in \mathbb{C}^n,$$

and is called positive definite if

$$x^*Mx > 0 \quad \textit{for all nonzero } x \in \mathbb{C}^n.$$

Let $M \in M_n(\mathbb{C})$ and $\alpha \subseteq [n]$ be an index set. We denote by $M[\alpha]$ the (sub)matrix of M whose entries lie in the rows and columns of M indexed by α. A matrix of the form $M[\alpha]$ is called a *principal submatrix* of M.

Example 2.4.2. *Let* $M = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}_{3\times 3}$ *and* $\alpha = \{1,3\}$. *Then* $M[\alpha] = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}_{2\times 2}$.

Proposition 2.4.3. [23, Observation 7.1.2] *Let* $M = M_n(\mathbb{C})$ *be a Hermitian matrix. If* M *is positive semidefinite (respectively, positive definite) then all its principal submatrices are positive semidefinite (respectively, positive definite).*

Proposition 2.4.4. [23, Observation 7.1.8] *Let* $M \in M_n(\mathbb{C})$ *be Hermitian and let* $E \in M_{n,m}$, *where* $M_{n,m}$ *is the set of* $n\times m$ *matrix with complex entries. If* M *is positive semidefinite then* E^*ME *is also positive semidefinite. If* M *is positive definite then* E^*ME *is positive definite if and only if* $\operatorname{rank} E = m$.

Let $\mathcal{R}_i$ and $\mathcal{C}_k$ denote the i^{th} row and k^{th} column of a matrix M, respectively. The elementary column operation $\mathcal{C}_j \pm (\mathcal{C}_{k_1} + \cdots + \mathcal{C}_{k_r})$ in M means the matrix M is transformed to a matrix M', where only j^{th} column $\mathcal{C}'_j$ of M' differs from the j^{th} column $\mathcal{C}_j$ of M and $\mathcal{C}'_j = \mathcal{C}_j \pm (\mathcal{C}_{k_1} + \cdots + \mathcal{C}_{k_r})$. The elementary row operations are also defined in a similar way.

Lemma 2.4.5. *Let $M \in M_n(\mathbb{C})$ be a Hermitian positive semidefinite matrix. Suppose M' is a matrix obtained from M by applying the elementary column and row operations $\mathcal{C}_{i_1} - \mathcal{C}_k, \mathcal{R}_{i_1} - \mathcal{R}_k, \ldots, \mathcal{C}_{i_r} - \mathcal{C}_k, \mathcal{R}_{i_r} - \mathcal{R}_k$, where $\{i_1, \ldots, i_r, k\} \subseteq [n]$. Then M' is a positive semidefinite matrix. If M is positive definite then M' is also positive definite.*

Proof. Let I_n be the identity matrix of order n and $\epsilon_{i,j}$ be the $n \times n$ matrix with 1 at $(i,j)^{th}$ place and zero elsewhere. Then the matrix $E = I_n - (\epsilon_{k,i_1} + \epsilon_{k,i_2} + \cdots + \epsilon_{k,i_r})$ has determinant $\det E = 1$ and $E^*ME = M'$. By Proposition 2.4.4, M' is positive semidefinite and since E has full rank, the matrix M' is positive definite if and only if M is positive definite. □

Theorem 2.4.6 (Hadamard)**.** *Let $M = [a_{i,j}] \in M_n(\mathbb{C})$ be positive definite. Then*

$$\det M \leq \prod_{i=1}^{n} a_{i,i} \tag{2.2}$$

with equality if and only if M is diagonal.

Proof. For a proof see [23, Theorem 7.8.1]. □

Theorem 2.4.7 (Fischer)**.** *Let $M \in M_n(\mathbb{C})$ be a positive semidefinite matrix having block decomposition $M = \left(\begin{array}{c|c} A & B \\ \hline B^* & C \end{array}\right)$ with square matrices A and C. Then*

$$\det M \leq \det(A)\det(C). \tag{2.3}$$

If M is positive definite then equality occurs in (2.3) *if and only if $B = 0$.*

Proof. Let $n = p + q$, i.e., $M \in M_{p+q}(\mathbb{C})$ with $A \in M_p(\mathbb{C})$ and $C \in M_q(\mathbb{C})$. We follow the proof of [23, Theorem 7.8.5]. First suppose that M is positive definite. Let $A = U_1 D U_1^*$ and $C = U_2 D' U_2^*$ be spectral decomposition, in which U_1 and U_2 are unitary and $D = \operatorname{diag}(d_1, \ldots, d_p)$ and

$D' = \operatorname{diag}(d'_1, \ldots, d'_q)$ are positive diagonal matrices. Let $U = U_1 \oplus U_2$ and then

$$U^*MU = \begin{bmatrix} D & U_1^*BU_2 \\ U_2^*B^*U_1 & D' \end{bmatrix}.$$

Then by Hadamard's inequality (2.2),

$$\det M = \det(U^*MU) \leq (d_1 \cdots d_p)(d'_1 \cdots d'_q) = (\det A)(\det C).$$

Now if M is positive semidefinite then we clearly have (2.3) as both A and C are positive semidefinite by Proposition 2.4.3. For the statement about equality, note that by Theorem 2.4.6, $\det M = \det(U^*MU) = (\det A)(\det C)$ if and only if $U_1^*BU_2 = 0$, i.e., $B = 0$ (as both U_1 and U_2 are unitary). □

Let $M \in M_n(\mathbb{C})$ be a Hermitian matrix and its real eigenvalues be arranged in a non-decreasing order $\lambda_1(M) \leq \lambda_2(M) \leq \cdots \leq \lambda_n(M)$. The Courant-Weyl inequalities compare eigenvalues of two Hermitian matrices with their sum.

Theorem 2.4.8 (Courant-Weyl). *Let $M_1, M_2 \in M_n(\mathbb{C})$ be Hermitian matrices. Then*

$$\lambda_i(M_1 + M_2) \leq \lambda_{i+j}(M_1) + \lambda_{n-j}(M_2) \text{ for } j = 0, 1, \ldots, n - i.$$

Proof. For a proof see [23, Theorem 4.3.1]. □

Lemma 2.4.9. *Let $M = (a_{i,j})_{n\times n} \in M_n(\mathbb{R})$ be a real symmetric positive semidefinite matrix. Suppose N is obtained from M by replacing one diagonal element, say $a_{i,i}$, with an element b such that $a_{i,i} \geq b$. If $\det N > 0$, then N is a positive definite matrix.*

Proof. Let $\epsilon_{i,j}$ be the $n \times n$ matrix with 1 at the $(i,j)^{th}$ place and zero elsewhere. Then $M = N + N'$, where $N' = (a_{i,i} - b)\epsilon_{i,i}$. Clearly, $\lambda_1(N') =$

$\cdots = \lambda_{n-1}(N') = 0$ and $\lambda_n(N') = a_{i,i} - b$. Since M is positive semidefinite, $0 \leq \lambda_1(M) \leq \cdots \leq \lambda_n(M)$. Taking $i = j = 1$ in the Courant-Weyl inequalities with $M = N + N'$, we obtain $\lambda_1(M) \leq \lambda_2(N) + \lambda_{n-1}(N') = \lambda_2(N)$. Thus $0 \leq \lambda_2(N) \leq \ldots \leq \lambda_n(N)$. As $\det N = \prod_{i=1}^n \lambda_i(N) > 0$, N must be positive definite. □

Chapter 3

Graphical parking function ideals and skeleton ideals

In this chapter we study the notion of k-skeleton ideals as introduced by Dochtermann [14]. In particular we focus on the k-skeleton ideal for $k = 1$. We show that for a (undirected) multigraph, number of standard monomials of 1-skeleton ideal is always bigger than or equal to the determinant of truncated signless Laplace matrix of the corresponding graph. Some of the results in this chapter are based on a joint work with C. Kumar and G. Lather [33].

3.1 Graphical parking function ideals

For a directed graph G, Postnikov and Shapiro associated a monomial ideal $\mathcal{M}_G$, and studied various algebraic and combinatorial properties of this ideal [45]. In this thesis we consider only loopless undirected graphs. The ideals $\mathcal{M}_G$ are called the G-parking function ideals (or the graphical parking function ideal for the graph G). Among others, they showed that the standard monomials of $\mathcal{M}_G$ are given by the G-parking functions, which are a natural generalization of the classical parking functions.

Definition 3.1.1 (Parking function)**.** [50] *A sequence* $(p_1, \ldots, p_n)$ *of nonnegative integers is said to be a parking function of length* n *if a rearrangement* $p_{i_1} \leq \cdots \leq p_{i_n}$ *satisfies* $p_{i_j} < j$ *for* $1 \leq j \leq n$*. Equivalently,* $\#\{i : p_i < r\} \geq r$ *for* $r = 1, \ldots, n$*.*

Example 3.1.2. *The parking functions of length* 2 *are* $\{(0,0),(0,1),(1,0)\}$*. The sequences* $(2,1,0)$ *and* $(1,0,1)$ *are examples of parking functions of length* 3*. The sequence* $(1,1,1)$ *is not a parking function.*

Parking functions were introduced by Konheim and Weiss [26] in relation to hashing problems.

Theorem 3.1.3. [26] *The number of parking functions of length* n *is* $(n+1)^{n-1}$*.*

Parking functions have appeared in many areas of mathematics. For more on parking functions, we refer to [52, 59]. One particular generalization of the classical parking functions is the G-parking functions or the graphical parking functions for a graph G.

Let G be a (multi)graph on the set of vertices $V = \{0, 1, \ldots, n\} = \{0\} \cup [n]$. The vertex 0 is considered to be the root of the graph. The graph G is determined by its *adjacency matrix* $A(G) = [a_{ij}]_{0 \leq i,j \leq n}$. Given a graph G, let $E(i,j)$ be the set of edges between $i, j \in V$. We assume $E(i,j) = E(j,i)$ and $|E(i,j)| = a_{ij} = a_{ji}$ for all i, j. We also assume that G is loopless, i.e., $a_{ii} = 0$ for all $i \in V$. For $\emptyset \neq A \subseteq [n] = \{1, 2, \ldots, n\}$, set $d_A(i) = \sum_{j \in V \setminus A} a_{ij}$, for $i \in A$. Then $d_i = d_{\{i\}}(i)$ is the degree of the vertex i in G.

Definition 3.1.4 (**G-parking function**)**.** *For a graph* G*, a sequence* $(p_1, \ldots, p_n)$ *of nonnegative integers is called a* G*-parking function, if for any nonempty subset* $A \subseteq \{1, \ldots, n\}$*, there exists* $i \in A$ *such that* $p_i < d_A(i)$*.*

Fix two nonnegative integers a and b. Let $K_{n+1}^{a,b}$ be the complete multigraph on the vertices $0, 1, \ldots, n$ with edges $(0, i)$, $i \neq 0$, of multiplicity a and the edges (i, j), $i, j \neq 0$, of multiplicity b. For $a = b = 1$, the graph $K_{n+1}^{1,1}$ is called the complete simple graph and is also denoted by K_{n+1}. Note that if $G = K_{n+1}$, then the G-parking functions are the ordinary parking functions. For a graph G, the set of all G-parking functions are denoted by $PF(G)$. We will also denote the set of all ordinary parking functions $PF(K_{n+1})$ as PF_n in Chapter 4.

We now define the Laplace matrix and signless Laplace matrix of a graph. Let $D_G = \text{diag}[d_0, d_1, \ldots, d_n]$ be the diagonal matrix of order $n+1$, where d_i is the degree of the vertex i. The *Laplace matrix* L_G and the *signless Laplace matrix* Q_G of G are given by

$$L_G = D_G - A(G) \quad \text{and} \quad Q_G = D_G + A(G).$$

By deleting rows and columns corresponding to the root 0 from L_G and Q_G, we respectively obtain *truncated (or reduced) Laplace matrix* $\widetilde{L}_G$ and *truncated (or reduced) signless Laplace matrix* $\widetilde{Q}_G$ of G. A *spanning tree* of a graph G is a subgraph $T \subseteq G$ such that the vertex set $V(T) = V(G)$ and T does not contain any cycle. The *Matrix-Tree Theorem* [50, Theorem 5.6.8] says that the number of spanning trees N_G of a graph G equals to $\det \widetilde{L}_G$. Although $\widetilde{L}_G$ is obtained from L_G by deleting the row and column corresponding to root vertex, it is well known that the number N_G does not depend on the vertex chosen. The number of G-parking functions are related to $\widetilde{L}_G$ by the following theorem:

Theorem 3.1.5. [16] *The number of G-parking functions of a graph G is given by* $\det \widetilde{L}_G$*, the number of spanning trees of G.*

Fix a field $\mathbb{K}$. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring on n variables. Sometimes, we write $R = R_n$ to indicate the number of variables in the polynomial ring. The G-parking function ideal is a monomial ideal in R_n.

Definition 3.1.6 (G-parking function ideal). [45] *For a graph G on the vertex set $V = \{0\} \cup [n]$ the G-parking function ideal $\mathcal{M}_G$ in R is given by*

$$\mathcal{M}_G = \left\langle m_A = \prod_{i \in A} x_i^{d_A(i)} : \emptyset \neq A \subseteq [n] \right\rangle.$$

Note that $R/\mathcal{M}_G$ is a finite dimensional vector space over the field $\mathbb{K}$. Also, a sequence $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is a G-parking function if $\mathbf{x}^{\mathbf{p}} = \prod_{i=1}^n x_i^{p_i}$ is a standard monomial of $R/\mathcal{M}_G$ (i.e., $\mathbf{x}^{\mathbf{p}} \notin \mathcal{M}_G$). Postnikov and Shapiro reproved Theorem 3.1.5 in an algebraic context.

Theorem 3.1.7. [45, Theorem 2.1] *For a graph G on the vertex set $V = \{0\} \cup [n]$,*

$$\dim_{\mathbb{K}} \left(\frac{R}{\mathcal{M}_G} \right) = \det \widetilde{L}_G = N_G.$$

Another important generalization of the parking function is the λ-parking function (or vector parking function) [17].

Definition 3.1.8 (λ-parking function)**.** *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{N}^n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 1$. A finite sequence $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is called a* λ-parking function *if a non-decreasing rearrangement $p_{j_1} \leq p_{j_2} \leq \cdots \leq p_{j_n}$ of $\mathbf{p}$ satisfies $p_{j_i} < \lambda_{n-i+1}$ for all i.*

A parking function is a λ-parking function for $\lambda = (n, n-1, n-2, \ldots, 1)$. Let $PF(\lambda)$ denote the set of λ-parking functions. The number $|PF(\lambda)|$ can be evaluated by the Steck determinant formula. For this, we first define the $n \times n$ Steck matrix $\Lambda(\lambda)$ whose $(i,j)^{th}$ entry is given by $\frac{\lambda_{n-i+1}^{j-i+1}}{(j-i+1)!}$ if $i \leq j+1$, and 0, otherwise (See [53]). In other words,

$$\Lambda(\lambda)=\Lambda(\lambda_1,\lambda_2,\ldots,\lambda_n)=\begin{pmatrix}\lambda_n & \frac{\lambda_n^2}{2!} & \frac{\lambda_n^3}{3!} & \cdots & \frac{\lambda_n^{n-1}}{(n-1)!} & \frac{\lambda_n^n}{n!}\\ 1 & \lambda_{n-1} & \frac{\lambda_{n-1}^2}{2!} & \cdots & \frac{\lambda_{n-1}^{n-2}}{(n-2)!} & \frac{\lambda_{n-1}^{n-1}}{(n-1)!}\\ 0 & 1 & \lambda_{n-2} & \cdots & \frac{\lambda_{n-2}^{n-3}}{(n-3)!} & \frac{\lambda_{n-2}^{n-2}}{(n-2)!}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \lambda_2 & \frac{\lambda_2^2}{2!}\\ 0 & 0 & 0 & \cdots & 1 & \lambda_1\end{pmatrix}_{n\times n}.$$

Theorem 3.1.9 (Steck)**.** $|PF(\lambda)|=n!\det(\Lambda(\lambda))$.

Proof. A proof of the above theorem can be found in [28, Theorem 2.8]. □

Given $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)\in\mathbb{N}^n$, consider the monomial ideal $\mathcal{M}_\lambda=\left\langle(\prod_{i\in A}x_i)^{\lambda_{|A|}}:\emptyset\neq A\subseteq[n]\right\rangle$ in R.

Proposition 3.1.10. *The standard monomials of $R/\mathcal{M}_\lambda$ are precisely the λ-parking functions for $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)$.*

Proof. Let $\mathbf{p}=(p_1,p_2,\ldots,p_n)$ be such that $\mathbf{x}^{\mathbf{p}}$ is a standard monomial of $R/\mathcal{M}_\lambda$. For each $\emptyset\neq A\subseteq[n]$, there exists some $i_A\in A$ such that $p_{i_A}<\lambda_{|A|}$. Take $A_1=[n]$. There exists $j_1\in A_1$ such that $p_{j_1}<\lambda_n$. Take $A_2=A_1\setminus\{j_1\}$. There exists $j_2\in A_2$ such that $p_{j_2}<\lambda_{n-1}$. Continuing this way we see that $p_{j_i}<\lambda_{n-i+1}$ for $1\le i\le n$. Thus $\mathbf{p}$ is a λ-parking function. □

By Theorem 3.1.9, we have

$$\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}_\lambda}\right)=|PF(\lambda)|=n!\det\left(\Lambda(\lambda)\right).$$

For particular values of λ, the numbers $\det(\Lambda(\lambda))$ are interesting. Let x be a variable and $b\in\mathbb{N}$. Suppose $f_{n,b}(x)=\det(\Lambda(x+(n-1)b,x+(n-2)b,\ldots,x+b,x))$ and $g_{n,b}(x)=\det\left(\Lambda(x+b,\underbrace{x,\ldots,x}_{n-1})\right)$. The functions $f_{n,b}(x)$ and $g_{n,b}(x)$ are polynomials in x of degree n.

Lemma 3.1.11. *$g_{n,b}(x)$ is a polynomial in x of degree n given by*

$$g_{n,b}(x) = \frac{x^{n-1}(x+nb)}{n!}.$$

Proof. We prove this by induction on n, following the proof of [32, Proposition 2.1]. Consider the Steck determinant

$$g_{n,b}(x) = \det \begin{bmatrix} x & \frac{x^2}{2!} & \frac{x^3}{3!} & \cdots & \frac{x^{n-1}}{(n-1)!} & \frac{x^n}{n!} \\ 1 & x & \frac{x^2}{2!} & \cdots & \frac{x^{n-2}}{(n-2)!} & \frac{x^{n-1}}{(n-1)!} \\ 0 & 1 & x & \cdots & \frac{x^{n-3}}{(n-3)!} & \frac{x^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & \frac{x^2}{2!} \\ 0 & 0 & 0 & \cdots & 1 & x+b \end{bmatrix}_{n\times n}.$$

Clearly, $g_{1,b}(x) = x+b$ and $g_{2,b}(x) = \frac{x(x+2b)}{2!}$. Assume that $g_{m,b}(x) = \frac{x^{m-1}(x+mb)}{m!}$ for $1 \le m < n$. Let $\mathcal{C}_i$ be the i^{th} column of the above matrix. Then $g_{n,b}(x) = \det[\mathcal{C}_1, \ldots, \mathcal{C}_n]$. Now $g'_{n,b}(x) = \sum_{i=1}^{n} \det[\mathcal{C}_1, \ldots, \mathcal{C}'_i, \ldots, \mathcal{C}_n]$, where $\mathcal{C}'_i$ is the derivative of $\mathcal{C}_i$ with respect to x. Since $\mathcal{C}'_i = \mathcal{C}_{i-1}$ for $i \ge 2$, $\det[\mathcal{C}_1, \ldots, \mathcal{C}'_i, \ldots, \mathcal{C}_n] = 0$ for $i = 2, \ldots, n$. Hence, $g'_{n,b}(x) = \det[\mathcal{C}'_1, \mathcal{C}_2, \ldots, \mathcal{C}_n]$. Note that $\mathcal{C}'_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Therefore, expanding along the first column we get $g'_{n,b}(x) = \det \Lambda(\lambda_1, \ldots, \lambda_{n-1})$, where $\lambda_1 = x+b$ and $\lambda_i = x$ for $2 \le i \le n-1$. Hence, $g'_{n,b}(x) = g_{n-1,b}(x) = \frac{x^{n-2}(x+(n-1)b)}{(n-1)!}$. Since $g_{n,b}(0) = 0$, upon integrating, we have $g_{n,b}(x) = \frac{x^{n-1}(x+nb)}{n!}$. $\square$

Lemma 3.1.12. [34, Proposition 2.5] *$f_{n,b}(x)$ is a polynomial in x of degree n given by*

$$f_{n,b}(x) = \frac{x(x+nb)^{n-1}}{n!}.$$

Proof. We have $f_{1,b}(x) = x$ and $f_{2,b}(x) = \frac{x(x+2b)}{2!}$. Our proof is by induction on n. Assume that $f_{j,b}(x) = \frac{x(x+jb)^{j-1}}{j!}$ for $j \in [n-1]$ and for all x. Using the properties of determinant we see that $f'_{n,b}(x) = f_{n-1,b}(x+b)$, where $f'_{n,b}(x)$ is the derivative of $f_{n,b}(x)$. Hence, $f'_{n,b}(x) = \frac{(x+b)(x+nb)^{n-2}}{(n-1)!}$. Since $f_{n,b}(0) = 0$, after integrating $f'_{n,b}(x)$, we have $f_{n,b}(x) = \frac{x(x+nb)^{n-1}}{n!}$. $\square$

The functions $f_{n,b}(x)$ and $g_{n,b}(x)$ also enumerate standard monomials of some G-parking function ideals for some particular values of x and b. For $G = K^{a,b}_{n+1}$, the complete multigraph on $n+1$ vertices,

$$\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}_{K^{a,b}_{n+1}}}\right) = |PF(\lambda)|, \tag{3.1}$$

for $\lambda = (a+(n-1)b, a+(n-2)b, \ldots, a+b, a)$. Therefore, by Lemma 3.1.12, $\dim_{\mathbb{K}}(R/\mathcal{M}_{K^{a,b}_{n+1}}) = a(a+nb)^{n-1}$, which is equal to the number of spanning trees of the complete multigraph $K^{a,b}_{n+1}$. In fact, Gaydarov and Hopkins [17] have completely characterized the overlap between G-parking functions and λ-parking functions. They have shown the following in [17, Theorem 2.5].

Theorem 3.1.13. *A G-parking function is a λ-parking function if and only if one of the following holds:*

- *G is an a-tree and $PF(G) = PF(\lambda)$, where $\lambda = (a, a, \ldots, a)$ for some $a \geq 1$;*
- *G is an a-cycle and $PF(G) = PF(\lambda)$, where $\lambda = (2a, \underbrace{a, \ldots, a}_{n-1})$ for some $a \geq 1$;*
- *$G = K^{a,b}_{n+1}$ and $PF(G) = PF(\lambda)$, where $\lambda = (a+(n-1)b, a+(n-2)b, \ldots, a)$ for some $a, b \geq 1$.*

Here an a-tree is a tree where every edge has multiplicity a and similarly an a-cycle is a cycle with multiplicity of each edge being a.

3.2 Skeleton ideals and the inequality

The ideals $\mathcal{M}_G$ have connections to *chip-firing* on G. Motivated by some constructions in 'hereditary set' chip-firing, Dochtermann [14] introduced the notion of k-skeleton ideals which are by definition subideals of the G-parking function ideal $\mathcal{M}_G$. Here we consider the graph G to be an undirected (multi) graph but the definitions can be extended to directed graphs also.

Definition 3.2.1 (**k-skeleton ideal**). *Let G be a graph on the vertex set $\{0\} \cup [n]$. For an integer k with $0 \leq k \leq n-1$, the k-skeleton ideals $\mathcal{M}_G^{(k)}$ are given by*

$$\mathcal{M}_G^{(k)} = \left\langle m_A = \prod_{i \in A} x_i^{d_A(i)} : \emptyset \neq A \subseteq [n],\ |A| \leq k+1 \right\rangle.$$

For example, if $G = C_5$, the cycle graph on five vertices $\{0,1,2,3,4\}$ and $k=1$, then $\mathcal{M}_{C_5}^{(1)} = \langle x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_2x_3, x_3x_4 \rangle$. Clearly, for $k = n-1$, we get $\mathcal{M}_G^{(n-1)} = \mathcal{M}_G$. Thus the k-skeleton ideals are in fact a generalization of the G-parking function ideals. For $k=0$, the ideal $\mathcal{M}_G^{(0)}$ is generated by the monomials $x_i^{d_i}$, where d_i is the degree of the vertex i. The standard monomials of $R/\mathcal{M}_G^{(0)}$ are easy to describe. By definition they are the monomials

$$\left\{ \prod_{i=1}^{n} x_i^{\alpha_i} : 0 \leq \alpha_i < d_i \right\}.$$

We proceed to study the standard monomials of $R/\mathcal{M}_G^{(1)}$. Recall that for $k = n-1$ the standard monomials are related to the truncated Laplace matrix $\widetilde{L}_G$. In fact, we have $\dim_{\mathbb{K}}\left(R/\mathcal{M}_G^{(n-1)}\right) = \det \widetilde{L}_G$ (Theorem 3.1.7), for any graph G. For one-skeleton ideals the truncated signless Laplace matrix $\widetilde{Q}_G$ makes an appearance. Let $G = K_{n+1}$, then the standard monomials of $R/\mathcal{M}_{K_{n+1}}^{(1)}$ and $\det \widetilde{Q}_{K_{n+1}}$ are related in the following way [14, Corollary 3.4]:

Theorem 3.2.2. $\dim_{\mathbb{K}}\left(R/\mathcal{M}_{K_{n+1}}^{(1)}\right) = \det \widetilde{Q}_{K_{n+1}}$.

One natural question to ask is what happens to the above equality if we remove one edge from a complete graph. Let us look at some examples:

Example 3.2.3. *Consider the graph G_1 on the vertex set $\{0,1,2,3\}$, obtained from the complete graph K_4 by removing the edge $(2,3)$. The one-skeleton ideal $\mathcal{M}_{G_1}^{(1)} = \langle x_1^3, x_2^2, x_3^2, x_1^2x_2, x_1^2x_3, x_2^2x_3^2\rangle$. By a simple calculation we see that $\dim_{\mathbb{K}}\left(R/\mathcal{M}_{G_1}^{(1)}\right) = 9$. The truncated signless Laplace matrix*

$$\widetilde{Q}_{G_1} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}_{3\times 3},$$

with $\det \widetilde{Q}_{G_1} = 8$. *Therefore, in this case we have* $\dim_{\mathbb{K}}\left(R/\mathcal{M}_{G_1}^{(1)}\right) > \det \widetilde{Q}_{G_1}$.

Example 3.2.4. *Let G_2 be the graph on the vertex set $\{0,1,2,3\}$ obtained from the complete graph K_4 by removing the edge $(0,3)$. We have $\mathcal{M}_{G_2}^{(1)} = \langle x_1^3, x_2^3, x_3^2, x_1^2x_2^2, x_1^2x_3, x_2^2x_3\rangle$ with $\dim_{\mathbb{K}}\left(R/\mathcal{M}_{G_2}^{(1)}\right) = 12$ and*

$$\widetilde{Q}_{G_2} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}_{3\times 3},$$

with $\det \widetilde{Q}_{G_2} = 12 = \dim_{\mathbb{K}}\left(R/\mathcal{M}_{G_2}^{(1)}\right)$.

Dochtermann asked the following question.

Question 3.2.5. [14, Question 3.7] *For any graph G is it true that*

$$\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}_G^{(1)}}\right) \geq \det \widetilde{Q}_G?$$

We show that the above inequality holds for any multigraph. For some results related to when the equality might occur in Question 3.2.5, see Section 3.3.

Let $n \geq 1$ and $M_n(\mathbb{N})$ be the set of all $n \times n$ matrices over nonnegative integers $\mathbb{N}$. Let

$$\mathcal{G}_n = \{H = [a_{i,j}] \in M_n(\mathbb{N}) : H^t = H \text{ and } a_{i,i} \geq \max_{j \neq i} a_{i,j} \text{ for } 1 \leq i \leq n\}.$$

For $H = [a_{i,j}]_{n \times n} \in \mathcal{G}_n$ with $\alpha_i = a_{i,i}$, we associate a monomial ideal

$$\mathcal{J}_H = \left\langle x_l^{\alpha_l}, x_i^{\alpha_i - a_{i,j}} x_j^{\alpha_j - a_{i,j}} : 1 \leq l \leq n, 1 \leq i < j \leq n \right\rangle$$

in the polynomial ring $R_n = \mathbb{K}[x_1, \ldots, x_n]$. If $H = \widetilde{Q}_G$, the truncated signless Laplace matrix of a multigraph G on $V = \{0, 1, \ldots, n\}$, then $\mathcal{J}_H = \mathcal{M}_G^{(1)}$. We will show that $\dim_{\mathbb{K}}(R_n/\mathcal{J}_H) \geq \det H$ for every positive semidefinite $H \in \mathcal{G}_n$. Therefore, as a corollary we get the inequality for any multigraph G. The proof uses the Courant-Weyl inequalities (Theorem 2.4.8) and Fischer's inequality (Theorem 2.4.7).

Theorem 3.2.6. *Let $H \in \mathcal{G}_n$ be positive semidefinite and $\mathcal{J}_H$ be the monomial ideal in the polynomial ring $R = R_n$ associated to H. Then*

$$\dim_{\mathbb{K}} \left(\frac{R_n}{\mathcal{J}_H} \right) \geq \det H.$$

Proof. We prove the theorem by induction on the order n of H. For $n = 1$, $H = [\alpha_1]_{1 \times 1}$ and $\mathcal{J}_H = \langle x_1^{\alpha_1} \rangle$, and thus $\dim_{\mathbb{K}} (R_1/\mathcal{J}_H) = \alpha_1 = \det H$. For $n = 2$,

$$H = \begin{bmatrix} \alpha_1 & a_{1,2} \\ a_{1,2} & \alpha_2 \end{bmatrix}_{2 \times 2} \text{ and } \mathcal{J}_H = \left\langle x_1^{\alpha_1}, x_2^{\alpha_2}, x_1^{\alpha_1 - a_{1,2}} x_2^{\alpha_2 - a_{1,2}} \right\rangle \subseteq R_2.$$

The standard monomials of $\mathcal{J}_H$ are $\{x_1^{t_1} x_2^{t_2} : t_1 < \alpha_1,\ t_2 < \alpha_2$ and, either $t_1 < \alpha_1 - a_{1,2}$ or $t_2 < \alpha_2 - a_{1,2}\}$. Thus $\dim_{\mathbb{K}} (R_2/\mathcal{J}_H) = \alpha_1 \alpha_2 - a_{1,2}^2 = \det H$.

Assume that $n \geq 3$ and the theorem holds for every positive semidefinite matrix in $\mathcal{G}_m$ for $m < n$. Let $H = [a_{i,j}]_{n \times n} \in \mathcal{G}_n$; $\alpha_i = a_{i,i}$, and $\max_{i \neq j} a_{i,j} = b$. On permuting rows and columns of H, we obtain $H' = \left[a'_{i,j}\right] \in \mathcal{G}_n$ similar to H such that there exists an integer r $(0 \leq r \leq n-2)$ satisfying $a'_{i,r+1} < b$ and $a'_{r+1,j} = b$ for $1 \leq i < r+1 < j \leq n$. The monomial ideal $\mathcal{J}_{H'}$ is obtained from $\mathcal{J}_H$ by renumbering variables. Thus $\dim_{\mathbb{K}} \left(R_n/\mathcal{J}_H\right) = \dim_{\mathbb{K}} \left(R_n/\mathcal{J}_{H'}\right)$ and $\det H = \det H'$. Hence, without loss of generality, assume that $H = H'$, i.e., there exists r $(0 \leq r \leq n-2)$ such that $a_{i,r+1} < b$ and $a_{r+1,j} = b$ for $1 \leq i < r+1 < j \leq n$. Now consider the short exact sequence of $\mathbb{K}$-vector spaces,

$$0 \to \frac{R_n}{\left(\mathcal{J}_H : x_{r+1}^{\alpha_{r+1}-b}\right)} \xrightarrow{\mu_{x_{r+1}^{\alpha_{r+1}-b}}} \frac{R_n}{\mathcal{J}_H} \xrightarrow{\nu} \frac{R_n}{\left\langle \mathcal{J}_H, x_{r+1}^{\alpha_{r+1}-b} \right\rangle} \to 0, \tag{3.2}$$

where $\mu_{x_{r+1}^{\alpha_{r+1}-b}}$ is the map induced by multiplication by $x_{r+1}^{\alpha_{r+1}-b}$ and ν is the quotient map.

Let $H_1 = \begin{bmatrix} \alpha_1 & a_{1,2} & \cdots & a_{1,r+1} \\ a_{1,2} & \alpha_2 & \cdots & a_{2,r+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,r+1} & a_{2,r+1} & \cdots & b \end{bmatrix}_{(r+1)\times(r+1)}$.

In other words, H_1 is the principal $(r+1) \times (r+1)$ submatrix of H consisting of the first $r+1$ rows and columns except the entry α_{r+1}, which is replaced by b. Then $H_1 \in \mathcal{G}_{r+1}$. If $\alpha_{r+1} = b$ in H, then H_1, being a principal submatrix of H, is positive semidefinite. In any case, we see that

$$\left(\mathcal{J}_H : x_{r+1}^{\alpha_{r+1}-b}\right) = \left\langle \mathcal{J}_{H_1}, x_l^{\alpha_l - b} : r+2 \leq l \leq n \right\rangle,$$

where $\mathfrak{I}_{H_1} \subseteq R_{r+1} = \mathbb{K}[x_1, \ldots, x_{r+1}]$. Thus

$$\dim_{\mathbb{K}}\left(\frac{R_n}{\left(\mathfrak{I}_H : x_{r+1}^{\alpha_{r+1}-b}\right)}\right) = \dim_{\mathbb{K}}\left(\frac{R_{r+1}}{\mathfrak{I}_{H_1}}\right) \cdot \left(\prod_{l=r+2}^{n}(\alpha_l - b)\right) \tag{3.3}$$

Let H_2 be the $(n-1)\times(n-1)$ submatrix of H obtained by deleting $(r+1)^{th}$ row and $(r+1)^{th}$ column. Since $H_2 \in \mathcal{G}_{n-1}$ is positive semidefinite, the monomial ideal $\mathfrak{I}_{H_2} \subseteq \mathbb{K}[x_1, \ldots, \hat{x}_{r+1}, \ldots, x_n] = R_{n-1}$ satisfies $\dim_{\mathbb{K}}(R_{n-1}/\mathfrak{I}_{H_2}) \geq \det H_2$, by induction assumption. Also, $\langle \mathfrak{I}_H, x_{r+1}^{\alpha_{r+1}-b}\rangle = \langle \mathfrak{I}_{H_2}, x_{r+1}^{\alpha_{r+1}-b}\rangle$. Thus

$$\dim_{\mathbb{K}}\left(\frac{R_n}{\left\langle \mathfrak{I}_H, x_{r+1}^{\alpha_{r+1}-b}\right\rangle}\right) = (\alpha_{r+1} - b)\dim_{\mathbb{K}}\left(\frac{R_{n-1}}{\mathfrak{I}_{H_2}}\right). \tag{3.4}$$

From (3.2), (3.3) and (3.4), we get

$$\dim_{\mathbb{K}}\left(\frac{R_n}{\mathfrak{I}_H}\right) = \left(\prod_{l=r+2}^{n}(\alpha_l - b)\right)\dim_{\mathbb{K}}\left(\frac{R_{r+1}}{\mathfrak{I}_{H_1}}\right) + (\alpha_{r+1} - b)\dim_{\mathbb{K}}\left(\frac{R_{n-1}}{\mathfrak{I}_{H_2}}\right). \tag{3.5}$$

As determinant is linear on columns, writing $\alpha_{r+1} = (\alpha_{r+1} - b) + b$ in H, we have

$$\det H = (\alpha_{r+1} - b)\det H_2 + \det T, \tag{3.6}$$

where T is the matrix obtained from H by replacing α_{r+1} with b. On applying elementary column and row operations, $\mathcal{C}_{r+2} - \mathcal{C}_{r+1}, \mathcal{R}_{r+2} - \mathcal{R}_{r+1}, \ldots, \mathcal{C}_n -$

$\mathcal{C}_{r+1}, \mathcal{R}_n - \mathcal{R}_{r+1}$ on T, it reduces to a $n \times n$ matrix T' described below

$$\begin{bmatrix} \alpha_1 & \cdots & a_{1,r} & a_{1,r+1} & a_{1,r+2}-a_{1,r+1} & \cdots & a_{1,n}-a_{1,r+1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1,r} & \cdots & \alpha_r & a_{r,r+1} & a_{r,r+2}-a_{r,r+1} & \cdots & a_{r,n}-a_{r,r+1} \\ a_{1,r+1} & \cdots & a_{r,r+1} & b & 0 & \cdots & 0 \\ a_{1,r+2}-a_{1,r+1} & \cdots & a_{r,r+2}-a_{r,r+1} & 0 & \alpha_{r+2}-b & \cdots & a_{r+2,n}-b \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1,n}-a_{1,r+1} & \cdots & a_{r,n}-a_{r,r+1} & 0 & a_{r+2,n}-b & \cdots & \alpha_n-b \end{bmatrix}.$$

Let $\epsilon_{i,j}$ be the $n \times n$ matrix with 1 at $(i,j)^{th}$ place and zero elsewhere. Then the matrix $P = I_n - (\epsilon_{r+1,r+2} + \cdots + \epsilon_{r+1,n})$ has determinant $\det P = 1$ and $P^tTP = T'$. Thus $\det T = \det T'$. Now we consider two cases:

Case I : $\det T \leq 0$. Then from (3.6), $\det H \leq (\alpha_{r+1} - b) \det H_2$. Thus by induction assumption and (3.5), we get

$$\det H \leq (\alpha_{r+1} - b) \dim_{\mathbb{K}} \left(\frac{R_{n-1}}{\mathcal{J}_{H_2}} \right) \leq \dim_{\mathbb{K}} \left(\frac{R_n}{\mathcal{J}_H} \right).$$

Case II : $\det T > 0$. If $\alpha_{r+1} = b$, then $H = T$ is positive definite. Otherwise, $H = T + S$, where $S = (\alpha_{r+1} - b)\epsilon_{r+1,r+1}$. Clearly, $\lambda_1(S) = \cdots = \lambda_{n-1}(S) = 0$ and $\lambda_n(S) = \alpha_{r+1} - b$. Since H is positive semidefinite, $0 \leq \lambda_1(H) \leq \lambda_2(H) \leq \cdots \leq \lambda_n(H)$. Taking $i = j = 1$ in the Courant-Weyl inequalities with $H = T + S$, we obtain $\lambda_1(H) \leq \lambda_2(T) + \lambda_{n-1}(S) = \lambda_2(T)$. Thus $0 \leq \lambda_2(T) \leq \cdots \leq \lambda_n(T)$. As $\det T = \prod_{i=1}^n \lambda_i(T) > 0$, T must be positive definite. Hence $T' = P^tTP$ is also positive definite. Thus by Fischer's inequality,

$$\det T = \det T' \leq \det(H_1) \det(C),$$

where $C = \begin{bmatrix} \alpha_{r+2} - b & \cdots & a_{r+2,n} - b \\ \vdots & \ddots & \vdots \\ a_{r+2,n} - b & \cdots & \alpha_n - b \end{bmatrix}$

is positive definite. By Fischer's inequality, $\det C \le \prod_{l=r+2}^{n}(\alpha_l - b)$. Hence,

$$\det(T) \le \left(\prod_{l=r+2}^{n}(\alpha_l - b)\right)\det(H_1). \tag{3.7}$$

From (3.6) and (3.7),

$$\det H \le \left(\prod_{l=r+2}^{n}(\alpha_l - b)\right)\det H_1 + (\alpha_{r+1} - b)\det H_2.$$

By (3.5) and induction assumption (T positive definite implies so is H_1), we have

$$\begin{aligned}\dim_{\mathbb{K}}\left(\frac{R_n}{\mathfrak{J}_H}\right) &\ge \left(\prod_{l=r+2}^{n}(\alpha_l - b)\right)\det H_1 + (\alpha_{r+1} - b)\det H_2\\ &\ge \det H.\end{aligned}$$

□

Corollary 3.2.7. *Let G be a multigraph on $V = \{0, 1, \ldots, n\}$. Then*

$$\dim_{\mathbb{K}}\left(\frac{R_n}{\mathcal{M}_G^{(1)}}\right) \ge \det \widetilde{Q}_G.$$

Proof. The matrix $\widetilde{Q}_G$ is a diagonally dominant matrix, i.e, if $\widetilde{Q}_G = [a_{i,j}]$, then $a_{i,i} \ge \sum_{j\ne i}|a_{i,j}|$ for each i. By Geršgorin disc theorem ([23, Theorem 6.1.1]), $\widetilde{Q}_G$ is a positive semidefinite matrix. Thus, taking $H = \widetilde{Q}_G$ in Theorem 3.2.6 we get our result. □

Remark 1. *It is natural to ask whether the condition in Theorem 3.2.6 is also necessary, i.e., if $H \in \mathcal{G}_n$ such that $\dim_{\mathbb{K}}(R_n/\mathfrak{J}_H) \ge \det H$ implies H must be positive semidefinite. As we always have $\dim_{\mathbb{K}}(R_n/\mathfrak{J}_H) \ge 0$, the answer to this question is obviously no if we take some $H \in \mathcal{G}_n$ for which $\det H < 0$. But in Example 3.2.8 below we provide a matrix $H \in \mathcal{G}_n$ such that H is not* *positive semidefinite but* $\dim_{\mathbb{K}}(R_n/\mathfrak{J}_H) > \det H > 0$*. In Example 3.2.9 we give an example of a matrix* H *in* $\mathcal{G}_n$ *for which* $\det H > \dim_{\mathbb{K}}(R_n/\mathfrak{J}_H) > 0$*. By the above theorem this matrix is not positive semidefinite, which can also be checked by showing that it has a negative eigenvalue.*

Example 3.2.8. *Consider the matrix*

$$H = \begin{bmatrix} 11 & 1 & 2 & 8 & 6 & 4 \\ 1 & 10 & 5 & 7 & 1 & 9 \\ 2 & 5 & 10 & 8 & 7 & 1 \\ 8 & 7 & 8 & 9 & 5 & 2 \\ 6 & 1 & 7 & 5 & 10 & 1 \\ 4 & 9 & 1 & 2 & 1 & 10 \end{bmatrix}_{6\times 6}$$

Using Macaulay2 [19] *we can check that* $\dim_{\mathbb{K}}(R_n/\mathfrak{J}_H) = 11301 > 2868 = \det H$*. If* $f(x)$ *is the characteristic polynomial of* H*, then* $f(0) > 0$*, while* $f(-1) < 0$*. So,* H *is not positive semidefinite.*

Example 3.2.9. *Let*

$$H = \begin{bmatrix} 11 & 10 & 2 & 8 & 6 & 4 \\ 10 & 10 & 9 & 7 & 9 & 9 \\ 2 & 9 & 10 & 8 & 7 & 8 \\ 8 & 7 & 8 & 9 & 5 & 2 \\ 6 & 9 & 7 & 5 & 10 & 9 \\ 4 & 9 & 8 & 2 & 9 & 10 \end{bmatrix}_{6\times 6}.$$

Again calculations using Macaulay2 [19] *show that* $\det H = 19216 > 355 = \dim_{\mathbb{K}}(R_n/\mathfrak{J}_H)$*. Also the matrix has an eigenvalue in the interval* $(-2, 0)$*. Consequently,* H *is not a positive semidefinite matrix.*

3.3 The equality condition

In this section we consider the graphs G such that $\dim_{\mathbb{K}}(R/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$. We give some examples of family of graphs G for which the above equality holds. Moreover, we characterize all subgraphs of $K_{n+1}^{a,1}$, in particular all simple graphs G which satisfies $\dim_{\mathbb{K}}(R/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$.

Recall that for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{N}^n$ we have defined $\mathcal{M}_\lambda = \left\langle (\prod_{i\in A} x_i)^{\lambda_{|A|}} : \emptyset \neq A \subseteq [n] \right\rangle$ in the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$. The standard monomials of $R/\mathcal{M}_\lambda$ are all the λ-parking functions. By definition, $\mathcal{M}_{K_{n+1}}^{(1)} = \mathcal{M}_\lambda$ for $\lambda = (n, n-1, \ldots, n-1)$ and $\mathcal{M}_{K_{n+1}^{a,b}}^{(1)} = \mathcal{M}_\lambda$ for $\lambda = (a+(n-1)b, a+(n-2)b, \ldots, a+(n-2)b)$.

We can easily see that $\det\left(\widetilde{Q}_{K_{n+1}^{a,b}}\right) = (a+(n-2)b)^{n-1}(a+(2n-2)b)$. Consequently, using Lemma 3.1.11 we have,

$$\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}_{K_{n+1}^{a,b}}^{(1)}}\right) = (n!)g_{n,b}(a+(n-2)b) = \det\left(\widetilde{Q}_{K_{n+1}^{a,b}}\right). \tag{3.8}$$

In case $a = b = 1$, i.e., for the complete simple graph K_{n+1} we have

$$\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}_{K_{n+1}}^{(1)}}\right) = (n!)g_{n,1}(n-1) = (n-1)^{n-1}(2n-1) = \det\left(\widetilde{Q}_{K_{n+1}}\right),$$

as shown also in Theorem 3.2.2.

We have seen that $\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}_G^{(1)}}\right) = \det\left(\widetilde{Q}_G\right)$ in case G is the complete simple graph or the complete multigraph. We show next that the equality in (3.8) also holds if we delete some edges through the root 0 from a complete multigraph $K_{n+1}^{a,b}$. We first check this for such simple graphs.

More precisely, let $0 \le r \le n$ and $G_{n,r}$ be the graph obtained from K_{n+1} by deleting exactly r edges through the root 0. We have $G_{n,0} = K_{n+1}$. On renumbering vertices, we assume that the deleted edges are between 0 and

i for $n-r+1 \leq i \leq n$. We proceed to verify that $\dim_{\mathbb{K}}\left(R/\mathcal{M}^{(1)}_{G_{n,r}}\right) = \det\left(\widetilde{Q}_{G_{n,r}}\right)$.

Let a be a fixed positive integer and let ω be a weight function (depending on $r \in [0,n]$) given by

$$\omega(i) = \begin{cases} a & \text{if} \quad i \in [n-r], \\ a-1 & \text{if} \quad i \in [n] \setminus [n-r]. \end{cases}$$

Let $I^{\langle a \rangle}_{n,r}$ be a monomial ideal in $R_n = \mathbb{K}[x_1, \ldots, x_n]$ given by

$$I^{\langle a \rangle}_{n,r} = \left\langle x_i^{\omega(i)}, x_i^{\omega(i)-1} x_j^{\omega(j)-1} : i,j \in [n] \quad \text{and} \quad i \neq j \right\rangle.$$

Clearly, $I^{\langle n \rangle}_{n,r} = \mathcal{M}^{(1)}_{G_{n,r}}$.

Lemma 3.3.1. *Let $r \geq 1$. Then $\left(I^{\langle a \rangle}_{n,r-1} : x_{n-r+1}\right) = I^{\langle a \rangle}_{n,r}$ and there exists a short exact sequence of $R = R_n$ modules (or $\mathbb{K}$-vector spaces)*

$$0 \to \frac{R_n}{I^{\langle a \rangle}_{n,r}} \xrightarrow{\mu_{x_{n-r+1}}} \frac{R_n}{I^{\langle a \rangle}_{n,r-1}} \xrightarrow{\nu} \frac{R_n}{\left\langle I^{\langle a \rangle}_{n,r-1}, x_{n-r+1} \right\rangle} \to 0, \tag{3.9}$$

where $\mu_{x_{n-r+1}}$ is the map induced by multiplication by x_{n-r+1} and ν is the natural projection.

Proof. Consider the map $\mu_{x_{n-r+1}} : R_n \to \frac{R_n}{I^{\langle a \rangle}_{n,r-1}}$ given by $\mu_{x_{n-r+1}}(f) = x_{n-r+1} f + I^{\langle a \rangle}_{n,r-1}$ for $f \in R_n$. Then $\ker \mu_{x_{n-r+1}} = \left(I^{\langle a \rangle}_{n,r-1} : x_{n-r+1}\right) = I^{\langle a \rangle}_{n,r}$ and this produces the short exact sequence in (3.9). □

Lemma 3.3.2. *Let $r \geq 1$. Then*

$$\dim_{\mathbb{K}}\left(\frac{R_n}{I^{\langle a \rangle}_{n,r}}\right) = \dim_{\mathbb{K}}\left(\frac{R_n}{I^{\langle a \rangle}_{n,r-1}}\right) - \dim_{\mathbb{K}}\left(\frac{R_{n-1}}{I^{\langle a \rangle}_{n-1,r-1}}\right).$$

Proof. We see that $\left\langle I^{\langle a \rangle}_{n,r-1}, x_{n-r+1} \right\rangle = \left\langle I^{\langle a \rangle}_{n,r}, x_{n-r+1} \right\rangle$. Also, $\frac{R_n}{\left\langle I^{\langle a \rangle}_{n,r}, x_{n-r+1} \right\rangle} \cong$

$\frac{R_{n-1}}{I^{\langle a\rangle}_{n-1,r-1}}$ as $\mathbb{K}$-vector spaces. From the short exact sequence of $\mathbb{K}$-vector spaces (3.9), we have

$$\begin{aligned}\dim_{\mathbb{K}}\left(\frac{R_n}{I^{\langle a\rangle}_{n,r}}\right) &= \dim_{\mathbb{K}}\left(\frac{R_n}{I^{\langle a\rangle}_{n,r-1}}\right) - \dim_{\mathbb{K}}\left(\frac{R_n}{\left\langle I^{\langle a\rangle}_{n,r-1}, x_{n-r+1}\right\rangle}\right)\\ &= \dim_{\mathbb{K}}\left(\frac{R_n}{I^{\langle a\rangle}_{n,r-1}}\right) - \dim_{\mathbb{K}}\left(\frac{R_{n-1}}{I^{\langle a\rangle}_{n-1,r-1}}\right).\end{aligned}$$

□

Lemma 3.3.3. (i) $\dim_{\mathbb{K}}\left(\frac{R_n}{I^{\langle a\rangle}_{n,0}}\right) = (a-1)^{n-1}(a+(n-1))$.

(ii) *For* $0 \leq r \leq n$,

$$\dim_{\mathbb{K}}\left(\frac{R_n}{I^{\langle a\rangle}_{n,r}}\right) = \sum_{i=0}^{r}(-1)^i\binom{r}{i}(a-1)^{n-i-1}(a+(n-i-1)).$$

Proof. We have $I^{\langle a\rangle}_{n,0} = \langle x_i^a, (x_ix_j)^{a-1} : i,j \in [n];\, i \neq j\rangle$. Thus $\dim_{\mathbb{K}}\left(\frac{R_n}{I^{\langle a\rangle}_{n,0}}\right)$ is equal to the number of λ-parking functions for $\lambda = (a, a-1, \ldots, a-1) \in \mathbb{N}^n$. Here $\lambda = (x+1, x, \ldots, x)$ for $x = a-1$ and $b = 1$. Therefore,

$$\begin{aligned}\dim_{\mathbb{K}}\left(\frac{R_n}{I^{\langle a\rangle}_{n,0}}\right) &= n!\det(\Lambda(\lambda)) = (n!)g_n(a-1)\\ &= (a-1)^{n-1}(a+n-1) \quad \text{(by Lemma 3.1.11)}.\end{aligned}$$

This proves (i).

We shall prove (ii) by induction on r.

For $r = 0$, it follows from (i).

Assume $r \geq 1$. We have

$$\dim_{\mathbb{K}}\left(\frac{R_n}{I^{\langle a\rangle}_{n,r}}\right) = \dim_{\mathbb{K}}\left(\frac{R_n}{I^{\langle a\rangle}_{n,r-1}}\right) - \dim_{\mathbb{K}}\left(\frac{R_{n-1}}{I^{\langle a\rangle}_{n-1,r-1}}\right).$$

Suppose $\theta_l(x) = x^{l-1}(x+l)$. Then by induction assumption, for $n \geq r \geq 1$, we have

$$\begin{aligned}
\dim_{\mathbb{K}}\left(\frac{R_n}{I^{\langle a\rangle}_{n,r-1}}\right) &= \sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\theta_{n-i}(a-1) \quad \text{and}\\
\dim_{\mathbb{K}}\left(\frac{R_{n-1}}{I^{\langle a\rangle}_{n-1,r-1}}\right) &= \sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\theta_{n-1-i}(a-1)\\
&= \sum_{i=1}^{r}(-1)^{i-1}\binom{r-1}{i-1}\theta_{n-i}(a-1).
\end{aligned}$$

Thus,

$$\begin{aligned}
\dim_{\mathbb{K}}\left(\frac{R_n}{I^{\langle a\rangle}_{n,r}}\right) &= \sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\theta_{n-i}(a-1) + \sum_{i=1}^{r}(-1)^i\binom{r-1}{i-1}\theta_{n-i}(a-1)\\
&= \theta_n(a-1) + \sum_{i=1}^{r-1}(-1)^i\left[\binom{r-1}{i}+\binom{r-1}{i-1}\right]\theta_{n-i}(a-1)\\
&\qquad + (-1)^r\theta_{n-r}(a-1)\\
&= \sum_{i=0}^{r}(-1)^i\binom{r}{i}\theta_{n-i}(a-1)\\
&= \sum_{i=0}^{r}(-1)^i\binom{r}{i}(a-1)^{n-i-1}(a+(n-i-1)).
\end{aligned}$$

□

Remark 2. *Note that* $I^{\langle a\rangle}_{n,n} = I^{\langle a-1\rangle}_{n,0}$ *for* $a \geq 2$. *Thus we obtain an interesting combinatorial identity:*

$$(a-2)^{n-1}(a+(n-2)) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(a-1)^{n-i-1}(a+(n-i-1)) \quad \textit{for} \quad n \geq 2.$$

Being a polynomial identity in a, *it is valid for all* $a \in \mathbb{R}$.

Proposition 3.3.4. $\dim_{\mathbb{K}}\left(\frac{R_n}{\mathcal{M}^{(1)}_{G_{n,r}}}\right) = \det\left(\widetilde{Q}_{G_{n,r}}\right)$.

Proof. The determinant of the truncated signless Laplace matrix $\widetilde{Q}_{G_{n,r}}$ of

$G_{n,r}$ is given by

$$\det\left(\widetilde{Q}_{G_{n,r}}\right) = (n-1)^{n-r-1}(n-2)^{r-1}\left[(2n-1)(n-2)+r\right]. \tag{3.10}$$

Indeed, on applying the column operation $\mathcal{C}_1 + (\mathcal{C}_2 + \cdots + \mathcal{C}_n)$ on $\widetilde{Q}_{G_{n,r}}$ followed by the row operations $\mathcal{R}_2 - \mathcal{R}_1, \mathcal{R}_3 - \mathcal{R}_1, \ldots, \mathcal{R}_n - \mathcal{R}_1$, $\widetilde{Q}_{G_{n,r}}$ reduces to the matrix

$$\begin{bmatrix} 2n-1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & n-1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n-1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & n-2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 0 & \cdots & n-2 \end{bmatrix}_{n\times n},$$

where the diagonal elements $n-2$ appear in last r rows. Now expanding the determinant along the first column, we get (3.10).

Also, $I_{n,r}^{\langle n\rangle} = \mathcal{M}_{G_{n,r}}^{(1)}$ and from Lemma 3.3.3, we have

$$\begin{aligned} &\dim_{\mathbb{K}}\left(\frac{R_n}{\mathcal{M}_{G_{n,r}}^{(1)}}\right) \\ &= \sum_{i=0}^{r}(-1)^i\binom{r}{i}(n-1)^{n-i-1}((2n-1)-i) \\ &= (n-1)^{n-r-1}(2n-1)\left\{\sum_{i=0}^{r}(-1)^i\binom{r}{i}(n-1)^{r-i}\right\} \\ &\qquad\qquad + (n-1)^{n-r-1}\left\{\sum_{i=0}^{r}(-1)^{i+1}\binom{r}{i}i(n-1)^{r-i}\right\}. \end{aligned}$$

Adding up we get,

$$\begin{aligned}\dim_{\mathbb{K}}\left(\frac{R_n}{\mathcal{M}^{(1)}_{G_{n,r}}}\right) &= (n-1)^{n-r-1}(n-2)^{r-1}\left[(2n-1)(n-2)+r\right]\\ &= \det\left(\widetilde{Q}_{G_{n,r}}\right).\end{aligned}$$

□

We now proceed to generalize Proposition 3.3.4 to multigraphs.

Theorem 3.3.5. *Let G be a multigraph on $V = \{0, 1, \ldots, n\}$ obtained from the complete multigraph $K^{a,b}_{n+1}$ by deleting some edges through the root 0. Then*

$$\dim_{\mathbb{K}}\left(\frac{R_n}{\mathcal{M}^{(1)}_{G}}\right) = \det \widetilde{Q}_G. \tag{3.11}$$

Proof. We shall prove this theorem by induction on n. For $n = 1$, $G = K^{a,0}_2$ for some $a \geq 0$. Then $\mathcal{M}^{(1)}_G = \langle x_1^a \rangle \subseteq R_1$ and $\widetilde{Q}_G = [a]$ and hence (3.11) holds. For $n = 2$, the adjacency matrix $A(G) = \begin{bmatrix} 0 & a_1 & a_2 \\ a_1 & 0 & b \\ a_2 & b & 0 \end{bmatrix}_{3\times 3}$ for some $a_1, a_2 \leq a$ and $b \geq 1$. Then $\mathcal{M}^{(1)}_G = \left\langle x_1^{a_1+b}, x_2^{a_2+b}, x_1^{a_1}x_2^{a_2} \right\rangle$ and $\widetilde{Q}_G = \begin{bmatrix} a_1+b & b \\ b & a_2+b \end{bmatrix}_{2\times 2}$. Again, $\dim_{\mathbb{K}}\left(R_2/\mathcal{M}^{(1)}_G\right) = (a_1+b)(a_2+b) - b^2 = \det \widetilde{Q}_G$ shows that (3.11) holds.

By induction assumption, suppose the theorem holds for multigraphs on the vertex set $\{0, 1, \ldots, m\}$; $m < n$, obtained from $K^{a,b}_{m+1}$ by deleting some edges through the root 0 for any $a, b \geq 1$. Let $n \geq 3$ and G be a multigraph on $V = \{0, 1, \ldots, n\}$ obtained from $K^{a,b}_{n+1}$ by deleting some edges through the root 0. The adjacency matrix $A(G) = [a_{ij}]_{(n+1)\times(n+1)}$ of G is of the form

$a_{0i} = a_{i0} = a_i \leq a$ and $a_{ij} = b$ for $i, j \in [n]$ with $i \neq j$. Then

$$\mathcal{M}_G^{(1)} = \left\langle x_l^{a_l+(n-1)b}, x_i^{a_i+(n-2)b} x_j^{a_j+(n-2)b} : 1 \leq l \leq n, 1 \leq i < j \leq n \right\rangle.$$

Let e_0 be a fixed edge from 0 to j in G ($1 \leq j \leq n$). Consider the multigraph $G_1 = G - e_0$ obtained from G by deleting the edge e_0. We see that $\mathcal{M}_{G_1}^{(1)} = \left(\mathcal{M}_G^{(1)} : x_j\right)$ and the sequence of $\mathbb{K}$-vector spaces

$$0 \rightarrow \frac{R_n}{\mathcal{M}_{G_1}^{(1)}} \xrightarrow{\mu_{x_j}} \frac{R_n}{\mathcal{M}_G^{(1)}} \xrightarrow{\nu} \frac{R_n}{\left\langle \mathcal{M}_G^{(1)}, x_j \right\rangle} \rightarrow 0 \tag{3.12}$$

is short exact, where as before μ_{x_j} is the map induced by multiplication by x_j and ν is the quotient map. Let G_2 be a multigraph on the vertex set $V \setminus \{j\}$ with adjacency matrix $A(G_2) = \left[a_{rs}^{(2)}\right]_{\substack{0 \leq r,s \leq n \\ r,s \neq j}}$; $a_{0,r}^{(2)} = a_r + b$, $a_{rs}^{(2)} = b$ for $r, s \in [n] \setminus \{j\}$, $r \neq s$. Then, writing $R_{n-1} = \mathbb{K}[x_1, \ldots, \hat{x}_j, \ldots, x_n]$ for the polynomial ring over $\mathbb{K}$ in $n-1$ variables $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$, we have

$$\frac{R_{n-1}}{\mathcal{M}_{G_2}^{(1)}} \cong \frac{R_n}{\left\langle \mathcal{M}_G^{(1)}, x_j \right\rangle}.$$

Thus from the short exact sequence (3.12), we get

$$\dim_{\mathbb{K}} \left(\frac{R_n}{\mathcal{M}_G^{(1)}} \right) = \dim_{\mathbb{K}} \left(\frac{R_n}{\mathcal{M}_{G_1}^{(1)}} \right) + \dim_{\mathbb{K}} \left(\frac{R_{n-1}}{\mathcal{M}_{G_2}^{(1)}} \right). \tag{3.13}$$

As determinant is linear on columns, we have

$$\det\left(\widetilde{Q}_G\right) = \det\left(\widetilde{Q}_{G_1}\right) + \det\left(\widetilde{Q}_{G_2}\right). \tag{3.14}$$

By induction assumption, $\dim_{\mathbb{K}} \left(\frac{R_{n-1}}{\mathcal{M}_{G_2}^{(1)}} \right) = \det\left(\widetilde{Q}_{G_2}\right)$. Thus from (3.13) and

(3.14), we see that

$$\dim_{\mathbb{K}}\left(\frac{R_n}{\mathcal{M}_G^{(1)}}\right)=\det\left(\widetilde{Q}_G\right) \iff \dim_{\mathbb{K}}\left(\frac{R_n}{\mathcal{M}_{G_1}^{(1)}}\right)=\det\left(\widetilde{Q}_{G_1}\right).$$

In other words, if the theorem holds for a multigraph G on V then it also holds for the multigraph $G_1 = G \setminus e_0$, and vice-versa. Since

$$\begin{aligned}\dim_{\mathbb{K}}\left(\frac{R_n}{\mathcal{M}_{K_{n+1}^{a,b}}^{(1)}}\right) &= (n!)g_n(a+(n-2)b)\\ &= (a+(n-2)b)^{n-1}(a+(2n-2)b)\\ &= \det\left(\widetilde{Q}_{K_{n+1}^{a,b}}\right),\end{aligned}$$

we see that (3.11) holds for G by deleting edges through the root from $K_{n+1}^{a,b}$, one by one. $\square$

We now describe an easy extension of Theorem 3.3.5. Let $\widetilde{K_n^b}$ denote the complete multigraph on vertex set $\{1, 2, \ldots, n\}$, i.e., between any two vertices of $\widetilde{K_n^b}$ there are exactly b edges. For a graph G and $W \subseteq V(G)$, the *induced subgraph* of G on the vertex set W, denoted by G_W, is the graph whose edge set consists of all the edges in G that have both endpoints in W.

Proposition 3.3.6. *Let G be a multigraph on $V = \{0, 1, \ldots, n\}$ and $\widetilde{G} = G_{V'}$ be the induced subgraph of G, where $V' = [n]$. Suppose $\widetilde{G} \cong \widetilde{K_{n_1}^{b_1}} \sqcup \widetilde{K_{n_2}^{b_2}} \sqcup \cdots \sqcup \widetilde{K_{n_r}^{b_r}}$. Then $\dim_{\mathbb{K}}(R/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$.*

Proof. Without loss of generality, suppose we can partition $[n] = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_r$ such that $G_{V_i} \cong \widetilde{K_{n_i}^{b_i}}$ for $1 \leq i \leq r$. Then the graphs $G_i = G_{V_i \cup \{0\}}$ are obtained from complete multigraphs $K_{n_i+1}^{a_i,b_i}$ for some $a_i > 0$, by deleting some edges through the root 0. Consequently, $\mathcal{M}_G^{(1)} = \sum_{i=1}^r \mathcal{M}_{G_i}^{(1)} R$ such that

$$\dim_{\mathbb{K}}\left(\frac{R}{\mathcal{M}_G^{(1)}}\right) = \prod_{i=1}^r \dim_{\mathbb{K}}\left(\frac{R_{n_i}}{\mathcal{M}_{G_i}^{(1)}}\right).$$

We see that the truncated signless Laplace matrix of G is a block diagonal matrix

$$\widetilde{Q}_G = \begin{bmatrix} \widetilde{Q}_{G_1} & & & \\ & \widetilde{Q}_{G_2} & & \\ & & \ddots & \\ & & & \widetilde{Q}_{G_r} \end{bmatrix}_{n\times n}.$$

Therefore,

$$\det \widetilde{Q}_G = \prod_{i=1}^{r} \det \widetilde{Q}_{G_i}.$$

Since by Theorem 3.3.5, $\dim_{\mathbb{K}}(R_{n_i}/\mathcal{M}_{G_i}^{(1)}) = \det \widetilde{Q}_{G_i}$, for each i, we see that $\dim_{\mathbb{K}}(R/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$. □

Recall that in Section 3.2 we have defined a monomial ideal $\mathfrak{J}_H$ for every $H \in \mathcal{G}_n$. Let G be a graph on the vertex set $\{0, 1, \ldots, n\}$. If we take $H = \widetilde{Q}_G$, then $\mathfrak{J}_H = \mathcal{M}_G^{(1)}$. The following theorem is a generalization of Theorem 3.3.5.

Theorem 3.3.7. *Let* $H = \begin{bmatrix} a_1 & b & b & \cdots & b \\ b & a_2 & b & \cdots & b \\ b & b & a_3 & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a_n \end{bmatrix}_{n\times n}$ *be a matrix over non-negative integers such that* $a_i \geq b$ *for* $1 \leq i \leq n$. *Then*

$$\dim_{\mathbb{K}}\left(\frac{R_n}{\mathfrak{J}_H}\right) = \det H.$$

Proof. We prove this by induction on n. For $n = 2$, the matrix $H = \begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix}_{2\times 2}$ and the ideal $\mathfrak{J}_H = \left\langle x_1^{a_1}, x_2^{a_2}, x_1^{a_1-b}x_2^{a_2-b}\right\rangle$. Thus $\dim_{\mathbb{K}}(R_2/\mathfrak{J}_H) = a_1a_2 - b^2 = \det H$.

Suppose $n \geq 3$ and the theorem is true for any m with $m < n$. If $b = 0$, then $\mathfrak{J}_H = \langle x_i^{a_i} : 1 \leq i \leq n\rangle$. Thus $\dim_{\mathbb{K}}(R_n/\mathfrak{J}_H) = \prod_{i=1}^n a_i = \det H$. If

$a_i = b$ for each i, then $\dim_{\mathbb{K}}(R_n/\mathfrak{J}_H) = 0 = \det H$. Hence, without loss of generality, assume that $a_1 > b > 0$. Let $r = a_1 - b > 0$. The ideal $\mathfrak{J}_H = \left\langle x_l^{a_l}, x_i^{a_i-b}x_j^{a_j-b} : 1 \leq l \leq n,\ 1 \leq i < j \leq n \right\rangle$.

Let $H_1 = \operatorname{diag}[b, a_2 - b, a_3 - b, \ldots, a_n - b]$ be the $n \times n$ diagonal matrix and H_2 be the matrix obtained from H by deleting row $\mathcal{R}_1$ and column $\mathcal{C}_1$. We see that $(\mathfrak{J}_H : x_1^r) = \mathfrak{J}_{H_1}$ and $\langle \mathfrak{J}_H, x_1^r\rangle = \langle \mathfrak{J}_{H_2}, x_1^r\rangle$. Therefore, $\dim_{\mathbb{K}}(R_n/\left(\mathfrak{J}_H : x_1^r\right)) = \dim_{\mathbb{K}}(R_n/\mathfrak{J}_{H_1}) = \det H_1$. Moreover, $\dim_{\mathbb{K}}(R_n/\left\langle \mathfrak{J}_H, x_1^r\right\rangle) = r\dim_{\mathbb{K}}(R_{n-1}/\mathfrak{J}_{H_2}) = r\det H_2$ (by induction hypothesis). Consider the short exact sequence of $\mathbb{K}$-vector spaces,

$$0 \to \frac{R_n}{(\mathfrak{J}_H : x_1^r)} \xrightarrow{\mu_{x_1^r}} \frac{R_n}{\mathfrak{J}_H}x \xrightarrow{\nu} \frac{R_n}{\langle \mathfrak{J}_H, x_1^r\rangle} \to 0, \tag{3.15}$$

where $\mu_{x_1^r}$ is the map induced by multiplication by x_1^r and ν is the natural quotient map. We have, $\dim_{\mathbb{K}}(R_n/\mathfrak{J}_H) = \dim_{\mathbb{K}}(R_n/\left(\mathfrak{J}_H : x_1^r\right)) + \dim_{\mathbb{K}}(R_n/\left\langle \mathfrak{J}_H, x_1^r\right\rangle) = \dim_{\mathbb{K}}(R_n/\mathfrak{J}_{H_1}) + r\dim_{\mathbb{K}}(R_{n-1}/\mathfrak{J}_{H_2})$. Hence,

$$\dim_{\mathbb{K}}(R_n/\mathfrak{J}_H) = \det H_1 + r\det H_2.$$

Writing $a_1 = b + r$ and using the additivity property of the determinant, we see that $\det H = r\det H_2 + \det A$, where A is the matrix obtained from H by replacing the element a_1 with b. On applying elementary column and row operations, $\mathcal{C}_2 - \mathcal{C}_1, \mathcal{R}_2 - \mathcal{R}_1, \ldots, \mathcal{C}_n - \mathcal{C}_1, \mathcal{R}_n - \mathcal{R}_1$ on A, it reduces to the matrix H_1. Thus, $\det A = \det H_1$. Hence, $\dim_{\mathbb{K}}(R_n/\mathfrak{J}_H) = \det H$.

$\square$

The following theorem is a generalization of Theorem 3.3.7.

Theorem 3.3.8. *Let* $H_i = \begin{bmatrix} \alpha_{i,1} & b_i & \cdots & b_i \\ b_i & \alpha_{i,2} & \cdots & b_i \\ \vdots & \vdots & \ddots & \vdots \\ b_i & b_i & \cdots & \alpha_{i,n_i} \end{bmatrix}_{n_i \times n_i}$ *with* $\alpha_{i,j} \geq b_i$ *and* A_i

be the $n_i \times n_i$ matrix with all entries equal to d_i. Consider the matrix

$$H = \begin{bmatrix} H_1 & A_1 & A_2 & \cdots & A_{r-1} \\ A_1 & H_2 & A_2 & \cdots & A_{r-1} \\ A_2 & A_2 & H_3 & \cdots & A_{r-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{r-1} & A_{r-1} & A_{r-1} & \cdots & H_r \end{bmatrix}_{\sum_{i=1}^{r} n_i \times \sum_{i=1}^{r} n_i} \tag{3.16}$$

with $b_1 \geq d_1$, $b_i \geq d_{i-1}$ for $2 \leq i \leq r$, and $d_i \geq d_{i+1}$ for $1 \leq i \leq r-2$. Assume that $\alpha_{i,j}$, b_i and d_i are all nonnegative integers. Suppose $n = \sum_{i=1}^{r} n_i$. Then $\dim_{\mathbb{K}}(R_n/\mathfrak{J}_H) = \det H$.

Proof. If $n = 2$, then we see that $\dim_{\mathbb{K}}(R_2/\mathfrak{J}_H) = \det H$. We prove the theorem by induction on the order n of H. Assume that $n \geq 3$ and the theorem holds for every $m \times m$ matrix of the above form for $m < n$. The monomial ideal $\mathfrak{J}_H$ is generated by the following monomials

$$\begin{aligned} x_{i,j}^{\alpha_{i,j}} &: 1 \leq i \leq r,\ 1 \leq j \leq n_i\,, \\ x_{i,u}^{\alpha_{i,u}-b_i} x_{i,v}^{\alpha_{i,v}-b_i} &: 1 \leq i \leq r,\ 1 \leq u < v \leq n_i\,, \\ x_{s,w}^{\alpha_{s,w}-d_{t-1}} x_{t,l}^{\alpha_{t,l}-d_{t-1}} &: 1 \leq s \leq r-1,\ 1 \leq w \leq n_s,\ s+1 \leq t \leq r,\ 1 \leq l \leq n_t\,. \end{aligned}$$

We now divide the proof into two cases:

Case I : $n_r = 1$. Let B_1 be the matrix obtained from H by deleting the row and column containing the diagonal element α_{r1} and B_2 be an $(n-1)\times(n-1)$ matrix whose all entries are d_{r-1}. Let $B_3 = B_1 - B_2$. We see that the ideals $\left(\mathfrak{J}_H : x_{r,1}^{\alpha_{r,1}-d_{r-1}}\right) = \left\langle \mathfrak{J}_{B_3}, x_{r,1}^{d_{r-1}} \right\rangle$ and $\left\langle \mathfrak{J}_H, x_{r,1}^{\alpha_{r,1}-d_{r-1}} \right\rangle = \left\langle \mathfrak{J}_{B_1}, x_{r,1}^{\alpha_{r,1}-d_{r-1}} \right\rangle$. By induction hypothesis,

$$\dim_{\mathbb{K}}\left(\frac{R_{n-1}}{\mathfrak{J}_{B_3}}\right) = \det B_3 \quad \text{and} \quad \dim_{\mathbb{K}}\left(\frac{R_{n-1}}{\mathfrak{J}_{B_1}}\right) = \det B_1.$$

Thus, $\dim_{\mathbb{K}}\left(R_n/(\mathcal{J}_H : x_{r,1}^{\alpha_{r,1}-d_{r-1}})\right) = d_{r-1}\dim_{\mathbb{K}}(R_{n-1}/\mathcal{J}_{B_3}) = d_{r-1}\det B_3$. Moreover, $\dim_{\mathbb{K}}\left(R_n/\left\langle \mathcal{J}_H, x_{r,1}^{\alpha_{r,1}-d_{r-1}}\right\rangle\right) = (\alpha_{r,1} - d_{r-1})\dim_{\mathbb{K}}(R_{n-1}/\mathcal{J}_{B_1}) = (\alpha_{r,1} - d_{r-1})\det B_1$. Using the short exact sequence of $\mathbb{K}$-vector spaces

$$0 \to \frac{R_n}{\left(\mathcal{J}_H : x_{r,1}^{\alpha_{r,1}-d_{r-1}}\right)} \xrightarrow{\mu_{x_{r,1}^{\alpha_{r,1}-d_{r-1}}}} \frac{R_n}{\mathcal{J}_H} \to \frac{R_n}{\left\langle \mathcal{J}_H, x_{r,1}^{\alpha_{r,1}-d_{r-1}}\right\rangle} \to 0,$$

we get $\dim_{\mathbb{K}}(R_n/\mathcal{J}_H) = d_{r-1}\det B_3 + (\alpha_{r,1} - d_{r-1})\det B_1$. As determinant is linear on columns, writing $\alpha_{r,1} = (\alpha_{r,1} - d_{r-1}) + d_{r-1}$, we have $\det H = (\alpha_{r,1} - d_{r-1})\det B_1 + \det B_4$, where B_4 is the matrix obtained from H by replacing the diagonal element $\alpha_{r,1}$ with d_{r-1}. Applying elementary column and row operations, $\mathcal{C}_1 - \mathcal{C}_n, \mathcal{R}_1 - \mathcal{R}_n, \ldots, \mathcal{C}_{n-1} - \mathcal{C}_n, \mathcal{R}_{n-1} - \mathcal{R}_n$ on B_4, we get $\det B_4 = d_{r-1}\det B_3$. Consequently, $\dim_{\mathbb{K}}(R_n/\mathcal{J}_H) = \det H$.

Case II : $n_r \geq 2$. Let B_5 be the matrix obtained from H by first deleting rows and columns containing the diagonal elements $\alpha_{r,1}, \alpha_{r,2}, \ldots, \alpha_{r,n_r-1}$ and then replacing the diagonal element α_{r,n_r} with b_r. Hence, B_5 is an $(n+1-n_r) \times (n+1-n_r)$ matrix. Note that the ideal $\left(\mathcal{J}_H : x_{r,n_r}^{\alpha_{r,n_r}-b_r}\right) = \left\langle \mathcal{J}_{B_5}, x_{r,1}^{\alpha_{r,1}-b_r}, \ldots, x_{r,n_r-1}^{\alpha_{r,n_r-1}-b_r}\right\rangle$. By induction hypothesis, we observe that $\dim_{\mathbb{K}}(R_{n+1-n_r}/\mathcal{J}_{B_5}) = \det B_5$. Thus

$$\begin{aligned}\dim_{\mathbb{K}}\left(\frac{R_n}{\left(\mathcal{J}_H : x_{r,n_r}^{\alpha_{r,n_r}-b_r}\right)}\right) &= \left(\prod_{i=1}^{n_r-1}(\alpha_{r,i} - b_r)\right)\dim_{\mathbb{K}}\left(\frac{R_{n+1-n_r}}{\mathcal{J}_{B_5}}\right) \\ &= \left(\prod_{i=1}^{n_r-1}(\alpha_{r,i} - b_r)\right)\det B_5.\end{aligned}$$

Let B_6 be the $(n-1) \times (n-1)$ matrix obtained from H by deleting the row and column containing the diagonal element α_{r,n_r}. We see that $\left\langle \mathcal{J}_H, x_{r,n_r}^{\alpha_{r,n_r}-b_r}\right\rangle = \left\langle \mathcal{J}_{B_6}, x_{r,n_r}^{\alpha_{rn_r}-b_r}\right\rangle$. Also, $\dim_{\mathbb{K}}(R_{n-1}/\mathcal{J}_{B_6}) = \det B_6$ (by induction hypothesis). Thus

$$\begin{aligned}\dim_{\mathbb{K}}\left(\frac{R_n}{\left\langle \mathfrak{J}_H, x_{r,n_r}^{\alpha_{r,n_r}-b_r}\right\rangle}\right) &= (\alpha_{r,n_r}-b_r)\dim_{\mathbb{K}}\left(\frac{R_{n-1}}{\mathfrak{J}_{B_6}}\right)\\ &= (\alpha_{r,n_r}-b_r)\det B_6.\end{aligned}$$

Now using the short exact sequence of $\mathbb{K}$-vector spaces

$$0 \to \frac{R_n}{\left(\mathfrak{J}_H : x_{r,n_r}^{\alpha_{r,n_r}-b_r}\right)} \xrightarrow{\mu_{x_{r,n_r}^{\alpha_{r,n_r}-b_r}}} \frac{R_n}{\mathfrak{J}_H} \to \frac{R_n}{\left\langle \mathfrak{J}_H, x_{r,n_r}^{\alpha_{r,n_r}-b_r}\right\rangle} \to 0,$$

we get $\dim_{\mathbb{K}}(R_n/\mathfrak{J}_H) = \left(\prod_{i=1}^{n_r-1}(\alpha_{r,i}-b_r)\right)\det B_5 + (\alpha_{r,n_r}-b_r)\det B_6$. Writing $\alpha_{r,n_r} = (\alpha_{r,n_r}-b_r)+b_r$ and using the additivity property of the determinant we see that $\det H = (\alpha_{r,n_r}-b_r)\det B_6 + \det B_7$, where B_7 is the matrix obtained from H by replacing the diagonal element α_{r,n_r} with b_r. Applying the elementary column and row operations, $\mathcal{C}_i - \mathcal{C}_n, \mathcal{R}_i - \mathcal{R}_n$ for $n-n_r < i < n$ on B_7, we get, $\det B_7 = \left(\prod_{i=1}^{n_r-1}(\alpha_{r,i}-b_r)\right)\det B_5$. Consequently, $\dim_{\mathbb{K}}(R_n/\mathfrak{J}_H) = \det H$. □

Definition 3.3.9. *Let G_1 be a multigraph on the vertex set $\{0,1,\dots,n\}$ and G_2 be a multigraph on the vertex set $\{0,1,\dots,m\}$. Let d be a nonnegative integer. Define the graph $G_1 *_d G_2$ on the vertex set $\{0,1,\dots,n,n+1,\dots,n+m\}$ as follows. If $1 \le i,j \le n$ then the number of edges between i and j in G is same as the number of edges between i and j in G_1. If $i,j \ge n+1$ then the number of edges between i and j in G is same as the number of edges between $i-n$ and $j-n$ in G_2. For $1 \le i \le n$ the number of edges between 0 and i in G is same as the number of edges between 0 and i in G_1. For $j \ge n+1$ the number of edges between 0 and j in G is same as the number of edges between 0 and $j-n$ in G_2. For each $1 \le i \le n$ and $j \ge n+1$ the number edges between i and j is exactly d.*

Example 3.3.10. *We give an example of the graphs constructed above in*

Figure 3.1.

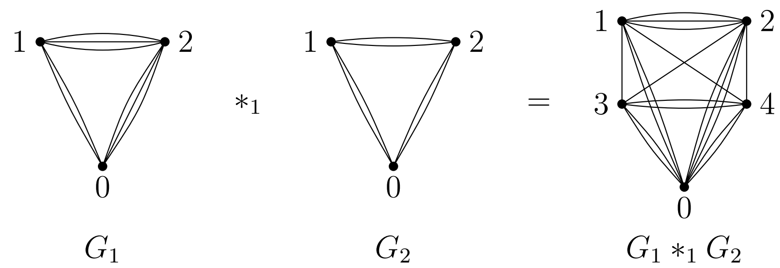


Figure 3.1: $G_1 *_d G_2$ for $d = 1$.

Corollary 3.3.11. *Let G_i be a multigraph on the vertex set $\{0, 1, \ldots, n_i\}$ obtained from a complete multigraph $K_{n_i+1}^{a_i,b_i}$ by removing some edges through the root* 0*, where $1 \leq i \leq r$. Suppose $n = \sum_{i=1}^{r} n_i$. Let $G = ((\cdots(G_1 *_{d_1} G_2) *_{d_2} \cdots *_{d_{r-2}} G_{r-1}) *_{d_{r-1}} G_r)$ be the multigraph on the vertex set $\{0, 1, \ldots, n\}$, where $d_1 \geq d_2 \geq \cdots \geq d_{r-1}$ are nonnegative integers, $b_1 \geq d_1$, and $b_i \geq d_{i-1}$ for $2 \leq i \leq r$. Then $\dim_{\mathbb{K}}(R_n/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$.*

Proof. The truncated signless Laplacian $\widetilde{Q}_G$ of the graph G is a matrix of the form (3.16). Taking $H = \widetilde{Q}_G$ in Theorem 3.3.8 we get our result. □

Lemma 3.3.12. *Let G be a multigraph on $\{0, 1, \ldots, t\}$ and $\widetilde{G}$ be the induced subgraph on the vertex set $[t]$. Suppose the connected components of $\widetilde{G}$ are $\widetilde{G}_1, \ldots, \widetilde{G}_r$ with $|V_i| = t_i$, where $V_i = V(\widetilde{G}_i)$ for $1 \leq i \leq r$. Consider the induced subgraphs $G_i = G_{V_i \cup \{0\}}$ of G. We have, $\dim_{\mathbb{K}}(R_t/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$ if and only if $\dim_{\mathbb{K}}(R_{t_i}/\mathcal{M}_{G_i}^{(1)}) = \det \widetilde{Q}_{G_i}$ for $1 \leq i \leq r$.*

Proof. We see that the ideal $\mathcal{M}_G^{(1)} = \sum_{i=1}^{r} \mathcal{M}_{G_i}^{(1)}$ such that $\dim_{\mathbb{K}}(R/\mathcal{M}_G^{(1)}) = \prod_{i=1}^{r} \dim_{\mathbb{K}}(R_{t_i}/\mathcal{M}_{G_i}^{(1)})$. Also, the truncated signless Laplace matrix $\widetilde{Q}_G$ is a block-diagonal matrix with blocks being the truncated signless Laplace matrices of the graphs G_i. Therefore, $\det \widetilde{Q}_G = \prod_{i=1}^{r} \det \widetilde{Q}_{G_i}$. By Theorem 3.2.6, $\dim_{\mathbb{K}}(R_{t_i}/\mathcal{M}_{G_i}^{(1)}) \geq \det \widetilde{Q}_{G_i}$ for $1 \leq i \leq r$. Hence, $\dim_{\mathbb{K}}(R_t/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$ if and only if $\dim_{\mathbb{K}}(R_{t_i}/\mathcal{M}_{G_i}^{(1)}) = \det \widetilde{Q}_{G_i}$ for $1 \leq i \leq r$. □

Next we proceed to give some necessary condition for a multigraph G to satisfy $\dim_{\mathbb{K}}(\frac{R}{\mathcal{M}_G}) = \det \widetilde{Q}_G$. Consequently we characterize all simple graphs G for which the equality between the number of standard monomials of the 1-skeleton ideal and the determinant of the truncated signless Laplace matrix holds. In order prove these results we introduce the notion of essentially connected subgraph of a multigraph below.

Definition 3.3.13. *Let G be a multigraph on the vertex set $\{0, 1, \ldots, t\}$ and $\widetilde{G}$ be the induced subgraph of G on the vertex set $\{1, \ldots, t\}$. If $\widetilde{G}_1$ is a connected component of $\widetilde{G}$ with vertex set V_1, then we call the induced subgraph $G_1 = G_{V_1 \cup \{0\}}$ of G an essentially connected component of G. Moreover, if $\widetilde{G}$ is connected, we say that G is essentially connected.*

Remark 3. *We see that a connected graph on vertex set $\{0, 1, \ldots, n\}$ may not be essentially connected. Further, an essentially connected graph may not be connected also. For example, let G be a simple graph on the vertex set $\{0, 1, 2\}$ obtained from the complete simple graph K_3 by removing all the edges attached to the root 0. Clearly, G is essentially connected but not connected. However, an essentially connected multigraph G is connected if and only if G has at least one edge attached to the root 0.*

Now we proceed to give the necessary condition for a graph G in terms of its essentially connected components. In order to do so we first prove analogous result for positive semidefinite matrices H and the monomial ideal $\mathcal{J}_H$ induced by H.

Discussion 3.3.14. Notice that by Lemma 3.3.12, to check whether for a multigraph G the equality $\dim_{\mathbb{K}}(R/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$ holds, it is enough to check this for its essentially connected components. Suppose G is an essentially connected multigraph on the vertex set $\{0, 1, \ldots, t\}$. Consider the induced subgraph $\widetilde{G} = G_{\{1,\ldots,t\}}$. Find two vertices in $\widetilde{G}$ such that the number of edges between them is maximum among the number of edges between any pair of vertices of $\widetilde{G}$. Rename these two vertices as 1 and 2. Suppose there are b edges between them. Now find some $i \in V(\widetilde{G})$, if it exists, such that $i \notin \{1,2\}$ and there are b edges from i to both the vertices 1 and 2. Rename the vertex i as 3 and continue this way to find a maximal *clique* (a complete multigraph) on the vertex set (say) $\{1,2,\ldots,n\}$ having b edges between any two vertices. Then, the truncated signless Laplace matrix of G will be of the form

$$H := \widetilde{Q}_G = \begin{bmatrix} \alpha_1 & \cdots & b & b & d_{1,1} & d_{1,2} & \cdots & d_{1,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b & \cdots & \alpha_{n-1} & b & d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,m} \\ b & \cdots & b & \alpha_n & d_{n,1} & d_{n,2} & \cdots & d_{n,m} \\ d_{1,1} & \cdots & d_{n-1,1} & d_{n,1} & \beta_1 & c_{1,2} & \cdots & c_{1,m} \\ d_{1,2} & \cdots & d_{n-1,2} & d_{n,2} & c_{1,2} & \beta_2 & \cdots & c_{2,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{1,m} & \cdots & d_{n-1,m} & d_{n,m} & c_{1,m} & c_{2,m} & \cdots & \beta_m \end{bmatrix}_{t\times t}, \tag{3.17}$$

where $n+m=t$ with $n \geq 2$. If $t=n$, then $\dim_{\mathbb{K}}(R_n/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$ by Theorem 3.3.7. Therefore, we may assume that $m \geq 1$. Note that if $\alpha_i = \alpha_j = b$ for some $i \neq j$, then the two vertices i and j will form a connected component of $\widetilde{G}$ and hence G will not be essentially connected. Similarly, for $i \neq j$ we cannot have $\beta_i = c_{i,j} = \beta_j$ or $\alpha_i = d_{i,j} = \beta_j$. Also, if for some $1 \leq j \leq m$, $d_{i,j} = b$ for each $1 \leq i \leq n$, then the set of vertices $\{1,\ldots,n\}$ will not form a maximal clique.

More precisely, the matrix H given in (3.17) satisfies the following conditions.

$$\text{For each } i \in [n] \text{ and } j \in [m],\ \alpha_i \geq b \geq 1,\ b \geq d_{i,j} \geq 0,\ \beta_j \geq 1, \text{ and } \beta_j \geq d_{i,j}. \tag{3.18}$$

$$\text{For each } i \in [m],\ k \in [m], \text{and } r < i < j,\ \beta_i \geq c_{i,j},\ \beta_i \geq c_{r,i} \text{ and } b \geq c_{i,k}. \tag{3.19}$$

$$\text{For each } i, j \in [n] \text{ and } i \neq j, \text{ either } \alpha_i > b \text{ or } \alpha_j > b. \tag{3.20}$$

$$\text{For each } i, j \in [m] \text{ and } i \neq j, \text{ either } \beta_i > c_{i,j} \text{ or } \beta_j > c_{i,j}. \tag{3.21}$$

$$\text{For each } i \in [n] \text{ and } j \in [m], \text{ either } \alpha_i > d_{i,j} \text{ or } \beta_j > d_{i,j}. \tag{3.22}$$

$$\text{For each } j \in [m],\ \text{there exists some } i \in [n] \text{ such that } b \neq d_{i,j}. \tag{3.23}$$

Our aim is to prove the following theorem. The proof uses Fischer's inequality (Theorem 2.4.7).

Theorem 3.3.15. *Consider the matrix H given in* (3.17)*. Assume that H satisfies the conditions* (3.18) *to* (3.23)*. Suppose H is a positive semidefinite matrix. If $\dim_{\mathbb{K}}(R_t/\mathfrak{J}_H) = \det H$, then for each $1 \leq l \leq m$ we have $d_{r,l} = d_{s,l}$ for $1 \leq r, s \leq n$.*

Proof. If possible, let $d_{r,l} \neq d_{s,l}$ for some l. Without loss of generality, assume that $l = 1$, i.e., $d_{r,1} \neq d_{s,1}$ for some $1 \leq r < s \leq n$. We first show that

$$\alpha_i > b \text{ for all } i \in [n]. \tag{3.24}$$

If possible, let $\alpha_i = b$ for some $i \in [n]$. Without loss of generality, let $\alpha_n = b$. Then, $\alpha_i > b$ for each $i < n$ (by the condition (3.20)). Moreover, the ideal

$\mathfrak{J}_H$ is generated by the following monomials

$$\begin{aligned} x_i^{\alpha_i - b}, x_n^b, y_j^{\beta_j} &: \ 1 \leq i < n, \ j \in [m], \\ x_n^{b-d_{n,j}} y_j^{\beta_j - d_{n,j}} &: \ j \in [m], \\ y_i^{\beta_i - c_{i,j}} y_j^{\beta_j - c_{i,j}} &: \ 1 \leq i < j \leq m. \end{aligned}$$

Consider the block-diagonal matrix $H_1 = \left(\begin{array}{c|c} D_{(n-1)\times(n-1)} & 0 \\ \hline 0 & A_{(m+1)\times(m+1)} \end{array}\right)$, where the matrix $D = \operatorname{diag}[\alpha_1 - b, \alpha_2 - b, \ldots, \alpha_{n-1} - b]$ is diagonal and the matrix A is obtained from H by deleting the rows and columns $\mathcal{R}_1, \mathcal{C}_1, \ldots, \mathcal{R}_{n-1}, \mathcal{C}_{n-1}$. We see that the ideal $\mathfrak{J}_{H_1} = \mathfrak{J}_H$. We also have $\dim_{\mathbb{K}}(R_t/\mathfrak{J}_H) > 0$ because of the conditions (3.18), (3.20) to (3.22). Thus $\det H > 0$ and hence, H is a positive definite matrix. Applying elementary column and row operations $\mathcal{C}_1 - \mathcal{C}_n, \mathcal{R}_1 - \mathcal{R}_n, \ldots, \mathcal{C}_{n-1} - \mathcal{C}_n, \mathcal{R}_{n-1} - \mathcal{R}_n$ on H we see that the reduced matrix

$$\begin{bmatrix} \alpha_1 - b & \cdots & 0 & 0 & d_{1,1} - d_{n,1} & d_{1,2} - d_{n,2} & \cdots & d_{1,m} - d_{n,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_{n-1} - b & 0 & d_{n-1,1} - d_{n,1} & d_{n-1,2} - d_{n,2} & \cdots & d_{n-1,m} - d_{n,m} \\ 0 & \cdots & 0 & b & d_{n,1} & d_{n,2} & \cdots & d_{n,m} \\ d_{1,1} - d_{n,1} & \cdots & d_{n-1,1} - d_{n,1} & d_{n,1} & \beta_1 & c_{1,2} & \cdots & c_{1,m} \\ d_{1,2} - d_{n,2} & \cdots & d_{n-1,2} - d_{n,2} & d_{n,2} & c_{1,2} & \beta_2 & \cdots & c_{2,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{1,m} - d_{n,m} & \cdots & d_{n-1,m} - d_{n,m} & d_{n,m} & c_{1,m} & c_{2,m} & \cdots & \beta_m \end{bmatrix}$$

is a positive definite matrix, by Lemma 2.4.5. By our assumption, $d_{r,1} \neq d_{s,1}$ for some $1 \leq r < s \leq n$, we see that $\det H_1 > \det H$ by Fischer's inequality. The matrix A is also positive definite, since H is positive definite. Hence, H_1 is a positive definite matrix because $\alpha_i > b$ for each $i < n$. By Theorem 3.2.6, $\dim_{\mathbb{K}}(R_t/\mathfrak{J}_{H_1}) \geq \det H_1$. Thus $\dim_{\mathbb{K}}(R_t/\mathfrak{J}_H) = \dim_{\mathbb{K}}(R_t/\mathfrak{J}_{H_1}) \geq \det H_1 > \det H$, a contradiction. Therefore, the matrix H given in (3.17) satisfies the condition (3.24).

Next we claim the following.

$$\text{For each } j \in [m], \text{ either } \beta_j > d_{n,j} \text{ or } b > d_{n,j}. \tag{3.25}$$

On the contrary, let $\beta_j = b = d_{n,j}$ for some $j \in [m]$. Without loss of generality, assume that $j = 1$, i.e., $\beta_1 = b = d_{n,1}$. Then, the ideal $\mathfrak{J}_H$ is generated by the following monomials

$$\begin{aligned} x_i^{\alpha_i}, x_n^{\alpha_n - b}, y_j^{\beta_j} &: 1 \le i < n,\ j \in [m], \\ x_i^{\alpha_i - b} x_j^{\alpha_j - b} &: 1 \le i < j < n, \\ x_i^{\alpha_i - d_{i,j}} y_j^{\beta_j - d_{i,j}} &: 1 \le i < n,\ j \in [m], \\ y_i^{\beta_i - c_{i,j}} y_j^{\beta_j - c_{i,j}} &: 1 \le i < j \le m. \end{aligned}$$

Let H_2 be the matrix obtained from H by replacing the diagonal element α_n with $\alpha_n - b$ and all other element of $\mathcal{R}_n$ and $\mathcal{C}_n$ with zero, i.e.,

$$H_2 = \begin{bmatrix} \alpha_1 & \cdots & b & 0 & d_{1,1} & d_{1,2} & \cdots & d_{1,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b & \cdots & \alpha_{n-1} & 0 & d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,m} \\ 0 & \cdots & 0 & \alpha_n - b & 0 & 0 & \cdots & 0 \\ d_{1,1} & \cdots & d_{n-1,1} & 0 & b & c_{1,2} & \cdots & c_{1,m} \\ d_{1,2} & \cdots & d_{n-1,2} & 0 & c_{1,2} & \beta_2 & \cdots & c_{2,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{1,m} & \cdots & d_{n-1,m} & 0 & c_{1,m} & c_{2,m} & \cdots & \beta_m \end{bmatrix}_{t \times t}.$$

Then, $\mathfrak{J}_H = \mathfrak{J}_{H_2}$. We have $\dim_{\mathbb{K}}(R_t/\mathfrak{J}_H) > 0$ because of the conditions (3.18), (3.20) to (3.22). Thus $\det H > 0$ and hence, H is a positive definite matrix. Applying elementary column and row operations $\mathcal{C}_n - \mathcal{C}_{n+1}$ and $\mathcal{R}_n - \mathcal{R}_{n+1}$

on H we see that the reduced matrix

$$\begin{bmatrix} \alpha_1 & \cdots & b & b-d_{1,1} & d_{1,1} & d_{1,2} & \cdots & d_{1,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b & \cdots & \alpha_{n-1} & b-d_{n-1,1} & d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,m} \\ b-d_{1,1} & \cdots & b-d_{n-1,1} & \alpha_n - b & 0 & d_{n,2}-c_{1,2} & \cdots & d_{n,m}-c_{1,m} \\ d_{1,1} & \cdots & d_{n-1,1} & 0 & b & c_{1,2} & \cdots & c_{1,m} \\ d_{1,2} & \cdots & d_{n-1,2} & d_{n,2}-c_{1,2} & c_{1,2} & \beta_2 & \cdots & c_{2,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{1,m} & \cdots & d_{n-1,m} & d_{n,m}-c_{1,m} & c_{1,m} & c_{2,m} & \cdots & \beta_m \end{bmatrix}_{t\times t}$$

is a positive definite matrix, by Lemma 2.4.5. Because of the condition (3.23) we have $b \neq d_{i,1}$, for some $1 \leq i < n$. Therefore, $\det H < \det H_2$ by Fischer's inequality. Since $\alpha_n > b$ and H is a positive definite matrix, H_2 is also positive definite. By Theorem 3.2.6, $\dim_{\mathbb{K}}(R_t/\mathfrak{J}_{H_2}) \geq \det H_2$. Thus, $\dim_{\mathbb{K}}(R_t/\mathfrak{J}_H) > \det H$, a contradiction. Hence, the matrix H given in (3.17) also satisfies condition (3.25).

The ideal $\mathfrak{J}_H$ is generated by the monomials

$$\begin{aligned} x_i^{\alpha_i}, y_j^{\beta_j} &: \ i \in [n],\ j \in [m], \\ x_i^{\alpha_i - b} x_j^{\alpha_j - b} &: \ 1 \leq i < j \leq n, \\ x_i^{\alpha_i - d_{i,j}} y_j^{\beta_j - d_{i,j}} &: \ i \in [n],\ j \in [m], \\ y_i^{\beta_i - c_{i,j}} y_j^{\beta_j - c_{i,j}} &: \ 1 \leq i < j \leq m. \end{aligned}$$

Consider the block-diagonal matrix $H_3 = \left(\begin{array}{c|c} \widehat{D}_{(n-1)\times(n-1)} & 0 \\ \hline 0 & \widehat{A}_{(m+1)\times(m+1)} \end{array}\right)$, where the matrix $\widehat{D} = \operatorname{diag}[\alpha_1 - b, \alpha_2 - b, \ldots, \alpha_{n-1} - b]$ is diagonal and the matrix $\widehat{A}$ is obtained from H by deleting the rows and columns $\mathcal{R}_1, \mathcal{C}_1, \ldots, \mathcal{R}_{n-1}, \mathcal{C}_{n-1}$ and then replacing the element α_n with b. We see that $(\mathfrak{J}_H : x_n^{\alpha_n - b}) = \mathfrak{J}_{H_3}$. Let H_4 be the matrix obtained from H by replacing the diagonal element α_n with $\alpha_n - b$ and every other elements in $\mathcal{R}_n$ and $\mathcal{C}_n$ with zero, i.e.,

$$H_4 = \begin{bmatrix} \alpha_1 & \cdots & b & 0 & d_{1,1} & d_{1,2} & \cdots & d_{1,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b & \cdots & \alpha_{n-1} & 0 & d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,m} \\ 0 & \cdots & 0 & \alpha_n - b & 0 & 0 & \cdots & 0 \\ d_{1,1} & \cdots & d_{n-1,1} & 0 & \beta_1 & c_{1,2} & \cdots & c_{1,m} \\ d_{1,2} & \cdots & d_{n-1,2} & 0 & c_{1,2} & \beta_2 & \cdots & c_{2,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{1,m} & \cdots & d_{n-1,m} & 0 & c_{1,m} & c_{2,m} & \cdots & \beta_m \end{bmatrix}_{t \times t}.$$

We have $\mathfrak{J}_{H_4} = \left\langle \mathfrak{J}_H, x_n^{\alpha_n - b} \right\rangle$. Now, from the short exact sequence of $\mathbb{K}$-vector spaces,

$$0 \to \frac{R_t}{(\mathfrak{J}_H : x_n^{\alpha_n - b})} \xrightarrow{\mu_{x_n^{\alpha_n - b}}} \frac{R_t}{\mathfrak{J}_H} \to \frac{R_t}{\langle \mathfrak{J}_H, x_n^{\alpha_n - b} \rangle} \to 0,$$

we have $\dim_{\mathbb{K}}(R_t/\mathfrak{J}_H) = \dim_{\mathbb{K}}(R_t/\mathfrak{J}_{H_3}) + \dim_{\mathbb{K}}(R_t/\mathfrak{J}_{H_4})$. Also, writing $\alpha_n = (\alpha_n - b) + b$ and using the additivity property of the determinant, we get $\det H = \det H_4 + \det B$, where B is obtained from the matrix H by replacing α_n with b. The matrix H_4 is positive semidefinite since H is positive semidefinite and $\alpha_n > b$. By Theorem 3.2.6, $\dim_{\mathbb{K}}(R_t/\mathfrak{J}_{H_4}) \geq \det H_4$. We also have $\dim_{\mathbb{K}}(R_t/\mathfrak{J}_{H_3}) > 0$ because of the conditions (3.18), (3.21), (3.24) and (3.25). Since $\dim_{\mathbb{K}}(R_t/\mathfrak{J}_H) = \det H$ we must have $\det B > 0$. By Lemma 2.4.9, B is a positive definite matrix since $\alpha_n \geq b$ and H is a positive semidefinite matrix . Applying elementary column and row operations $\mathcal{C}_1 - \mathcal{C}_n, \mathcal{R}_1 - \mathcal{R}_n, \ldots, \mathcal{C}_{n-1} - \mathcal{C}_n, \mathcal{R}_{n-1} - \mathcal{R}_n$ on B we see that the reduced matrix

$$\begin{bmatrix} \alpha_1 - b & \cdots & 0 & 0 & d_{1,1}-d_{n,1} & d_{1,2}-d_{n,2} & \cdots & d_{1,m}-d_{n,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_{n-1}-b & 0 & d_{n-1,1}-d_{n,1} & d_{n-1,2}-d_{n,2} & \cdots & d_{n-1,m}-d_{n,m} \\ 0 & \cdots & 0 & b & d_{n,1} & d_{n,2} & \cdots & d_{n,m} \\ d_{1,1}-d_{n,1} & \cdots & d_{n-1,1}-d_{n,1} & d_{n,1} & \beta_1 & c_{1,2} & \cdots & c_{1,m} \\ d_{1,2}-d_{n,2} & \cdots & d_{n-1,2}-d_{n,2} & d_{n,2} & c_{1,2} & \beta_2 & \cdots & c_{2,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{1,m}-d_{n,m} & \cdots & d_{n-1,m}-d_{n,m} & d_{n,m} & c_{1,m} & c_{2,m} & \cdots & \beta_m \end{bmatrix}$$

is a positive definite matrix, by Lemma 2.4.5. Since $d_{r,1} \neq d_{s,1}$ for some $1 \leq r < s \leq n$, we have $\det B < \det H_3$, by Fischer's inequality. The matrix H_3 is positive definite since B is a positive definite matrix and $\alpha_i > b$ for $1 \leq i < n$. By Theorem 3.2.6, $\dim_{\mathbb{K}}(R_t/\mathfrak{J}_{H_3}) \geq \det H_3$. Thus, $\dim_{\mathbb{K}}(R_t/\mathfrak{J}_H) \geq \det H_3 + \det H_4 > \det B + \det H_4 = \det H$, a contradiction, and this proves the theorem. □

By Theorem 3.3.5, if G is a simple graph on the vertex set $\{0, 1, \ldots, n\}$, obtained from a complete simple graph K_{n+1} by deleting some edges through the root 0, then $\dim_{\mathbb{K}}(R_n/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$. Furthermore, by Lemma 3.3.12, in order to check for a graph G when the equality $\dim_{\mathbb{K}}(R_n/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$ holds, we just need to check this for its essentially connected components. Now as a consequence of Theorem 3.3.15 we can characterize all simple graphs G which satisfy the property $\dim_{\mathbb{K}}(R_n/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$. More generally,

Theorem 3.3.16. *Let G be a subgraph of the complete multigraph $K_{n+1}^{a,1}$ on the vertex set $\{0, 1, \ldots, n\}$. The graph G satisfies $\dim_{\mathbb{K}}(R_n/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$ if and only if each essentially connected component G_i of G with $|V(G_i)| = n_i$, is obtained from a complete multigraph $K_{n_i+1}^{a_i,1}$ by deleting some edges through the root 0.*

Proof. First suppose that G is an essentially connected subgraph of the complete multigraph $K_{n+1}^{a,1}$ such that $\dim_{\mathbb{K}}(R_n/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$. We proceed along the same lines as in Discussion 3.3.14. Suppose $\widetilde{G}$ is the induced subgraph $G_{\{1,\dots,n\}}$ of G. We find two vertices in $\widetilde{G}$ such that there is an edge between them and rename these two vertices as 1 and 2. Now find another vertex (if it exists) which has edges connecting both 1 and 2. Rename the new vertex as 3 and continue this way to find a maximal clique on the vertex set say $\{1, 2, \dots, r\}$ for $r \leq n$. The truncated signless Laplace matrix of G will be of the form

$$\widetilde{Q}_G = \begin{bmatrix} \alpha_1 & \cdots & 1 & 1 & d_{1,1} & d_{1,2} & \cdots & d_{1,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \alpha_{r-1} & 1 & d_{r-1,1} & d_{r-1,2} & \cdots & d_{r-1,m} \\ 1 & \cdots & 1 & \alpha_r & d_{r,1} & d_{r,2} & \cdots & d_{r,m} \\ d_{1,1} & \cdots & d_{r-1,1} & d_{r,1} & \beta_1 & c_{1,2} & \cdots & c_{1,m} \\ d_{1,2} & \cdots & d_{r-1,2} & d_{r,2} & c_{1,2} & \beta_2 & \cdots & c_{2,m} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{1,m} & \cdots & d_{r-1,m} & d_{r,m} & c_{1,m} & c_{2,m} & \cdots & \beta_m \end{bmatrix}_{n\times n},$$

where $r + m = n$, and for each $1 \leq j \leq m$ there exists some $i \in [r]$ such that $d_{i,j} = 0$. Since $\dim_{\mathbb{K}}(R_n/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$, we must have $d_{i,j} = 0$ for all i and j, by Theorem 3.3.15. Hence, we have $r = n$ since G is essentially connected. Thus G is obtained from a complete multigraph $K_{n+1}^{a,1}$ by deleting some edges through the root 0.

If G is a subgraph of $K_{n+1}^{a,1}$ and $\dim_{\mathbb{K}}(R_n/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$, then by Lemma 3.3.12 and the above discussion we see that each essential component G_i of G is obtained from a complete multigraph $K_{n_i+1}^{a_i,1}$ by deleting some edges through the root 0.

The converse follows from Lemma 3.3.12 and Theorem 3.3.5. □

Corollary 3.3.17. *Let G be a simple graph on $n + 1$ vertices $\{0, 1, \dots, n\}$.*

For the graph G, $\dim_{\mathbb{K}}(R_n/\mathcal{M}_G^{(1)}) = \det \widetilde{Q}_G$ holds if and only if each essentially connected component G_i of G with $|V(G_i)| = n_i$, is obtained from a complete simple graph K_{n_i+1} by deleting some edges through the root 0.

Proof. Taking $a = 1$ in Theorem 3.3.16 we get our result. $\square$

Chapter 4

Monomial ideals induced by permutation avoiding patterns

In this chapter we consider some Artinian monomial ideals induced by permutation avoiding patterns. We count the number of standard monomials of their quotient rings. The results here are based on a joint work with C. Kumar [35].

4.1 Permutation avoiding patterns and monomial ideals

Let $\mathfrak{S}_n$ be the set of all permutations of $[n] = \{1, 2, \ldots, n\}$. We write elements of $\mathfrak{S}_n$ in word notation. For example, the permutation 132 means $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$. Sometimes we also write permutations, for example 132, as $(1, 3, 2)$. For $r \leq n$, consider a $\tau \in \mathfrak{S}_r$, called a *pattern*. A permutation $\sigma \in \mathfrak{S}_n$ is said to *avoid a pattern* τ if there do not exist integers $1 \leq j_1 < \cdots < j_r \leq n$ such that for all $1 \leq a < b \leq r$, we have $\tau(a) < \tau(b)$ if and only if $\sigma(j_a) < \sigma(j_b)$. For example, the permutation $165432 \in \mathfrak{S}_6$ avoids the pattern 312. Let $\mathfrak{S}_n(\tau)$ be the subset consisting of permutations $\sigma \in \mathfrak{S}_n$ that avoid pattern τ. If $r > n$, then $\mathfrak{S}_n(\tau) = \mathfrak{S}_n$. Also, if $\tau^{(i)} \in \mathfrak{S}_{r_i}$ for $1 \leq i \leq s$, then $\mathfrak{S}_n(\tau^{(1)}, \ldots, \tau^{(s)}) = \bigcap_{j=1}^{s} \mathfrak{S}_n(\tau^{(j)})$. Enumeration and combinatorial properties of the set of permutations avoiding patterns are obtained in [47].

For a nonempty set $S \subseteq \mathfrak{S}_n$, consider the monomial ideal $I_S = \left\langle \mathbf{x}^{\sigma} = \prod_{i=1}^{n} x_i^{\sigma(i)} : \sigma \in S \right\rangle$ in R induced by S, where $R = \mathbb{K}[x_1, \ldots, x_n]$ is the polynomial ring in n variables over a field $\mathbb{K}$. Throughout this chapter we use R to denote the polynomial ring in n variables. The monomial ideal $I_{\mathfrak{S}_n}$ is called a *permutohedron ideal* and the Alexander dual $I_{\mathfrak{S}_n}^{[\mathbf{n}]}$ of $I_{\mathfrak{S}_n}$ with respect to $\mathbf{n} = (n, \ldots, n)$ is the *tree ideal* $\mathcal{M}_{K_{n+1}}$ (i.e., $I_{\mathfrak{S}_n}^{[\mathbf{n}]}$ is the G-parking function ideal $\mathcal{M}_G$ for $G = K_{n+1}$). Hence, the i^{th} Betti number $\beta_i(I_{\mathfrak{S}_n}^{[\mathbf{n}]})$ of $I_{\mathfrak{S}_n}^{[\mathbf{n}]}$ is given by

$$\beta_i(I_{\mathfrak{S}_n}^{[\mathbf{n}]}) = \beta_{i+1}\left(\frac{R}{I_{\mathfrak{S}_n}^{[\mathbf{n}]}}\right) = (i!)S(n+1, i+1); \quad (0 \leq i \leq n-1),$$

where $S(n, r)$ is the Stirling number of the second kind, i.e., the number of set-partitions of $[n]$ into r blocks (see [45, Corollary 6.9]). Further, by the theorem of Konheim and Weiss [26] (see Theorem 3.1.3), we know that the standard monomials of $\frac{R}{I_{\mathfrak{S}_n}^{[\mathbf{n}]}}$ is given by $\dim_{\mathbb{K}}\left(\frac{R}{I_{\mathfrak{S}_n}^{[\mathbf{n}]}}\right) = |\mathrm{PF}_n| = (n+1)^{n-1}$ (see also Theorem 3.1.13), where PF_n is the set of ordinary parking functions $PF(K_{n+1})$. Note that for $S \subseteq \mathfrak{S}_n$, we have $I_{\mathfrak{S}_n}^{[\mathbf{n}]} \subseteq I_S^{[\mathbf{n}]}$. Thus standard monomials of $\frac{R}{I_S^{[\mathbf{n}]}}$ are always ordinary parking functions.

Recall that, a sequence $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is called a G-parking function if $\mathbf{x}^{\mathbf{p}} = \prod_{i=1}^{n} x_i^{p_i}$ is a standard monomial of $\frac{R}{\mathcal{M}_G}$ (i.e., $\mathbf{x}^{\mathbf{p}} \notin \mathcal{M}_G$), where $\mathcal{M}_G$ is the G-parking function ideal of the graph G (see Section 3.1). Let $\mathrm{SPT}(G)$ be the set of spanning trees of G rooted at 0 and $\mathrm{PF}(G)$ be the set of G-parking functions of G (cf. Section 3.1). Then $|\mathrm{PF}(G)| =$

$|\mathrm{SPT}(G)|$ (see [45, Theorem 2.1]). A recursively defined bijection $\phi : \mathrm{PF}_n \longrightarrow \mathrm{SPT}(K_{n+1})$ has been constructed by Kreweras [27]. An algorithmic bijection $\phi : \mathrm{PF}(G) \longrightarrow \mathrm{SPT}(G)$, called *DFS-burning algorithm*, is given by Perkinson et. al. [44] for a simple graph G and by Gaydarov and Hopkins [17] for multigraph G.

For various subsets $S \subseteq \mathfrak{S}_n$, the Alexander dual $I_S^{[\mathbf{n}]}$ of I_S with respect to $\mathbf{n} = (n, \ldots, n)$ has many interesting properties similar to the Alexander dual of permutohedron ideal. The Betti numbers and enumeration of standard monomials of the Alexander dual $I_S^{[\mathbf{n}]}$ for subsets $S = \mathfrak{S}_n(132, 231)$, $\mathfrak{S}_n(123, 132)$ and $\mathfrak{S}_n(123, 132, 213)$ are obtained in [30, 31].

Let $W = \mathfrak{S}_n(132, 312)$. The monomial ideal I_W of R is called a *hypercubic ideal* in [29]. The standard monomials of $\frac{R}{I_W^{[\mathbf{n}]}}$ correspond bijectively to a subset $\widetilde{\mathrm{PF}}_n$ of PF_n. An element $\mathbf{p} \in \widetilde{\mathrm{PF}}_n$ is called a *restricted parking function* of length n. In this chapter we show that (see Theorem 4.2.11) the number of restricted parking functions of length n is given by

$$\dim_{\mathbb{K}} \left(\frac{R}{I_W^{[\mathbf{n}]}} \right) = |\widetilde{\mathrm{PF}}_n| = \sum_{r=1}^{n} (r!)\ s(n, r),$$

where $s(n, r)$ is the (signless) Stirling number of the first kind, i.e., the number of permutations of $[n]$ having exactly r cycles in its cyclic decomposition. Thus the n^{th} term of integer sequence (A007840) in OEIS [48] can be interpreted as the number of restricted parking functions of length n, or equivalently, as the number of standard monomials of the Artinian quotient $\frac{R}{I_W^{[\mathbf{n}]}}$.

The concept of pattern avoiding permutations has been generalized to many combinatorial objects. A notion of rooted forests that avoids a set of permutations is introduced and many classes of such objects are enumerated in [3]. Let F_n be the set of rooted-labeled forests on $[n]$. Let $F_n(\tau)$ (or more generally, $F_n(\tau^{(1)}, \ldots, \tau^{(r)})$) be the subset of F_n consisting of rooted-labeled forests avoiding a pattern τ (or a set of patterns $\{\tau^{(1)}, \dots, \tau^{(r)}\}$). Anders and Archer [3] have shown that

$$|F_n(213, 312)| = \sum_{r=1}^{n} (r!)\ s(n, r) = |\widetilde{\mathrm{PF}}_n|.$$

It is surprising that enumeration of standard monomials of $\frac{R}{I_W^{[\mathbf{n}]}}$ and enumeration of rooted-labeled forests $F_n(213, 312)$ avoiding 213 and 312-patterns are related. It is an interesting problem to construct an algorithmic bijection $\phi : \widetilde{\mathrm{PF}}_n \longrightarrow F_n(213, 312)$, analogous to DFS-burning algorithm that could explain the relationship between these objects.

In the last section of this chapter we have considered the monomial ideal I_S for the subsets $\mathfrak{S}_n(123, 132, 312)$, $\mathfrak{S}_n(123, 213, 231)$, $\mathfrak{S}_n(132, 213, 231)$, $\mathfrak{S}_n(123, 132, 231)$, $\mathfrak{S}_n(213, 312, 321)$ and $\mathfrak{S}_n(123, 231, 312)$.

4.2 Hypercubic ideals and restricted parking functions

Consider the subset $W = \mathfrak{S}_n(132, 312)$ of permutations of $[n]$ that avoid 132 and 312-patterns. For $\sigma \in \mathfrak{S}_n$, it can be checked that $\sigma \in W$ if and only if $\sigma(1) \in [n]$ is arbitrary, and $\sigma(j) = \ell$ for $j > 1$ if either $\sigma(i) = \ell + 1$ or $\sigma(i) = \ell - 1$ for some $i < j$, and hence, $|W| = 2^{n-1}$ (see, for example [47, Proposition 10]). The monomial ideal I_W appeared in [29], where it is called a *hypercubic ideal*. Many properties of I_W and its Alexander dual $I_W^{[\mathbf{n}]}$ with respect to $\mathbf{n} = (n, \dots, n) \in \mathbb{N}^n$ have been obtained in [29]. We proceed to enumerate the standard monomials of $\frac{R}{I_W^{[\mathbf{n}]}}$. For this purpose, we consider a generalization.

Let $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{N}^n$ with $1 \le u_1 < u_2 < \cdots < u_n$. For $\sigma \in \mathfrak{S}_n$, let $\sigma\mathbf{u} = (u_{\sigma(1)}, \dots, u_{\sigma(n)})$ and $\mathbf{x}^{\sigma\mathbf{u}} = \prod_{i=1}^{n} x_i^{u_{\sigma(i)}}$. For any nonempty subset $S \subseteq$

$\mathfrak{S}_n$, we consider the monomial ideal $I_S(\mathbf{u}) = \langle \mathbf{x}^{\sigma\mathbf{u}} : \sigma \in S \rangle$ in the polynomial ring R. We see that $I_S((1,2,\ldots,n)) = I_S$. The ideals $I_{\mathfrak{S}_n}(\mathbf{u})$ and $I_W(\mathbf{u})$ are also called a *permutohedron ideal* and a *hypercubic ideal*, respectively.

The minimal generators of the Alexander dual $I_W(\mathbf{u})^{[\mathbf{u_n}]}$ of the ideal $I_W(\mathbf{u})$ with respect to $\mathbf{u_n} = (u_n,\ldots,u_n) \in \mathbb{N}^n$ are as follows:

Theorem 4.2.1. [29, Theorem 3.3] *The minimal generators of the Alexander dual $I_W(\mathbf{u})^{[\mathbf{u_n}]}$ are given by*

$$I_W(\mathbf{u})^{[\mathbf{u_n}]} = \left\langle \prod_{j\in T} x_j^{\mu_{j,T}} : \emptyset \neq T = \{j_1,\ldots,j_t\} \subseteq [n]; j_1 < \cdots < j_t \right\rangle,$$

where $\mu_{j_1,T} = u_n - u_t + 1$ and $\mu_{j_i,T} = u_n - u_{t+j_i-i} + 1$ for $2 \leq i \leq t$.

For an integer $c \geq 1$, we consider the Alexander dual $I_W(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ of the hypercubic ideal $I_W(\mathbf{u})$ with respect to $\mathbf{u_n}+\mathbf{c}-\mathbf{1} = (u_n+c-1,\ldots,u_n+c-1) \in \mathbb{N}^n$. Replacing $[\mathbf{u_n}]$ by $[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]$ in the above Theorem 4.2.1 we get the following.

Proposition 4.2.2. *The minimal generators of $I_W(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ are given by*

$$I_W(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]} = \left\langle \prod_{j\in T} x_j^{\mu^{\mathbf{u}}_{j,T}} : \emptyset \neq T = \{j_1,\ldots,j_t\} \subseteq [n];\ j_1 < \cdots < j_t \right\rangle,$$

where $\mu^{\mathbf{u}}_{j_1,T} = u_n - u_t + c$ and $\mu^{\mathbf{u}}_{j_i,T} = u_n - u_{t+j_i-i} + c$ for $2 \leq i \leq t$.

The ideal $I_W(\mathbf{u})^{[\mathbf{u_n}+\mathbf{c}-\mathbf{1}]}$ is an example of a class of ideal called *order monomial ideal* ([45]).

Definition 4.2.3 (Order monomial ideal)**.** [45] *Let $(P,\preceq)$ be a finite partially ordered set, or poset. Let $\mathcal{M} = \{m_v \mid v \in P\}$ be a collection of monomials in the polynomial ring $\mathbb{K}[x_1,\ldots,x_n]$ labeled by elements of the poset P. Also let $\mathcal{M}_v$ denote the set of all monomials divisible by m_v. Let us say that the ideal $\mathcal{I} = \langle \mathcal{M} \rangle$ generated by the monomials m_v is an order monomial ideal, if the following condition is satisfied.*

- *For any pair $v_1, v_2 \in P$, there exists an upper bound $v \in P$ of v_1 and v_2 such that $\mathcal{M}_{v_1} \cap \mathcal{M}_{v_2} \subseteq \mathcal{M}_v$, i.e., m_v divides* $\operatorname{lcm}(m_{v_1}, m_{v_2})$.

Here an upper bound means an element v such that $v \succeq v_1$ and $v \succeq v_2$ in P. In particular, this condition implies that the poset P has a unique maximal element. Postnikov and Shapiro [45] have given an example of a free resolution for any order monomial ideal and determined when such a free resolution is minimal as follows.

For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, let $R(-\mathbf{a})$ denote the free $\mathbb{N}^n$-graded R-submodule in R generated by the monomial $\mathbf{x}^{\mathbf{a}}$. If $\mathbf{a} \geq \mathbf{b}$ componentwise, then $R(-\mathbf{a})$ is a submodule of $R(-\mathbf{b})$ and we write $R(-\mathbf{a}) \hookrightarrow R(-\mathbf{b})$ to denote the natural multidegree preserving embedding of R-modules.

Let $\mathcal{M} = \{m_v \mid v \in P\}$ and $\mathcal{I} = \langle \mathcal{M} \rangle$ be an order monomial ideal. For any subset $V \subseteq P$, let $m_V = \operatorname{lcm}(m_v \mid v \in V)$ be the least common multiple of the monomials m_v, $v \in V$. Assume that $m_\emptyset = 1$. Let $\mathbf{a}_V \in \mathbb{N}^n$ be the exponent vector of the monomial m_V.

The *homological order complex* $C_*(\mathcal{M})$ for an order monomial ideal $\mathcal{I} = \langle \mathcal{M} \rangle$ is the sequence of $\mathbb{N}^n$-graded R-modules

$$\cdots \xrightarrow{\delta_4} C_3 \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 = R \to R/\mathcal{I} \to 0$$

whose k^{th} component is

$$C_k = \bigoplus_{v_1 \precneq \cdots \precneq v_k} R(-\mathbf{a}_{\{v_1, \ldots, v_k\}}),$$

where the direct sum is over strictly increasing k-chains $v_1 \prec \cdots \prec v_k$ in P. The differential $\delta_k : C_k \to C_{k-1}$ is defined on the component $R(-\mathbf{a}_{\{v_1, \ldots, v_k\}})$ as the alternating sum $\delta_k = \sum_{i=1}^{k} (-1)^i E_i$ of the multidegree preserving embeddings $E_i : R(-\mathbf{a}_{\{v_1, \ldots, v_k\}}) \hookrightarrow R(-\mathbf{a}_{\{v_1, \ldots, \widehat{v_i}, \ldots, v_k\}})$ of R-modules, where $v_1, \ldots, \widehat{v_i}, \ldots, v_k$ denotes the sequence with skipped i^{th} element.

Theorem 4.2.4. [45, Theorem 6.1] *The homological order complex $C_*(\mathcal{M})$ is a free resolution of the order monomial ideal $\mathcal{I} = \langle \mathcal{M} \rangle$.*

If $m_{\{v_1,\ldots,v_k\}} \neq m_{\{v_1,\ldots,\widehat{v_i},\ldots,v_k\}}$, for any increasing chain $v_1 \prec \cdots \prec v_k$ in the poset P and $i = 1, \ldots, k$, then the homological order complex $C_*(\mathcal{M})$ is a minimal free resolution of the order monomial ideal $\mathcal{I} = \langle \mathcal{M} \rangle$.

The above construction of $C_*(\mathcal{M})$ is an example of the cellular complexes due to Bayer and Sturmfels [6] (see also Section 2.3.3). Here the cell complex is the *geometrical order complex* $\Delta(P)$ of the poset P. The faces of the simplicial complex $\Delta(P)$ correspond to nonempty strictly increasing chains in P:

$$\Delta(P) = \{\{v_1, \ldots, v_k\} \subseteq P \mid v_1 \prec \cdots \prec v_k,\, k \geq 1\}.$$

Suppose $P = \Sigma_n$, the set of all nonempty subsets in $[n]$ ordered by inclusion. Then $\Delta(\Sigma_n)$ is the barycentric subdivision of the $(n-1)$-dimensional simplex.

Proposition 4.2.5. *The ideal $I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ is an order monomial ideal.*

Proof. The minimal generators of $I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ are given in Proposition 4.2.2. Let $T_1 = \{j_1, \ldots, j_s\} \in \Sigma_n$ and $T_2 = \{j'_1, \ldots, j'_t\} \in \Sigma_n$, where $j_1 < \cdots < j_s$ and $j'_1 < \cdots < j'_t$. The monomial labeling $m_{T_i} = \prod_{j \in T_i} x_j^{\mu_{j,T_i}}$ for $i = 1, 2$, where μ_{j,T_i} are as in Proposition 4.2.2. Since $u_1 < \cdots < u_n$, we see that m_T divides $\mathrm{lcm}(m_{T_1}, m_{T_2})$, where $T = T_1 \cup T_2 \in \Sigma_n$. $\square$

Let $T = \{\{j_1, j_2, \ldots, j_t\} \subseteq [n] \mid j_1 < j_2 < \cdots < j_t\}$. The monomial labeling m_T for T is $x_{j_1}^{u_n - u_t + c} \left(\prod_{i=2}^{t} x_{j_i}^{u_n - u_{t+j_i-i}+c}\right)$. Since $c \geq 1$ and $u_n > u_r$ for all $r \leq n-1$, we see that the exponent of x_{j_i} in m_T is nonzero for all i. Hence, $m_{T'} \neq m_T$ for any $T' \subsetneq T$. By Theorem 4.2.4, the minimal resolution of $I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ is the cellular resolution supported on the order complex $\Delta(\Sigma_n)$ of Σ_n. Thus, the i^{th} Betti number

$$\beta_i(I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}) = (i!)S(n+1, i+1); \quad (0 \leq i \leq n-1),$$

where $S(n+1, i+1)$ is the Stirling number of the second kind, i.e., the number of partition of the set $\{0, 1, \ldots, n\}$ into $(i+1)$ nonempty blocks (see [45, Equation 6.4]).

We now describe standard monomials of $\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$. Since $I_W(\mathbf{u}) \subseteq I_{\mathfrak{S}_n}(\mathbf{u})$, we have $I_{\mathfrak{S}_n}(\mathbf{u})^{[\mathbf{u_n+c-1}]} \subseteq I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}$. Hence, standard monomials of $\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$ are of the form $\mathbf{x}^{\mathbf{p}}$ for some $\mathbf{p} \in \mathrm{PF}_n(\lambda)$. Thus the standard monomials are given by a subset of the set of λ-parking functions $\mathrm{PF}(\lambda)$ for $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i = u_n - u_i + c$. Recall that, a sequence $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ is called a λ-*parking function* of length n, if a nondecreasing rearrangement $p_{i_1} \leq p_{i_2} \leq \cdots \leq p_{i_n}$ of $\mathbf{p}$ satisfies $p_{i_j} < \lambda_{n-j+1}$ for $1 \leq j \leq n$ (see also Definition 3.1.8).

Definition 4.2.6. *A λ-parking function $\mathbf{p} = (p_1, \ldots, p_n) \in \mathrm{PF}_n(\lambda)$ is said to be a restricted λ-parking function of length n if there exists a permutation $\alpha \in \mathfrak{S}_n$ such that $p_{\alpha_i} < \mu^{\mathbf{u}}_{\alpha_i, T_i}$ for all $1 \leq i \leq n$, where $\alpha_i = \alpha(i)$, $T_1 = [n]$, $T_i = [n] \setminus \{\alpha_1, \ldots, \alpha_{i-1}\}$; $(i \geq 2)$ and $\mu^{\mathbf{u}}_{j,T}$ is as in Proposition 4.2.2.*

Let $\widetilde{\mathrm{PF}}_n(\lambda)$ be the set of restricted λ-parking functions of length n. For $\mathbf{u} = (1, 2, \ldots, n)$ and $c = 1$, we have $\lambda = (n, n-1, \ldots, 1)$. In this case, a restricted λ-parking function is called a *restricted parking function* of length n and we simply write $\widetilde{\mathrm{PF}}_n$ for $\widetilde{\mathrm{PF}}_n(\lambda)$. Also, $\mu_{j,T} = \mu^{\mathbf{u}}_{j,T}$ is given by $\mu_{j_1,T} = n-t+1$ and $\mu_{j_i,T} = (n-t+1)-(j_i-i)$; $i \geq 2$, where $\emptyset \neq T = \{j_1, \ldots, j_t\} \subseteq [n]$ with $j_1 < \cdots < j_t$.

Proposition 4.2.7. *A monomial $\mathbf{x}^{\mathbf{p}}$ is a standard monomial of $\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$ if and only if $\mathbf{p} \in \widetilde{\mathrm{PF}}_n(\lambda)$ is a restricted λ-parking function of length n, with $\lambda_i = u_n - u_i + c\,; (1 \leq i \leq n)$. In particular, a monomial $\mathbf{x}^{\mathbf{p}}$ is a standard monomial of $\frac{R}{I_W^{[\mathbf{n}]}}$ if and only if $\mathbf{p} \in \widetilde{\mathrm{PF}}_n$ is a restricted parking function of length n.*

Proof. Standard monomials of $\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n}]}}$ are characterized in [29, Theorem

4.3]. Replacing u_n by $u_n + c - 1$ and proceeding on similar lines, we get the desired result. $\square$

Using the cellular resolution of $I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}$ supported on the order complex $\Delta(\Sigma_n)$, we obtain the multigraded Hilbert series $H\left(\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}}\right)$ of $\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$. Proceeding as in the proof of [29, Proposition 4.5], we get a combinatorial formula

$$\begin{aligned} |\widetilde{\mathrm{PF}}_n(\lambda)| &= \dim_{\mathbb{K}}\left(\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}}\right) \qquad (4.1)\\ &= \sum_{i=1}^{n}(-1)^{n-i}\sum_{\emptyset=A_0\subsetneq A_1\subsetneq\cdots\subsetneq A_i=[n]}\prod_{q=1}^{i}\left(\prod_{j\in A_q\setminus A_{q-1}}\mu^{\mathbf{u}}_{j,A_q}\right)\end{aligned}$$

for enumeration of standard monomials of $\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}}$, where $\mu^{\mathbf{u}}_{j,A_q}$ is as in Proposition 4.2.2. Let $\mathcal{C}$ be a chain in Σ_n of the form

$$\mathcal{C} : A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_i = [n]$$

of length $\ell(\mathcal{C}) = i-1$ and let $\mu^{\mathbf{u}}(\mathcal{C}) = \prod_{q=1}^{i}\left(\prod_{j\in A_q\setminus A_{q-1}}\mu^{\mathbf{u}}_{j,A_q}\right)$, where $A_0 = \emptyset$. Suppose $\mathfrak{Ch}([n])$ is the set of such chains $\mathcal{C}$ in Σ_n. Then formula (4.1) can be expressed compactly as

$$|\widetilde{\mathrm{PF}}_n(\lambda)| = \dim_{\mathbb{K}}\left(\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}}\right) = \sum_{\mathcal{C}\in\mathfrak{Ch}([n])}(-1)^{n-\ell(\mathcal{C})-1}\mu^{\mathbf{u}}(\mathcal{C}). \qquad (4.2)$$

We now take $u_i = i$ in (4.2). For $c \geq 1$, let $\dim_{\mathbb{K}}\left(\frac{R}{I_W^{[\mathbf{n+c-1}]}}\right) = a_n(c)$. Then we see that $a_n(c)$ is a polynomial expression in c of degree n for $n \geq 1$. In fact, $a_1(c) = c$ and $a_2(c) = c^2 + 2c$.

Lemma 4.2.8. *Let $n \geq 3$, $\mathbf{u} = (1, 2, \ldots, n)$ and $c \geq 1$. For a chain $\mathcal{C} \in \mathfrak{Ch}[n]$ of length $i-1$ of the form $A_1 \subsetneq \cdots \subsetneq A_r \subsetneq A_{r+1} \subsetneq \cdots \subsetneq A_i = [n]$ with $n \in A_{r+1} \setminus A_r$ and $|A_{r+1} \setminus A_r| \geq 2$, there exists a unique chain, namely $\widetilde{\mathcal{C}} : A_1 \subsetneq \cdots \subsetneq A_r \subsetneq A_r \cup \{n\} \subsetneq A_{r+1} \subsetneq \cdots \subsetneq A_i = [n]$ in $\mathfrak{Ch}[n]$ of length i*

such that $\mu^{\mathbf{u}}(\mathcal{C}) = \mu^{\mathbf{u}}(\widetilde{\mathcal{C}})$.

Proof. Since $\mu^{\mathbf{u}}(\mathcal{C}) = \prod_{q=1}^{i}\left(\prod_{j\in A_q\setminus A_{q-1}} \mu^{\mathbf{u}}_{j,A_q}\right)$, the equality $\mu^{\mathbf{u}}(\mathcal{C}) = \mu^{\mathbf{u}}(\widetilde{\mathcal{C}})$ holds if $\mu^{\mathbf{u}}_{n,A_r\cup\{n\}} = \mu^{\mathbf{u}}_{n,A_{r+1}}$. We see that $\mu^{\mathbf{u}}_{n,A_r\cup\{n\}} = n - (|A_r| + 1 + n - (|A_r|+1)) + c = c$ and $\mu^{\mathbf{u}}_{n,A_{r+1}} = n - (|A_{r+1}| + n - |A_{r+1}|) + c = c$. $\square$

Let $\mathfrak{Ch}'[n]$ be the set of chains in Σ_n obtained from $\mathfrak{Ch}[n]$ by deleting chains $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ appearing in Lemma 4.2.8. Then

$$a_n(c) = \sum_{\mathcal{C}\in\mathfrak{Ch}([n])} (-1)^{n-\ell(\mathcal{C})-1}\mu^{\mathbf{u}}(\mathcal{C}) = \sum_{\mathcal{C}\in\mathfrak{Ch}'([n])} (-1)^{n-\ell(\mathcal{C})-1}\mu^{\mathbf{u}}(\mathcal{C}).$$

For $\mathbf{u} = (1, 2, \ldots, n)$ and $c \geq 1$, the value $\mu^{\mathbf{u}}(\mathcal{C})$ depends on the chain $\mathcal{C}$ and c. Thus, we write $\mu^{c}(\mathcal{C})$ for $\mu^{\mathbf{u}}(\mathcal{C})$. Hence, $a_n(c) = \sum_{\mathcal{C}\in\mathfrak{Ch}([n])}(-1)^{n-\ell(\mathcal{C})-1}\mu^{c}(\mathcal{C}) = \sum_{\mathcal{C}\in\mathfrak{Ch}'([n])}(-1)^{n-\ell(\mathcal{C})-1}\mu^{c}(\mathcal{C})$.

For $n \geq 3$, the chains in $\mathfrak{Ch}'[n]$ can be divided into three types.

- A chain $\mathcal{C} : A_1 \subsetneq \cdots \subsetneq A_i = [n]$ in $\mathfrak{Ch}'[n]$ is called a *Type-I* chain if $A_1 = \{n\}$. The Type-I chains in $\mathfrak{Ch}'[n]$ are in one-to-one correspondence with chains in $\mathfrak{Ch}[n-1]$. This correspondence is given by

$$\mathcal{C} \mapsto \mathcal{C} \setminus A_1 : A_2 \setminus \{n\} \subsetneq \cdots \subsetneq A_i \setminus \{n\} = [n-1].$$

  As $\ell(\mathcal{C}) - 1 = \ell(\mathcal{C} \setminus A_1)$ and $\mu^{c}(\mathcal{C}) = (n-1+c)\ \mu^{c}(\mathcal{C} \setminus A_1)$, we have

$$\sum_{\substack{\mathcal{C}\in\mathfrak{Ch}'[n];\\ \text{Type-I}}} (-1)^{n-\ell(\mathcal{C})-1}\ \mu^{c}(\mathcal{C}) = (n-1+c)\ a_{n-1}(c).$$

- A chain $\mathcal{C} : A_1 \subsetneq \cdots \subsetneq A_i = [n]$ in $\mathfrak{Ch}'[n]$ is called a *Type-II* chain if $A_{i-1} = [n-1]$. The Type-II chains in $\mathfrak{Ch}'[n]$ are in one-to-one correspondence with chains in $\mathfrak{Ch}[n-1]$. This correspondence is given by

$$\mathcal{C} \mapsto \mathcal{C}|_{[n-1]} : A_1 \subsetneq \cdots \subsetneq A_{i-1} = [n-1].$$

As $\ell(\mathcal{C})-1=\ell(\mathcal{C}|_{[n-1]})$ and $\mu^c(\mathcal{C})=(c)\ \mu^{c+1}(\mathcal{C}|_{[n-1]})$, we have

$$\sum_{\substack{\mathcal{C}\in\mathfrak{Ch}'[n];\\ \text{Type-II}}}(-1)^{n-\ell(\mathcal{C})-1}\ \mu^c(\mathcal{C})=(c)\ a_{n-1}(c+1).$$

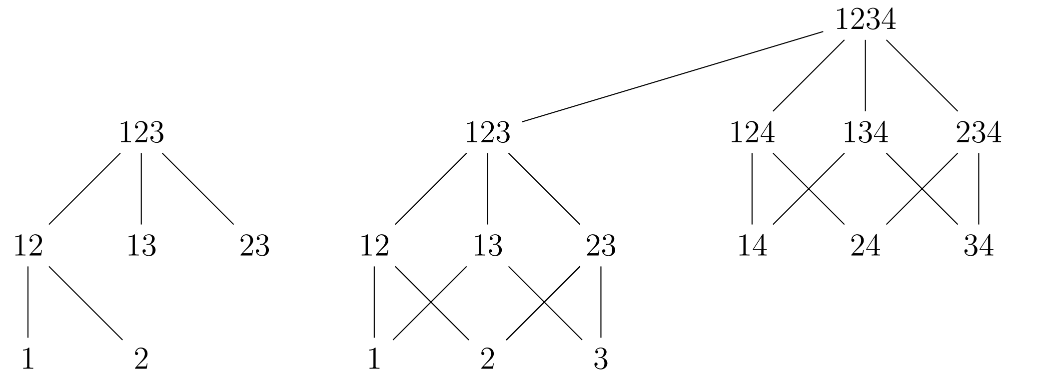
- A chain $\mathcal{C}:A_1\subsetneq\cdots\subsetneq A_i=[n]$ in $\mathfrak{Ch}'[n]$ is called a *Type-III* chain if $n\in A_1$ and $|A_1|\geq 2$. The Type-III chains in $\mathfrak{Ch}'[n]$ are in one-to-one correspondence with chains in $\mathfrak{Ch}[n-1]$. This correspondence is given by
$$\mathcal{C}\mapsto\mathcal{C}\setminus\{n\}:A_1\setminus\{n\}\subsetneq\cdots\subsetneq A_i\setminus\{n\}=[n-1].$$
As $\ell(\mathcal{C})=\ell(\mathcal{C}\setminus\{n\})$ and $\mu^c(\mathcal{C})=(c)\ \mu^c(\mathcal{C}\setminus\{n\})$, we have
$$\sum_{\substack{\mathcal{C}\in\mathfrak{Ch}'[n];\\ \text{Type-III}}}(-1)^{n-\ell(\mathcal{C})-1}\ \mu^c(\mathcal{C})=(-c)\ a_{n-1}(c).$$

Consider the poset Σ_n and form the poset $\Lambda_n=\Sigma_{n-1}\coprod(\Sigma_{n-1}*\{n\})$; for $n\geq 2$, where $\Sigma_{n-1}*\{n\}=\{A\cup\{n\}:A\in\Sigma_{n-1}\}$ is a subposet of Σ_n. Two elements $A,B\in\Lambda_n$ are comparable if either $A,B\in\Sigma_{n-1}$ are comparable or $A,B\in\Sigma_{n-1}*\{n\}$ are comparable or the comparable pair $(A,B=[n])$, where $A\in\Sigma_{n-1}$. The Hasse diagram of Λ_n for $n=3,4$ are given in Figure 4.1.

We see that Type-II chains in $\mathfrak{Ch}'[n]$ are chains in Λ_n with an edge $[n-1]\subsetneq[n]$, while Type-III chains in $\mathfrak{Ch}'[n]$ are chains in Λ_n containing $[n]$ but not $[n-1]$.

Proposition 4.2.9. *For $n\geq 3$ and $c\geq 1$, $a_n(c)=\dim_{\mathbb{K}}\left(\frac{R}{I_W^{[\mathbf{n+c-1}]}}\right)$ satisfies the recurrence relation*

$$a_n(c)=(n-1)a_{n-1}(c)+c\ a_{n-1}(c+1).$$



Λ_3 Λ_4

Figure 4.1: Hasse diagram for Λ_3 and Λ_4

Proof. As $a_n(c) = \sum_{\mathcal{C}\in\mathfrak{Ch}([n])}(-1)^{n-\ell(\mathcal{C})-1}\mu^c(\mathcal{C}) = \sum_{\mathcal{C}\in\mathfrak{Ch}'([n])}(-1)^{n-\ell(\mathcal{C})-1}\mu^c(\mathcal{C})$, we have

$$\begin{aligned}
a_n(c) &= \left[\sum_{\substack{\mathcal{C}\in\mathfrak{Ch}'[n];\\ \text{Type-I}}} + \sum_{\substack{\mathcal{C}\in\mathfrak{Ch}'[n];\\ \text{Type-II}}} + \sum_{\substack{\mathcal{C}\in\mathfrak{Ch}'[n];\\ \text{Type-III}}}\right] (-1)^{n-\ell(\mathcal{C})-1}\ \mu^c(\mathcal{C}) \\
&= (n-1+c)\ a_{n-1}(c) + (c)\ a_{n-1}(c+1) + (-c)\ a_{n-1}(c) \\
&= (n-1)\ a_{n-1}(c) + (c)\ a_{n-1}(c+1).
\end{aligned}$$

□

Replacing c by an indeterminate x, we consider polynomial $a_n(x)$. The recurrence relation in Proposition 4.2.9 holds for all $c \geq 1$, thus there exists a polynomial identity

$$a_n(x) = (n-1)\ a_{n-1}(x) + x\ a_{n-1}(x+1) \quad \text{for} \ \ n \geq 3. \tag{4.3}$$

Since $a_1(x) = x$ and $a_2(x) = x^2 + 2x$, on setting $a_0(x) = 1$, the recurrence relation (4.3) is valid for $n \geq 1$. Note that $a_n(0) = 0$ for $n \geq 1$.

Proposition 4.2.10. *For $n \geq 1$, $a_n(x) = \sum_{r=1}^{n} s(n,r)\; x(x+1)\cdots(x+r-1)$, where $s(n,r)$ is the (signless) Stirling number of the first kind.*

Proof. Let $x^{\bar{r}} = x(x+1)\cdots(x+r-1)$ be the r^{th} *rising power* of x. Then $\{x^{\bar{r}} : r = 0, 1, \ldots\}$ is a $\mathbb{Q}$-basis of $\mathbb{Q}[x]$, where $x^{\bar{0}} = 1$. As $a_n(0) = 0$ for $n \geq 1$, we can express $a_n(x) = \sum_{r=1}^{n} \alpha_n(r) x^{\bar{r}}$. By recurrence relation (4.3)

$$\begin{aligned}
\sum_{r=1}^{n} \alpha_n(r) x^{\bar{r}} &= (n-1) \sum_{r=1}^{n-1} \alpha_{n-1}(r) x^{\bar{r}} + x \sum_{r=1}^{n-1} \alpha_{n-1}(r)(x+1)^{\bar{r}} \\
&= \sum_{r=1}^{n-1} (n-1)\alpha_{n-1}(r) x^{\bar{r}} + \sum_{r=1}^{n-1} \alpha_{n-1}(r) x^{\overline{r+1}} \\
&= \sum_{r=1}^{n-1} (n-1)\alpha_{n-1}(r) x^{\bar{r}} + \sum_{r=2}^{n} \alpha_{n-1}(r-1) x^{\bar{r}} \\
&= \alpha_{n-1}(n-1) x^{\bar{n}} + \sum_{r=1}^{n-1} \left[(n-1)\alpha_{n-1}(r) + \alpha_{n-1}(r-1)\right] x^{\bar{r}}.
\end{aligned}$$

Therefore, $\alpha_n(n) = \alpha_{n-1}(n-1)$ and $\alpha_n(r) = (n-1)\alpha_{n-1}(r) + \alpha_{n-1}(r-1)$. Moreover, since $a_1(x) = x$, we have $\alpha_1(1) = 1$. Taking $\alpha_0(0) = 1$, we see that $\alpha_n(r)$ and the (signless) Stirling number $s(n,r)$ of the first kind satisfy the same recurrence relation with the same initial conditions; see [51, Lemma 1.3.3]. Thus $\alpha_n(r) = s(n,r)$. $\square$

Theorem 4.2.11. *For $n \geq 1$, $\dim_{\mathbb{K}} \left(\frac{R}{I_W^{[\mathbf{n}]}} \right) = a_n = \sum_{r=1}^{n} (r!)\; s(n,r)$.*

Proof. Since $a_n = a_n(1)$, the statement follows from Proposition 4.2.10. $\square$

Consider the integer sequence (A007840) in OEIS [48]. The n^{th} term b_n of this sequence is the number of factorization of permutations of $[n]$ into ordered cycles and $b_n = \sum_{r=1}^{n}(r!)\ s(n,r)$. It can be verified that

$$b_n = \operatorname{Per}([m_{ij}]_{n\times n}) = \operatorname{Per}\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 1 & \dots & 1 \\ 1 & 1 & 3 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & n \end{bmatrix},$$

where $m_{ii} = i$ and $m_{ij} = 1$ for $i \neq j$ (see A007840 in [48]). We recall that *permanent* $\operatorname{Per}([m_{ij}]_{n\times n})$ of the matrix $[m_{ij}]_{n\times n}$ is given by $\sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^{n} m_{i\sigma(i)}$. There are many combinatorial interpretations of this integer sequence. Theorem 4.2.11 gives a description of the integer sequence (A007840) in terms of enumeration of standard monomials of $\frac{R}{I_W^{[\mathbf{n}]}}$, or equivalently, in terms of the number $|\widetilde{\mathrm{PF}}_n|$ of restricted parking functions of length n.

We now show that enumeration of standard monomials of $\frac{R}{I_W^{[\mathbf{n}]}}$ is related to enumeration of rooted-labeled unimodal forests on $[n]$. The concept of permutations avoiding patterns has been extended to many combinatorial objects; such as, trees, graphs and posets. Let F_n be the set of (unordered) rooted-labeled forests on the vertex set $[n]$. By Cayley's formula, $|F_n| = (n+1)^{n-1}$ (cf. [45]). A rooted-labeled forest on $[n]$ is said to *avoid a pattern* $\tau \in \mathfrak{S}_r$ if along each path from a root to a vertex, the sequence of labels do not contain a subsequence with the same relative order as in the patterns $\tau = \tau(1)\tau(2)\dots\tau(r)$ (see [3]). Let $F_n(\tau)$ be the set of rooted-labeled forests on $[n]$ that avoid pattern τ. For example, if $\tau = 21$ is a transposition, then $F_n(21)$ is the set of rooted-labeled increasing forests on $[n]$. In other words, labels on any path from a root to a vertex for a forest in $F_n(21)$ form an increasing sequence. Let $F_n(\tau^{(1)},\dots,\tau^{(s)})$ be the set of rooted-labeled forests on $[n]$ that avoid a set $\{\tau^{(1)},\dots,\tau^{(s)}\}$ of patterns. The enumeration of rooted-labeled forests on $[n]$ that avoid various patterns are obtained in [3].

In particular, it is shown that $|F_n(213, 312)| = \sum_{r=1}^{n}(r!)\ s(n, r)$ for $n \geq 1$. The rooted-labeled forests on $[n]$ avoiding 213 and 312-patterns are precisely the *unimodal* forests. Since $|\widetilde{\mathrm{PF}}_n| = |F_n(213, 312)|$, an explicit or algorithmic bijection $\phi : \widetilde{\mathrm{PF}}_n \longrightarrow F_n(213, 312)$ is desired.

Before we end this section, we describe an easy extension of Theorem 4.2.11.

Let $b, c \geq 1$ and $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ with $u_i = u_1 + (i-1)b$. Let $\mathrm{PF}_n(\lambda)$ be the set of λ-parking function of length n where $\lambda_i = u_n - u_i + c = (n-i)b+c$. Then $|\mathrm{PF}_n(\lambda)| = c(c+nb)^{n-1}$ (see, for example, [17, Theorem 2.5] or Theorem 3.1.13). Let $|\widetilde{\mathrm{PF}}_n(\lambda)| = \dim_{\mathbb{K}}\left(\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}}\right) = \widetilde{a_n}(c)$. Actually, $\widetilde{a_n}(c)$ depends on b also, but we are treating b to be a fixed constant. Also, $\widetilde{a_n}(c)$ is a polynomial expression in c.

Proposition 4.2.12. *For $n \geq 3,\ b, c \geq 1,\ \widetilde{a_n}(c)$ satisfies a recurrence relation*

$$\widetilde{a_n}(c) = ((n-1)b)\ \widetilde{a_{n-1}}(c) + (c)\ \widetilde{a_{n-1}}(c+b).$$

Proof. From equation (4.2), we have

$$\widetilde{a_n}(c) = \dim_{\mathbb{K}}\left(\frac{R}{I_W(\mathbf{u})^{[\mathbf{u_n+c-1}]}}\right) = \sum_{\mathcal{C}\in\mathfrak{Ch}([n])} (-1)^{n-\ell(\mathcal{C})-1}\ \mu^{\mathbf{u}}(\mathcal{C}),$$

where $u_i = u_1 + (i-1)b$. For such $\mathbf{u}$, Lemma 4.2.8 holds. Thus

$$\widetilde{a_n}(c) = \sum_{\mathcal{C}\in\mathfrak{Ch}([n])} (-1)^{n-\ell(\mathcal{C})-1}\ \mu^{\mathbf{u}}(\mathcal{C}) = \sum_{\mathcal{C}\in\mathfrak{Ch}'([n])} (-1)^{n-\ell(\mathcal{C})-1}\ \mu^{\mathbf{u}}(\mathcal{C}).$$

Now proceeding as in the proof of Proposition 4.2.9 we get our result. $\square$

Replacing c with an indeterminate x, we consider polynomial $\widetilde{a_n}(x)$. Thus there is a polynomial identity

$$\widetilde{a_n}(x) = ((n-1)b)\ \widetilde{a_{n-1}}(x) + x\ \widetilde{a_{n-1}}(x+b) \quad \text{for} \ \ n \geq 3. \tag{4.4}$$

Since $\widetilde{a_1}(x) = x$ and $\widetilde{a_2}(x) = x^2 + 2bx$, on setting $\widetilde{a_0}(x) = 1$, the recurrence relation (4.4) is valid for $n \geq 1$. Again, we have $\widetilde{a_n}(0) = 0$ for $n \geq 1$.

Theorem 4.2.13. *For $n \geq 1$, $\widetilde{a_n}(x) = \sum_{r=1}^{n}(b^{n-r}\ s(n,r))\ x(x+b)\cdots(x+(r-1)b)$. In particular, for $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_i = (n-i)b + c$*

$$|\widetilde{\mathrm{PF}}_n(\lambda)| = \widetilde{a_n}(c) = b^n \sum_{r=1}^{n} s(n,r)\ \frac{\Gamma(\frac{c}{b}+r)}{\Gamma(\frac{c}{b})},$$

where Γ is the gamma function, i.e., $\Gamma(x+1) = x\ \Gamma(x)$ for $x > 0$ and $\Gamma(1) = 1$.

Proof. As in the proof of Theorem 4.2.11, let

$$\widetilde{a_n}(x) = \sum_{r=1}^{n} \widetilde{\alpha_n}(r)\ x(x+b)\cdots(x+(r-1)b).$$

Then from recurrence relation (4.4), $\widetilde{\alpha_n}(r)$ satisfies the recurrence relation

$$\widetilde{\alpha_n}(r) = (n-1)b\ \widetilde{\alpha_{n-1}}(r) + \widetilde{\alpha_{n-1}}(r-1); \quad \text{for } 1 \leq r \leq n,$$

with initial conditions $\widetilde{\alpha_0}(1) = 0$ and $\widetilde{\alpha_1}(1) = 1$. Therefore, $\widetilde{\alpha_n}(r) = b^{n-r}\ s(n,r)$. $\square$

4.3 Some other cases

The Betti numbers and enumeration of standard monomials of the Artinian quotient $\frac{R}{I_S^{[\mathbf{n}]}}$ for $S = \mathfrak{S}_n(132,231), \mathfrak{S}_n(123,132)$ and $\mathfrak{S}_n(123,132,213)$ are given in [30,31]. In this section, the monomial ideal I_S and its Alexander dual $I_S^{[\mathbf{n}]}$ are studied for various other subsets $S \subseteq \mathfrak{S}_n$ consisting of permutations avoiding patterns. For clarity of presentation, we divide the study into three cases.

Case 1. $S_1 = \mathfrak{S}_n(123, 132, 312)$, $S_2 = \mathfrak{S}_n(123, 213, 231)$, $S_3 = \mathfrak{S}_n(132, 213, 231)$.

Case 2. $T_1 = \mathfrak{S}_n(123, 132, 231)$, $T_2 = \mathfrak{S}_n(213, 312, 321)$.

Case 3. $U = \mathfrak{S}_n(123, 231, 312)$.

We have, $|S_a| = |T_b| = |U| = n$ for $1 \leq a \leq 3$ and $1 \leq b \leq 2$ (see [47, Lemma 6, Proposition 16*]).

Lemma 4.3.1. *The minimal generators of the Alexander dual $I_S^{[\mathbf{n}]}$ for $S = S_a$, T_b or U are given as follows.*

(i) $I_{S_1}^{[\mathbf{n}]} = \left\langle x_\ell^{\ell+1},\ x_i^i \left(\prod_{j>i} x_j\right) : 1 \leq \ell \leq n-1;\ 1 \leq i \leq n \right\rangle$.

(ii) $I_{S_2}^{[\mathbf{n}]} = \left\langle x_\ell^n,\ x_i^i x_j^{j-1} : 1 \leq \ell \leq n;\ 1 \leq i < j \leq n \right\rangle$.

(iii) $I_{S_3}^{[\mathbf{n}]} = \left\langle x_\ell^n,\ x_i^i x_j^{n-(j-i)} : 1 \leq \ell \leq n;\ 1 \leq i < j \leq n \right\rangle$.

(iv) $I_{T_1}^{[\mathbf{n}]} = \left\langle x_\ell^{\ell+1},\ x_n^n,\ x_i^i x_n^i : 1 \leq \ell \leq n-1;\ 1 \leq i < n \right\rangle$.

(v) $I_{T_2}^{[\mathbf{n}]} = \left\langle x_\ell^{n-\ell+1},\ x_n^n,\ x_i^{n-i} x_n^{n-i} : 1 \leq \ell \leq n-1;\ 1 \leq i < n \right\rangle$.

(vi) $I_U^{[\mathbf{n}]} = \left\langle \prod_{j \in A} x_j^{\nu_{j,A}} : A = \{j_1, \ldots, j_t\} \in \Sigma_n \right\rangle$, *where* $\nu_{j_1,A} = n - (j_{|A|} - j_1)$ *and* $\nu_{j_i,A} = j_i - j_{i-1}$ *for* $i \geq 2$, *provided* $j_1 < j_2 < \cdots < j_t$.

Proof. We recall that a vector $\mathbf{b} \in \mathbb{N}^n$ satisfying $\mathbf{b} \leq \mathbf{n}$ (i.e., $b_i \leq n$) is maximal with $\mathbf{x}^{\mathbf{b}} \notin I_S$ if and only if $\mathbf{x}^{\mathbf{n}-\mathbf{b}}$ is a minimal generator of $I_S^{[\mathbf{n}]}$ (see Proposition 2.3.13). Now proceeding as in the proof of [31, Lemma 2.1, 2.2], we can get the minimal generators of the Alexander duals. We sketch a proof of part (i), (v) and (vi) as the proof for other parts is on similar lines.

For any two integers r, s with $r \leq s$, let $[r, s]$ denote the set $\{r, r+1, \ldots, s\}$.

(i) For $\ell \in [n-1]$, let $\mathbf{b}_\ell = (n, \ldots, n-\ell-1, \ldots, n)$ (ℓ^{th} coordinate $n-\ell-1$, elsewhere n). We claim that $\mathbf{x}^{\mathbf{b}_\ell} \notin I_{S_1}$. If not, then there is a $\sigma \in S_1$ such that $\mathbf{x}^\sigma$ divides $\mathbf{x}^{\mathbf{b}_\ell}$. Thus $1 \leq \sigma(\ell) \leq n-\ell-1$. This

implies that $\ell \leq n-2$. Moreover, $|\{a \in [n] : a < \sigma(\ell)\}| \leq n-\ell-2$ and $|[\ell+1, n]| = n-\ell$ together ensure that there exists some $u, v \in [\ell+1, n]$ such that $\sigma(\ell) < \sigma(u) < \sigma(v)$. Hence, σ contains either a 123 pattern or a 132 pattern, a contradiction. Further, for any $\mathbf{b}'_\ell$ with $\mathbf{b}_\ell < \mathbf{b}'_\ell \leq \mathbf{n}$, $\mathbf{x}^{\sigma'}$ divides $\mathbf{x}^{\mathbf{b}'_\ell}$ for $\sigma' = (n-1, n-2, \ldots, 1, n) \in S_1$. This gives the minimal generator $x_\ell^{\ell+1} \in I_{S_1}^{[\mathbf{n}]}$. For $i \in [n]$, let $\mathbf{b}_{i,n} = (n, \ldots, n, n-i, n-1, \ldots, n-1) \in \mathbb{N}^n$ (i.e., i^{th} coordinate is $n-i$, first $i-1$ coordinates are n, and the last $n-i$ coordinates are $n-1$). We claim that $\mathbf{x}^{\mathbf{b}_{i,n}} \notin I_{S_1}$. If not, then there exists $\sigma \in S_1$ such that $\mathbf{x}^\sigma$ divides $\mathbf{x}^{\mathbf{b}_{i,n}}$. If $i = 1$ or n, we see that such a σ cannot exist. Let $2 \leq i \leq n-1$. We have $\sigma(i) \leq n-i$ and $\sigma(u) \leq n-1$ for $i+1 \leq u \leq n$. Moreover, $|[1, n-i]| = n-i$ and $|[i, n]| = n-i+1$ together ensure that there exists some $v \in [i+1, n]$ such that $n-i < \sigma(v) \leq n-1$. Since $\sigma(j) = n$ for some $j < i$ we see that σ contains a 312 pattern, a contradiction. We now show that such a $\mathbf{b}_{i,n}$ is maximal. Let $\mathbf{b}_{i,n} < \mathbf{b}' \leq \mathbf{n}$. Then the i^{th} coordinate of $\mathbf{b}'$ is r for some $r \geq n-i$. If $r \geq n-i+1$, then $\mathbf{x}^{\sigma'}$ divides $\mathbf{x}^{\mathbf{b}'}$ for $\sigma' = (n, n-1, \ldots, 1) \in S_1$. When $r = n-i$, there exists some j with $i+1 \leq j \leq n$ such that the j^{th} coordinate of $\mathbf{b}'$ is n. In this case take $\sigma' \in S_1$, where $\sigma'(t) = \begin{cases} n-t & \text{for} \quad t < j, \\ n-t+1 & \text{for} \quad t > j, \\ n & \text{otherwise.} \end{cases}$
Then $\mathbf{x}^{\sigma'}$ divides $\mathbf{x}^{\mathbf{b}'}$. This gives the minimal generators $x_i^i(\prod_{j>i} x_j) \in I_{S_1}^{[\mathbf{n}]}$. Next we show that $\mathbf{b}_\ell$ and $\mathbf{b}_{i,n}$ are the only possible maximal $\mathbf{b}(\leq \mathbf{n})$ such that $\mathbf{x}^{\mathbf{b}} \notin I_{S_1}$. If not, let $\mathbf{b} = (\alpha_1, \ldots, \alpha_n) \leq \mathbf{n}$ be maximal, not be equal to $\mathbf{b}_\ell$ or $\mathbf{b}_{i,n}$, and satisfy $\mathbf{x}^{\mathbf{b}} \notin I_{S_1}$. Then $\alpha_n \geq 1$ and $\alpha_i \geq n-i$ for $1 \leq i \leq n-1$. We have $\alpha_n \leq n-1$ because otherwise $\mathbf{x}^\sigma$ divides $\mathbf{x}^{\mathbf{b}}$, where $\sigma = (n-1, n-2, \ldots, 1, n) \in S_1$. If $\alpha_{n-1} = 1$, then $\mathbf{b} \leq \mathbf{b}_{n-1,n}$. If $\alpha_{n-1} = n$, then $\mathbf{x}^\sigma$ divides $\mathbf{x}^{\mathbf{b}}$, where $\sigma = (n-1, n-2, \ldots, 2, n, 1) \in S_1$. Thus $2 \leq \alpha_{n-1} \leq n-1$. Similarly, we can show that $i+1 \leq \alpha_{n-i} \leq n-1$ for

$0 \leq i \leq n-2$. If $\alpha_1 = n$, then $\mathbf{x}^{\sigma}$ divides $\mathbf{x}^{\mathbf{b}}$, where $\sigma = (n, n-1, \ldots, 1) \in S_1$. Thus $\alpha_1 = n-1$. But, in that case $\mathbf{b} \leq \mathbf{b}_{1,n}$, a contradiction.

(v) Let $\mathbf{b}_n = (n, \ldots, n, 0)$ (n^{th} coordinate 0, elsewhere n). We see that $\mathbf{x}^{\mathbf{b}_n} \notin I_{T_2}$. Further for any $\mathbf{b}'$ with $\mathbf{n} \geq \mathbf{b}' > \mathbf{b}_n$, $\mathbf{x}^{\sigma'}$ divides $\mathbf{x}^{\mathbf{b}'}$ where $\sigma' = (2, 3, \ldots, n, 1) \in T_2$. This gives the minimal generator $\mathbf{x}_n^n \in I_{T_2}^{[\mathbf{n}]}$. For $\ell \in [n-1]$, let $\mathbf{b}_\ell = (n, \ldots, \ell-1, \ldots, n)$ (ℓ^{th} coordinate $\ell-1$, elsewhere n). We claim that $\mathbf{x}^{\mathbf{b}_\ell} \notin I_{T_2}$. If not, then there is a $\sigma \in T_2$ such that $\mathbf{x}^{\sigma}$ divides $\mathbf{x}^{\mathbf{b}_\ell}$. Thus $1 \leq \sigma(\ell) \leq \ell - 1$. Hence $\ell \geq 2$. Moreover, $|\{a \in [n] : a < \sigma(\ell)\}| \leq \ell - 2$ and $|[1, \ell-1]| = \ell - 1$ together ensure that there exists some $u \in [1, \ell-1]$ such that $\sigma(u) > \sigma(\ell)$. Since $\sigma(n) = i$ for some $i \in [n]$, we see that σ contains 213 or 321 or 312 pattern, a contradiction. Further, for any $\mathbf{b}'_\ell$ with $\mathbf{b}_\ell < \mathbf{b}'_\ell \leq \mathbf{n}$, $\mathbf{x}^{\sigma'}$ divides $\mathbf{x}^{\mathbf{b}'_\ell}$ for $\sigma' = (1, 2, \ldots, n) \in T_2$. This gives the minimal generator $x_\ell^{n-\ell+1} \in I_{T_2}^{[\mathbf{n}]}$. For $1 \leq i < n$ let $\mathbf{b}_{i,n} = (n, \ldots, n, i, n, \ldots, n, i) \in \mathbb{N}^n$ (i.e., i^{th} coordinate and n^{th} coordinate are i and elsewhere n). We claim that $\mathbf{x}^{\mathbf{b}_{i,n}} \notin I_{T_2}$. If not, then there exists $\sigma \in T_2$ such that $\mathbf{x}^{\sigma}$ divides $\mathbf{x}^{\mathbf{b}_{i,n}}$. Thus $\sigma(i) \leq i$ and $\sigma(n) \leq i$. This implies that $2 \leq i \leq n-1$. If $\sigma(j) = n$ for some $j < i$, then σ has either a 312 pattern or a 321 pattern, a contradiction. Let $\sigma(j) = n$ for some $j > i$. Since $|[1, i]| = i$ and $\sigma(i) \leq i$, $\sigma(n) \leq i$, there exists some $u < i$ such that $\sigma(u) > \sigma(i)$. Thus σ contains a 213 pattern which is a contradiction. We now show that such a $\mathbf{b}_{i,n}$ is maximal. Let $\mathbf{b}_{i,n} < \mathbf{b}' \leq \mathbf{n}$. Then the i^{th} coordinate or the n^{th} coordinate of $\mathbf{b}'$ is strictly greater than i.

In the first case take $\sigma' \in T_2$, where $\sigma'(t) = \begin{cases} t & \text{if} \quad t < i, \\ t+1 & \text{if} \quad t \geq i,\, t \neq n, \\ i & \text{if} \quad t = n, \end{cases}$

so that $\mathbf{x}^{\sigma'}$ divides $\mathbf{x}^{\mathbf{b}'}$. In the second case take $\sigma' \in T_2$, where $\sigma'(t) = \begin{cases} t & \text{if} \quad t \leq i, \\ t+1 & \text{if} \quad t \geq i+1,\, t \neq n, \\ i+1 & \text{if} \quad t = n, \end{cases}$

so that $\mathbf{x}^{\sigma'}$ divides $\mathbf{x}^{\mathbf{b}'}$. Next we show that $\mathbf{b}_n$, $\mathbf{b}_\ell$ and $\mathbf{b}_{i,n}$ are the only possible maximal $\mathbf{b}(\leq \mathbf{n})$ such that $\mathbf{x}^{\mathbf{b}} \notin I_{T_2}$. If not, let $\mathbf{b} = (\alpha_1, \ldots, \alpha_n) \leq \mathbf{n}$ maximal, not equal to $\mathbf{b}_n$ or $\mathbf{b}_\ell$ or $\mathbf{b}_{i,n}$, and satisfy $\mathbf{x}^{\mathbf{b}} \notin I_{T_2}$. Then $\alpha_n \geq 1$ and $\alpha_i \geq i$ for $1 \leq i \leq n-1$. If $\alpha_n = 1$, then $\alpha_i \geq i+1$ for $1 \leq i \leq n-1$. In that case, $\mathbf{x}^\sigma$ divides $\mathbf{x}^{\mathbf{b}}$, where $\sigma = (2, 3, \ldots, n, 1) \in T_2$. Thus $\alpha_n \geq 2$. If $\alpha_n = 2$, then $\alpha_i \geq i+1$ for $2 \leq i \leq n-1$. In that case, $\mathbf{x}^\sigma$ divides $\mathbf{x}^{\mathbf{b}}$, where $\sigma = (1, 3, \ldots, n, 2) \in T_2$. Thus $\alpha_n \geq 3$. Continuing this way we can show that $\alpha_n = n$. But, in that case $\mathbf{x}^\sigma$ divides $\mathbf{x}^{\mathbf{b}}$, where $\sigma = (1, 2, \ldots, n) \in T_2$, a contradiction.

(vi) If $A = \{\ell\} \in \Sigma_n$, then taking $\widehat{\mathbf{b}}_\ell = (n, \ldots, 0, \ldots, n)$ (i.e., 0 at ℓ^{th} place and elsewhere n), we see that $\mathbf{x}^{\widehat{\mathbf{b}}_\ell} \notin I_U$. Further, for any $\mathbf{b}'$ with $\widehat{\mathbf{b}}_\ell < \mathbf{b}' \leq \mathbf{n}$, $\mathbf{x}^{\sigma'}$ divides $\mathbf{x}^{\mathbf{b}'}$ for $\sigma' = (\ell, \ldots, 2, 1, n, n-1, \ldots, \ell+1)$. Thus we get the minimal generator $x_\ell^n \in I_U^{[\mathbf{n}]}$.
For $A = \{j_1, \ldots, j_t\} \in \Sigma_n$ with $t \geq 2$ and $j_1 < \cdots < j_t$, let $\widehat{\mathbf{b}}_A = (b_1, \ldots, b_n)$, where $b_{j_1} = j_t - j_1,\ b_{j_i} = n - (j_i - j_{i-1})$ (for $i \geq 2$) and $b_r = n$ (for $r \notin A$).
Claim : $\mathbf{x}^{\widehat{\mathbf{b}}_A} \notin I_U$.

Otherwise, there exists a $\sigma \in U$ such that $\mathbf{x}^\sigma$ divides $\mathbf{x}^{\widehat{\mathbf{b}}_A}$. Thus $\sigma(j_1) \leq j_t - j_1$ and $\sigma(j_i) \leq n - (j_i - j_{i-1})$ for $2 \leq i \leq t$. We show that

$$\sigma(j_1) > \sigma(j_2) > \cdots > \sigma(j_t).$$

If $\sigma(j_{i-1}) < \sigma(j_i)$ for $1 < i \leq t$, then $\sigma(j_{i-1}), \sigma(j_i) \in [n - (j_i - j_{i-1})]$. But $|[n - (j_i - j_{i-1})]| = n - (j_i - j_{i-1})$ and $|[j_{i-1}] \coprod [j_i, n]| = n - (j_i - j_{i-1}) + 1$. Thus there exists $\ell \in [n] \setminus [j_{i-1}, j_i]$ such that $\sigma(\ell) \notin [n - (j_i - j_{i-1})]$. This shows that $\sigma(j_{i-1}) < \sigma(j_i) < \sigma(\ell)$. Hence, σ has a 123 or a 312-pattern, a contradiction to $\sigma \in U$. Now $\sigma(j_t) < \sigma(j_1) \leq j_t - j_1$ implies that $j_t - j_1 \geq 2$. Again, $\sigma(j_1), \sigma(j_t) \in [j_t - j_1]$, but $|[j_t - j_1]| = j_t - j_1 < |[j_1, j_t]| = j_t - j_1 + 1$. Thus there exists $\ell \in [j_1 + 1, j_t - 1]$ such that $\sigma(\ell) > j_t - j_1$. This shows that, $\sigma(j_t) < \sigma(j_1) < \sigma(\ell)$ with $j_1 < \ell < j_t$. Thus σ has a 231-pattern, a contradiction. This proves our claim. It can be shown that $\widehat{\mathbf{b}}_A$ has the desired maximality property and hence $\mathbf{x}^{\mathbf{n}-\widehat{\mathbf{b}}_A}$ is a minimal generator of $I_U^{[\mathbf{n}]}$. $\square$

We shall show that all monomial ideals in Lemma 4.3.1 are order monomial ideals.

It is convenient to study the monomial ideal I_S in Lemma 4.3.1 according to the three cases already described.

Case-1. To each monomial ideal $I_{S_a}^{[\mathbf{n}]}$, we associate a poset $\Sigma_n(S_a)$ (for $1 \leq a \leq 3$) as follows.

(i) Let $\Sigma_n(S_1) = \{\{\ell\} : 1 \leq \ell \leq n-1\} \cup \{[i,n] : 1 \leq i \leq n\}$, where $[i,n] = \{a \in \mathbb{N} : i \leq a \leq n\}$ and $[n,n] = \{n\}$. We define a poset structure on $\Sigma_n(S_1)$ by describing cover relations. For $\ell, \ell' \in [n-1]$ and $i, i' \in [n]$, $\{\ell\}$ covers $\{\ell'\}$ (or $[i',n]$), if $\ell' = \ell + 1$ (respectively, $i' = \ell + 2$). Also, $[i,n]$ covers $\{\ell'\}$ (or $[i',n]$) if $i = \ell'$ (respectively, $i' = i+1$). The monomial labels $\omega_{\{\ell\}} = x_\ell^{\ell+1}$ and $\omega_{[i,n]} = x_i^i x_{i+1} \ldots x_n$. Set $\mu^1_{j,C}$ for $C \in \Sigma_n(S_1)$ so that $\omega_C = \prod_{j \in C} x_j^{\mu^1_{j,C}}$. The finite poset $\Sigma_n(S_1)$ appeared in [31].

(ii) Let $\Sigma_n(S_2) = \{\{\ell\} : 1 \leq \ell \leq n\} \cup \{\{i,j\} : 1 \leq i < j \leq n\}$. A poset structure on $\Sigma_n(S_2)$ is given by the following cover relations. For $i, j, i', j' \in [n]$ with $i < j$ and $i' < j'$, $\{i,j\}$ covers $\{i',j'\}$, if either ($i = i'$ and $j' = j+1$) or ($j = i'$ and $j' = j+1$). Also, $\{i,j\}$ covers $\{i'\}$ if either ($i = i'$ and $j = n$) or ($i' = j = n$). In this case, the monomial labels $\omega_{\{\ell\}} = x_\ell^n$ and $\omega_{\{i,j\}} = x_i^i x_j^{j-1}$. Set $\mu^2_{j,C}$ for $C \in \Sigma_n(S_2)$ so that $\omega_C = \prod_{j \in C} x_j^{\mu^2_{j,C}}$.

(iii) Let $\Sigma_n(S_3) = \{\{\ell\} : 1 \leq \ell \leq n\} \cup \{\{i,j\} : 1 \leq i < j \leq n\}$. Again, a poset structure on $\Sigma_n(S_3)$ is given by the following cover relations. For $i, j, i', j' \in [n]$ with $i < j$ and $i' < j'$, $\{i,j\}$ covers $\{i',j'\}$, if either ($i = i'$ and $j = j'+1$) or ($i = i'-1$ and $j' = j$). Also, $\{i,j\}$ covers $\{i'\}$

if either ($i = i'$ and $j = i+1$) or ($i' = j = i+1$). Again, the monomial labels $\omega_{\{\ell\}} = x_\ell^n$ and $\omega_{\{i,j\}} = x_i^i x_j^{n-(j-i)}$. Set $\mu_{j,C}^3$ for $C \in \Sigma_n(S_3)$ so that $\omega_C = \prod_{j \in C} x_j^{\mu_{j,C}^3}$.

The Hasse diagrams of $\Sigma_4(S_1)$, $\Sigma_4(S_2)$ and $\Sigma_4(S_3)$ are given in Figure 4.2.

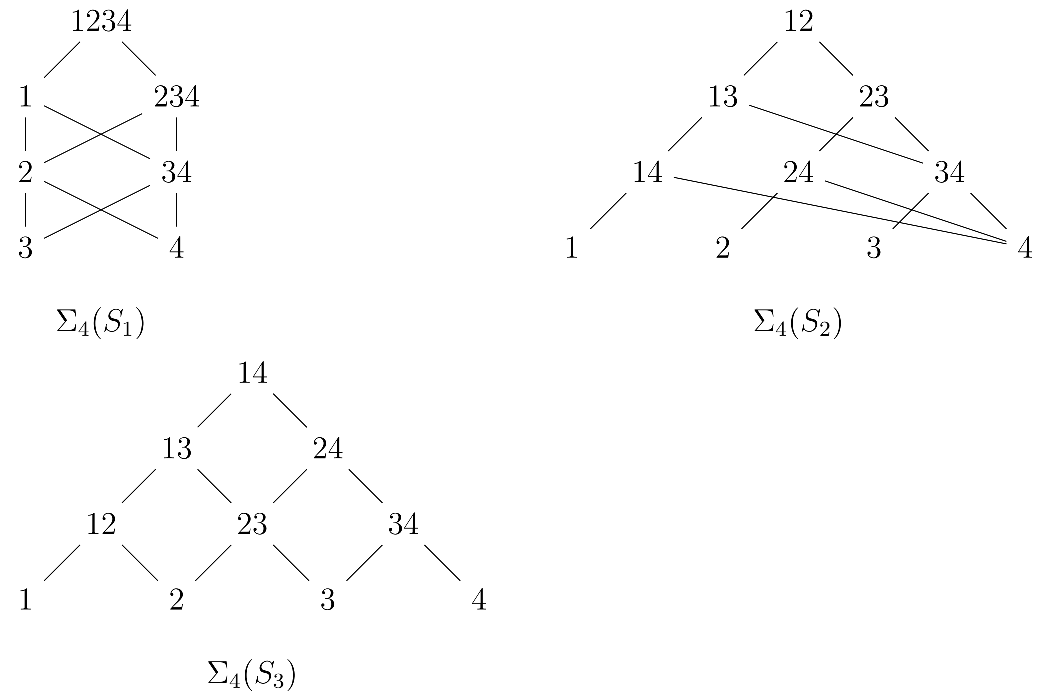


Figure 4.2: Hasse diagram for $\Sigma_4(S_1)$, $\Sigma_4(S_2)$ and $\Sigma_4(S_3)$.

Proposition 4.3.2. (i). *The ideal* $I_{S_a}^{[\mathbf{n}]}$ *is an order monomial ideal for* $1 \leq a \leq 3$.
(ii). *The free complex* $\mathbb{F}_*(\Delta(\Sigma_n(S_a)))$ *is a cellular resolution of* $I_{S_a}^{[\mathbf{n}]}$ *supported on the order complex* $\Delta(\Sigma_n(S_a))$ *for* $1 \leq a \leq 3$.

Proof. Given the poset structure on $\Sigma_n(S_a)$ as above, it can be directly verified that $I_{S_a}^{[\mathbf{n}]}$ is an order monomial ideal. We show for example, that $I_{S_2}^{[\mathbf{n}]}$ is an order monomial ideal (cf. Definition 4.2.3).
Let $P = \Sigma_n(S_2)$. If $v_1,\ v_2 \in P$ such that $v_1 \preceq v_2$, then we take $v = v_2$ so that m_v divides $\operatorname{lcm}(m_{v_1}, m_{v_2})$. Let $v_1,\ v_2 \in P$ such that v_1 and v_2 are not comparable. The following three cases arise:

(a) $v_1 = \{i, j\}$ and $v_2 = \{r, s\}$, where $i < j$, $i < r < j$ and $r < s \leq n$. In this case take $v = \{i, r\}$ so that m_v divides $\mathrm{lcm}(m_{v_1}, m_{v_2})$.

(b) $v_1 = \{i, j\}$ and $v_2 = \{r\}$, where $i < j$ and $i, j \neq r < n$. In this case take $v = \{i, r\}$ so that m_v divides $\mathrm{lcm}(m_{v_1}, m_{v_2})$.

(c) $v_1 = \{i\}$ and $v_2 = \{j\}$, where $i < j$. In this case take $v = \{i, j\}$ so that m_v divides $\mathrm{lcm}(m_{v_1}, m_{v_2})$.

Thus $I_{S_2}^{[\mathbf{n}]}$ is an order monomial ideal.

In view of Theorem 4.2.4 we see that the free complex $\mathbb{F}_*(\Delta(P))$ is a cellular resolution of the order monomial ideal $I = \langle \omega_u : u \in P \rangle$ (see Theorem 4.2.4), where P is $\Sigma_n(S_a)$ for $1 \leq a \leq 3$. □

Remark 4. *The cellular resolution $\mathbb{F}_*(\Delta(\Sigma_n(S_a)))$ is minimal for $a = 1$, but nonminimal for $a = 2, 3$. Also, the r^{th} Betti number $\beta_r(I_{S_1}^{[\mathbf{n}]})$ is given by (see* [31, Theorem 2.7]*)*

$$\beta_r(I_{S_1}^{[\mathbf{n}]}) = \sum_{s=0}^{r+1} \binom{n-1}{s} \binom{n-s}{r+1-s}; \quad (0 \leq r \leq n-1).$$

We now identify standard monomials of $\frac{R}{I_{S_a}^{[\mathbf{n}]}}$. Consider the following subsets of the set PF_n of parking functions $\mathbf{p} = (p_1, \ldots, p_n)$ of length n.

(i) $\mathrm{PF}_n^1 = \{\mathbf{p} \in \mathrm{PF}_n : p_t \leq t,$ for all t and if $p_i = i,$ then $p_j = 0$ for some $j \in [i, n]\}$.

(ii) $\mathrm{PF}_n^2 = \{\mathbf{p} \in \mathrm{PF}_n :$ if $p_i \geq i,$ then $p_j < j - 1$ for all $j \in [i+1, n]\}$.

(iii) $\mathrm{PF}_n^3 = \{\mathbf{p} \in \mathrm{PF}_n :$ if $p_i \geq i,$ then $p_j < n - (j - i)$ for all $j \in [i+1, n]\}$.

In view of Lemma 4.3.1, $\mathbf{x}^{\mathbf{p}} \notin I_{S_a}^{[\mathbf{n}]}$ if and only if $\mathbf{p} \in \mathrm{PF}_n^a$ for $1 \leq a \leq 3$. Thus (fine) Hilbert series $H\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}}, \mathbf{x}\right)$ of $\frac{R}{I_{S_a}^{[\mathbf{n}]}}$ is given by $H\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}}, \mathbf{x}\right) = \sum_{\mathbf{p} \in \mathrm{PF}_n^a} \mathbf{x}^{\mathbf{p}}$. In particular, $|\mathrm{PF}_n^a| = \dim_{\mathbb{K}} \left(\frac{R}{I_{S_a}^{[\mathbf{n}]}}\right) = H\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}}, \mathbf{1}\right)$, where $\mathbf{1} =$

$(1,\ldots,1)$. Using the cellular resolution $\mathbb{F}_*(\Delta(\Sigma_n(S_a)))$ supported on the order complex $\Delta(\Sigma_n(S_a))$, the (fine) Hilbert series $H\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}},\mathbf{x}\right)$ is given by

$$H\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}},\mathbf{x}\right)=\frac{\sum_{i=0}^{n}(-1)^i\sum_{(C_1,\ldots,C_i)\in\mathcal{F}_{i-1}^a}\prod_{q=1}^{i}\left(\prod_{j\in C_q\setminus C_{q-1}}x_j^{\mu_{j,C_q}^a}\right)}{(1-x_1)\cdots(1-x_n)},\qquad(4.5)$$

where $\mathcal{F}_{i-1}^a$ is the set of $i-1$-dimensional faces of $\Delta(\Sigma_n(S_a))$, $(C_1,\ldots,C_i)\in\mathcal{F}_{i-1}^a$ is a (strict) chain $C_1\prec\ldots\prec C_i$ of length $i-1$, $C_0=\emptyset$ and $\mu_{j,C}^a$ is as in the definition of poset $\Sigma_n(S_a)$.

Proposition 4.3.3. *The number of standard monomials of $\frac{R}{I_{S_a}^{[\mathbf{n}]}}$ is given by*

$$\dim_{\mathbb{K}}\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}}\right)=\sum_{i=1}^{n}(-1)^{n-i}\sum_{\substack{(C_1,\ldots,C_i)\in\mathcal{F}_{i-1}^a\\ C_1\cup\cdots\cup C_i=[n]}}\prod_{q=1}^{i}\left(\prod_{j\in C_q\setminus C_{q-1}}\mu_{j,C_q}^a\right),$$

where summation is carried over all $(i-1)$-dimensional faces $(C_1,\ldots,C_i)\in\mathcal{F}_{i-1}^a$ of $\Delta(\Sigma_n(S_a))$ with $C_1\cup\cdots\cup C_i=[n]$ and $C_0=\emptyset$. Also,

$$\dim_{\mathbb{K}}\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}}\right)=\sum_{\substack{0\le i\le n;\\ (C_1,\ldots,C_i)\in\mathcal{F}_{i-1}^a}}(-1)^i\left(\prod_{q=1}^{i}\Big(\prod_{j\in C_q\setminus C_{q-1}}(\mu_{j,\{j\}}^a-\mu_{j,C_q}^a)\Big)\right)\left(\prod_{l\notin C_i}\mu_{l,\{l\}}^a\right),$$

where summation is carried over all faces $(C_1,\ldots,C_i)\in\mathcal{F}_{i-1}^a$ including the empty face $C_0=\emptyset$.

Proof. As $|\mathrm{PF}_n^a|=\dim_{\mathbb{K}}\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}}\right)=H\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}},\mathbf{1}\right)$, letting $\mathbf{x}\to\mathbf{1}$ in the rational function expression (4.5) of $H\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}},\mathbf{x}\right)$, and applying L'Hospital's rule, we get the first formula. For more details, see the proof of [29, Proposition 4.5]. In order to get the second formula, put $y_j=\frac{1}{x_j}$ in (4.5) to get a rational function, say $\tilde{H}\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}},\mathbf{y}\right)$. Now letting $\mathbf{y}\to\mathbf{1}$ in the product $\left(\prod_{j=1}^{n}y_j^{\mu_{j,\{j\}}^a-1}\right)\tilde{H}\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}},\mathbf{y}\right)$, we get the second formula, which is due to Postnikov and Shapiro [45, Proposition 8.4]. $\square$

Theorem 4.3.4. *The number of standard monomials of* $\frac{R}{I^{[\mathbf{n}]}_{S_a}}$ *is given by*

$$\dim_{\mathbb{K}}\left(\frac{R}{I^{[\mathbf{n}]}_{S_a}}\right) = |\mathrm{PF}^a_n| = \frac{(n+1)!}{2}, \quad (1 \leq a \leq 3).$$

Proof. As $\dim_{\mathbb{K}}\left(\frac{R}{I^{[\mathbf{n}]}_{S_a}}\right) = 1$ for $n = 1$, we assume that $n > 1$. The cases $a \in \{1, 2, 3\}$ are treated separately as follows:

(i) Let $a = 1$. Using the second formula

$$\dim_{\mathbb{K}}\left(\frac{R}{I^{[\mathbf{n}]}_{S_1}}\right) = \sum_{\substack{0\leq i\leq n;\\ (C_1,\ldots,C_i)\in\mathcal{F}^1_{i-1}}} (-1)^i \left(\prod_{q=1}^{i}\Big(\prod_{j\in C_q\setminus C_{q-1}} (\mu^1_{j,\{j\}} - \mu^1_{j,C_q})\Big)\right)\left(\prod_{l\notin C_i} \mu^1_{l,\{l\}}\right)$$

in Proposition 4.3.3, we shall show that

$$\dim_{\mathbb{K}}\left(\frac{R}{I^{[\mathbf{n}]}_{S_1}}\right) = n(n!) + (n-1)((n-1)!) \sum_{\substack{1\leq i\leq n;\\ 0=j_0<j_1<\cdots<j_i<n}} (-1)^i \frac{1}{\prod_{q=2}^{i} j_q}. \quad (4.6)$$

The term corresponding to the empty chain is $n(n!)$. Also, for a (strict) chain $C_1 \prec \cdots \prec C_i$ in $\mathcal{F}^1_{i-1}$, the corresponding term in the second formula is zero if the chain has a singleton member. Thus surviving terms are of the form $C_l = [j_{i-l+1}, n]$ for some sequence $0 = j_0 < j_1 < \cdots < j_i < n$. Note that the term corresponding to such a chain is precisely, $(-1)^i \frac{(n-1)((n-1)!)}{\prod_{q=2}^{i} j_q}$. This proves (4.6). Let $\alpha_n = \sum_{i\geq 1}(-1)^{i+1} \sum_{0=j_0<j_1<\ldots<j_i<n} \frac{1}{\prod_{q=2}^{i} j_q}$. Clearly, $\alpha_1 = 0$. For $n > 1$, we claim that $\alpha_n = \frac{n}{2}$. We have,

$$\begin{aligned}
\alpha_n &= \sum_{i\geq 1}(-1)^{i+1} \sum_{0=j_0<j_1<\cdots<j_i<n-1} \frac{1}{\prod_{q=2}^{i} j_q} + \sum_{i\geq 1}(-1)^{i+1} \sum_{0=j_0<j_1<\cdots<j_i=n-1} \frac{1}{\prod_{q=2}^{i} j_q} \\
&= \alpha_{n-1} + \frac{1}{n-1}\sum_{i\geq 2}(-1)^{i+1} \sum_{0=j_0<j_1<\cdots<j_{i-1}<n-1} \frac{1}{\prod_{q=2}^{i-1} j_q} + 1 \\
&= \alpha_{n-1} - \frac{1}{n-1}\alpha_{n-1} + 1 = \frac{n-2}{n-1}\alpha_{n-1} + 1.
\end{aligned}$$

On solving this recurrence relation, we get $\alpha_n = \frac{n}{2}$ for $n > 1$. Now in view of (4.6),

$$\dim_{\mathbb{K}}\left(\frac{R}{I_{S_1}^{[\mathbf{n}]}}\right) = n(n!) + (n-1)((n-1)!)\left(\frac{-n}{2}\right) = \frac{(n+1)!}{2}.$$

(ii) Let $a = 2$. Since $\dim_{\mathbb{K}}\left(\frac{R}{I_{S_a}^{[\mathbf{n}]}}\right) = 1$ or 3 for $n = 1$ or 2, respectively, we assume that $n > 2$. Suppose $\mathcal{F}^2[n] = \cup_{i=1}^{n}\{(C_1,\ldots,C_i) \in \mathcal{F}^2_{i-1} : \cup_{j=1}^{i} C_j = [n]\}$. For $\mathcal{C} = (C_1,\ldots,C_i) \in \mathcal{F}^2[n]$, we write $\mu^2(\mathcal{C}) = \prod_{q=1}^{i}\left(\prod_{j\in C_q\setminus C_{q-1}} \mu^2_{j,C_q}\right)$. In view of the first formula in Proposition 4.3.3, we have

$$\tilde{\alpha}_n = \dim_{\mathbb{K}}\left(\frac{R}{I_{S_2}^{[\mathbf{n}]}}\right) = \sum_{\mathcal{C}\in\mathcal{F}^2[n]} (-1)^{n-\ell(\mathcal{C})-1}\mu^2(\mathcal{C}).$$

Now decompose $\mathcal{F}^2[n] = \mathcal{F}^2[n]' \coprod \mathcal{F}^2[n]''$, where $\mathcal{C} = (C_1,\ldots,C_i) \in \mathcal{F}^2[n]'$ if $|C_1| = 1$ and $\mathcal{C} \in \mathcal{F}^2[n]''$ if $|C_1| = 2$. Then $\tilde{\alpha}_n = \tilde{\alpha}'_n + \tilde{\alpha}''_n$, where

$$\tilde{\alpha}'_n = \sum_{\mathcal{C}\in\mathcal{F}^2[n]'} (-1)^{n-\ell(\mathcal{C})-1}\mu^2(\mathcal{C}) \quad \text{and} \quad \tilde{\alpha}''_n = \sum_{\mathcal{C}\in\mathcal{F}^2[n]''} (-1)^{n-\ell(\mathcal{C})-1}\mu^2(\mathcal{C}).$$

A chain $\mathcal{C} = (C_1,\ldots,C_i) \in \mathcal{F}^2[n]'$ is called a *Type-I, Type-II or Type-III* chain, if $(C_1, C_2) = (\{i\},\{i,n\})$ for $i < n$, $(C_1, C_2) = (\{n\},\{i,n\})$ for $i < n$ or $(C_1, C_2) = (\{n\},\{i,n-1\})$ for $i < n-1$, respectively. Now

$$\begin{aligned}
\tilde{\alpha}'_n &= \left[\sum_{\substack{\mathcal{C}\in\mathcal{F}^2[n]';\\ \text{Type-I}}} + \sum_{\substack{\mathcal{C}\in\mathcal{F}^2[n]';\\ \text{Type-II}}} + \sum_{\substack{\mathcal{C}\in\mathcal{F}^2[n]';\\ \text{Type-III}}}\right] (-1)^{n-\ell(\mathcal{C})-1}\ \mu^2(\mathcal{C}) \\
&= n\tilde{\alpha}'_{n-1} - \frac{n}{n-1}\tilde{\alpha}''_n + n\tilde{\alpha}''_{n-1} = n\tilde{\alpha}_{n-1} - \frac{n}{n-1}\tilde{\alpha}''_n.
\end{aligned}$$

Claim : $\tilde{\alpha}''_n = -\frac{(n-1)(n!)}{2}$.

For $1 \le t \le n-1$, consider saturated chains $\mathcal{C}^{(t)}$ in $\mathcal{F}^2[n]''$ of the form

$$\mathcal{C}^{(t)} : \{t,n\} \prec \{t,n-1\} \prec \cdots \prec \{t,t+1\} \prec \{t-1,t\} \prec \cdots \prec \{1,2\}.$$

Then $\mu^2(\mathcal{C}^{(t)}) = t((n-1)!)$. Any other chain in $\mathcal{F}^2[n]''$ is either of the form

$$\mathcal{C} : \{r, n\} \prec \cdots \prec \{r, r+1\} \prec \cdots \prec \{s-1, s\} \prec \{l, s-1\} \prec \{l, s-2\} \prec \cdots$$

or

$$\mathcal{C}' : \{r, n\} \prec \cdots \prec \{r, r+1\} \prec \cdots \prec \{s'-1, s'\} \prec \{l', s'-2\} \prec \cdots \prec \cdots, \qquad (\text{for } 3 \leq r \leq n-1),$$

where s (or s') is the largest integer such that $\{l, s-1\}$ covers $\{s-1, s\}$ in $\mathcal{C}$ (or $\{l', s'-1\}$ is not in $\mathcal{C}'$) for some $l < s-2$ (or $l' < s'-2$). Let $\tilde{\mathcal{C}} = \mathcal{C} \backslash \{\{l, s-1\}\}$ be the chain obtained from $\mathcal{C}$ by deleting $\{l, s-1\}$ and $\tilde{\mathcal{C}}' = \mathcal{C}' \cup \{\{l', s'-1\}\}$ be the chain obtained from $\mathcal{C}'$ on adjoining $\{l', s'-1\}$. We see that, $\mu^2(\mathcal{C}) = \mu^2(\tilde{\mathcal{C}})$ and $\mu^2(\mathcal{C}') = \mu^2(\tilde{\mathcal{C}}')$. As length $\ell(\mathcal{C}) = \ell(\tilde{\mathcal{C}}) + 1$ and $\ell(\mathcal{C}') = \ell(\tilde{\mathcal{C}}') - 1$, the terms in $\tilde{\alpha}_n'' = \sum_{\mathcal{C} \in \mathcal{F}^2[n]''} (-1)^{n-\ell(\mathcal{C})-1} \mu^2(\mathcal{C})$ corresponding to chains $\mathcal{C} \in \mathcal{F}^2[n]''$ different from $\mathcal{C}^{(t)}$ cancel out. Thus

$$\tilde{\alpha}_n'' = \sum_{t=1}^{n-1} (-1)^{n-\ell(\mathcal{C}^{(t)})-1} \, \mu^2(\mathcal{C}^{(t)}) = \sum_{t=1}^{n-1} (-1)^{n-(n-2)-1} \, t((n-1)!) = -\frac{(n-1)(n!)}{2}.$$

Now $\tilde{\alpha}_n = \tilde{\alpha}_n' + \tilde{\alpha}_n'' = n\tilde{\alpha}_{n-1} - \frac{n}{n-1}\tilde{\alpha}_n'' + \tilde{\alpha}_n'' = n\tilde{\alpha}_{n-1} + \frac{n!}{2}$. On solving this recurrence, we get $\tilde{\alpha}_n = \frac{(n+1)!}{2}$, as desired.

(iii) Let $a = 3$ and assume that $n > 2$. Proceeding as in part (ii), we write

$$\dim_{\mathbb{K}} \left(\frac{R}{I_{S_3}^{[\mathbf{n}]}} \right) = \sum_{\mathcal{C} \in \mathcal{F}^3[n]} (-1)^{n-\ell(\mathcal{C})-1} \mu^3(\mathcal{C}),$$

where $\mathcal{F}^3[n]$ is the collection of all chains $\bar{\mathcal{C}} = (C_1, \ldots, C_i)$ in $\mathcal{F}_{i-1}^3$ (for some i) with $\cup_{j=1}^i C_j = [n]$ and $\mu^3(\bar{\mathcal{C}}) = \prod_{q=1}^i \left(\prod_{j \in C_q \backslash C_{q-1}} \mu_{j, C_q}^3 \right)$. For $1 \leq t \leq n-1$, let $\bar{\mathcal{C}}^{(t)}$ be the chain in $\mathcal{F}^3[n]$ of the form

$$\bar{\mathcal{C}}^{(t)} : \{t\} \prec \{t, t+1\} \prec \cdots \prec \{t, n-1\} \prec \{t, n\} \prec \{t-1, n\} \prec \cdots \prec \{1, n\}$$

and $\bar{\mathcal{C}}^{(t)} \setminus \{\{t\}\}$ is the chain obtained from $\bar{\mathcal{C}}^{(t)}$ by deleting the first element $\{t\}$. Now $\mu^3(\bar{\mathcal{C}}^{(t)}) = n!$ and $\mu^3(\bar{\mathcal{C}}^{(t)} \setminus \{\{t\}\}) = t((n-1)!)$. There is one more chain $\bar{\mathcal{C}} : \{n\} \prec \{n-1, n\} \prec \ldots \prec \{1, n\}$ in $\mathcal{F}^3[n]$, with $\mu^3(\bar{\mathcal{C}}) = n!$. As in part (ii), it can be shown that the terms corresponding to remaining chains cancel out. Thus

$$\dim_{\mathbb{K}} \left(\frac{R}{I_{S_3}^{[\mathbf{n}]}} \right) = n(n!) - (1 + 2 + \cdots + (n-1))((n-1)!) = \frac{(n+1)!}{2}.$$

□

Theorem 4.3.4 shows that the integer sequence $\left\{ \dim_{\mathbb{K}} \left(\frac{R}{I_{S_a}^{[\mathbf{n}]}} \right) = \frac{(n+1)!}{2} \right\}_{n=1}^{\infty}$ for $1 \leq a \leq 3$ is the integer sequence (A001710) in OEIS [48]. As $|\mathrm{PF}_n^a| = \frac{(n+1)!}{2}$, it is expected that the set PF_n^a could be easily enumerated. Let $\mathbf{p} \in \mathrm{PF}_n^1$. Then $p_t \leq t$; for all t and $p_i = i$ implies that $p_j = 0$ for some $j \in [i+1, n]$. We count $\mathbf{p} \in \mathrm{PF}_n^1$ according to the value s of the largest $t \in [n]$ with $p_t = t$. If $p_t < t$; for all $t \in [n]$, then we take $s = 0$. As $p_n < n$, we have $0 \leq s \leq n-1$. For $s = 0$, any $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbf{N}^n$ such that $p_t < t$; for all t is a parking function and number of such $\mathbf{p} \in \mathrm{PF}_n^1$ is precisely $\prod_{t=1}^{n}(t) = n!$. Now let $s \geq 1$. Any sequence $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{N}^n$ satisfying the following conditions

$$p_t \leq t \text{ for all } t < s, \;\; p_s = s, \;\; \text{and } p_j < j \text{ for all } j > s, \text{with at least one } p_j = 0, \tag{4.7}$$

is always a parking function. The number of $\mathbf{p}$ satisfying conditions (4.7) is

$$\prod_{t=1}^{s-1}(t+1) \left[\prod_{j=s+1}^{n} j - \prod_{j'=s+1}^{n} (j'-1) \right] = (n-s)((n-1)!).$$

This shows that $|\mathrm{PF}_n^1| = \sum_{s=0}^{n-1}(n-s)((n-1)!) = \frac{(n+1)!}{2}$. Similarly, PF_n^a for $a = 2, 3$ can also be enumerated. However, it is still an interesting problem to construct an (explicit) bijection $\phi : \mathrm{PF}_n^a \longrightarrow F_{n+1}(21)$, where $F_{n+1}(21)$ is the set of rooted-labeled increasing forests on $[n+1]$.

Case-2 : To monomial ideals $I_{T_1}^{[\mathbf{n}]}$ and $I_{T_2}^{[\mathbf{n}]}$, we associate finite posets $\Sigma_n(T_1)$ and $\Sigma_n(T_2)$ respectively, as below.

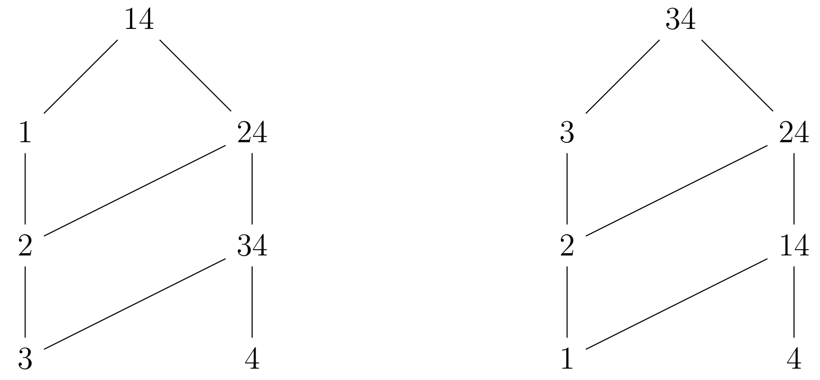
(i) Let $\Sigma_n(T_1) = \{\{\ell\}, \{i,n\} : 1 \leq \ell \leq n-1;\ 1 \leq i \leq n\}$, where $\{n,n\} = \{n\}$. We define a poset structure on $\Sigma_n(T_1)$ by describing cover relations. For $\ell, \ell' \in [n-1]$ and $i, i' \in [n]$, $\{\ell\}$ covers $\{\ell'\}$, if $\ell' = \ell + 1$. Also, $\{i,n\}$ covers $\{\ell'\}$ (or $\{i',n\}$) if $i = \ell'$ (respectively, $i' = i+1$). The monomial labels $\omega_{\{\ell\}} = x_\ell^{\ell+1}$, $\omega_{\{n\}} = x_n^n$ and $\omega_{\{i,n\}} = x_i^i x_n^i$ for $\ell, i \in [n-1]$. Set $\hat{\mu}_{j,C}^1$ for $C \in \Sigma_n(T_1)$ so that $\omega_C = \prod_{j \in C} x_j^{\hat{\mu}_{j,C}^1}$.

(ii) Let $\Sigma_n(T_2) = \Sigma_n(T_1)$. But the poset structure on $\Sigma_n(T_2)$ is obtained by interchanging $\{i\}$ with $\{n-i\}$ (and also, $\{i,n\}$ with $\{n-i,n\}$)(for $1 \leq i < n$) in the poset $\Sigma_n(T_1)$. The cover relations of the poset $\Sigma_n(T_2)$ are given as follows. For $\ell, \ell', i, i' \in [n-1]$, $\{\ell\}$ covers $\{\ell'\}$, if $\ell' = \ell - 1$ and $\{i,n\}$ covers $\{\ell'\}$ (or $\{i',n\}$) if $i = \ell'$ (respectively, $i' = i-1$). In addition, $\{1,n\}$ covers $\{n\}$. The monomial labels $\omega_{\{\ell\}} = x_\ell^{n-\ell+1}$, $\omega_{\{n\}} = x_n^n$ and $\omega_{\{i,n\}} = x_i^{n-i} x_n^{n-i}$ for $\ell, i \in [n-1]$. Set $\hat{\mu}_{j,C}^2$ for $C \in \Sigma_n(T_1)$ so that $\omega_C = \prod_{j \in C} x_j^{\hat{\mu}_{j,C}^2}$.

The Hasse diagram of $\Sigma_4(T_1)$ and $\Sigma_4(T_2)$ are given in Figure 4.3.

Proposition 4.3.5. (i). *The ideals $I_{T_1}^{[\mathbf{n}]}$ and $I_{T_2}^{[\mathbf{n}]}$ are order monomial ideals.* (ii). *The free complex $\mathbb{F}_*(\Delta(\Sigma_n(T_b)))$ is the minimal cellular resolution of $I_{S_a}^{[\mathbf{n}]}$ supported on the order complex $\Delta(\Sigma_n(T_b))$ for $1 \leq b \leq 2$. Thus the r^{th} Betti number $\beta_r(I_{T_b}^{[\mathbf{n}]})$ is given by*

$$\beta_r(I_{T_b}^{[\mathbf{n}]}) = \binom{n}{r+1} + (r+1)\binom{n-1}{r+1} + r\binom{n-1}{r}, \quad (0 \leq r \leq n-1).$$



$\Sigma_4(T_1)$ $\Sigma_4(T_2)$

Figure 4.3: Hasse diagram for $\Sigma_4(T_1)$ and $\Sigma_4(T_2)$.

Proof. From the definitions of the poset $\Sigma_n(T_b)$, it is clear that the ideal $I_{T_b}^{[\mathbf{n}]}$ is an order monomial ideal. Further, the cellular resolution $\mathbb{F}_*(\Delta(\Sigma_n(T_b)))$ is the minimal resolution of $I_{S_a}^{[\mathbf{n}]}$ supported on the order complex $\Delta(\Sigma_n(T_b))$ because monomial label on any face of $\Delta(\Sigma_n(T_b))$ is different from the monomial label on subfaces. Thus the r^{th} Betti number $\beta_r(I_{T_b}^{[\mathbf{n}]})$ equals the number of (strict) chains of length r in the poset $\Sigma_n(T_b)$. Since $\Sigma_n(T_2)$ is obtained from $\Sigma_n(T_1)$ by changing i to $n-i$ for $i \in [n]$, number of chains of length r in both the posets are same. We count chains of length r in $\Sigma_n(T_1)$ for $0 \leq r \leq n-1$. Consider a (strict) chain

$$\mathcal{C} : C_1 \prec C_2 \prec \ldots \prec C_s \prec C_{s+1} \prec \ldots \prec C_{r+1}.$$

If all C_j are of the form $\{t_j, n\}$ for $t_j \in [n]$, then the chain $\mathcal{C}$ can be identified with an $(r+1)$-subset $\{t_1, \ldots, t_{r+1}\}$ of $[n]$. Thus the number of such chains is $\binom{n}{r+1}$. If $C_s = \{t_s\}$ and $C_{s+1} = \{t_{s+1}, n\}$ for some s with $t_{s+1} < t_s$, then the chain $\mathcal{C}$ can be identified with an $(r+1)$-subset $\{t_1, \ldots, t_{r+1}\}$ of $[n-1]$ with a chosen element t_s. Any $j \in \{t_1, \ldots, t_{r+1}\}$ represents singleton $\{j\}$ if $j \geq t_s$, while it represents $\{j, n\}$ for $j < t_s$. The number of such chains is precisely $(r+1)\binom{n-1}{r+1}$. Now we count chains $\mathcal{C}$ with $C_s = \{t_s\}$ and $C_{s+1} = \{t_s, n\}$

(i.e., $t_s = t_{s+1}$). In this case, chain $\mathcal{C}$ can be identified with an r-subset $\{t_1, \ldots, t_s = t_{s+1}, \ldots, t_{r+1}\}$ of $[n-1]$ with a chosen element t_s. Thus the number of such chains is $r\,\binom{n-1}{r}$. Since any r-chain $\mathcal{C}$ in $\Sigma_n(T_1)$ is a chain of one of the three types, we get the desired result. $\square$

Consider the following subsets of PF_n of parking function $\mathbf{p} = (p_1, \ldots, p_n)$.

(i) $\widehat{\mathrm{PF}}_n^1 = \{\mathbf{p} \in \mathrm{PF}_n : p_t \leq t, \text{ for all } t \text{ and if } p_i = i, \text{ then } p_n < i\}$.

(ii) $\widehat{\mathrm{PF}}_n^2 = \{\mathbf{p} \in \mathrm{PF}_n : p_{n-t} \leq t, \text{ for all } t \text{ and if } p_{n-i} = i, \text{ then } p_n < i\}$.

In view of Lemma 4.2.8, $\mathbf{x}^{\mathbf{p}} \notin I_{T_b}^{[\mathbf{n}]}$ if and only if $\mathbf{p} \in \widehat{\mathrm{PF}}_n^b$ for $b = 1, 2$. Thus, $|\widehat{\mathrm{PF}}_n^b| = \dim_{\mathbb{K}}\left(\frac{R}{I_{T_b}^{[\mathbf{n}]}}\right)$. Also, the mapping $(p_1, p_2, \ldots, p_{n-1}, p_n) \mapsto (p_{n-1}, p_{n-2}, \ldots, p_1, p_n)$ induces a bijection between $\widehat{\mathrm{PF}}_n^1$ and $\widehat{\mathrm{PF}}_n^2$.

Theorem 4.3.6. *The number of standard monomials of $\frac{R}{I_{T_b}^{[\mathbf{n}]}}$ is given by*

$$|\widehat{\mathrm{PF}}_n^b| = \dim_{\mathbb{K}}\left(\frac{R}{I_{T_b}^{[\mathbf{n}]}}\right) = s(n+1, 2); \quad (b = 1, 2),$$

where $s(n+1, 2)$ is the (signless) Stirling number of the first kind.

Proof. We take $b = 1$. Proceeding as in Proposition 4.3.3, we get

$$\dim_{\mathbb{K}}\left(\frac{R}{I_{T_1}^{[\mathbf{n}]}}\right) = \sum_{\mathcal{C} \in \widehat{\mathcal{F}}^1[n]} (-1)^{n-\ell(\mathcal{C})-1}\, \widehat{\mu}^1(\mathcal{C}),$$

where $\widehat{\mathcal{F}}^1[n]$ is the collection of all chains $\mathcal{C} = (C_1, \ldots, C_i)$ in $\Sigma_n(T_1)$ such that $C_1 \cup \cdots \cup C_i = [n]$ and $\widehat{\mu}^1(\mathcal{C}) = \prod_{q=1}^{i}\left(\prod_{j \in C_q \setminus C_{q-1}} \widehat{\mu}^1_{j,C_q}\right)$. For $1 \leq t \leq n$, let $\widehat{\mathcal{C}}^{(n)} : \{n\} \prec \{n-1, n\} \prec \cdots \prec \{1, n\}$,

$$\widehat{\mathcal{C}}^{(t)} : \{n-1\} \prec \cdots \prec \{t\} \prec \{t, n\} \prec \{t-1, n\} \prec \cdots \prec \{1, n\}; \quad (1 \leq t \leq n-1)$$

and $\widehat{\mathcal{C}}'^{(t)}$ be the chain obtained from $\widehat{\mathcal{C}}^{(t)}$ by deleting $\{t, n\}$. For $t = n$, we have $\{n, n\} = \{n\}$. It is clear that $\widehat{\mathcal{F}}^1[n] = \{\widehat{\mathcal{C}}^{(t)}, \widehat{\mathcal{C}}'^{(t)} : 1 \leq t \leq n\}$. Also,

$\widehat{\mu}^1(\widehat{\mathcal{C}}^{(t)}) = n!$ and $\widehat{\mu}^1(\widehat{\mathcal{C}}'^{(t)}) = \frac{t-1}{t}(n!)$ for $1 \leq t \leq n$. As $\ell(\widehat{\mathcal{C}}^{(t)}) = \ell(\widehat{\mathcal{C}}'^{(t)}) + 1 = n-1$, we see that

$$\begin{aligned}\dim_{\mathbb{K}}\left(\frac{R}{I_{T_1}^{[\mathbf{n}]}}\right) &= \sum_{t=1}^{n}\left(\widehat{\mu}^1(\widehat{\mathcal{C}}^{(t)}) - \widehat{\mu}^1(\widehat{\mathcal{C}}'^{(t)})\right) = \sum_{t=1}^{n}\left(n! - \frac{t-1}{t}n!\right) \\ &= \sum_{t=1}^{n}\frac{n!}{t} = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)n! = s(n+1,2).\end{aligned}$$

$\square$

A nice formula $|\widehat{\mathrm{PF}}_n^1| = |\widehat{\mathrm{PF}}_n^2| = s(n+1,2)$, deserves a combinatorial proof. We count parking functions $\mathbf{p} = (p_1, \ldots, p_n)$ in $\widehat{\mathrm{PF}}_n^1$ according to the value of p_n. Clearly, $0 \leq p_n \leq n-1$. For any $0 \leq t \leq n-1$, we see that $p_n = t$ implies that $p_i < i$ for all $i \leq t$ and $p_j \leq j$ for $j > t$. Also, any $(p_1, \ldots, p_n)$ with $p_n = t$ and $p_i < i$ for all $i \leq t$, while $p_j \leq j$ for all $t < j \leq n-1$ is always a parking function of length n. Thus the number of $\mathbf{p} = (p_1, \ldots, p_n) \in \widehat{\mathrm{PF}}_n^1$ with $p_n = t$ is $\left(\prod_{i=1}^{t} i\right)\left(\prod_{j=t+1}^{n-1}(j+1)\right) = \frac{n!}{t+1}$. Hence, $|\widehat{\mathrm{PF}}_n^1| = \sum_{t=0}^{n-1}\frac{n!}{t+1}$.

By Theorem 4.3.6 we see that the integer sequence (A000254) in OEIS [48] is the integer sequence $\left\{\dim_{\mathbb{K}}\left(\frac{R}{I_{T_b}^{[\mathbf{n}]}}\right) = s(n+1,2)\right\}_{n=1}^{\infty}$ for $b = 1, 2$.

Case-3 : We finally consider the monomial ideal $I_U^{[\mathbf{n}]}$. The minimal generators $\prod_{j\in A} x_j^{\nu_{j,A}}$ of $I_U^{[\mathbf{n}]}$ are parametrized by the poset Σ_n. Again, it can be directly verified that the ideal $I_U^{[\mathbf{n}]}$ is an order monomial ideal and the cellular resolution $\mathbb{F}_*(\Delta(\Sigma_n))$ supported on the order complex $\Delta(\Sigma_n)$ is the minimal free resolution of $I_U^{[\mathbf{n}]}$. Thus the r^{th} Betti number $\beta_r(I_U^{[\mathbf{n}]}) = (r!)S(n+1, r+1)$ for $0 \leq r \leq n-1$.

Now we describe standard monomials of $\frac{R}{I_U^{[\mathbf{n}]}}$. Let $\overline{\mathrm{PF}}_n = \{\mathbf{p} \in \mathrm{PF}_n : \mathbf{x}^{\mathbf{p}} \notin I_U^{[\mathbf{n}]}\}$.

Lemma 4.3.7. *Let* $\mathbf{p} = (p_1, \ldots, p_n) \in \mathrm{PF}_n$. *Then* $\mathbf{p} \in \overline{\mathrm{PF}}_n$ *if and only if there exists a permutation* $\alpha \in \mathfrak{S}_n$ *such that* $p_{\alpha_i} < \nu_{\alpha_i, T_i}$ *for all* i, *where* $\alpha_i = \alpha(i)$, $T_1 = [n]$ *and* $T_j = [n] \setminus \{\alpha_1, \ldots, \alpha_{j-1}\}$ *for* $j \geq 2$. *Also,* $\nu_{j,T}$ *is in* *the Lemma 4.3.1.*

Proof. The proof can be obtained by proceeding along the same lines as [29, Theorem 4.3]. ☐

Proceeding as in Proposition 4.3.3, we get a combinatorial formula for the number of standard monomials of $\frac{R}{I_U^{[\mathbf{n}]}}$.

Proposition 4.3.8. *The number of standard monomials of* $\frac{R}{I_U^{[\mathbf{n}]}}$ *is given by*

$$|\overline{\mathrm{PF}}_n| = \dim_{\mathbb{K}}\left(\frac{R}{I_U^{[\mathbf{n}]}}\right) = \sum_{i=1}^{n}(-1)^{n-i} \sum_{\emptyset=C_0\subsetneq C_1\subsetneq\cdots\subsetneq C_i=[n]} \prod_{q=1}^{i}\left(\prod_{j\in C_q\setminus C_{q-1}} \nu_{j,C_q}\right),$$

where summation is carried over all strict chains $\emptyset = C_0 \subsetneq C_1 \subsetneq \cdots \subsetneq C_i = [n]$.

Neither using Proposition 4.3.8, nor by any combinatorial tricks, we could determine the $|\overline{\mathrm{PF}}_n| = \dim_{\mathbb{K}}\left(\frac{R}{I_U^{[\mathbf{n}]}}\right)$. Computations for smaller values of n suggest that $\left\{\dim_{\mathbb{K}}\left(\frac{R}{I_U^{[\mathbf{n}]}}\right)\right\}_{n=1}^{\infty}$ could be the combinatorially interesting integer sequence (A003319) in OEIS [48].

Chapter 5

Edge ideals of some circulant graphs

In this chapter we consider the edge ideals of three families of circulant graphs $C_n(1,2,\ldots,\widehat{j},\ldots,\lfloor\frac{n}{2}\rfloor)$, $C_{lm}(1,2,\ldots,\widehat{2l},\ldots,\widehat{3l},\ldots,\lfloor\frac{lm}{2}\rfloor)$ and $C_{lm}(1,2,\ldots,\widehat{l},\ldots,\widehat{2l},\ldots,\widehat{3l},\ldots,\lfloor\frac{lm}{2}\rfloor)$. We explicitly compute all the $\mathbb{N}$-graded Betti numbers as well as the Castelnuovo-Mumford regularity of these ideals. Other algebraic and combinatorial properties like regularity, projective dimension, induced matching number and when such graphs are well-covered, Cohen-Macaulay, sequentially Cohen-Macaulay, Buchsbaum and S_2 are also discussed. The results in this chapter are based on a joint work with S. Anand [2].

5.1 Circulant graph and its edge ideal

Let $n \geq 1$ be a positive integer and $S \subseteq \{1,2,\ldots,\lfloor\frac{n}{2}\rfloor\}$. The *circulant graph* $G = C_n(S)$ is a finite simple graph (i.e., a subgraph of the complete simple graph K_n) with vertex set $V(G) = \{0,1,\ldots,n-1\}$ and edge set $E(G) = \{\{i,j\} \mid |i-j|_n \in S\}$, where $|i|_n = \min\{|i|, n-|i|\}$ is the *circular distance* modulo n. For $S = \{a_1, a_2, \ldots, a_s\}$ we write $C_n(a_1, a_2, \ldots, a_s)$ instead of $C_n(\{a_1, a_2, \ldots, a_s\})$. Circulant graphs are examples of undirected Cayley graphs [18].

Example 5.1.1. *Let $S = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Then $C_n(S)$ is the complete simple graph K_n. Note that $C_n(1)$ is just the cycle. In fact, $C_n(i)$ is a cycle whenever* $\gcd(n, i) = 1$. *Circulant graphs are considered to be a generalization of cycles.*

Given a circulant graph $G = C_n(S)$ we can associate it with a square-free monomial ideal in the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$. More generally, let $G = (V(G), E(G))$ denote a finite simple graph with vertex set $V(G) = \{x_1, x_2, \ldots, x_n\}$ and edge set $E(G)$. By identifying the vertices of G with the indeterminates in the polynomial ring $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$, we can associate G to the quadratic square-free monomial ideal

$$I(G) = \langle x_i x_j \mid \{x_i, x_j\} \in E(G) \rangle \subseteq R,$$

called the *edge ideal* of G. Edge ideals were first introduced by Villarreal [56]. They are mainly studied to investigate relations between algebraic properties of the ideals and combinatorial properties of the corresponding graphs. The main focus is on describing invariants of $I(G)$ in terms of G. For the edge ideal $I(G)$ in $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$ there exists an $\mathbb{N}$-graded minimal free resolution

$$\mathcal{F} : 0 \to F_p \to F_{p-1} \to \cdots \to F_0 \to R/I(G) \to 0,$$

where $p \leq n$ (by Hilbert's syzygy theorem; see Section 2.1) and $F_i = \oplus_j R(-j)^{\beta_{i,j}}$. The numbers $\beta_{i,j}$ are the i^{th} $\mathbb{N}$-graded Betti numbers of $R/I(G)$ in degree j and we write $\beta_{i,j}(G)$ for $\beta_{i,j}(R/I(G))$. Also, the length p of the resolution is the *projective dimension* of $R/I(G)$ as an R-module and is denoted by $\mathrm{pd}(R/I(G))$ (we write $\mathrm{pd}(G)$ for $\mathrm{pd}(R/I(G))$), i.e.,

$$\mathrm{pd}(G) = \max\{i \mid \beta_{i,j}(G) \neq 0 \text{ for some } j\}.$$

Betti numbers and projective dimension are among the most important invariants in a graded minimal free resolution. Moreover, the Castelnuovo-Mumford regularity (or simply the regularity) of $R/I(G)$ is another important invariant encoded in the minimal free resolution of $R/I(G)$. Denoted by $\mathrm{reg}(R/I(G))$ (or simply $\mathrm{reg}(G)$), the *regularity* of $R/I(G)$ (see also Definition 2.1.5) is defined as

$$\mathrm{reg}(G) = \max\{j - i \mid \beta_{i,j}(G) \neq 0\}.$$

The edge ideal $I(G)$ is said to have a *linear resolution* if for all $i \geq 0$, $\beta_{i,j} = 0$ for all $j \neq i + 2$. Thus $I(G)$ has a linear resolution if and only if $\mathrm{reg}(I(G)) = 2$.

Let $G = (V(G), E(G))$ be a finite simple graph. The *complement* of G, denoted by G^c, is the graph $(V(G^c), E(G^c))$ where $V(G^c) = V(G)$ and $E(G^c) = \{\{x, y\} \mid \{x, y\} \notin E(G)\}$. The *neighborhood* of $x \in V(G)$ is the set $N_G(x) = \{y \mid \{x, y\} \in E(G)\}$. The closed neighborhood of x is $N_G[x] = N_G(x) \cup \{x\}$. The *degree* (or *valency*) of x is $\deg(x) = |N_G(x)|$. A graph $H = (V(H), E(H))$ is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For $W \subseteq V(G)$, the *induced subgraph* of G on the vertex set W, denoted by G_W, is the graph whose edge set consists of all the edges in G that have both endpoints in W (i.e., $E(G_W) = \{\{x, y\} \in E(G) \mid x, y \in W\}$). Let G and H be two simple graphs with disjoint set of vertices, i.e., $V(G) \cap V(H) = \emptyset$. The *join* of G and H, denoted by $G * H$, is the graph on the vertex set $V(G) \cup V(H)$ with edge set given by $E(G * H) = E(G) \cup E(H) \cup \{\{x, y\} \mid x \in V(G) \text{ and } y \in V(H)\}$. If $t \geq 2$ is an integer, then for a graph G, the

t-times join $\underbrace{G * \cdots * G}_{t\text{-times}}$ is denoted as $G^{*(t)}$.

Suppose G and H are two simple graphs on disjoint vertex sets and $G*H$ is their join. Mousivand [40] has given a formula to compute all the $\mathbb{N}$-graded Betti numbers of $G * H$ in terms of G and H.

Proposition 5.1.2. [40, Corollary 3.4] *Let G and H be two simple graphs with disjoint vertex sets having m and n vertices, respectively. Then the $\mathbb{N}$-graded Betti numbers $\beta_{i,d}(G * H)$ may be expressed as*

$$\begin{cases} \sum_{j=0}^{d-2} \left\{ \binom{n}{j} \beta_{i-j,d-j}(G) + \binom{m}{j} \beta_{i-j,d-j}(H) \right\} & \text{if} \quad d \neq i+1, \\ \sum_{j=0}^{d-2} \left\{ \binom{n}{j} \beta_{i-j,d-j}(G) + \binom{m}{j} \beta_{i-j,d-j}(H) \right\} + \sum_{j=1}^{d-1} \binom{m}{j}\binom{n}{d-j} & \text{if} \quad d = i+1. \end{cases}$$

This, in particular, determines the regularity of $G * H$.

Proposition 5.1.3. [40, Proposition 3.12] *Let G and H be two simple graphs on disjoint vertex sets with one of them having at least one edge. Then*

$$\operatorname{reg}(G * H) = \max\{\operatorname{reg}(G), \operatorname{reg}(H)\}.$$

A multipartite graph G is a simple graph such that $V(G) = \sqcup_{i=0}^{d} V_i$ and for each i there is no edge between any two vertices in V_i. If G and H are two discrete graphs, i.e., both G and H do not have any edge, then $G * H$ is a complete bipartite graph. The Betti numbers of a complete bipartite graph are known. In fact, Betti numbers of a complete multipartite graph and hence, its regularity is also known.

Theorem 5.1.4. [24, Theorem 5.3.8] *The $\mathbb{N}$-graded Betti numbers of the complete multipartite graph $K_{n_1,\ldots,n_t}$ are independent of the characteristic of* *the field* $\mathbb{K}$ *and may be written as*

$$\beta_{i,d}(K_{n_1,\ldots,n_t}) = \begin{cases} \sum_{l=2}^{t}(l-1)\sum_{\substack{\alpha_1+\cdots+\alpha_l=d,\\ j_1<\cdots<j_l,\\ \alpha_1,\ldots,\alpha_l\geq 1}} \binom{n_{j_1}}{\alpha_1}\cdots\binom{n_{j_l}}{\alpha_l} & \text{if} \quad d=i+1, \\ 0 & \text{if} \quad d\neq i+1. \end{cases}$$

Given a graph G, an *induced matching* of G is an induced subgraph consisting of pairwise disjoint edges. The maximum number of edges in an induced matching is called the *induced matching number* of G, and is denoted by $\nu(G)$. We say that a subset $W \subset V(G)$ is a *vertex cover* of G if $e \cap W \neq \emptyset$ for all edges $e \in E(G)$. The complement of a vertex cover is an *independent set.* A graph G is called *well-covered* if every minimal vertex cover (with respect to the partial order of inclusion) has the same cardinality. Via the duality between vertex covers and independent sets, being well-covered is equivalent to the property that every maximal independent set has the same cardinality. The cardinality of a largest independent set in G is denoted by $\alpha(G)$. The family of all independent sets of G is a simplicial complex on the vertex set $V(G)$, called the *independence complex* of G, and is denoted by Δ_G.

$$\Delta_G = \{W \subset V(G) \mid \text{W is an independent set of G}\}.$$

Note that $\dim \Delta_G = \alpha(G) - 1$. A graph is well-covered if and only if Δ_G is a pure simplicial complex (see Section 2.3.1).

Various algebraic properties of the graph G is defined in terms of its independence complex. Recall that, a simplicial complex Δ is called Cohen-Macaulay/sequentially Cohen-Macaulay/Buchsbaum or it satisfies Serre's condition S_2 if the Stanley-Reisner ring $\mathbb{K}[\Delta]$ has the corresponding property (cf. Section 2.3.1). We say a graph G is *Cohen-Macaulay/sequentially Cohen-Macaulay/Buchsbaum* or it satisfies *Serre's condition* S_2 if its independence complex Δ_G satisfies the corresponding property. Further recall that a simplicial complex Δ is called a Buchsbaum simplicial complex if $\mathrm{lk}_\Delta(x)$ is Cohen-Macaulay for all $x \in V(\Delta)$. Since $\mathrm{lk}_{\Delta_G}(x) = \Delta_{G\setminus N_G[x]}$, a graph G is Buchsbaum if $G \setminus N_G[x]$ is Cohen-Macaulay for all $x \in V(G)$. A graph G is said to be *vertex decomposable* (respectively, *shellable*) if Δ_G is vertex decomposable (respectively, shellable) (see Section 2.3.1). Since Δ_G is pure if and only if G is well-covered, we remark that in order for a graph G to be vertex decomposable/shellable/Cohen-Macaulay/Buchsbaum or to satisfy Serre's condition S_2, it must be a well-covered graph.

5.2 Betti numbers and regularity

In this section we study the following three families of circulant graphs.

- $C_n(1, 2, \ldots, \widehat{j}, \ldots, \lfloor \frac{n}{2} \rfloor)$,
- $C_{lm}(1, 2, \ldots, \widehat{2l}, \ldots, \widehat{3l}, \ldots, \lfloor \frac{lm}{2} \rfloor)$,
- $C_{lm}(1, 2, \ldots, \widehat{l}, \ldots, \widehat{2l}, \ldots, \widehat{3l}, \ldots, \lfloor \frac{lm}{2} \rfloor)$.

We compute all the $\mathbb{N}$-graded Betti numbers of edge ideals for these graphs. This, in turn, gives the Castelnuovo-Mumford regularity for these graphs. Various other algebraic and combinatorial properties are also determined in the next Section 5.3.

Let H_1 be the circulant graph $C_n(1, \ldots, \widehat{j}, \ldots, \lfloor \frac{n}{2} \rfloor)$, where $1 \le j \le \lfloor \frac{n}{2} \rfloor$. We begin with the observation that H_1 can be written as joins of complement of cycles (see Lemma 5.2.1). Then we compute the Betti numbers and regularity using Propositions 5.1.2 and 5.1.3, respectively.

Lemma 5.2.1. *Let $H_1 = C_n(1, \ldots, \widehat{j}, \ldots, \lfloor \frac{n}{2} \rfloor)$ and $d = \gcd(j, n)$. Then $H_1 = G_1^{*(d)}$,where G_1 is a graph on $\frac{n}{d}$ number of vertices with $G_1^c = C_{\frac{n}{d}}$, the cycle of length $\frac{n}{d}$ (if $\frac{n}{d} = 2$, then C_2 denotes the complete graph on 2 vertices).*

Proof. Suppose the vertices of H_1 are labeled as $\{0, 1, \ldots, n-1\}$. We partition the set $V(H_1)$ into d components $V_i = \{i, j+i, 2j+i, \ldots, (\frac{n}{d}-1)j+i\}$ for $0 \leq i < d$, where the indices of vertices are computed modulo n. Note that for each $r \neq s$, there is an edge from every vertex of $H_{1_{V_r}}$ to the vertices of $H_{1_{V_s}}$. In other words, $H_1 = H_{1_{V_0}} * \cdots * H_{1_{V_{d-1}}}$. Also, if we consider the complement graph $H^c_{1_{V_i}}$ on the vertex set V_i, then $H^c_{1_{V_i}}$ is a cycle of length $\frac{n}{d}$ for each i. Therefore, $H_{1_{V_i}} = G_1$ for $0 \leq i < d$, and this proves the statement. $\square$

Example 5.2.2. *The graph $C_{12}(1, \widehat{2}, 3, 4, 5, 6)$ is isomorphic to $C^c_6 * C^c_6$ in Figure 5.1.*

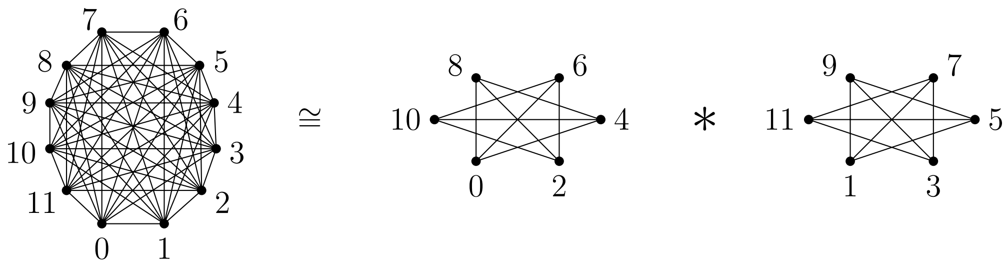


Figure 5.1: $C_{12}(1, \widehat{2}, 3, 4, 5, 6) \cong C^c_6 * C^c_6$

Let $\frac{n}{d} = k$. As $j \leq \frac{n}{2}$, we have $d < n$ and hence, $k \geq 2$. Recall that, $\operatorname{reg}(R/I(H_1))$ is also denoted as $\operatorname{reg}(H_1)$. By Proposition 5.1.3, in order to determine $\operatorname{reg}(H_1)$, it is enough to find $\operatorname{reg}(G_1)$, where G_1 is a graph on k vertices with $G^c_1 = C_k$, the cycle of length k. We compute $\operatorname{reg}(G_1)$ by using the Hochster's formula (2.1).

Theorem 5.2.3. *Let G_1 be a graph on k number of vertices such that G^c_1 is a cycle of length $k \geq 4$. Then* $\operatorname{reg}(R/I(G_1)) = 2$.

Proof. Let Δ_{G_1} be the independence complex of G_1 on the vertex set $\{0, 1, \ldots, k-1\}$. Then the Stanley-Reisner ideal $I_{\Delta_{G_1}} = I(G_1)$, where $I(G_1)$

is the edge ideal of G_1. For each $k \geq 4$, the facets of Δ_{G_1} are

$$\{0,1\},\{1,2\},\ldots,\{k-2,k-1\},\{k-1,0\}.$$

Recall that the Betti numbers $\beta_{i,j}(R/I(G_1))$ are also denoted by $\beta_{i,j}(G_1)$. By Hochster's formula (see (2.1)),

$$\beta_{i,r}(G_1) = \sum_{\substack{V\subseteq\{0,1,\ldots,k-1\}\\ |V|=r}} \dim_{\mathbb{K}} \widetilde{H}_{|V|-i-1}(\Delta[V];\mathbb{K}), \tag{5.1}$$

where $\Delta[V] = \{\tau \in \Delta_{G_1} \mid \tau \subseteq V\}$ is a subcomplex of Δ_{G_1}. Since Δ_{G_1} is a one dimensional simplicial complex, $\widetilde{H}_{|V|-i-1}$ is possibly non-zero for the cases $|V| = i+1$ and $|V| = i+2$, i.e., $r = i+1$ and $r = i+2$, respectively. Therefore, we have possibly non-zero Betti numbers $\beta_{i,i+1}$ for $i = 0,1,\ldots,k-1$, and $\beta_{i,i+2}$ for $i = 0,1,\ldots,k-2$.

Claim: $\beta_{i,i+2}(G_1) = \begin{cases} 1 & \text{if} \quad i = k-2, \\ 0 & \text{otherwise.} \end{cases}$

Proof of the claim: Note that by Hochster's formula

$$\beta_{i,i+2}(G_1) = \sum_{\substack{V\subseteq\{0,1,\ldots,k-1\}\\ |V|=i+2}} \dim_{\mathbb{K}} \widetilde{H}_1(\Delta[V];\mathbb{K}).$$

We have $\widetilde{H}_1(\Delta[V];\mathbb{K}) = 0$ if $|V| \neq k$ as in this case the connected components of $\Delta[V]$ are contractible. Now, if $|V| = k$, i.e., $i+2 = k$, then $\Delta[V] = \Delta_{G_1}$. The claim now follows by noting that Δ_{G_1} is a triangulation of the 1-dimensional sphere $\mathbb{S}^1$. Consequently, $\operatorname{reg}(R/I(G_1)) = 2$ for $k \geq 4$. $\square$

Corollary 5.2.4. *Let $H_1 = C_n(1,\ldots,\widehat{j},\ldots,\lfloor \frac{n}{2} \rfloor)$ and $d = \gcd(j,n)$. Then*

$$\operatorname{reg}\left(\frac{R}{I(H_1)}\right) = \begin{cases} 1 & \text{if} \quad n = 2j \quad \textit{or} \quad n = 3d, \\ 2 & \textit{otherwise.} \end{cases}$$

Proof. If $\frac{n}{d} = 2$ or 3, then $n = 2j$ or $n = 3d$, respectively and in these cases, by Lemma 5.2.1, the graph H_1 is a multipartite (d-partite) graph with each partition set having $\frac{n}{d}$ number of vertices. By Theorem 5.1.4, $\operatorname{reg}(R/I(H_1)) = 1$ if $n = 2j$ or $n = 3d$. If $k = \frac{n}{d} \geq 4$, then $\operatorname{reg}(R/I(H_1)) = 2$, by Lemma 5.2.1, Proposition 5.1.3 and Theorem 5.2.3. □

Remark 5. *Since* $\operatorname{reg}(I(H_1)) = \operatorname{reg}(R/I(H_1)) + 1$, *Corollary 5.2.4 and* [54, Theorem 3.3] *are essentially equivalent. Here we give a slightly different proof. Of course, the formula for regularity can be deduced once we give all the Betti numbers for* $R/I(H_1)$ *(see Theorem 5.2.8).*

A sequence of real numbers $\{a_1, a_2, \ldots, a_r\}$ is said to be *palindromic* if $a_i = a_{r-i+1}$ for $1 \leq i \leq r$. In the next proposition we show that the Betti numbers $\beta_{i,i+1}(G_1)$ are palindromic in the following sense:

Proposition 5.2.5. *For* $k \geq 4$, *we have* $\beta_{i,i+1}(G_1) = 0$ *if* $i \notin \{1, 2, \ldots, k-3\}$, *and* $\beta_{i,i+1}(G_1) = \beta_{k-i-2,k-i-1}(G_1)$ *for* $1 \leq i \leq k-3$.

Proof. We make use of the (Hochster's) formula (5.1) once again. Clearly, $\beta_{0,1}(G_1) = 0$. If $i \geq k-2$, then $|V| \geq k-1$ and hence, $\Delta[V]$ has only one connected component. Therefore,

$$\beta_{i,i+1}(G_1) = 0 \quad \text{for} \quad i \geq k-2,$$

and this proves the first part of the proposition. For the second part, by Hochster's formula,

$$\beta_{i,i+1}(G_1) = \sum_{\substack{V \subseteq \{0,1,\ldots,k-1\} \\ |V| = i+1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta[V]; \mathbb{K})$$

and,

$$\beta_{k-i-2,k-i-1}(G_1) = \sum_{\substack{V \subseteq \{0,1,\ldots,k-1\} \\ |V| = k-i-1}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta[V]; \mathbb{K}),$$

for $1 \leq i \leq k-3$. Note that if $|V| = i+1$, then $|V^c| = k-i-1$ and vice-versa. Since $\dim_{\mathbb{K}} \widetilde{H}_0(\Delta[V];\mathbb{K}) =$ (number of connected components of $\Delta[V]$) $- 1$, we just need to show that number of connected components of $\Delta[V]$ with $|V| = i+1$ is same as number of connected components of $\Delta[V^c]$, and this follows by a close inspection of the structure of $\Delta = \Delta_{G_1}$. □

Remark 6. *It can be checked that the rings $R/I(G_1)$ are Gorenstein of dimension 2. The total Betti numbers of $R/I(G_1)$ are palindromic (see* [43, Theorem 25.6]*). Here we have shown the palindromicity by using Hochster's formula.*

Our aim now is to explicitly calculate the Betti numbers of the circulant graph H_1 in Lemma 5.2.1. But first we state below the Betti numbers for the graph G_1. Note that all the Betti numbers of G_1 has already been calculated by Dochtermann [13] which we indicate in the proof.

Proposition 5.2.6. *For $k \geq 2$, the nonzero Betti numbers of $R/I(G_1)$ are given by*

$$\beta_{i,i+1}(G_1) = \begin{cases} \binom{k}{i+1}\frac{i(k-i-2)}{k-1} + 1 & \text{if } i = k-1, \\ \binom{k}{i+1}\frac{i(k-i-2)}{k-1} & \text{otherwise,} \end{cases} \tag{5.2}$$

and,

$$\beta_{i,i+2}(G_1) = \begin{cases} 1 & \text{if } k \geq 4 \text{ and } i = k-2, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

Proof. First observe that the ideal $I(G_1)$ is the zero ideal, when $k = 2$ or 3. Thus the above formulas are valid in this case. For $k \geq 4$ the Betti numbers $\beta_{i,i+2}$ are computed in the above claim (in proof of Theorem 5.2.3). When

$k \geq 5$ the formulas for Betti numbers $\beta_{i,i+1}$ are given in [13, Remark 7]. Using Macaulay2 [19] we can directly check $\beta_{i,i+1}$ for the case $k = 4$. $\square$

Our aim now is to compute all the Betti numbers of the joins $G_1^{*(d)}$. We will essentially use the formula of Betti numbers of G_1 given by Dochtermann and the formula to calculate Betti numbers of join of graphs in terms of the Betti numbers of the original graph given by Mousivand. This will enable us to find the Betti numbers of the graph H_1 since $H_1 = G_1^{*(d)}$, where G_1 is the complement of the cycle of length $k = \frac{n}{d}$.

Theorem 5.2.7. *Let G_1 be the complement of a cycle of length $k \geq 2$. For $d \geq 2$,*

$$\beta_{i,i+1}(G_1^{*(d)}) = \binom{dk}{i+1}\frac{i(dk-i-2)}{dk-1} + d\binom{(d-1)k}{i-k+1} \tag{5.4}$$

and,

$$\beta_{i,i+2}(G_1^{*(d)}) = \begin{cases} d\binom{(d-1)k}{i-k+2} & \text{if} \quad k \geq 4, \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

Proof. The following two identities can be verified directly.

$$\sum_{j=0}^{t}\binom{u}{j}\binom{v}{t-j} = \binom{u+v}{t} \tag{5.6}$$

and,

$$\binom{m}{i+1}\frac{i(m-i-2)}{m-1} = m\binom{m-1}{i} - m\binom{m-2}{i-1} - \binom{m}{i+1}. \tag{5.7}$$

We prove the formulas of Betti numbers by induction on d. Using the formula of Mousivand [40] (see Proposition 5.1.2) we have,

$$
\begin{aligned}
&\beta_{i,i+1}(G_1 * G_1)\\
&= 2\sum_{j=0}^{i-1}\binom{k}{j}\beta_{i-j,i-j+1}(G_1) + \sum_{j=1}^{i}\binom{k}{j}\binom{k}{i-j+1}\\
&= 2\sum_{j=0}^{i-1}\binom{k}{j}\binom{k}{i-j+1}\frac{(i-j)(k-i+j-2)}{k-1} + 2\binom{k}{i-k+1}\\
&\qquad\qquad + \sum_{j=1}^{i}\binom{k}{j}\binom{k}{i-j+1} \quad \text{(by (5.2))}\\
&= 2\sum_{j=0}^{i-1}\binom{k}{j}\left[k\binom{k-1}{i-j} - k\binom{k-2}{i-j-1} - \binom{k}{i-j+1}\right]\\
&\qquad\qquad + 2\binom{k}{i-k+1} + \sum_{j=1}^{i}\binom{k}{j}\binom{k}{i-j+1} \quad \text{(by (5.7)).}
\end{aligned}
$$

Using Equation (5.6), we get the following:

- $\sum_{j=0}^{i-1}\binom{k}{j}\binom{k-1}{i-j} = \binom{2k-1}{i} - \binom{k}{i}$.
- $\sum_{j=0}^{i-1}\binom{k}{j}\binom{k-2}{i-j-1} = \binom{2k-2}{i-1}$.
- $\sum_{j=0}^{i-1}\binom{k}{j}\binom{k}{i-j+1} = \binom{2k}{i+1} - k\binom{k}{i} - \binom{k}{i+1}$.
- $\sum_{j=1}^{i}\binom{k}{j}\binom{k}{i-j+1} = \binom{2k}{i+1} - 2\binom{k}{i+1}$.

Hence,

$$
\begin{aligned}
\beta_{i,i+1}(G_1 * G_1) &= 2k\binom{2k-1}{i} - 2k\binom{2k-2}{i-1} - \binom{2k}{i+1} + 2\binom{k}{i-k+1}\\
&= \binom{2k}{i+1}\frac{i(2k-i-2)}{2k-1} + 2\binom{k}{i-k+1} \qquad \text{(by (5.7)).}
\end{aligned}
$$

Therefore, Equation (5.4) is valid for $d = 2$. Assuming the formula is true for $d \geq 2$, we verify it for $d+1$. Once again, the formula of Mousivand [40] (see Proposition 5.1.2) gives

$$\begin{aligned}
&\beta_{i,i+1}\left(G_1^{*(d+1)}\right)\\
&=\sum_{j=0}^{i-1}\binom{k}{j}\beta_{i-j,i-j+1}\left(G_1^{*(d)}\right)+\sum_{j=0}^{i-1}\binom{dk}{j}\beta_{i-j,i-j+1}(G_1)+\sum_{j=1}^{i}\binom{dk}{j}\binom{k}{i-j+1}\\
&=\sum_{j=0}^{i-1}\binom{k}{j}\left[\binom{dk}{i-j+1}\frac{(i-j)(dk-i+j-2)}{dk-1}+d\binom{(d-1)k}{i-j-k+1}\right]\\
&\qquad+\sum_{j=0}^{i-1}\binom{dk}{j}\binom{k}{i-j+1}\frac{(i-j)(k-i+j-2)}{k-1}+\binom{dk}{i-k+1}\\
&\qquad\qquad+\sum_{j=1}^{i}\binom{dk}{j}\binom{k}{i-j+1}.
\end{aligned}$$

By using Equations (5.7) and (5.6) in a similar way as for the $d=2$ case, we see that

$$\beta_{i,i+1}\left(G_1^{*(d+1)}\right)=\binom{(d+1)k}{i+1}\frac{i((d+1)k-i-2)}{(d+1)k-1}+(d+1)\binom{dk}{i-k+1}.$$

This completes the induction. Equation (5.5) can also be verified by induction in a similar way.

□

Lemma 5.2.1, Corollary 5.2.4, Proposition 5.2.6, and Theorem 5.2.7 all together yield the following result.

Theorem 5.2.8. *Suppose $n\geq 5$ is an integer. Let H_1 be the circulant graph $C_n(1,\ldots,\widehat{j},\ldots,\lfloor\frac{n}{2}\rfloor)$ with $d=\gcd(j,n)$ and $k=\frac{n}{d}$.*

Case I: *For $d=1$,*

$$\beta_{i,i+1}(H_1)=\begin{cases}\binom{n}{i+1}\frac{i(n-i-2)}{n-1} & \textit{for}\quad 1\leq i\leq n-2,\\ 0 & \textit{otherwise.}\end{cases}$$

$$\beta_{i,i+2}(H_1)=\begin{cases}1 & \textit{if}\quad i=n-2,\\ 0 & \textit{otherwise.}\end{cases}$$

Case II: *For $d \geq 2$,*

$$\beta_{i,i+1}(H_1) = \binom{n}{i+1}\frac{i(n-i-2)}{n-1} + d\binom{n-k}{i-k+1} \text{ for } 1 \leq i \leq n-1,$$

$$\beta_{i,i+2}(H_1) = \begin{cases} d\binom{n-k}{i-k+2} & \text{if } \quad n \geq 4d \text{ and } 1 \leq i \leq n-2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have $G = G_1^{*(d)}$, where G_1 is a graph on k number of vertices with $G_1^c = C_k$. If $d = 1$, then the result follows from Proposition 5.2.6. Therefore, we can assume that $d \geq 2$. But in that case, the formulas are proved in Theorem 5.2.7. Note here that $k = \frac{n}{d}$. □

We now focus on the graph $H_2 = C_{lm}(1, \ldots, \{\widehat{il}\}_{i\geq 2}, \ldots, \lfloor \frac{lm}{2} \rfloor)$. which we simply write it as $C_{lm}(1, \ldots, \widehat{2l}, \ldots, \widehat{3l}, \ldots, \lfloor \frac{lm}{2} \rfloor)$ throughout the rest of the chapter. We first show that this circulant graph can be written as a join of cycles. More generally, we determine when a join (also called as product) of cycles is a circulant graph. Using this structure result we compute all the Betti numbers of the circulant graph H_2.

Lemma 5.2.9. *Let $m_1, \ldots, m_l \geq 3$ be integers. Then the join $C_{m_1} * \cdots * C_{m_l}$ is a circulant graph if and only if $m_i = m$ for some $m \geq 3$, and for all i. In addition, if this is the case, then $C_m^{*(l)} = C_{lm}(1, \ldots, \widehat{2l}, \ldots, \widehat{3l}, \ldots, \lfloor \frac{lm}{2} \rfloor)$.*

Proof. First notice that, if $m_i \neq m_j$ for some $i \neq j$, then $C_{m_1} * \cdots * C_{m_l}$ is not a regular graph (in regular graph all vertices have same degree) and hence, cannot be a circulant graph. Now we show that $C_m^{*(l)}$ is the circulant graph $H_2 = C_{lm}(1, \ldots, \widehat{2l}, \ldots, \widehat{3l}, \ldots, \lfloor \frac{lm}{2} \rfloor)$. We partition the set $V(H_2)$ into l components: $V_i = \{i, l+i, \ldots, (m-1)l+i\}$, for $0 \leq i < l$. Note that, the induced subgraphs $H_{2_{V_i}}$ are the cycles C_m for all i. Also, for all $i \neq j$, there is an edge between each vertex of V_i to the vertices of V_j, thus proving the statement. □

Example 5.2.10. *The circulant graph* $C_{18}(1,2,3,4,5,\widehat{6},7,8,\widehat{9})$ *is isomorphic to* $C_6 * C_6 * C_6$ *in Figure 5.2.*

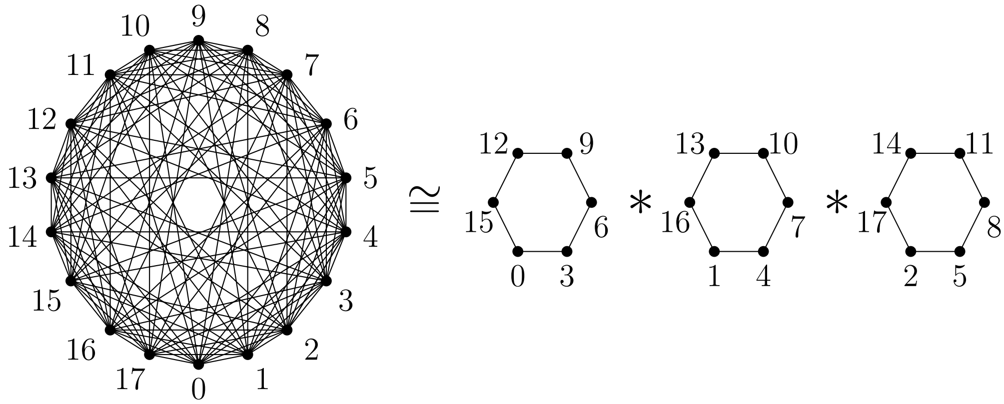


Figure 5.2: $C_{18}(1,2,3,4,5,\widehat{6},7,8,\widehat{9}) \cong C_6 * C_6 * C_6$.

We remark that the following lemma is entirely a collection of results by other authors which we indicate in the proof.

Lemma 5.2.11. *Let* $m \geq 5$ *be an integer. Then*

(i) $\operatorname{pd}(C_m) = \lfloor \frac{2m+1}{3} \rfloor$ *and* $\operatorname{reg}(C_m) = \lfloor \frac{m+1}{3} \rfloor$ *so that* $\operatorname{pd}(C_m) + \operatorname{reg}(C_m) = m$.

(ii) The initial Betti numbers

$$\beta_{i,i+1}(C_m) = \begin{cases} m & \text{if} \quad i = 1,2, \\ 0 & \text{if} \quad i > 2. \end{cases}$$

(iii) For $2 \leq r < \operatorname{reg}(C_m)$*, the nonzero Betti numbers*

$$\beta_{i,i+r}(C_m) = \frac{m}{m-2r}\binom{r}{i-r}\binom{m-2r}{r}.$$

(iv) Let $r = \operatorname{reg}(C_m)$ *and* $p = \operatorname{pd}(C_m)$.

(a) For $m \equiv 0 \pmod 3$ *the nonzero Betti numbers*

$$\beta_{i,i+r}(C_m) = \begin{cases} 3\binom{r}{i-r} & \textit{if} \quad i \neq p, \\ 3\binom{r}{i-r} - 1 & \textit{otherwise.} \end{cases}$$

(b) For $m \equiv 1 \pmod 3$ *the nonzero Betti numbers*

$$\beta_{i,i+r}(C_m) = \begin{cases} m\binom{r}{i-r} & \textit{if} \quad i \neq p, \\ m\binom{r}{i-r} + 1 & \textit{otherwise.} \end{cases}$$

(c) For $m \equiv 2 \pmod 3$ *the nonzero Betti numbers*

$$\beta_{i,i+r}(C_m) = \begin{cases} 1 & \textit{if} \quad i = p, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The proof follows from [24, Theorem 7.6.28]. See also [1, Corollary 4.4] for (i) and (iii). For (ii) see [1, Lemma 4.7]. The formulas in (iv) can be deduced by a close inspection of [1, Corollary 4.4]. For example, when $m \equiv 0 \pmod 3$ we can assume that $m = 3t$ for some $t \geq 1$ and then, $p = 2t$ and $r = t$. By [1, Corollary 4.4],

$$\begin{aligned} \beta_{i,i+r}(C_m) &= \frac{m}{m-2r}\binom{r}{i-r}\binom{m-2r}{r} \quad \text{(for } i < p\text{)} \\ &= 3\binom{r}{i-r}. \end{aligned}$$

Also, for $i = p$, we have $i + r = m$ and as $3\binom{r}{i-r} = 3$ in this case, we see that $\beta_{p,p+r}(C_m) = 3\binom{r}{i-r} - 1$. The formulas for $m \equiv 1 \pmod 3$ and $m \equiv 2 \pmod 3$ can be deduced in a similar way. $\square$

We proceed to compute the initial Betti numbers $\beta_{i,i+1}(H_2)$.

Theorem 5.2.12. *Let $m \geq 5$ and $l \geq 2$ be integers. Then for $1 \leq i \leq lm-1$,*

$$\beta_{i,i+1}(H_2) = lm\binom{(l-1)m+1}{i-1} + (l-1)\binom{lm}{i+1} - l\binom{(l-1)m}{i+1}.$$

Proof. Note that $C_{lm}(1,\ldots,\widehat{2l},\ldots,\widehat{3l},\ldots,\lfloor\frac{lm}{2}\rfloor) = C_m^{*(l)}$. The following identity can be verified directly:

$$\binom{u}{t} + \binom{u}{t-1} = \binom{u+1}{t}. \tag{5.8}$$

We prove the above expression of $\beta_{i,i+1}(H_2)$ by induction on l. For $l = 2$,

$$\begin{aligned}
&\beta_{i,i+1}(C_m * C_m)\\
&= 2\sum_{j=0}^{i-1}\binom{m}{j}\beta_{i-j,i-j+1}(C_m) + \sum_{j=1}^{i}\binom{m}{j}\binom{m}{i-j+1} && \text{(by Proposition 5.1.2)}\\
&= 2m\left[\binom{m}{i-1} + \binom{m}{i-2}\right] + \sum_{j=0}^{i+1}\binom{m}{j}\binom{m}{i+1-j} - 2\binom{m}{i+1} && \text{(by Lemma 5.2.11 (ii))}\\
&= 2m\binom{m+1}{i-1} + \binom{2m}{i+1} - 2\binom{m}{i+1} && \text{(by (5.8) and (5.6))}.
\end{aligned}$$

Therefore, the statement is true for $l = 2$. Assuming it is true for $l \geq 2$, we prove it for $l+1$. Now the formula of Mousivand [40] (see Proposition 5.1.2) yields

$$\begin{aligned}
&\beta_{i,i+1}(C_m^{*(l+1)})\\
&= \sum_{j=0}^{i-1}\left[\binom{m}{j}\beta_{i-j,i-j+1}(C_m^{*(l)}) + \binom{lm}{j}\beta_{i-j,i-j+1}(C_m)\right] + \sum_{j=1}^{i}\binom{lm}{j}\binom{m}{i-j+1}\\
&= \sum_{j=0}^{i-1}\binom{m}{j}\left[lm\binom{(l-1)m+1}{i-j-1} + (l-1)\binom{lm}{i-j+1} - l\binom{(l-1)m}{i-j+1}\right]\\
&\qquad + m\left[\binom{lm}{i-1} + \binom{lm}{i-2}\right] + \sum_{j=0}^{i+1}\binom{lm}{j}\binom{m}{i-j+1}\\
&\qquad - \binom{m}{i+1} - \binom{lm}{i+1} \qquad \text{(by Lemma 5.2.11(ii))}.
\end{aligned}$$

The induction is completed by applying the identities (5.8) and (5.6) in the same way as for the $l = 2$ case. $\square$

We next determine the nonlinear Betti numbers $\beta_{i,i+r}(H_2)$, where $2 \leq r < \operatorname{reg}(H_2)$. Note that, by Lemma 5.2.9 and Proposition 5.1.3, $\operatorname{reg}(H_2) = \operatorname{reg}(C_m)$.

Theorem 5.2.13. *Let $m \geq 5$ and $l \geq 2$ be integers. Then for $2 \leq r < \operatorname{reg}(H_2)$,*

$$\beta_{i,i+r}(H_2) = \frac{lm}{m-2r}\binom{m-2r}{r}\binom{(l-1)m+r}{i-r}.$$

Proof. We prove this again by induction on l. For $l = 2$, we use the formula of Mousivand [40] (see Proposition 5.1.2) to get

$$\begin{aligned}
\beta_{i,i+r}(C_m * C_m) &= 2\sum_{j=0}^{i+r-2}\binom{m}{j}\beta_{i-j,i-j+r}(C_m) \\
&= 2\sum_{j=0}^{i-r}\binom{m}{j}\frac{m}{m-2r}\binom{r}{i-j-r}\binom{m-2r}{r} \quad \text{(by Lemma 5.2.11, (iii))} \\
&= \frac{2m}{m-2r}\binom{m-2r}{r}\binom{m+r}{i-r} \quad \text{(by (5.6))}.
\end{aligned}$$

Assuming the formula is true for $l \geq 2$, we calculate it for $l+1$. Using the formula of Mousivand [40] (see Proposition 5.1.2) again we get

$$\begin{aligned}
&\beta_{i,i+r}(C_m^{*(l+1)}) \\
&= \sum_{j=0}^{i+r-2}\left[\binom{m}{j}\beta_{i-j,i-j+r}(C_m^{*(l)}) + \binom{lm}{j}\beta_{i-j,i-j+r}(C_m)\right] \\
&= \sum_{j=0}^{i+r-2}\binom{m}{j}\frac{lm}{m-2r}\binom{m-2r}{r}\binom{(l-1)m+r}{i-j-r} \\
&\qquad + \sum_{j=0}^{i+r-2}\binom{lm}{j}\frac{m}{m-2r}\binom{r}{i-j-r}\binom{m-2r}{r} \\
&= \frac{m}{m-2r}\binom{m-2r}{r}\left[l\sum_{j=0}^{i-r}\binom{m}{j}\binom{(l-1)m+r}{i-r-j} + \sum_{j=0}^{i-r}\binom{lm}{j}\binom{r}{i-j-r}\right] \\
&= \frac{(l+1)m}{m-2r}\binom{m-2r}{r}\binom{lm+r}{i-r} \quad \text{(by (5.6))}.
\end{aligned}$$

□

Finally, we would like to calculate $\beta_{i,i+r}(H_2)$, where $r = \operatorname{reg}(R/I(H_2))$. By Proposition 5.1.2 we see that $r = \operatorname{reg}(R/I(H_2)) = \operatorname{reg}(R/I(C_m))$ since $H_2 = C_m^{*(l)}$. There are three cases depending on the value of m modulo 3.

Theorem 5.2.14. *Let $m \geq 5$ be an integer and $r = \operatorname{reg}(R/I(H_2)) = \operatorname{reg}(R/I(C_m))$. Then for $l \geq 2$ and $1 \leq i \leq lm - 1$,*

$$\beta_{i,i+r}(H_2) = \begin{cases} 3l\binom{(l-1)m+r}{i-r} - l\binom{(l-1)m}{i-m+r} & \text{for} \quad m \equiv 0 \pmod 3, \\ lm\binom{(l-1)m+r}{i-r} + l\binom{(l-1)m}{i-m+r} & \text{for} \quad m \equiv 1 \pmod 3, \\ l\binom{(l-1)m}{i-m+r} & \text{for} \quad m \equiv 2 \pmod 3. \end{cases}$$

Proof. We prove this by induction on l. Let $p = \operatorname{pd}(C_m)$ and $r = \operatorname{reg}(C_m)$ so that $p + r = m$. Assume that $m \equiv 0 \pmod 3$, i.e., $m = 3k$ for some $k \geq 2$. First we check the formula for $l = 2$. We have by the formula of Mousivand [40] (see Proposition 5.1.2),

$$\begin{aligned} \beta_{i,i+r}(C_m * C_m) &= 2\sum_{j=0}^{i+r-2}\binom{m}{j}\beta_{i-j,i-j+r}(C_m) \\ &= 2\sum_{j=0}^{i+r-2}3\binom{m}{j}\binom{r}{i-j-r} - 2\binom{m}{i-p} \quad \text{(by Lemma 5.2.11 (iv) (a))} \\ &= 6\binom{m+r}{i-r} - 2\binom{m}{i-m+r} \quad \text{(by Lemma 5.2.11 (i)).} \end{aligned}$$

We now verify the formula for $l+1$ assuming it is true for $l \geq 2$.

$$\begin{aligned} &\beta_{i,i+r}(C_m^{*(l+1)}) \\ &= \sum_{j=0}^{i+r-2}\binom{m}{j}\beta_{i-j,i-j+r}(C_m^{*(l)}) + \sum_{j=0}^{i+r-2}\binom{lm}{j}\beta_{i-j,i-j+r}(C_m) \quad \text{(by Proposition 5.1.2)} \\ &= \sum_{j=0}^{i-r}\binom{m}{j}3l\binom{(l-1)m+r}{i-j-r} - \sum_{j=0}^{i-p}\binom{m}{j}l\binom{(l-1)m}{i-j-p} + \sum_{j=0}^{i-r}3\binom{lm}{j}\binom{r}{i-j-r} \\ &\qquad - \binom{lm}{i-p} \quad \text{(by Lemma 5.2.11, (iv)(a)).} \end{aligned}$$

The induction is completed by using Equation (5.6) and the fact that $p+r =$

m.

The formulas for the cases $m \equiv 1 \pmod 3$ and $m \equiv 2 \pmod 3$ can be verified similarly using Lemma 5.2.11 (iv) (b) and (c), respectively.

□

Remark 7. *The formulas for $\beta_{i,i+j}(C_m * C_m)$ are obtained in* [1], *which is the $l = 2$ case in Lemma 5.2.9. Our formulas in Theorem 5.2.14 for $l = 2$ are slightly different than theirs. The calculations done here are inspired by those in* [1].

Remark 8. *For calculating Betti numbers we have assumed that $m \geq 5$. However, for $m = 3$ we have $G = K_{3l}$, the complete graph on $3l$ vertices and hence it has a linear resolution. For $m = 4$, we see that $G = C_{4l}(1, 2, \ldots, 2l-1, \widehat{2l})$ and thus Betti numbers of $R/I(G)$ can be calculated using Theorem 5.2.8.*

We now consider the graph $H_3 = C_n(1, \ldots, \widehat{l}, \ldots, \widehat{2l}, \ldots, \widehat{3l}, \ldots, \lfloor \frac{n}{2} \rfloor)$, where $n = lm$ is any composite number. We show that H_3 is a multipartite graph. Moreover, we determine which multipartite graphs are circulant.

Lemma 5.2.15. *Let $l \geq 2$ and $m_1, \ldots, m_l \geq 2$ be integers. The complete multipartite graph $K_{m_1,\ldots,m_l}$ is a circulant graph if and only if $m_i = m$ for $1 \leq i \leq l$. In addition, if this is the case, then $K_{\underbrace{m, m, \ldots, m}_{l\text{-times}}} = C_{lm}(1, \ldots, \widehat{l}, \ldots, \widehat{2l}, \ldots, \lfloor \frac{lm}{2} \rfloor)$.*

Proof. If $m_i \neq m_j$ for some $i \neq j$, then $K_{m_1,\ldots,m_l}$ is not a regular graph and hence, cannot be a circulant graph. To prove the second statement, we partition the set $V(H_3)$ into l components: $V_i = \{i, l+i, \ldots, (m-1)l+i\}$, for $0 \leq i < l$. Note that, the induced subgraphs ${H_3}_{V_i}$ consists of only isolated vertices. Also, for all $i \neq j$ there is an edge between each vertex of V_i and V_j, thus proving the statement. □

Example 5.2.16. *The circulant graph* $C_{12}(1,\widehat{2},3,\widehat{4},5,\widehat{6})$ *is isomorphic to* $K_{6,6}$ *in Figure 5.3.*

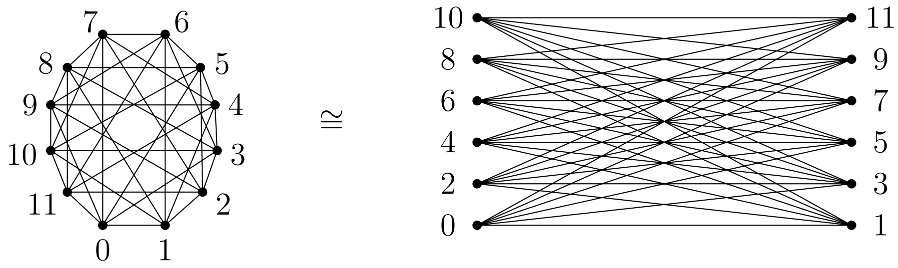


Figure 5.3: $C_{12}(1,\widehat{2},3,\widehat{4},5,\widehat{6})$

Theorem 5.2.17. *The* $\mathbb{N}$*-graded Betti numbers of* $R/I(H_3)$ *can be written as*

$$\beta_{i,i+1}(H_3)=\sum_{r=2}^{l}(r-1)\sum_{\substack{\alpha_1+\cdots+\alpha_r=i+1,\\ 1\le\alpha_1,\ldots,\alpha_r\le m}}\binom{l}{r}\binom{m}{\alpha_1}\cdots\binom{m}{\alpha_r},$$

and $\beta_{i,d}(H_3)=0$ *for* $d\neq i+1$.

Proof. By Lemma 5.2.15, the circulant graph H_3 is the complete multipartite graph $K_{\underbrace{m,m,\ldots,m}_{l\text{-times}}}$ and hence, Δ_{H_3} is a disjoint union of l number of simplices each of dimension m. The proof now follows by applying Theorem 5.1.4. □

Remark 9. *As we can write* $H_3=G_2^{*(l)}$, *where* G_2 *is a graph consisting of* m *number of isolated vertices, the result in Theorem 5.2.17 can also be deduced by applying the formula of Mousivand* [40] *(see Proposition 5.1.2).*

5.3 Other algebraic and combinatorial properties

In this section we determine when the graphs H_1, H_2 and H_3 from Section 5.2 are vertex decomposable, shellable, well-covered, Cohen-Macaulay, sequentially Cohen-Macaulay, Buchsbaum and S_2. We also calculate the induced matching number and projective dimension of these graphs. Moreover it is shown that if a graph G can be written as $G_1 * \cdots * G_d$, then the above mentioned properties of G can be described in terms of G_i's.

First, we state some known results in this direction.

Theorem 5.3.1. [15, Theorem 2.3] *Let Δ be a pure simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$.*

(i) The following implications hold for Δ :

$$\begin{aligned} \textit{vertex decomposable} \implies \textit{shellable} \implies \textit{Cohen-Macaulay} \\ \implies \textit{Buchsbaum.} \end{aligned}$$

(ii) If $\dim \Delta = 0$, then Δ is vertex decomposable (and thus, shellable, Cohen-Macaulay, and Buchsbaum).

(iii) If $\dim \Delta = 1$, then Δ is vertex decomposable/shellable/Cohen-Macaulay if and only if Δ is connected. If Δ is not connected, then Δ is Buchsbaum but not Cohen-Macaulay.

(iv) If $\dim \Delta \geq 2$ and Δ is Cohen-Macaulay, then Δ is connected.

Recall that for a graph G, the projective dimension of $R/I(G)$ is denoted by $\mathrm{pd}(G)$.

Theorem 5.3.2. [24, Theorem 4.2.6] *If G is a graph such that G^c is disconnected then*

$$\mathrm{pd}(G) = |V(G)| - 1.$$

In the following proposition we record a number of properties for the cycle graphs C_m. We remark that the following proposition is entirely a collection of results by other authors which we indicate in the proof.

Proposition 5.3.3. *Let $m \geq 3$ be an integer and C_m denote the cycle of length m. Then*

(i) *C_m is well-covered/Buchsbaum if and only if $m \leq 5$ or $m = 7$.*

(ii) *C_m is vertex decomposable/shellable/Cohen-Macaulay or sequentially Cohen-Macaulay if and only if $m \in \{3, 5\}$.*

(iii) *C_m satisfies Serre's condition S_2 if and only if $m \in \{3, 5, 7\}$.*

Proof. The statements regarding when C_m is well-covered, Buchsbaum or sequentially Cohen-Macaulay can be deduced from the proof of [1, Proposition 3.2]. Statements regarding Cohen-Macaulay and S_2 properties are the content of [57, Corollary 7.3.19] and [21, Proposition 1.6], respectively. The criteria for vertex decomposability and shellability can be deduced from [55, Theorem 3.4]. □

Let $G_1, \ldots, G_d$ are finite simple graphs on disjoint set of vertices for $d \geq 2$, and $G = G_1 * \cdots * G_d$. In the following two propositions we describe some properties of G in terms of G_i's.

Proposition 5.3.4. *Let $d \geq 2$ be an integer and $G_1, \ldots, G_d$ are finite simple graphs with disjoint vertex sets. If $G = G_1 * \cdots * G_d$, then*

(i) *the induced matching number*

$$\nu(G) = \begin{cases} \max_i\{\nu(G_i)\} & \text{if } \nu(G_i) \neq 0 \text{ for some } i, \\ 1 & \text{otherwise.} \end{cases}$$

(ii) *G is well-covered if and only if all G_j's are well-covered and for each $i \neq j$, $\alpha(G_i) = \alpha(G_j)$.*

(iii) *G is vertex decomposable/shellable/Cohen-Macaulay (or S_2) if and only if G_j's are complete graphs for all j and this is equivalent to* $\dim \Delta_{G_j} = 0$ *for all j, where Δ_{G_j} is the independence complex of the graph G_j.*

(iv) *G is sequentially Cohen-Macaulay if and only if G_t is sequentially Cohen-Macaulay for some $1 \leq t \leq d$ and G_j's are complete for all $j \neq t$.*

(v) *G is Buchsbaum if and only if each G_i is Buchsbaum for $1 \leq i \leq d$.*

Proof. We see that $\Delta_G = \sqcup_{i=1}^{d} \Delta_{G_i}$, where Δ_G is the independence complex of G.

(i) Note that $\nu(G_i) = 0$ if and only if G_i consists of isolated vertices. Thus, if $\nu(G_i) = 0$ for all i, then G is a complete multipartite graph. In that case, we have $\nu(G) = 1$. Therefore, we can assume that $\nu(G_i) \neq 0$ for some i. Since every induced matching of G is an induced subgraph of G, we see that for each i the maximal (with respect to inclusion) induced matchings of G_i are also maximal induced matchings of G. Hence, if $\nu(G) = 1$ then $\nu(G_i) \leq 1$ for all i and consequently, $\nu(G) = \max_i\{\nu(G_i)\}$. Therefore, assume that $\nu(G) \geq 2$. In that case, we see that every maximal induced matching of G is an induced matching of G_i for some i. The statement now follows.

(ii) Let G be a well-covered graph. Suppose $A \subseteq V(G_i)$ is a maximal independent set of G_i. Then observe that A is also a maximal independent set in G. Thus $|A| = \alpha(G)$. Hence each G_i is well-covered and for each $i \neq j$, $\alpha(G_i) = \alpha(G_j) = \alpha(G)$. The converse part follows easily by observing that if $A \subseteq V(G)$ is a maximal independent set of G, then $A \subseteq V(G_i)$ for some i.

(iii) First note that, for S_2 property the statement directly follows from Theorem 2.3.8 since $\mathrm{lk}_{\Delta_G}(\emptyset) = \Delta_G$. Now we consider the Cohen-Macaulay property. If G_i is not complete for some i, then $\dim \Delta_{G_i} \geq 1$ and consequently, $\dim \Delta_G \geq 1$. Since Cohen-Macaulay simplicial complexes of positive dimension are connected, we have that G is not Cohen-Macaulay. The converse follows from the fact that all 0-dimensional simplicial complexes are Cohen-Macaulay (see Theorem 5.3.1). The statement about vertex-decomposability and shellability follows from the fact that all 0-dimensional simplicial complexes are vertex decomposable/shellable and also from the well-known hierarchy of conditions:

$$\text{vertex decomposable} \implies \text{shellable} \implies \text{Cohen-Macaulay}.$$

(iv) Let G be a sequentially Cohen-Macaulay graph. Suppose G_r and G_s are not complete for some $r \neq s$. Then $\Delta_G^{[1]}$ is 1-dimensional and disconnected and hence, not Cohen-Macaulay, which is a contradiction. Therefore, we must have at most one G_j that is not complete. In that case, $\Delta_G^{[l]} = \Delta_{G_j}^{[l]}$ for all $l > 0$. Hence, G_j is sequentially Cohen-Macaulay. Converse part follows from the fact that $\Delta_G^{[l]} = \Delta_{G_j}^{[l]}$ for all $l > 0$.

(v) Let G be a Buchsbaum graph. Let $x \in V(G_i)$ for some i. Then $G \setminus N_G[x] = G_i \setminus N_{G_i}[x]$ and hence, $G_i \setminus N_{G_i}[x]$ is Cohen-Macaulay for all $x \in V(G_i)$. Thus G_i is Buchsbaum for each i. Conversely, take $x \in V(G)$, then $x \in V(G_i)$ for some i. Since $G \setminus N_G[x] = G_i \setminus N_{G_i}[x]$, we have that $G \setminus N_G[x]$ is Cohen-Macaulay. Hence G is Buchsbaum.

□

We remark that, for $d = 2$, some of the properties in the above proposition are proved in [1, Proposition 3.2]. Our proof is inspired by the proof of that proposition.

Proposition 5.3.5. *Let $t \geq 2$ and $n_1, \ldots, n_t \geq 2$ be integers. If G denotes the complete multipartite graph $K_{n_1,\ldots,n_t}$, then*

(i) $\operatorname{reg}(R/I(G)) = 1$, $\operatorname{pd}(R/I(G)) = \sum_{i=1}^{t} n_i - 1$ *and* $\nu(G) = 1$.

(ii) G is a Buchsbaum graph. Moreover, G is well-covered if and only if $n_i = n_j$ for all i and j.

(iii) G does not satisfy any of the following properties: vertex decomposability, shellability, Cohen-Macaulayness, sequentially Cohen-Macaulay property and Serre's condition S_2.

Proof. For (i), note that the statement about regularity follows from Theorem 5.1.4. Since G^c is disjoint union of complete graphs K_{n_i}, $\operatorname{pd}(R/I(G)) = \sum_{i=1}^{t} n_i - 1$ (see Theorem 5.3.2). Also, $\nu(G) = 1$ follows from the definition. As for (ii) and (iii), note that $G = G_1 * \cdots * G_t$, where G_i's are graphs consisting of n_i number of isolated vertices. Now the statements quickly follow from Proposition 5.3.4. $\square$

In the next theorem we calculate projective dimension and induced matching number of H_1. Further, we describe other combinatorial properties such as when H_1 is Cohen-Macaulay, well-covered, Buchsbaum etc.

Theorem 5.3.6. *Let $n \geq 4$ be an integer and $H_1 = C_n(1, \ldots, \widehat{j}, \ldots, \lfloor \frac{n}{2} \rfloor)$. Then*

(i) $\operatorname{pd}(H_1) = \begin{cases} n-2 & \textit{if} \quad \gcd(j,n) = 1, \\ n-1 & \textit{otherwise.} \end{cases}$

(ii) The induced matching number $\nu(H_1) = \begin{cases} 2 & \textit{when} \quad k = 4, \\ 1 & \textit{otherwise,} \end{cases}$

where $k = \frac{n}{\gcd(j,n)}$.

(iii) H_1 is well-covered as well as a Buchsbaum graph.

(iv) H_1 *is vertex decomposable/shellable/Cohen-Macaulay or sequentially Cohen-Macaulay or* S_2 *if and only if* $\gcd(j, n) = 1$.

Proof. Let $\gcd(j, n) = d$ and $k = \frac{n}{d} \geq 2$. Recall that by Lemma 5.2.1, $H_1 = G_1^{*(d)}$, where G_1 is a graph on k number of vertices with $G_1^c = C_k$, the cycle of length k. First consider the case $d = 1$. In that case, $n = k$ and $H_1 = G_1$. Let $V(G_1) = \{0, 1, \ldots, k-1\}$. Then the facets of the simplicial complex Δ_{G_1} are $\{0, 1\}, \{1, 2\}, \ldots, \{k-1, 0\}$.

(i) This follows from Theorem 5.2.8.

(ii) When $n = 4$, we have $\nu(H_1) = \nu(C_4^c) = 2$. For $n \geq 5$, recall that an induced matching is an induced subgraph of H_1. Hence the fact $\nu(H_1) = \nu(C_n^c) = 1$ follows from a direct inspection of the structure of C_n^c.

(iii) G_1 is well-covered as all maximal independent sets have cardinality 2. Also, for $x \in V(G_1)$, $G_1 \setminus N_{G_1}[x]$ is the complete graph on 2 vertices and hence, Cohen-Macaulay. Therefore, G_1 is Buchsbaum.

(iv) G_1 is vertex decomposable, shellable and Cohen-Macaulay as Δ_{G_1} is a pure 1-dimensional connected simplicial complex (see Theorem 5.3.1). Since Cohen-Macaulay simplicial complexes of dimension 1 are also sequentially Cohen-Macaulay, G_1 is sequentially Cohen-Macaulay. The S_2 property follows from Theorem 2.3.8.

We now consider the case $d \geq 2$. Then by Theorem 5.3.2, $\mathrm{pd}(H_1) = n-1$ (it can also be checked by Theorem 5.2.8). If $k = 2$ or 3, H_1 is a multipartite graph and hence $\nu(H_1) = 1$. For $k \geq 4$, the statement in (ii) is deduced by applying Proposition 5.3.4. Also by Proposition 5.3.4, H_1 is well-covered (resp. Buchsbaum) if and only if G_1 is well-covered (resp. Buchsbaum). Recall that G_1^c is a cycle of length k. If $k \geq 4$ then the statement in (iii) follows from the $d = 1$ case. For $k = 2$ and 3, G_1 consists of isolated vertices and hence G_1 is well-covered as well as Buchsbaum.

For H_1 to be vertex decomposable/shellable/Cohen-Macaulay or sequentially Cohen-Macaulay or S_2, G_1 needs to be a complete graph (by Proposition 5.3.4). But G_1 can never be complete and this completes the proof of the proposition. $\square$

Remark 10. *For S_2 and the Cohen-Macaulay properties the statements in Proposition 5.3.6 are proved in* [41, Theorem 4.1]*. Also, except the projective dimension, induced matching number, the S_2 and sequentially Cohen-Macaulay properties, the statements for all other properties have been proved in* [15, Theorem 4.2]*. Here we give an alternative proof using Proposition 5.3.4 and Lemma 5.2.1.*

In the next theorem we determine projective dimension and the induced matching number of H_2 as well as when the graph H_2 is well-covered, Buchsbaum, vertex-decomposable, shellable, Cohen-Macaulay, sequentially Cohen-Macaulay or S_2.

Theorem 5.3.7. *Let $l \geq 1$ and $m \geq 3$ be integers. Suppose $H_2 = C_{lm}(1, \ldots, \widehat{2l}, \ldots, \widehat{3l}, \ldots, \lfloor \frac{lm}{2} \rfloor)$. Then*

(i) $\mathrm{pd}(H_2) = \lfloor \frac{2m+1}{3} \rfloor$ *when $l = 1$, and* $\mathrm{pd}(H_2) = lm - 1$ *when $l \geq 2$. Also, the induced matching number $\nu(H_2) = \lfloor \frac{m}{3} \rfloor$.*

(ii) H_2 is well-covered/Buchsbaum if and only if $m \in \{3, 4, 5, 7\}$.

(iii) H_2 is vertex decomposable/shellable/Cohen-Macaulay or sequentially Cohen-Macaulay if and only if either $l = 1$ and $m \in \{3, 5\}$ or $l \geq 2$ and $m = 3$.

(iv) H_2 satisfies Serre's condition S_2 if and only if either $l = 1$ and $m \in \{3, 5, 7\}$ or $l \geq 2$ and $m = 3$.

Proof. By Lemma 5.2.9, the circulant graph $H_2 = C_m^{*(l)}$. Statement regarding the projective dimension follows from Theorem 5.3.2 and Lemma 5.2.11. For the induced matching number, note that if $l = 1$, then $\nu(H_2) = \nu(C_m) = \lfloor \frac{m}{3} \rfloor$. When $l \geq 2$, by Proposition 5.3.4, $\nu(H_2) = \nu(C_m) = \lfloor \frac{m}{3} \rfloor$. For the remaining statements we may subdivide the proof into two cases: $l = 1$ and $l \geq 2$. The case of $l = 1$ can be deduced from Proposition 5.3.3 and the $l \geq 2$ case is obtained by applying Proposition 5.3.4. □

The following theorem is a direct application of Proposition 5.3.5.

Theorem 5.3.8. *For the graph $H_3 = C_n(1, \ldots, \widehat{l}, \ldots, \widehat{2l}, \ldots, \widehat{3l}, \ldots, \lfloor \frac{n}{2} \rfloor)$,*

(i) $\operatorname{reg}(R/I(H_3)) = 1$, $\operatorname{pd}(R/I(H_3)) = n - 1$ *and* $\nu(H_3) = 1$.

(ii) H_3 is well-covered and Buchsbaum.

(iii) H_3 does not satisfy any of the following properties: vertex decomposability, shellability, Cohen-Macaulayness, sequentially Cohen-Macaulay property or Serre's condition S_2.

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